Tutorial 7

1. The velocity of an object, travelling along a straight line, was measured at various times as follows:

Time (min)	0	1	2	3	4	5	6	7	8	9	10	11	12
Velocity (cm/min)	0.00	0.65	1.72	3.48	6.39	11.18	19.09	32.12	53.60	89.02	147.41	243.69	402.43

Estimate the distance travelled in 12 minutes using (i) Simpson's 1/3 rule, (ii) Simpson's 3/8 rule, and (iii) Romberg integration, applied to Trapezoidal rule estimates with h=1, 2, and 4.

Solution:

t	v		Simp 1/3	Simp 3/8	Trap h=1	Trap h=2	Trap h=4
0	0				0.33	1.72	12.78
1	0.65		1.44	3.97	1.19		
2	1.72				2.60	8.11	
3	3.48		7.34		4.94		
4	6.39			28.23	8.79	25.48	119.98
5	11.18		23.40		15.14		
6	19.09				25.61	72.69	
7	32.12		67.06	136.98	42.86		
8	53.6				71.31	201.01	912.06
9	89.02		185.70		118.22		
10	147.41			624.28	195.55	549.84	
11	243.69		508.20		323.06		
12	402.43						
			793.14	793.46	809.57	858.85	1044.82
				Romberg $O(h^4)$	793.14	796.86	
				Romberg $O(h^6)$	792.89		

The distance travelled in 12 minutes is: 793.14 cm using Simpson's 1/3 rule, 793.46 cm using 3/8 rule, and 792.89 cm using Romberg integration with three trapezoidal estimates (809.57 using h=1, 858.85 using h=2, 1044.82 using h=4). Note that the Romberg, $O(h^4)$, with h=1 and 2, is same as Simpson's 1/3. Also note that 1/3 rule is more accurate than 3/8 (assuming Romberg to be the best estimate).

2. The flow rate through a circular pipe is given by $Q = \int_0^{r_0} 2\pi r\, v\, dr$, where v is the velocity at a distance of r from the centre of pipe and r_0 is the radius of the pipe. If the velocity is approximated by $v = 2\left(1 - \frac{r}{r_0}\right)^{1/7}$ (in m/s), and the pipe radius is 12 cm, compute Q using

Gauss-Legendre quadrature with 2, 3, and 4 Gauss points. Perform an error analysis for the results, using the true value of the flow rate as 0.07389025921170497 m^3/s.

Solution:

Conversion to standard domain: $z=r/0.06-1 \Rightarrow I=0.06\, I_z$

The table below shows the computation of I_z and I: $f = 2\pi r \times 2\left(1-\frac{r}{0.12}\right)^{1/7}$

2-point	z	w	r	f	w.f		T.V.	0.07389
	-0.57735	1	0.025359	0.308044	0.308044			
	0.57735	1	0.094641	0.952475	0.952475		I	Error (%)
				Iz	**1.260519**		0.075631	-2.35605
3-point	z	w	r	f	w.f			
	-0.77460	0.555556	0.013524	0.167072	0.092818			
	0	0.888889	0.060000	0.682900	0.607022			
	0.77460	0.555556	0.106476	0.979540	0.544189		I	Error (%)
				Iz	**1.244028**		0.074642	-1.01697
4-point	z	w	r	f	w.f			
	-0.86114	0.347855	0.008332	0.103630	0.036048			
	-0.33998	0.652145	0.039601	0.469970	0.306489			
	0.33998	0.652145	0.080399	0.862339	0.56237			
	0.86114	0.347855	0.111668	0.958623	0.333462		I	Error (%)
				Iz	**1.238368**		0.074302	-0.55737

Tutorial 8

1. Solve the differential equation $dy/dt = -100\,y + 99\,e^{-t}$ with the initial condition $y(0)=2$ using, (a) Euler's forward (explicit) method, and (b) Euler backward (implicit) method, to obtain the value of y at t=0.1. Use time steps of 0.01, 0.02 and 0.025. Find the analytical solution and compare the errors for these time steps.

Solution:

The analytical solution is $y= e^{-t}+ e^{-100t}$. Formulae used:

Euler Forward: $y_{n+1} = y_n + hf(t_n, y_n)$ Euler Backward: $y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$

Euler Explicit:						Implicit Euler:			
Delta t	0.01					Delta t	0.01		
t	y	f	Exact y	Error		t	y	Exact y	Error
0	2	-101	2	0		0	2	2	0
0.01	0.99	-0.98507	1.35793	0.36793		0.01	1.49007	1.35793	-0.13215
0.02	0.98015	-0.97526	1.11553	0.13538		0.02	1.23024	1.11553	-0.1147
0.03	0.9704	-0.96556	1.02023	0.04984		0.03	1.09549	1.02023	-0.07526
0.04	0.96074	-0.95595	0.97911	0.01836		0.04	1.02333	0.97911	-0.04423
0.05	0.95118	-0.94644	0.95797	0.00679		0.05	0.98253	0.95797	-0.02456
0.06	0.94172	-0.93702	0.94424	0.00253		0.06	0.95744	0.94424	-0.01319
0.07	0.93235	-0.9277	0.93331	0.00096		0.07	0.94025	0.93331	-0.00695
0.08	0.92307	-0.91847	0.92345	0.00038		0.08	0.92707	0.92345	-0.00362
0.09	0.91389	-0.90933	0.91405	0.00017		0.09	0.91593	0.91405	-0.00188
0.1	0.90479	-0.90028	0.90488	9.1E-05		0.1	0.90586	0.90488	-0.00098
Delta t	0.02					Delta t	0.02		
t	y	f	Exact y	Error		t	y	Exact y	Error
0	2	-101	2	0		0	2	2	0
0.02	-0.02	99.0397	1.11553	1.13553		0.02	1.3136	1.11553	-0.19806
0.04	1.96079	-100.961	0.97911	-0.98169		0.04	1.07199	0.97911	-0.09288
0.06	-0.05843	99.0777	0.94424	1.00267		0.06	0.97889	0.94424	-0.03465
0.08	1.92312	-100.924	0.92345	-0.99967		0.08	0.93555	0.92345	-0.0121
0.1	-0.09535	99.1143	0.90488	1.00024		0.1	0.90904	0.90488	-0.00416
Delta t	0.025					Delta t	0.025		
t	y	f	Exact y	Error		t	y	Exact y	Error
0	2	-101	2	0		0	2	2	0
0.025	-0.525	149.056	1.05739	1.58239		0.025	1.26111	1.05739	-0.20372
0.05	3.20139	-225.967	0.95797	-2.24342		0.05	1.03297	0.95797	-0.07501
0.075	-2.4478	336.626	0.9283	3.37609		0.075	0.95118	0.9283	-0.02289
0.1	5.96786	-507.207	0.90488	-5.06298		0.1	0.91162	0.90488	-0.00673

Clearly, explicit method is unstable for step size of 0.025, and is oscillating in a finite band for 0.02.

2. The amount of lowering of water level, s, in a well at a time t, due to pumping from groundwater is governed by an equation of the form $s=C\,W(u)$, where C is a constant (proportional to the discharge), W is called the Well Function, and u is inversely proportional to t. The well function is given by the equation $dW(u)/du = -Exp(-u)/u$. If the value of $W(1)$ is 0.21938393, find the value of $W(0.5)$ using (a) Romberg integration algorithm with accuracy $O(h^6)$ (b) Modified Euler with $h= -0.5$ (c) Heun's

method with $h = -0.25$ and (d) Fourth-order Runge-Kutta method with $h = -0.25$. Perform an error analysis using the true value of $W(0.5)$ as 0.55977359.

Solution:

Romberg: Use trapezoidal with h=0.125, 0.25, 0.5 to estimate $\int_{0.5}^{1} -\frac{e^{-u}}{u}\,du$

Modified Euler: $y_{n+1} = y_n + hf\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\right)$

Heun's method: $y_{n+1} = y_n + h\dfrac{f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))}{2}$

4th order R-K method:

$$k_1 = f(t_n, y_n); k_2 = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right)$$

$$k_3 = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right); k_4 = f(t_n + h, y_n + hk_3)$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

f(u)=dW/du=-Exp(-u)/u			Romberg	I=From 0.5 to 1			Modified Euler	
u	f(u)		O(h^2)				W(0.75)	0.3113538
1	-0.36788		-0.39524	h=0.5			W(0.5)	**0.534295**
0.875	-0.47641		-0.35507	h=0.25			Error	0.02548
0.75	-0.62982		-0.34414	h=0.125				
0.625	-0.85642						Heun's	
0.5	-1.21306		O(h^4)					
			-0.34169	h=0.5, 0.25			W(0.75)0	0.3113538
W(1.0)=	0.219384		-0.34050	h=0.25, 0.125			W(0.75)	0.3440966
W(0.5)=	**0.559774**							
W(0.75)=	**0.340332**		O(h^6)	W(0.5)=	**0.559801**		W(0.5)0	0.50155
			-0.34042	Error	-0.00003		W(0.5)	**0.574457**
							Error	-0.01468
4th order R-K		h=	-0.25					
k1	-0.36788	-0.62982						
k2	-0.47641	-0.85642						
k3	-0.47641	-0.85642						
k4	-0.62982	-1.21306						
W(0.75)	0.340357	**0.559880**						
	Error	-0.00011						

Tutorial 9

1. Solve the differential equation $dy/dx = x^2y - 2y$ with $y(0)=1$ over the interval $x=0$ to 0.5, using (a) Heun's method without iteration with $h=0.25$ and 0.125, (b) Heun's method with iteration (with $h=0.25$ and stopping error criterion of 1%), and (c) 4th order Runge-Kutta method with $h=0.125$ and 0.25. Obtain the exact value of y at $x=0.5$ and perform an error analysis.

Solution: $f(x,y) = x^2y - 2y$. Exact solution is $y=\exp(x^3/3-2x)$ and value at 0.5 is **0.383531573.**

Heun's	Without iteration							
h=	0.25							
xi	yi	f(xi,yi)	y0	xi+1	f(xi+1,y0)	yi+1	Error	
0	1	-2	0.5	0.25	-0.968750	0.62890625		
0.25	0.6289063	-1.21851	0.32428	0.5	-0.567490	**0.40565681**	**-0.02213**	
h=	0.125							
xi	yi	f(xi,yi)	y0	xi+1	f(xi+1,y0)	yi+1		
0	1	-2	0.75	0.125000	-1.488281	0.78198242		
0.125000	0.7819824	-1.55175	0.58801	0.250000	-1.139277	0.61379344		
0.250000	0.6137934	-1.18922	0.46514	0.375000	-0.864870	0.48541249		
0.375000	0.4854125	-0.90256	0.37259	0.500000	-0.652036	**0.388250**	**-0.004718**	
Heun's	With iteration							
h=	0.25							
xi	yi	f(xi,yi)	y0	xi+1	f(xi+1,y0)	yi+1	Error (%)	
0	1	-2	0.5	0.25	-0.968750	0.62890625		
0	1	-2	0.62891	0.25	-1.218506	0.59768677	-5.223385	
0	1	-2	0.59769	0.25	-1.158018	0.60524774	1.2492353	
0	1	-2	0.60525	0.25	-1.172667	0.60341656	-0.303467	
0.25	0.6034166	-1.16912	0.31114	0.5	-0.544489	0.38921547	20.06056	
0.25	0.6034166	-1.16912	0.38922	0.5	-0.681127	0.37213573	-4.589653	
0.25	0.6034166	-1.16912	0.37214	0.5	-0.651238	**0.37587192**	0.9940069	**0.007660**

4th order R-K							
h=	0.25						
x	0	0.125	0.125	0.25			
y	1.000000	0.75	0.81396	0.5962			
k	-2.000000	-1.48828	-1.61521	-1.15513	y(0.25)=	0.609912	
x	0.25	0.375	0.375	0.5			
y	0.609912	0.4621988	0.50249	0.37633			Error
k	-1.181704	-0.85940	-0.93431	-0.65858	y(0.5)=	**0.383757**	**-0.0002255**

4th order R-K						
h=	0.125					
x	0	0.0625	0.0625	0.125		
y	1.000000	0.875	0.89084	0.77773		
k	-2.000000	-1.74658	-1.77820	-1.54330	y(0.125)=	0.779315
x	0.125	0.1875	0.1875	0.25		
y	0.779315	0.6826621	0.69548	0.6085		
k	-1.546454	-1.34132	-1.36651	-1.17897	y(0.25)=	0.609709
x	0.25	0.3125	0.3125	0.375		
y	0.609709	0.5358772	0.546	0.47988		
k	-1.181311	-1.01942	-1.03867	-0.89227	y(0.375)=	0.480756
x	0.375	0.4375	0.4375	0.5		
y	0.480756	0.4248866	0.43273	0.38293		
k	-0.893905	-0.76845	-0.78263	-0.67012	y(0.5)=	**0.383544**	**-0.0000120**

2. Solve the differential equation $dy/dx = 10 \sin(\pi x)$ with the initial condition $y(0)=0$ and step length of 0.2 using (a) the 4th order R-K method, (b) the Milne's method and (c) 4th order Adams method to obtain the value of y at t=0.2, 0.4, 0.6, 0.8 and 1.0. (For the multi-step methods use the values obtained from the R-K method for start-up)

Solution: $f(x,y) = 10 \sin(\pi x)$. Exact solution is $(10-10 \cos \pi x)/\pi$ and the values are:

x	0.2	0.4	0.6	0.8	1
y	0.607918	2.199467	4.166731	5.758280	6.366198

4th order R-K	h=	0.2			
x=	0.2	0.4	0.6	0.8	1
k1	0.0000	5.8778	9.5106	9.5106	5.8779
k2	3.0902	8.0902	10.0000	8.0902	3.0902
k3	3.0902	8.0902	10.0000	8.0902	3.0902
k4	5.8778	9.5106	9.5106	5.8779	0.0000
y=	0.6080	2.1996	4.1670	5.7586	6.3666

Milne's	h=	0.2
Find y(0.8), y(1)		
		f(x,y)
y(0.8)^0	5.6710	5.8779
y(0.8)^1	5.7616	5.8779
y(0.8)^2	5.7616	5.8779
y(1)^0	6.2790	0.0000
y(1)^1	6.3684	0.0000
y(1)^2	6.3684	0.0000

Adams'	h=	0.2
Find y(0.8), y(1)		
		f(x,y)
y(0.8)^0	5.6623	5.8779
y(0.8)^1	5.7663	5.8779
y(0.8)^2	5.7663	5.8779
y(1)^0	6.2759	0.0000
y(1)^1	6.3800	0.0000
y(1)^2	6.3800	

Tutorial 10

1. Solve the differential equation $d^2y/dx^2 - dy/dx - 2y + 2x = 3$ with the boundary conditions $y(0)=0$ and $y(0.5)=0.6967$ using (a) the shooting method and (b) the direct method (use $\Delta x=0.25$ for both).

Solution:

(a) Note that Ralston's method has been used to solve the IVP

f1=dy1/dx=y2										
f2=dy2/dx=3+2y1-2x+y2										
h=	0.25									
Assume y2(0)=0						Assume y2(0)=1				
x	y1	y2	f1	f2		x	y1	y2	f1	f2
0	0	0	0	3		0	0	1	1	4
0.1875	0	0.5625	0.5625	3.1875		0.1875	0.1875	1.75	1.75	4.75
0.25	0.09375	0.78125				0.25	0.375	2.125		
x	y1	y2	f1	f2		x	y1	y2	f1	f2
0.25	0.09375	0.78125	0.78125	3.46875		0.25	0.375	2.125	2.125	5.375
0.4375	0.24023	1.43164	1.43164	4.03711		0.4375	0.77344	3.13281	3.13281	6.80469
0.5	**0.39746**	1.74316				0.5	**1.07422**	3.70703		
Therefore, y2(0)=		0.44217								
x	y1	y2	f1	f2						
0	0	0.44217	0.44217	3.44217						
0.1875	0.08291	1.08757	1.08757	3.87838						
0.25	0.21811	1.37541								
x	y1	y2	f1	f2						
0.25	0.21811	1.37541	1.37541	4.31163						
0.4375	0.476	2.18384	2.18384	5.26084						
0.5	0.6967	2.61152								
Note: 0.21811 may be directly obtained by linear interpolation between 0.09375 and 0.375										

(b) There are three nodes: at 0, 0.25, and 0.5. Only one unknown, y(0.25).

The finite difference equation at x=0.25 is

$$\frac{y_2 - 2y_1 + y_0}{0.25^2} - \frac{y_2 - y_0}{0.5} - 2y_1 + 2 \times x_1 = 3 => 18y_0 - 34y_1 + 14y_2 = 3 - 2 \times x_1$$

With the known values of $x_1=0.25$, $y_0=0$ and $y_2=0.6967$, we get $y_1=0.21335$ (compared to 0.21811)

(If we use h=0.125: 5 nodes, 3 unknowns, the equations are

$$-130y_1 + 60y_2 = 3 - 2 \times 0.125 = 2.75$$

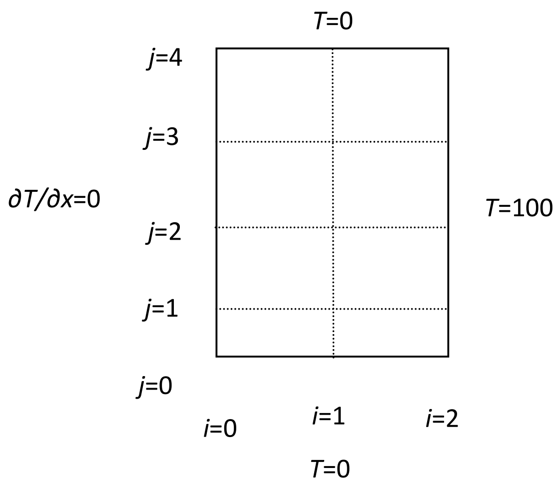
$$68y_1 - 130y_2 + 60y_3 = 3 - 2 \times 0.25 = 2.5$$

$$68y_2 - 130y_3 = 3 - 2 \times 0.375 - 60 \times 0.6967 = -39.552$$

and the solution is 0.07713, 0.21294, 0.41563)

2. For a plate of size L_x=2 cm and L_y=4 cm, with the boundary conditions as T=0 for y=0 and y=4; T=100 for x=2; and insulated boundary at x=0, find the steady state temperature at the centre using the Finite Difference Method (use Δx=1 cm and Δy=1 cm).

Solution:



The discretization is shown above. We need to find $T_{1,2}$. (Six unknowns are $T_{0,1}$, $T_{0,2}$, $T_{0,3}$, $T_{1,1}$, $T_{1,2}$, and $T_{1,3}$)

The governing equation is $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$. The nodal equation, using central difference approximation for the second derivative is $\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2} = 0$. Since Δx and Δy are equal, $-T_{i,j-1} - T_{i-1,j} + 4T_{i,j} - T_{i,j+1} - T_{i+1,j} = 0$.

Using ghost nodes (i=−1), $T_{-1,j} = T_{1,j}$. The system of equation is:

$$\begin{bmatrix} 4 & -2 & -1 & & & \\ -1 & 4 & 0 & -1 & & \\ -1 & 0 & 4 & -2 & -1 & \\ & -1 & -1 & 4 & 0 & -1 \\ & & -1 & 0 & 4 & -2 \\ & & & -1 & -1 & 4 \end{bmatrix} \begin{Bmatrix} T_{0,1} \\ T_{1,1} \\ T_{0,2} \\ T_{1,2} \\ T_{0,3} \\ T_{1,3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 100 \\ 0 \\ 100 \\ 0 \\ 100 \end{Bmatrix}$$

The solution is $T_{0,1}$=37.5, $T_{1,1}$=50, $T_{0,2}$=50, $T_{1,2}$=62.5, $T_{0,3}$=37.5, and $T_{1,3}$=50.

3. Toxic pollutant transport in a river is governed by the following equation:

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} - D\frac{\partial^2 c}{\partial x^2} + kc = 0;\;\; 0 \le x \le 1;\;\; c(0,t) = c_0;\;\; \left.\frac{\partial c}{\partial x}\right|_{(1,t)} = 0;\;\; c(x,0) = x^2 e^{-x}$$

Set-up the matrix equations for the solution of the above equation in terms of Courant Number (or CFL number) and Grid Peclet Number for general Δx and Δt :

$$\text{Courant or CFL No. } C = \frac{v\Delta t}{\Delta x}$$

$$\text{Grid Peclet No. } P_e = \frac{v\Delta x}{D}$$

Solution:

Using forward difference for time derivative and central for the space (time is denoted by superscript, *n*, and space by subscript, *i*), we get (for spatial variables, using a weight of μ for the known time step, *n*, and a weight of 1−μ for the unknown time step, *n*+1)

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} + v\left[(1-\mu)\frac{c_{i+1}^{n+1} - c_{i-1}^{n+1}}{2\Delta x} + \mu\frac{c_{i+1}^n - c_{i-1}^n}{2\Delta x}\right] - D\left[(1-\mu)\frac{c_{i+1}^{n+1} - 2c_i^{n+1} + c_{i-1}^{n+1}}{\Delta x^2} + \mu\frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{\Delta x^2}\right] + k\left[(1-\mu)c_i^{n+1} + \mu c_i^n\right] = 0$$

Which may be written as,

$$\left[(1-\mu)\left(-\frac{C}{2} - \frac{C}{P_e}\right)\right]c_{i-1}^{n+1} + \left[1 + (1-\mu)\left(k\Delta t + 2\frac{C}{P_e}\right)\right]c_i^{n+1} + \left[(1-\mu)\left(\frac{C}{2} - \frac{C}{P_e}\right)\right]c_{i+1}^{n+1} = \left[\mu\left(\frac{C}{2} + \frac{C}{P_e}\right)\right]c_{i-1}^n + \left[1 - \mu\left(k\Delta t + 2\frac{C}{P_e}\right)\right]c_i^n + \left[\mu\left(-\frac{C}{2} + \frac{C}{P_e}\right)\right]c_{i+1}^n$$

Or,

$$a_{i,i-1}c_{i-1}^{n+1} + a_{i,i}c_i^{n+1} + a_{i,i+1}c_{i+1}^{n+1} = b_i$$

The matrix form is (given c_0=0 and derivative=zero at the n[th] node)

$$\begin{bmatrix} a_{1,1} & a_{1,2} & & & & \\ a_{2,1} & a_{2,2} & a_{2,3} & & & \\ & a_{3,2} & a_{3,3} & a_{3,4} & & \\ & & . & . & . & \\ & & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & & a_{n,n-1} + a_{n,n+1} & a_{n,n} \end{bmatrix} \begin{Bmatrix} c_1^{n+1} \\ c_2^{n+1} \\ c_3^{n+1} \\ . \\ c_{n-1}^{n+1} \\ c_n^{n+1} \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ b_{n-1} \\ b_n \end{Bmatrix}$$