ESO 208A: Computational Methods in Engineering

Partial Differential Equation: Introduction, Parabolic Equation

Saumyen Guha

Department of Civil Engineering
IIT Kanpur

Introduction

A general 2nd Order PDE:

$$\alpha\phi_{xx} + 2\beta\phi_{xy} + \gamma\phi_{yy} + \theta\phi_x + \omega\phi_y + \rho(\phi, x, y) = 0$$

- ✓ $\beta^2 - \alpha\gamma = 0$: Parabolic PDE, *e.g.*, Diffusion and Advection-Diffusion Equation
- ✓ $\beta^2 - \alpha\gamma < 0$: Elliptic PDE, *e.g.*, Laplace Equation
- ✓ $\beta^2 - \alpha\gamma > 0$: Hyperbolic PDE, *e.g.*, Wave equation

We will learn a few *Finite Difference* methods for most common PDEs in Engineering Problems!

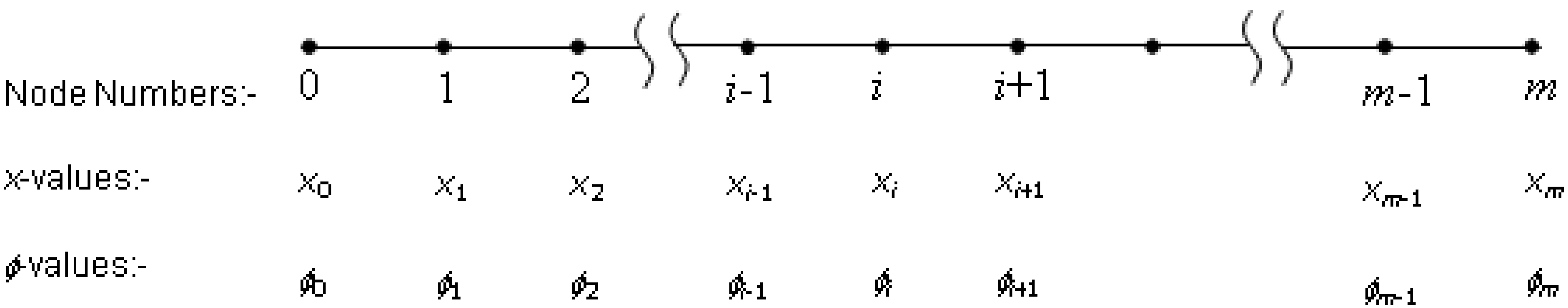
Parabolic PDE: semi-discretization

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} \qquad\qquad \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \alpha \frac{\partial^2 \phi}{\partial x^2}$$

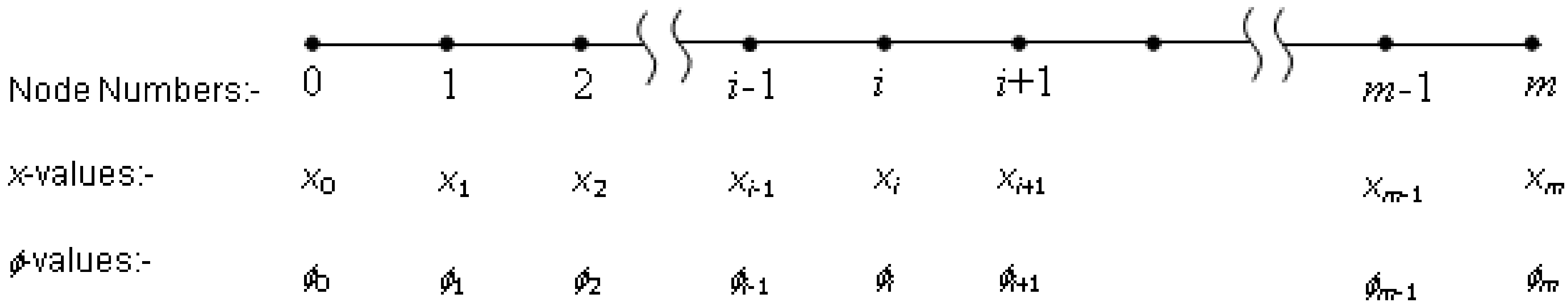
$$\phi(0,t) = c_0;\ \phi(L,t) = c_L;\ \ \phi(x,0) = f(x)$$

$$\frac{d\phi_i}{dt} = \alpha_i \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2}$$

$$\frac{d\phi_i}{dt} = -u_i \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} + \alpha_i \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2}$$



Parabolic PDE: semi-discretization

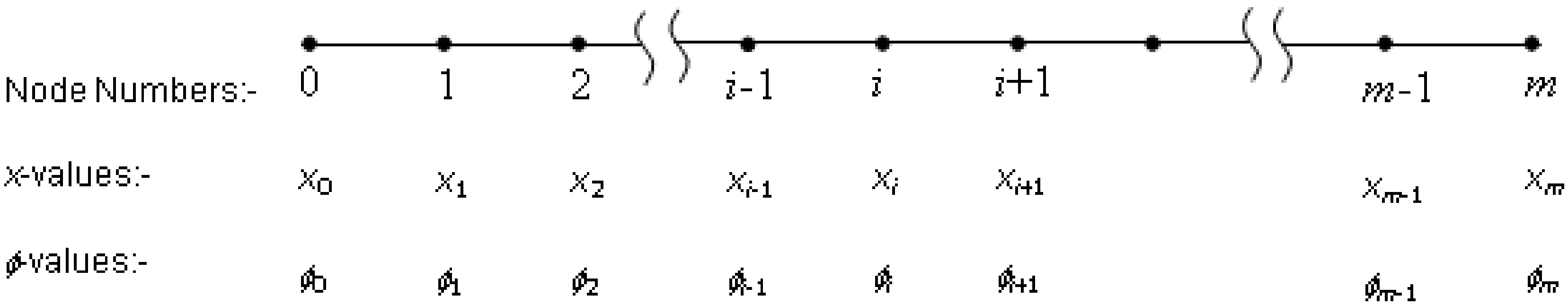


$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} \qquad \frac{d\phi_i}{dt} = \alpha_i \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2}$$

$$\phi(0,t) = c_0;\ \ \phi(L,t) = c_L;\ \ \phi(x,0) = f(x);\ \ m = 4$$

$$\begin{bmatrix} \dfrac{d\phi_1}{dt} \\ \dfrac{d\phi_2}{dt} \\ \dfrac{d\phi_3}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{2\alpha_1}{\Delta x^2} & \dfrac{\alpha_1}{\Delta x^2} & 0 \\ \dfrac{\alpha_2}{\Delta x^2} & -\dfrac{2\alpha_2}{\Delta x^2} & \dfrac{\alpha_2}{\Delta x^2} \\ 0 & \dfrac{\alpha_3}{\Delta x^2} & -\dfrac{2\alpha_3}{\Delta x^2} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} + \begin{bmatrix} \dfrac{c_0 \alpha_1}{\Delta x^2} \\ 0 \\ \dfrac{c_L \alpha_3}{\Delta x^2} \end{bmatrix}$$

Parabolic PDE: semi-discretization

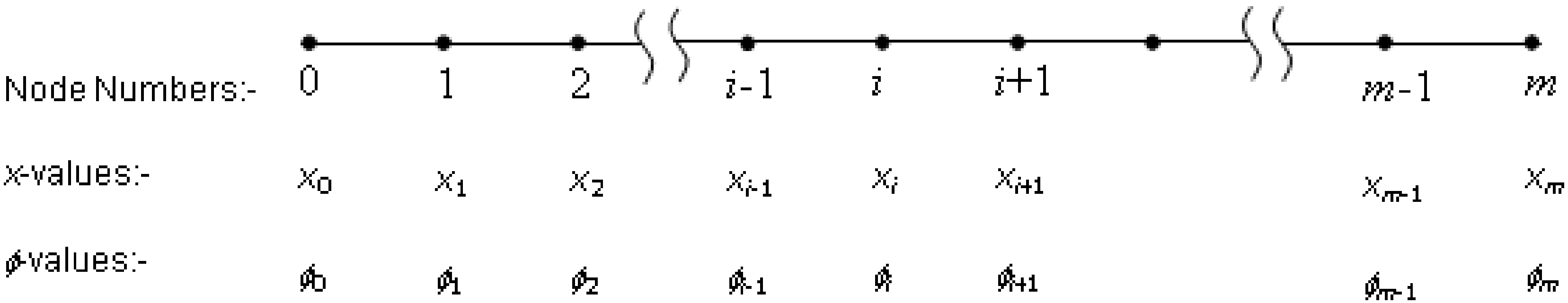


$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \alpha \frac{\partial^2 \phi}{\partial x^2} \qquad \frac{d\phi_i}{dt} = -u_i \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} + \alpha_i \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2}$$

$$\phi(0,t) = c_0; \;\; \phi(L,t) = c_L; \;\; \phi(x,0) = f(x)$$

$$\begin{bmatrix} \dfrac{d\phi_1}{dt} \\ \dfrac{d\phi_2}{dt} \\ \dfrac{d\phi_3}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{2\alpha_1}{\Delta x^2} & -\dfrac{u_1}{2\Delta x} + \dfrac{\alpha_1}{\Delta x^2} & 0 \\ \dfrac{u_2}{2\Delta x} + \dfrac{\alpha_2}{\Delta x^2} & -\dfrac{2\alpha_2}{\Delta x^2} & -\dfrac{u_2}{2\Delta x} + \dfrac{\alpha_2}{\Delta x^2} \\ 0 & \dfrac{u_3}{2\Delta x} + \dfrac{\alpha_3}{\Delta x^2} & -\dfrac{2\alpha_3}{\Delta x^2} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} + \begin{bmatrix} \dfrac{u_1 c_0}{2\Delta x} + \dfrac{c_0 \alpha_1}{\Delta x^2} \\ 0 \\ -\dfrac{u_3 c_L}{2\Delta x} + \dfrac{c_L \alpha_3}{\Delta x^2} \end{bmatrix}$$

Parabolic PDE: semi-discretization



$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} \qquad\qquad \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \alpha \frac{\partial^2 \phi}{\partial x^2}$$

$$\phi(0,t) = c_0; \;\; \phi(L,t) = c_L; \;\; \phi(x,0) = f(x)$$

$$\frac{\mathrm{d}\bar{\phi}}{\mathrm{dt}} = \mathbf{A}\bar{\phi} + \mathbf{b} \qquad \bar{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \qquad \bar{\phi}(0) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}$$

Parabolic PDE: semi-discretization

$$\frac{\mathrm{d}\bar{\phi}}{\mathrm{dt}} = \mathbf{A}\bar{\phi} + \mathbf{b} \qquad \bar{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \qquad \bar{\phi}(0) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}$$

$$\bar{\phi}^{n+1} = \bar{\phi}^n + \Delta t[\mu\{\mathbf{A}^n\bar{\phi}^n + \mathbf{b}^n\} + (1-\mu)\{\mathbf{A}^{n+1}\bar{\phi}^{n+1} + \mathbf{b}^{n+1}\}]$$

μ = 0: Euler Backward; μ = 1: Euler Forward

μ = 1/2: Trapezoidal

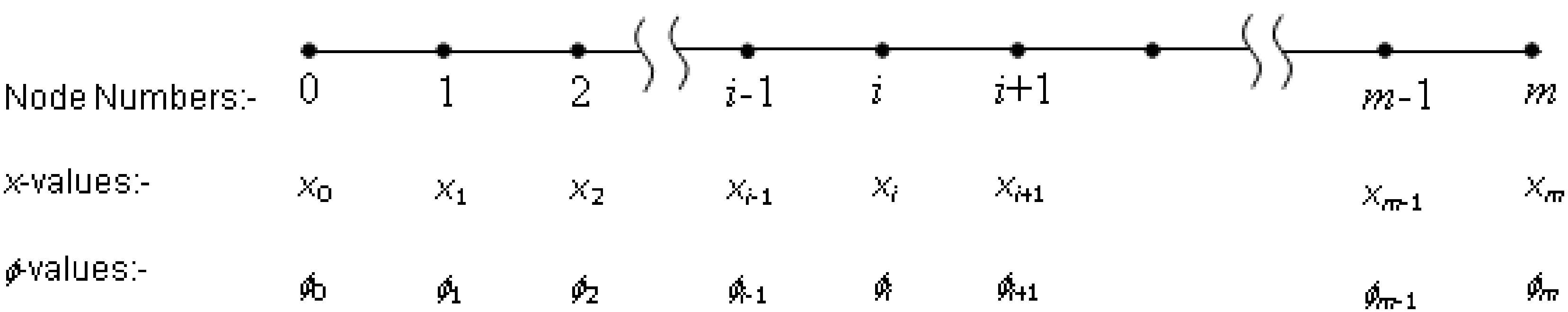
$$[I - \Delta t(1-\mu)\mathbf{A}^{n+1}]\bar{\phi}^{n+1}$$
$$= [I + \mu\Delta t\mathbf{A}^n]\bar{\phi}^n + \Delta t[(1-\mu)\mathbf{b}^{n+1} + \mu\mathbf{b}^n]$$

Parabolic PDE: full-discretization

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2}$$

$$\phi(0,t) = c_0;\ \phi(L,t) = c_L;\ \ \phi(x,0) = f(x)$$

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \mu \alpha_i^n \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2} + (1-\mu)\alpha_i^{n+1} \frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}$$



Parabolic PDE: full-discretization

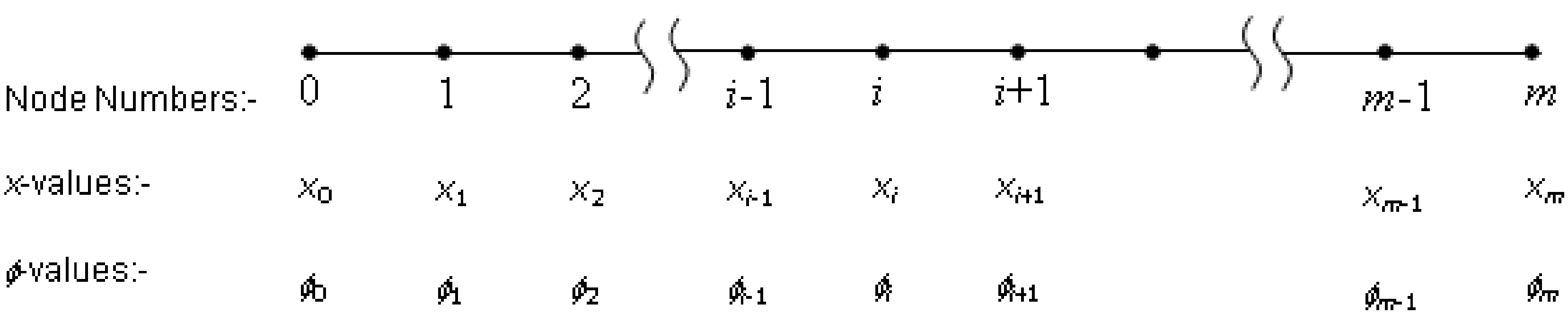
$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \alpha\frac{\partial^2 \phi}{\partial x^2}$$

$$\phi(0,t) = c_0;\ \phi(L,t) = c_L;\ \ \phi(x,0) = f(x)$$

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t}$$

$$= \mu\left[-u_i^n\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} + \alpha_i^n\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}\right]$$

$$+ (1-\mu)\left[-u_i^{n+1}\frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2\Delta x} + \alpha_i^{n+1}\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}\right]$$



Types of Boundary Condition

- ✓ Dirichlet Condition (1st Type):
  - ✓ Variable value is specified

$$\phi(0,t) = c_0;\ \phi(L,t) = c_L$$

- ✓ Neumann Condition (2nd Type):
  - ✓ Gradient is specified

$$\left.\frac{dj}{dx}\right|_{(0,t)\text{ and/or }(L,t)} = c$$

- ✓ Robin Condition (3rd Type):
  - ✓ A linear combination of the variable and gradient is specified at $(0, t)$ and/or (L, t)

$$a\frac{dj}{dx} + bf = c$$

Example

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}$$

$u = 0.1;\ \alpha = 0.01; T(0,t) = 0;\ T(1,t) = 0;\ T(x,0) = 50\sin\pi x$

Solve Using:

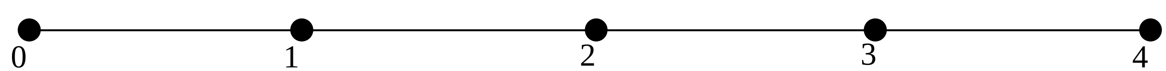
(a) Euler Forward in time and Centeral Difference in space (EF-CD)

(b) Euler Backward in time and Central Difference in space (EB-CD)

(c) Crank Nicholson method

(d) A 2nd order R-K method in time and Central Difference approximation in space.

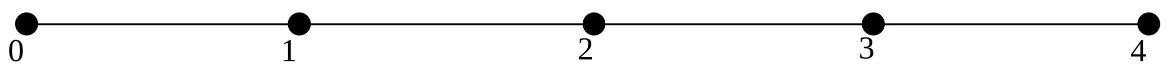
Use $\Delta x = 0.25$ and $\Delta t = 0.5$



Example

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}$$

$u = 0.1;\ \alpha = 0.01; T(0,t) = 0;\ T(1,t) = 0;\ T(x,0) = 50\sin\pi x$



$$\frac{dT_j}{dt} + u\frac{T_{j+1} - T_{j-1}}{2\Delta x} = \alpha\frac{T_{j+1} - 2T_j + T_{j-1}}{\Delta x^2}$$

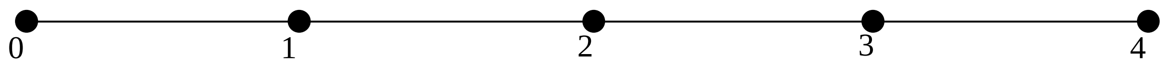
$$\frac{dT_j}{dt} = \left(\frac{u}{2\Delta x} + \frac{\alpha}{\Delta x^2}\right)T_{j-1} + \left(-\frac{2\alpha}{\Delta x^2}\right)T_j + \left(-\frac{u}{2\Delta x} + \frac{\alpha}{\Delta x^2}\right)T_{j+1}$$

$$\frac{dT_j}{dt} = 0.36T_{j-1} - 0.32T_j - 0.04T_{j+1}$$

Example

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}$$

$u = 0.1;\ \alpha = 0.01; T(0,t) = 0;\ T(1,t) = 0;\ \ T(x,0) = 50\sin\pi x$



$$\frac{dT_j}{dt} = 0.36T_{j-1} - 0.32T_j - 0.04T_{j+1}$$

$$\frac{d\mathbf{T}}{dt} = \mathbf{A}\mathbf{T}$$

$$\mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} -0.32 & -0.04 & 0 \\ 0.36 & -0.32 & -0.04 \\ 0 & 0.36 & -0.32 \end{bmatrix} \qquad \mathbf{T}^0 = \begin{bmatrix} T_1^0 \\ T_2^0 \\ T_2^0 \end{bmatrix} = \begin{bmatrix} 35.3553 \\ 50.0 \\ 35.3553 \end{bmatrix}$$

Example

$$\frac{d\mathbf{T}}{dt} = \mathbf{AT} \quad \mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} -0.32 & -0.04 & 0 \\ 0.36 & -0.32 & -0.04 \\ 0 & 0.36 & -0.32 \end{bmatrix} \qquad \mathbf{T}^0 = \begin{bmatrix} T_1^0 \\ T_2^0 \\ T_2^0 \end{bmatrix} = \begin{bmatrix} 35.3553 \\ 50.0 \\ 35.3553 \end{bmatrix}$$

Euler Forward:

$$\mathbf{T^{n+1}} = \left[I + \Delta t\mathbf{A} \right] \mathbf{T^n}$$

$$\mathbf{T}^{0.5} = \begin{bmatrix} T_1^{0.5} \\ T_2^{0.5} \\ T_2^{0.5} \end{bmatrix} = \begin{bmatrix} 0.84 & -0.02 & 0 \\ 0.18 & 0.84 & -0.02 \\ 0 & 0.18 & 0.84 \end{bmatrix} \begin{bmatrix} 35.3553 \\ 50.0 \\ 35.3553 \end{bmatrix} = \begin{bmatrix} 28.6985 \\ 47.6568 \\ 38.6985 \end{bmatrix}$$

Example

$$\frac{d\mathbf{T}}{dt} = \mathbf{AT} \quad \mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} -0.32 & -0.04 & 0 \\ 0.36 & -0.32 & -0.04 \\ 0 & 0.36 & -0.32 \end{bmatrix} \qquad \mathbf{T}^0 = \begin{bmatrix} T_1^0 \\ T_2^0 \\ T_2^0 \end{bmatrix} = \begin{bmatrix} 35.3553 \\ 50.0 \\ 35.3553 \end{bmatrix}$$

Euler Backward:

$$\left[I - \Delta t\mathbf{A} \right] \mathbf{T}^{\mathbf{n+1}} = \mathbf{T}^{\mathbf{n}}$$

$$\begin{bmatrix} 1.16 & 0.02 & 0 \\ -0.18 & 1.16 & 0.02 \\ 0 & -0.18 & 1.16 \end{bmatrix} \begin{bmatrix} T_1^{0.5} \\ T_2^{0.5} \\ T_2^{0.5} \end{bmatrix} = \begin{bmatrix} 35.3553 \\ 50.0 \\ 35.3553 \end{bmatrix}$$

$$\mathbf{T}^{0.5} = \begin{bmatrix} T_1^{0.5} \\ T_2^{0.5} \\ T_2^{0.5} \end{bmatrix} = \begin{bmatrix} 29.6674 \\ 47.0556 \\ 37.7804 \end{bmatrix}$$

Example

$$\frac{d\mathbf{T}}{dt} = \mathbf{AT} \quad \mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} -0.32 & -0.04 & 0 \\ 0.36 & -0.32 & -0.04 \\ 0 & 0.36 & -0.32 \end{bmatrix} \qquad \mathbf{T}^0 = \begin{bmatrix} T_1^0 \\ T_2^0 \\ T_2^0 \end{bmatrix} = \begin{bmatrix} 35.3553 \\ 50.0 \\ 35.3553 \end{bmatrix}$$

Crank-Nicholson:

$$\left[I - 0.5\Delta t\mathbf{A} \right] \mathbf{T^{n+1}} = \left[I + 0.5\Delta t\mathbf{A} \right] \mathbf{T^n}$$

$$\begin{bmatrix} 1.08 & 0.01 & 0 \\ -0.09 & 1.08 & 0.02 \\ 0 & -0.09 & 1.08 \end{bmatrix} \begin{bmatrix} T_1^{0.5} \\ T_2^{0.5} \\ T_2^{0.5} \end{bmatrix} = \begin{bmatrix} 0.92 & -0.01 & 0 \\ 0.09 & 0.92 & -0.01 \\ 0 & 0.09 & 0.92 \end{bmatrix} \begin{bmatrix} 35.3553 \\ 50.0 \\ 35.3553 \end{bmatrix} = \begin{bmatrix} 32.0269 \\ 48.8284 \\ 37.0269 \end{bmatrix}$$

$$\mathbf{T}^{0.5} = \begin{bmatrix} T_1^{0.5} \\ T_2^{0.5} \\ T_2^{0.5} \end{bmatrix} = \begin{bmatrix} 29.2166 \\ 47.2923 \\ 38.2252 \end{bmatrix}$$

Example

$$\frac{d\mathbf{T}}{dt} = \mathbf{AT} \quad \mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} -0.32 & -0.04 & 0 \\ 0.36 & -0.32 & -0.04 \\ 0 & 0.36 & -0.32 \end{bmatrix} \qquad \mathbf{T}^0 = \begin{bmatrix} T_1^0 \\ T_2^0 \\ T_2^0 \end{bmatrix} = \begin{bmatrix} 35.3553 \\ 50.0 \\ 35.3553 \end{bmatrix}$$

2nd Order Runge-Kutta (Heun's Predictor-Corrector Form):

$$\mathbf{T}_p^{n+1} = \left[I + \Delta t\mathbf{A} \right] \mathbf{T}^n$$

This is same as Euler-Forward. EF solution is the Predictor.

$$\mathbf{T}_c^{n+1} = \mathbf{T}^n + \frac{\Delta t}{2}\left[\mathbf{AT}_p^{n+1} + \mathbf{AT}^n \right] = \mathbf{T}^n + \frac{\Delta t}{2}\mathbf{A}\left[\mathbf{T}_p^{n+1} + \mathbf{T}^n \right]$$

$$\mathbf{T}_c^{0.5} = \begin{bmatrix} T_{1c}^{0.5} \\ T_{2c}^{0.5} \\ T_{2c}^{0.5} \end{bmatrix} = \begin{bmatrix} 35.3553 \\ 50.0 \\ 35.3553 \end{bmatrix} + \begin{bmatrix} -0.08 & -0.01 & 0 \\ 0.09 & -0.08 & -0.01 \\ 0 & 0.09 & -0.08 \end{bmatrix} \left[\begin{Bmatrix} 28.6985 \\ 47.6568 \\ 38.6985 \end{Bmatrix} + \begin{Bmatrix} 35.3553 \\ 50.0 \\ 35.3553 \end{Bmatrix} \right] = \begin{bmatrix} 29.2544 \\ 47.2118 \\ 38.2201 \end{bmatrix}$$

Convergence

What about Consistency, Stability, Convergence?

How does one choose Δx and Δt? Are they interdependent?

Diffusion Equation:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \mu \alpha_i^n \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2} + (1-\mu)\alpha_i^{n+1} \frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}$$

$$\left[-(1-\mu)\alpha_i^{n+1}\frac{\Delta t}{\Delta x^2}\right]\phi_{i+1}^{n+1} + \left[1+2(1-\mu)\alpha_i^{n+1}\frac{\Delta t}{\Delta x^2}\right]\phi_i^{n+1} + \left[-(1-\mu)\alpha_i^{n+1}\frac{\Delta t}{\Delta x^2}\right]\phi_{i-1}^{n+1}$$

$$= \left[\mu\alpha_i^n\frac{\Delta t}{\Delta x^2}\right]\phi_{i+1}^n + \left[1-2\mu\alpha_i^n\frac{\Delta t}{\Delta x^2}\right]\phi_i^n + \left[\mu\alpha_i^n\frac{\Delta t}{\Delta x^2}\right]\phi_{i-1}^n$$

Convergence

Advection-Diffusion Equation:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \mu\left[-u_i^n \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} + \alpha_i^n \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}\right] + (1-\mu)\left[-u_i^{n+1} \frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2\Delta x} + \alpha_i^{n+1} \frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}\right]$$

$$\left[(1-\mu)\left(u_i^{n+1}\frac{\Delta t}{2\Delta x} - \alpha_i^{n+1}\frac{\Delta t}{\Delta x^2}\right)\right]\phi_{i+1}^{n+1} + \left[1 + 2(1-\mu)\alpha_i^{n+1}\frac{\Delta t}{\Delta x^2}\right]\phi_i^{n+1} + \left[(1-\mu)\left(-u_i^{n+1}\frac{\Delta t}{2\Delta x} - \alpha_i^{n+1}\frac{\Delta t}{\Delta x^2}\right)\right]\phi_{i-1}^{n+1}$$

$$= \left[\mu\left(-u_i^n\frac{\Delta t}{2\Delta x} + \alpha_i^n\frac{\Delta t}{\Delta x^2}\right)\right]\phi_{i+1}^n + \left[1 - 2\mu\alpha_i^n\frac{\Delta t}{\Delta x^2}\right]\phi_i^n + \left[\mu\left(u_i^n\frac{\Delta t}{2\Delta x} + \alpha_i^n\frac{\Delta t}{\Delta x^2}\right)\right]\phi_{i-1}^n$$

Groups $u\frac{\Delta t}{\Delta x}$ and $\alpha\frac{\Delta t}{\Delta x^2}$ govern the equations.

Convergence

✓ Peclet Number:

$$P_e = \frac{uL}{\alpha} = \frac{uL}{D}$$

✓ Grid Peclet Number:

$$P_g = \frac{u\Delta x}{\alpha} = \frac{u\Delta x}{D}$$

✓ CFL (Courant-Friedrich-Lewy) Number:

$$C = u\frac{\Delta t}{\Delta x}$$

Therefore,

$$\frac{C}{P_g} = \alpha\frac{\Delta t}{\Delta x^2}$$

Convergence

✓ If u and α are constants (not function of x):

$$\left[(1-\mu)\left(u\frac{\Delta t}{2\Delta x}-\alpha\frac{\Delta t}{\Delta x^2}\right)\right]\phi_{i+1}^{n+1}+\left[1+2(1-\mu)\alpha\frac{\Delta t}{\Delta x^2}\right]\phi_i^{n+1}+\left[(1-\mu)\left(-u\frac{\Delta t}{2\Delta x}-\alpha\frac{\Delta t}{\Delta x^2}\right)\right]\phi_{i-1}^{n+1}$$
$$=\left[\mu\left(-u\frac{\Delta t}{2\Delta x}+\alpha\frac{\Delta t}{\Delta x^2}\right)\right]\phi_{i+1}^{n}+\left[1-2\mu\alpha\frac{\Delta t}{\Delta x^2}\right]\phi_i^{n}+\left[\mu\left(u\frac{\Delta t}{2\Delta x}+\alpha\frac{\Delta t}{\Delta x^2}\right)\right]\phi_{i-1}^{n}$$

$$\left[(1-\mu)\left(\frac{C}{2}-\frac{C}{P_g}\right)\right]\phi_{i+1}^{n+1}+\left[1+2(1-\mu)\frac{C}{P_g}\right]\phi_i^{n+1}+\left[(1-\mu)\left(-\frac{C}{2}-\frac{C}{P_g}\right)\right]\phi_{i-1}^{n+1}$$
$$=\left[\mu\left(-\frac{C}{2}+\frac{C}{P_g}\right)\right]\phi_{i+1}^{n}+\left[1-2\mu\frac{C}{P_g}\right]\phi_i^{n}+\left[\mu\left(\frac{C}{2}+\frac{C}{P_g}\right)\right]\phi_{i-1}^{n}$$

The the solutions depend on these two dimensionless groups or numbers. Therefore, stability and convergence will also depend on these two!

Consistency

Diffusion Equation (μ-CD scheme):

$$\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = \mu\alpha_i^n \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2} + (1-\mu)\alpha_i^{n+1}\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}$$

$$\left[-(1-\mu)\alpha_i^{n+1}\frac{\Delta t}{\Delta x^2}\right]\phi_{i+1}^{n+1} + \left[1+2(1-\mu)\alpha_i^{n+1}\frac{\Delta t}{\Delta x^2}\right]\phi_i^{n+1} + \left[-(1-\mu)\alpha_i^{n+1}\frac{\Delta t}{\Delta x^2}\right]\phi_{i-1}^{n+1}$$

$$= \left[\mu\alpha_i^n\frac{\Delta t}{\Delta x^2}\right]\phi_{i+1}^n + \left[1-2\mu\alpha_i^n\frac{\Delta t}{\Delta x^2}\right]\phi_i^n + \left[\mu\alpha_i^n\frac{\Delta t}{\Delta x^2}\right]\phi_{i-1}^n$$

There are four kinds of terms that which to be expanded in Taylor's series:

$\phi_{i\pm1}^{n+1}$ ϕ_i^{n+1} $\phi_{i\pm1}^n$ ϕ_i^n

Consistency

$$\left[-\left(1-\mu\right)\alpha\frac{\Delta t}{\Delta x^2}\right]\phi_{i+1}^{n+1}+\left[1+2\left(1-\mu\right)\alpha\frac{\Delta t}{\Delta x^2}\right]\phi_{i}^{n+1}+\left[-\left(1-\mu\right)\alpha\frac{\Delta t}{\Delta x^2}\right]\phi_{i-1}^{n+1}$$
$$=\left[\mu\alpha\frac{\Delta t}{\Delta x^2}\right]\phi_{i+1}^{n}+\left[1-2\mu\alpha\frac{\Delta t}{\Delta x^2}\right]\phi_{i}^{n}+\left[\mu\alpha\frac{\Delta t}{\Delta x^2}\right]\phi_{i-1}^{n}$$

$$\phi_i^{n+1}=\phi_i^n+\Delta t\left.\frac{\partial\phi}{\partial t}\right|_i^n+\frac{\Delta t^2}{2!}\left.\frac{\partial^2\phi}{\partial t^2}\right|_i^n+\frac{\Delta t^3}{3!}\left.\frac{\partial^3\phi}{\partial t^3}\right|_i^n+HOT$$

$$\phi_{i\pm1}^n=\phi_i^n\pm\Delta x\left.\frac{\partial\phi}{\partial x}\right|_i^n+\frac{\Delta x^2}{2!}\left.\frac{\partial^2\phi}{\partial x^2}\right|_i^n\pm\frac{\Delta x^3}{3!}\left.\frac{\partial^3\phi}{\partial x^3}\right|_i^n+\frac{\Delta x^4}{4!}\left.\frac{\partial^4\phi}{\partial x^4}\right|_i^n+HOT$$

$$\phi_{i\pm1}^{n+1}=\phi_i^n+\left.\left(\Delta t\frac{\partial}{\partial t}\pm\Delta x\frac{\partial}{\partial x}\right)\phi\right|_i^n+\frac{1}{2!}\left.\left(\Delta t\frac{\partial}{\partial t}\pm\Delta x\frac{\partial}{\partial x}\right)^2\phi\right|_i^n+\frac{1}{3!}\left.\left(\Delta t\frac{\partial}{\partial t}\pm\Delta x\frac{\partial}{\partial x}\right)^3\phi\right|_i^n+HOT$$

$$
\left[-(1-\mu)\alpha\frac{\Delta t}{\Delta x^{2}}\right]\left\{\begin{array}{l}\phi_i^n+\left(\Delta t\frac{\partial}{\partial t}+\Delta x\frac{\partial}{\partial x}\right)\phi\Big|_i^n+\frac{1}{2!}\left(\Delta t\frac{\partial}{\partial t}+\Delta x\frac{\partial}{\partial x}\right)^{2}\phi\Big|_i^n+\frac{1}{3!}\left(\Delta t\frac{\partial}{\partial t}+\Delta x\frac{\partial}{\partial x}\right)^{3}\phi\Big|_i^n\\+\frac{1}{4!}\left(\Delta t\frac{\partial}{\partial t}+\Delta x\frac{\partial}{\partial x}\right)^{4}\phi\Big|_i^n+HOT\end{array}\right\}
$$

$$
+\left[1+2(1-\mu)\alpha\frac{\Delta t}{\Delta x^{2}}\right]\left\{\phi_i^n+\Delta t\frac{\partial\phi}{\partial t}\Big|_i^n+\frac{\Delta t^{2}}{2!}\frac{\partial^{2}\phi}{\partial t^{2}}\Big|_i^n+\frac{\Delta t^{3}}{3!}\frac{\partial^{3}\phi}{\partial t^{3}}\Big|_i^n+HOT\right\}+
$$

$$
\left[-(1-\mu)\alpha\frac{\Delta t}{\Delta x^{2}}\right]\left\{\begin{array}{l}\phi_i^n+\left(\Delta t\frac{\partial}{\partial t}-\Delta x\frac{\partial}{\partial x}\right)\phi\Big|_i^n+\frac{1}{2!}\left(\Delta t\frac{\partial}{\partial t}-\Delta x\frac{\partial}{\partial x}\right)^{2}\phi\Big|_i^n+\frac{1}{3!}\left(\Delta t\frac{\partial}{\partial t}-\Delta x\frac{\partial}{\partial x}\right)^{3}\phi\Big|_i^n\\+\frac{1}{4!}\left(\Delta t\frac{\partial}{\partial t}-\Delta x\frac{\partial}{\partial x}\right)^{4}\phi\Big|_i^n+HOT\end{array}\right\}
$$

$$
=\left[\mu\alpha\frac{\Delta t}{\Delta x^{2}}\right]\left\{\phi_i^n+\Delta x\frac{\partial\phi}{\partial x}\Big|_i^n+\frac{\Delta x^{2}}{2!}\frac{\partial^{2}\phi}{\partial x^{2}}\Big|_i^n+\frac{\Delta x^{3}}{3!}\frac{\partial^{3}\phi}{\partial x^{3}}\Big|_i^n+\frac{\Delta x^{4}}{4!}\frac{\partial^{4}\phi}{\partial x^{4}}\Big|_i^n+HOT\right\}+\left[1-2\mu\alpha\frac{\Delta t}{\Delta x^{2}}\right]\phi_i^n
$$

$$
+\left[\mu\alpha\frac{\Delta t}{\Delta x^{2}}\right]\left\{\phi_i^n-\Delta x\frac{\partial\phi}{\partial x}\Big|_i^n+\frac{\Delta x^{2}}{2!}\frac{\partial^{2}\phi}{\partial x^{2}}\Big|_i^n-\frac{\Delta x^{3}}{3!}\frac{\partial^{3}\phi}{\partial x^{3}}\Big|_i^n+\frac{\Delta x^{4}}{4!}\frac{\partial^{4}\phi}{\partial x^{4}}\Big|_i^n+HOT\right\}
$$

Consistency

Diffusion Equation (μ-CD scheme):

$$\left.\frac{\partial \phi}{\partial t}\right|_i^n - \alpha \left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^n = -\frac{\Delta t}{2}\left.\frac{\partial^2 \phi}{\partial t^2}\right|_i^n + (1-\mu)\Delta t \alpha \left.\frac{\partial^3 \phi}{\partial t\, \partial x^2}\right|_i^n - \frac{\Delta t^2}{6}\left.\frac{\partial^3 \phi}{\partial t^3}\right|_i^n$$

$$+(1-\mu)\frac{\Delta t^2}{2}\alpha \left.\frac{\partial^4 \phi}{\partial t^2\, \partial x^2}\right|_i^n + \alpha \frac{\Delta x^2}{12}\left.\frac{\partial^4 \phi}{\partial x^4}\right|_i^n + HOT$$

Truncation Error:

$$TE = - + \frac{\Delta t}{2}\left.\frac{\partial^2 \phi}{\partial t^2}\right|_i^n - (1-\mu)\Delta t \alpha \left.\frac{\partial^3 \phi}{\partial t\, \partial x^2}\right|_i^n + \frac{\Delta t^2}{6}\left.\frac{\partial^3 \phi}{\partial t^3}\right|_i^n$$

$$-(1-\mu)\frac{\Delta t^2}{2}\alpha \left.\frac{\partial^4 \phi}{\partial t^2\, \partial x^2}\right|_i^n - \alpha \frac{\Delta x^2}{12}\left.\frac{\partial^4 \phi}{\partial x^4}\right|_i^n + HOT$$

Consistency

$$TE = - + \frac{\Delta t}{2}\frac{\partial^2 \phi}{\partial t^2}\bigg|_i^n - (1 - \mu)\Delta t\,\alpha\frac{\partial^3 \phi}{\partial t\,\partial x^2}\bigg|_i^n + \frac{\Delta t^2}{6}\frac{\partial^3 \phi}{\partial t^3}\bigg|_i^n$$

$$- (1 - \mu)\frac{\Delta t^2}{2}\alpha\frac{\partial^4 \phi}{\partial t^2\,\partial x^2}\bigg|_i^n - \alpha\frac{\Delta x^2}{12}\frac{\partial^4 \phi}{\partial x^4}\bigg|_i^n + HOT$$

Identity from the original equation:

$$\alpha\frac{\partial^3 \phi}{\partial t\,\partial x^2} = \frac{\partial}{\partial t}\left(\alpha\frac{\partial^2 \phi}{\partial x^2}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right) = \frac{\partial^2 \phi}{\partial t^2}$$

$$TE = \left(\mu - \tfrac{1}{2}\right)\Delta t\frac{\partial^2 \phi}{\partial t^2}\bigg|_i^n + \left(\mu - \tfrac{2}{3}\right)\frac{\Delta t^2}{2}\frac{\partial^3 \phi}{\partial t^3}\bigg|_i^n - \alpha\frac{\Delta x^2}{12}\frac{\partial^4 \phi}{\partial x^4}\bigg|_i^n + HOT$$

Method is O(Δt, Δx^2). For $\mu = ½$, the method is O(Δt^2, Δx^2)

Consistency

$$TE = \left(\mu - \tfrac{1}{2}\right)\Delta t \left.\frac{\partial^2 \phi}{\partial t^2}\right|_i^n + \left(\mu - \tfrac{2}{3}\right)\frac{\Delta t^2}{2}\left.\frac{\partial^3 \phi}{\partial t^3}\right|_i^n - \alpha\frac{\Delta x^2}{12}\left.\frac{\partial^4 \phi}{\partial x^4}\right|_i^n + HOT$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right) = \frac{\partial}{\partial t}\left(\alpha\frac{\partial^2 \phi}{\partial x^2}\right) = \alpha\frac{\partial^2}{\partial x^2}\left(\frac{\partial \phi}{\partial t}\right) = \alpha\frac{\partial^2}{\partial x^2}\left(\alpha\frac{\partial^2 \phi}{\partial x^2}\right) = \alpha^2\frac{\partial^4 \phi}{\partial x^4}$$

$$TE = \left\{\left(\mu - \tfrac{1}{2}\right)\Delta t\,\alpha^2 - \alpha\frac{\Delta x^2}{12}\right\}\left.\frac{\partial^4 \phi}{\partial x^4}\right|_i^n + \left(\mu - \tfrac{2}{3}\right)\frac{\Delta t^2}{2}\left.\frac{\partial^3 \phi}{\partial t^3}\right|_i^n + HOT$$

In order to make the first term zero, one may choose:

$$\frac{\alpha \Delta t}{\Delta x^2} = \frac{1}{\left(\mu - \tfrac{1}{2}\right)12}$$

Consistency

For $\mu > ½$, it is also possible to make the method $O(\Delta t^2, \Delta x^2)$ by carefully choosing Δt and Δx, *e.g.*, for Euler Forward, $\mu = 1$:

$$\frac{\alpha \Delta t}{\Delta x^2} = \frac{1}{6}$$

Advection-Dispersion Equation:

$$\left[(1-\mu)\left(u\frac{\Delta t}{2\Delta x}-\alpha\frac{\Delta t}{\Delta x^2}\right)\right]f_{i+1}^{n+1}+\left[1+2(1-\mu)\alpha\frac{\Delta t}{\Delta x^2}\right]f_{i}^{n+1}+\left[(1-\mu)\left(-u\frac{\Delta t}{2\Delta x}-\alpha\frac{\Delta t}{\Delta x^2}\right)\right]f_{i-1}^{n+1}$$
$$=\left[\mu\left(-u\frac{\Delta t}{2\Delta x}+\alpha\frac{\Delta t}{\Delta x^2}\right)\right]f_{i+1}^{n}+\left[1-2\mu\alpha\frac{\Delta t}{\Delta x^2}\right]f_{i}^{n}+\left[\mu\left(u\frac{\Delta t}{2\Delta x}+\alpha\frac{\Delta t}{\Delta x^2}\right)\right]f_{i-1}^{n}$$

Putting the values of the Taylor’s series expansion of the terms (like in diffusion equation) and simplifying

Consistency

$$\left.\frac{\partial \phi}{\partial t}\right|_i^n + u\left.\frac{\partial \phi}{\partial x}\right|_i^n - \alpha\left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^n = -u^2\Delta t\left(\mu - \tfrac{1}{2}\right)\left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^n + 2u\alpha\Delta t\left(\mu - \tfrac{1}{2}\right)\left.\frac{\partial^3 \phi}{\partial x^3}\right|_i^n - \alpha^2\Delta t\left(\mu - \tfrac{1}{2}\right)\left.\frac{\partial^4 \phi}{\partial x^4}\right|_i^n$$

$$+\frac{u^3\Delta t^2}{6}\left(\tfrac{\mu}{2} - \tfrac{1}{3}\right)\left.\frac{\partial^3 \phi}{\partial x^3}\right|_i^n - \frac{u\Delta x^2}{6}\left.\frac{\partial^3 \phi}{\partial x^3}\right|_i^n + HOT$$

No surprises in the order of accuracy! The method is $O(\Delta t, \Delta x^2)$. For $\mu = ½$, the method is $O(\Delta t^2, \Delta x^2)$

New surprise is:

$$\left.\frac{\partial \phi}{\partial t}\right|_i^n + u\left.\frac{\partial \phi}{\partial x}\right|_i^n - \left[\alpha - u^2\Delta t\left(\mu - \tfrac{1}{2}\right)\right]\left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^n = 2u\alpha\Delta t\left(\mu - \tfrac{1}{2}\right)\left.\frac{\partial^3 \phi}{\partial x^3}\right|_i^n - \alpha^2\Delta t\left(\mu - \tfrac{1}{2}\right)\left.\frac{\partial^4 \phi}{\partial x^4}\right|_i^n$$

$$+\frac{u^3\Delta t^2}{6}\left(\tfrac{\mu}{2} - \tfrac{1}{3}\right)\left.\frac{\partial^3 \phi}{\partial x^3}\right|_i^n - \frac{u\Delta x^2}{6}\left.\frac{\partial^3 \phi}{\partial x^3}\right|_i^n + HOT$$

Numerical Diffusion!

Consistency

Strategies to compensate for Numerical Diffusion:

- ✓ Use modified Diffusion/Dispersion coefficient:

$$a' = a + u^2 \Delta t \left(m - \tfrac{1}{2} \right)$$

- ✓ Use Backward difference approximation for the advection term

If you thought that combining schemes that are consistent independently will always give you a consistent scheme for PDE, think again!

Let us look at one scheme that combines two consistent schemes!

Consistency: inconsistent scheme

Consider the following approximation for the Diffusion equation:

$$\frac{\phi_i^{n+1} - \phi_i^{n-1}}{\Delta t} = \alpha \frac{\phi_{i+1}^{n} - 2\phi_i^{n} + \phi_{i-1}^{n}}{\Delta x^2} + O(\Delta t^2, \Delta x^2)$$

Replace:

$$\phi_i^n = \frac{\phi_i^{n+1} + \phi_i^{n-1}}{2} + O(\Delta t^2)$$

Resulting Scheme:

$$\left(1 + 2\alpha \frac{\Delta t}{\Delta x^2}\right)\phi_i^{n+1} = \left(1 - 2\alpha \frac{\Delta t}{\Delta x^2}\right)\phi_i^{n-1} + 2\alpha \frac{\Delta t}{\Delta x^2}\phi_{i+1}^{n} + 2\alpha \frac{\Delta t}{\Delta x^2}\phi_{i-1}^{n}$$

Consistency: inconsistent scheme

$$\left(1 + 2\alpha\frac{\Delta t}{\Delta x^2}\right)\phi_i^{n+1} = \left(1 - 2\alpha\frac{\Delta t}{\Delta x^2}\right)\phi_i^{n-1} + 2\alpha\frac{\Delta t}{\Delta x^2}\phi_{i+1}^{n} + 2\alpha\frac{\Delta t}{\Delta x^2}\phi_{i-1}^{n}$$

It turns out that, this method is Unconditionally Stable!

Now, substitute the Taylor's series expansions of the terms!

$$\frac{\partial\phi}{\partial t} - \alpha\frac{\partial^2\phi}{\partial x^2} = -\frac{\Delta t^2}{6}\frac{\partial^3\phi}{\partial t^3} + \alpha\frac{\Delta x^2}{12}\frac{\partial^4\phi}{\partial x^4} - \alpha\frac{\Delta t^2}{\Delta x^2}\frac{\partial^3\phi}{\partial t^3} - \frac{\alpha}{12}\frac{\Delta t^4}{\Delta x^2}\frac{\partial^4\phi}{\partial t^4}\cdots$$

It is an inconsistent scheme!

This is the Du Fort–Frankel scheme for the diffusion equation!

Let us now see brief stability analysis for the diffusion equation!

Stability: Diffusion Equation

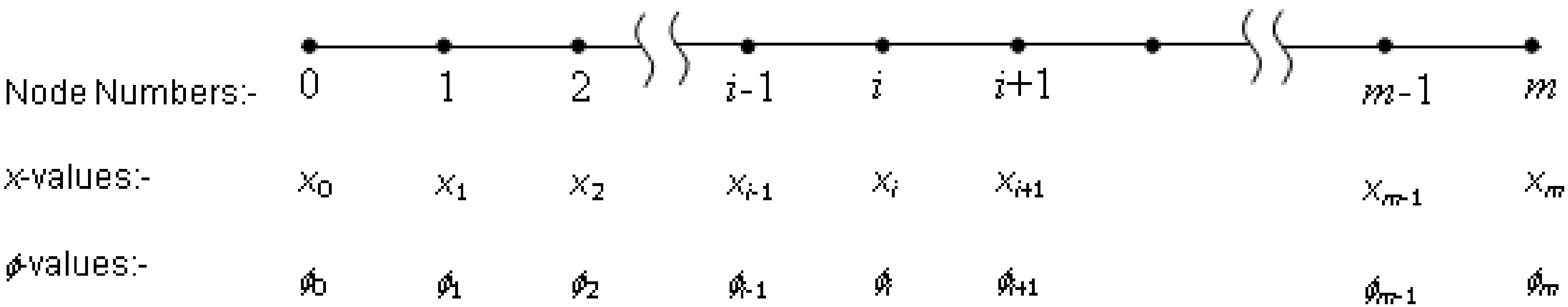
$$\frac{d\phi_i}{dt} = \alpha \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} \quad \Longrightarrow \quad \frac{\mathrm{d}\bar{\phi}}{\mathrm{dt}} = \mathbf{A}\bar{\phi} + \mathbf{b}$$

Assuming Dirichlet type zero boundary conditions!

$$\mathbf{A} = \frac{\alpha}{\Delta x^2} \begin{bmatrix} -2 & 1 & 0 & 0 & \cdot\cdot & \cdot\cdot & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdot\cdot & \cdot\cdot & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdot\cdot & \cdot\cdot & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdot\cdot & \cdot\cdot & \cdot\cdot & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & -2 & 1 & 0 \\ 0 & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & 1 & -2 & 1 \\ 0 & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & 0 & 1 & -2 \end{bmatrix}$$

Stability: Diffusion Equation

Consider the grid:



Size of the matrix **A** is $(m - 1)$. It is a tri-diagonal matrix of the form **A**$[l, d, u]$. Eigenvalues of such a matrix is given by:

$$\lambda_k = \frac{\alpha}{\Delta x^2}\left(d + 2\sqrt{lu}\cos\frac{k\pi}{m}\right) = \frac{\alpha}{\Delta x^2}\left(-2 + 2\cos\frac{k\pi}{m}\right)$$

Boundary conditions only affects the first and last row entries. For large m, that has little effect on the eigenvalues!

Stability: Diffusion Equation

$$\lambda_k = \frac{\alpha}{\Delta x^2}\left(-2 + 2\cos\frac{k\pi}{m}\right) \qquad k = 1, 2 \ldots, m-1$$

✓ For large *m*, the largest (absolute) eigenvalue is:

$$\lambda_{m-1} = -\frac{4\alpha}{\Delta x^2}$$

✓ For large *m*, the smallest (absolute) eigenvalue is:

$$\lambda_1 = \frac{2\alpha}{\Delta x^2}\left(\cos\frac{\pi}{m} - 1\right)$$

✓ The ratio:

$$\left|\frac{\lambda_{m-1}}{\lambda_1}\right| = \left|\frac{2}{\left(\cos\frac{\pi}{m} - 1\right)}\right| \approx \frac{4m^2}{\pi^2}$$

Larger the *m*, *stiffer* the system becomes!

Stability: Diffusion Equation

Recall stability limits for such systems:

✓ Euler Forward:

$$h \leq \frac{2}{|\lambda_{\mathrm{max}}|} \qquad \Longrightarrow \qquad \Delta t \leq \frac{2}{|\lambda_{\mathrm{m-1}}|} = \frac{\Delta x^2}{2\alpha}$$

$$\frac{C}{P_g} = \alpha \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

✓ For 4th order R-K:

$$h \leq \frac{2.785}{|\lambda_{\mathrm{max}}|} \qquad \Longrightarrow \qquad \Delta t \leq \frac{2.785}{|\lambda_{\mathrm{m-1}}|} = 0.7\frac{\Delta x^2}{\alpha}$$

$$\frac{C}{P_g} = \alpha \frac{\Delta t}{\Delta x^2} \leq 0.7$$

Stability analysis of all numerical methods for linear PDE is done by *von-Neumann Analysis* (aka *Fourier Analysis*)!

ESO 208A: Computational Methods in Engineering

Partial Differential Equation: Elliptic Equation

Saumyen Guha

Department of Civil Engineering
IIT Kanpur

Laplace Equation: 1st Type BC

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

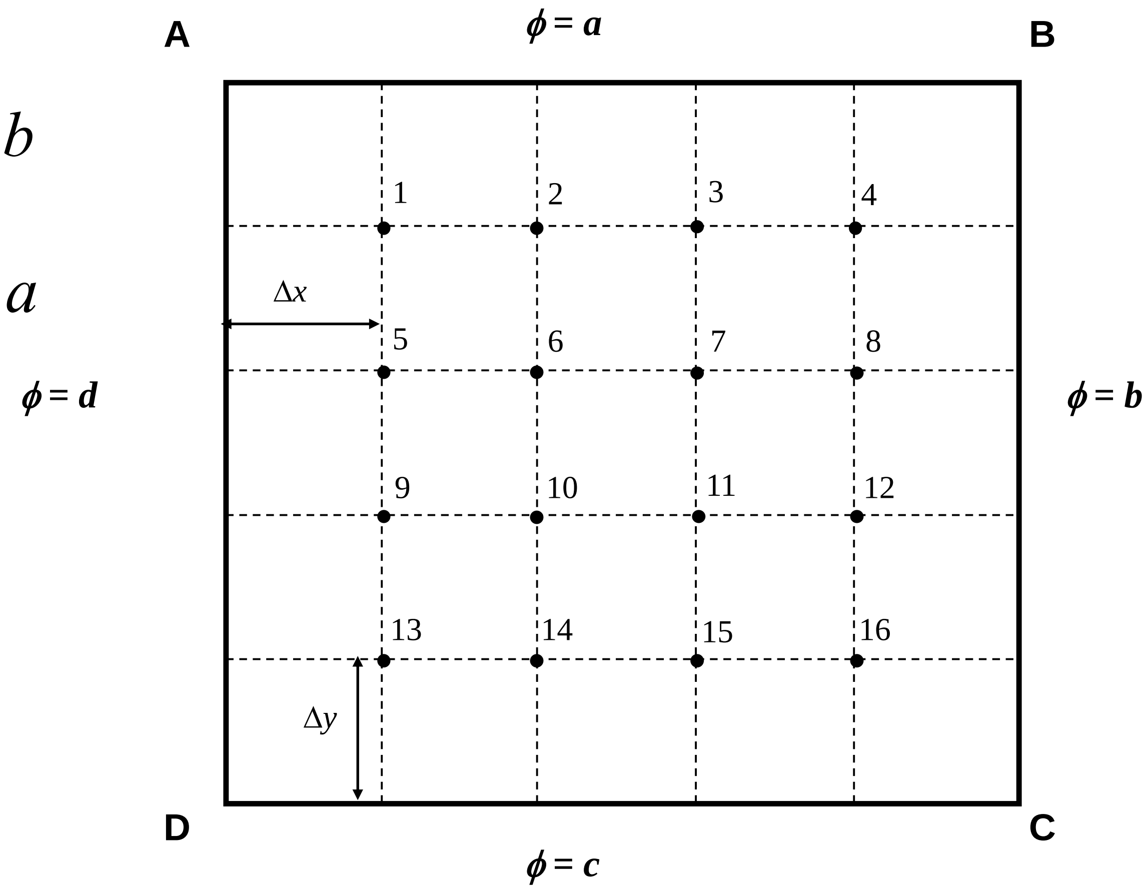
$$x \in \left(0, L_x\right) \text{ and } y \in \left(0, L_y\right)$$

$$\phi\left(0, y\right) = d,\ \phi\left(L_x, y\right) = b$$

$$\phi\left(x, 0\right) = c,\ \phi\left(x, L_y\right) = a$$

$$\Delta x = \frac{L_x}{5}$$

$$\Delta y = \frac{L_y}{5}$$



Laplace Equation

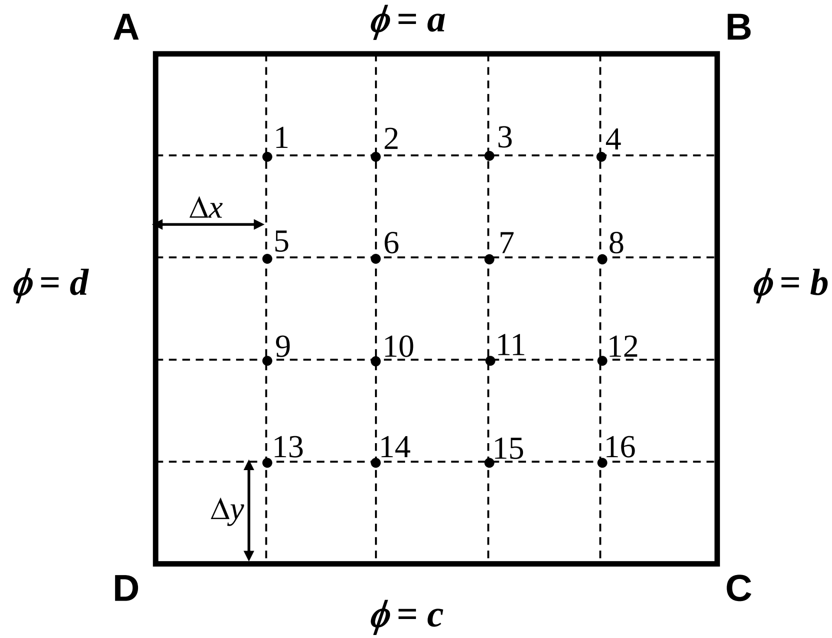
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} = \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2}$$

$$\left.\frac{\partial^2 \phi}{\partial y^2}\right|_{i,j} = \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2}$$

Transform the equation to the form:

$\boldsymbol{A\phi} = \boldsymbol{b}$



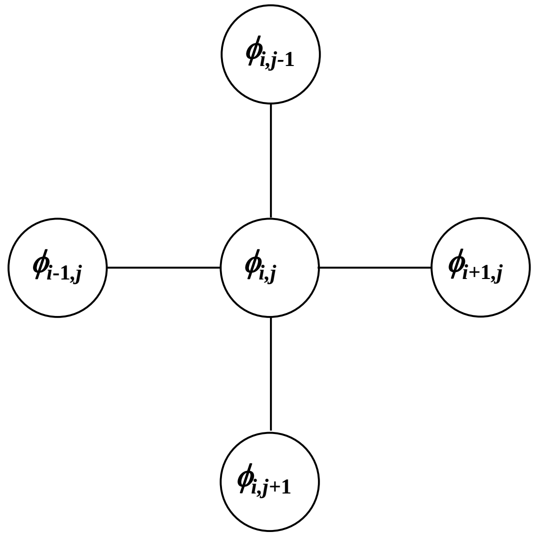
- ✓ Create a node number vs. co-ordinate look-up table
- ✓ Initialize a null matrix ($\boldsymbol{A}$) of size $N\times N$ and a vector ($\boldsymbol{b}$) of size N
- ✓ N is the total number of unknown nodes.

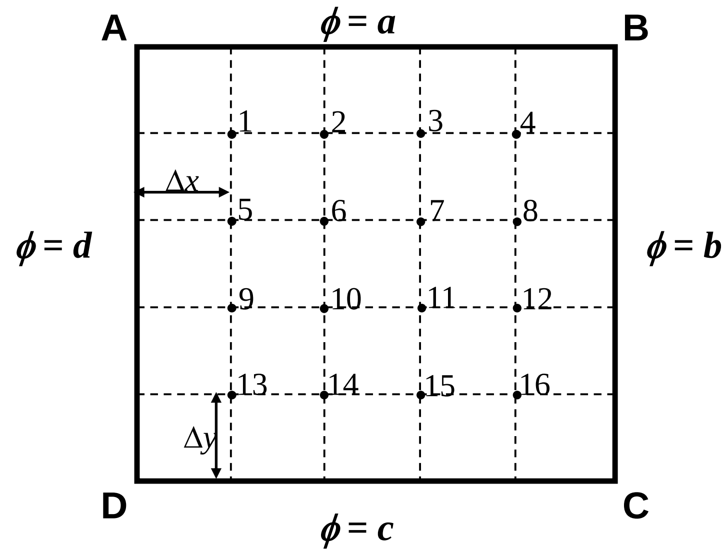
Laplace Equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} = \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2}$$

$$\left.\frac{\partial^2 \phi}{\partial y^2}\right|_{i,j} = \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2}$$

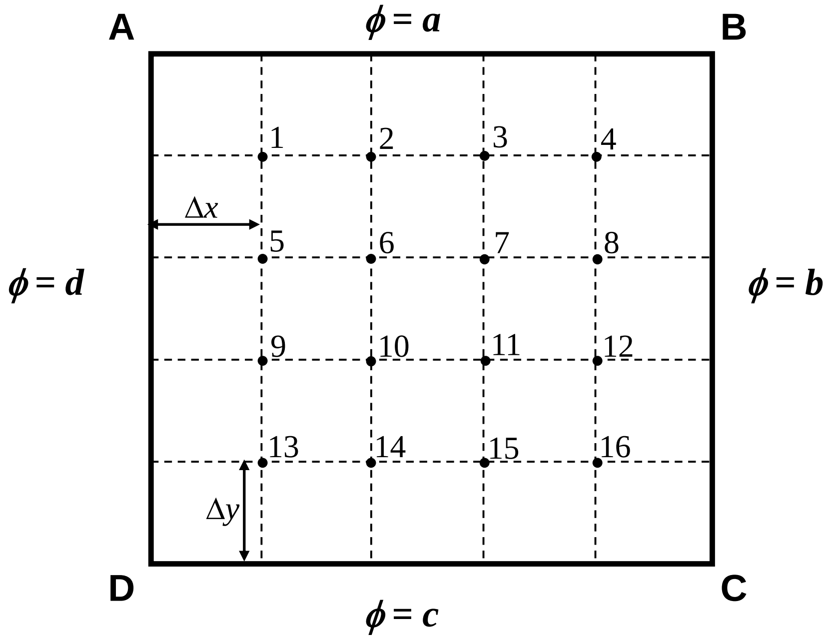




$$\left.\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)\right|_{i,j} = \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2} = 0$$

$$\left(\frac{1}{\Delta y^2}\right)\phi_{i,j-1} + \left(\frac{1}{\Delta x^2}\right)\phi_{i-1,j} + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_{i,j} + \left(\frac{1}{\Delta x^2}\right)\phi_{i+1,j} + \left(\frac{1}{\Delta y^2}\right)\phi_{i,j+1} = 0$$

Laplace Equation



Assuming the origin at D

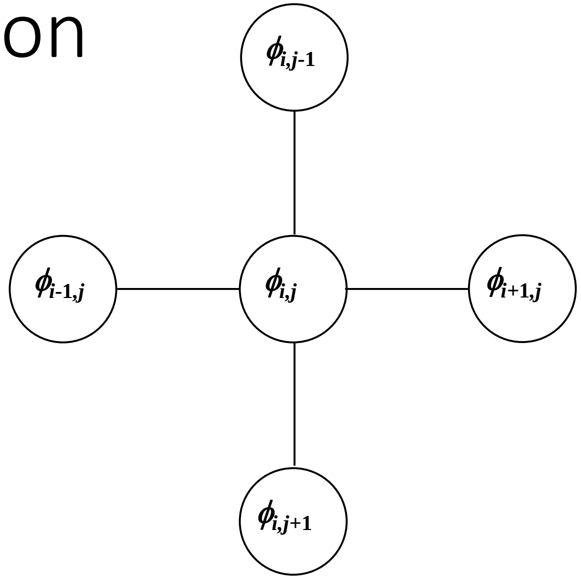
Node No.	x	y	x-neighbour	y-neighbour
1	Δx	4Δy	$\phi = d$, 2	5, $\phi = a$
⋮	⋮	⋮	⋮	⋮
3	3Δx	4Δy	2, 4	7, $\phi = a$
⋮	⋮	⋮	⋮	⋮
6	2Δx	3Δy	5, 7	10, 2
⋮	⋮	⋮	⋮	⋮
9	Δx	2Δy	$\phi = d$, 10	13, 5

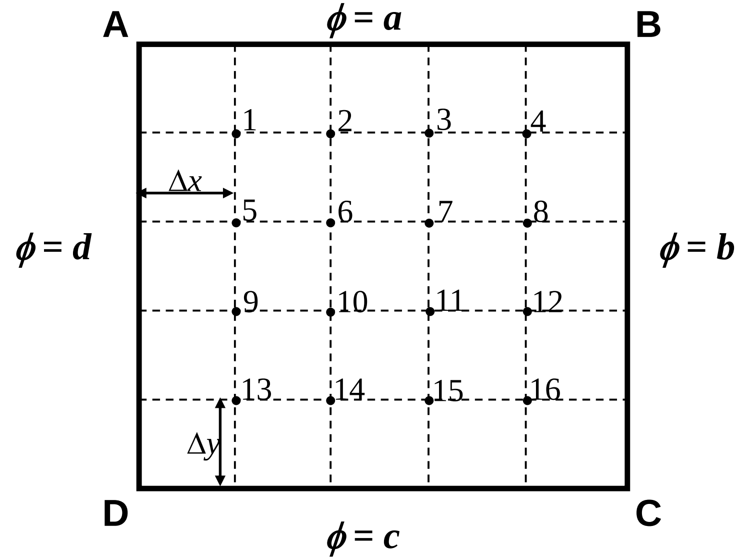
$$
\underbrace{\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} f_1\\ f_2\\ f_3\\ f_4\\ f_5\\ f_6\\ f_7\\ f_8\\ f_9\\ f_{10}\\ f_{11}\\ f_{12}\\ f_{13}\\ f_{14}\\ f_{15}\\ f_{16} \end{bmatrix}}_{\boldsymbol{\phi}}
=
\underbrace{\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}}_{\boldsymbol{b}}
$$

One equation for each node!

Laplace Equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

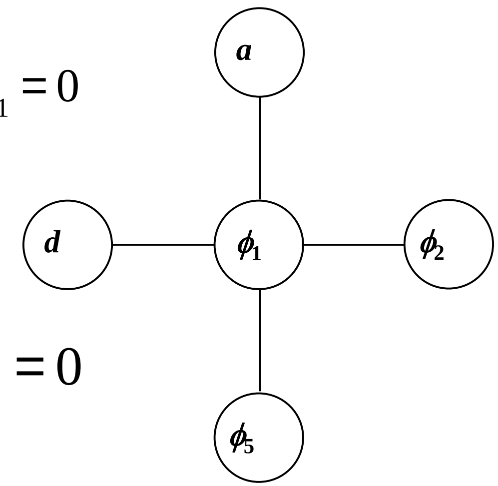




Denote:
$\alpha = 1/\Delta x^2\,; \beta = 1/\Delta y^2$

$$\left(\frac{1}{\Delta y^2}\right)\phi_{i,j-1} + \left(\frac{1}{\Delta x^2}\right)\phi_{i-1,j} + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_{i,j} + \left(\frac{1}{\Delta x^2}\right)\phi_{i+1,j} + \left(\frac{1}{\Delta y^2}\right)\phi_{i,j+1} = 0$$

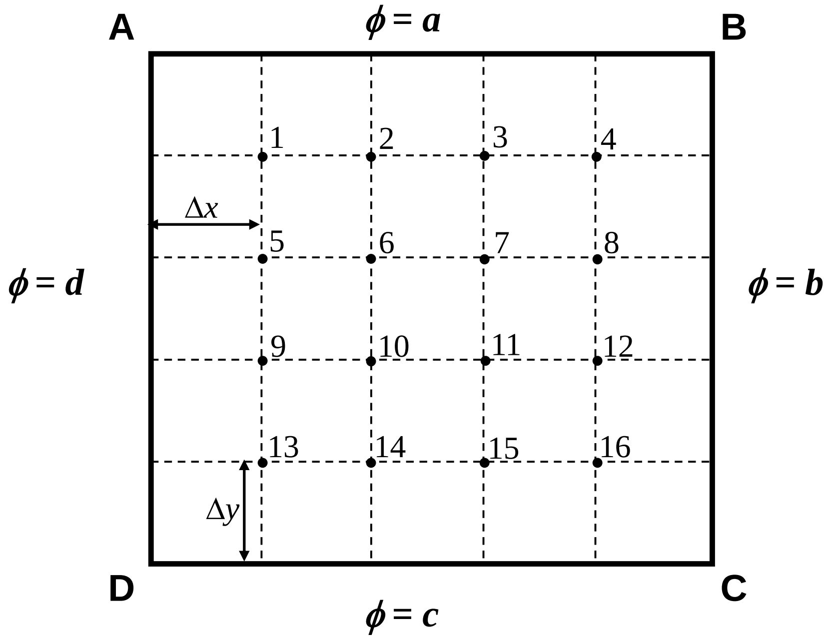
Node 1



$$\left(\frac{1}{\Delta y^2}\right)a + \left(\frac{1}{\Delta x^2}\right)d + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_1 + \left(\frac{1}{\Delta x^2}\right)\phi_2 + \left(\frac{1}{\Delta y^2}\right)\phi_5 = 0$$

$$\left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_1 + \left(\frac{1}{\Delta x^2}\right)\phi_2 + \left(\frac{1}{\Delta y^2}\right)\phi_5 = -\left(\frac{1}{\Delta y^2}\right)a - \left(\frac{1}{\Delta x^2}\right)d$$

Laplace Equation



Assuming the origin at D

Node No.	x	y	x-neighbour	y-neighbour
1	Δx	4Δy	$\phi = d$, 2	5, $\phi = a$
⋮	⋮	⋮	⋮	⋮
3	3Δx	4Δy	2, 4	7, $\phi = a$
⋮	⋮	⋮	⋮	⋮
6	2Δx	3Δy	5, 7	10, 2
⋮	⋮	⋮	⋮	⋮
9	Δx	2Δy	$\phi = d$, 10	13, 5

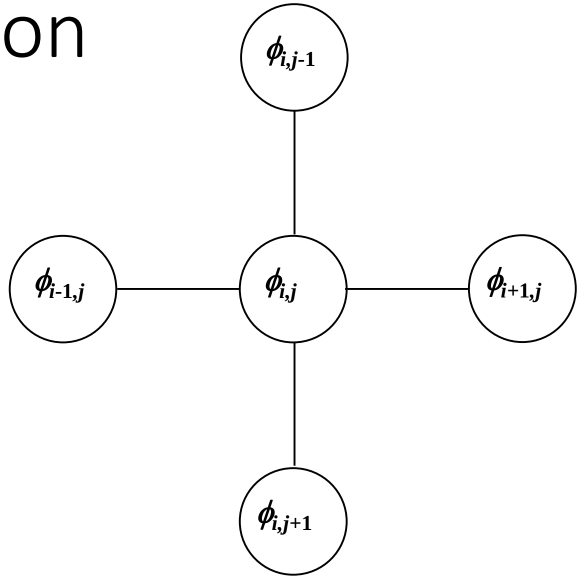
$$
\overset{A}{\begin{bmatrix}
-2(a+b) & a & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}}
\overset{\boldsymbol{\phi}}{\begin{bmatrix}
f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \\ f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{16}
\end{bmatrix}}
=
\overset{\boldsymbol{b}}{\begin{bmatrix}
-ba - ad \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}}
$$

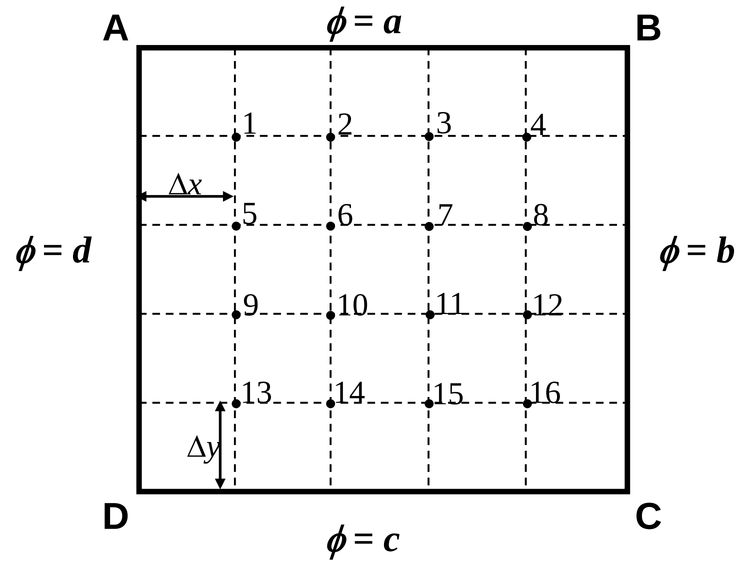
One equation for each node!

Laplace Equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

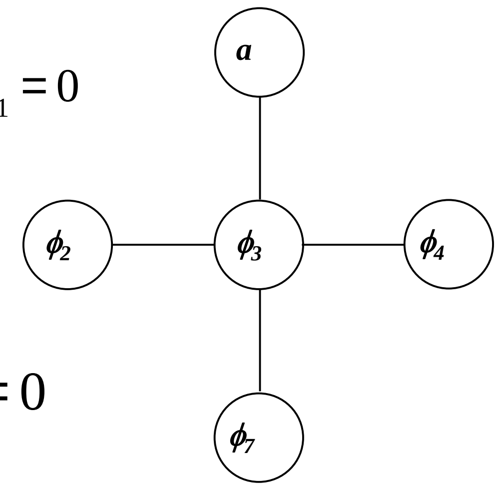
Denote:
$\alpha = 1/\Delta x^2\,;\beta = 1/\Delta y^2$





$$\left(\frac{1}{\Delta y^2}\right)\phi_{i,j-1} + \left(\frac{1}{\Delta x^2}\right)\phi_{i-1,j} + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_{i,j} + \left(\frac{1}{\Delta x^2}\right)\phi_{i+1,j} + \left(\frac{1}{\Delta y^2}\right)\phi_{i,j+1} = 0$$

Node 3

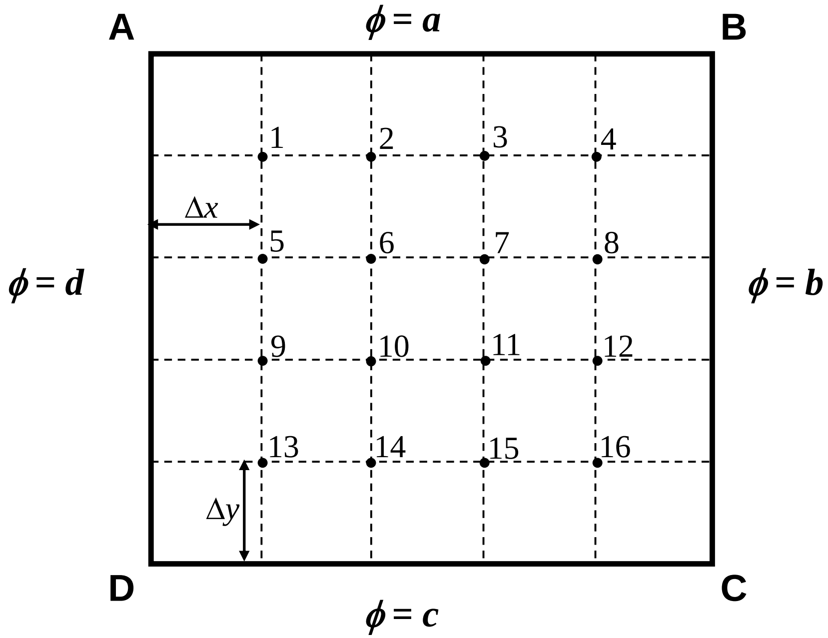


$$\left(\frac{1}{\Delta y^2}\right)a + \left(\frac{1}{\Delta x^2}\right)\phi_2 + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_3 + \left(\frac{1}{\Delta x^2}\right)\phi_4 + \left(\frac{1}{\Delta y^2}\right)\phi_7 = 0$$

$$\left(\frac{1}{\Delta x^2}\right)\phi_2 + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_3 + \left(\frac{1}{\Delta x^2}\right)\phi_4 + \left(\frac{1}{\Delta y^2}\right)\phi_7 = -\left(\frac{1}{\Delta y^2}\right)a$$

Laplace Equation

Assuming the origin at D

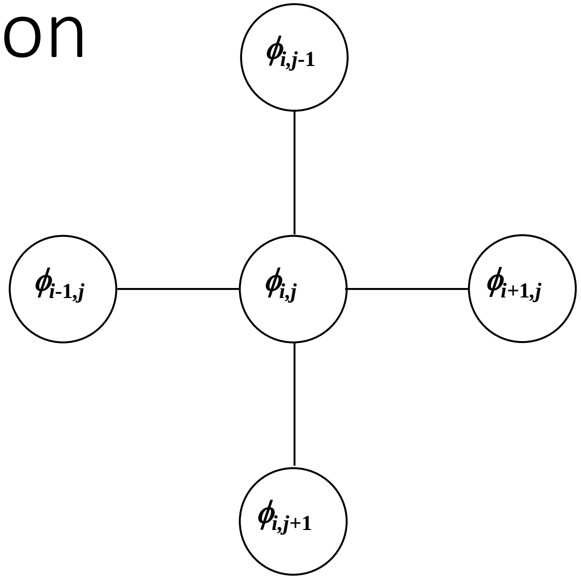


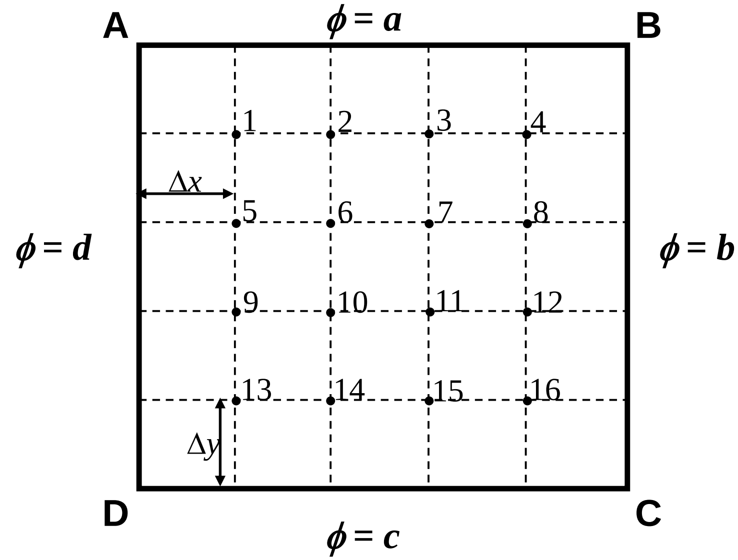
Node No.	x	y	x-neighbour	y-neighbour
1	Δx	$4\Delta y$	$\phi = d$, 2	5, $\phi = a$
⋮	⋮	⋮	⋮	⋮
3	$3\Delta x$	$4\Delta y$	2, 4	7, $\phi = a$
⋮	⋮	⋮	⋮	⋮
6	$2\Delta x$	$3\Delta y$	5, 7	10, 2
⋮	⋮	⋮	⋮	⋮
9	Δx	$2\Delta y$	$\phi = d$, 10	13, 5

$$
\overset{A}{\begin{bmatrix}
-2(\alpha+\beta) & \alpha & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha & -2(\alpha+\beta) & \alpha & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}}
\overset{\boldsymbol{\phi}}{\begin{bmatrix}
\phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \\ \phi_{10} \\ \phi_{11} \\ \phi_{12} \\ \phi_{13} \\ \phi_{14} \\ \phi_{15} \\ \phi_{16}
\end{bmatrix}}
=
\overset{\boldsymbol{b}}{\begin{bmatrix}
-\beta a - \alpha d \\ 0 \\ -\beta a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}}
$$

One equation for each node!

Laplace Equation





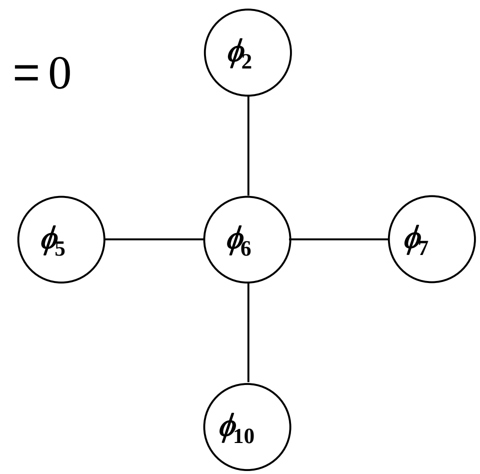
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Denote:

$\alpha = 1/\Delta x^2\,; \beta = 1/\Delta y^2$

$$\left(\frac{1}{\Delta y^2}\right)\phi_{i,j-1} + \left(\frac{1}{\Delta x^2}\right)\phi_{i-1,j} + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_{i,j} + \left(\frac{1}{\Delta x^2}\right)\phi_{i+1,j} + \left(\frac{1}{\Delta y^2}\right)\phi_{i,j+1} = 0$$

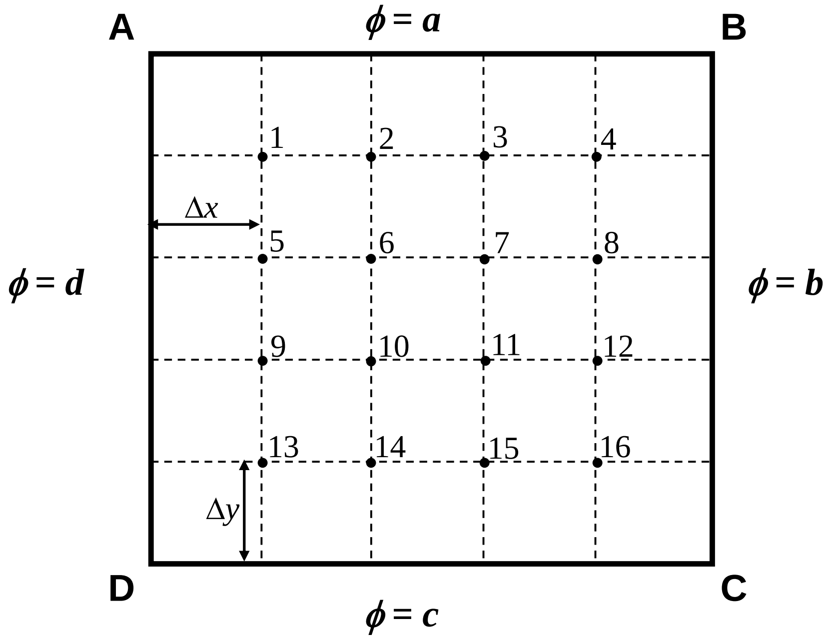
Node 6



$$\left(\frac{1}{\Delta y^2}\right)\phi_2 + \left(\frac{1}{\Delta x^2}\right)\phi_5 + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_6 + \left(\frac{1}{\Delta x^2}\right)\phi_7 + \left(\frac{1}{\Delta y^2}\right)\phi_{10} = 0$$

Laplace Equation

Assuming the origin at D

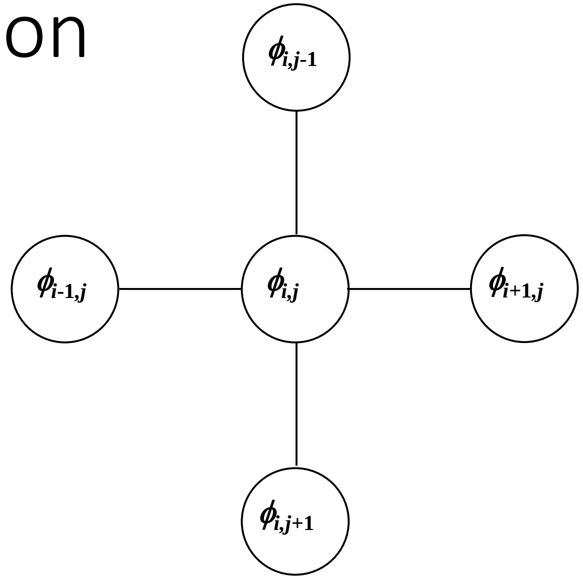


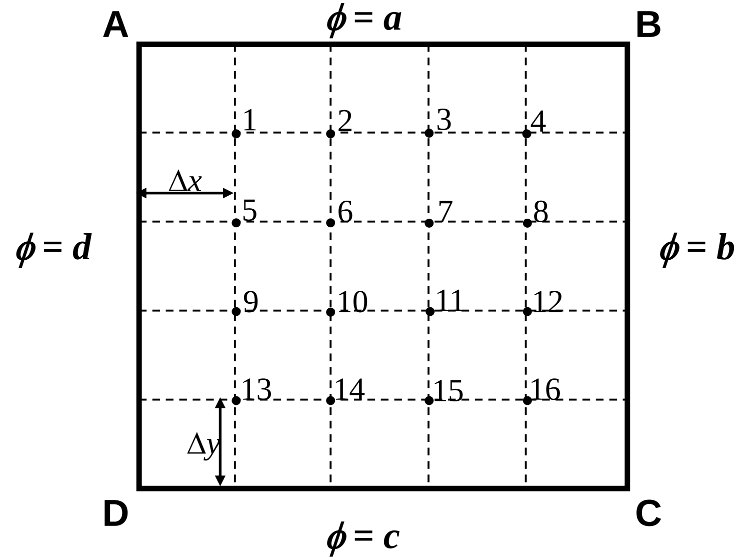
Node No.	x	y	x-neighbour	y-neighbour
1	Δx	4Δy	$\phi = d$, 2	5, $\phi = a$
⋮	⋮	⋮	⋮	⋮
3	3Δx	4Δy	2, 4	7, $\phi = a$
⋮	⋮	⋮	⋮	⋮
6	2Δx	3Δy	5, 7	10, 2
⋮	⋮	⋮	⋮	⋮
9	Δx	2Δy	$\phi = d$, 10	13, 5

$$
\overset{A}{\begin{bmatrix}
-2(a+b) & a & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & -2(a+b) & a & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & 0 & 0 & a & -2(a+b) & a & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}}
\overset{\boldsymbol{\phi}}{\begin{bmatrix}
f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \\ f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{16}
\end{bmatrix}}
=
\overset{\boldsymbol{b}}{\begin{bmatrix}
-ba - ad \\ 0 \\ -ba \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}}
$$

One equation for each node!

Laplace Equation



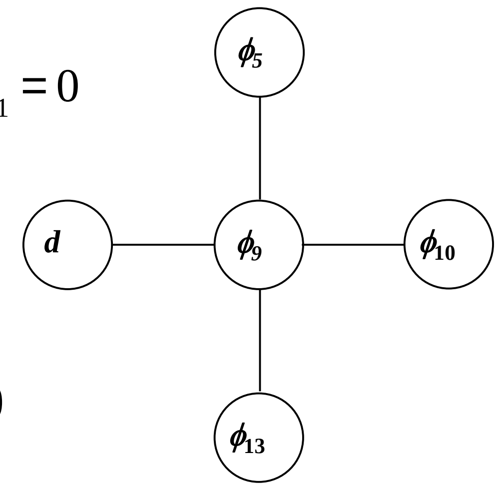


$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Denote:
$\alpha = 1/\Delta x^2 \,;\, \beta = 1/\Delta y^2$

$$\left(\frac{1}{\Delta y^2}\right)\phi_{i,j-1} + \left(\frac{1}{\Delta x^2}\right)\phi_{i-1,j} + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_{i,j} + \left(\frac{1}{\Delta x^2}\right)\phi_{i+1,j} + \left(\frac{1}{\Delta y^2}\right)\phi_{i,j+1} = 0$$

Node 9

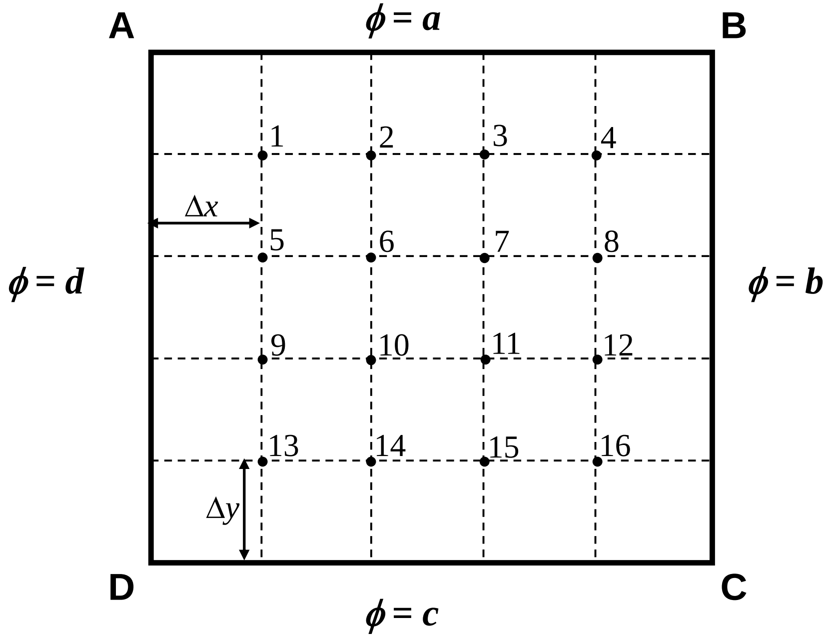


$$\left(\frac{1}{\Delta y^2}\right)\phi_{13} + \left(\frac{1}{\Delta x^2}\right)d + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_9 + \left(\frac{1}{\Delta x^2}\right)\phi_{10} + \left(\frac{1}{\Delta y^2}\right)\phi_5 = 0$$

$$\left(\frac{1}{\Delta y^2}\right)\phi_5 + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_9 + \left(\frac{1}{\Delta x^2}\right)\phi_{10} + \left(\frac{1}{\Delta y^2}\right)\phi_{13} = -\left(\frac{1}{\Delta x^2}\right)d$$

Laplace Equation

Assuming the origin at D



Node No.	x	y	x-neighbour	y-neighbour
1	Δx	4Δy	$\phi = d$, 2	5, $\phi = a$
⋮	⋮	⋮	⋮	⋮
3	3Δx	4Δy	2, 4	7, $\phi = a$
⋮	⋮	⋮	⋮	⋮
6	2Δx	3Δy	5, 7	10, 2
⋮	⋮	⋮	⋮	⋮
9	Δx	2Δy	$\phi = d$, 10	13, 5

$$
\underbrace{\begin{bmatrix}
-2(a+b) & a & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & -2(a+b) & a & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & 0 & 0 & a & -2(a+b) & a & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 & 0 & 0 & -2(a+b) & a & 0 & 0 & b & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \\ f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{16} \end{bmatrix}}_{\boldsymbol{\phi}}
=
\underbrace{\begin{bmatrix} -ba-ad \\ 0 \\ -ba \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -ad \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{b}}
$$

One equation for each node!

Laplace Equation: 1st and 2nd Type BC

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

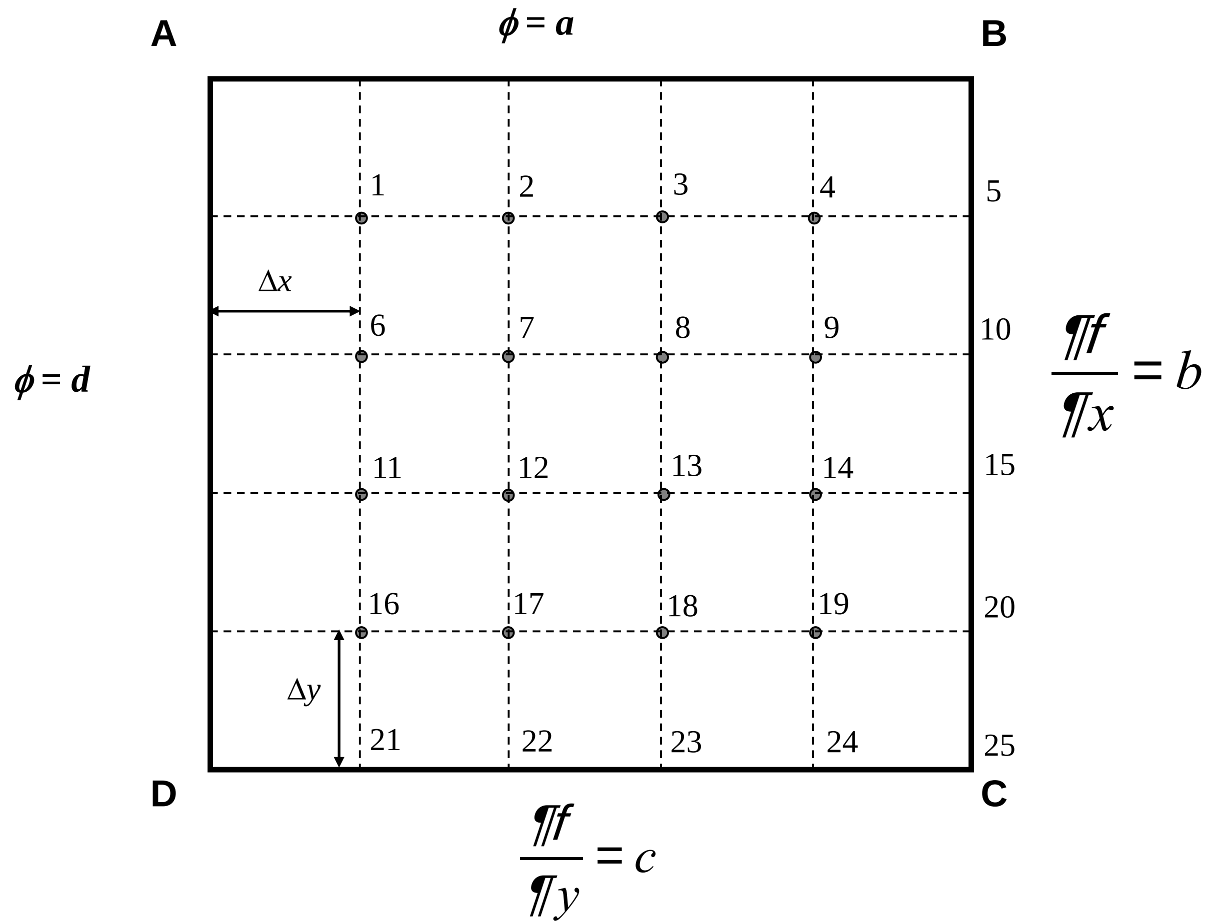
$$x \in (0, L_x) \text{ and } y \in (0, L_y)$$

$$\left.\frac{\partial \phi}{\partial y}\right|_{(x,0)} = c, \ \phi(x, L_y) = a$$

$$\phi(0, y) = d \text{ and } \left.\frac{\partial \phi}{\partial x}\right|_{(L_x, y)} = b$$

$$\Delta x = \frac{L_x}{5} \qquad \Delta y = \frac{L_y}{5}$$

Number of unknowns increased from 16 to 25



Neumann and Robin BC

Three Options for implementation:

- ✓ Backward Difference approximation with increased size of the matrix
  - ✓ asymmetric backward difference approximation
  - ✓ size of the matrix is increased
  - ✓ Solution at the boundary nodes are obtained together
- ✓ Ghost Node
  - ✓ Symmetric central difference approximation
  - ✓ Size of the matrix is increased
  - ✓ Solution at the boundary nodes are obtained together
- ✓ Backward Difference approximation without increasing the size of the matrix
  - ✓ asymmetric backward difference approximation
  - ✓ size of the matrix remains unaltered
  - ✓ Unknowns at the boundary nodes to be computed separately using the approximation of the BC after the solution have been computed for the interior nodes

Backward Difference

Number of equations is now 24 and the size of the matrix $\boldsymbol{A}$ is 24×24

For Node 5, the 5th equation is:

$$\frac{\phi_3 - 4\phi_4 + 3\phi_5}{2\Delta x} = b \quad \text{or} \quad \left(\frac{1}{2\Delta x}\right)\phi_3 + \left(-\frac{2}{\Delta x}\right)\phi_4 + \left(\frac{3}{2\Delta x}\right)\phi_5 = b$$

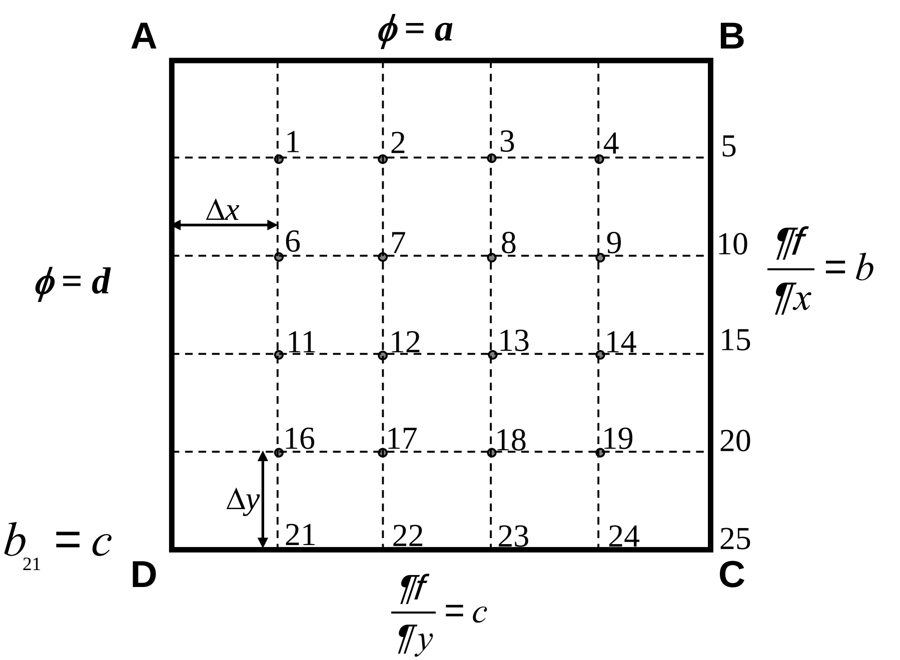
$$a_{53} = \left(\frac{1}{2\Delta x}\right), \quad a_{54} = \left(-\frac{2}{\Delta x}\right), \quad a_{55} = \left(\frac{3}{2\Delta x}\right), \quad \text{and} \quad b_5 = b$$

For Node 21, the 21st equation is:

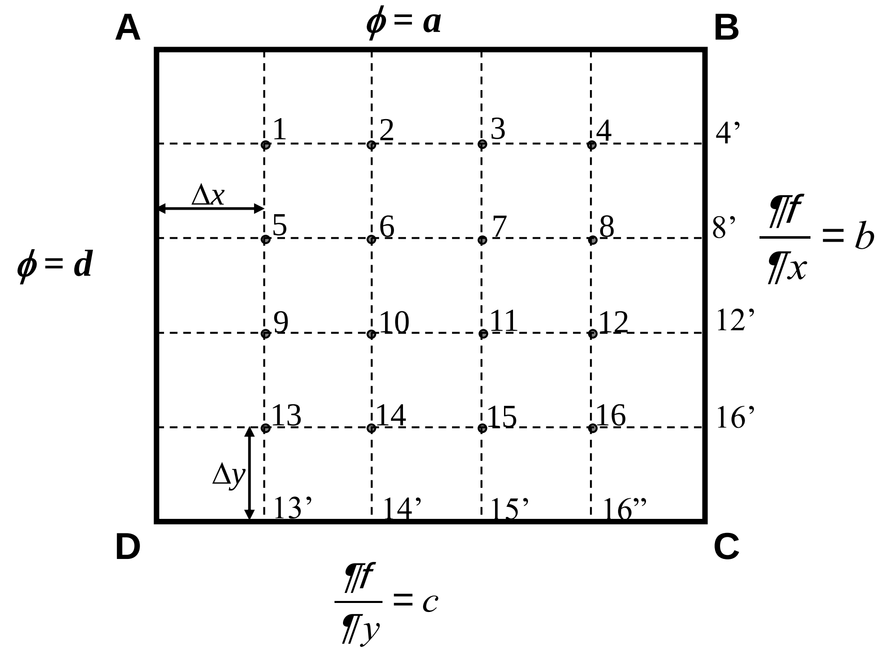
$$\frac{\phi_{11} - 4\phi_{16} + 3\phi_{21}}{2\Delta y} = c$$

$$\left(\frac{1}{2\Delta y}\right)\phi_{11} + \left(-\frac{2}{\Delta y}\right)\phi_{16} + \left(\frac{3}{2\Delta y}\right)\phi_{21} = c$$

$$a_{21\ 11} = \left(\frac{1}{2\Delta y}\right), \quad a_{21\ 16} = \left(-\frac{2}{\Delta y}\right), \quad a_{21\ 21} = \left(\frac{3}{2\Delta x}\right), \quad \text{and} \quad b_{21} = c$$



Backward Difference



Number of equations will remain at 16 and the size of the matrix ***A*** is 16×16

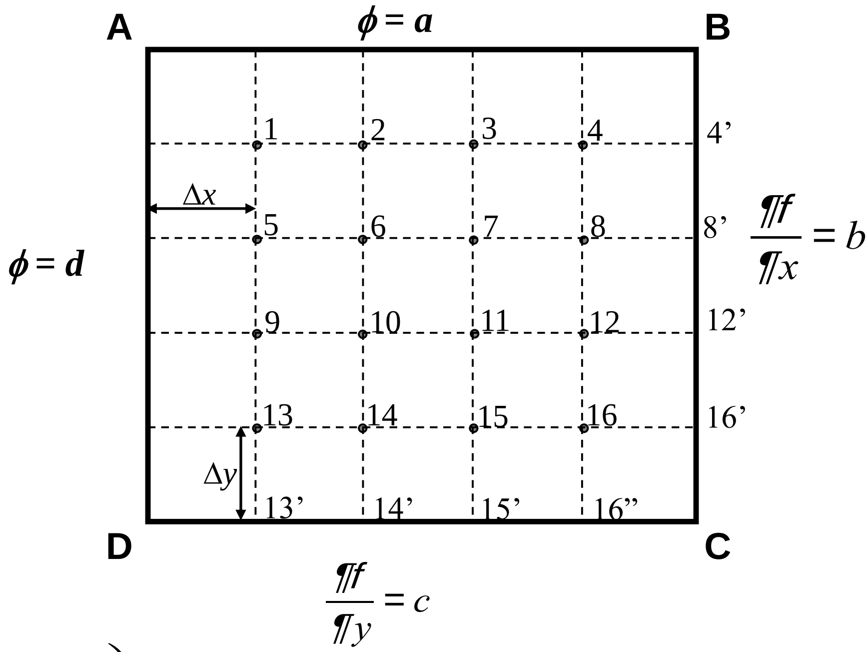
For Node 16, the 16th equation is:

$$\left(\frac{1}{\Delta y^2}\right)\phi_{12} + \left(\frac{1}{\Delta x^2}\right)\phi_{15} + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_{16} + \left(\frac{1}{\Delta x^2}\right)\phi_{16'} + \left(\frac{1}{\Delta y^2}\right)\phi_{16''} = 0$$

$$\frac{\phi_{15} - 4\phi_{16} + 3\phi_{16'}}{2\Delta x} = b \quad \text{or} \quad \left(\frac{1}{2\Delta x}\right)\phi_{15} + \left(-\frac{2}{\Delta x}\right)\phi_{16} + \left(\frac{3}{2\Delta x}\right)\phi_{16'} = b$$

$$\frac{\phi_{12} - 4\phi_{16} + 3\phi_{16''}}{2\Delta y} = c \quad \text{or} \quad \left(\frac{1}{2\Delta y}\right)\phi_{12} + \left(-\frac{2}{\Delta y}\right)\phi_{16} + \left(\frac{3}{2\Delta y}\right)\phi_{16''} = c$$

Backward Difference



Number of equations will remain at 16 and the size of the matrix ***A*** is 16×16

For Node 16, the 16th equation is:

$$\left(\frac{2}{3\Delta y^2}\right)\phi_{12}+\left(\frac{2}{3\Delta x^2}\right)\phi_{15}+\left(-\frac{2}{3\Delta x^2}-\frac{2}{3\Delta y^2}\right)\phi_{16}=-\frac{2b}{3\Delta x}-\frac{2c}{3\Delta y}$$

Recall, for Node 16, the 16th equation for the 1st type BC was:

$$\left(\frac{1}{\Delta y^2}\right)\phi_{12}+\left(\frac{1}{\Delta x^2}\right)\phi_{15}+\left(-\frac{2}{\Delta x^2}-\frac{2}{\Delta y^2}\right)\phi_{16}=-\frac{b}{\Delta x^2}-\frac{c}{\Delta y^2}$$