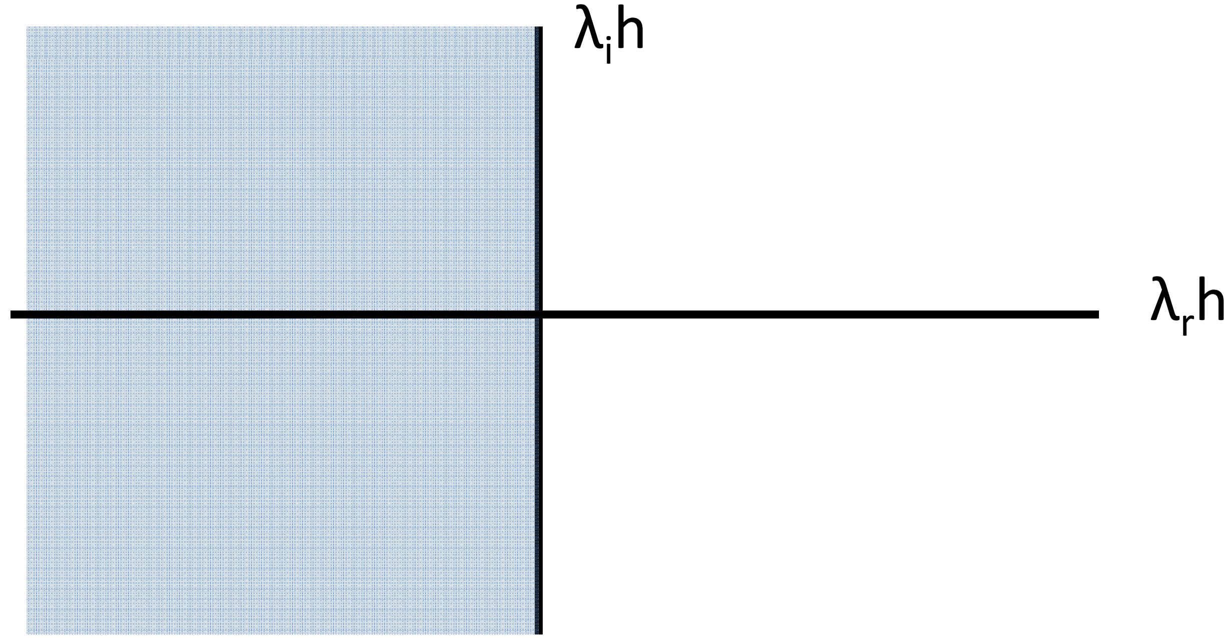
Linear Stability Analysis

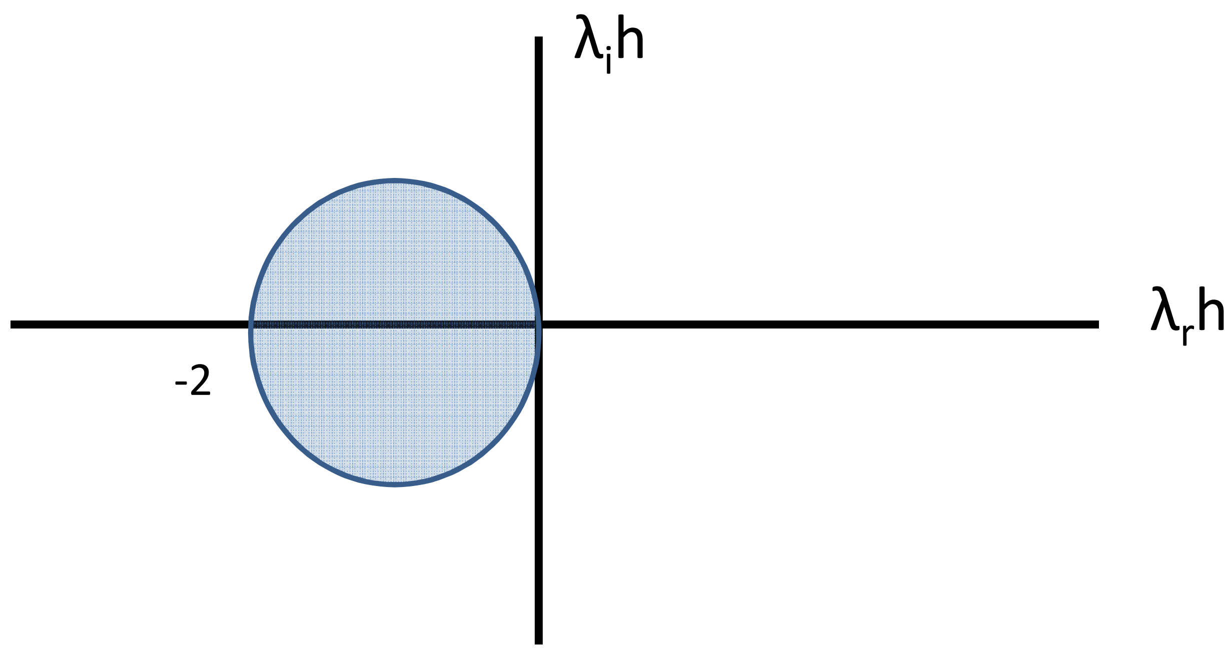
$$y = y_0 e^{n(\lambda_r h + i\lambda_i h)}$$

- The analytical solution is bounded for all negative λ_r
- Stability Region



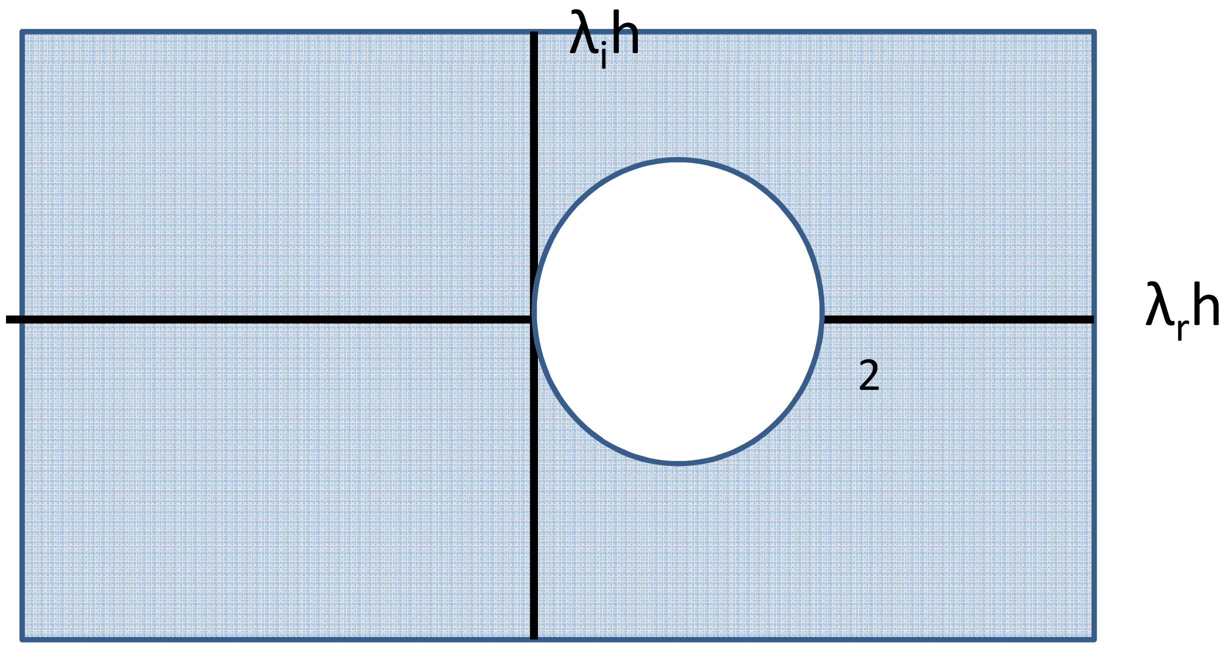
Linear Stability Analysis: Euler Forward

- The stability region is shown below: a circle of radius 1, centered at (-1,0)
- For real negative values of λ, the condition is $|\lambda h| \leq 2$



Linear Stability Analysis: Euler Backward

- The stability region is shown below: outside a circle of radius 1, centered at (-1,0)
- For real negative values of λ, the method is unconditionally stable



Linear Stability Analysis: Trapezoidal method

- For Trapezoidal method

$$y_{n+1} = y_n + h\frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2} \Rightarrow y_{n+1} = y_n \frac{1 + \lambda_r h/2 + i\lambda_i h/2}{1 - \lambda_r h/2 - i\lambda_i h/2}$$

- The stability region is, therefore, given by

$$\left|\frac{1 + \lambda_r h/2 + i\lambda_i h/2}{1 - \lambda_r h/2 - i\lambda_i h/2}\right| \le 1$$

- Which implies $\lambda_r h \le 0$
- Same as that for the exact solution.
- Unconditionally stable, does not give bounded solution when the exact is not bounded!

Linear Stability Analysis: R-K method

- For 2nd order R-K method (Heun's)

$$y_{n+1} = y_n + h\frac{f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))}{2}$$

$$\Rightarrow y_{n+1} = y_n\left(1 + \lambda h + \lambda^2 h^2 / 2\right)$$

- The stability region is, therefore, the region inside the shape whose boundary is given by

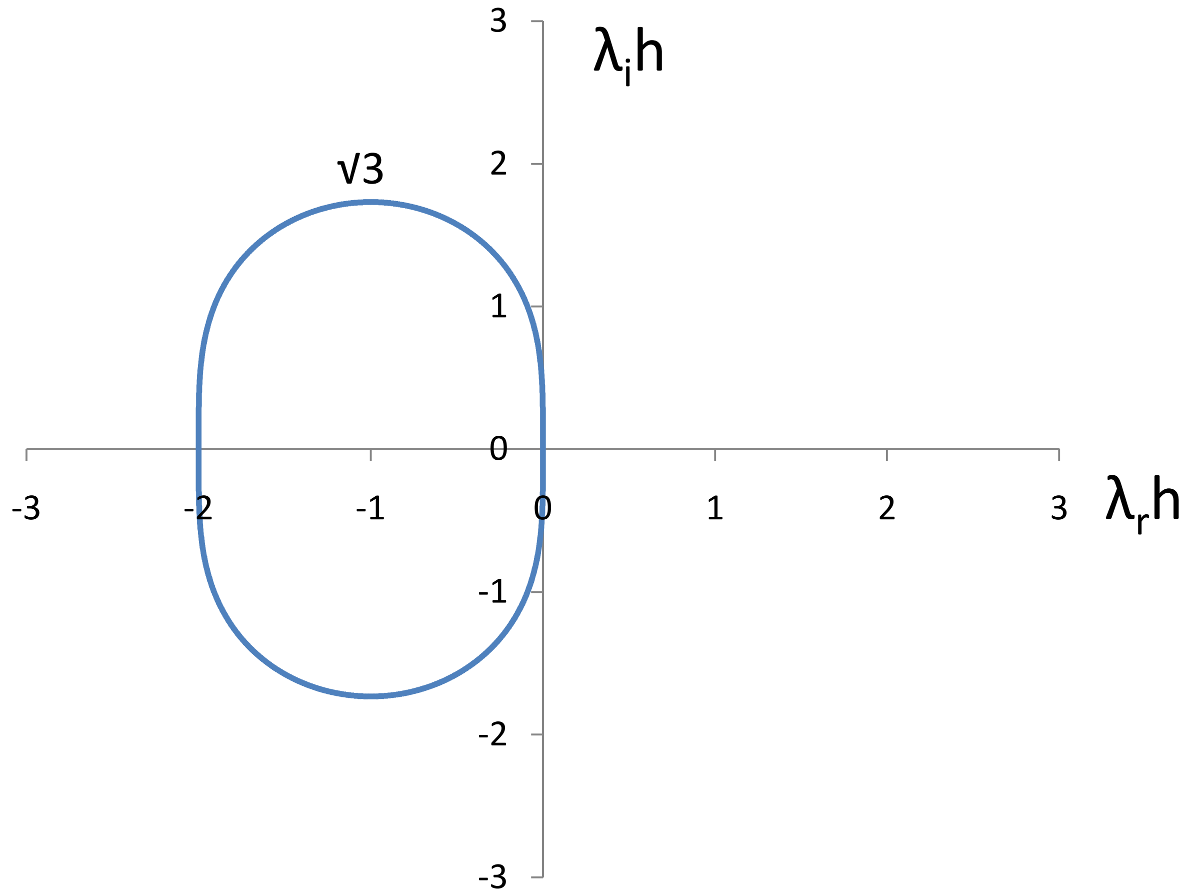
$$\left|1 + \lambda h + \lambda^2 h^2 / 2\right| = 1$$

- Or:

$$\lambda_i h = \pm\sqrt{-\lambda_r h(2 + \lambda_r h) + 2\sqrt{-\lambda_r h(2 + \lambda_r h)}}$$

Stability Region for Heun's method

- The stability region is shown below: centered at (-1,0), major axis = $2\sqrt{3}$, minor =2
- For real negative values of λ, the condition is $|\lambda h| \leq 2$



Predictor-Corrector methods

- Implicit methods are stable but require solution of a nonlinear equation at each step
- Explicit methods require less computational effort per step but may need a very small time-step for stability
- Avoid the nonlinear equation solution, by predicting the “unknown” value using explicit method and then correcting it using implicit

Predictor-Corrector methods

- For example, Heun's method:
  - Predictor: $y_{n+1}^{p} = y_n + hf\left(t_n, y_n\right)$
  - Corrector: $y_{n+1}^{c} = y_n + \frac{h}{2}\left[f\left(t_n, y_n\right) + f\left(t_{n+1}, y_{n+1}^{p}\right)\right]$
- Why stop at one step only? Iterate using the corrected value in the implicit step.

$$y_{n+1}^{(0)} = y_n + hf\left(t_n, y_n\right)$$

$$y_{n+1}^{(i)} = y_n + \frac{h}{2}\left[f\left(t_n, y_n\right) + f\left(t_{n+1}, y_{n+1}^{(i-1)}\right)\right]$$

- Repeat till convergence

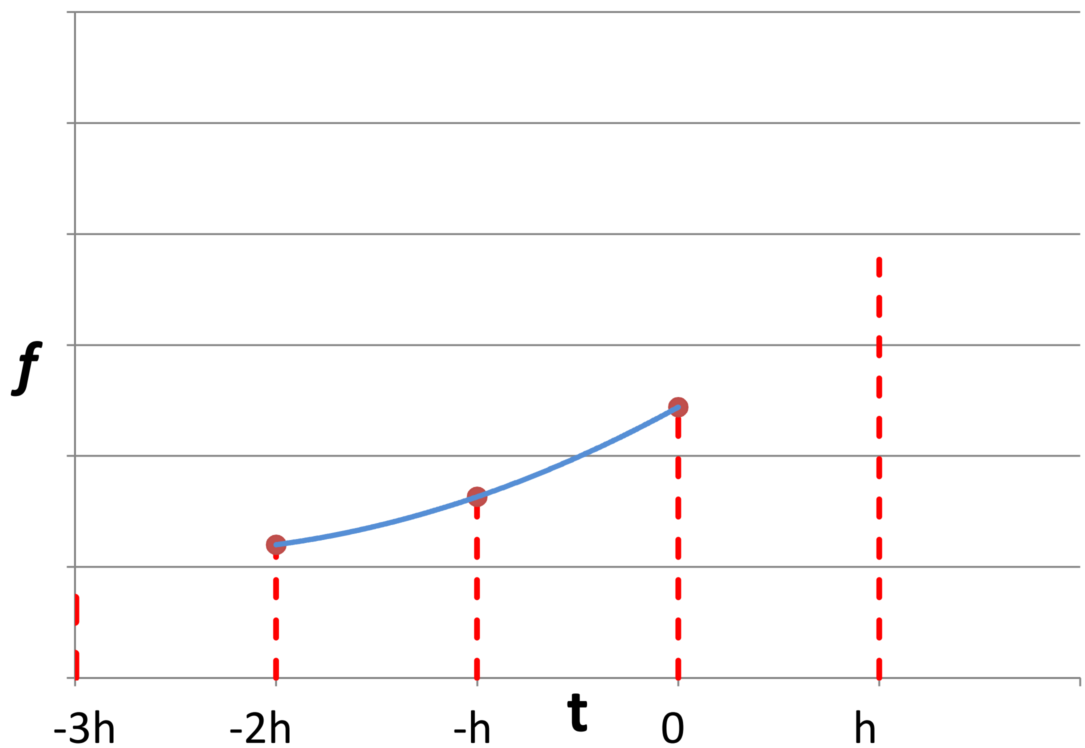
Predictor-Corrector : Milne's method

- Milne's method (multi-step):
- Non-self starting
- Uses Simpson's 1/3 methodology
- Predictor: interpolate a quadratic using n-2, n-1, and n; integrate over n-3 to n+1
- Corrector: interpolate a quadratic using n-1, n, and n+1; integrate over n-1 to n+1

Milne’s method: Predictor

- Approximate f by a quadratic function:

$$f = \frac{(t+h)t}{(-2h+h)(-2h)} f_{n-2} + \frac{(t+2h)t}{(-h+2h)(-h)} f_{n-1} + \frac{(t+2h)(t+h)}{(2h)(h)} f_n$$



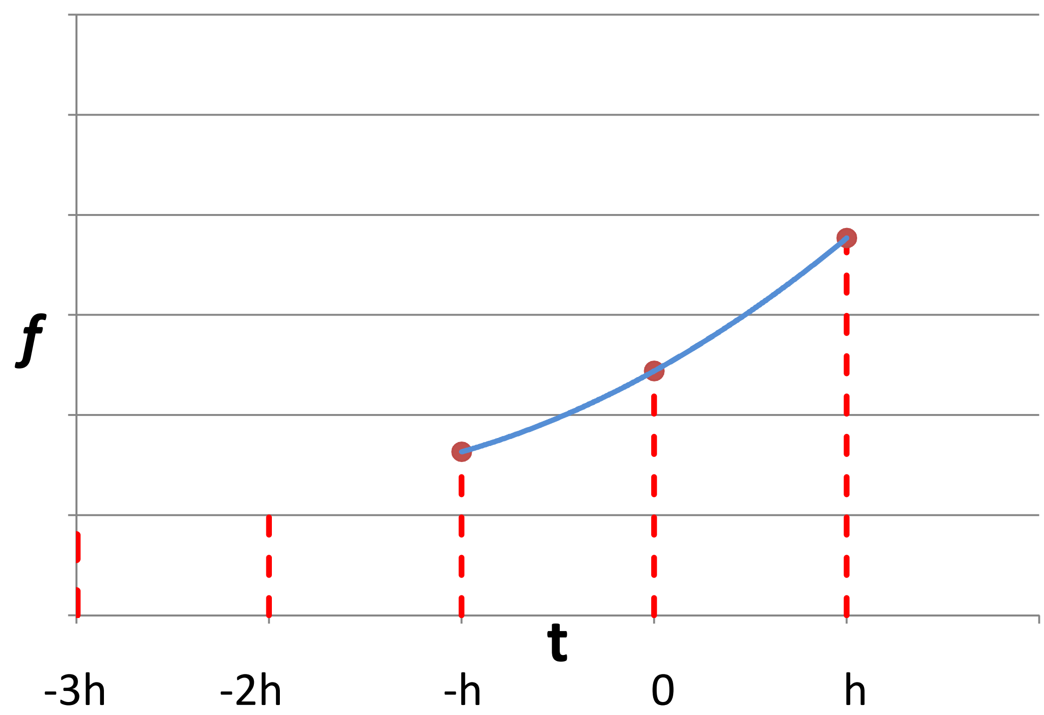
- Integrate from -3h to h:

$$y_{n+1}^{(0)} = y_{n-3} + \int_{-3h}^{h} f dt = y_{n-3} + \frac{4h}{3}\left(2f_{n-2} - f_{n-1} + 2f_n\right)$$

Milne's method: Corrector

- Approximate f by a quadratic function:

$$f = \frac{t(t-h)}{(-h)(-2h)} f_{n-1} + \frac{(t+h)(t-h)}{(h)(-h)} f_n + \frac{(t+h)t}{(2h)(h)} f_{n+1}^{(i-1)}$$



- Integrate from -h to h:

$$y_{n+1}^{(i)} = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + f\left(t_{n+1}, y_{n+1}^{(i-1)}\right)\right]$$

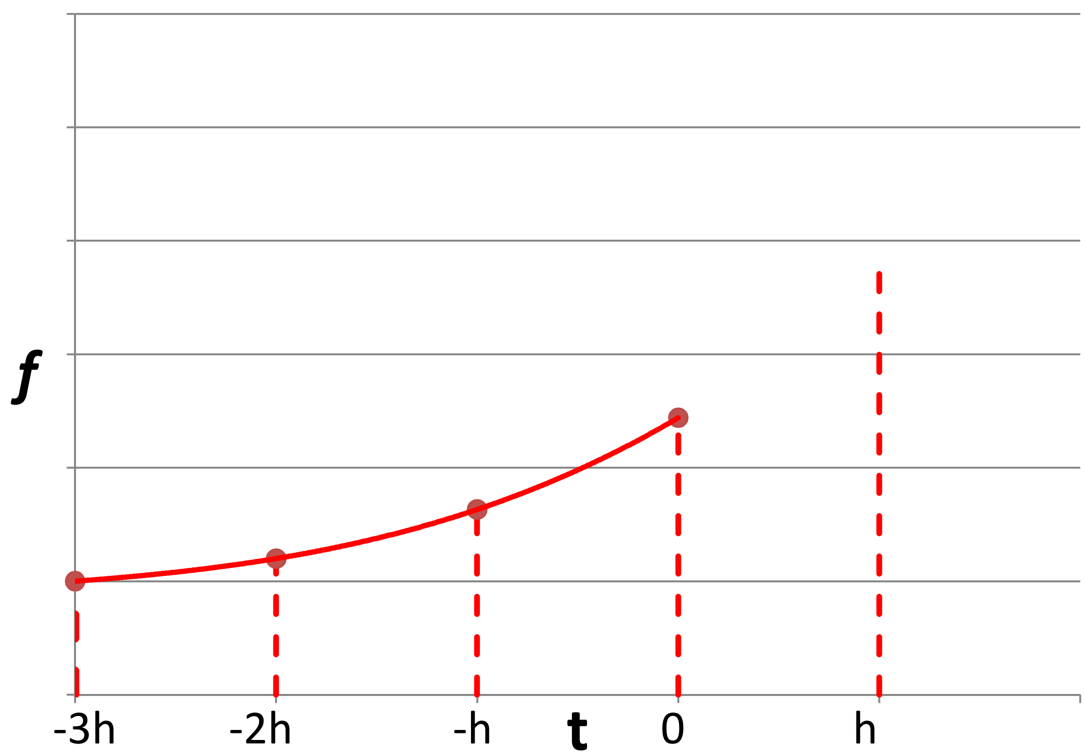
Predictor-Corrector : Adams method

- Adams method:
- Uses Adams-Bashforth (explicit) and Adams-Moulton (implicit)
- For Example, take the 4^{th} order method
- Predictor: interpolate a cubic using n-3, n-2, n-1, and n; integrate over n to n+1
- Corrector: interpolate a cubic using n-2, n-1, n, and n+1; integrate over n to n+1

Adams method: Predictor

- Approximate f by a cubic function:

$$f = \frac{(t+2h)(t+h)t}{(-h)(-2h)(-3h)} f_{n-3} + \frac{(t+3h)(t+h)t}{(-h)(-h)(-2h)} f_{n-2} + \frac{(t+3h)(t+2h)t}{(2h)(h)(-h)} f_{n-1} + \frac{(t+3h)(t+2h)(t+h)}{(3h)(2h)(h)} f_n$$



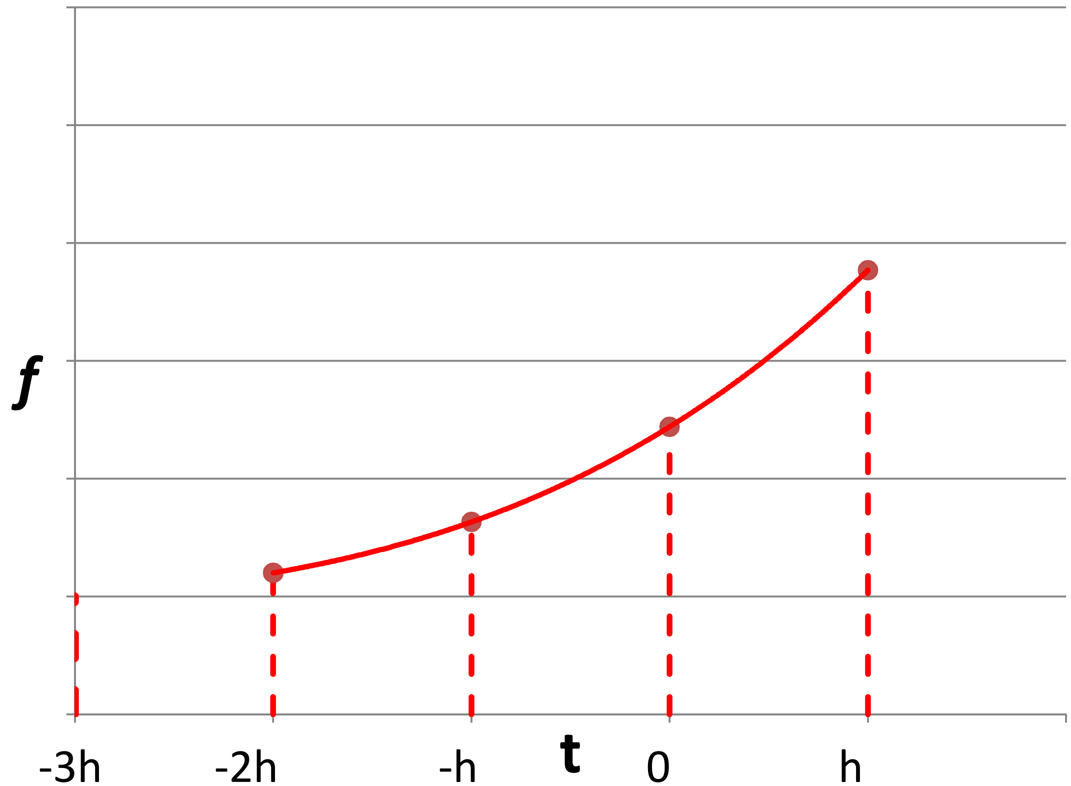
- Integrate from 0 to h:

$$y_{n+1}^{(0)} = y_n + \int_0^h f\,dt = y_n + h\left(-\frac{3}{8} f_{n-3} + \frac{37}{24} f_{n-2} - \frac{59}{24} f_{n-1} + \frac{55}{24} f_n\right)$$

Adams method: Corrector

- Approximate f by a cubic function:

$$f = \frac{(t+h)t(t-h)}{(-h)(-2h)(-3h)} f_{n-2} + \frac{(t+2h)t(t-h)}{(h)(-h)(-2h)} f_{n-1} + \frac{(t+2h)(t+h)(t-h)}{(2h)(h)(-h)} f_n + \frac{(t+2h)(t+h)t}{(3h)(2h)(h)} f_{n+1}^{(i-1)}$$



- Integrate from 0 to h:

$$y_{n+1}^{(i)} = y_n + h\left(\frac{1}{24} f_{n-2} - \frac{5}{24} f_{n-1} + \frac{19}{24} f_n + \frac{3}{8} f_{n+1}^{(i-1)}\right)$$

System of ODEs

- If we have several dependent variables, y_i, i from 1 to m
- Derivatives could be functions of time and one or more y^s
- Initial conditions on all y^s should be given
- The system may be expressed as

$$\frac{dy_1}{dt} = f_1(t, y_1, y_2, \ldots, y_m)$$

$$\frac{dy_2}{dt} = f_2(t, y_1, y_2, \ldots, y_m)$$

$$\ldots$$

$$\frac{dy_m}{dt} = f_m(t, y_1, y_2, \ldots, y_m)$$

$$y_1\big|_{(t=0)} = y_{1,0}; y_2\big|_{(t=0)} = y_{2,0}; \ldots; y_m\big|_{(t=0)} = y_{m,0}$$

Higher order ODEs

- If we have a higher order ODE, it could be converted into a system of ODEs
- For example, $$c_2 \frac{d^2 y}{dt^2} + c_1 \frac{dy}{dt} + c_0 y = f(t)$$
- Could be expressed as (using y_1=y and y_2=dy/dt):

$$\frac{dy_1}{dt} = f_1(t, y_1, y_2) = y_2$$

$$\frac{dy_2}{dt} = f_2(t, y_1, y_2) = \frac{f(t) - c_0 y_1 - c_1 y_2}{c_2}$$