ESO 208A: Computational Methods in Engineering
Problem Set 8

1. Consider the non-dimensional form of 1-D diffusion equation $\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial x^2}$ where $0 \le x \le 1$, with constant boundary conditions; $C(0,t)$ = $C(1,t)$ = C_0 and the initial condition: $C(x,0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$; with μ = 0.5 & σ = 0.5 (Gaussian narrow enough, such that the initial condition satisfies the boundary conditions at x = 0 and 1, for all practical purposes).

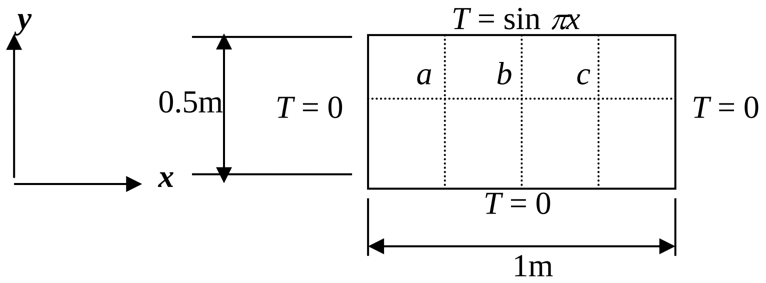
(i) Write the discretized form of the diffusion equation using an explicit scheme with forward difference in time and 2nd order central difference in spatial domain.

(ii) Use five uniformly spaced points for discretization of the spatial domain. Compute the maximum value of Δt that can be used for stable solution. Use a Δt equal to half of the maximum value and calculate the concentration profile $C(x,t)$ after one time step with C_0=0. Graphically show the expected trends of solutions as time progresses.

(iii) From the physical consideration, mention the $C(x, t)$ values as $t \to \infty$.

2. The transport of a pollutant in groundwater is described by the advection-dispersion equation $\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2}$ with boundary conditions $C(0, t) = 0$ and $C(1, t) = 0$; and initial condition $C(x,0) = \begin{cases} 2.5x & 0 \le x \le 0.5 \\ 0 & 0.5 < x \le 1.0 \end{cases}$. Solve using 2nd order Runge Kutta scheme in time and central difference approximation in space for one time step. The data given are: D = 0.4 (L^2/T); u = 1 (L/T); Δx = 0.25 (L); and Δt = 0.05 (T). L and T are dimensions. Prepare an approximate sketch of the solution $C(x)$ v/s x at t = 0.05. For the given Δx, determine the upper limit of Δt from the stability considerations of the 2nd order Runge Kutta method applied to this problem (Stability Region of 2nd order Runge Kutta method is given in the formula sheet).

3a. Steady state temperature (T) distribution in a plate is governed by the Laplace equation. The dimensions and boundary conditions are as shown in the figure below. Calculate the temperatures at points a, b and c. Given $\Delta x = \Delta y$ = 0.25 m.



b) The 1-D Heat equation with a source term is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + S(x)$$

where,

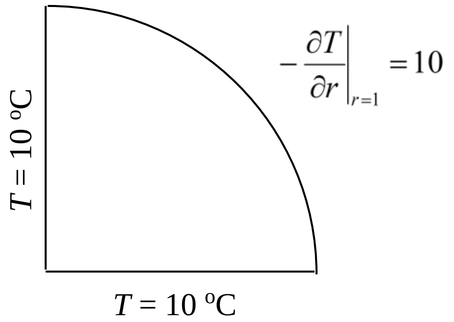
$$T(x,0) = 0;\ T(0,t) = 0;\ T(1,t) = T_{Steady}(1);\ S(x) = -(x^2 - 4x + 2)e^{-x};\ T_{Steady}(x) = x^2 e^{-x}$$

Discretize the above equation using Crank-Nicholson scheme (θ-method in time with $\theta = ½$ and central difference in space). Formulate the matrix equation using $\alpha = 1$, $\Delta x = 0.25$ and $\Delta t = 0.1$. Solve it for one time step.

4. Temperature distribution in the quarter circular disc of unit radius shown in the figure is governed by the Laplace equation. In the polar co-ordinates, it is written as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

The boundary conditions are T(r, 0) = 10, T(r, π/2)=10, T(0, θ)=10 and $-\frac{\partial T(1,\theta)}{\partial r} = 10$.



Using $\Delta r = 1/2$, $\Delta\theta = \pi/6$, compute the temperatures at the interior grid points.

5. Consider the following non-homogeneous heat equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \left(\pi^2 - 1\right)e^{-t} \sin \pi x \qquad 0 \le x \le 1;\ t \ge 0$$

with initial and boundary conditions $T(0,t) = T(1,t) = 0$ and $T(x,0) = \sin \pi x$

a) Write a <u>computer program</u> to solve the equation using Euler explicit-Central difference approximations, for $\alpha = 1$, $\Delta x = 0.05$ and $\Delta t = 0.001$. Plot $T(x)$ vs. x at t = 0.0, 0.5, 1.0, 1.5 and 2.0 in one plot.

b) Take new $\Delta t = 0.002$ and solve the equation for the same α and Δx. Plot $T(x)$ vs. x in the 2nd plot at $t = (0.0, 0.05, 0.1, 0.15, 0.2)$.

c) Explain the results obtained in (a) and (b).

6. Temperature distribution in a plate is governed by the following equation:
$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$ in $x \in (0,1), y \in (0,1)$, subject to the boundary conditions $T(0,y) = T(1,y) = T(x,0) = 0$ and $T(x,1) = \sin \pi x$. The exact solution of the problem is given by $T(x,y) = \sin \pi x \frac{\sinh \pi y}{\sinh \pi}$. Develop a computer code for the numerical solution of the problem using central difference approximations and graphically compare the numerical solution with the exact solution at $x = 0.5$ for $\Delta x = \Delta y = 0.1$.