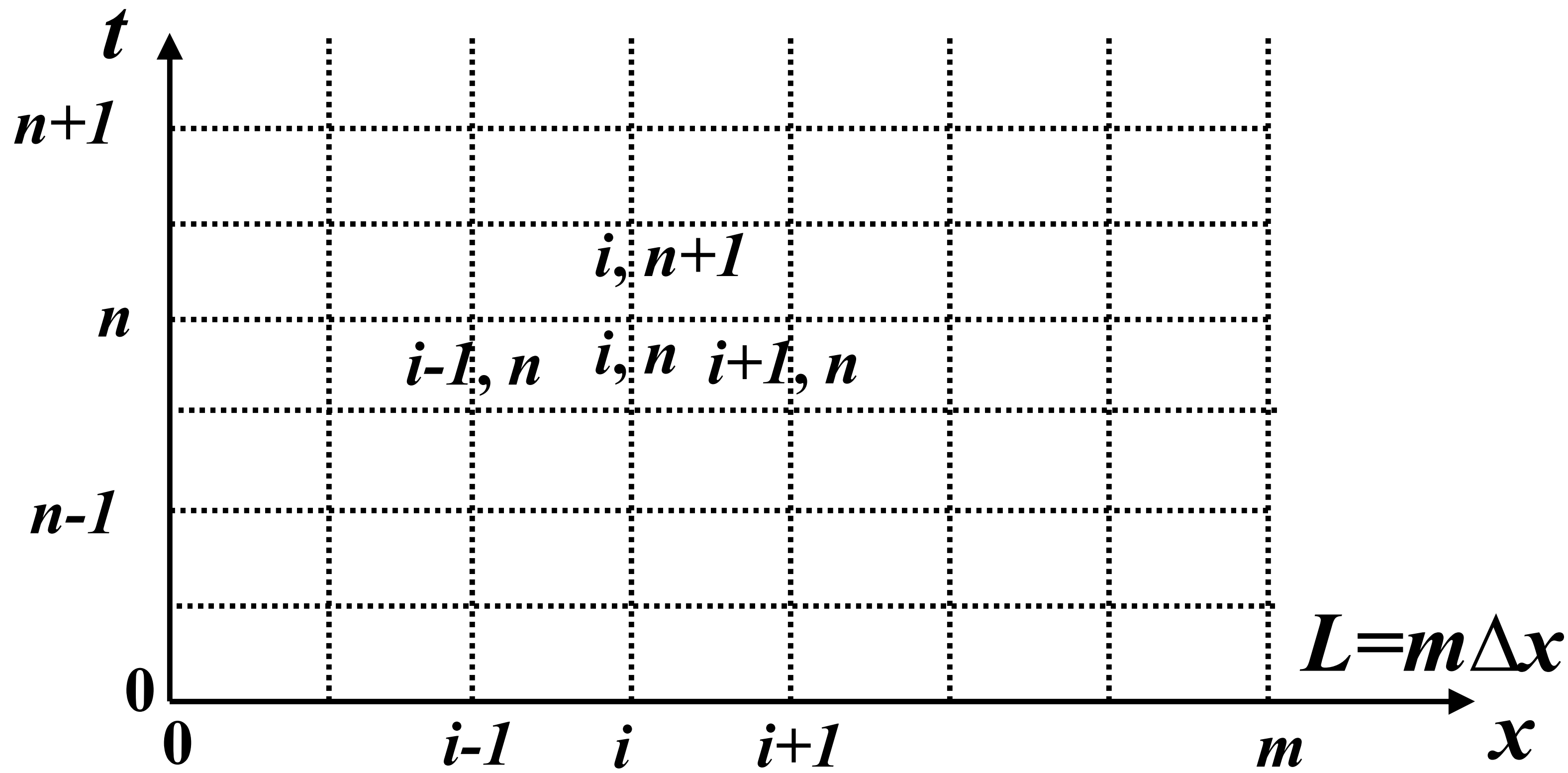
Advection-Diffusion Equation

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial x^2}$$

$$c_i^{n+1} = \left(\frac{u\Delta t}{2\Delta x} + \frac{D\Delta t}{\Delta x^2}\right)c_{i-1}^n + \left(1 - \frac{2D\Delta t}{\Delta x^2}\right)c_i^n + \left(-\frac{u\Delta t}{2\Delta x} + \frac{D\Delta t}{\Delta x^2}\right)c_{i+1}^n$$



Time-Weighting

$$c_i^{n+1-\mu} = \mu c_i^n + (1-\mu)c_i^{n+1}$$

$$-(1-\mu)\left(\frac{C}{2}+\frac{C}{P_g}\right)c_{i-1}^{n+1} + \left(1+\frac{2(1-\mu)C}{P_g}\right)c_i^{n+1} + (1-\mu)\left(\frac{C}{2}-\frac{C}{P_g}\right)c_{i+1}^{n+1} =$$

$$\mu\left(\frac{C}{2}+\frac{C}{P_g}\right)c_{i-1}^{n} + \left(1-\frac{2\mu C}{P_g}\right)c_i^{n} + \mu\left(-\frac{C}{2}+\frac{C}{P_g}\right)c_{i+1}^{n}$$

- For Neumann B.C. (explicit, with zero derivative)

$$c_m^{n+1} = \frac{2D\Delta t}{\Delta x^2}c_{m-1}^n + \left(1-\frac{2D\Delta t}{\Delta x^2}\right)c_m^n$$

Wave Equation

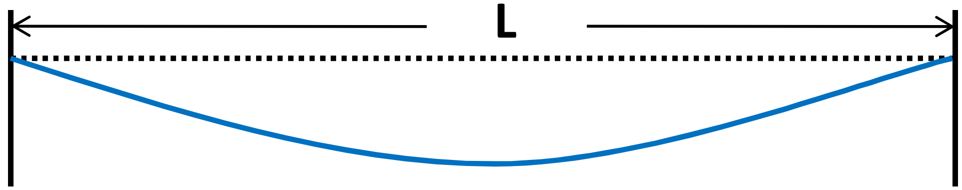
- Waves: Seismic, Sound, Electromagnetic..
- We consider only 1-D case

$$\frac{\partial^2 \phi}{\partial t^2} = u(x)^2 \frac{\partial^2 \phi}{\partial x^2}$$

- u is velocity, may vary with x (we assume it to be constant), ϕ is a scalar (could be displacement, electric/magnetic field…)
- Need two initial and two boundary conditions: E.g. $\phi(0,x)$ and $\partial\phi/\partial t(0,x)$; $\phi(t,0)$ and $\phi(t,L)$.

Wave Equation

- Consider the vibration of a string fixed between two supports



- $\phi(x,t)$ represents the vertical displacement
- u is the wave velocity, depends on tension in the string and density
- The two initial conditions are: $\phi(0,x)=f(x)$ and $\partial\phi/\partial t(0,x)=0$; and the two boundary conditions are $\phi(t,0)=0$ and $\phi(t,L)=0$.

Wave Equation

- One method to solve is by reducing into two first order equations:

$$\frac{\partial \phi_1}{\partial t} = \phi_2$$

$$\frac{\partial \phi_2}{\partial t} = u^2 \frac{\partial^2 \phi_1}{\partial x^2}$$

- We get a system of IVPs by using semi-discretization w.r.t. x
- Solve by any of the previously discussed techniques

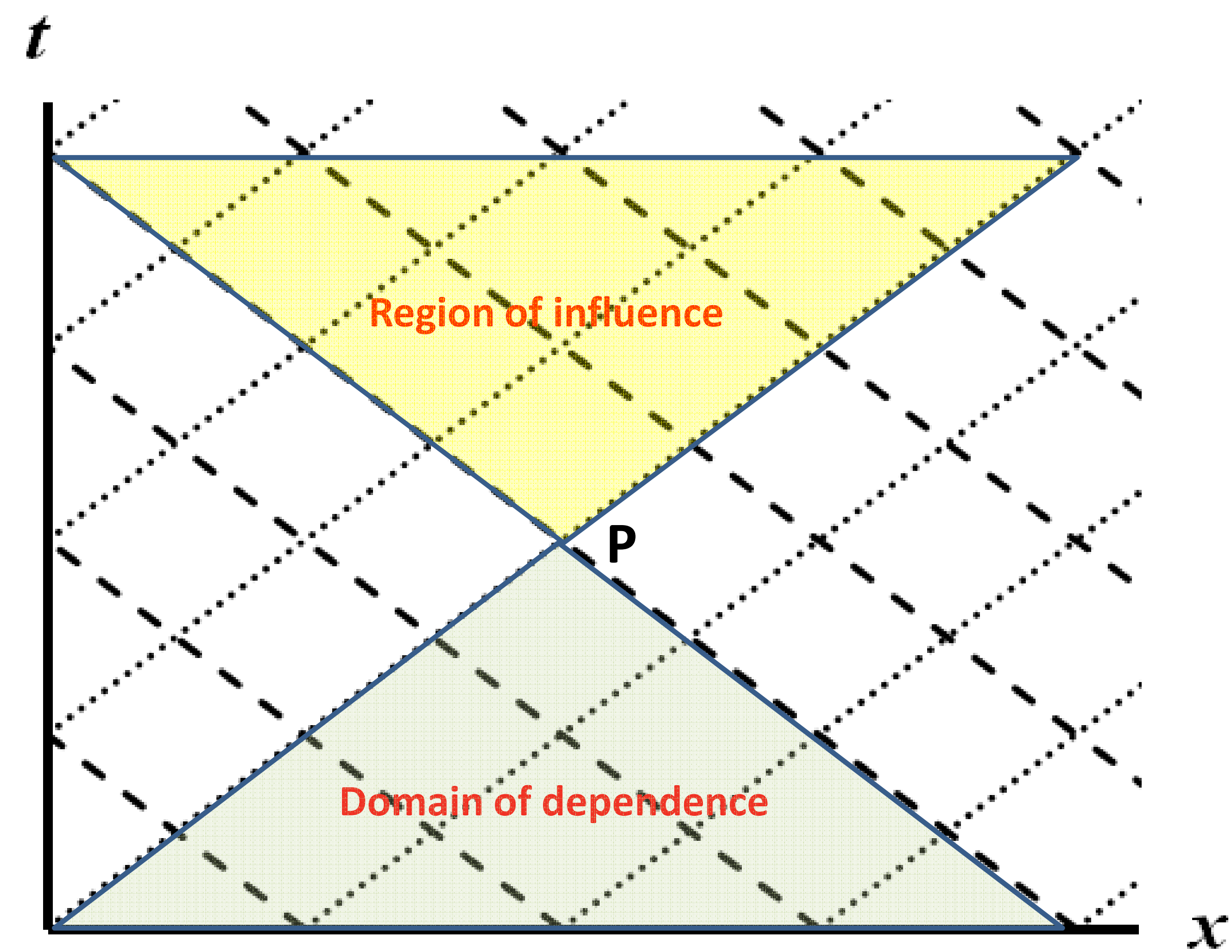
Wave Equation

- Or, we could use full-discretization:

$$\frac{\partial^2 \phi}{\partial t^2} = u^2 \frac{\partial^2 \phi}{\partial x^2}$$

$$\frac{\phi_i^{n-1} - 2\phi_i^n + \phi_i^{n+1}}{\Delta t^2} = u^2 \frac{\phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n}{\Delta x^2}$$

- How to decide the step-size?
- Need to look at characteristics
- a=1, b=0, c= $-u^2$. b^2 – ac= u^2 .
- Hyperbolic equation: Two sets of characteristics with slope $1/u$ and $-1/u$

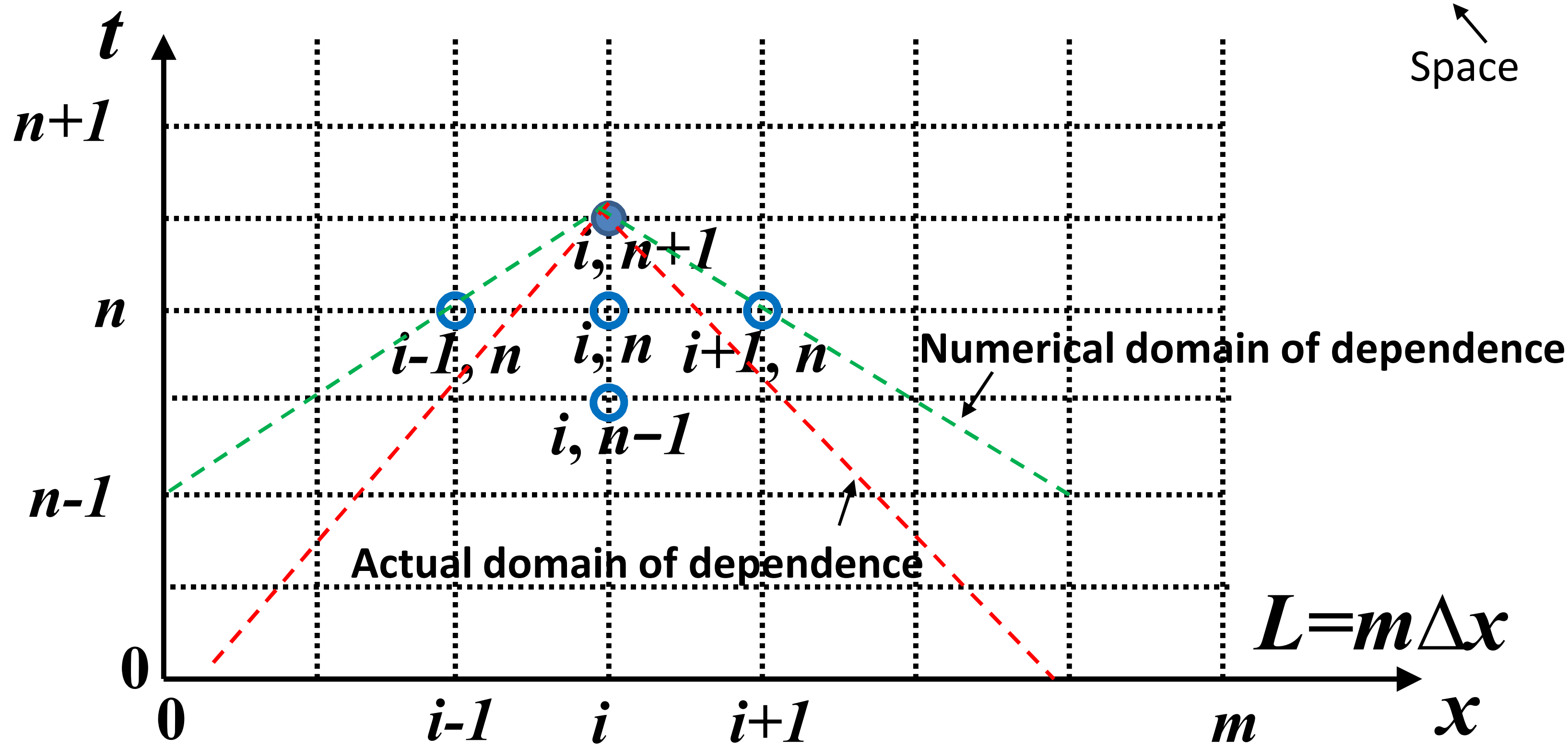

t
Region of influence
P
Domain of dependence
x

Wave Equation

- Dirichlet B.C. : $\phi(t,0)=0,\ \phi(t,L)=0$
- Initial displacement: e.g. $\phi(0,x)= \sin(\pi x/L)$
- Initial velocity, e.g., $\partial\phi/\partial t(0,x)=0$
- Uniform grid, space Δx and time, Δt: ϕ_i^n

Time

Space



Hyperbolic Equation

- A necessary condition for convergence: the numerical domain of dependence must contain the physical domain of dependence
- Known as the Courant-Friedrichs-Lewy (CFL) condition
- $u\Delta t \leq \Delta x$ for the explicit scheme
- The Courant number, C≤ 1
- Implicit schemes, no limit on C for convergence (should be small for accuracy)

Wave Equation: Implicit Scheme

- The spatial derivative may be written as a weighted average at different times:

$$\frac{\phi_i^{n-1} - 2\phi_i^{n} + \phi_i^{n+1}}{\Delta t^2} = u^2 \left[\begin{array}{l} \dfrac{1}{4}\dfrac{\phi_{i-1}^{n-1} - 2\phi_i^{n-1} + \phi_{i+1}^{n-1}}{\Delta x^2} + \\ \dfrac{1}{2}\dfrac{\phi_{i-1}^{n} - 2\phi_i^{n} + \phi_{i+1}^{n}}{\Delta x^2} + \\ \dfrac{1}{4}\dfrac{\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1}}{\Delta x^2} \end{array} \right]$$

Wave Equation: Implicit Scheme

- Results in a tridiagonal system:

$$-\frac{C^2}{4}\phi_{i-1}^{n+1}+\left(1+\frac{C^2}{2}\right)\phi_i^{n+1}-\frac{C^2}{4}\phi_{i+1}^{n+1}$$

$$=\frac{C^2}{4}\phi_{i-1}^{n-1}-\left(1+\frac{C^2}{2}\right)\phi_i^{n-1}+\frac{C^2}{4}\phi_{i+1}^{n-1}$$

$$+\frac{C^2}{2}\phi_{i-1}^{n}+\left(2-C^2\right)\phi_i^{n}+\frac{C^2}{2}\phi_{i+1}^{n}$$

- Thomas algorithm. Non-self starting!
- If the initial velocity is zero, at the first time step, we could apply the central difference and write $\phi_i^{n+1}=\phi_i^{n-1}$

Implicit Scheme: Start-up

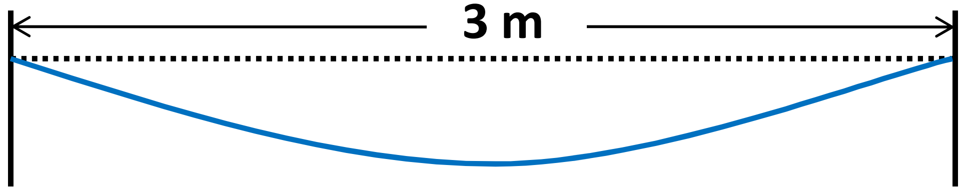
- The equations at the first time-step are:

$$-\frac{C^2}{2}\phi_{i-1}^{1}+\left(2+C^2\right)\phi_i^1-\frac{C^2}{2}\phi_{i+1}^{1}=\frac{C^2}{2}\phi_{i-1}^{0}+\left(2-C^2\right)\phi_i^0+\frac{C^2}{2}\phi_{i+1}^{0}$$

- For further steps, no problem since time level n−1 and n are available

Wave Equation: Example

- A string between two supports is given an initial displacement of $-0.05\sin(\pi x/3)$



- The wave velocity in string is 100 m/s
- Initial velocity is zero
- Using Δx=1 m and Δt=0.01 s, find the displacement at 0.01 s and 0.02 s at 1 m and 2 m
- Courant Number $C = u\Delta t/\Delta x = 1$

Implicit Scheme

- The equations at the first time-step are:

$$-0.5\phi_{i-1}^{1}+3\phi_{i}^{1}-0.5\phi_{i+1}^{1}=0.5\phi_{i-1}^{0}+\phi_{i}^{0}+0.5\phi_{i+1}^{0}$$

- The initial displacement at 1 and 2 are equal to −0.0433 m
- At Node 1:

$$-0.5\phi_{0}^{1}+3\phi_{1}^{1}-0.5\phi_{2}^{1}=0.5\phi_{0}^{0}+\phi_{1}^{0}+0.5\phi_{2}^{0}\Rightarrow 3\phi_{1}^{1}-0.5\phi_{2}^{1}=-0.06495$$

- At Node 2:

$$-0.5\phi_{1}^{1}+3\phi_{2}^{1}-0.5\phi_{3}^{1}=0.5\phi_{1}^{0}+\phi_{2}^{0}+0.5\phi_{3}^{0}\Rightarrow -0.5\phi_{1}^{1}+3\phi_{2}^{1}=-0.06495$$

- Solution is: both equal to −0.02598 m

Implicit Scheme

- The equations at the second time-step are:

$$-0.25\phi_{i-1}^{n+1} + 1.5\phi_i^{n+1} - 0.25\phi_{i+1}^{n+1}$$
$$= 0.25\phi_{i-1}^{n-1} - 1.5\phi_i^{n-1} + 0.25\phi_{i+1}^{n-1}$$
$$+0.5\phi_{i-1}^{n} + \phi_i^{n} + 0.5\phi_{i+1}^{n}$$

- At Node 1: $-0.25\phi_0^2 + 1.5\phi_1^2 - 0.25\phi_2^2 = 0.25\phi_0^0 - 1.5\phi_1^0 + 0.25\phi_2^0$
$+0.5\phi_0^1 + \phi_1^1 + 0.5\phi_2^1 \Rightarrow 1.5\phi_1^2 - 0.25\phi_2^2 = 0.01516$

- At Node 2: $-0.25\phi_1^2 + 1.5\phi_2^2 - 0.25\phi_3^2 = 0.25\phi_1^0 - 1.5\phi_2^0 + 0.25\phi_3^0$
$+0.5\phi_1^1 + \phi_2^1 + 0.5\phi_3^1 \Rightarrow -0.25\phi_1^2 + 1.5\phi_2^2 = 0.01516$

- Solution is: both equal to 0.01212 m

PDE with three independent variables

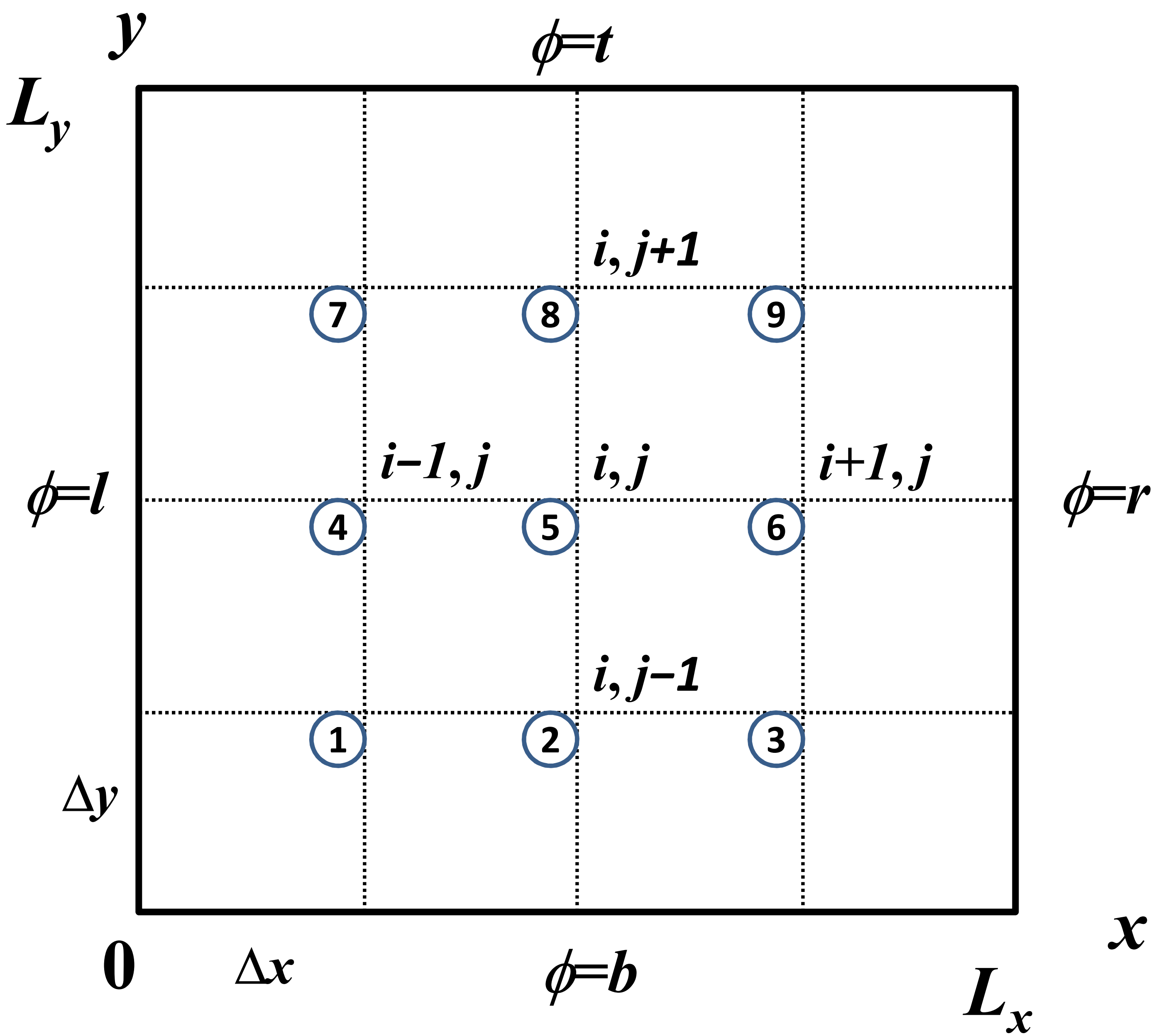
- Till now, only x and t (or x and y)
- 2-D transient diffusion

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right)$$

- Discretized form (Implicit):

$$\frac{c_{i.j}^{n+1} - c_{i,j}^{n}}{\Delta t} = D\left[\frac{c_{i-1,j}^{n+1} - 2c_{i,j}^{n+1} + c_{i+1,j}^{n+1}}{\Delta x^2} + \frac{c_{i,j-1}^{n+1} - 2c_{i,j}^{n+1} + c_{i,j+1}^{n+1}}{\Delta y^2}\right]$$

- Initial and boundary conditions needed



9 unknowns: At each time step a banded matrix is formed

Can we make it tridiagonal for faster solution?

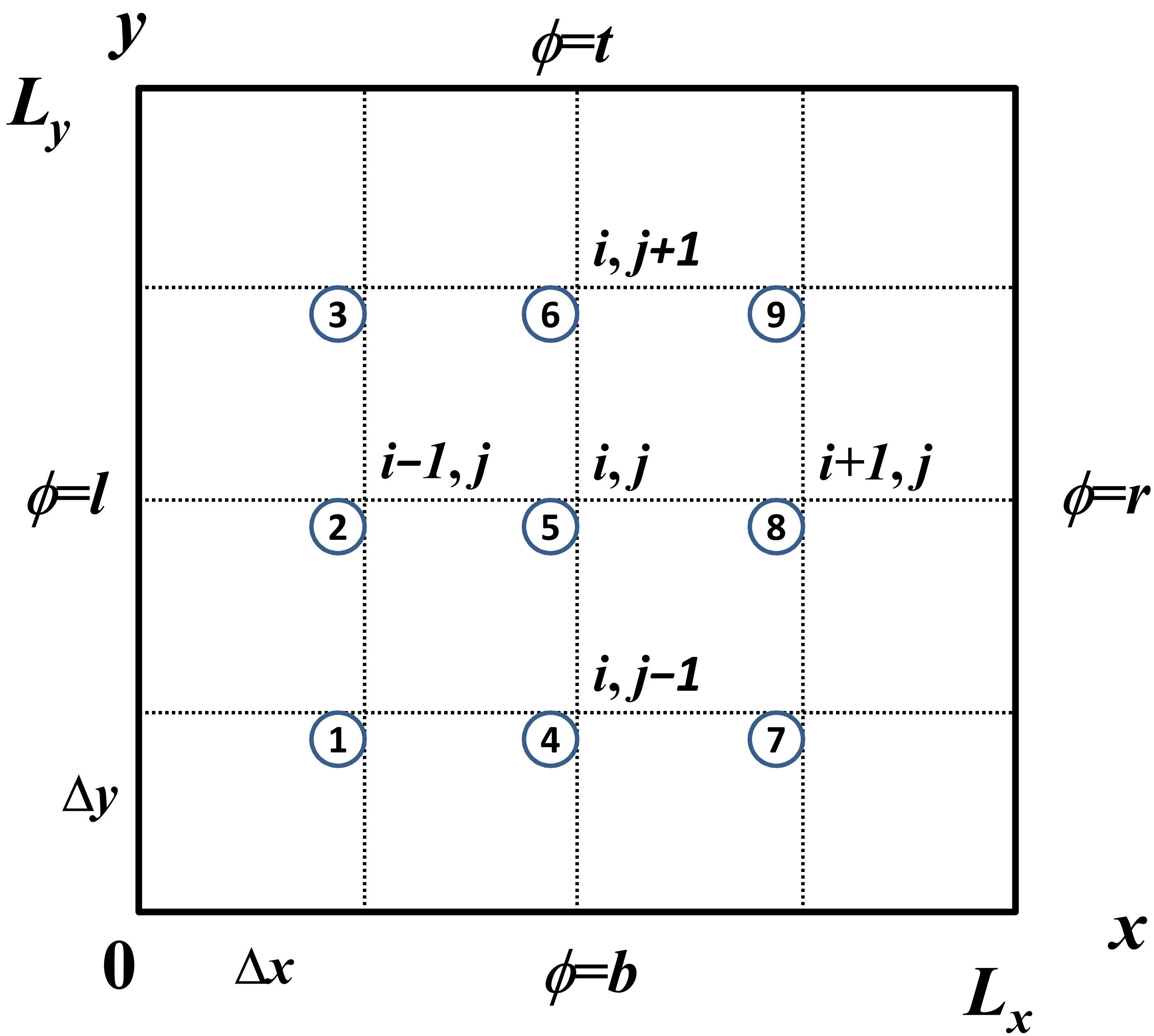
Alternating Direction Implicit Scheme

- Use implicit in one direction (say, x) at "half time step" and explicit in other (y)
- Use implicit in the other direction (y) at the next "half time step" and explicit in x

$$\frac{c_{i.j}^{n+1/2} - c_{i,j}^{n}}{\Delta t / 2} = D\left[\frac{c_{i-1,j}^{n+1/2} - 2c_{i,j}^{n+1/2} + c_{i+1,j}^{n+1/2}}{\Delta x^2} + \frac{c_{i,j-1}^{n} - 2c_{i,j}^{n} + c_{i,j+1}^{n}}{\Delta y^2}\right]$$

$$\frac{c_{i.j}^{n+1} - c_{i,j}^{n+1/2}}{\Delta t / 2} = D\left[\frac{c_{i-1,j}^{n+1/2} - 2c_{i,j}^{n+1/2} + c_{i+1,j}^{n+1/2}}{\Delta x^2} + \frac{c_{i,j-1}^{n+1} - 2c_{i,j}^{n+1} + c_{i,j+1}^{n+1}}{\Delta y^2}\right]$$

- Could reverse the order in the next time step (implicit in y for the first half)



Node numbers need to be modified accordingly in the two half-steps