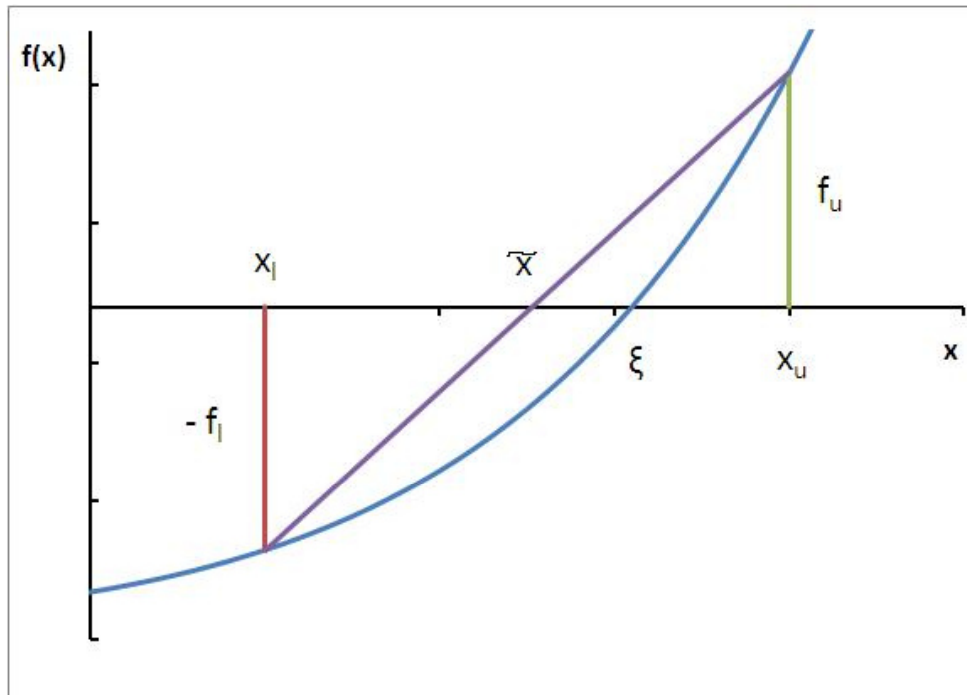


Linear Interpolation



$$\frac{\tilde{x} - x_l}{-f_l} = \frac{x_u - \tilde{x}}{f_u} \Rightarrow \tilde{x} = x_l - f_l \frac{x_u - x_l}{f_u - f_l} = x_u - f_u \frac{x_u - x_l}{f_u - f_l}$$

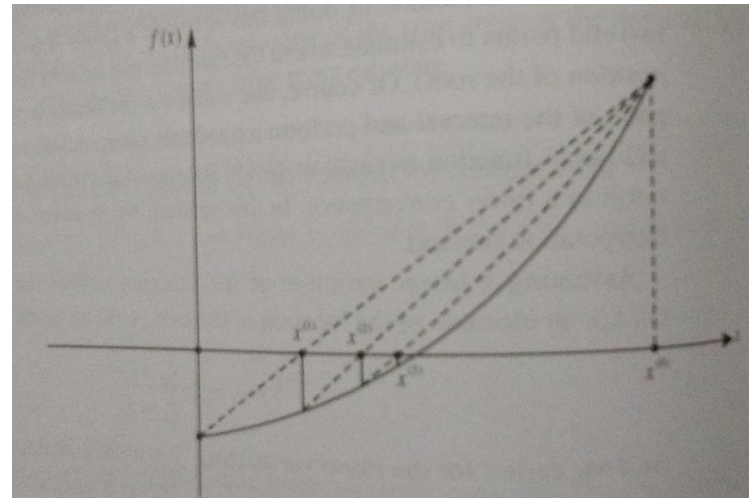
Linear Interpolation: Error Analysis

- Linear interpolation between two points, x_l and x_u ,

$$f(x) = f(x_u) + (x - x_u) \frac{f_u - f_l}{x_u - x_l} + (x - x_u)(x - x_l) \frac{f''(\zeta)}{2}; x, \zeta \in (x_l, x_u)$$

- If we assume the function to be uniformly concave/convex, one end of the interval, x_u (or x_l), remains fixed, say, $x^{(0)}$, and

$$x^{(i+1)} = x^{(0)} - f_0 \frac{x^{(0)} - x^{(i)}}{f_0 - f_i}$$



Linear Interpolation: Error Analysis

From the interpolation equation, applied at $x=\xi$:

$$f(\xi) = 0 = f_0 + (\xi - x^{(0)}) \frac{f_0 - f_i}{x^{(0)} - x^{(i)}} + (\xi - x^{(0)}) (\xi - x^{(i)}) \frac{f''(\zeta_1)}{2}; \quad \zeta_1 \in (x^{(i)}, x^{(0)})$$
$$\Rightarrow f_0 \frac{x^{(0)} - x^{(i)}}{f_0 - f_i} = -e^{(0)} - e^{(0)} e^{(i)} \frac{f''(\zeta_1)}{2f'(\zeta_2)}; \quad \zeta_1, \zeta_2 \in (x^{(i)}, x^{(0)})$$

And, from the iteration equation

$$\xi - x^{(i+1)} = \xi - x^{(0)} + f_0 \frac{x^{(0)} - x^{(i)}}{f_0 - f_i} \Rightarrow e^{(i+1)} = -e^{(0)} e^{(i)} \frac{f''(\zeta_1)}{2f'(\zeta_2)}$$

In the limit, the iterations converge to the root, and

$$\lim_{i \rightarrow \infty} \left| \frac{e^{(i+1)}}{e^{(i)}} \right| = \left| e^{(0)} \frac{f''(\zeta_3)}{2f'(\zeta_4)} \right|; \quad \zeta_3, \zeta_4 \in (\xi, x^{(0)})$$

Linear Interpolation: Error Analysis

- Linearly convergent
- **Asymptotic error constant** is not constant (was $\frac{1}{2}$ for bisection), but depends on the nature of the function

$$C = \left| e^{(0)} \frac{f''(\zeta_3)}{2f'(\zeta_4)} \right|$$

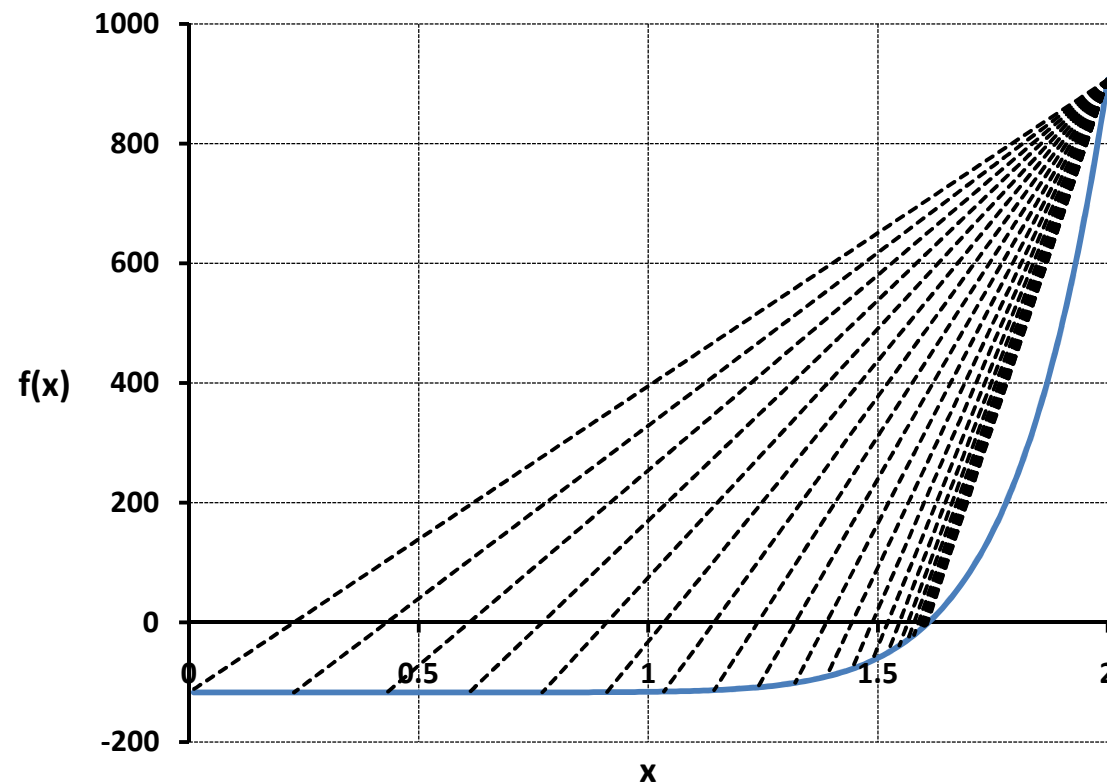
For the function used in the example,

$$f(x) = x^3 - 1.2500000x^2 - 1.5625250x + 1.9530938$$

The iterations show one end fixed at -2 , the root is -1.25 ; $e^{(0)}=0.75$.
 $|f''/2f'|$ varies from 0.5 to 0.8 over $(-2, -1.25)$, indicating C between 0.35 and 0.6.

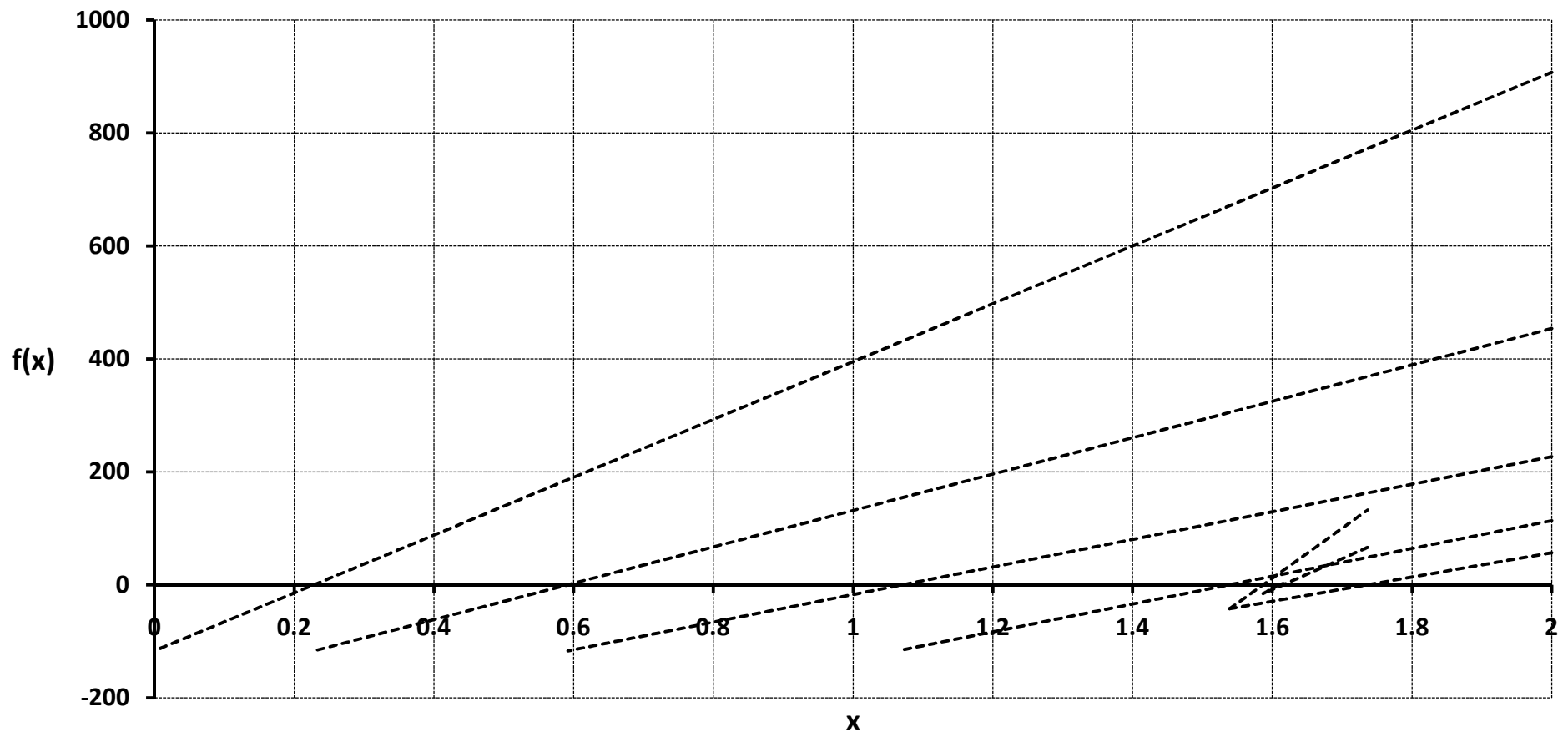
Linear Interpolation: Modification

- Called Modified False Position or Modified Regula Falsi Method
- To avoid slow convergence (E.g. : $x^{10} - 117.02 = 0$)



Modified Regula Falsi

- If one end is fixed for a few iterations (generally 2-3), reduce the function value to some fraction (usually half).



Roots of Nonlinear Equations: Open methods

- Because of linear convergence, bracketing methods are generally used only as starting methods
- Once we are close to the root, we may switch to “open methods” which generally converge faster
- However, since these do not bracket the root, convergence is not guaranteed

Different open methods are:

Fixed-point, Newton-Raphson, Secant, Muller ...

Fixed-point Method

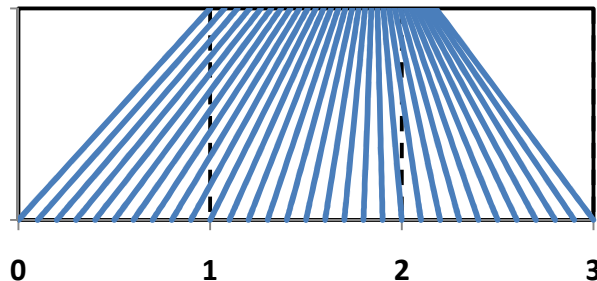
$f(x) = 0$ can always be written as $x = \phi(x)$

E.g., $f(x) = x^2 - 2 = 0$ could be written as $x = \frac{2}{x}$ or $x = \frac{x}{2} + \frac{1}{x}$

If nothing else, $x = x + f(x)$

$\phi(x)$ may be thought of as a mapping which maps each point on the x-axis to a different point, except for some points which are mapped on themselves [i.e., $\phi(x)=x$]. These are called the fixed-points of the function $\phi(x)$.

- We start with an initial guess of the root as $x^{(0)}$, and then obtain the next guess as $x^{(1)} = \phi(x^{(0)})$. Continue till $x^{(i+1)} \approx x^{(i)}$



Fixed-point Method: Example

- Flow depth in an open channel (T.V.=1.86)

$$f(x) = x^{2.5} - 2x - 1 = 0$$

- We could write, $\phi_1(x) = (1 + 2x)^{0.4}$ or $\phi_2(x) = (x^{2.5} - 1)/2$
- Iterations shown below:

x	Phi ₁ (x)
2.000000	1.903654
1.903654	1.873962
1.873962	1.864668
1.864668	1.861744
1.861744	1.860823
1.860823	1.860533
1.860533	1.860442

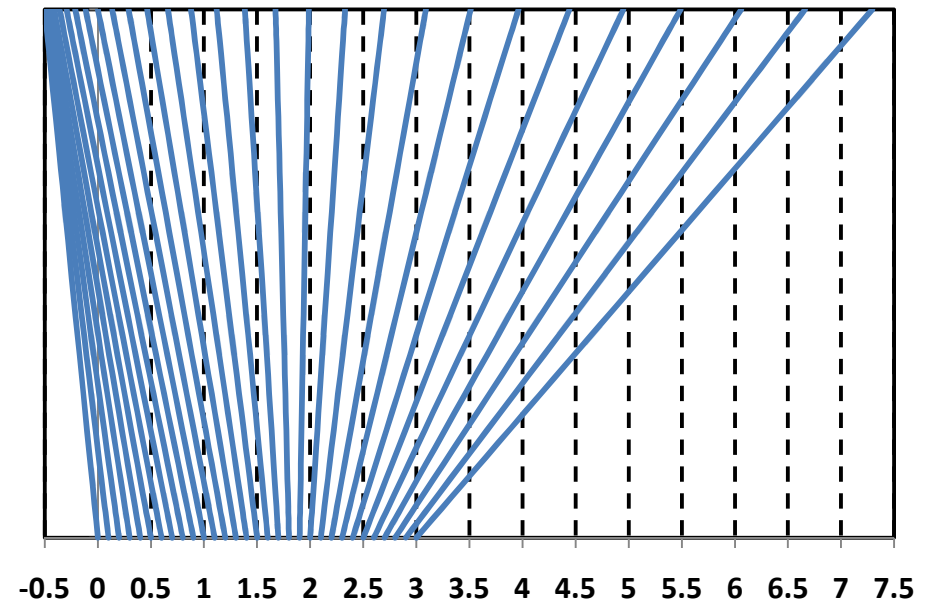
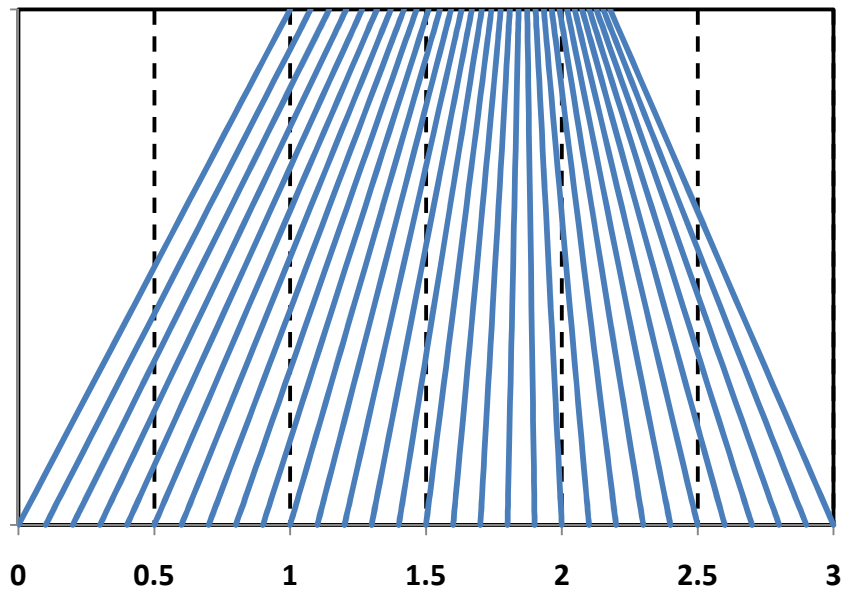
x	Phi ₂ (x)
2.000	2.328
2.328	3.636
3.636	12.108
12.108	254.587
254.587	517084.882

Fixed-point Method: Example

- The mappings are shown below:

$$\phi_1(x) = (1+2x)^{0.4}$$

$$\phi_2(x) = (x^{2.5}-1)/2$$



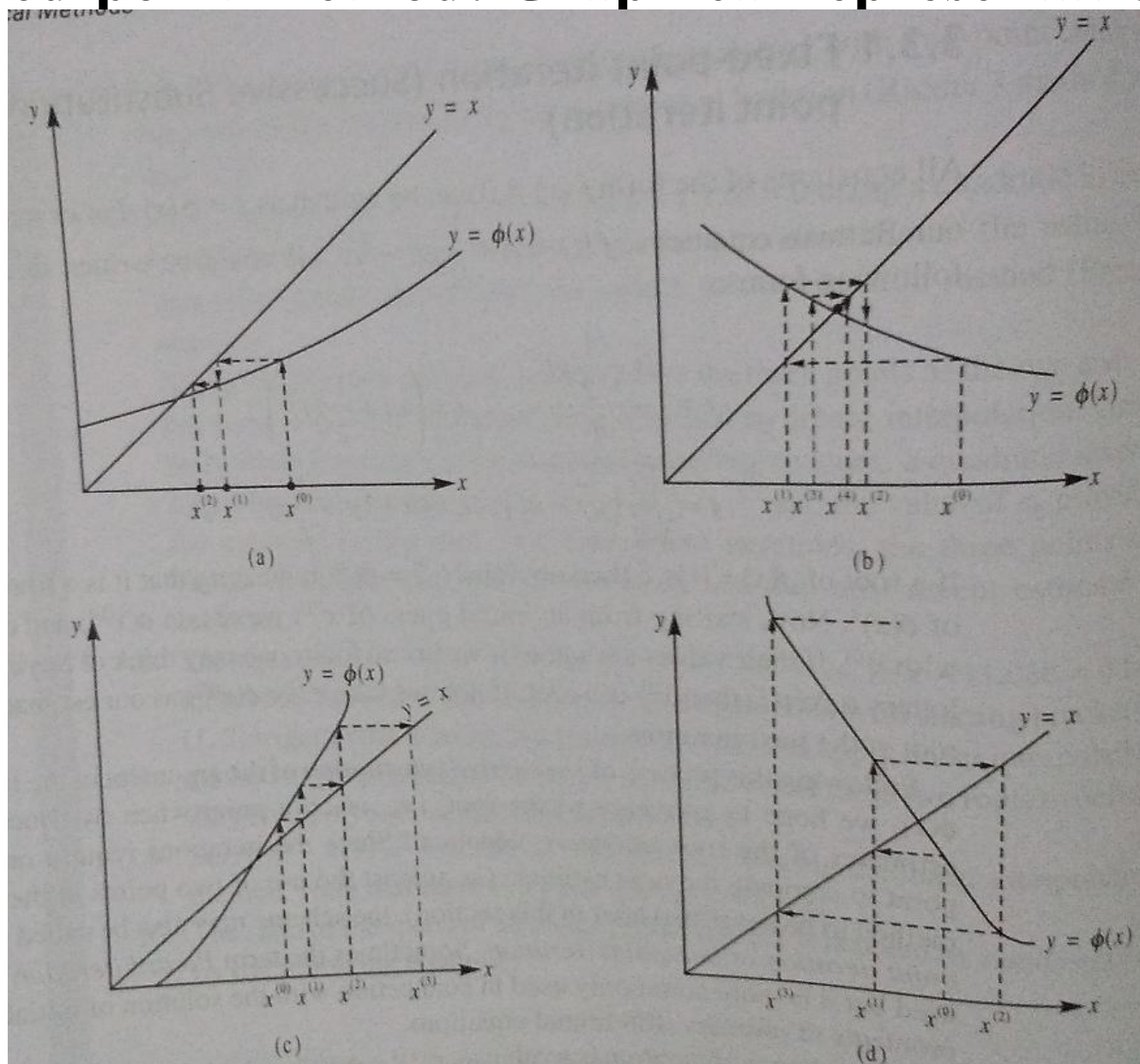
Fixed-point Method: Error Analysis

- True Error at iteration (i+1)

$$e^{(i+1)} = \xi - x^{(i+1)} = \phi(\xi) - \phi(x^{(i)})$$

- Using Mean Value Theorem $e^{(i+1)} = (\xi - x^{(i)})\phi'(\zeta)$ where $\zeta \in (x^{(i)}, \xi)$
- Linearly convergent with the Error Constant $C = \phi'(\zeta)$
- For convergence, C should be less than 1

Fixed-point Method: Graphical representation

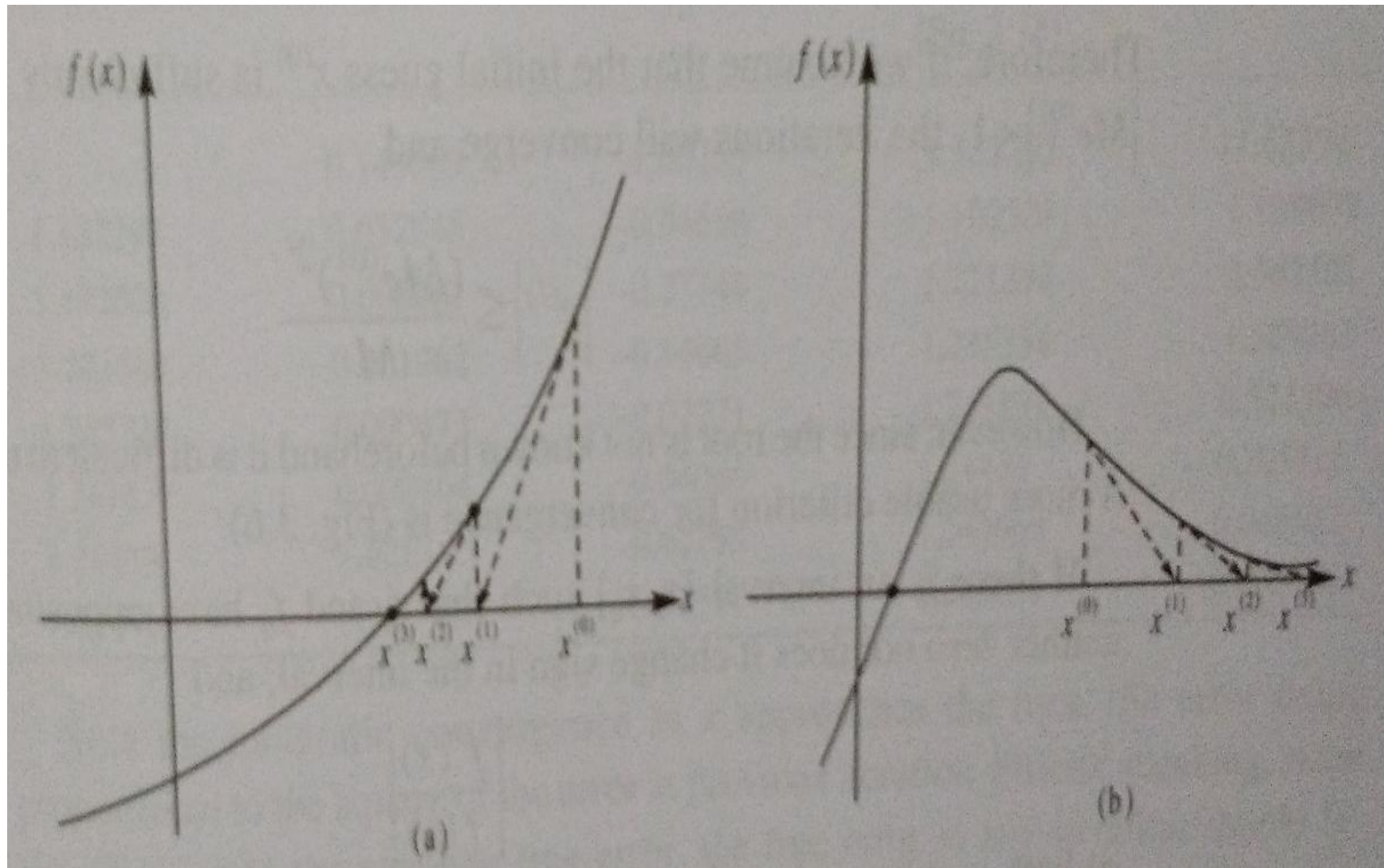


Roots of Nonlinear Equations: Newton Method

- Fixed point method has linear convergence, hence a better method is desirable
- Newton (or Newton-Raphson) method is one such method, which has a second-order convergence
- In linear interpolation, function is approximated by a linear function, with the slope estimated from the difference of the function values at the two points.
- In Newton method also, the function is approximated by a linear function, but the slope is obtained analytically

$$f(x) \approx f(x^{(i)}) + (x - x^{(i)})f'(x^{(i)})$$

Newton Method: Graphical Representation



Newton Method: Algorithm and Error Analysis

- The straight-line approximation is

$$f(x) \approx f(x^{(i)}) + (x - x^{(i)})f'(x^{(i)})$$

- It will intersect the x-axis at $x^{(i+1)}$, hence

$$0 = f(x^{(i)}) + (x^{(i+1)} - x^{(i)})f'(x^{(i)}) \Rightarrow x^{(i+1)} = x^{(i)} - \frac{f(x^{(i)})}{f'(x^{(i)})}$$

- To write the error, use Taylor's series

$$f(x) = f(x^{(i)}) + (x - x^{(i)})f'(x^{(i)}) + \frac{(x - x^{(i)})^2}{2} f''(\zeta) \quad \zeta \in (x, x^{(i)})$$

- As x approaches the root,

$$f(\xi) = 0 = f(x^{(i)}) + (\xi - x^{(i)})f'(x^{(i)}) + \frac{(\xi - x^{(i)})^2}{2} f''(\zeta)$$

Newton Method: Error Analysis

$$x^{(i+1)} = x^{(i)} - \frac{f(x^{(i)})}{f'(x^{(i)})}$$

$$f(\xi) = 0 = f(x^{(i)}) + (\xi - x^{(i)})f'(x^{(i)}) + \frac{(\xi - x^{(i)})^2}{2} f''(\zeta)$$

From these equations,

$$e^{(i+1)} = \xi - x^{(i+1)} = \xi - x^{(i)} + \frac{f(x^{(i)})}{f'(x^{(i)})} = -\frac{f''(\zeta)}{2f'(x^{(i)})} (\xi - x^{(i)})^2$$

$$\lim_{i \rightarrow \infty} \left| \frac{e^{(i+1)}}{[e^{(i)}]^2} \right| = \left| \frac{f''(\xi)}{2f'(\xi)} \right|$$

Second-order convergence.