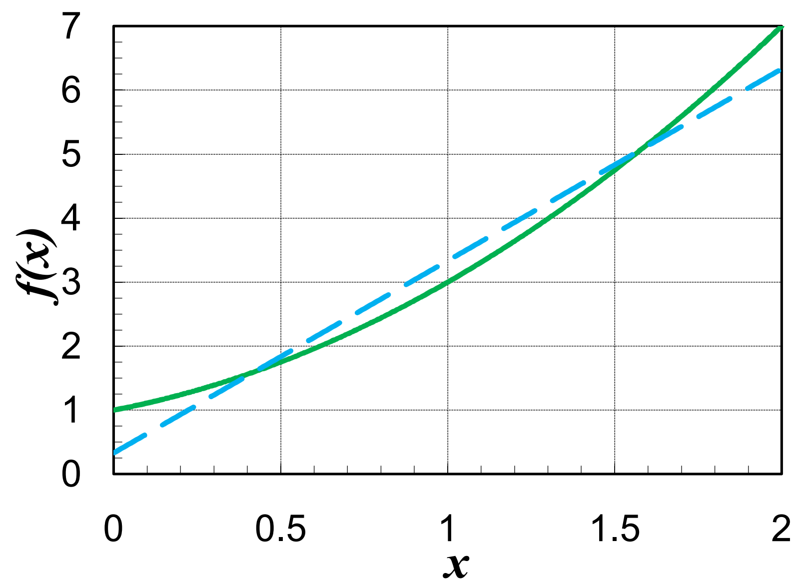
Orthogonal polynomials: Tchebycheff

- Legendre: More error near end points.
- In this case, 2/3 near 0 & 2, and 1/3 at 1



- If we shift the line up by 1/6, it will make both the errors equal (1/2)
- Tchebycheff: If we could assign some “suitable” weights to the error, the “maximum” error could be minimized.

Orthogonal polynomials: Tchebycheff

- Tchebycheff (or Chebyshev): Instead of minimizing the square of the error, use weighted errors such that weights are larger near the end points.
- The weight is taken as $1/\sqrt{1-x^2}$
- Approximating polynomial $f_m(x) = \sum_{j=0}^{m} c_j T_j(x)$
- *T's* are known *as Tchebycheff polynomials and the* coefficients are chosen such that the weighted error is minimum:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left(f(x) - \sum_{j=0}^{m} c_j T_j(x) \right)^2 dx$$

Tchebycheff polynomials

- Tchebycheff polynomials are orthogonal over $(-1,1)$ w.r.t. the weight $1/\sqrt{1-x^2}$ and have a maximum magnitude of unity.
- Therefore, $T_0(x)=1$, and assuming $T_1(x)=d_0+d_1 x$

$$\int_{-1}^{1} \frac{d_0+d_1x}{\sqrt{1-x^2}}dx = 0 \Rightarrow d_0 = 0 \Rightarrow T_1(x) = x$$

- Similarly, for $T_2(x)$:

$$\int_{-1}^{1} \frac{d_0+d_1x+d_2x^2}{\sqrt{1-x^2}}dx = 0; \int_{-1}^{1} \frac{x(d_0+d_1x+d_2x^2)}{\sqrt{1-x^2}}dx$$

$$\Rightarrow d_0 + \frac{d_2}{2} = 0; d_1 = 0 \Rightarrow T_2(x) = -1+2x^2$$

Tchebycheff polynomials

- General form

$$T_n(x) = \cos\left(n\cos^{-1}x\right)$$

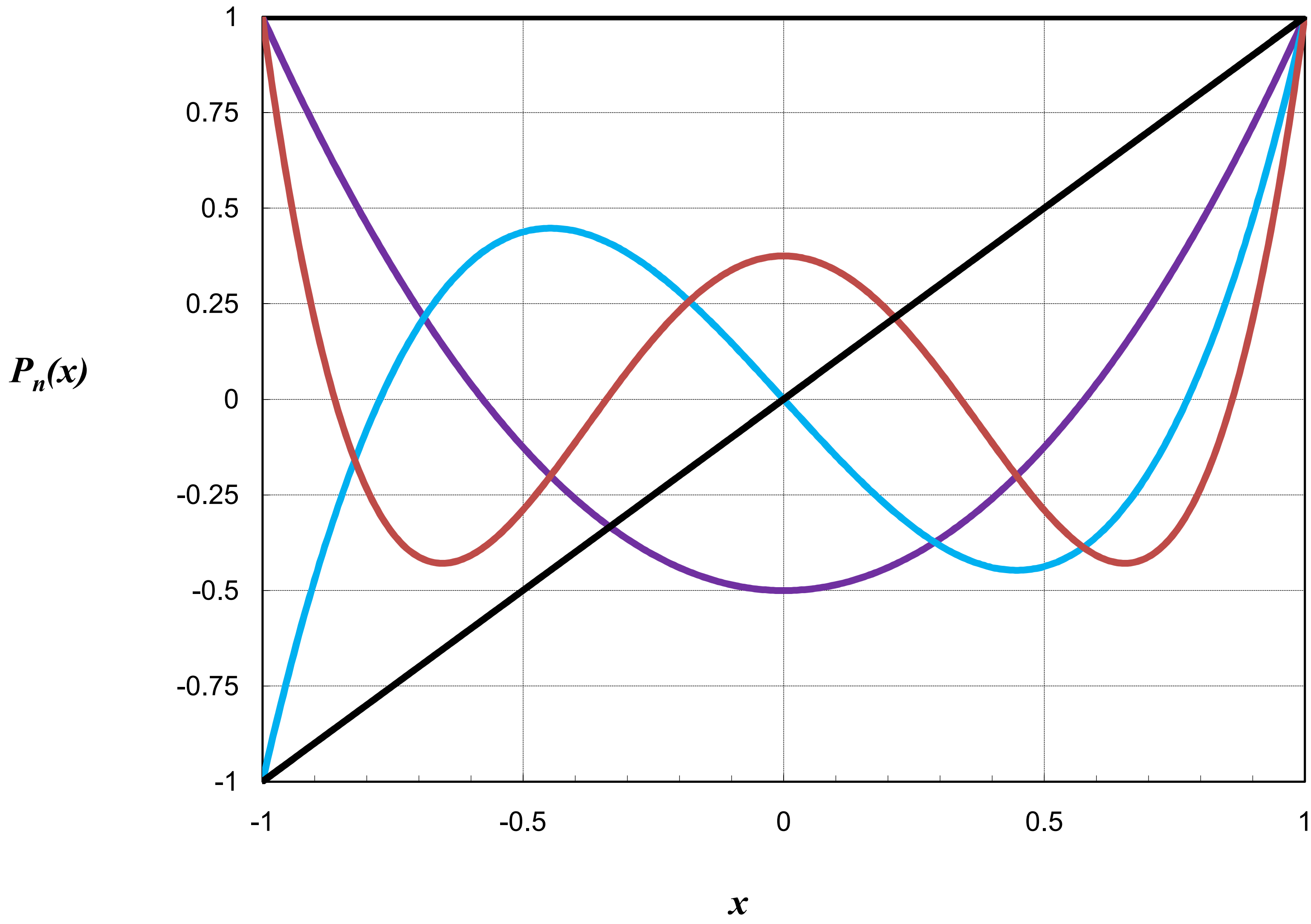
- Recursive Formula

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

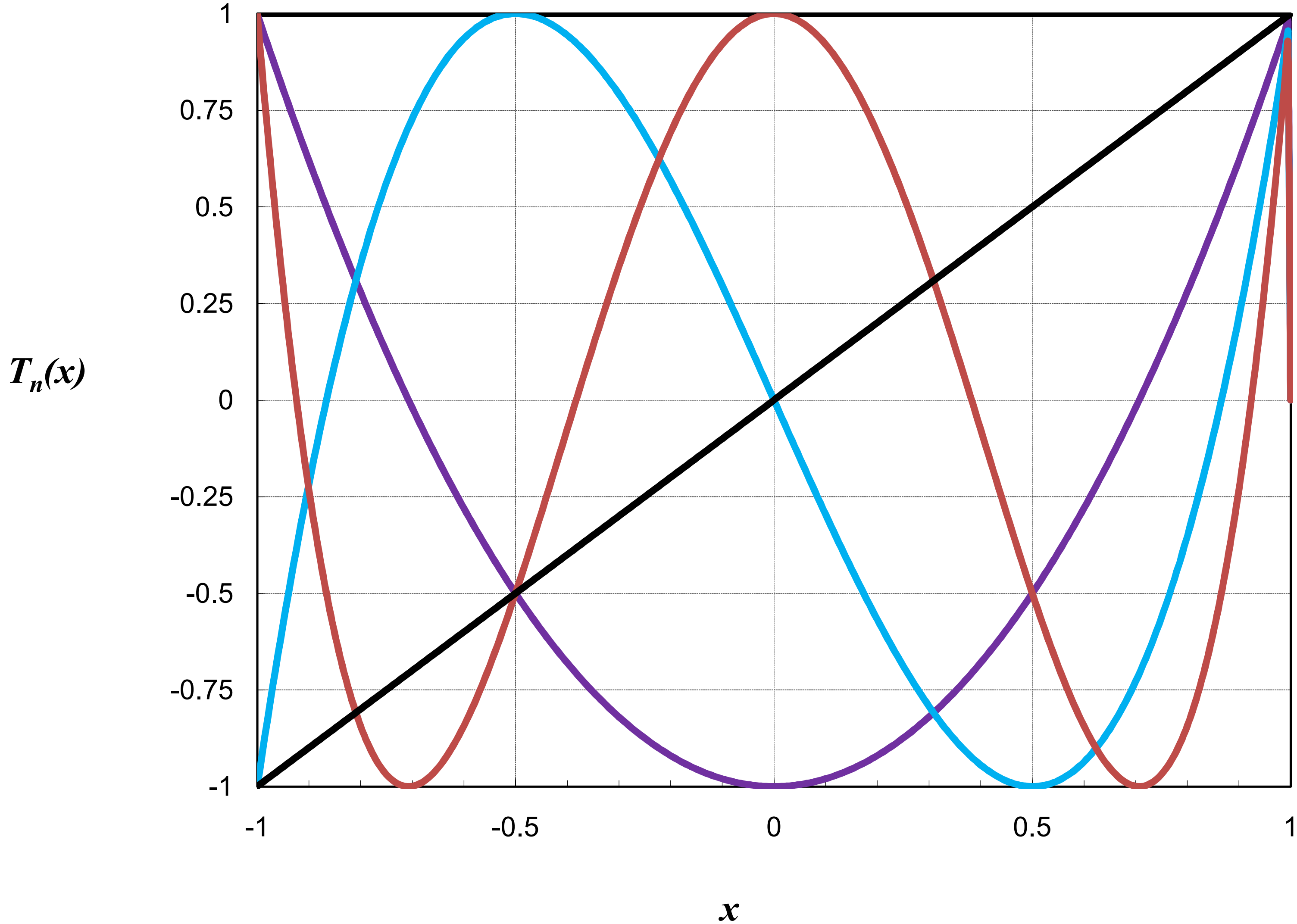
- Orthogonality

$$\left\langle T_i(x), T_j(x)\right\rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_i(x)T_j(x)dx = \begin{cases} 0 & i \neq j \\ \pi & i = j = 0 \\ \dfrac{\pi}{2} & i = j \neq 0 \end{cases}$$

$$P_0(x)=1,\ P_1(x) = x,\ P_2(x) = (-1+3x^2)/2,\ P_3(x) = (-3x+5x^3)/2,\ P_4(x) = (3-30x^2+35x^4)/8$$



$$T_0(x)=1,\ T_1(x) = x,\ T_2(x) = (-1+2x^2),\ T_3(x) = (-3x+4x^3),\ T_4(x) = (1-8x^2+8x^4)$$

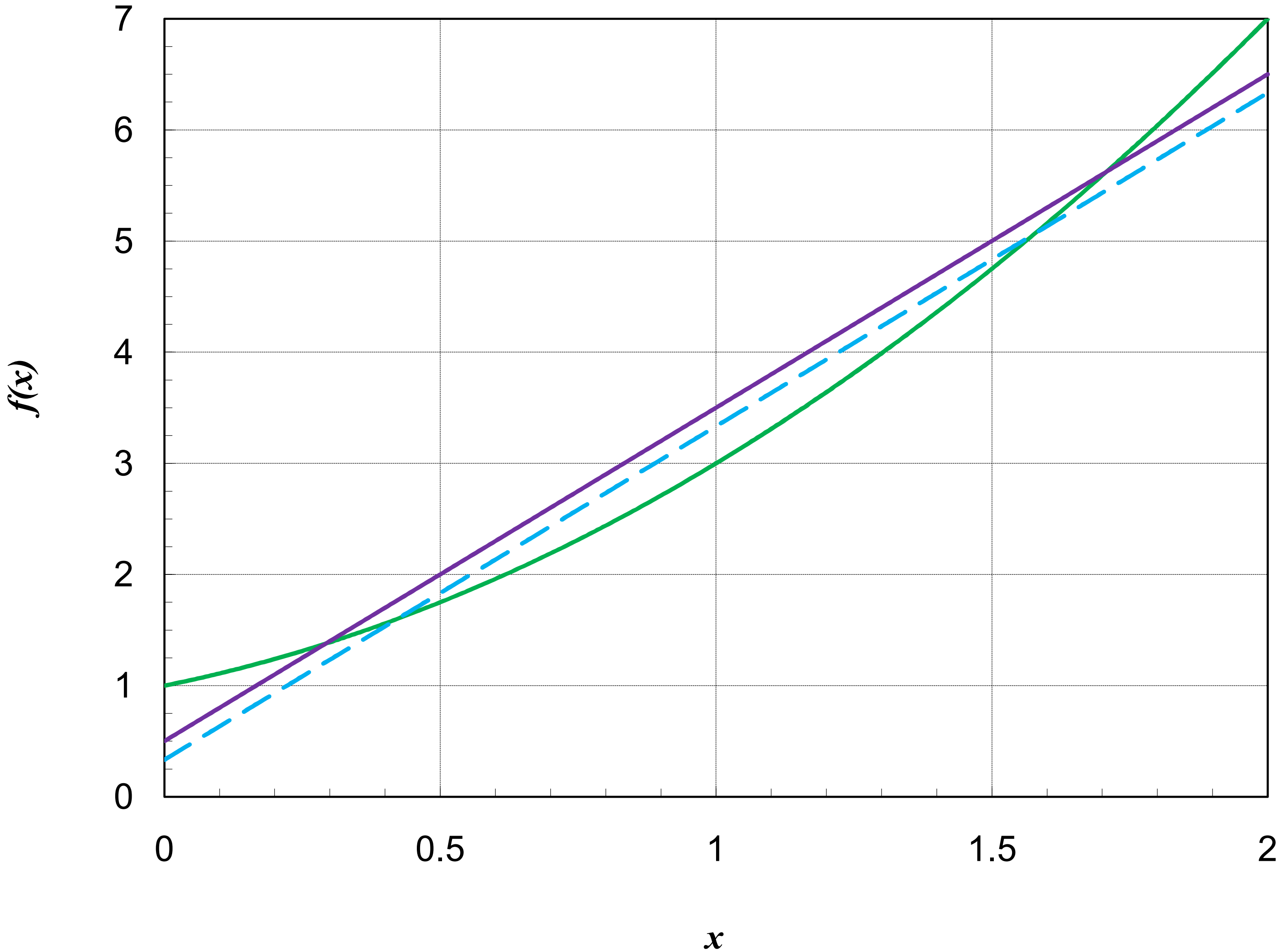


Tchebycheff polynomials: Example

- Fit straight line to $1+x^*+x^{*2}$ over (0,2)
- Normalize: $x=x^*-1$
- $f(x)=3+3x+x^2$

$$A = \begin{bmatrix} \pi & 0 \\ 0 & \pi/2 \end{bmatrix}; b = \begin{Bmatrix} 7\pi/2 \\ 3\pi/2 \end{Bmatrix}$$

- Solution: $f_1(x^*)= 7/2 + 3\ (x^*-1) = ½ + 3\ x^*$
- The maximum error is reduced: known as the *Minimax approximation*



7
6
5
4
3
2
1
0
0
0.5
1
1.5
2
f(x)
x

Least Squares Method : Recap

- Obtain the best mth degree polynomial fit to the function $f(x)$ over the interval (a,b)

➢ e.g., best straight line, m=1, which fits $1+x+x^2$, i.e., $f(x)$, over the interval (0,2)

- Formulation: $$\text{Minimize} \int_a^b \left(f(x) - f_m(x)\right)^2 dx$$

➢ e.g., $$\text{Minimize} \int_0^2 \left(1 + x + x^2 - f_1(x)\right)^2 dx$$

Least Squares Method : Recap

- *General* form of approximating polynomial:

$$f_m(x) = \sum_{j=0}^{m} c_j \phi_j(x)$$

➢ e.g., $f_1(x) = c_0 + c_1 x$

- Minimization of $\displaystyle\int_a^b \left(f(x) - \sum_{j=0}^{m} c_j \phi_j(x) \right)^2 dx$

➢ e.g., $\displaystyle\int_0^2 \left(1 + x + x^2 - (c_0 + c_1 x)\right)^2 dx$

Least Squares Method : Recap

- Stationary Point theorem

$$\int_a^b \left(f - \sum_{j=0}^{m} c_j \phi_j \right) \phi_i dx = 0 \quad \text{for } i = 0,1,2,\ldots,m$$

➢ e.g.,

$$\int_a^b \left(1 + x + x^2 - (c_0 + c_1 x)\right) dx = 0 \quad \text{w.r.t. } c_0$$

$$\int_a^b \left(1 + x + x^2 - (c_0 + c_1 x)\right) x dx = 0 \quad \text{w.r.t. } c_1$$

Least Squares Method : Recap

- Inner product: $\langle f, g \rangle = \int_a^b f . g \; dx$

$$\left\langle \sum_{j=0}^{m} c_j \phi_j, \phi_i \right\rangle = \langle f, \phi_i \rangle \quad \text{for } i = 0,1,2,\ldots,m$$

➢ e.g.,

$$\langle c_0 + c_1 x, 1 \rangle = \left\langle 1 + x + x^2, 1 \right\rangle$$

$$\langle c_0 + c_1 x, x \rangle = \left\langle 1 + x + x^2, x \right\rangle$$

Least Squares Method : Recap

- Normal Equations: $[A]\{c\}=\{b\}$

$$a_{ij}=\left\langle \phi_i,\phi_j \right\rangle; b_i=\left\langle \phi_i, f \right\rangle;\ i,j=0,1,2,\ldots,m$$

➢ e.g.,

$$\begin{bmatrix} \int_0^2 1.1dx & \int_0^2 1.xdx \\ \int_0^2 x.1dx & \int_0^2 x.xdx \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \end{Bmatrix} = \begin{Bmatrix} \int_0^2 1.(1+x+x^2)dx \\ \int_0^2 x.(1+x+x^2)dx \end{Bmatrix}$$

Least Squares Method : Recap

$$\begin{bmatrix} 2 & 2 \\ 2 & 8/3 \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \end{Bmatrix} = \begin{Bmatrix} 20/3 \\ 26/3 \end{Bmatrix}$$

$$\text{Solution}: c_0 = \frac{1}{3}; c_1 = 3$$

$f_1(x)$=1/3 + 3 x

Least Squares Method : Recap

- Using orthogonal polynomials $\phi_0 = 1$ and $\phi_1 = 1 - x$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \end{Bmatrix} = \begin{Bmatrix} 20/3 \\ -2 \end{Bmatrix}$$

$$\text{Solution}: c_0 = \frac{10}{3}; c_1 = -3$$

$$f_1(x) = 10/3 - 3\,(1-x) = 1/3 + 3\,x$$

Least Squares Method : Recap

- Using Legendre polynomials $\phi_0 = 1$ and $\phi_1 = x$ with the domain changed to $(-1,1)$ using

$$x = \frac{x^* - \frac{b+a}{2}}{\frac{b-a}{2}} = x^* - 1$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \end{Bmatrix} = \begin{Bmatrix} 20/3 \\ 2 \end{Bmatrix} \Rightarrow c_0 = \frac{10}{3}; c_1 = 3$$

$$f_1(x) = 10/3 + 3\,x = 1/3 + 3\,x^*$$

Least Squares Method : Recap

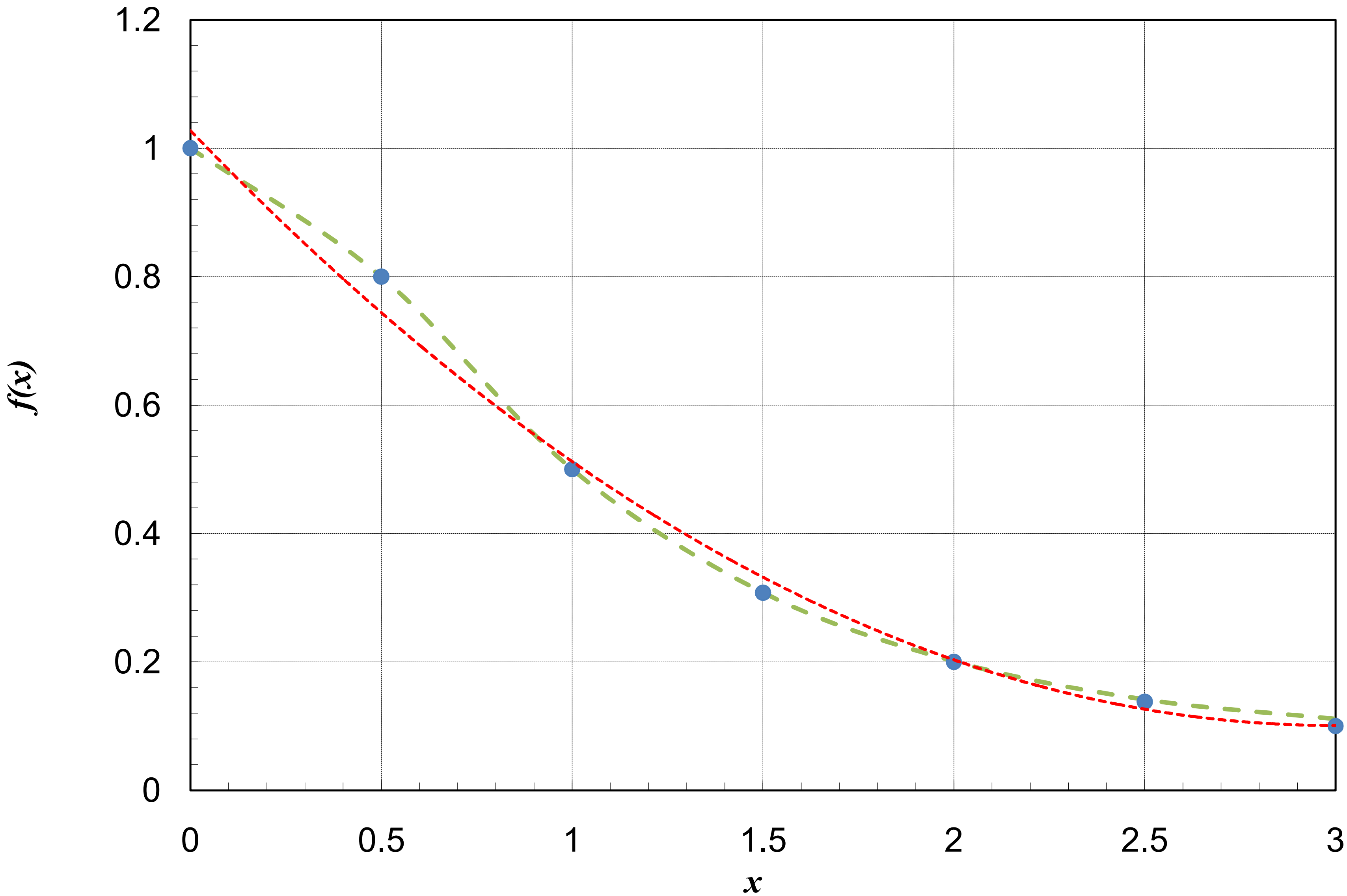
- Using Tchebycheff polynomials $\phi_0 = 1$ and $\phi_1 = x$ with the domain changed to $(-1,1)$ using weight of $1/\sqrt{(1-x^2)}$

$$A = \begin{bmatrix} \pi & 0 \\ 0 & \pi/2 \end{bmatrix}; b = \begin{Bmatrix} 7\pi/2 \\ 3\pi/2 \end{Bmatrix} \Rightarrow c_0 = \frac{7}{2}; c_1 = 3$$

$$f_1(x) = 7/2 + 3x = 1/2 + 3x^*$$

Approximation of Data

- Data denoted by $(x_k, f(x_k)) \quad k = 0,1,2,\ldots,n$
- n+1 data points
- Approximating polynomial: $f_m(x)$
- If m=n, unique polynomial passing through all the data points - Interpolation
- If m<n, best-fit polynomial capturing the trend of data – Regression : depends on the definition of “best-fit”
- If m>n, non-unique polynomial passing through all the points



1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
x
f(x)

Interpolation

- There is a *unique* n^th^ degree polynomial passing through the *n*+1 data points

$$\left(x_k, f\left(x_k\right)\right) \quad k = 0,1,2,\ldots,n$$

- Represent it as $$f_n(x) = \sum_{j=0}^{n} c_j \phi_j\left(x\right)$$
- As discussed before, the basis functions may be taken in several different forms.
- Conventional form, $\phi_j\left(x\right) = x^j$, i.e.,

$$f_n(x) = c_0 + c_1 x + \ldots + c_n x^n$$

Interpolation

- The polynomial must pass through all the n+1 data points: the coefficients are given by

$$\begin{bmatrix} 1 & x_0 & x_0^2 & . & x_0^n \\ 1 & x_1 & x_1^2 & . & x_1^n \\ 1 & x_2 & x_2^2 & . & x_2^n \\ . & . & . & . & . \\ 1 & x_n & x_n^2 & . & x_n^n \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ c_n \end{Bmatrix} = \begin{Bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ . \\ f(x_n) \end{Bmatrix} \Rightarrow Ac = b$$

- Solve by any of the linear equation methods
- The A matrix is called Vandermonde matrix
- Unique solution if all x's are distinct
- Ill- conditioned for large n: Not recommended

Interpolation: Example

- Find the interpolating polynomial for the given data points: (0,1), (1,3), (2,7)
- 3 data points => n=2
- Second degree interpolating polynomial
- $x_0=0, f(x_0)=0; x_1=1, f(x_1)=3; x_2=2, f(x_2)=7$

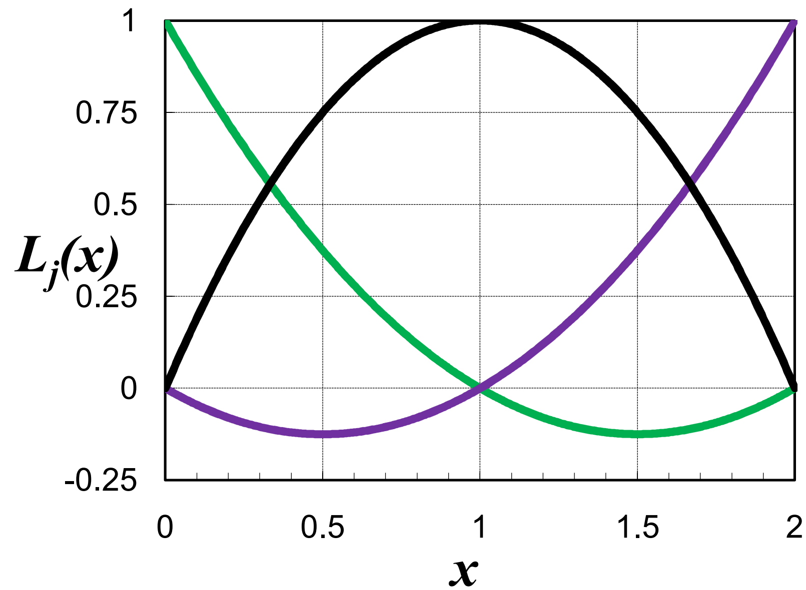
$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \\ 7 \end{Bmatrix} \Rightarrow c_0 = 1; c_1 = 1, c_2 = 1$$

- Interpolating polynomial is $1+x+x^2$

Interpolation: Lagrange polynomials

$$f_n(x) = \sum_{j=0}^{n} c_j \phi_j(x)$$

- Each of the basis functions, ϕ_j, is an nth-degree polynomial, such that its value is 1 at $x = x_j$ and zero at all other data points, denoted by $L_j(x)$
- For example, using the same data points as before (x=0,1,2):



Interpolation: Lagrange polynomials

$$f_n(x) = \sum_{j=0}^{n} c_j L_j(x) \qquad \text{with } L_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

- What will be the value of the coefficient, c_i ?
- SAME AS $f(x_i)$!
- How to obtain the L_i ?

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Lagrange Polynomial: Example

- Find the interpolating polynomial for the given data points: (0,1), (1,3), (2,7)
- Second degree interpolating polynomial

$$L_0 = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{2-3x+x^2}{2}$$

$$L_1 = \frac{(x-0)(x-2)}{(1-0)(1-2)} = 2x - x^2$$

$$L_2 = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{-x+x^2}{2}$$

- Interpolating polynomial is

$$1 \times L_0 + 3 \times L_1 + 7 \times L_2 = 1 + x + x^2$$

Lagrange Polynomial: Example

- Useful when the grid points are fixed but function values may be changing
- For example, estimating the temperature at a point using the measured temperatures at a few nearby points
- The value of the Lagrange polynomials at the desired point need to be calculated only once
- Then, we just need to multiply these values with the corresponding temperatures.
- What if a new measurement is added?