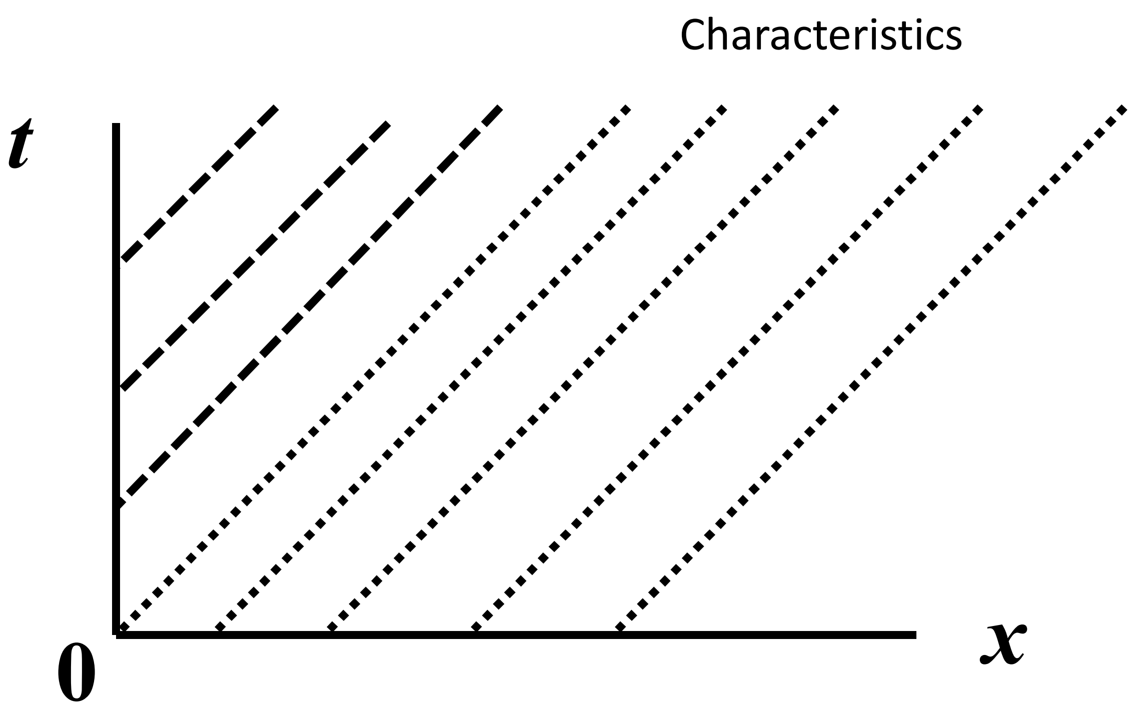
Partial Differential Equations

- Two or more independent variables
- Classifications of PDEs to identify appropriate IC/BC
- Characteristics: hyper-planes (or line, or plane), along which "information" (and discontinuity) propagates
- Help in identifying the "domain (or region) of influence" and the "domain (or region) of dependence"

Partial Differential Equations: Characteristics

- "Pure advection"

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = 0$$



Partial Differential Equations: Characteristics

- Characteristics are defined: ξ=constant, where $\xi = x - ut$
- c is constant along a characteristic line (known as Riemann Invariant)
- For "channel flow"

$$\frac{\partial y}{\partial t} + V\frac{\partial y}{\partial x} + y\frac{\partial V}{\partial x} = 0$$

$$\frac{\partial y}{\partial x} + \frac{V}{g}\frac{\partial V}{\partial x} + \frac{1}{g}\frac{\partial V}{\partial t} = 0$$

Characteristic Lines

- Characteristics are ξ=constant, and η=constant

$$\xi = x - \left(V + \sqrt{gy}\right)t \qquad \eta = x - \left(V - \sqrt{gy}\right)t$$

- Along these lines, $\frac{d}{dt}\left(V \pm 2\sqrt{gy}\right) = 0$

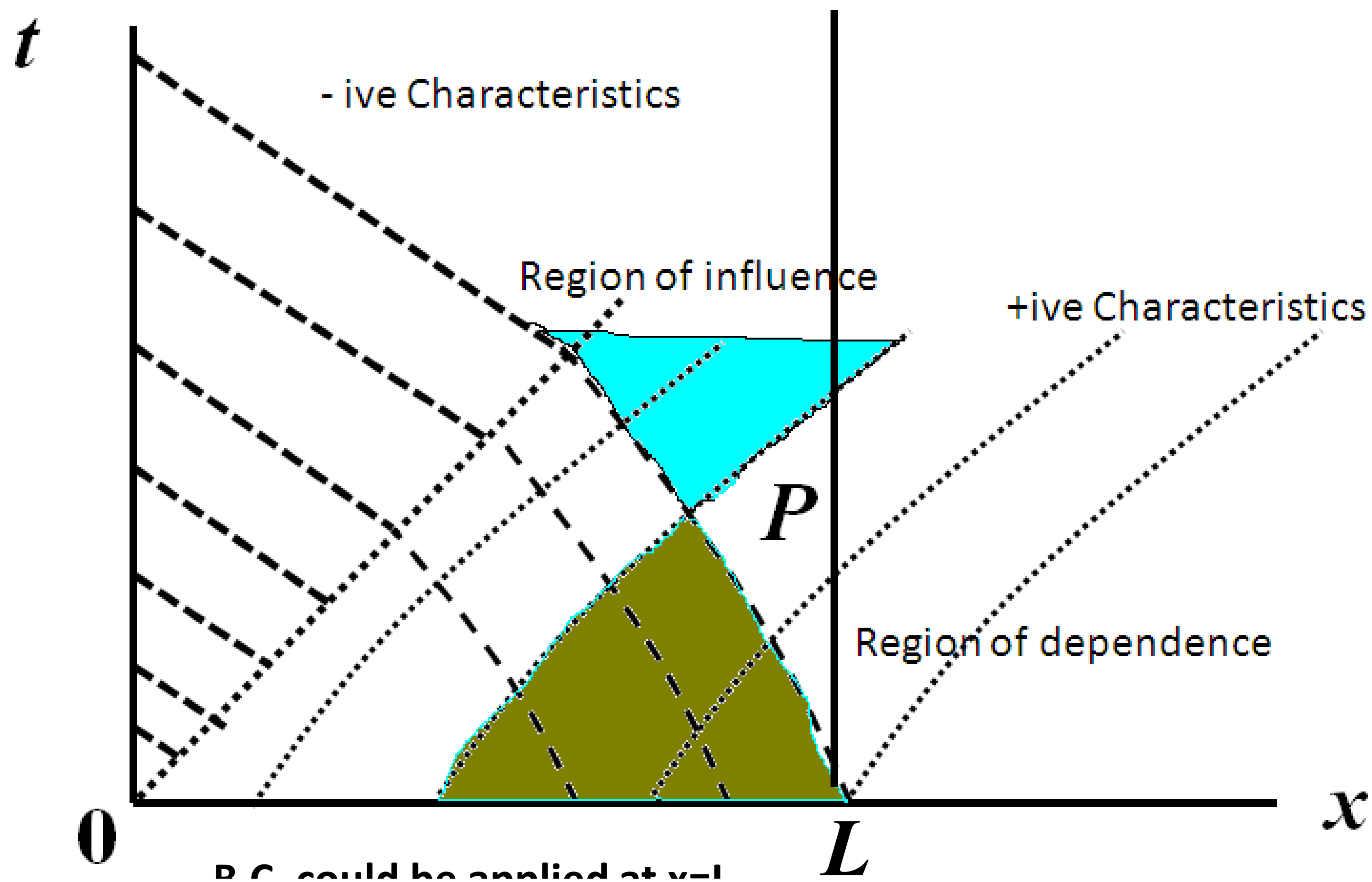
- Along the characteristic $\xi = x - \left(V + \sqrt{gy}\right)t$

$V + 2\sqrt{gy}$ is constant, along $\eta = x - \left(V - \sqrt{gy}\right)t$

$V - 2\sqrt{gy}$ is constant.

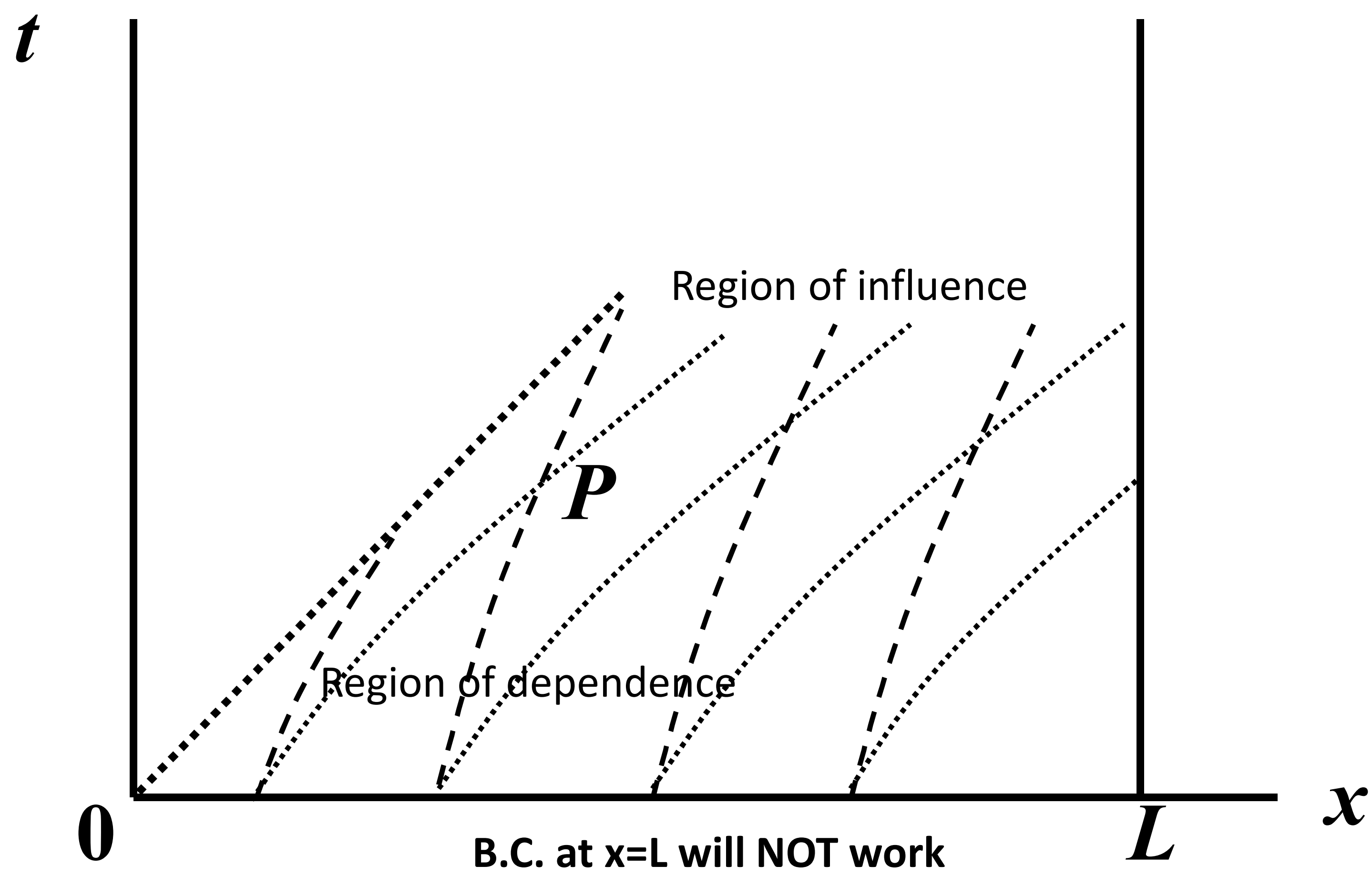
Pair of Characteristic Lines

- Two sets of characteristics
- (Assuming $V < \sqrt{gy}$)



Pair of Characteristic Lines

- What if $V > \sqrt{gy}$
- Both characteristics have +ive slope



Partial Differential Equations: Characteristics

- Most of the physical problems are governed by a 2nd-order PDE: use ξ and η
- A general second-order PDE is (note 2b)

$$a\frac{\partial^2\phi}{\partial x^2}+2b\frac{\partial^2\phi}{\partial x\partial y}+c\frac{\partial^2\phi}{\partial y^2}+d\frac{\partial\phi}{\partial x}+e\frac{\partial\phi}{\partial y}+f\phi=g$$

- We consider only Linear: all the coefficients may be *f* (x,y only)
- We switch to the compact notation

$$a\phi_{xx}+2b\phi_{xy}+c\phi_{yy}+d\phi_x+e\phi_y+f\phi=g$$

Partial Differential Equations: Characteristics

- With $\xi(x,y)$ and $\eta(x,y)$

$$\phi_x = \phi_\xi \xi_x + \phi_\eta \eta_x$$

$$\phi_{xx} = \xi_x \left(\phi_\xi \xi_x + \phi_\eta \eta_x\right)_\xi + \eta_x \left(\phi_\xi \xi_x + \phi_\eta \eta_x\right)_\eta$$

- And so on...
- The PDE is then written as

$$A\phi_{\xi\xi} + 2B\phi_{\xi\eta} + C\phi_{\eta\eta} + D\phi_\xi + E\phi_\eta + F\phi = G$$

- with

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2$$

$$B = a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + c\xi_y\eta_y$$

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$$

Partial Differential Equations: Characteristics

- Other, lower order terms, are not critical
- The aim is to choose ξ and η in such a way as to eliminate higher order terms

$$A\phi_{\xi\xi} + 2B\phi_{\xi\eta} + C\phi_{\eta\eta} + D\phi_{\xi} + E\phi_{\eta} + F\phi = G$$

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2$$

$$B = a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + c\xi_y\eta_y$$

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$$

- A and C are similar, let us make these 0
- Slope of the characteristic ξ =constant:

$$d\xi = 0 \Rightarrow \frac{\partial\xi}{\partial x}dx + \frac{\partial\xi}{\partial y}dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$$

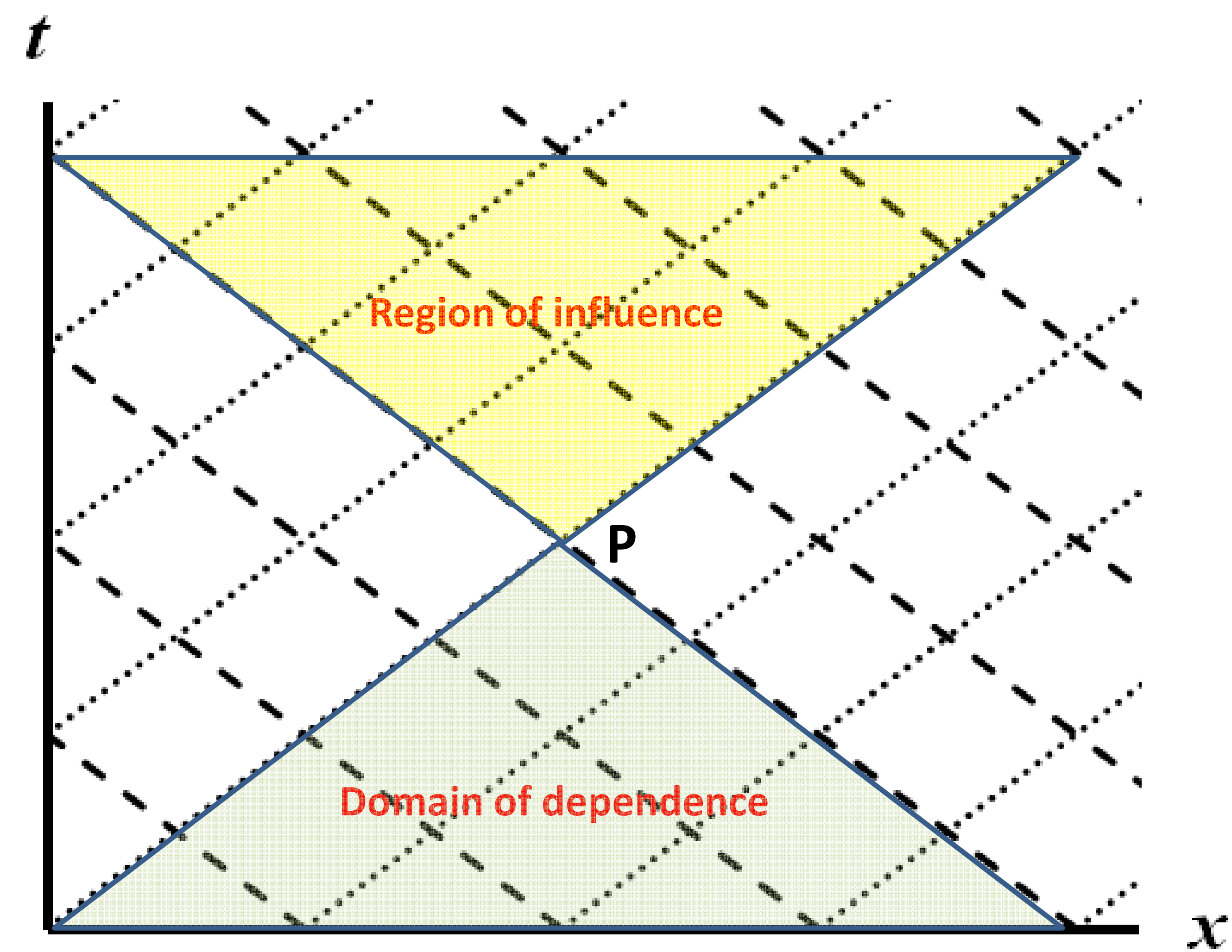
Partial Differential Equations: Characteristics

- To have A=0, $a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0$
- In terms of the slope, $-\xi_x/\xi_y$, $a\left(\frac{\xi_x}{\xi_y}\right)^2 + 2b\frac{\xi_x}{\xi_y} + c = 0$

- Same equation (but with η) to make C=0
- Two roots, one relates to ξ and the other to η: $$\text{Slope} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$
- Classification of PDE: based on $b^2 - ac$
- Similar to classification of conic sections as parabola, hyperbola, and ellipse

Partial Differential Equations: Classification

- If $b^2 - ac > 0$, Hyperbolic PDE
- Two sets of characteristic lines
- Slopes are $\dfrac{b + \sqrt{b^2 - ac}}{a}$ and $\dfrac{b - \sqrt{b^2 - ac}}{a}$
- Depending on the values of a, b, and c, the slopes could be both positive, both negative, or one positive and one negative
- Example: Wave equation $\dfrac{\partial^2 \psi}{\partial t^2} = u^2 \dfrac{\partial^2 \psi}{\partial x^2}$
- $a = -u^2, b = 0, c = 1$; $b^2 - ac = u^2$
- Slopes +1/u and −1/u: $\xi = x - ut \quad \eta = x + ut$


t
Region of influence
P
Domain of dependence
x

Partial Differential Equations: Classification

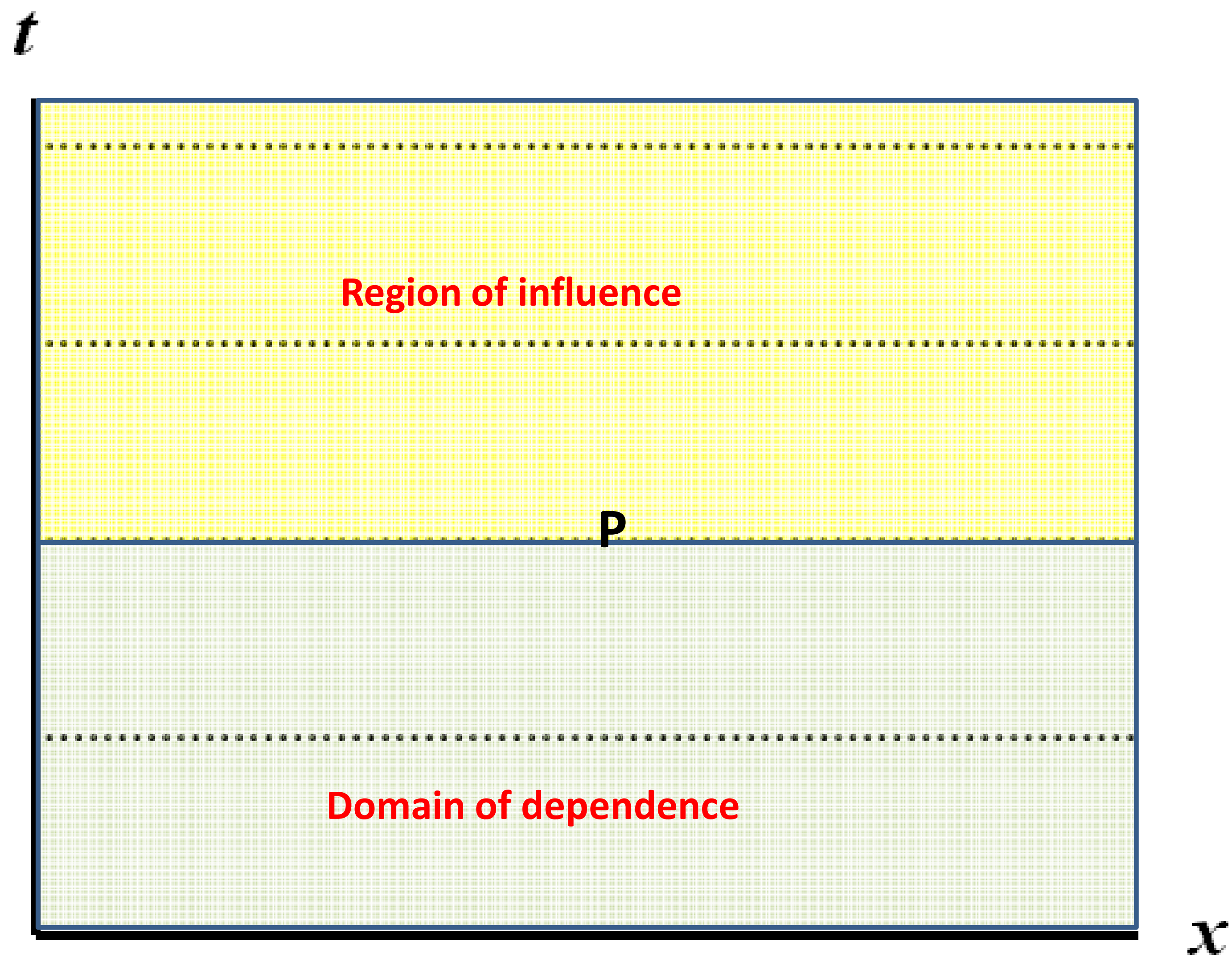
- If $b^2 - ac = 0$, Parabolic PDE
- Single set of characteristic lines, Slope b/a
- Example: Diffusion equation $\frac{\partial c}{\partial t} = D_0 \frac{\partial^2 c}{\partial x^2}$
- a= D_0,b=0,c=0
- Slope is zero, lines parallel to x-axis
- ξ=t, η arbitrary, may be taken as x

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0$$

$$B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 0$$

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = D_0$$

- $D_0\phi_{xx} + D\phi_t + E\phi_x + F\phi = G$ Not very useful!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Partial Differential Equations: Classification

- If $b^2 - ac < 0$, Elliptic PDE

- No "real" characteristic lines. Solution at a point is affected by ALL points

- Example: Laplace equation $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$

- a= 1,b=0,c=1

Partial Differential Equations: Numerical Solution

- Should be consistent with the physics of the problem. E.g., value of the dependent variable at a point should not depend on a point outside its domain of dependence
- Finite difference approximations of the derivatives are used
- Since there are two (or more) independent variables, we could use semi- or full-discretization

Numerical Solution: Semi- and full-discretization

- Semi-discretization uses finite difference approximations for derivatives w.r.t. one or more variables (generally spatial) and keeps the derivative w.r.t. the other variable (usually time) in original form
- This converts the PDE into a system of IVPs, and is solved by, say, R-K method
- The full-discretization uses Finite Difference approximation for all

Semi-discretization

- For example, the diffusion equation

$$\frac{\partial c}{\partial t} = D_0 \frac{\partial^2 c}{\partial x^2}$$

over x=(0,L), with given initial condition $c(0,x)=c_0$; and two boundary conditions, $c(t,0)=c_a$ and $c(t,L)=c_b$

- For semi-discretization, use a spatial grid with generally uniform spacing of Δx, and approximate the second derivative with central difference

Semi-discretization

- At the ith node

$$\frac{dc_i}{dt} = D_0 \frac{c_{i-1} - 2c_i + c_{i+1}}{\Delta x^2}$$

- Partial derivative w.r.t. time is converted to ordinary, as c_i is function of time only
- The resulting system of IVPs is

$$\frac{d\{c\}}{dt} = [A]\{c\} + \{b\}$$

- with tridiagonal [A]; and {b} coming from boundary conditions (details later)

Full-discretization

- For full-discretization, use a spatial grid with spacing of Δx, and use a temporal grid, with generally uniform spacing Δt
- If we use forward difference for the time-derivative and central for space
- Using superscript for time, at the i^{th} node

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} = D_0 \frac{c_{i-1}^n - 2c_i^n + c_{i+1}^n}{\Delta x^2}$$

- Or, $c_i^{n+1} = \alpha c_{i-1}^n + (1 - 2\alpha) c_i^n + \alpha c_{i+1}^n; \quad \alpha = \frac{D_0 \Delta t}{\Delta x^2}$
- Explicit (may use implicit or trapezoidal)