Maths Test
Question 1

Let $h(x) = \dfrac{f(x)}{g(x)}$, where

$$f(1) = 2 \qquad f'(1) = 4$$
$$g(1) = 8 \qquad g'(1) = 12.$$

What is the gradient of the tangent to the graph of $y = h(x)$ at $x = 1$?

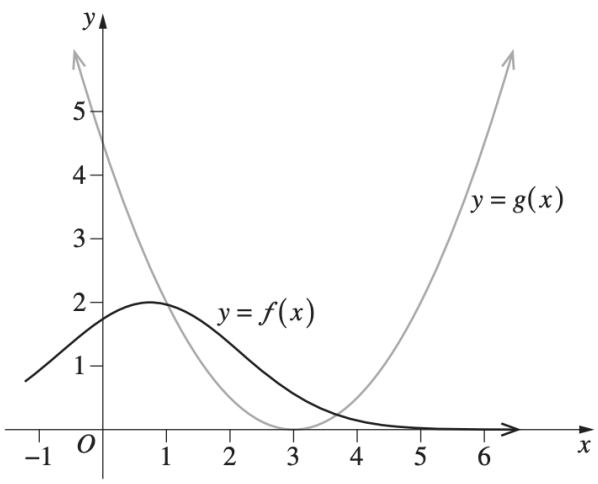
A. -8

B. 8

C. $-\dfrac{1}{8}$

D. $\dfrac{1}{8}$

Question 2
The graph shows two functions $y = f(x)$ and $y = g(x)$.



Define $h(x) = f(g(x))$.

How many stationary points does $y = h(x)$ have for $1 \le x \le 5$?

A. 0

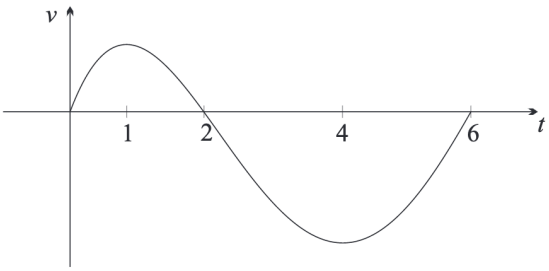
B. 1

C. 2

D. 3

Question 3

A ball moves in a straight line along a horizontal track. The graph below shows the ball's velocity v at time t.



For what values of t is the speed of the particle decreasing?

(A) $0 < t < 1$ and $2 < t < 4$

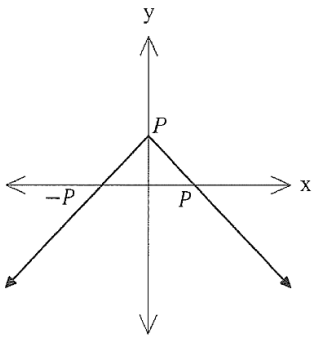
(B) $1 < t < 2$ and $4 < t < 6$

(C) $1 < t < 4$

(D) $2 < t < 6$

Question 4

The graph shows the function $y = P - |x|$.



Find the value of P, given:

$$\int_{-P}^{P} (P - |x|)\, dx = 16$$

A. $P = 4$

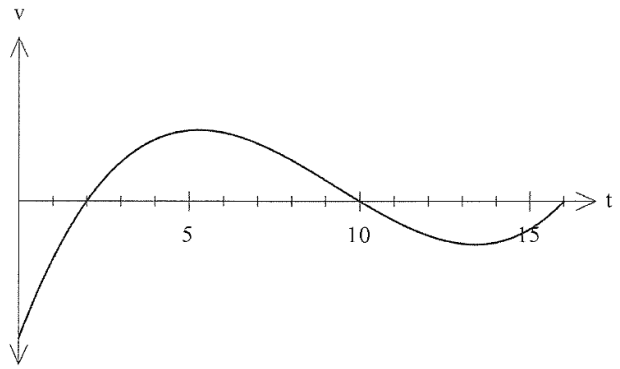
B. $P = 8$

C. $P = -4$

D. $P = -8$

Question 5

The graph below shows the velocity function of a particle during its travel for the first 16 seconds.



Given that the particle begins at the origin, which of the following statements is true?

A. At $t = 2$, the particle is to the left of the origin, and travelling towards the origin.

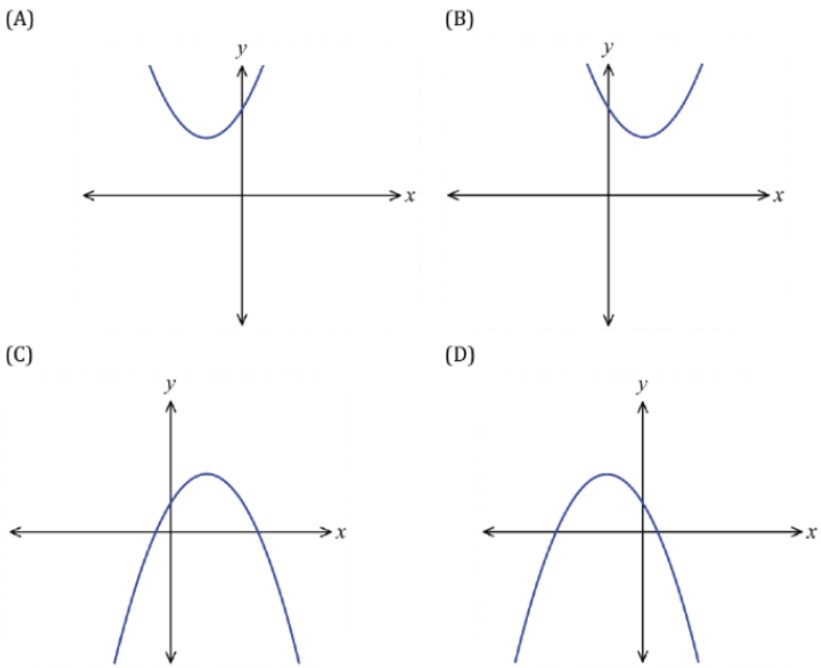
B. At $t = 5$, the particle is at the origin, and at rest.

C. At $t = 10$, the particle is to the right of the origin, and at rest.

D. At $t = 15$, the particle is at the origin, and travelling from left to right.

Question 6

Which diagram could be the graph of the parabola $y = 2 - (x + 1)^2$?



Question 7

What is the solution to the inequation $x^2 + 4x + 3 \geq 0$?

A. $x \leq -1$ or $x \leq -3$

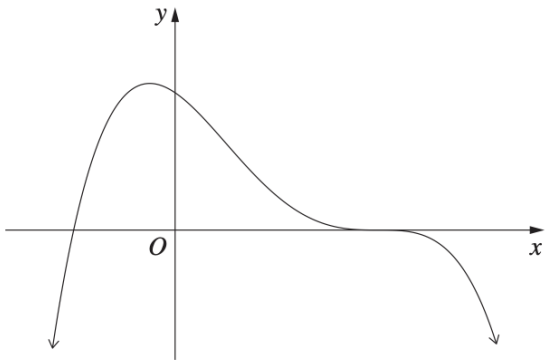
B. $x \leq -1$ or $x \geq -3$

C. $x \geq -1$ or $x \geq -3$

D. $x \geq -1$ or $x \leq -3$

Question 8

The graph of $y = f(x)$ is shown.



Which of the following could be the equation of this graph?

A. $y = (1 - x)(2 + x)^3$

B. $y = (x + 1)(x - 2)^3$

C. $y = (x + 1)(2 - x)^3$

D. $y = (x - 1)(2 + x)^3$

Question 9

The only stationary point on the graph $y = f(x)$ is the point (a,b).

What are the coordinates of the only stationary point on the graph $y = -f(2x)$?

A. $\left(\frac{1}{2}a, -b\right)$

B. $(2a, -b)$

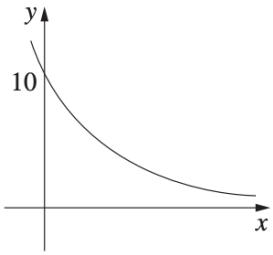
C. $\left(-\frac{1}{2}a, b\right)$

D. $(-2a, b)$

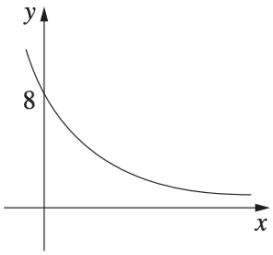
Question 10

Which of the following best represents the graph of $y = 10(0.8)^x$?

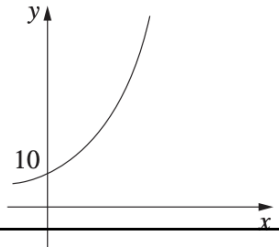
A.



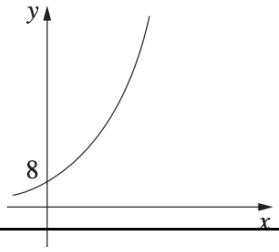
B.



C.

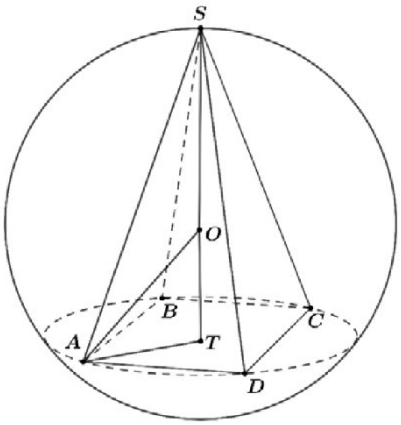


D.



Maths Test
Question 11

The diagram below shows a sphere of radius R and centre O circumscribing a square pyramid. The points A, B, C, D and S all lie on the surface of the sphere and form the vertices of the pyramid with $ABCD$ the base. The point T lies directly beneath O and lies in the base of the pyramid such that $AT \perp OT$.



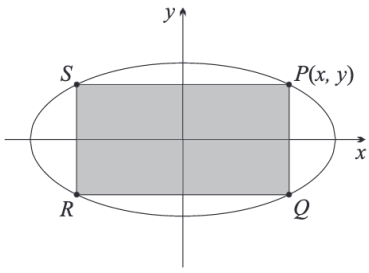
a) Let $x = OT$. Show that the volume of the pyramid is **3**

$$V = \frac{2}{3}(R + x)^2(R - x)$$

b) Hence find the maximum volume a square pyramid may take when its vertices are circumscribed by a sphere of radius R. **3**

Question 12

Rectangle $PQRS$ has been inscribed inside an ellipse with equation $x^2 + 4y^2 = 36$, as shown in the diagram below. The point P has coordinates (x, y).



(a) Show that the area of the rectangle is $A = 2x\sqrt{36 - x^2}$. $\boxed{2}$

(b) Show that $\dfrac{dA}{dx} = \dfrac{72 - 4x^2}{\sqrt{36 - x^2}}$. $\boxed{2}$

(c) Hence, find the maximum possible area of the rectangle. $\boxed{3}$

Question 13

Consider the curve $y = x(x - 2)(x + 1)$.

a) Find the coordinates of any stationary point, to 1 decimal place, and determine their nature. **3**

b) Find the coordinates of any points of inflection. **3**

c) Sketch the curve showing all important features. **2**

d) For what values of c will $x(x-2)(x+1) = c$ have exactly 1 solution? Give your answer correct to 1 decimal place. **1**

Question 14

Without using calculus, sketch the graph of $y = 2 + \frac{1}{x+4}$, showing the asymptotes and the x and y intercepts. **3**

Question 15

Consider the function $f(x) = 2x^2(k - x)$, where k is a constant.

(a) If $f(x)$ has a maximum turning point at $(2, 8)$, show that $k = 3$. **2**

(b) Find the minimum turning point (you do not need to test for this) and any inflexion points (prove by testing or otherwise). **3**

(c) Sketch the function $y = f(x)$, showing all features. **2**

(d) Hence sketch the function $y = -f(x) + 3$ **2**

Question 16

The temperature T inside an oven, measured in degrees Celsius, m minutes after it is switched on, can be modelled by the equation $T = 200 - 175e^{-km}$, where k is a constant.

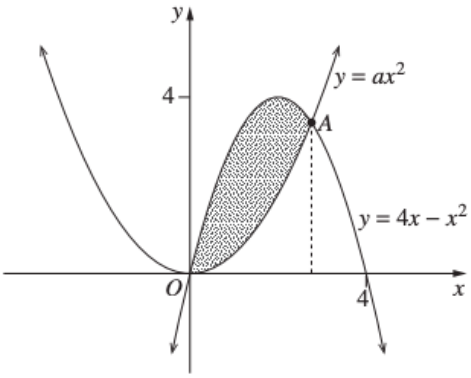
(a) Write down the initial temperature of the oven. **1**

(b) If the oven reached a temperature of 150°C after 16 minutes and 15 seconds, find the value of k correct to three significant figures. **2**

(c) What will be the temperature in the oven after 10 minutes? **1**

Question 17

The diagram shows two parabolas $y = 4x - x^2$ and $y = ax^2$, where $a > 0$. The two parabolas intersect at the origin, O, and at A.

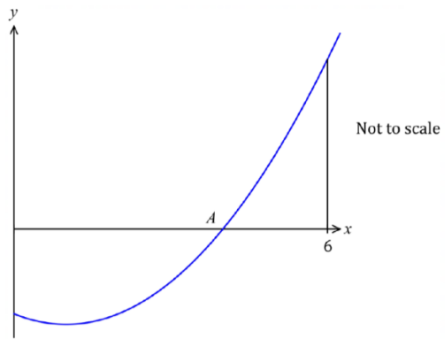


(a) Show that the x-coordinate of A is $\dfrac{4}{a+1}$. **2**

(b) Find the value of a such that the shaded area is $\dfrac{16}{3}$. **4**

Question 18

The diagram below shows the graph of $y = x^2 - 2x - 8$



(a) What are the coordinates of A? 1

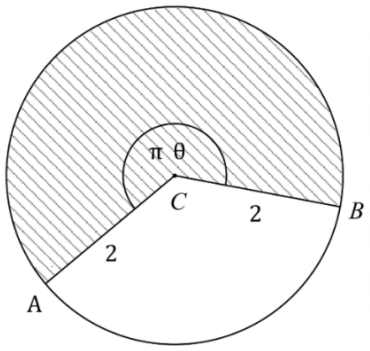
(b) Find the area bounded by the x-axis and the curve $y = x^2 - 2x - 8$ between $0 \le x \le 6$ 3

Question 19

Solve: **2**

$$\frac{243}{6\sqrt{3} - \sqrt{27}} = 3^x$$

Question 20



The angle at the centre C of a circle of radius 2 cm is $\pi\theta$ radians, $0 < \theta < 2$, as shown on the diagram above.

(a) Write down the length of the arc of the shaded sector 1

(b) The sector is cut from the circle along the radii CA and CB and folded to make a cone. Find the radius of the cone. 1

(c) Show that the volume of the cone is given by $V = \frac{\pi}{3}\sqrt{4\theta^4 - \theta^6}$ 1

(The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2\sqrt{l^2 - r^2}$ where l is the slant height)

(d) Find the value of θ, to 2 decimal places, for which the volume of the cone is maximised. 4

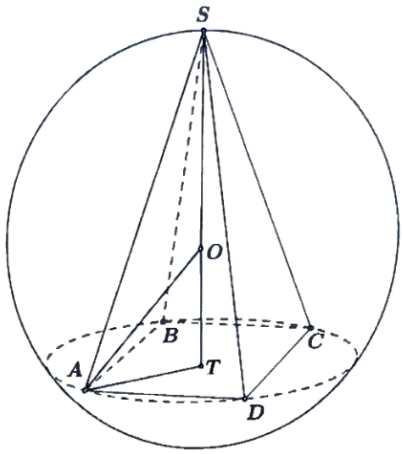
Multiple Choice Answers

1: D
2: D
3: B
4: A
5: C
6: D
7: D
8: C
9: A
10: A

Short Answer Solutions

Question 11

The diagram below shows a sphere of radius R and centre O circumscribing a square pyramid. The points A, B, C, D and S all lie on the surface of the sphere and form the vertices of the pyramid with $ABCD$ the base. The point T lies directly beneath O and lies in the base of the pyramid such that $AT \perp OT$.



a) Let $x = OT$. Show that the volume of the pyramid is **3**

$$V = \frac{2}{3}(R+x)^2(R-x)$$

Base $= 2\sqrt{R^2-x^2}\sqrt{R^2-x^2}$ — 1

$= 2(R^2-x^2)$

Height $= x+R$ — 1

$\therefore V = \frac{1}{3} \times 2 \times (R^2-x^2) \times (x+R)$

$= \frac{2}{3}(R+x)(R-x)(R+x)$ — 1

$V = \frac{2}{3}(R+x)^2(R-x)$.

b) Hence find the maximum volume a square pyramid may take when its vertices are circumscribed by a sphere of radius R. **3**

$$V = \frac{2}{3}(R+x)^2(R-x)$$

$$\frac{dV}{dx} = \frac{2}{3}\left[2(R+x)(R-x) - (R+x)^2\right]$$

Stationary Pt. when $\frac{dV}{dx} = 0$

$$0 = \frac{2}{3}\left[2(R+x)(R-x) - (R+x)^2\right]$$

$$0 = (R+x)\left[2(R-x) - (R+x)\right]$$

$$0 = 2R - 2x - R - x \qquad (R \neq -x;\ R, x > 0 \text{ as lengths}).$$

$$0 = R - 3x$$

$$x = \frac{R}{3} \quad \text{—— (1)}$$

$$\frac{d^2V}{dx^2} = \frac{2}{3}\left[2(-2x) - 2(R+x)\right]$$

$$= -\frac{4}{3}(3x+R) \quad \text{—— (1) test}$$

when $x = \frac{R}{3}$ $\quad \frac{d^2V}{dx^2} = -\frac{4}{3}(R+R) < 0$ as $R > 0$

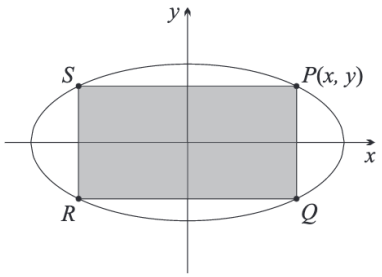
So local max at $x = \frac{R}{3}$

Hence max volume : $V\left(\frac{R}{3}\right) = \frac{2}{3}\left(R+\frac{R}{3}\right)^2\left(R-\frac{R}{3}\right)$

$$= \frac{2}{3} \times \frac{16R^2}{9} \times \frac{2R}{3} = \frac{64}{81}R^3 \quad \text{—— (1)}$$

Question 12

Rectangle $PQRS$ has been inscribed inside an ellipse with equation $x^2 + 4y^2 = 36$, as shown in the diagram below. The point P has coordinates (x, y).



(a) Show that the area of the rectangle is $A = 2x\sqrt{36 - x^2}$. $\boxed{2}$

$x^2 + 4y^2 = 36$

$y^2 = \dfrac{36 - x^2}{4}$

$y = \dfrac{\sqrt{36 - x^2}}{2}$ ✓

reject -ve as P is in 1st quadrant

$A = PS \times PQ$

$= 2x \times 2y$

$= 2x \times \dfrac{2\sqrt{36 - x^2}}{2}$

$= 2x\sqrt{36 - x^2}$ ✓

(b) Show that $\dfrac{dA}{dx} = \dfrac{72 - 4x^2}{\sqrt{36 - x^2}}$. $\boxed{2}$

let $u = 2x$ $\quad v = (36 - x^2)^{\frac{1}{2}}$

$u' = 2$ $\quad v' = \frac{1}{2}(36 - x^2)^{-\frac{1}{2}} \times (-2x)$

$= -\dfrac{x}{\sqrt{36 - x^2}}$

$\dfrac{dA}{dx} = vu' + uv'$

$= 2\sqrt{36 - x^2} - \dfrac{2x^2}{\sqrt{36 - x^2}}$ ✓

$= \dfrac{2(36 - x^2) - 2x^2}{\sqrt{36 - x^2}} = \dfrac{72 - 4x^2}{\sqrt{36 - x^2}}$ ✓

(c) Hence, find the maximum possible area of the rectangle. $\boxed{3}$

max area when $\frac{dA}{dx} = 0$

$$0 = \frac{72 - 4x^2}{\sqrt{36 - x^2}}$$

$$0 = 72 - 4x^2$$

$$4x^2 = 72$$

$$x^2 = 18$$

$$x = \pm 3\sqrt{2}$$ ✓ find x-values at stationary points

reject $-3\sqrt{2}$, as x is in 1st quadrant

x	1	$3\sqrt{2}$	5
$\frac{dA}{dx}$	$\frac{68}{\sqrt{35}}$	0	$-\frac{28}{\sqrt{11}}$
	/	—	\

$\therefore$ maximum area when $x = 3\sqrt{2}$ ✓ show that $x = 3\sqrt{2}$ gives max area

max area is $A = 2 \times 3\sqrt{2} \times \sqrt{36 - (3\sqrt{2})^2}$

$$= 6\sqrt{2} \times \sqrt{36 - 18}$$

$$= 6\sqrt{2} \times 3\sqrt{2}$$

$$= 36\,u^2$$ ✓ area

note: possible to show that $x = 3\sqrt{2}$ gives maximum area by using $\frac{d^2A}{dx^2}$, but this method required substantially more algebra and was much more error prone

Question 13

Consider the curve $y = x(x-2)(x+1)$.

a) Find the coordinates of any stationary point, to 1 decimal place, and determine their nature. **3**

$y = x^3 - x^2 - 2x$

$\frac{dy}{dx} = 3x^2 - 2x - 2$

$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 2x - 2 = 0$ — 1 set $\frac{dy}{dx} = 0$

$\therefore x = \frac{2 \pm \sqrt{4+24}}{6}$, $\therefore x \approx 1.2$ or -0.5

$\frac{d^2y}{dx^2} = 6x - 2$

When $x \approx 1.2$ $y = -2.1$ $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} \approx 5.2$
> 0

$\therefore$ local min at $\approx (1.2, -2.1)$ — 1

when $x \approx -0.5$, $y \approx 0.6$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} \approx -5$
< 0

$\therefore$ local max at $(-0.5, 0.6)$ — 1

MUST have a POINT

Find the coordinates of any points of inflection. **3**

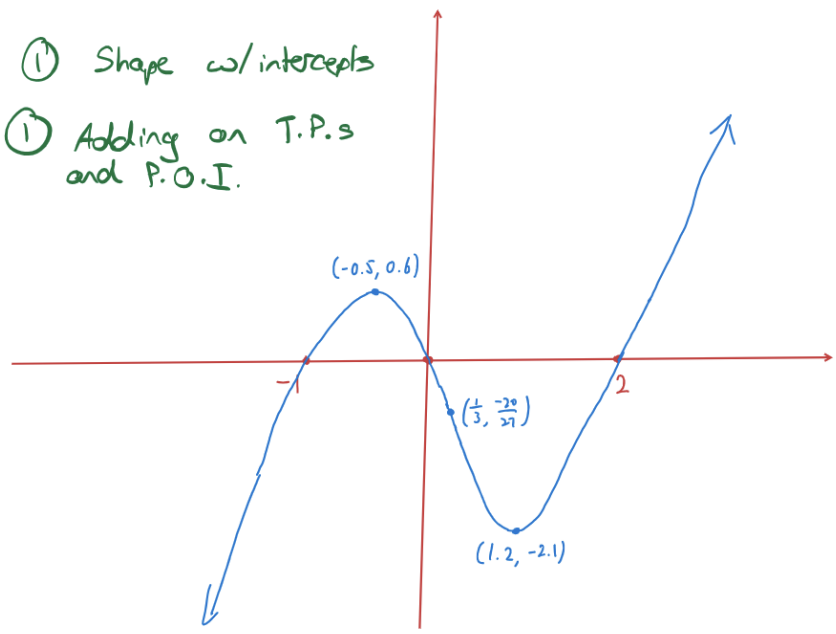
$\frac{d^2y}{dx^2} = 0 \Rightarrow 6x - 2 = 0$
$\therefore x = \frac{1}{3}$ — 1 x-value

x	0	$\frac{1}{3}$	1
y''	-2	0	1
	−		+

When $x = \frac{1}{3}$, $y = \frac{-20}{27}$, $\frac{dy}{dx} \neq 0$, $\frac{d^2y}{dx^2} = 0$ and change of concavity before and after — 1 Checking change of concavity

$\therefore$ point of inflection at $\left(\frac{1}{3}, \frac{-20}{27}\right)$ — 1 Final POINT

c) Sketch the curve showing all important features. **2**



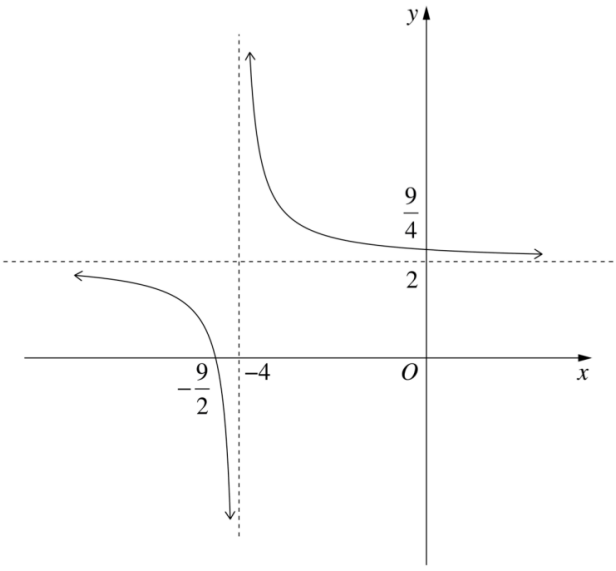
d) For what values of c will $x(x-2)(x+1) = c$ have exactly 1 solution? Give your answer correct to 1 decimal place. **1**

$c > 0.6$ or $c < -2.1$

Question 14

Criteria	Marks
• Provides correct solution	3
• Provides correct graph without x and y intercepts, or equivalent merit	2
• Draws a hyperbola, or equivalent merit	1

Sample answer:



y-intercept: $y = 2 + \dfrac{1}{4}$

$= \dfrac{9}{4}$

x-intercept: $0 = 2 + \dfrac{1}{x+4}$

$x = -\dfrac{9}{2}$

Question 15

Consider the function $f(x) = 2x^2(k - x)$, where k is a constant.

(a) If $f(x)$ has a maximum turning point at $(2, 8)$, show that $k = 3$. **2**

$f'(2) = 0$ $\quad f'(x) = 4kx - 6x^2$ $\quad$ OR $\quad f(2) = 8$

$0 = 4k \times 2 - 6 \times 2^2$ $\quad 8 = 2 \times 2^2(k-2)$

$0 = 2k - 6$ $\quad 8 = 8(k-2)$

$2k = 6$ $\quad 1 = k - 2$

$k = 3$ $\quad k = 3$

(b) Find the minimum turning point (you do not need to test for this) and any inflexion points (prove by testing or otherwise). **3**

$f'(x) = 12x - 6x^2$

$f'(x) = 6x(2-x)$

since $(2,8)$ is max

$6x = 0 \Rightarrow x = 0$ $y=0$ is min

$f''(x) = 12 - 12x$

Possible inflexion at $f''(x) = 0$.

$0 = 12 - 12x$

$x = 1$

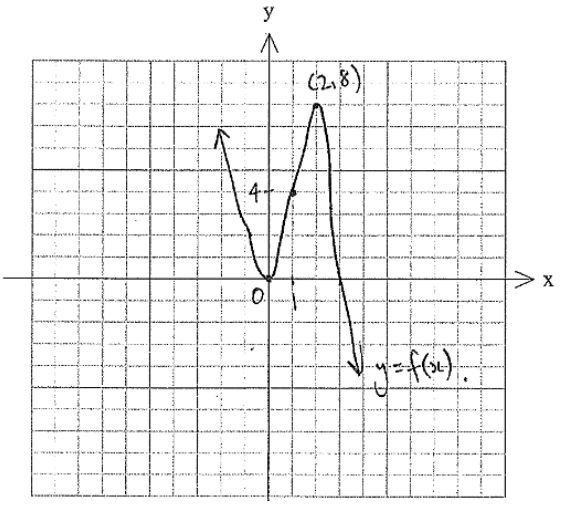
Since $x=1$ is between $(2,8)$ max turning and $(0,0)$ which is min turning point $(1,4)$ is an inflexion point

OR

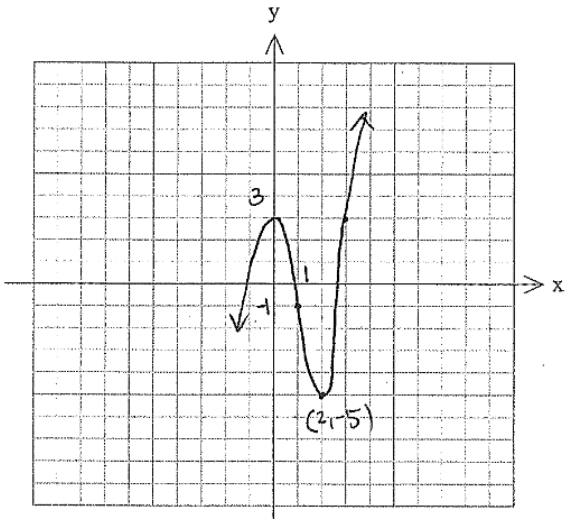
x	0	1	2
$f''(x)$	12	0	-12
	$\cup$		$\cap$

Concavity changes: $(1,4)$ is an inflexion point.

(c) Sketch the function $y = f(x)$, showing all features. **2**



(d) Hence sketch the function $y = -f(x) + 3$ **2**



Question 16

The temperature T inside an oven, measured in degrees Celsius, m minutes after it is switched on, can be modelled by the equation $T = 200 - 175e^{-km}$, where k is a constant.

(a) Write down the initial temperature of the oven. **1**

When $m = 0$,

$T = 200 - 175e^{-0}$

$T = 25°$

Pretty well done. In part (a) a few students could not successfully evaluate the initial temperature as they incorrectly used $e^0 = 0$. Be careful with this, which also arises when evaluating definite integrals involving exponentials.

(b) If the oven reached a temperature of 150°C after 16 minutes and 15 seconds, find the value of k correct to three significant figures. **2**

When $m = 16.25$, $T = 150$,

$150 = 200 - 175e^{-16.25k}$

$\frac{-50}{-175} = e^{-16.25k}$

$-16.25k = \ln\frac{2}{7}$

$k = \frac{-\ln\left(\frac{2}{7}\right)}{16.25} = \frac{4}{65}\ln\left(\frac{7}{2}\right) \approx 0.0771$ to 3 s.f.

In part (b) most students could manipulate the resulting exponential equation correctly. Common errors tended to be in using the calculator incorrectly or in rounding correctly to three significant figures. Revisit significant figures if required.

(c) What will be the temperature in the oven after 10 minutes? **1**

When $m = 10$, $T = 200 - 175e^{-10 \times \frac{4}{65} \ln\left(\frac{7}{2}\right)}$ $= 119.05°\text{C}$
Part (c) was generally well done barring a few calculator errors. Exercise care to enter expressions correctly into the calculator. Using the ANS key or a pronumeral using the STO function on your calculator can make it easier to enter complex expressions.

Question 17

Criteria	Marks
• Provides the correct solution	2
• Attempts to solve a quadratic equation to find the x-coordinate of A, or equivalent merit	1

Sample answer:

$$ax^2 = 4x - x^2$$

$$(a+1)x^2 - 4x = 0$$

$$x[(a+1)x - 4] = 0$$

$$x = 0, \quad (a+1)x - 4 = 0$$

$$(a+1)x = 4$$

$$x = \frac{4}{a+1}$$

Criteria	Marks
• Provides the correct solution	4
• Correctly substitutes the limits into the anti-derivative, or equivalent merit	3
• Finds the anti-derivative, or equivalent merit	2
• Writes an expression for the area involving an integral of the difference between two functions, or equivalent merit	1

Sample answer:

$$\int_0^{\frac{4}{a+1}} (4x - x^2 - ax^2)\,dx = \frac{16}{3}$$

$$\therefore \quad \int_0^{\frac{4}{a+1}} (4x - (a+1)x^2)\,dx = \frac{16}{3}$$

$$\left[2x^2 - \frac{(a+1)x^3}{3}\right]_0^{\frac{4}{a+1}} = \frac{16}{3}$$

$$\frac{2\times 16}{(a+1)^2} - \frac{(a+1)}{3}\times\frac{64}{(a+1)^3} - 0 = \frac{16}{3}$$

$$\frac{32}{(a+1)^2} - \frac{64}{3(a+1)^2} = \frac{16}{3}$$

$$\frac{96-64}{3(a+1)^2} = \frac{16}{3}$$

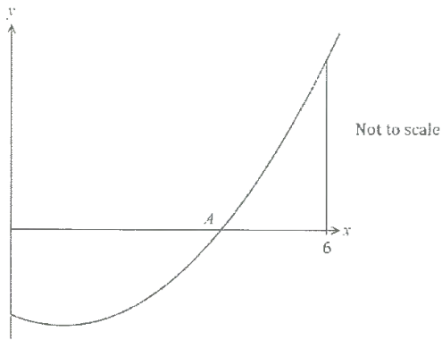
$$\frac{32}{3(a+1)^2} = \frac{16}{3}$$

$$\therefore \quad (a+1)^2 = 2$$

$$a = \sqrt{2} - 1 \qquad (\text{since } a > 0)$$

Question 18

The diagram below shows the graph of $y = x^2 - 2x - 8$



(a) What are the coordinates of A? 1

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = 4, -2$

$\therefore A(4,0)$

(b) Find the area bounded by the x-axis and the curve $y = x^2 - 2x - 8$ between $0 \le x \le 6$ 3

$A = \left|\int_0^4 x^2 - 2x - 8\, dx\right| + \int_4^6 x^2 - 2x - 8\, dx$

$= \left|\left[\frac{x^3}{3} - x^2 - 8x\right]_0^4\right| + \left[\frac{x^3}{3} - x^2 - 8x\right]_4^6$

$= \left|\left[\frac{4^3}{3} - 4^2 - 8(4)\right]\right| + \left[\left(\frac{6^3}{3} - 6^2 - 8(6)\right) - \left(\frac{4^3}{3} - 4^2 - 8(4)\right)\right]$

$= \left|-\frac{80}{3}\right| + \left[-12 - \left(-\frac{80}{3}\right)\right]$

$= \frac{124}{3}$

$= 41\frac{1}{3}$ units^2

Question 19

Solve: **2**

$$\frac{243}{6\sqrt{3} - \sqrt{27}} = 3^x$$

$\frac{243}{6\sqrt{3} - 3\sqrt{3}}$

$= \frac{3^5}{3\sqrt{3}}$

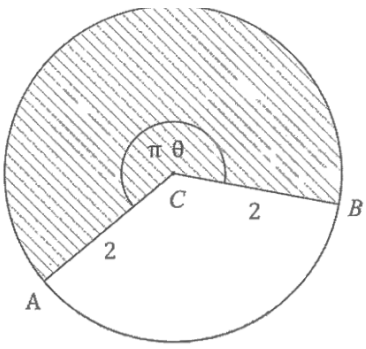
$= \frac{3^4}{\sqrt{3}}$

$= 3^3\sqrt{3}$

$= 3^{3\frac{1}{2}}$

$\therefore\ x = 3\frac{1}{2}$

Question 20



The angle at the centre C of a circle of radius 2 cm is $\pi\theta$ radians, $0 < \theta < 2$, as shown on the diagram above.

(a) Write down the length of the arc of the shaded sector 1

$l = r\theta$

$= 2\pi\theta$

(b) The sector is cut from the circle along the radii CA and CB and folded to make a cone. Find the radius of the cone. 1

circumference $= 2\pi\theta$

$C = 2\pi r$

$\therefore 2\pi r = 2\pi\theta$

$r = \theta$

(c) Show that the volume of the cone is given by $V = \frac{\pi}{3}\sqrt{4\theta^4 - \theta^6}$ 1

(The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2\sqrt{l^2 - r^2}$ where l is the slant height)

$V = \frac{1}{3}\pi r^2\sqrt{l^2 - r^2}$ $\quad l = 2$

$= \frac{1}{3}\pi\theta^2\sqrt{2^2 - \theta^2}$

$= \frac{\pi}{3}\theta^2\sqrt{4 - \theta^2}$

$= \frac{\pi}{3}\sqrt{\theta^4(4 - \theta^2)}$

$= \frac{\pi}{3}\sqrt{4\theta^4 - \theta^6}$ as required

(d) Find the value of θ, to 2 decimal places, for which the volume of the cone is maximised. 4

$V = \frac{\pi}{3}(4\theta^4 - \theta^6)^{\frac{1}{2}}$

Maximum when $V' = 0$

$V' = \frac{\pi}{6}(4\theta^4 - \theta^6)^{-\frac{1}{2}} \cdot (16\theta^3 - 6\theta^5)$

$0 = \frac{\pi(16\theta^3 - 6\theta^5)}{6\sqrt{4\theta^4 - \theta^6}}$

$0 = \pi(16\theta^3 - 6\theta^5)$

$16\theta^3 - 6\theta^5 = 0$

$2\theta^3(8 - 3\theta^2) = 0$

$\theta = 0$ or $8 - 3\theta^2 = 0$

$\theta \neq 0$ $\quad 3\theta^2 = 8$

$\theta^2 = \frac{8}{3}$ $\quad \theta > 0$

$\theta = \sqrt{\frac{8}{3}}$

$\theta = 1.63$ (2 d.p.)

Check nature:

θ	1	1.63	1.8
V'	3.02	0	-3.72
	/	—	\

$\therefore$ Maximised when $\theta = 1.63$