1. Find the quotient and remainder when x^4 is divided by $x^2 + 2x - 1$. [3]

$$
\begin{array}{r}
x^2 - 2x + 5 \\
x^2 + 2x - 1 \overline{) \, x^4 \qquad\qquad\qquad} \\
x^4 + 2x^3 - x^2 \\
\hline
-2x^3 + x^2 \\
-2x^3 - 4x^2 + 2x \\
\hline
5x^2 - 2x \\
5x^2 + 10x - 5 \\
\hline
-12x + 5
\end{array}
$$

$$\frac{x^4}{x^2+2x-1} = (x^2 - 2x + 5) + \frac{-12x+5}{x^2+2x-1}$$

quotiene is $x^2 - 2x + 5$

remainder is ~~-12~~ $-12x + 5$

2. Expand $(2-x)(1+2x)^{-\frac{3}{2}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

-3

$(1+2x)^{-\frac{3}{2}} = 1 + -\frac{3}{2}x + \frac{-\frac{3}{2}(-\frac{3}{2}-1)}{2}x^2 \ldots$

$(2-x)(1+2x)^{-\frac{3}{2}} = (2-x)(1 - \frac{3}{2}x + 1.875x^2)$ M1

$= 2 - 3x + 3.75x^2 - (x - \frac{3}{2}x^2 + 1.875x^3)$

$= 2 - 3x - x + 3.75x^2 + \frac{3}{2}x^2 - 1.875x^3$

$= \underline{2 - 4x + 5.25x^2}$

3. The equation of a curve is $y = \overset{u}{x}\overset{v}{\tan^{-1}}(\frac{1}{2}x)$.

(a) Find $\frac{dy}{dx}$. [3]

$u = x$ $\quad v = \tan^{-1}(\frac{1}{2}x)$ $\quad t = \frac{1}{2}x$ $\quad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$\frac{du}{dx} = 1$ $\quad \frac{dv}{dx} = \frac{1}{1+t^2}\times\frac{1}{2}$ $\quad \frac{dt}{dx} = \frac{1}{2}$ $\quad = \tan^{-1}(\frac{1}{2}x) + x\cdot\frac{1}{2+\frac{1}{2}x^2}$

$= \frac{1}{2+\frac{1}{2}x^2}$

$\frac{dy}{dx} = \tan^{-1}(\frac{1}{2}x) + \frac{x}{2+\frac{1}{2}x^2}$

(b) The tangent to the curve at the point where $x = 2$ meets the y-axis at the point with coordinates $(0, p)$.

Find p. [3]

when $x = 2$

$\frac{dy}{dx} = \tan^{-1}(1) + \frac{1}{2}$ $\quad y = 2\tan^{-1}(1)$

$y = (\tan^{-1}(1) + \frac{1}{2})x + C$

$x = 2,\ y = 2\tan^{-1}(1)$

$2\tan^{-1}(1) = \tan^{-1}(1)\cdot 2 + 1 + C$

$C = -1$

$p = -1$

4. Using the substitution $u = \sqrt{x}$, find the exact value of

$$\int_3^{\infty} \frac{1}{(x+1)\sqrt{x}}\, dx.$$ [6]

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x}\, du$

$\int \frac{1}{(u^2+1)\sqrt{x}} \cdot 2\sqrt{x}\, du$ $\quad u = \sqrt{x}$ $\quad$ when $x = \infty$, $u = \infty$

when $x = 3$, $u = \sqrt{3}$

$= 2\int_{\sqrt{3}}^{\infty} \frac{1}{u^2+1}\, du$

$[2\tan^{-1}(u)] = {}_{\sqrt{3}}^{\infty}[2\tan^{-1}(u)]$

~~$[2\tan^{-1}(\frac{\sqrt{x}}{1})]$~~

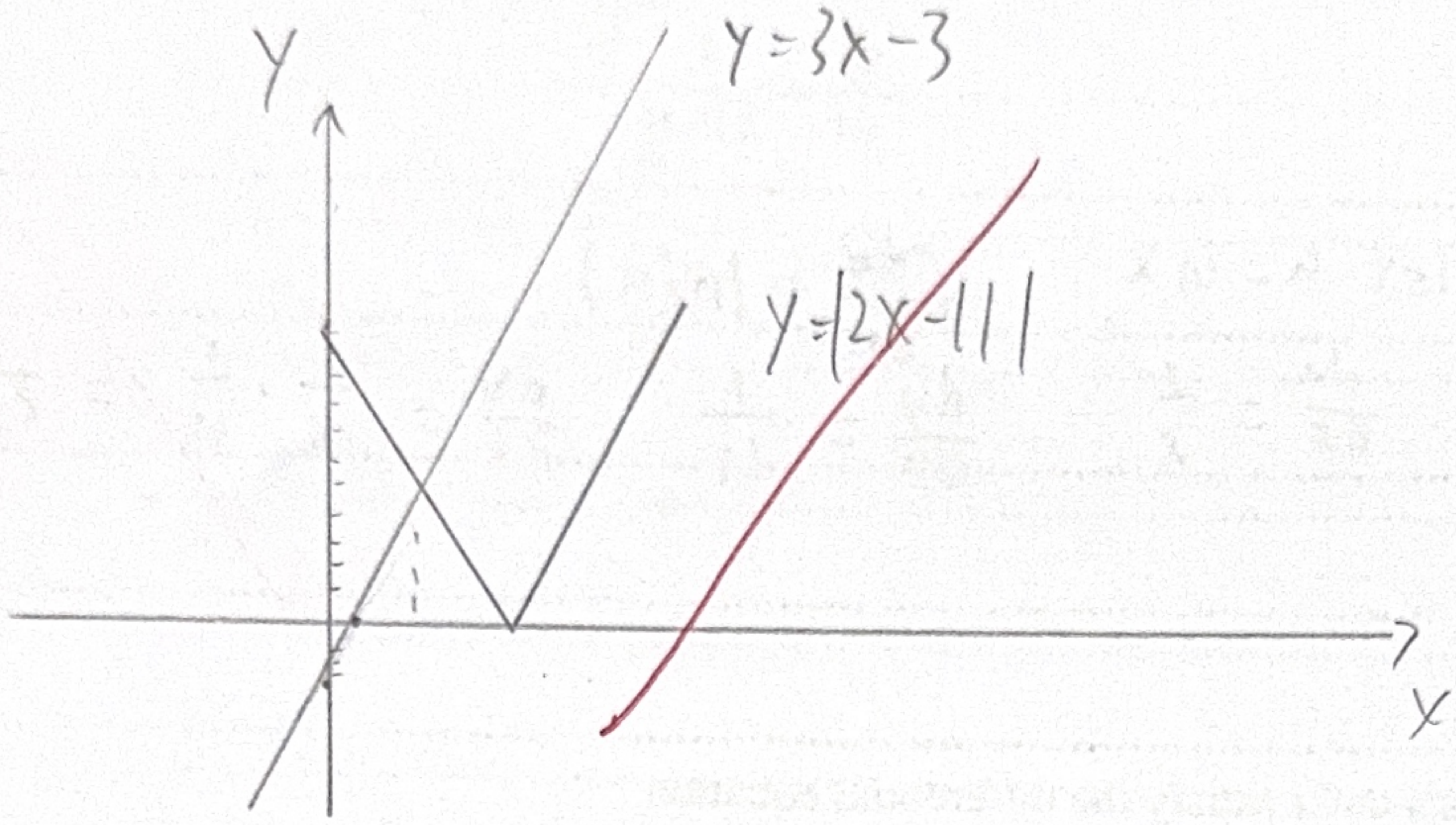
$= 2\tan^{-1}(\infty) - 2\tan^{-1}(\sqrt{3})$

$= 2 \times \frac{1}{2}\pi - 2 \cdot \frac{1}{3}\pi$

$= \pi - \frac{2}{3}\pi$

$= \frac{1}{3}\pi$

5. (a) Sketch, on the same diagram, the graphs of $y = |2x - 11|$ and $y = 3x - 3$. [2]



x	y_1	y_2
1	9	0
0	11	−3
2	7	3
6	1	15
5.5	0	13.5
10	9	27

(b) Solve the inequality $|2x - 11| < 3x - 3$. [3]

$\cancel{(2x-11)^2 < (3x-3)^2}$

$\cancel{4x^2 - 44x + 121 < 9x^2 - 18x + 9}$

$\cancel{-44x < 5x^2 - 18x - 112}$

$y = -2x + 11$

$y > 3x - 3$

$2x - 11 = 3x - 3$

$- = x$

$3x - 3 = -2x + 11$

$5x = 14$

$x = 2.8$

$2.8 < x$

(c) Find the smallest integer N satisfying the inequality $|2\ln N - 11| < 3\ln N - 3$. [2]

$2.8 < \ln N$

$e^{2.8} < N$

$16.4 < N$ $N = 17$

6. (a) Given that $y = \ln(\ln x)$, show that

$$\frac{dy}{dx} = \frac{1}{x \ln x}.$$

[1]

let $u = \ln x$ $\quad y = \ln(u)$

$\frac{du}{dx} = \frac{1}{x}$ $\quad \frac{dy}{du} = \frac{1}{u}$ $\quad \frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{u} = \frac{1}{x \ln x}$

The variables x and t satisfy the differential equation

$$x \ln x + t\frac{dx}{dt} = 0.$$

It is given that $x = e$ when $t = 2$.

(b) Solve the differential equation obtaining an expression for x in terms of t, simplifying your answer. [7]

$-t\frac{dx}{dt} = x\ln x$

$-t\,dx = x\ln x\,dt$

$\int \frac{1}{-t}dt = \int \frac{1}{x\ln x}dx$ $\quad$ ~~let $u = x\ln x$~~

$-1\int t^{-1}dt = \ln(\ln x)$

$-\ln(t) = \ln(\ln x) + C$

when $t = 2$ and $x = e$

$-\ln(2) = 0 + C$

$C = -\ln(2)$

$-\ln(t) = \ln(\ln x) - \ln 2$ $\quad \ln 2 = \ln(\ln x) + \ln(t)$

~~$0 = \ln\left(\frac{\ln x}{2}\right) + \ln(t)$~~ $\quad \ln 2 = \ln(\ln(x) \cdot t)$

~~$= \ln$~~ $\quad 2 = t\ln(x)$

$\cancel{\frac{2}{\ln x} = t}$

$\ln(x) = \frac{2}{t}$

$x = e^{\frac{2}{t}}$

(c) Hence state what happens to the value of x as t tends to infinity. [1]

as $t \to \infty$

$\frac{2}{t} \to 0$

$e^{\frac{2}{t}} \to 1$

$x \to 1$ x will become 1

7. (a) The complex number u is given by $u = 8 - 15\mathrm{i}$. Showing all necessary working, find the two square roots of u. Give answers in the form $a + \mathrm{i}b$, where the numbers a and b are real and exact. [5] -2

$u = 8 - 15i$

$|u| = \sqrt{8^2 + 15^2}$ $\quad \arg(u) = -\tan^{-1}\left(\frac{15}{8}\right)$ $\quad \arg(\sqrt{u}) = -0.5404195$

$= 17$ $\quad = -1.080839$ $\quad |\sqrt{u}| = \sqrt{17}$

$u = |u|(\cos(\arg u) + i\sin(\arg u))$

$u = 17(\cos(-1.080839) + i\sin(-1.080839))$

$\sqrt{u} = \sqrt{17}(\cos(-0.5404195) + i\sin(-0.5404195))$

$= 3.53 - 2.12i$

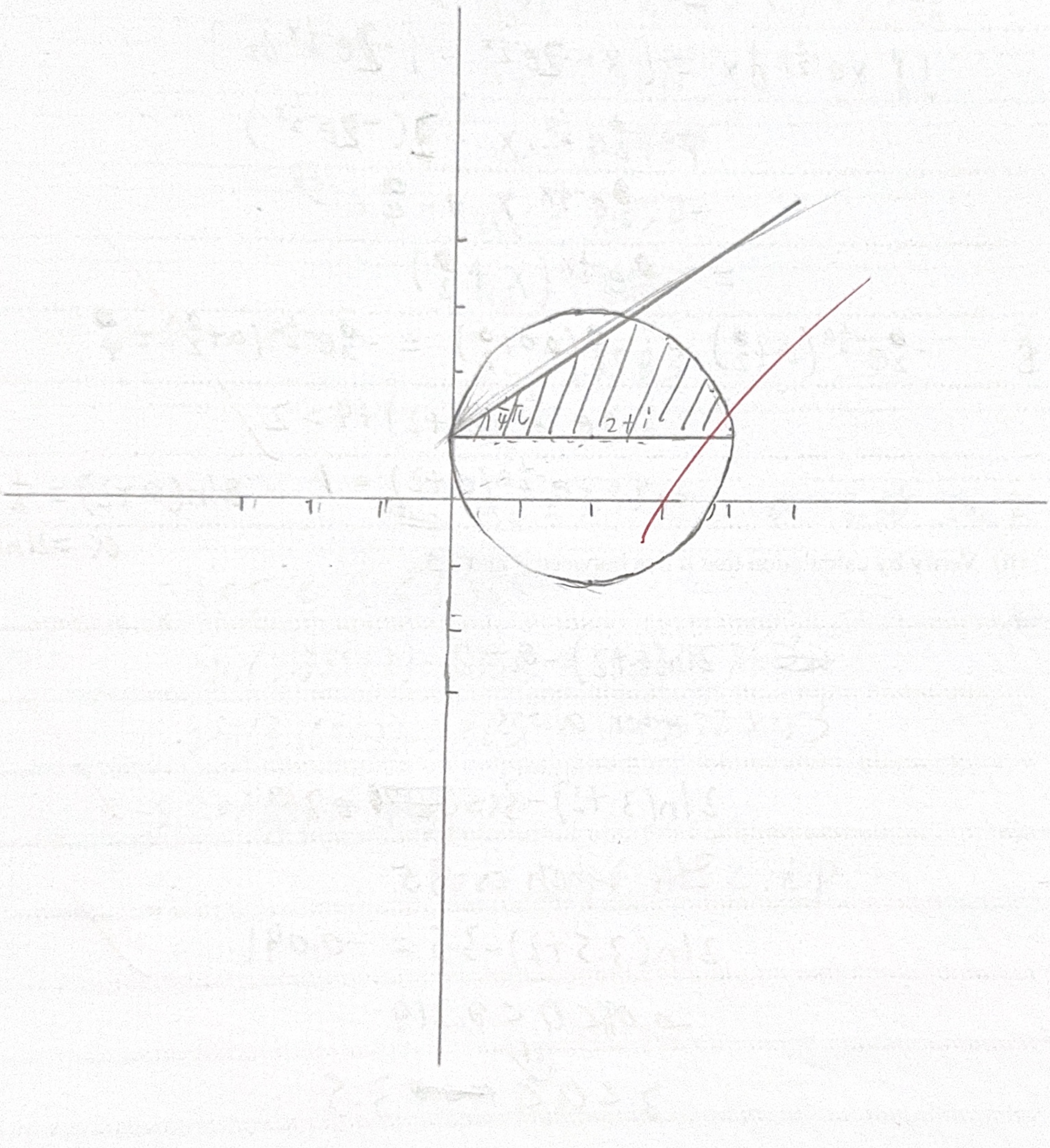
$\cancel{\sqrt{u} = \sqrt{8 - 15i}}$ $\quad \cancel{v^2 = 8 - 15i}$

$\cancel{v = \sqrt{8 - 15i}}$

$\cancel{v =}$

(b) On an Argand diagram, shade the region whose points represent complex numbers satisfying both the inequalities $|z - 2 - i| \leqslant 2$ and $0 \leqslant \arg(z - i) \leqslant \frac{1}{4}\pi$. [4]

$|z-(2+i)| \leq 2$



8. The positive constant a is such that $\int_0^a x e^{-\frac{1}{2}x}\,dx = 2$.

$u \quad \frac{dv}{dx}$

(i) Show that a satisfies the equation $a = 2\ln(a+2)$. [5]

$u = x \qquad \frac{dv}{dx} = e^{-\frac{1}{2}x} \qquad t = -\frac{1}{2}x$

$\frac{du}{dx} = 1 \qquad v = -2e^{-\frac{1}{2}x} \qquad \frac{dt}{dx} = -\frac{1}{2} \qquad dx = -2dt$

$\int x e^{-\frac{1}{2}x} dx = [x \cdot -2e^{-\frac{1}{2}x} - \int -2e^{-\frac{1}{2}x} dx]$

$= -2e^{-\frac{1}{2}x} \cdot x + 2(-2e^{-\frac{1}{2}x})$

$= -2e^{-\frac{1}{2}x} \cdot x - 4e^{-\frac{1}{2}x}$

$= -2e^{-\frac{1}{2}x}(x+2)$

$-2e^{-\frac{1}{2}a}(a+2) - (-2(0+2)) = -2e^{-\frac{1}{2}a}(a+2) + 4$

$-2e^{-\frac{1}{2}a}(a+2) + 4 = 2$

$e^{-\frac{1}{2}a}(a+2) = 1 \qquad \ln(a+2) = \frac{1}{2}a$

$a = 2\ln(a+2)$

(ii) Verify by calculation that a lies between 3 and 3.5. [2]

$2\ln(a+2) - a = 0$

when $a = 3$

$2\ln(3+2) - 3 = 0.219$

when $a = 3.5$

$2\ln(3.5+2) - 3.5 = -0.091$

$-0.091 < 0 < 0.219$

$3 < a < 3.5$

(iii) Use an iteration based on the equation in part **(i)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$a = 2\ln(a+2)$ $\quad a_0 = 3$

$a_1 = 2\ln(3+2) = 3.2189$

$a_2 = 2\ln(a_1+2) = 3.3046$

$a_3 = 3.3371$

$a_4 = 3.3494$

$a_5 = 3.3540$

$a_6 = 3.3557$

$a_7 = 3.3563$

$a_8 = 3.3566$

$a_9 = 3.3566$ $\quad a = 3.36$ to 2 d.p.

$3.355 < a < 3.365$

$2\ln(3.355+2) - 3.355 = 1.0614 \times 10^{-3}$

$2\ln(3.365+2) - 3.365 = -5.2072 \times 10^{-3}$

$-5.2072 \times 10^{-3} < 0 < 1.0614 \times 10^{-3}$

a is 3.36 to 2 d.p.

9. Let $f(x) = \frac{4x^2 + 7x + 4}{(2x+1)(x+2)}$.

(i) Express $f(x)$ in partial fractions. [5]

$\cancel{4x^2+}$

$2x^2+5x+2 \overline{)\,4x^2+7x+4}$ quotient 2

$4x^2+10x+4$

$-3x$

$(2x+1)(x+2) = \cancel{2x^2+4x+x+2}\; 2x^2+x+4x+2$

$= 2x^2+5x+2$

$\frac{4x^2+7x+4}{(2x+1)(x+2)} = 2 + \frac{-3x}{(2x+1)(x+2)}$

$\frac{-3x}{(2x+1)(x+2)} = \frac{A}{(2x+1)} + \frac{B}{(x+2)}$

$-3x = A(x+2) + B(2x+1)$

$A + 2B = -3$

$2A + B = 0$

$4A + 2B = 0$

$3A = 3$

$A = 1$

$B = -2$

$f(x) = 2 + \frac{1}{2x+1} - \frac{2}{x+2}$

(ii) Show that $\int_0^4 f(x)\,dx = 8 - \ln 3$. [5] 2

$\int_0^4 2 + \frac{1}{2x+1} - \frac{2}{x+2}\,dx$

$= \left[2x + \overset{\frac{1}{}}{2}\ln(2x+1) - 2\ln(x+2)\right]_0^4$ B2 B0

$= 8 + 2\ln(9) - 2\ln(4) - (0 + 2\ln(1) - 2\ln(2))$

$= 8 + 2\ln(9) - 2\ln(4) + 2\ln(2)$ M1

$= 8 + \ln\left(\frac{81 \times 4}{16}\right)$ A0

$=$

10. (i) Show that $\sin 2x \cot x \equiv 2\cos^2 x$. [2]

LHS

$\sin 2x \cot x$

$= \sin 2x \frac{1}{\tan x}$

$= \sin 2x \frac{\cos x}{\sin x}$

$= 2\cancel{\sin(x)}\cos(x) \cdot \frac{\cos x}{\cancel{\sin x}}$

$= 2\cos^2 x$

(ii) Using the identity in part **(i)**,

(a) find the least possible value of

$$3\sin 2x \cot x + 5\cos 2x + 8$$

as x varies, [4]

$3 \cdot 2\cos^2 x + 5\cos 2x + 8$

$\downarrow$ $\downarrow$

~~least 0 least 5~~

~~$\cos x(6\cos x + 5) + 8$~~

$5\cos 2x = 5[2(\cos^2 x - 1)]$

$= 10\cos^2 x - 5$

$6\cos^2 x + 10\cos^2 x + 3$

$16\cos^2 x + 3 \Rightarrow$ least is 3

$\downarrow$

0

(b) find the exact value of $\int_{\frac{1}{8}\pi}^{\frac{1}{6}\pi} \operatorname{cosec} 4x \tan 2x \, dx$. [5]

$\operatorname{cosec} 4x \tan 2x = \frac{1}{\sin 4x} \cdot \frac{\sin 2x}{\cos 2x}$

$= \cancel{2\sin 2x \cos 2x} \cdot \cancel{\frac{\sin 2x}{\cos 2x}}$

$= 2\sin^2 2x$

$2\int (\sin 2x)^2 dx$ let $u = \sin 2x$

$2\int u^2 \cdot \frac{1}{2\cos 2x} du$ $\frac{du}{dx} = 2\cos 2x$

$\cancel{2\int 2\left(\frac{1}{3}\right.}$ $dx = \frac{1}{2\cos 2x} du$

$\sin 2x \cdot \frac{\sin 2x}{\cos 2x}$