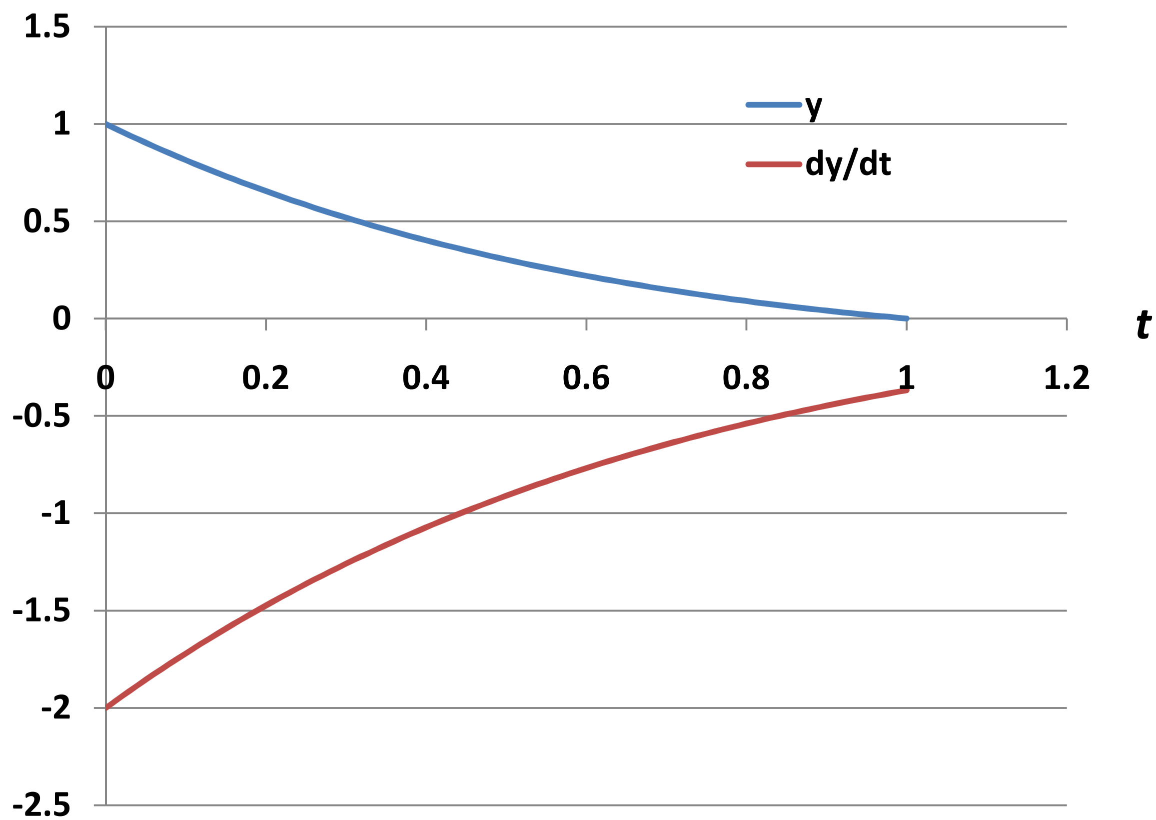
First Order ODE's: Example

- Given: $dy/dt = -y - e^{-t}$; $y(0)=1$

➢ Find: y at t=0.1, 0.2, 0.3, 0.4, 0.5 (using h=0.1)

➢ Exact Solution: $y = e^{-t}(1-t)$



Example

- For t=0.1 (TV = 0.814354):

➢ Euler Forward : $y_{n+1} = y_n + hf(t_n, y_n)$

$$y_{0.1} = 1 + 0.1(-2) = 0.8$$

➢ Euler Backward : $y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$

$$y_{0.1} = 1 + 0.1\left(-y_{0.1} - e^{-0.1}\right) => y_{0.1} = 0.826833$$

➢ Trapezoidal or Implicit Heun's :

$$y_{n+1} = y_n + h\frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}$$

$$y_{0.1} = 1 + 0.1\frac{-2 + \left(-y_{0.1} - e^{-0.1}\right)}{2} => y_{0.1} = 0.814055$$

Example

- For t=0.2 (TV = 0.654985):

➢ Euler Forward : $y_{n+1} = y_n + hf\left(t_n, y_n\right)$

$$y_{0.2} = 0.8 + 0.1\left(-1.704837\right) = 0.629516$$

➢ Euler Backward : $y_{n+1} = y_n + hf\left(t_{n+1}, y_{n+1}\right)$

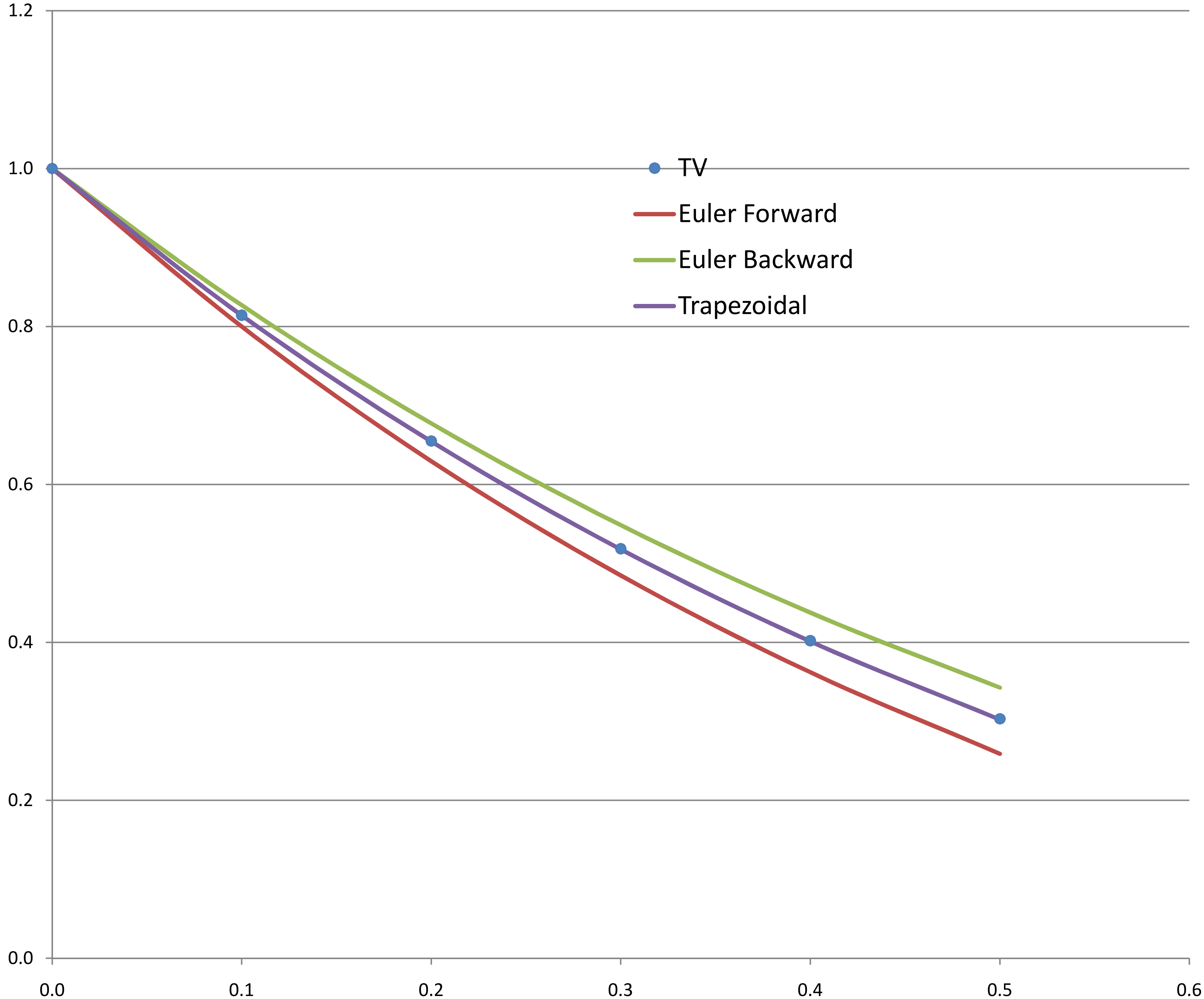
$$y_{0.2} = 0.826833 + 0.1\left(-y_{0.2} - e^{-0.2}\right) \Rightarrow y_{0.2} = 0.677236$$

➢ Trapezoidal or Implicit Heun's :

$$y_{n+1} = y_n + h\frac{f\left(t_n, y_n\right) + f\left(t_{n+1}, y_{n+1}\right)}{2}$$

$$y_{0.2} = 0.814055 + 0.1\frac{-1.718893 + \left(-y_{0.2} - e^{-0.2}\right)}{2} \Rightarrow y_{0.2} = 0.654452$$

t	TV	Euler Forward	Euler Backward	Trapezoidal
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.814354	0.800000	0.826833	0.814055
0.2	0.654985	0.629516	0.677236	0.654452
0.3	0.518573	0.484692	0.548322	0.517859
0.4	0.402192	0.362141	0.437537	0.401342
0.5	0.303265	0.258895	0.342621	0.302316



1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
TV
Euler Forward
Euler Backward
Trapezoidal

Example

- Explicit multi-step method, with *k*=2:

$$y_{n+1} = y_n + h\left(\frac{23}{12} f_n - \frac{4}{3} f_{n-1} + \frac{5}{12} f_{n-2}\right)$$

- Use the values obtained from Trapezoidal
- At t=0.3 (TV= 0.518573):

$$y_{0.3} = y_{0.2} + h\left(\frac{23}{12} f_{0.2} - \frac{4}{3} f_{0.1} + \frac{5}{12} f_0\right)$$

$$= 0.654452 - 0.1\left(\frac{23}{12} 1.473182 - \frac{4}{3} 1.718893 + \frac{5}{12} 2\right)$$

$$= 0.517944$$

Example

- Implicit multi-step method, with *k*=1:

$$y_{n+1} = y_n + h\left(\frac{5}{12} f_{n+1} + \frac{2}{3} f_n - \frac{1}{12} f_{n-1}\right)$$

- Use the values obtained from Trapezoidal
- At t=0.2 (TV= 0.654985):

$$y_{0.2} = y_{0.1} + h\left(\frac{5}{12} f_{0.2} + \frac{2}{3} f_{0.1} - \frac{1}{12} f_0\right)$$

$$y_{0.2} = y_{0.1} + 0.1\left(\frac{5}{12}\left[- y_{0.2} - e^{-0.2}\right] - \frac{2}{3} 1.718893 + \frac{1}{12} 2\right)$$

- => $y_{0.2}$ = 0.654735

Example

- Backward Difference method, k=2:

$$hf_{n+1} = \frac{3}{2} y_{n+1} - 2y_n + \frac{1}{2} y_{n-1}$$

- At t=0.2 (TV= 0.654985):

$$0.1\left(-y_{0.2} - e^{-0.2}\right) = \frac{3}{2} y_{0.2} - 2y_{0.1} + \frac{1}{2} y_0$$

$$0.1\left(-y_{0.2} - e^{-0.2}\right) = \frac{3}{2} y_{0.2} - 2 \times 0.814055 + \frac{1}{2} \times 1$$

=> $y_{0.2} = 0.653899$

Most Common: Runge-Kutta methods

- The "average slope" over the interval (t_n, t_{n+1}) is approximated by a weighted mean of slopes at "a few" intermediate points in the interval (t_n, t_{n+1}).
- Explicit, since the "intermediate points" are obtained directly from "known" values at "previous" points: No start-up problem.
- ***General Form:***

$$y_{n+1} = y_n + h \sum_{i=1}^{m} w_i k_i$$

where, w are the weights and k are slopes

Runge-Kutta methods

$$y_{n+1} = y_n + h\sum_{i=1}^{m} w_i k_i$$

- For example, with m=1, we have the Explicit Euler method:

$$w_1 = 1; k_1 = f\left(t_n, y_n\right); y_{n+1} = y_n + hf\left(t_n, y_n\right)$$

- With m=2, we may write

$$k_1 = f\left(t_n, y_n\right); k_2 = f\left(t_n + \alpha h, y_n + \alpha h k_1\right); y_{n+1} = y_n + h\left(w_1 k_1 + w_2 k_2\right)$$

- How to obtain the 3 parameters w_1, w_2, α
- To make it more general, we could use k_2 at $t_n+\alpha_1 h$, $y_n+\alpha_2 h k_1$: Discussed later.

Runge-Kutta methods

- m=2,

$$k_1 = f\left(t_n, y_n\right); k_2 = f\left(t_n + \alpha h, y_n + \alpha h k_1\right); y_{n+1} = y_n + h\left(w_1 k_1 + w_2 k_2\right)$$

- Taylor's series:

$$y_{n+1} = y_n + h f_n + \frac{h^2}{2!} f_n' + \frac{h^3}{3!} f_n'' + \frac{h^4}{4!} f_n''' + \ldots$$

$$\begin{aligned}
k_2 &= f\left(t_n + \alpha h, y_n + \alpha h k_1\right) \\
&= f_n + \alpha h \frac{\partial f}{\partial t} + \alpha h f_n \frac{\partial f}{\partial y} \\
&\quad + \alpha^2 h^2 \frac{\partial^2 f}{\partial t^2} + 2\alpha^2 h^2 f_n \frac{\partial f}{\partial t} \frac{\partial f}{\partial y} + \alpha^2 h^2 f_n^2 \frac{\partial^2 f}{\partial y^2} + \ldots
\end{aligned}$$

$$y_n + hf_n + \frac{h^2}{2!} f_n' + \frac{h^3}{3!} f_n'' + \frac{h^4}{4!} f_n''' + \ldots = y_n + w_1 h f_n +$$

$$w_2 h \left(\begin{array}{l} f + \alpha h \dfrac{\partial f}{\partial t} + \alpha h f \dfrac{\partial f}{\partial y} + \alpha^2 h^2 \dfrac{\partial^2 f}{\partial t^2} + \\ 2\alpha^2 h^2 f \dfrac{\partial f}{\partial t} \dfrac{\partial f}{\partial y} + \alpha^2 h^2 f^2 \dfrac{\partial^2 f}{\partial y^2} + \ldots \end{array} \right)_n$$

$$f' = \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dy}{dt} \frac{\partial f}{\partial y} = \frac{\partial f}{\partial t} + f \frac{\partial f}{\partial y}$$

$$f'' = \frac{d}{dt} \frac{df}{dt} = \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial t} + f \frac{\partial f}{\partial y} \right) + f \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial t} + f \frac{\partial f}{\partial y} \right)$$

Runge-Kutta methods

- Equating coefficients:

$$h: \quad 1 = w_1 + w_2$$

$$h^2: \quad \frac{1}{2}\left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right) = w_2\alpha\frac{\partial f}{\partial t} + w_2\alpha f\frac{\partial f}{\partial y} \Rightarrow w_2\alpha = \frac{1}{2}$$

- If we use h^3 also, there would be 3 more equations – No solution
- Therefore, the best we can do is Error O(h^3)
- Infinite combinations possible with

$$w_1; w_2 = 1 - w_1; \alpha = \frac{1}{2w_2}$$

2^{nd} order Runge-Kutta methods

- Known as the second-order R-K method
- For example, $w_1 = 0; w_2 = 1; \alpha = \dfrac{1}{2}$

  known as the Mid-point method or Modified Euler method:

$$y_{n+1} = y_n + hf\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f\left(t_n, y_n\right)\right)$$

- $w_1 = \dfrac{1}{2}; w_2 = \dfrac{1}{2}; \alpha = 1$: Heun's or Improved Euler

$$y_{n+1} = y_n + \frac{h}{2}\left(f\left(t_n, y_n\right) + f\left(t_n + h, y_n + hf\left(t_n, y_n\right)\right)\right)$$

2^{nd} order Runge-Kutta methods

$$w_1 = \frac{1}{3}; w_2 = \frac{2}{3}; \alpha = \frac{3}{4}$$

Ralston's method

$$y_{n+1} = y_n + \frac{h}{3}\left(f(t_n, y_n) + 2f\left(t_n + \frac{3}{4}h, y_n + \frac{3}{4}hf(t_n, y_n) \right) \right)$$

Similarly, third-, fourth-, and higher order methods could be derived. General form is:

$$y_{n+1} = y_n + h\sum_{i=1}^{m} w_i k_i$$

$$k_1 = f(t_n, y_n); k_2 = f(t_n + \beta_1 h, y_n + \alpha_{11} h k_1)$$

$$k_3 = f(t_n + \beta_2 h, y_n + \alpha_{12} h k_1 + \alpha_{22} h k_2)$$

$$\ldots$$

$$k_m = f\left(t_n + \beta_{m-1} h, y_n + \sum_{i=1}^{m-1} \alpha_{i,m-1} h k_i \right)$$

Higher order Runge-Kutta methods

- Third Order (one of several possible forms)

$$k_1 = f(t_n, y_n); k_2 = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right)$$

$$k_3 = f(t_n + h, y_n - hk_1 + 2hk_2)$$

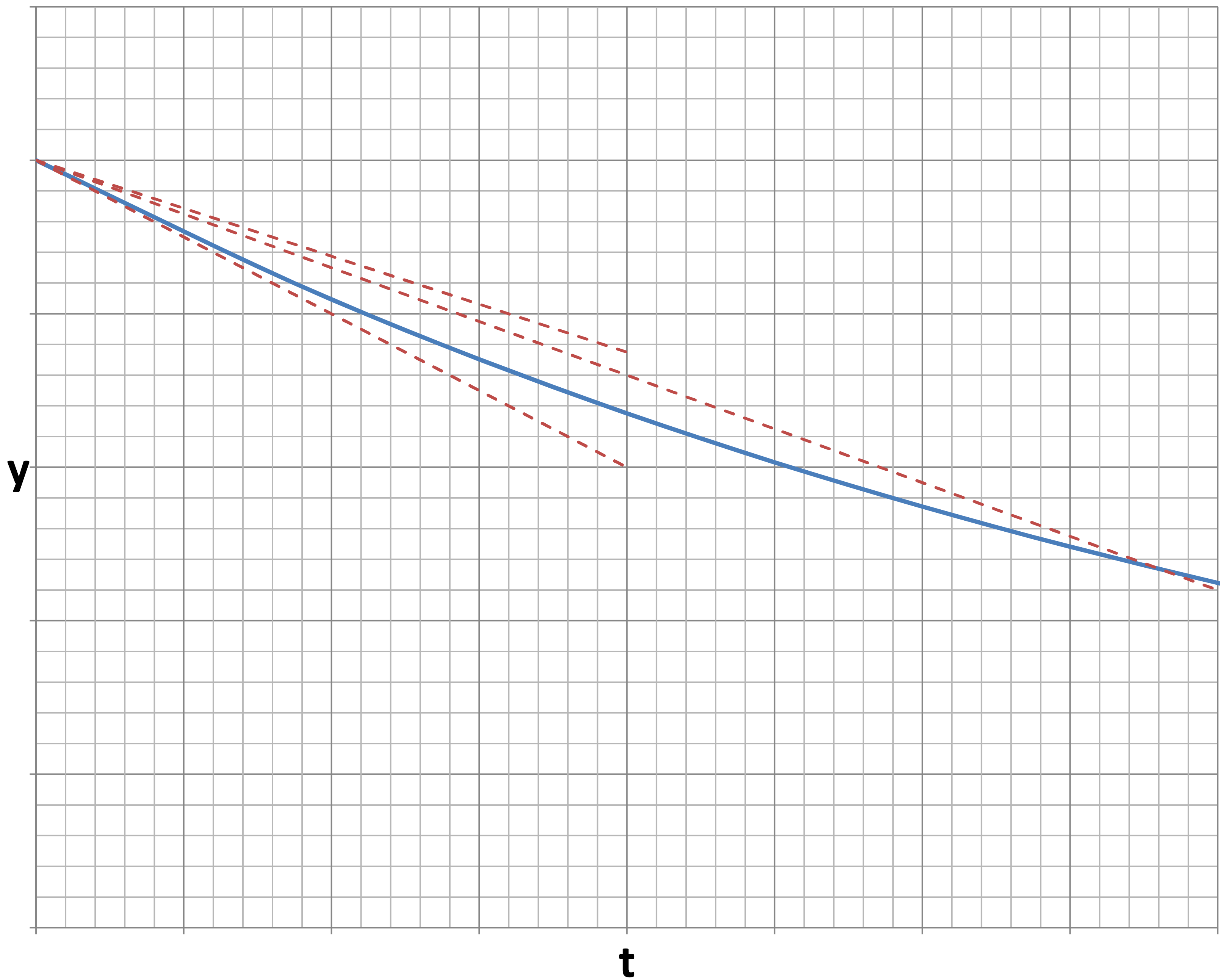
$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 4k_2 + k_3)$$

- Most common: fourth-order

$$k_1 = f(t_n, y_n); k_2 = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right)$$

$$k_3 = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right); k_4 = f(t_n + h, y_n + hk_3)$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$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