Boundary Value problems: Shooting Method

- Convert into two first-order ODEs

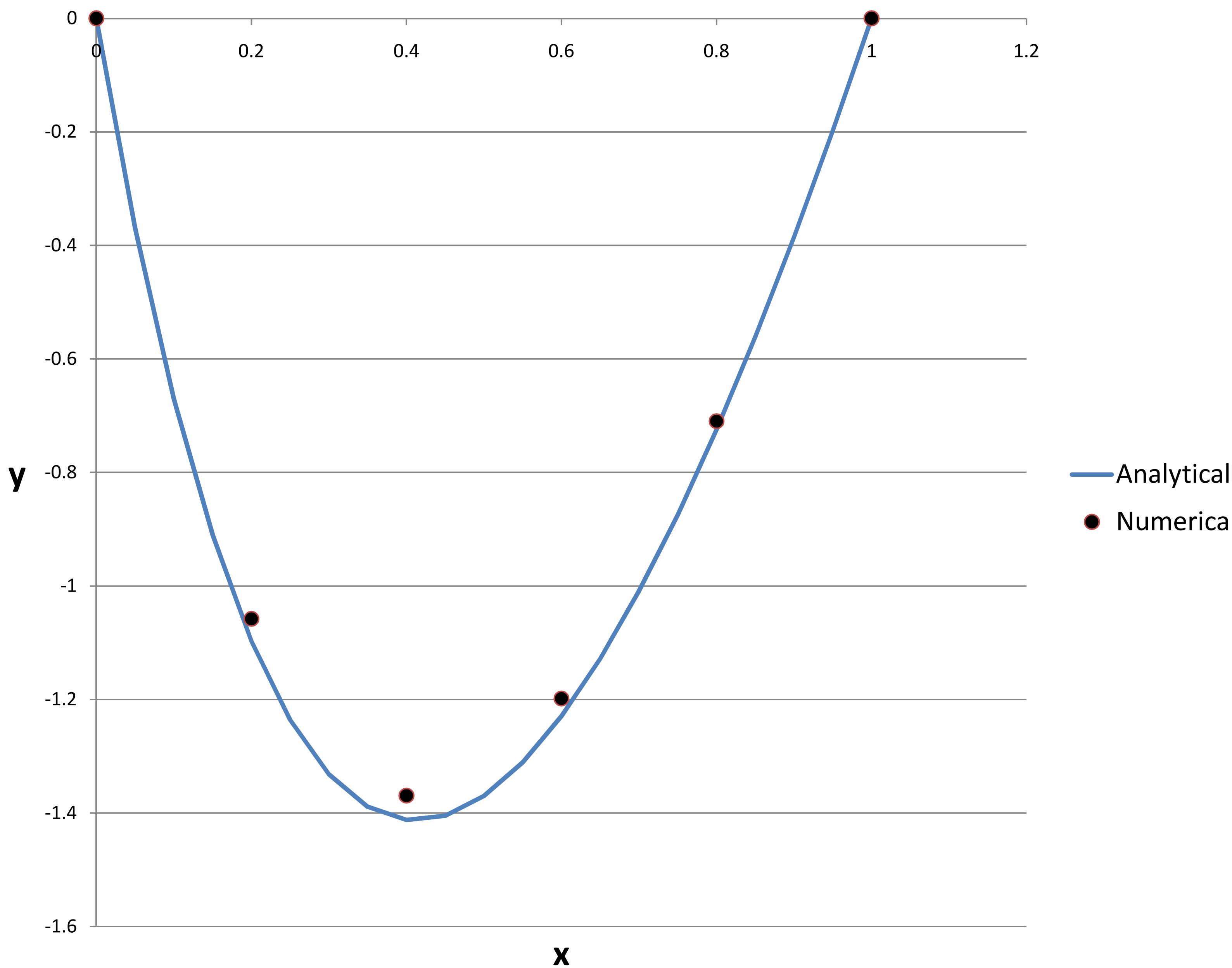
$$p(x)\frac{d^2y}{dx^2}+q(x)\frac{dy}{dx}+r_1(x)y=r_0(x)$$

- y_1=>y; y_2=>dy/dx

$$\frac{dy_1}{dx}=f_1(x,y_1,y_2)=y_2$$

$$\frac{dy_2}{dx}=f_2(x,y_1,y_2)=\frac{r_0(x)-r_1(x)y_1-q(x)y_2}{p(x)}$$

- Boundary conditions: $y_1(a)=y_a; y_1(b)=y_b$
- For IVP, we need y_2(a), which is not given
- Assume y_2(a), solve IVP, compare y_1(b)



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Analytical
Numerical

Boundary Value problems: Direct Method

- Approximate the derivatives by finite differences using a grid of points (generally equally spaced)
- Take linear equation:

$$p(x)\frac{d^2y}{dx^2}+q(x)\frac{dy}{dx}+r_1(x)y=r_0(x)$$

- with the boundary conditions

$$y(a)=y_a; y(b)=y_b$$

- Let (a,b) be divided into n equal intervals [h=(b-a)/n]

Direct Method

- At each node, we get an equation relating the y values at nodes i-1, i, and i+1 (or more, if higher order finite difference formula is used)

$$a_{i,i-1}y_{i-1} + a_{i,i}y_i + a_{i,i+1}y_{i+1} = b_i$$

- where:

$$a_{i,i-1} = \frac{p(x_i)}{h^2} - \frac{q(x_i)}{2h}; a_{i,i} = -2\frac{p(x_i)}{h^2} - r_1(x_i);$$

$$a_{i,i+1} = \frac{p(x_i)}{h^2} + \frac{q(x_i)}{2h}; b_i = r_0(x_i)$$

Direct Method: Boundary Conditions

- Virtual, Imaginary, or Ghost Node:
  - Add a fictitious node (n+1)
  - The equation at node n can now be written
  - Write central difference approximation as

$$\frac{y_{n+1} - y_{n-1}}{2h} = y'_b \Rightarrow y_{n+1} = y_{n-1} + 2hy'_b$$

- The equation at node n becomes

$$\left(a_{n,n-1} + a_{n,n+1}\right)y_{n-1} + a_{n,n}y_n = b_1 - a_{n,n+1}2hy'_b$$

Direct Method: Example

- Second-order equation:

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = 6e^x$$

- Boundary conditions: y(0)=y(1)=0
- Use h=0.2 (n=5), 6 nodes, 4 unknowns (y_0 and y_5 are given to be 0)
- Nodal equation:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{0.2^2} + 3\frac{y_{i+1} - y_{i-1}}{0.4} + 2y_i = 6e^{x_i}$$

$$17.5y_{i-1} - 48y_i + 32.5y_{i+1} = 6e^{x_i}$$

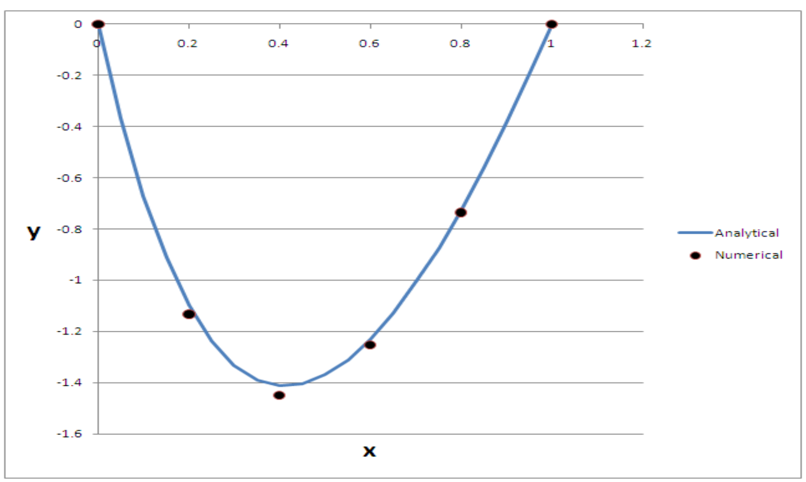
Direct Method: Example

- Equations at the 0^{th} and 5^{th} nodes are not needed
- At the 1^{st} node: $17.5\times 0-48y_1+32.5y_2=6e^{0.2}$
- At the 4^{th} node: $17.5y_3-48y_4+32.5\times 0=6e^{0.8}$
- The tridiagonal system is:

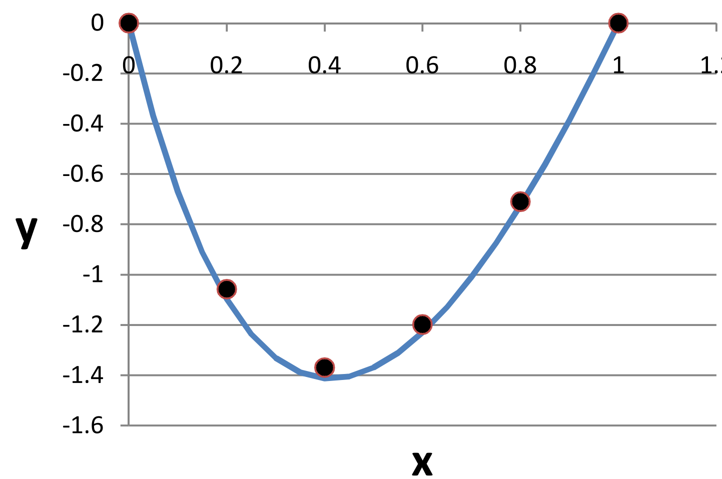
$$\begin{bmatrix} -48 & 32.5 & & \\ 17.5 & -48 & 32.5 & \\ & 17.5 & -48 & 32.5 \\ & & 17.5 & -48 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}=\begin{Bmatrix} 6e^{0.2} \\ 6e^{0.4} \\ 6e^{0.6} \\ 6e^{0.8} \end{Bmatrix}$$

Direct Method: Example

- The solution is {-1.1335, -1.4487, -1.2538, -0.7353}
- Almost similar results as compared to Shooting method (in general, it is difficult to say which method will be better)



Direct



Shooting

Direct Method: Example

- Let us change the right boundary condition to y'(1)=4.0687
- Using ghost node (node 6) at x=1.2,

$$y_6 = y_4 + 2\times 0.2\times 4.0687 = y_4 + 1.6275$$

- At the 5th node:

$$17.5y_4 - 48y_5 + 32.5\times\left(y_4 + 1.6275\right) = 6e^1$$

$$50y_4 - 48y_5 = 6e^1 - 32.5\times 1.6275 = -36.5841$$

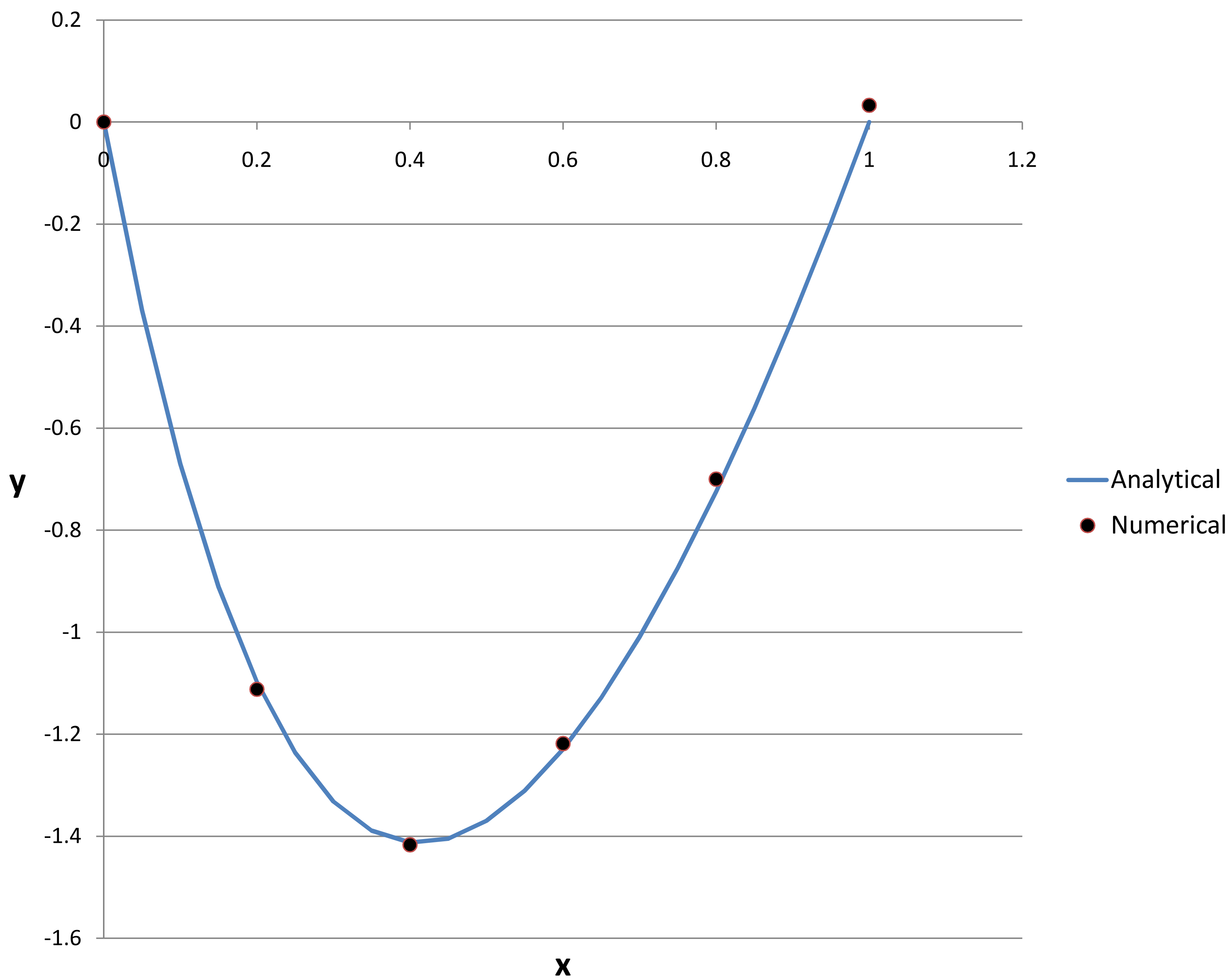
- The tridiagonal system is:

Direct Method: Example

$$\begin{bmatrix} -48 & 32.5 & & & \\ 17.5 & -48 & 32.5 & & \\ & 17.5 & -48 & 32.5 & \\ & & 17.5 & -48 & 32.5 \\ & & & 50 & -48 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{Bmatrix} = \begin{Bmatrix} 6e^{0.2} \\ 6e^{0.4} \\ 6e^{0.6} \\ 6e^{0.8} \\ -36.5841 \end{Bmatrix}$$

And the solution is

{-1.1120,-1.4168,-1.2183,-0.7001, 0.0329}



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Partial Differential Equations

- Two or more independent variables
  - Vibration of a string: y=f(x,t)
  - Steady-state temperature of a plate, T=f(x,y)
  - Transient temperature in a cube, T=f(t,x,y,z)
- Need Initial and/or Boundary conditions
  - Diffusion Equation: $$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$$
  - Advection-Diffusion Equation:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial x^2}$$

Partial Differential Equations: Examples

- Diffusion Equation in 3D:

$$\frac{\partial c}{\partial t} = D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} + D_z \frac{\partial^2 c}{\partial z^2}$$

- 3D Advection-Diffusion Equation:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} + D_z \frac{\partial^2 c}{\partial z^2}$$

- Laplace Equation (for 2D potential)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Partial Differential Equations: Examples

- Wave equation:

$$\frac{\partial^2 \psi}{\partial t^2} = u^2 \frac{\partial^2 \psi}{\partial x^2}$$

  - Needs two initial conditions and two b.c.
- Classifications of PDEs helps us in identifying the appropriate IC/BC
- On the basis of Characteristics
- These are the hyper-planes (line, if 2 independent variables; plane if 3), along which "information" propagates

Partial Differential Equations: Characteristics

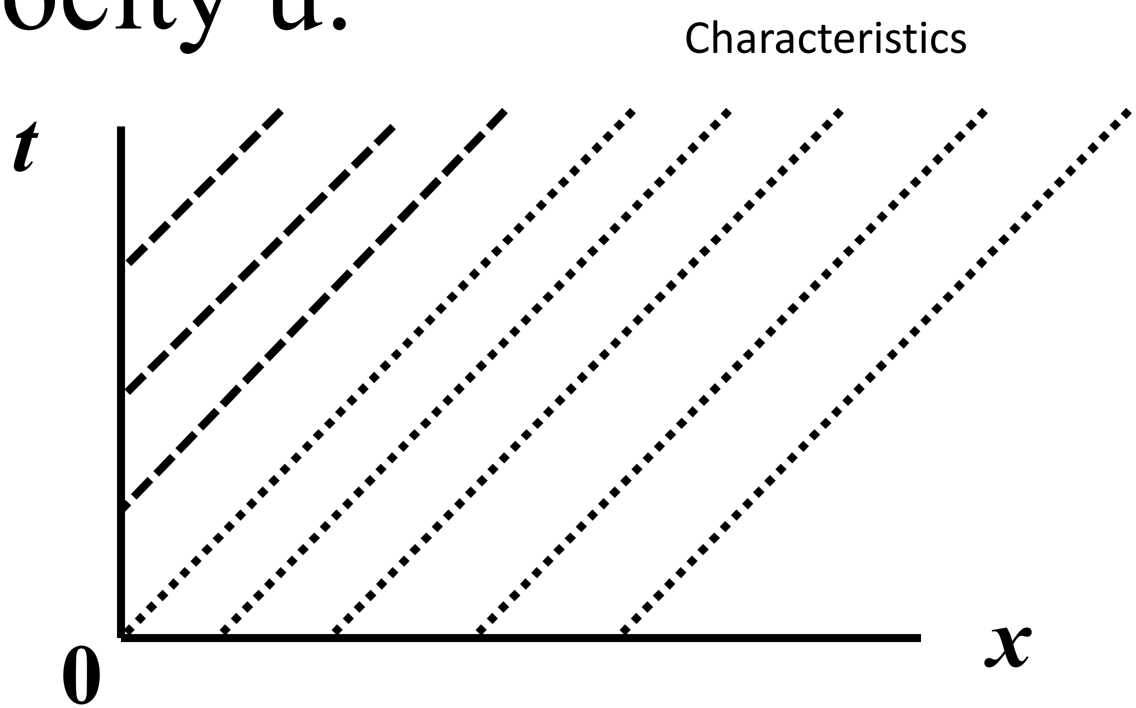
- The governing equations become simpler along the characteristics
- For example, a first-order PDE in 2 independent variables reduces to an ODE along the characteristic lines
- These also help in identifying the "domain (or region) of influence" and the "domain (or region) of dependence"
- Which helps in proper selection of initial/boundary conditions

Partial Differential Equations: Characteristics

- Consider the "pure advection"

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = 0$$

- Clearly, the information propagates at the velocity u:



Partial Differential Equations: Characteristics

- c is constant along these lines
- How do we find the characteristics?
- Define a new variable

$$\xi = \xi(t, x)$$

- Partial derivatives:

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial \xi}\frac{\partial \xi}{\partial t}$$

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \xi}\frac{\partial \xi}{\partial x}$$

Partial Differential Equations: Characteristics

- From the governing equation:

$$\frac{\partial c}{\partial \xi}\left(\frac{\partial \xi}{\partial t}+u\frac{\partial \xi}{\partial x}\right)=0$$

- Resulting in $\xi = x - ut$
- Along the lines, ξ=constant, $dx/dt=u$
- Governing equation becomes dc/dt=0
- c is constant along a characteristic line (known as Riemann Invariant)

Characteristic Lines

- Let us now consider a set of two first-order nonlinear equations: "channel flow"

$$\frac{\partial y}{\partial t} + V\frac{\partial y}{\partial x} + y\frac{\partial V}{\partial x} = 0$$

$$\frac{\partial y}{\partial x} + \frac{V}{g}\frac{\partial V}{\partial x} + \frac{1}{g}\frac{\partial V}{\partial t} = f(x,t)$$

- Multiply 2nd eqn. by α and add to 1st

$$\frac{\partial y}{\partial t} + V\frac{\partial y}{\partial x} + y\frac{\partial V}{\partial x} + \alpha\left(\frac{\partial y}{\partial x} + \frac{V}{g}\frac{\partial V}{\partial x} + \frac{1}{g}\frac{\partial V}{\partial t}\right) = \alpha f(x,t)$$

Characteristic Lines

- Write it as

$$\frac{g}{\alpha}\left[\frac{\partial y}{\partial t}+(V+\alpha)\frac{\partial y}{\partial x}\right]+\left[\frac{\partial V}{\partial t}+\left(V+\frac{gy}{\alpha}\right)\frac{\partial V}{\partial x}\right]=\alpha f(x,t)$$

- For conversion to ODE, should have

$$V+\alpha=V+\frac{gy}{\alpha}\Rightarrow\alpha=\pm\sqrt{gy}$$

- i.e., $\xi=x-\left(V+\sqrt{gy}\right)t$ and $\eta=x-\left(V-\sqrt{gy}\right)t$
- Along these lines,
- With f=0

$$\pm\sqrt{\frac{g}{y}}\frac{dy}{dt}+\frac{dV}{dt}=0$$

Characteristic Lines

- Or $$\frac{d}{dt}\left(V \pm 2\sqrt{gy}\right) = 0$$

- $V \pm 2\sqrt{gy}$ is the Riemann invariant

- Along the characteristic $\xi = x - \left(V + \sqrt{gy}\right)t$

$V + 2\sqrt{gy}$ is constant, along $\eta = x - \left(V - \sqrt{gy}\right)t$

$V - 2\sqrt{gy}$ is constant