First Order ODE's: Solution Algorithm

- Single-step methods:

➢ Euler Forward or Explicit method:

$$y_{n+1} = y_n + hf\left(t_n, y_n\right)$$

➢ Euler Backward or Implicit method:

$$y_{n+1} = y_n + hf\left(t_{n+1}, y_{n+1}\right)$$

- Multi-step methods:

➢ Implicit Heun's : $y_{n+1} = y_n + h\dfrac{f\left(t_n, y_n\right) + f\left(t_{n+1}, y_{n+1}\right)}{2}$

➢ Explicit Heun's (or just Heun's method):

$$y_{n+1} = y_n + h\frac{f\left(t_n, y_n\right) + f\left(t_{n+1}, y_n + hf\left(t_n, y_n\right)\right)}{2}$$

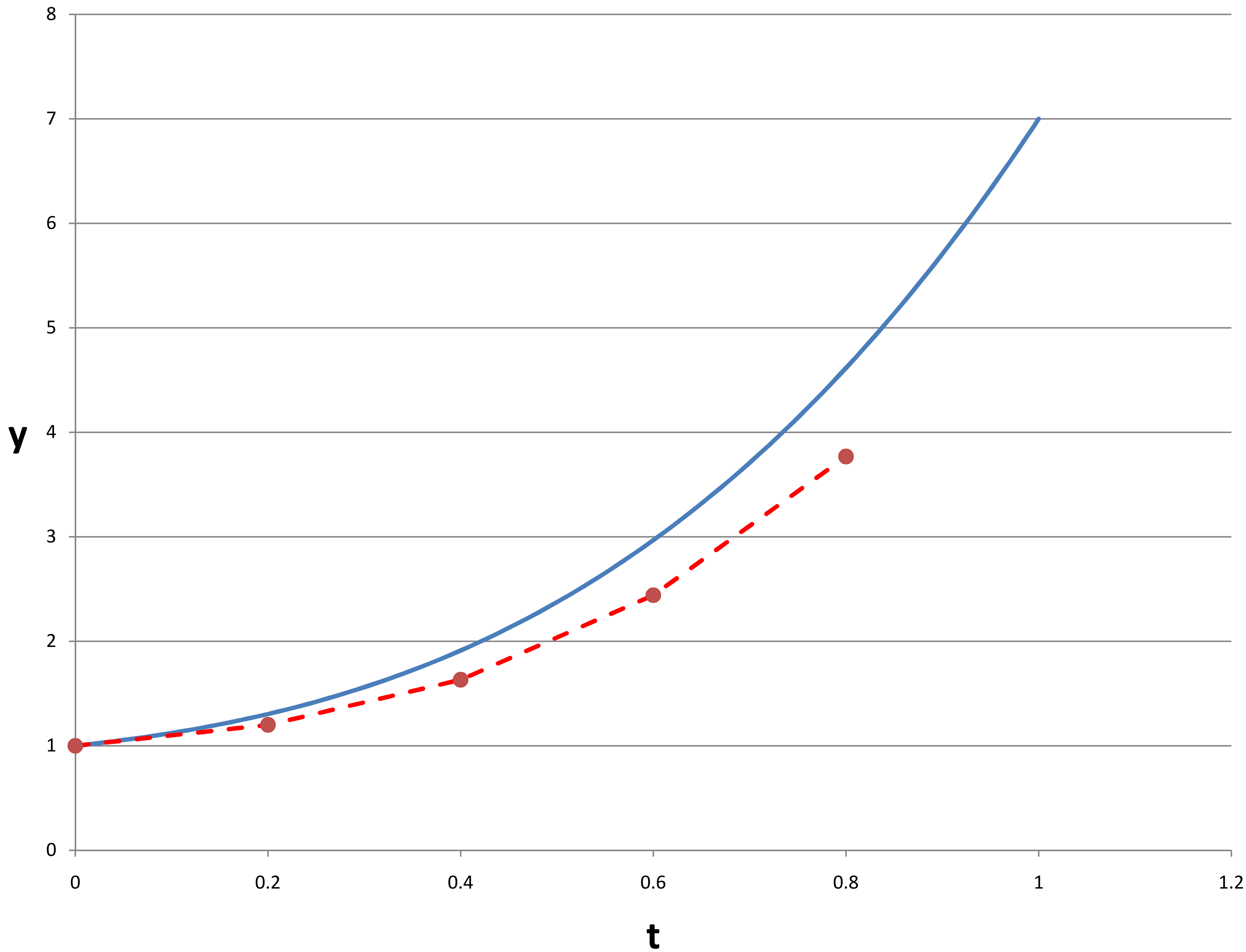
Derivation of multi-step methods

- Given: $$\frac{dy}{dt} = f(t,y) \qquad y\ _{\text{at } t=t_0} = y_0$$
  - subscript *n* is for "known" point and *n*+1 for the "desired" point: given t_n, y_n, t_{n+1}, find y_{n+1}
  - All previous points, 0,1,2…,*n*−1 are "known"
- Linear: We write the desired value, y_{n+1}, in terms of a linear combination of y_n and the "slopes"
$$y_{n+1} = y_n + h\left(\beta f_{n+1} + \sum_{i=0}^{k} \alpha_i f_{n-i}\right)$$
- Explicit if $\beta=0$, implicit otherwise. $k = 0,1,2,\ldots,n$

Derivation of multi-step methods

$$y_{n+1} = y_n + h\left(\beta f_{n+1} + \sum_{i=0}^{k} \alpha_i f_{n-i}\right)$$

- The term in the (..) may be thought of as an "average slope" over the interval (t_n, t_{n+1})
- For explicit methods, the average slope is obtained from a weighted average of a few (=k+1) "previous (i.e., known)" slopes
- For implicit methods, the average slope is obtained from a weighted average of a few "previous" slopes and the "unknown" slope


y
t
0
1
2
3
4
5
6
7
8
0
0.2
0.4
0.6
0.8
1
1.2

Explicit multi-step methods: Adams Bashforth

- Consider k=2:

$$y_{n+1} = y_n + h\left(\alpha_0 f_n + \alpha_1 f_{n-1} + \alpha_2 f_{n-2}\right)$$

- Use Taylor's series ($t_{n-1}=t_n - h$; $t_{n-2}=t_n - 2h$)

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2!} f_n' + \frac{h^3}{3!} f_n'' + \frac{h^4}{4!} f_n''' + \ldots$$

$$f_{n-1} = f_n - hf_n' + \frac{h^2}{2!} f_n'' - \frac{h^3}{3!} f_n''' + \ldots$$

$$f_{n-2} = f_n - 2hf_n' + \frac{4h^2}{2!} f_n'' - \frac{8h^3}{3!} f_n''' + \ldots$$

Explicit multi-step methods

- Combine:

$$y_n + hf_n + \frac{h^2}{2!} f_n' + \frac{h^3}{3!} f_n'' + \frac{h^4}{4!} f_n''' + \ldots = y_n + h\alpha_0 f_n$$

$$+ h\alpha_1 \left(f_n - hf_n' + \frac{h^2}{2!} f_n'' - \frac{h^3}{3!} f_n''' + \ldots \right)$$

$$+ h\alpha_2 \left(f_n - 2hf_n' + \frac{4h^2}{2!} f_n'' - \frac{8h^3}{3!} f_n''' + \ldots \right)$$

- Match the coefficients:

$$\alpha_0 + \alpha_1 + \alpha_2 = 1; -\alpha_1 - 2\alpha_2 = \frac{1}{2}; \frac{\alpha_1}{2} + 2\alpha_2 = \frac{1}{6}$$

- And get:

$$\alpha_0 = \frac{23}{12}; \alpha_1 = -\frac{4}{3}; \alpha_2 = \frac{5}{12}$$

Explicit multi-step methods

- Therefore, for k=2:

$$y_{n+1} = y_n + h\left(\frac{23}{12} f_n - \frac{4}{3} f_{n-1} + \frac{5}{12} f_{n-2}\right)$$

- Non-self starting, since at the start we do not have the values of $f_{n\text{-}1}$ and $f_{n\text{-}2}$
- May use single-step method for first two-steps and then switch to the above formula
- The lowest order error term for this method is

$$\frac{h^4}{4!} f_n''' + h\alpha_1\left(\frac{h^3}{3!} f_n'''\right) + h\alpha_2\left(\frac{8h^3}{3!} f_n'''\right) = \frac{3h^4}{8} f'''\left(= \frac{3h^4}{8} y''''\right)$$

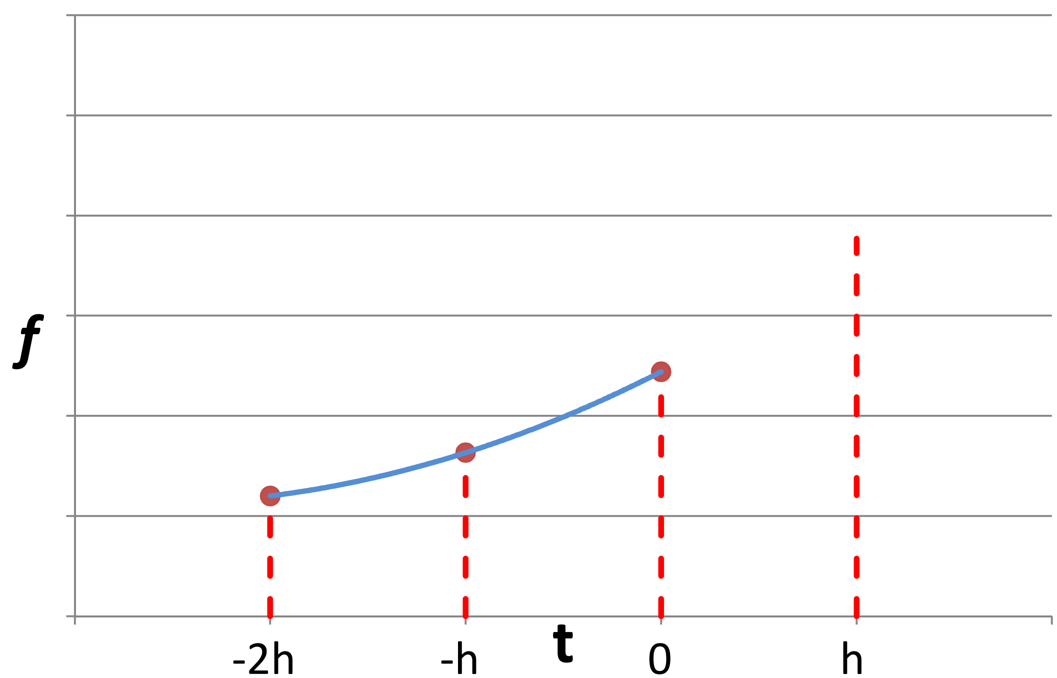
Adams Bashforth: Alternative formulation

- For k=2:

$$y_{n+1} = y_n + h\left(\alpha_0 f_n + \alpha_1 f_{n-1} + \alpha_2 f_{n-2}\right)$$

- Approximate f by a quadratic function:

$$f = \frac{(t+2h)(t+h)}{(2h)(h)} f_n + \frac{(t+2h)t}{(-h+2h)(-h)} f_{n-1} + \frac{(t+h)t}{(-2h+h)(-2h)} f_{n-2}$$



Adams Bashforth: Alternative formulation

- Write

$$y_{n+1} = y_n + \int_0^h f\,dt$$

- Integrate the quadratic f:

$$\int_0^h \frac{(t+2h)(t+h)}{(2h)(h)}dt = \frac{23h}{12}$$

$$\int_0^h \frac{(t+2h)t}{(-h+2h)(-h)}dt = -\frac{4h}{3}$$

$$\int_0^h \frac{(t+h)t}{(-2h+h)(-2h)}dt = \frac{5h}{12}$$

- Same formula:

$$y_{n+1} = y_n + h\left(\frac{23}{12}f_n - \frac{4}{3}f_{n-1} + \frac{5}{12}f_{n-2}\right)$$

Implicit multi-step methods: Adams Moulton

- Consider k=1:

$$y_{n+1} = y_n + \beta h f_{n+1} + h\left(\alpha_0 f_n + \alpha_1 f_{n-1}\right)$$

- Use Taylor's series:

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2!} f_n' + \frac{h^3}{3!} f_n'' + \frac{h^4}{4!} f_n''' + \ldots$$

$$f_{n-1} = f_n - hf_n' + \frac{h^2}{2!} f_n'' - \frac{h^3}{3!} f_n''' + \ldots$$

$$f_{n+1} = f_n + hf_n' + \frac{h^2}{2!} f_n'' + \frac{h^3}{3!} f_n''' + \ldots$$

Implicit multi-step methods

- Combine:

$$y_n + hf_n + \frac{h^2}{2!}f_n' + \frac{h^3}{3!}f_n'' + \frac{h^4}{4!}f_n''' + \ldots = y_n + h\alpha_0 f_n$$
$$+h\alpha_1\left(f_n - hf_n' + \frac{h^2}{2!}f_n'' - \frac{h^3}{3!}f_n''' + \ldots\right)$$
$$+h\beta\left(f_n + hf_n' + \frac{h^2}{2!}f_n'' + \frac{h^3}{3!}f_n''' + \ldots\right)$$

- Match the coefficients:

$$\beta + \alpha_0 + \alpha_1 = 1; \beta - \alpha_1 = \frac{1}{2}; \frac{\beta}{2} + \frac{\alpha_1}{2} = \frac{1}{6}$$

- And get:

$$\beta = \frac{5}{12}; \alpha_0 = \frac{2}{3}; \alpha_1 = -\frac{1}{12}$$

Implicit multi-step methods

- Therefore, for k=1:

$$y_{n+1} = y_n + h\left(\frac{5}{12} f_{n+1} + \frac{2}{3} f_n - \frac{1}{12} f_{n-1}\right)$$

- Non-self starting, since at the start we do not have the value of $f_{n\text{-}1}$
- Implicit since f_{n+1} on the RHS
- The lowest order error term for this method is

$$\frac{h^4}{4!} f_n''' + h\alpha_1\left(\frac{h^3}{3!} f_n'''\right) - h\beta\left(\frac{h^3}{3!} f_n'''\right) = -\frac{h^4}{24} f'''\left(= -\frac{h^4}{24} y''''\right)$$

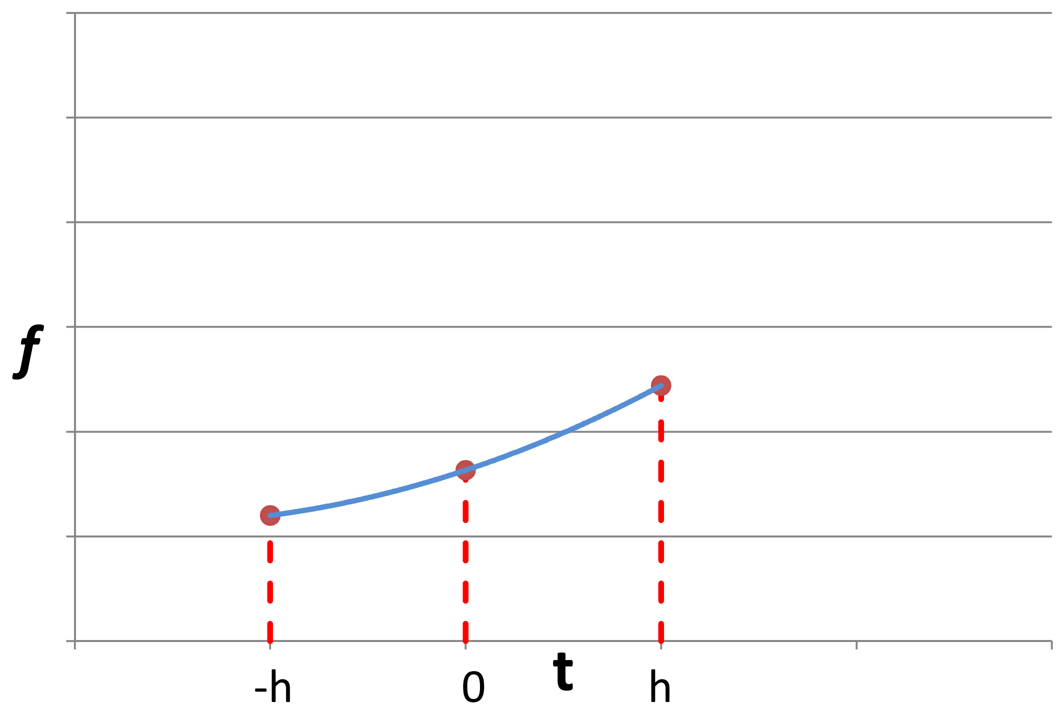
Adams Moulton: Alternative formulation

- For k=1:

$$y_{n+1} = y_n + \beta h f_{n+1} + h\left(\alpha_0 f_n + \alpha_1 f_{n-1}\right)$$

- Approximate f by a quadratic function:

$$f = \frac{(t+h)t}{(2h)(h)} f_{n+1} + \frac{(t+h)(t-h)}{(h)(-h)} f_n + \frac{t(t-h)}{(-h)(-2h)} f_{n-1}$$



Adams Moulton: Alternative formulation

- Write

$$y_{n+1} = y_n + \int_0^h f\,dt$$

$$\int_0^h \frac{(t+h)t}{(2h)(h)} dt = \frac{5h}{12}$$

- Integrate the quadratic f:

$$\int_0^h \frac{(t+h)(t-h)}{(h)(-h)} dt = \frac{2h}{3} \qquad \int_0^h \frac{t(t-h)}{(-h)(-2h)} dt = -\frac{h}{12}$$

- Same formula:

$$y_{n+1} = y_n + h\left(\frac{5}{12} f_{n+1} + \frac{2}{3} f_n - \frac{1}{12} f_{n-1}\right)$$

Another option: Backward Difference methods

- We write the "unknown" slope, $f(t_{n+1},y_{n+1})$, in terms of a linear combination of y_{n+1} and the "known" y^s $(y_n, y_{n-1}\dots)$

- Always implicit:

$$hf_{n+1} = \sum_{i=0}^{k} \alpha_i y_{n+1-i}$$

  - $k=0,1,2,\dots,n+1$

- Derivation is similar to the multi-step method
- E.g., for k=2:

$$hf_{n+1} = \alpha_0 y_{n+1} + \alpha_1 y_n + \alpha_2 y_{n-1}$$

Backward Difference methods

- Use Taylor's series: $hf_{n+1} = \alpha_0 y_{n+1} + \alpha_1 y_n + \alpha_2 y_{n-1}$

$$f_{n+1} = f_n + hf_n' + \frac{h^2}{2!} f_n'' + \frac{h^3}{3!} f_n''' + \ldots$$

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2!} f_n' + \frac{h^3}{3!} f_n'' + \frac{h^4}{4!} f_n''' + \ldots$$

$$y_{n-1} = y_n - hf_n + \frac{h^2}{2!} f_n' - \frac{h^3}{3!} f_n'' + \frac{h^4}{4!} f_n''' - \ldots$$

- Match the coefficients:

$$\alpha_0 + \alpha_1 + \alpha_2 = 0; \alpha_0 - \alpha_2 = 1; \frac{\alpha_0}{2} + \frac{\alpha_2}{2} = 1$$

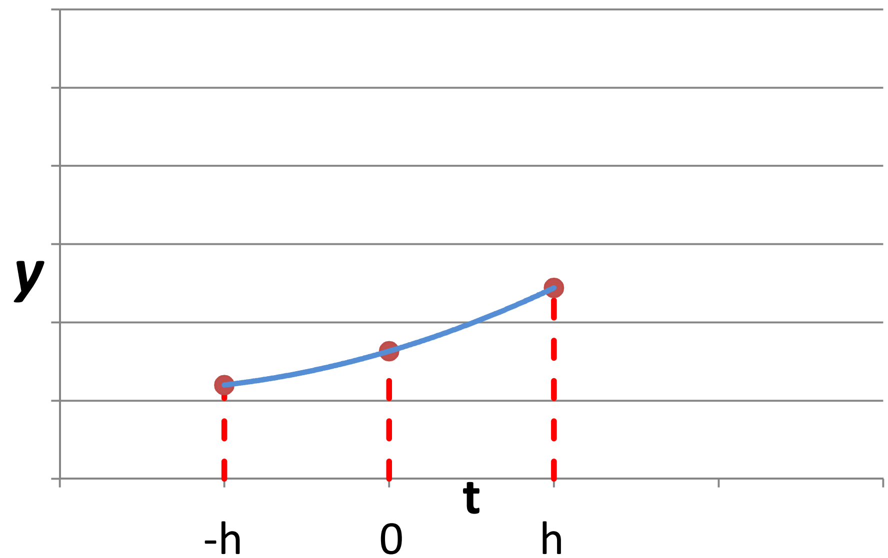
- And get:

$$\alpha_0 = \frac{3}{2}; \alpha_1 = -2; \alpha_2 = \frac{1}{2}$$

Backward Difference methods: **Alternative View**

- Approximate y by a quadratic:

$$y = \frac{(t+h)t}{(2h)(h)} y_{n+1} + \frac{(t+h)(t-h)}{(h)(-h)} y_n + \frac{t(t-h)}{(-h)(-2h)} y_{n-1}$$



- Estimate the derivative at n+1 (i.e., h):
  - e.g.: $$\frac{d}{dt}\frac{(t+h)t}{(2h)(h)}\bigg|_{t=h} = \frac{3}{2h}$$
- And get:

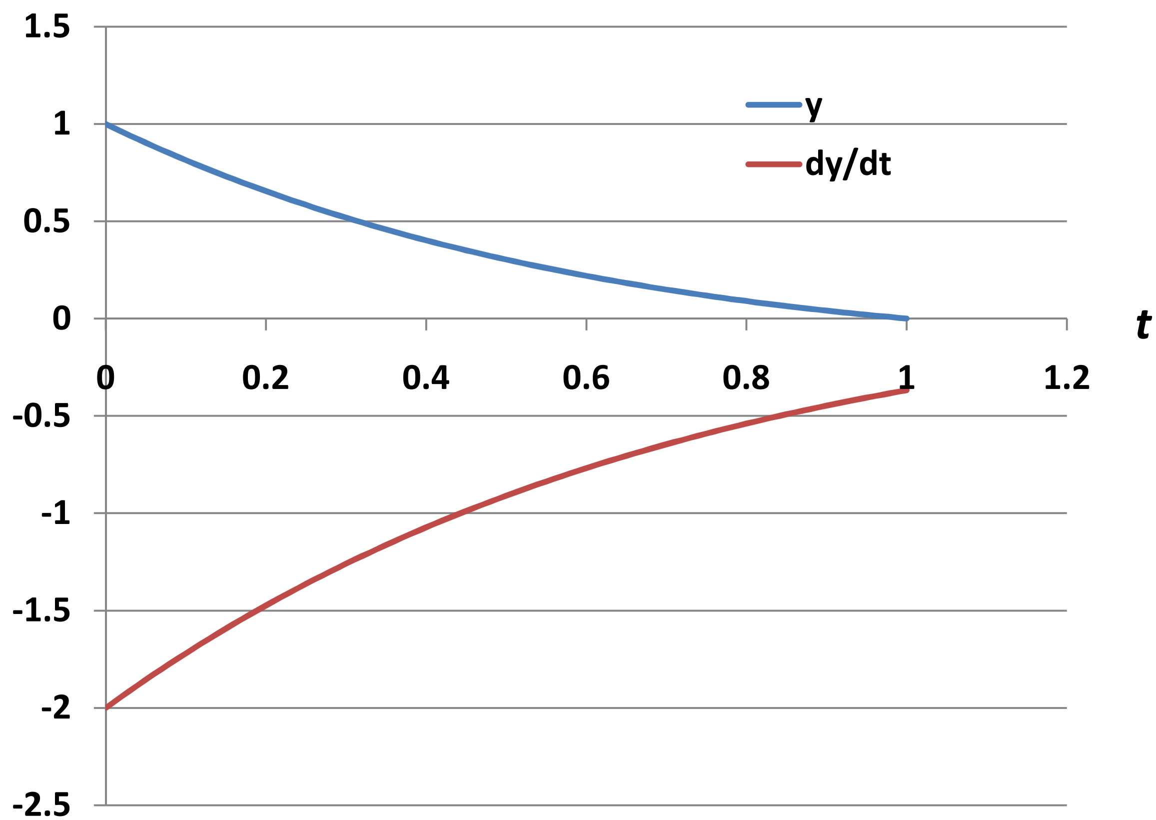
$$\alpha_0 = \frac{3}{2}; \alpha_1 = -2; \alpha_2 = \frac{1}{2} \quad \Rightarrow \quad hf_{n+1} = \frac{3}{2} y_{n+1} - 2y_n + \frac{1}{2} y_{n-1}$$

First Order ODE's: Example

- Given: $dy/dt = -y - e^{-t}$; $y(0)=1$

➢ Find: y at t=0.1, 0.2, 0.3, 0.4, 0.5 (using h=0.1)

➢ Exact Solution: $y = e^{-t}(1-t)$



Example

- For t=0.1 (TV = 0.814354):

➢ Euler Forward : $y_{n+1} = y_n + hf(t_n, y_n)$

$$y_{0.1} = 1 + 0.1(-2) = 0.8$$

➢ Euler Backward : $y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$

$$y_{0.1} = 1 + 0.1\left(-y_{0.1} - e^{-0.1}\right) => y_{0.1} = 0.826833$$

➢ Trapezoidal or Implicit Heun's :

$$y_{n+1} = y_n + h\frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}$$

$$y_{0.1} = 1 + 0.1\frac{-2 + \left(-y_{0.1} - e^{-0.1}\right)}{2} => y_{0.1} = 0.814055$$