Acceleration in Special Relativity: Rindler Space time

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July 2021

*The diagrams have been prepared by C.K. Siddhartha

1 Introduction

Most of the new-learners of Special Relativity (SR) often think that SR deals only with inertial reference frames and they often believe that "acceleration" is a forbidden word in SR. So, whenever they see that something is accelerating they outrightly say that "it is beyond the purview of Special Relativity and one should invoke General Relativity". But this is actually wrong and Special Relativity can handle accelerated frames too. Here, in this article uniformly accelerated reference frames in SR and Born Rigid motion are considered and some important properties of hyperbolic motion are stated and proved ultimately leading to a curvillinear coordinate system called Rindler coordinates (named after the physicist Wolfgang Rindler). Alongside, Rindler Horizon is introduced, a common misconception about the horizon is removed by explicit arguments. We also have carried on some original investigations regarding analysis of twin paradox by considering the situation from the reference frame of the moving observer using Rindler coordinates. We then briefly discuss the relationship between the Rindler and Schwarzschild metric and what this implies for the event horizon of a black hole and the Rindler horizon for an accelerating observer.

1.1 A Note on convention

In this article, we shall be using the "mostly minuses" convention for Minkowski metric of flat space-time, i.e. the Minkowski metric tensor will be considered as

$$\eta_{\mu\nu} = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

A large part of the article shall deal with 1D motion, for which the metric tensor will be $\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$

2 Concept of an Instantaneously Co-moving Inertial Reference Frame (ICRF):

When a body is under an accelerated motion w.r.t. some inertial frame S, then there exists an (actually a host of) inertial reference frame at any instant such that the body is instantaneously at rest w.r.t that frame. This frame basically has the same velocity as that of the body at that instant. Note that this frame is inertial (as it moves at a constant velocity w.r.t S). This frame is called an "instantaneously co-moving inertial reference frame" which shall be abbreviated as ICRF in this article. Also note that the t' axis of this ICRF will be parallel to the tangent to the spacetime trajectory of the particle in the S frame.

3 A Few De nitions

A few definitions shall be made explicit at the beginning of the article itself.

4 Acceleration: From an inertial frame, a moving observer has a 4-velocity (which is also called proper 4-velocity). The derivative of 4-velocity w.r.t. the proper time (time measured by the moving observer) is called 4-acceleration.

Proper Acceleration: The 4-acceleration vector as seen by the ICRF is called proper acceleration.

4 Motion with a constant proper acceleration: Hyperbolic trajectories in space-time

Let us consider the 1-D case since firstly 1D case is enough to understand the main essence of Rindler motion and secondly the worldline can be easily sketched on a paper for this.
Let us consider a body of some rest mass be accelerated with a constant proper acceleration (proper acceleration means the 4-acceleration vector as seen by the ICRF) $\vec{A} = \ \vec{e_x}$ where $\vec{e_x}$ is a basis vector along x' axis of the ICRF at any spacetime point along the trajectory of the particle in Minkowski space of an inertial frame S (x, ct).

Let $\vec{U}$ be the 4-velocity vector of the particle.

It is known that

$$\vec{U}.\vec{U} = c^2 \tag{1}$$

Now, $\frac{d\vec{U}}{d\tau} = \vec{A}$

Note that τ is the proper time for the accelerated observer.

Hence,

$\frac{d}{d\tau}(\vec{U}.\vec{U}) = 0$ (as $\|\overrightarrow{U}\| = c^2$,see (1)) or,

$$\vec{U}.\vec{A} = 0 \tag{2}$$

As (2) is a Lorentz invariant scalar product hence it holds in the ICRF also and hence $\vec{U}$ and $\vec{A}$ are Minkowski orthogonal. Since $\vec{A} = \quad \vec{e_x}$, hence $\vec{U} \| \vec{e_t}$ and by virtue of (1), $\vec{U} = c\vec{e_t}$

Also note that

$$\vec{A}.\vec{A} = \quad ^2 \tag{3}$$

Now, from S (x, ct) frame,

$\overrightarrow{U} = U^t\vec{e_t} + U^x\vec{e_x}$

$\overrightarrow{A} = A^t\vec{e_t} + A^x\vec{e_x}$

Now using (1), (2), (3) we get,

$$(U^t)^2 \quad (U^x)^2 = c^2 \tag{4}$$

$$(U^t)(A^t) \quad (U^x)(A^x) = 0 \tag{5}$$

$$(A^t)^2 \quad (A^x)^2 = \quad ^2 \tag{6}$$

From 5,

$(U^t)^2(A^t)^2 = (U^x)^2(A^x)^2$

or, $((U^x)^2 + c^2)(A^t)^2 = (U^x)^2((A^t)^2 + \quad ^2)$ (using 4 and 6)

or,

$$c^2(A^t)^2 = \quad ^2(U^x)^2 \tag{7}$$

Similarly we can get,

$$^2(U^t)^2 = c^2(U^x)^2 \tag{8}$$

Now ignoring negative solutions of (7) and (8) on the basis of the fact that A^x, U^t, U^x should be positive (the body accelerates towards positive x) and the solutions should satisfy (5), we get -

$$A^t = (\frac{}{c})U^x \tag{9}$$

$$A^x = (\frac{}{c})U^t \tag{10}$$

Again,

$A^t = \frac{dU^t}{d\tau}$ and $A^x = \frac{dU^x}{d\tau}$

Hence, $U^x = (\frac{c}{})\frac{dU^t}{d\tau} = (\frac{c}{})^2\frac{d^2U^x}{d\tau^2}$

Thus, $U^x(\tau) = k_1 e^{\ \frac{}{c})\tau} + k_2 e^{\quad \frac{}{c})\tau}$ We suppose that $U^x(\tau = 0) = 0$ and hence $k_1 = \quad k_2$, which implies

$$U^x(\tau) = k_1(e^{\ \frac{}{c})\tau} \quad e^{\quad \frac{}{c})\tau}) \tag{11}$$

Again $(\frac{}{c})U^t = \frac{dU^x}{d\tau}$ and hence

$$U^t(\tau) = k_1(e^{\ \frac{}{c})\tau} + e^{\quad \frac{}{c})\tau}) \tag{12}$$

We apply (4) for the particular case of $\tau = 0$, we get $k_1 = \frac{c}{2}$
Putting that in (11) and (12) we get,

$$U^t = c.\cosh(\frac{}{c}\tau) \tag{13}$$

$$U^x = c.\sinh(\frac{}{c}\tau) \tag{14}$$

Hence,
$ct = \quad U^t d\tau = (\frac{c^2}{})\sinh((\frac{}{c})\tau) + c_1$
$x = \quad U^x d\tau = (\frac{c^2}{})\cosh((\frac{}{c})\tau) + c_2$
Let at t=0,we have $\tau = 0$ and we set the origin of S such that at t=0, we have $x = \frac{c^2}{}$. Hence, we have $c_1 = c_2 = 0$ and thus, the equation of the trajectory is

$$(x)^2 \quad (ct)^2 = (\frac{c^2}{})^2 \tag{15}$$

It is the equation of a hyperbola and hence the motion under constant proper acceleration is called hyperbolic motion.

5 Rindler Horizon: An Event Horizon for an accelerated observer

The asymptotes to the hyperbola (15) are $ct = \pm x$. Light from the spacetime points to the left of the asymptote $ct = x$ shall not reach the accelerated observer. Thus the asymptote $ct = x$ will form an event horizon called "Rindler Horizon".

5.1 Does the space beyond Rindler Horizon truly appear "black" to the accelerated observer ?

Most of the treatments on the topic of Rindler Horizon generally do not address to this question and the analogy of this event horizon with a black hole's event horizon results in formation of an idea in the minds of most people that as if the accelerated observer doesn't see anything beyond the Rindler Horizon and the point x=0 (according to S frame) is the "end of the observable universe" for the accelerated observer. But, it is not the case.

When the observer is at P_1 (refer to Figure-1), the space-time points from which light reaches the observer corresponding to the spatial locations x_A, x_B, x_C are P_A, P_B, P_C respectively. When the observer is at P_2, the spacetime points from which light reaches the observer corresponding to x_A, x_B, x_C are P_{A1}, P_{A2}, P_{A3}. Hence the observer can see beyond the origin (the Rindler Horizon), but the clocks at space locations x_A, x_B, x_C tick more and more slowly and any event happening at shaded space-time region can't be seen by the accelerated observer.

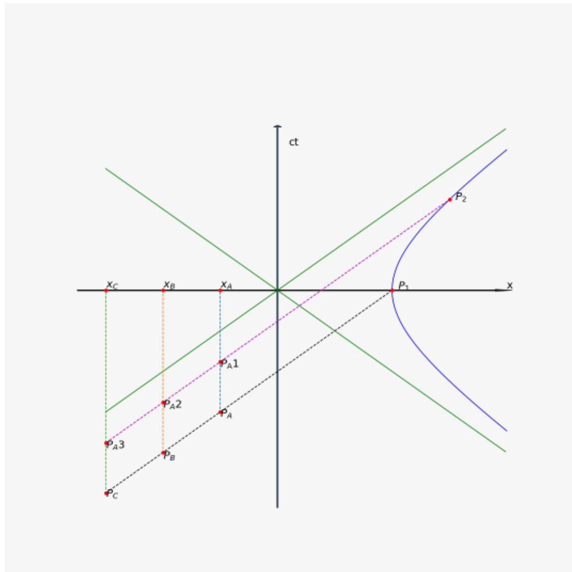

Figure 1:

6 What causes a Rindler motion ?

Rindler motion is the motion of a body under constant proper acceleration and not under constant 4-acceleration. We have seen in the derivation of Section 4 that the components of the 4-acceleration vector is changing from the viewpoint of S. The 4-acceleration vector from the ICRF is viewed as proper acceleration. Actually a Rindler motion is not caused by a constant 4-force, but by a constant ordinary force (a force which causes constant rate of change of spatial components of 4-momentum w.r.t the inertial frame S). The proof is as follows:

Let ordinary force be $\vec{F}$ and the (rest) mass of the particle be m.
Now, $\vec{F} = m\frac{d\vec{U^r}}{dt}$, where $\vec{U^r}$ refers to the spatial components of 4-velocity $\vec{U}$ and t is the time as measured by the inertial frame S Here in 1.D case,
$\vec{F} = m\frac{d\vec{U^x}}{dt} = m\frac{d\vec{U^x}}{d\tau}\frac{d\tau}{dt}$

$$F^x = \frac{m}{\gamma}A^x \tag{16}$$

Now, from S frame (using (14)), we get

$$A^x = \frac{dU^x}{d\tau} = \quad \cosh(\frac{}{c}\tau) \tag{17}$$

If v is the normal velocity of the particle at any instant w.r.t S, then,
$\gamma^2 = \frac{1}{1 \quad \frac{v^2}{c^2}} = \frac{1}{1 \quad \frac{U^x)^2}{\gamma^2 c^2}}$
or, $\gamma^2 = 1 + (\frac{U^x}{c})^2$

Hence using (14),

$$\gamma = \cosh(\frac{}{c}\tau) \tag{18}$$

Putting (17) and (18) in equation (16), we get:

$$F^x = \frac{m}{\gamma}A^x = m \quad = constant \tag{19}$$

Hence, from (19) we conclude that to produce a Rindler motion, we need a constant ordinary force acting on the particle.

7 A Few facts about a series of accelerated observers sharing the same Rindler Horizon

Let us take an inertial frame S (x, ct). Consider a whole series of observers, each moving with a constant proper acceleration throughout (note that acceleration of one observer is different from the other one) and at rest w.r.t. S when t=0. The observer moving with $\vec{A} = \quad \vec{e_x}$ ($\vec{e_x}$ is unit vector along x' axis of the ICRF) is at a distance of $x = \frac{c^2}{}$ at t=0 w.r.t S. Clearly different observers shall have different values of and hence different trajectories, but however the trajectory of each observer is described by the general equation of motion (15).

Fact 1: Consider any observer in this series. Let at some time t_1 (according to S), it has some velocity v and the observer is at $x = x_1$ (according to S). Let the ICRF of that observer at that spacetime point be $S'(x', ct')$. If we set $x' = 0$ and $t' = 0$ at $x = 0$ and $t = 0$; then (x_1, ct_1) will be transformed to $(\frac{c^2}{_1}, 0)$ where $\quad ^2_1$ is the norm of the 4-acceleration vector.

Proof: The space-time hyperbola will be $x^2 \quad (ct)^2 = (\frac{c^2}{_1})^2$. Consider the point $A(x_1, ct_1)$ on it.

The ct' axis of it's ICRF will be parallel to the tangent line to the hyperbola at this point A.

Tangent line equation:

$$\frac{ct \quad x_1}{x \quad x_1} = \frac{x}{ct_1} \tag{20}$$

Rearranging (20) and using the fact that point A lies on the spacetime hyperbola, the tangent line equation will be finally:

$$(x_1)x \quad (ct_1)ct = (\frac{c^2}{_1})^2 \tag{21}$$

The ct' axis of the ICRF has the same slope as (21) but passes through the origin of S.

Hence, it's equation is:

$$(x_1)x \quad (ct_1)ct = 0 \tag{22}$$

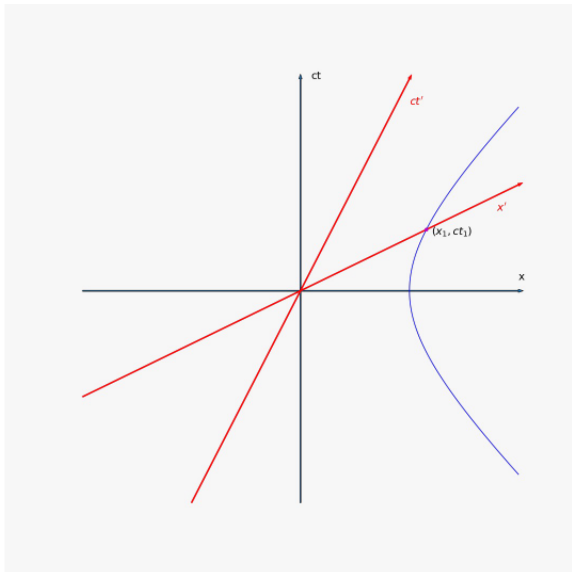

Figure 2:

The x' axis of the ICRF forms the same angle with $x = ct$ light cone as ct' axis does.
Thus, equation of the x' axis will be:

$$(ct_1)x \quad (x_1)ct = 0 \tag{23}$$

We see that the point A lies on the line described by (23).
Hence the space-time point $A(x_1, ct_1)$ lies on the x' axis.
Thus

$$t' = 0 \tag{24}$$

Now the space-time interval is Lorentz invariant. Hence:

$$(ct_1)^2 \quad (x_1)^2 = (ct')^2 \quad (x')^2 \tag{25}$$

Putting (24) in (25) we get

$$x_1 = \frac{c^2}{} \tag{26}$$

Noting (24) and (26) we conclude the proof.

Terminology: Any observer at any instant of time is at rest w.r.t it's ICRF. Now, there can be numerous such ICRF with different origins. The ICRF for which the origin of it coincides with that of the frame S shall be called (in this article) by an "aligned ICRF" or "AICRF" in short.

Fact 2: The distance between one observer and the spatial location of the Rindler horizon is fixed from the accelerated observer's point of view.
Proof: At any instant of time, consider the ICRF $S''(x'', ct'')$ for which the observer is at origin.
Now, this ICRF is at rest w.r.t the AICRF $S'(x', ct')$. If for this observer $= \sqrt{\vec{A}.\vec{A}}$, then, according to Fact-1, $x' = \frac{c^2}{}$.
Note that $x = 0$ is the Rindler Horizon location for the observer.
Thus from S'', the horizon is at $x'' = \quad \frac{c^2}{}$ at any time, hence at a constant distance.

Fact 3: According to each observer, all other observers are at rest.
Proof: Consider two observers A and B with $_1$ and $_2$ as magnitudes of proper accelerations respectively. Consider the AICRF of observer A at space-time point $E_1(x_0, ct_0)$.

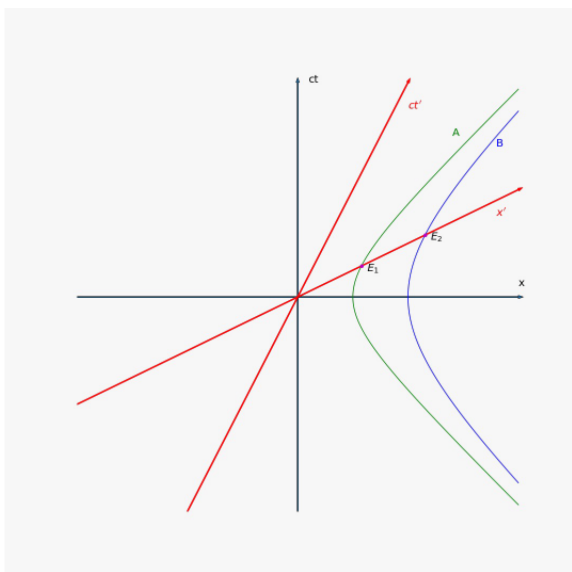

Figure 3:

We know that slope of the curve A at E_1 is $\frac{x_0}{ct_0}$.
Now, the equation of the x' axis or $t' = 0$ axis is (according to (23) of Fact-1):

$$(ct_0)x \quad (x_0)ct = 0 \tag{27}$$

This $t' = 0$ is the line of simultaneity of observer A Let x' axis intersects curve B at E_2. Let $E_2 = (x_2, ct_2)$ Now, as E_2 lies on $t' = 0$,

$$(x_2)ct_0 = (ct_2)x_0 \tag{28}$$

Now E_2 lies on the hyperbola $(x)^2 \quad (ct)^2 = (\frac{c^2}{\ _2})^2$, hence, the slope of tangent to the curve B at E_2 will be $\frac{x_2}{ct_2}$. Now, from (28),

$$\frac{x_2}{ct_2} = \frac{x_0}{ct_0} \tag{29}$$

Thus, along the line of simultaneity of observer A, the velocities of both observers A and B are equal to:

$$c\frac{dx}{d(ct)} = c^2\frac{t_0}{x_0} \tag{30}$$

Hence, w.r.t each other, A and B are at rest. Since they are any two general observers, so, all observers are at rest w.r.t. each other

8 Born Rigid Motion

If the series of observers as described in Section-7 form a perfectly rigid body, then the way as described in the previous section is the way it should move, different parts moving with different proper accelerations, but all of the observers sharing a common Rindler horizon. By Fact-3, this way of motion is the only way such that the proper distance between the observers remain constant although to the inertial observer in S, it appears that they are moving closer. However this inertial observer in S can still say that the body is not experiencing any stress as the "coming closer" of the points can be attributed to length contraction.
Note that this is also the resolution to Bell's spaceship paradox (we shall not discuss the paradox in detail here).

9 Rindler Coordinates

Rindler coordinate system is a curvillinear coordinate system used to describe non-inertial observers undergoing Born rigid motion (see Section-8).
From Section-4, if the particle has a constant proper acceleration , then, the equation of motion is:

$$ct = \frac{c^2}{\ }\sinh(\frac{\ }{c}\tau) \tag{31}$$

$$x = \frac{c^2}{\ }\cosh(\frac{\ }{c}\tau) \tag{32}$$

Now, if we consider a host of such accelerated observers moving with different values of , but undergoing a Born rigid motion. Then, along each observer, $\frac{c^2}{\ }$ is constant.
Hence, we characterize each hyperbola by some coordinate,

$$\frac{c^2}{\ } = \widetilde{x} \tag{33}$$

Now, by Fact-1 (Section-7), the line of simultaneity of one observer along any such hyperbola passes through the origin and hence if we construct radial lines from the origin into the Rindler wedge (the wedge described by the asymptotic lines $ct = \pm x$ in the positive x direction), then each of these lines will act as a line of simultaneity for all of the Rindler observers. Again, each of these lines has a constant hyperbolic angle.
Before going further, we choose a fixed value of proper acceleration as ${}_0$ and we express the other values of in terms of it as

$$= \lambda \ {}_0 \tag{34}$$

Hence using (34), we write the hyperbolic angles in (31) and (32) as:

$$\frac{}{c}\tau = \frac{{}_0}{c}\lambda\tau = \frac{\lambda c\tau}{(\frac{c^2}{{}_0})} \tag{35}$$

Now, as described before, since each radial line has the same value of hyperbolic angle, hence to remove λ (which varies from one hyperbola to another) we define

$$\lambda\tau = \widetilde{t} \tag{36}$$

We also let:

$$\frac{c^2}{{}_0} = D \tag{37}$$

Hence, we get:

$$ct = \widetilde{x}\sinh(\frac{c\widetilde{t}}{D}) \tag{38}$$

$$x = \widetilde{x}\cosh(\frac{c\widetilde{t}}{D}) \tag{39}$$

(38) and (39) describe the Rindler coordinate system $(\widetilde{x}, c\widetilde{t})$ and it serves as a conversion formula from Rindler coordinate system to inertial $S(x.ct)$ system. Analogously, the reverse transformation equation can be obtained from (38) and (39) and they will be:

$$c\widetilde{t} = D\,\mathrm{arctanh}(\frac{ct}{x}) \tag{40}$$

$$\widetilde{x} = \sqrt{(x)^2 \quad (ct)^2} \tag{41}$$

9.1 Rindler Metric

Note that the space-time is yet flat described by Minkowski metric if Cartesian coordinates are used. Hence,

$$ds^2 = c^2dt^2 \quad dx^2 \tag{42}$$

Using (38) and (39), we get:

$$ds^2 = \frac{\widetilde{x}^2}{D^2}c^2d\widetilde{t}^2 \quad d\widetilde{x}^2 \tag{43}$$

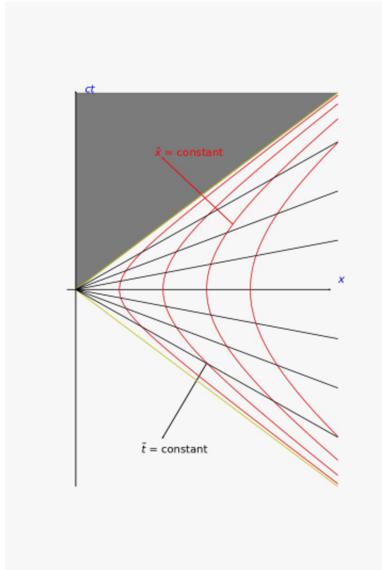

Figure 4: Rindler coordinates: The shaded portion corresponds to the space-time points that the Rindler observers cannot see in the $x \geq 0$ region. Note that this shaded portion extends to the left of ct-axis also, actually the entire region above $x = ct$ line is shaded

(43) describes the Rindler metric in (1+1)D In matrix form it can be written as :

$$\eta_{Rindler} = \begin{bmatrix} \frac{x^2}{D^2} & 0 \\ 0 & 1 \end{bmatrix} \tag{44}$$

10 The Twin Paradox

It involves two observes, A who stays on Earth and B who goes on a round trip journey close to the speed of light. Both carry physical clocks that are synchronized.When the twins are back together, they compare their clocks and they find that the clock of B shows less time to have elapsed than the clock of A, thus B is younger. This is due to time dilation.The argument continues that in special relativity, since everything is relative, all frames of reference are equally valid. Therefore the situation can also be viewed as B being stationary and A going on a round trip at speeds comparable to that of light. In this case clock of A ticks more slowly than B and when A returns, we find that A is younger than B. The argument says that this is a contradiction that disproves special relativity.

Figure-5 shows the trajectories of A and B through spacetime using reference frame of A i.e. the coordinates in which A remains stationary at the origin. In these coordinates A remain at x=0 and simply travels up the time axis from the

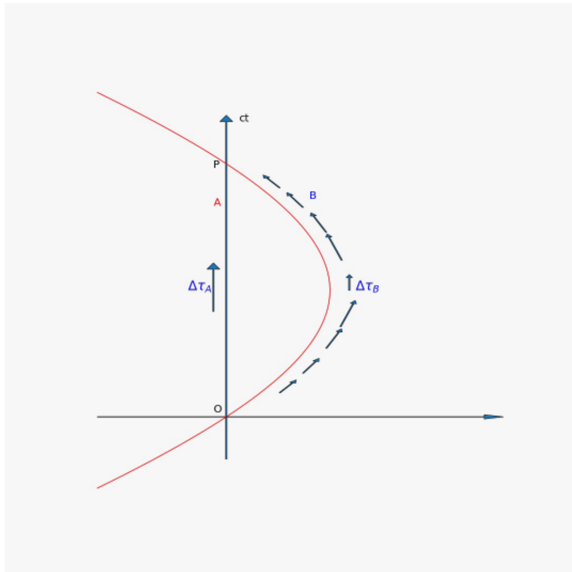

Figure 5:

starting point O to the finishing point P as shown by the black arrow. B travels along the x axis from point O with an initial velocity and a constnt negative proper acceleration, which causes it to stops and then reverse it's direction to meet A again at point P, as shown by the red arrows. So the red line shows trajectory of B through spacetime as measured using coordinates of A.

We know that proper time τ is related to the path length of observer's trajectory s as

$$s^2 = c^2 \quad \tau^2 \tag{45}$$

s in flat spacetime is calculated using the Minkowski metric and it tells us that if you move a distance dx along the x axis, dy along the y axis and dz along the z axis in a time dt, then the total distance you have moved in spacetime is given by the Minkowski metric:

$$ds^2 = c^2 dt^2 \quad dx^2 \quad dy^2 \quad dz^2 \tag{46}$$

Assume all the motion to be along the x axis,so dy=dz=0,and the metric simplifies to

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 \quad dx^2 \tag{47}$$

To calculate the length of the red curve we say velocity is defined by $v = dx/dt$

so $dx = vdt$, and if we take equation (47) and substitute for dx we end up with

$$d\tau = \sqrt{1 \quad \frac{v^2(t)}{c^2}}dt \tag{48}$$

So the elapsed time τ_B is given by the integral:

$$\tau_B = \int_O^P \sqrt{1 \quad \frac{v^2(t)}{c^2}}dt \tag{49}$$

where $v(t)$ is velocity of B as a function of time. Since v^2 is always positive that means the term inside the square root is always less than one. Hence for observer B,

$$1 \quad \frac{v^2(t)}{c^2} < 1 \tag{50}$$

And therefore the integral from t_O to t_P must be less than t_P- t_O. This means elapsed time for B, τ_B must be less than elapsed time τ_A for A i.e. when they meet again B has aged less than A has.

Now coming to the paradox,we can draw the space-time diagram using coordinates of B , i.e. the coordinates in which B is at rest, to give something like:

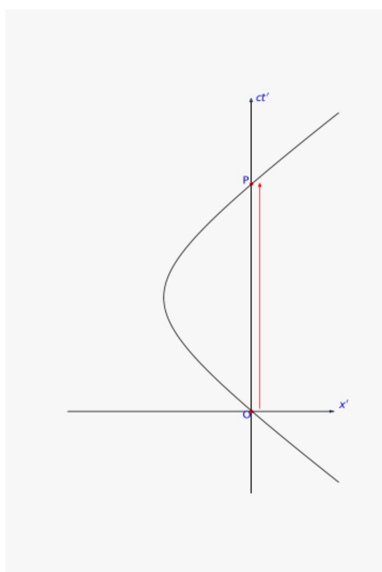

Figure 6:

In these coordinates, B remains stationary and its trajectory is given by the red line going straight up along the time axis, while the trajectory of A is shown by the black line which heads off in the negative x direction before returning. If we use the same argument as above we would conclude that A should have aged less than B, but we can't have both A and B aged less than each other.

This is the paradox.

11 Resolution to the Twin Paradox

Actually, the paradox can be practically resolved in only one statement: we can't do the same analysis from the reference frame of B as the first postulate of special relativity says "all inertial frames are equivalent" and surely the frame of B is a non-inertial one.
But, here we dig in deeper and using Rindler coordinates we shall analyze the situation from the reference frame B and show that even if analyzed from the perspective of the non-inertial observer B, the proper time elapsed in the frame of A is greater than that in the frame of B.
Thus, to set up the Rindler coordinates, at first we have to figure out the equation of the hyperbola B in the space-time frame of A. Let the constant proper acceleration of the spaceship be ${}_0$.

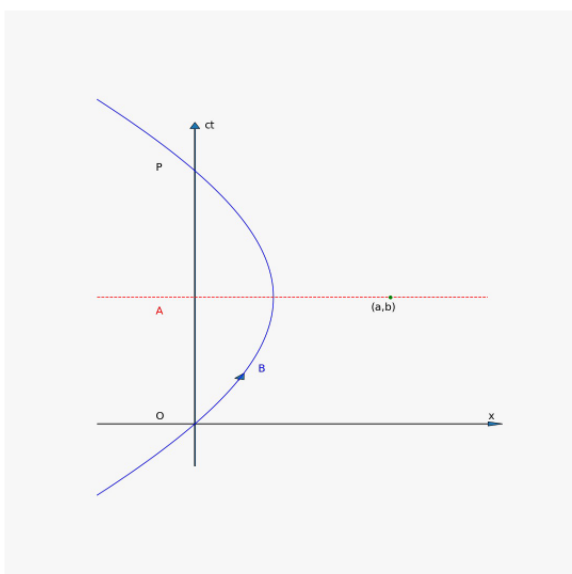

Figure 7:

Let the equation of curve B be:

$$(x \quad a)^2 \quad (ct \quad b)^2 = (\frac{c^2}{\ _0})^2 \tag{51}$$

Now, this hyperbola passes through the points $O(0,0)$ and $P(0, ct_p)$ Hence we get

$$a^2 \quad b^2 = (\frac{c^2}{\ _0})^2 \tag{52}$$

$$a^2 \quad (ct_p \quad b)^2 = (\frac{c^2}{\ _0})^2 \tag{53}$$

Solving (52) and (53), we get

$$a = \frac{\sqrt{4c^4 + c^2 t_p^2 \ _0^2}}{2 \ _0} \tag{54}$$

$$b = \frac{ct_p}{2} \tag{55}$$

Let

$$\frac{c^2}{\ _0} = D \tag{56}$$

Note that as $\ _0 < 0$, hence $D < 0$. So, the train of observers undergoing Born Rigid Rindler motion in this way would be:

$$(x \quad a)^2 \quad (ct \quad b)^2 = (\frac{c^2}{\ })^2 \tag{57}$$

where a and b are given by (54) and (55). Also note that here $\ < 0$.
[Note that although we have found out the values of a and b, but we shall continue to use a and b as symbols for some time in this analysis.]
Thus, we can write the space-time points using the Rindler coordinates as:

$$x = a + \widetilde{x} \cosh \frac{c\widetilde{t}}{D} \tag{58}$$

$$ct = b + \widetilde{x} \sinh \frac{c\widetilde{t}}{D} \tag{59}$$

To keep things clear and to avoid confusions, we make clear that the motion of B is given by:

$$\widetilde{x} = D \tag{60}$$

and the motion of A is given by

$$x = 0 \tag{61}$$

Now, the motion of the earth from the Rindler coordinates can be found out by putting (61) in (58) to get:

$$\widetilde{x} = \quad a \,\mathrm{sech}(\frac{c\widetilde{t}}{D}) \tag{62}$$

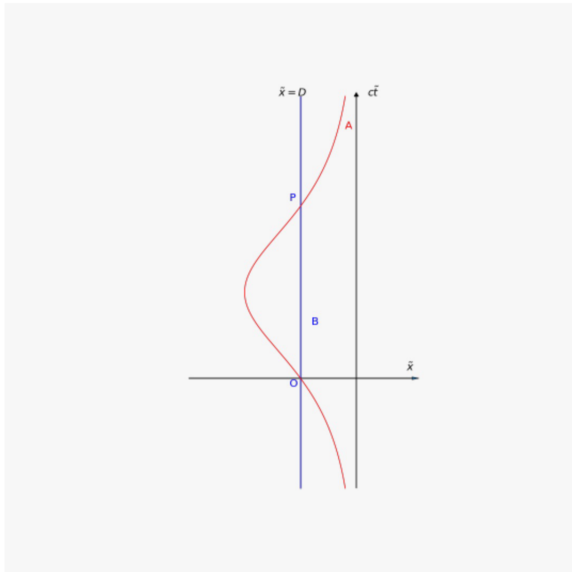

Figure 8: The motion of A in Rindler frame of B.

Now, the Rindler metric is:

$$ds^2 = c^2dt^2 \quad dx^2 = \frac{\widetilde{x}^2}{D^2}c^2\widetilde{dt}^2 \quad \widetilde{dx}^2 \tag{63}$$

So, once we have set up the Rindler coordinates and found out the equations of motion of both the B and A in the Rindler frame of the B (as well as being equipped with the metric), we can now proceed with the calculations of proper time for both the observers. Now, we attempt to express O and P in Rindler coordinates.
Clearly, $\widetilde{x}$ = D in both the cases. We put $t = 0$ and $\widetilde{x} = D$ in (59) and also put the value of b, to get the relationship between $\widetilde{t_0}$ and t_p as:

$$\frac{ct_p}{D} = \sinh(\frac{c\widetilde{t_0}}{D}) \tag{64}$$

Again, we put $t = t_p$ and $\widetilde{x} = D$ in (59) and also put the value of b, to get the relationship between $\widetilde{t_p}$ and t_p as:

$$\frac{ct_p}{D} = \sinh(\frac{c\widetilde{t_p}}{D}) \tag{65}$$

Hence from (64) and (65), we infer that:

$$\widetilde{t_0} = \widetilde{\quad t_p} \tag{66}$$

where clearly $t_p > 0$ Now, we calculate proper time for B using the Rindler coordinates.
We note that

$$ds^2 = c^2 d\tau^2 \tag{67}$$

Hence

$$\tau_B = \frac{.s_B}{c} = \int_{t_0}^{t_p} \widetilde{d}t \tag{68}$$

Thus, we get from (68) as:

$$\tau_B = (\widetilde{t_p} \quad \widetilde{t_0}) = 2\widetilde{t_p} \tag{69}$$

Now, we calculate proper time for the earth-bound observer A using the Rindler coordinates.
Using the metric in (63), we say,

$$.s_A = \int_{t_0}^{t_p} \sqrt{\frac{\widetilde{x}^2}{D^2} c^2 \quad (\frac{d\widetilde{x}}{d\widetilde{t}})} \widetilde{d}t \tag{70}$$

where $\widetilde{x}(\widetilde{t})$ is given by (62).
Performing the integration, we get

$$.s_A = \frac{a}{2}[\sinh(\frac{2c\widetilde{t_p}}{D}) \operatorname{sech}^2(\frac{c\widetilde{t_p}}{D}) \quad \sinh(\frac{2c\widetilde{t_0}}{D}) \operatorname{sech}^2(\frac{c\widetilde{t_0}}{D})] \tag{71}$$

where a is given by (54) Noting that $\widetilde{t_0} = \widetilde{\quad t_p}$, from (71) we get:

$$.s_A = a \sinh(\frac{2c\widetilde{t_p}}{D}) \operatorname{sech}^2(\frac{c\widetilde{t_p}}{D}) \tag{72}$$

Now, we put t_p from (65) into (54) and noting (56), we get:

$$a = D \cosh(\frac{ct_p}{D}) \tag{73}$$

Putting (73) in (72)

$$\begin{aligned} .s_A &= D \cosh(\frac{ct_p}{D}) \sinh(\frac{2c\widetilde{t_p}}{D}) \operatorname{sech}^2(\frac{c\widetilde{t_p}}{D}) \\ &= D \sinh(\frac{2c\widetilde{t_p}}{D}) \operatorname{sech}(\frac{c\widetilde{t_p}}{D}) \\ &= 2D \sinh(\frac{c\widetilde{t_p}}{D}) \end{aligned} \tag{74}$$

By Taylor expansion, we get:

$$.s_A = 2D[\frac{c\widetilde{t_p}}{D} + \frac{1}{3!}(\frac{c\widetilde{t_p}}{D})^3 + ...] \tag{75}$$

Clearly from (75),

$$.s_A > 2c\widetilde{t_p} \tag{76}$$

Hence,

$$\begin{aligned} \tau_A &= \frac{.s_A}{c} \\ &> 2\widetilde{t_p} \end{aligned} \tag{77}$$

From (69) and (77), we conclude that even in the frame of the spaceship of B, using Rindler coordinates,

$$\tau_A > \quad \tau_B \tag{78}$$

12 Schwarzschild metric and the Rindler metric

In Einstein's theory of general relativity, the Schwarzschild metric is the solution to the Einstein field equations that describes the gravitational field outside a spherical mass, on the assumption that the electric charge of the mass, and angular momentum of the mass is 0. The metric is given by

$$ds^2 = (1 \quad \frac{2GM}{c^2 r})c^2dt^2 \quad \frac{1}{(1 \quad \frac{2GM}{c^2 r})}dr^2 \quad r^2d\theta^2 \quad r^2 sin^2(\theta)d\phi^2 \tag{79}$$

There are two values of r where the Schwarzschild metric goes bad:

$$r = 0 \tag{80}$$

$$r = \frac{2GM}{c^2} \tag{81}$$

At each of these values, one of the components of the metric diverges but, as we will see, the interpretation of this divergence is rather different in the two cases. We will learn that the divergence at the point r = 0 is because the space-time is sick: this point is called the singularity. The theory of general relativity breaks down as we get close to the singularity and to make sense of what's happening there we need to turn to a quantum theory of space-time.
In contrast, nothing so dramatic happens at the surface $r = \frac{2GM}{c^2}$ and the divergence in the metric is merely because we've made a poor choice of coordinates: this surface is referred to as the event horizon, usually called simply the horizon. Many of the surprising properties of black holes lie in interpreting the event horizon.
In the following discussion, we will take c = 1. Note that this will not affect our discussion since c is a known constant, it can always be added back into out equations using dimensional analysis.

12.1 The Near Horizon Limit: Rindler Space

To understand what's happening near the horizon r = 2GM (using equation 62 and substituting c = 1), we can zoom in and look at the metric in the vicinity of the horizon. To do this, we write

$$r = 2GM + \eta \tag{82}$$

where we take $0 < \eta << 2GM$. This means that we're looking at the region of space-time just outside the horizon. We then approximate the components of the metric as

$$1 \quad \tfrac{2GM}{r} \approx \tfrac{\eta}{2GM} \qquad \text{and} \qquad r^2 = (2GM + \eta)^2 \approx (2GM)^2$$

To this order, the Schwarzschild metric becomes

$$ds^2 = \frac{\eta}{2GM}dt^2 \quad \frac{2GM}{\eta}d\eta^2 \quad (2GM)^2 d\theta^2 \quad (2GM)^2 sin^2(\theta)d\phi^2 \tag{83}$$

We notice that the metric has been expressed as a direct product of a 2-sphere of radius 2GM(θ, ϕ) and 2 dimensional Lorentzian geometry(t, η). We'll direct our attention exclusively to the 2 dimensional Lorentzian geometry. We make the change of variables

$$\rho^2 = 8GM\eta \tag{84}$$

The 2d metric now becomes

$$ds^2 = (\frac{\rho}{4GM})^2 dt^2 \quad d\rho^2 \tag{85}$$

This is nothing but the Rindler metric given by equation(43). The corresponding Rindler coordinates are given by

$$T = \rho \sinh(\frac{t}{4GM}) \tag{86}$$

$$X = \rho \cosh(\frac{t}{4GM}) \tag{87}$$

after which the metric becomes

$$ds^2 = dT^2 \quad dX^2 \tag{88}$$

Equations (86) and (87) are the coordinates of an observer undergoing constant acceleration $a = \frac{1}{4GM}$, where t is the proper time of this observer. This makes sense: an observer who sits at a constant ρ value, corresponding to a constant r value, must accelerate in order to avoid falling into the black hole.
We can now start to map out what part of Minkowski space corresponds to the outside of the black hole horizon. This is $\rho > 0$ and t ϵ (-∞, ∞). From the

change of variables given by equations (86) and (87), we see that this corresponds to the region $X > |T|$.
We can also see what becomes of the horizon itself. This sits at r = 2GM, or $\rho = 0$. For any finite t, the horizon $\rho = 0$ gets mapped to the origin of Minkowski space, X = T = 0. However, since the time coordinate part of the metric vanishes, scaling $t \to \infty$ and $\rho \to 0$ keeping the $\rho e^{\pm \frac{t}{4GM}}$ combination fixed, we see that the event horizon actually gets mapped to the lines $X = \pm T$ This implies that the event horizon of a blackhole is not a time like surface like the surface of a star or planet. It is rather a null surface. [1]

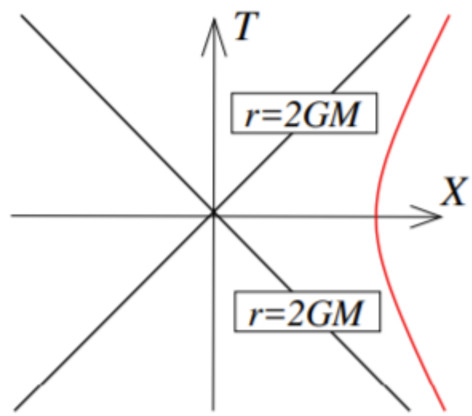

Figure 9: The near horizon limit of a black hole is Rindler spacetime, with the null lines $X = \pm T$ corresponding to the horizon at $r = 2GM$. Also shown in red is a line of constant $r > 2GM$ outside the black hole.[1]

However, we know that the lines $X = T$ corresponds to the Rindler horizon. Therefore, the Rindler horizon and the event horizon of a black hole have identical worldlines in the space-time diagram(in the near horizon limit).

13 Conclusion

We have covered a lot of material so far. We first started of with a discussion of motion with constant acceleration. This results in hyperbolic trajectories in space-time. We then saw how these hyperbolic trajectories results in the formation of a Rindler horizon. Subsequently, we dealt with Born Rigid Motion and dived deeper into the mathematics describing Rindler motion. This was then used to analyze the twin's paradox and later deal with black holes. We showed that a change in variable of the Schwarzschild metric yields the Rindler metric and there exists a map between the Rindler and event horizon. This results

Image taken from: http://www damtp cam ac uk/user/tong/gr/six pdf in some interesting phenomenon. The most bizarre is the apparent switching of roles time and space below the event horizon of a black hole. Space-time somehow turns into time-space. This mysterious flip gives us some fascinating insight on how time and space blend together in what is perhaps the strangest place in space-time.