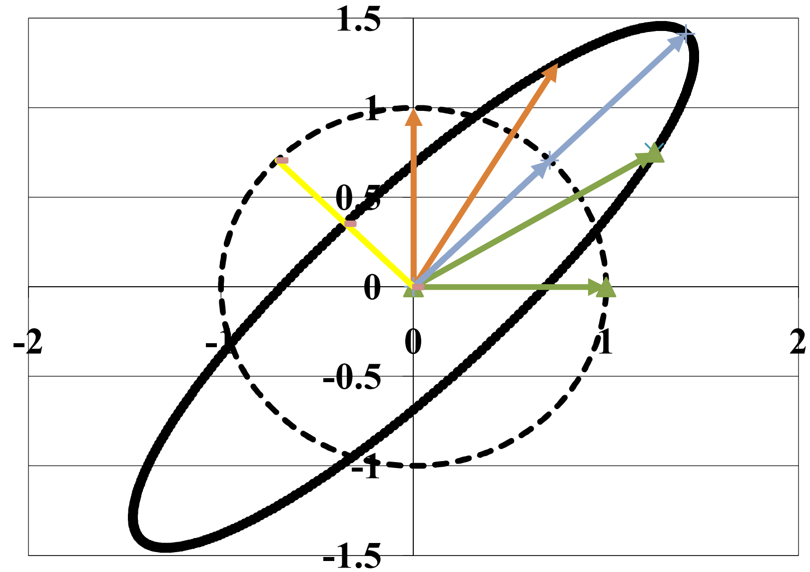
Eigenvalues and Eigenvectors

- The system [A]{x}={b}: [A] operating on vector {x} to transform it to another vector, {b}.
- It will, in general, lead to a change in "direction" as well as the "length" of the vector {x}. Example, for a unit vector {x}:

$$[A]=\begin{bmatrix}1.25 & 0.75\\ 0.75 & 1.25\end{bmatrix}$$

$$\{x\}=\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix}$$

$$\{b\}=\begin{Bmatrix}1.25\cos\theta+0.75\sin\theta\\ 0.75\sin\theta+1.25\cos\theta\end{Bmatrix}$$



- For a particular vector, {x}, if [A]{x}=λ{x}, there is no rotation.
- λ is called an Eigenvalue of [A] and {x} is the corresponding Eigenvector (it will only give a direction of the eigenvector, the magnitude is arbitrary). λ denotes the change in "length" of {x}

Eigenvalues: Some properties

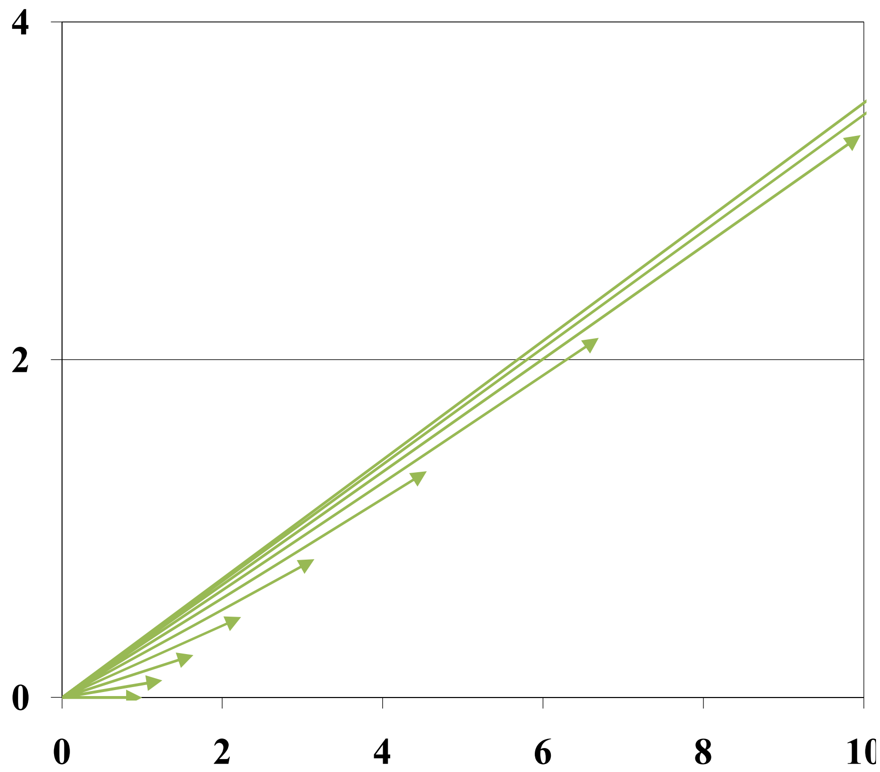
- $Ax=\lambda x$ => $A^2x=A\,Ax=A\,\lambda x = \lambda^2 x$. Eigenvalue of A^k will be λ^k
- $Ax=\lambda x$ => $A^{-1}x= (1/\lambda)\,x$. Eigenvalue of A^{-1} will be $1/\lambda$
- Since $(A- \lambda I)x=0$, I being the n x n identity matrix, determinant of $(A- \lambda I)$ must be zero
- Det $(A- \lambda I)$ is a polynomial of degree n: “Characteristic Polynomial” of A (German word Eigen means inherent, characteristic). The n eigenvalues may be obtained by using methods described earlier, e.g., Bairstow
- Symmetric matrices: maximum length of Ax is along one of the eigenvectors [the 2-norm of A is equal to its *eigenvalue of the maximum magnitude* (also known as the ***spectral radius*** of A, $\rho(A)$, since the set of eigenvalues is called the spectrum of the matrix. $\rho(A) \le \|A\|$, for all consistent norms]
- On the other hand, the 2-norm of A^{-1} will be $1/\lambda_{min}$

Eigenvalues: Some properties

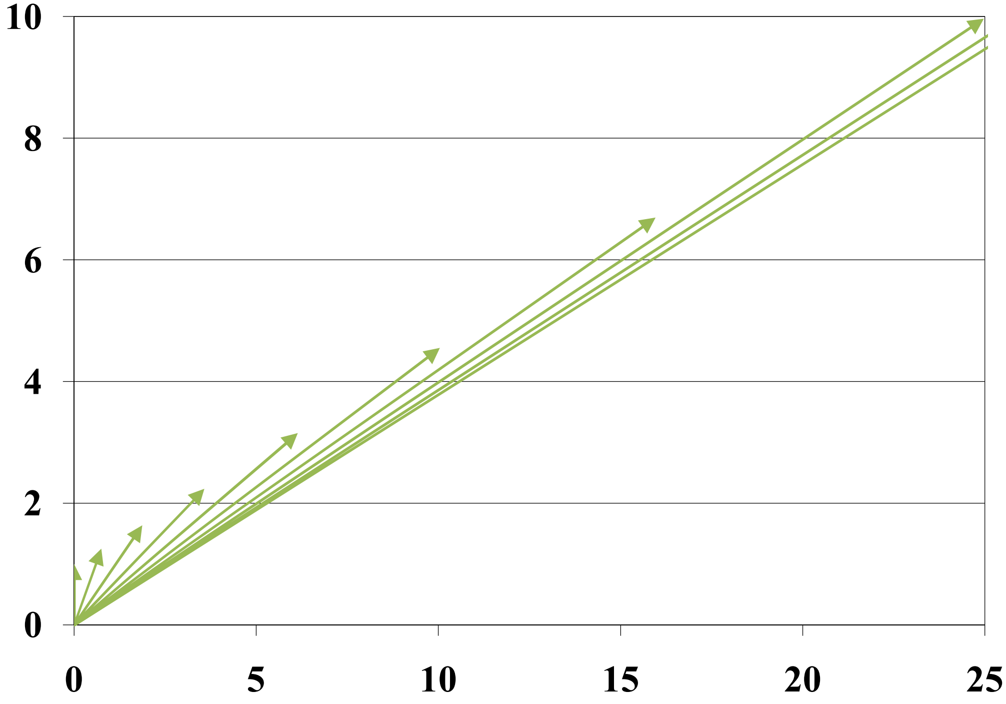
- For symmetric matrices, therefore, if we use the 2-norm the condition number of A, C(A)= $\|A\|.\|A^{-1}\|$, is equal to the ratio $\lambda_{max} / \lambda_{min}$ (Indicating that it is always ≥ 1, the lower the better!)
- For a general matrix, the condition number is equal to the square-root of the ratio $\lambda_{max} / \lambda_{min}$, where the eigenvalues correspond to the symmetric matrix $A^T A$.
- It means that finding the largest and smallest eigenvalues of a matrix has great practical significance.
- We first look at the methods of finding these and then look at the methods for finding ALL eigenvalues.

Eigenvalues: Finding the largest

- If we assume that A has *n* independent eigenvectors, x_i, i=1,…,n; any vector, say $z^{(0)}$, may be written as $z^{(0)} = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$
- Then, $Az^{(0)} = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \ldots + c_n \lambda_n x_n$
- And, $A^k z^{(0)} = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \ldots + c_n \lambda_n^k x_n$
- We now assume that A has a single dominant eigenvalue, say, λ_1 ($|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots |\lambda_{n-1}| \geq |\lambda_n|$)
- As *k* becomes large, the resulting vector tends to the dominant eigenvector. However, its length tends to become infinite ($|\lambda_1|>1$) or zero ($|\lambda_1|<1$). Therefore, we normalize the length at each step to make it of unit norm (typically L_2, but L_1 or L_∞ may be used, L_∞ being the most convenient)



4
2
0
0
2
4
6
8
10



10
8
6
4
2
0
0
5
10
15
20
25

Largest Eigenvalue : Power method

- The algorithm is written as:
  - Choose an arbitrary unit vector $z^{(0)}$
  - Multiply $z^{(0)}$ by A and normalize to get $z^{(1)}$
  - Repeat till $z^{(i)}$ and $z^{(i+1)}$ are the same ($z^{(i+1)} = Az^{(i)} / \left\| Az^{(i)} \right\|$)
  - z is the eigenvector and the normalization factor is the corresponding eigenvalue
- Example: $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Choose starting vector as $\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$
- Iterations: $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}; \frac{1}{\sqrt{5}} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}; \frac{1}{\sqrt{5}} \begin{Bmatrix} 5 \\ 4 \end{Bmatrix}; \frac{1}{\sqrt{41}} \begin{Bmatrix} 5 \\ 4 \end{Bmatrix}; \frac{1}{\sqrt{41}} \begin{Bmatrix} 14 \\ 13 \end{Bmatrix}; \frac{1}{\sqrt{365}} \begin{Bmatrix} 14 \\ 13 \end{Bmatrix};$

$$\frac{1}{\sqrt{365}} \begin{Bmatrix} 41 \\ 40 \end{Bmatrix}; \frac{1}{\sqrt{3281}} \begin{Bmatrix} 41 \\ 40 \end{Bmatrix}; \frac{1}{\sqrt{3281}} \begin{Bmatrix} 122 \\ 121 \end{Bmatrix}; \frac{1}{\sqrt{29525}} \begin{Bmatrix} 122 \\ 121 \end{Bmatrix}; \ldots; \frac{1}{\sqrt{265721}} \begin{Bmatrix} 365 \\ 364 \end{Bmatrix}$$

Power Method

- Converging towards the Eigenvector $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$
- The eigenvalue is approximately $\sqrt{\dfrac{265721}{29525}} = 3$
- If we use L_∞ norm:

$$\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}; \begin{Bmatrix} 1 \\ 0.5 \end{Bmatrix}; \begin{Bmatrix} 2.5 \\ 3 \end{Bmatrix}; \begin{Bmatrix} 5/6 \\ 1 \end{Bmatrix}; \begin{Bmatrix} 8/3 \\ 17/6 \end{Bmatrix}; \begin{Bmatrix} 16/17 \\ 1 \end{Bmatrix}; \begin{Bmatrix} 49/17 \\ 50/17 \end{Bmatrix}$$

$$\begin{Bmatrix} 148/17 \\ 149/17 \end{Bmatrix}; \begin{Bmatrix} 148/149 \\ 1 \end{Bmatrix}; \begin{Bmatrix} 445/149 \\ 446/149 \end{Bmatrix}; \begin{Bmatrix} 445/446 \\ 1 \end{Bmatrix}$$

- Again, same eigenvector and eigenvalue=446/149, nearly 3

Smallest Eigenvalue – Inverse power method

- Inverse of A has eigenvalues which are reciprocal of those of A. Power method to get the largest eigenvalue of A^{-1} will give us the smallest eigenvalue of A, provided it is unique.
- Example: Inverse of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is $\begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$; starting vector $\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$
- Iterations:

$$\begin{Bmatrix} 2/3 \\ -1/3 \end{Bmatrix}; \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix}; \begin{Bmatrix} 5/6 \\ -2/3 \end{Bmatrix}; \begin{Bmatrix} 1 \\ -4/5 \end{Bmatrix}; \begin{Bmatrix} 14/15 \\ -13/15 \end{Bmatrix}; \begin{Bmatrix} 41/45 \\ -40/45 \end{Bmatrix}$$

- Converging towards $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ with eigenvalue 1. Smallest eigenvalue for A is, therefore, 1/1=1.
- Since computation of inverse is time-consuming, it is more efficient to write $z^{(i+1)} = A^{-1} z^{(i)} / \left\| A^{-1} z^{(i)} \right\|$ as $Az^{(i+1)} = z^{(i)}$ followed by normalization. The system is efficiently solved by using LU decomposition!

Eigenvalue “closest to θ“ – Inverse power with shift

- The eigenvalues of A−θI are (λ−θ). Applying inverse power method to this matrix gives us the smallest eigenvalue of A−θI, which implies that by adding θ to it, we get the eigenvalue of A which is closest to θ.
- Example: Find the eigenvalue of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ closest to 2.5
- A−θI = $\begin{bmatrix} -1/2 & 1 \\ 1 & -1/2 \end{bmatrix}$ and the inverse is $\begin{bmatrix} 2/3 & 4/3 \\ 4/3 & 2/3 \end{bmatrix}$. Use $\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$
- Iterations: $\begin{Bmatrix} 2/3 \\ 4/3 \end{Bmatrix}; \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}; \begin{Bmatrix} 5/3 \\ 4/3 \end{Bmatrix}; \begin{Bmatrix} 1 \\ 4/5 \end{Bmatrix}; \begin{Bmatrix} 26/15 \\ 28/15 \end{Bmatrix}; \begin{Bmatrix} 13/14 \\ 1 \end{Bmatrix}; \begin{Bmatrix} 41/21 \\ 40/21 \end{Bmatrix}$
- Converges to (1,1), eigenvalue is about 2. Smallest eigenvalue of A−θI is ½=0.5. Closest to 2.5 is, therefore, 3.