Tutorial 1

1. The computation of the expression

$$f(x) = \frac{\sqrt{1+8x^2}-1}{2}$$

involves the difference of small numbers when $x << 1$. Obtain the value of $f(x)$ for $x = 0.002$, and also the relative error (True Value is $0.7999936001023980 \times 10^{-5}$), performing operations by rounding all mantissas to six decimals: (a) using the expression above, (b) employing a Taylor's series expansion and using the first three terms, and (c) using the equivalent expression $f(x) = \frac{4x^2}{\sqrt{1+8x^2}+1}$. For the case (a), perform a backward error analysis to find the relative error in x required to make the computed result *exact*.

Solution:

(a) For x=0.002, $1 + 8x^2 = 0.100003 \times 10^1$, $\sqrt{1+8x^2} = 0.100001 \times 10^1$, and $f(x) = 0.500000 \times 10^{-5}$. True Value is $0.7999936001023980 \times 10^{-5}$. Relative error = 37.5%.

(b) $f'(x) = \frac{4x}{\sqrt{1+8x^2}}$; $f''(x) = \frac{4}{\sqrt{1+8x^2}} - \frac{32x^2}{(1+8x^2)^{3/2}}$; Expanding about 0, $f(x) = 0 + x \times 0 + \frac{x^2}{2} \times 4$. For x=0.002, $f(x) = 0.800000 \times 10^{-5}$. Relative error = $-8\text{x}10^{-4}$%.

(c) For x=0.002, $1 + 8x^2 = 0.100003 \times 10^1$, $\sqrt{1+8x^2} = 0.100001 \times 10^1$, $\sqrt{1+8x^2} + 1 = 0.200001 \times 10^1$ and $f(x) = 0.799996 \times 10^{-5}$. Relative error = $-3\text{x}10^{-4}$%.

For the case (a), for 0.500000×10^{-5} to be *exact* result, we need $x = \sqrt{\frac{(1+2\times 0.500000\times 10^{-5})^2-1}{8}} = 0.00158114$ (the given value is 0.002). Relative error = 20.9%.

NOTE: In (a), 16-digit accuracy was used to perform the computations and then the values were rounded-off. If the computations are performed on a 8-digit scientific calculator, $\sqrt{1+8x^2} = 0.100002 \times 10^1$, and $f(x) = 0.100000 \times 10^{-4}$. Relative error = −25%. For this value to be the exact answer, we need $x = \sqrt{\frac{(1+2\times 0.100000\times 10^{-4})^2-1}{8}} = 0.00223608$ (the given value is 0.002). Relative error = −11.8%.

2. On a plot of land, which is in the shape of a right-angled triangle, the two perpendicular sides were measured as $a = 300.0 \pm 0.1$ m and $b = 400.0 \pm 0.1$ m. How accurately is it possible to estimate the hypotenuse c?

Solution:

$c = \sqrt{a^2+b^2}$; $\frac{\partial c}{\partial a} = \frac{a}{\sqrt{a^2+b^2}}$; $\frac{\partial c}{\partial b} = \frac{b}{\sqrt{a^2+b^2}}$. For given values, a=300 m, b=400 m, $\frac{\partial c}{\partial a} = 0.6$; $\frac{\partial c}{\partial b} = 0.8$. **Using first order approximation** (Δa=Δb=±0.1 m)

$$\Delta c = \Delta a \frac{\partial c}{\partial a} + \Delta b \frac{\partial c}{\partial b} = \pm 0.14 \text{ m}$$

3. The following set of equations is to be solved to get the value of x for a given δ. For what values of δ will this problem be well-conditioned?

$$x + y = 2$$
$$x + (1-\delta)y = 1$$

Solution:

Solving for x, we get $x = f(\delta) = \frac{2\delta-1}{\delta}$. The condition number of the problem is $C_P = \left|\frac{\delta f\prime(\delta)}{f(\delta)}\right| = \left|\frac{\delta\frac{1}{\delta^2}}{\frac{2\delta-1}{\delta}}\right| = \left|\frac{1}{2\delta-1}\right|$. Well conditioned for δ>1 or δ<0.

(You may see the values of x for a δ value of 2 and 2.02 and another for 0.6 and 0.606 to illustrate the relative change in x for a 1% change in δ)

Tutorial 2

1. Find a root (one root is, obviously, $x = 0$) of the equation: $f(x) = \sin x - (x/2)^2 = 0$ using Bisection method, Regula-Falsi method, Fixed Point method, Newton-Raphson method and Secant method. In each case, calculate true relative error and approximate relative error at each iteration (the true root may be taken as 1.933753762827021). Plot both of these errors (on log scale) vs. iteration number for each of the methods. Terminate the iterations when the approximate relative error is less than 0.01 %. Use starting points for Bisection, Regula-Falsi and Secant methods as $x = 1$ and $x= 2$ and for Fixed Point and Newton methods, $x = 1.5$.

Solution:

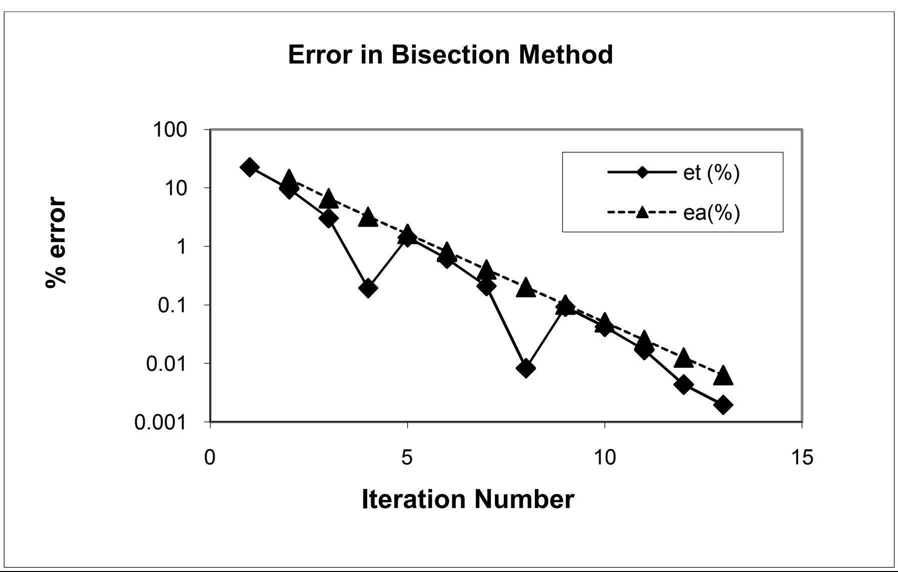
True Value=1.93375376
e_r=absolute(((True Value - $x^{(i)}$)/True Value)*100) %
ε_r=absolute((($x^{(i)}$-$x^{(i-1)}$)/$x^{(i)}$)*100) %

<u>Bisection Method</u>

Iteration 1: x=(1+2)/2=1.5, f(1.5)= 0.434994987 thus root lies between 1.5 and 2 so x for next iteration x=(1.5+2)/2=1.75.

Calculate e_r and ε_r with the formula given above. Note that in figures et and ea are used for true and approximate relative errors.

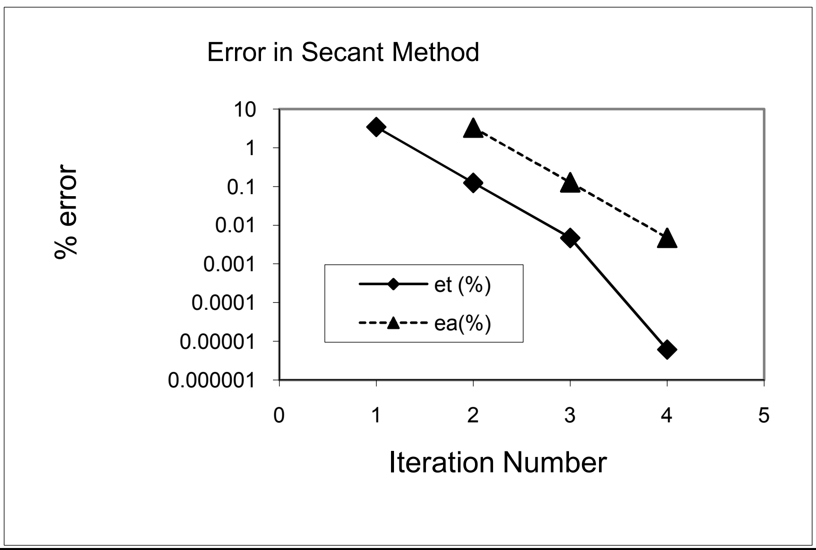
Iteration	x	sinx-(x/2)^2	e_r (%)	ε_r (%)
	1	0.591470985		
	2	-0.090702573		
1	1.5	0.434994987	22.43066	
2	1.75	0.218360947	9.502439	14.28571
3	1.875	0.075179532	3.038327	6.666667
4	1.9375	-0.004962282	0.193729	3.225806
5	1.90625	0.035813793	1.422299	1.639344
6	1.921875	0.015601413	0.614285	0.813008
7	1.929688	0.005363397	0.210278	0.404858
8	1.933594	0.000211505	0.008275	0.20202
9	1.935547	-0.002372653	0.092727	0.100908
10	1.93457	-0.00107989	0.042226	0.05048
11	1.934082	-0.000434021	0.016976	0.025246
12	1.933838	-0.000111215	0.00435	0.012625
13	1.933716	5.01558E-05	0.001962	0.006313



Secant Method

Iteration 1: $x^{(3)}=x^{(2)}-((x^{(2)}-x^{(1)})/(f_2-f_1))*(f_2)=2-((2-1)/(-0.090702573-0.591470985))*(-0.090702573)=1.867039$ and so on.

Iteration	x	sinx-(x/2)^2	e_r (%)	ε_r (%)
	1	0.591470985		
	2	-0.090702573		
1	1.867039	0.084981622	3.450020524	
2	1.931355	0.003167407	0.124069284	3.330083
3	1.933845	-0.000119988	0.004693665	0.128757
4	1.933754	1.56453E-07	6.12036E-06	0.0047

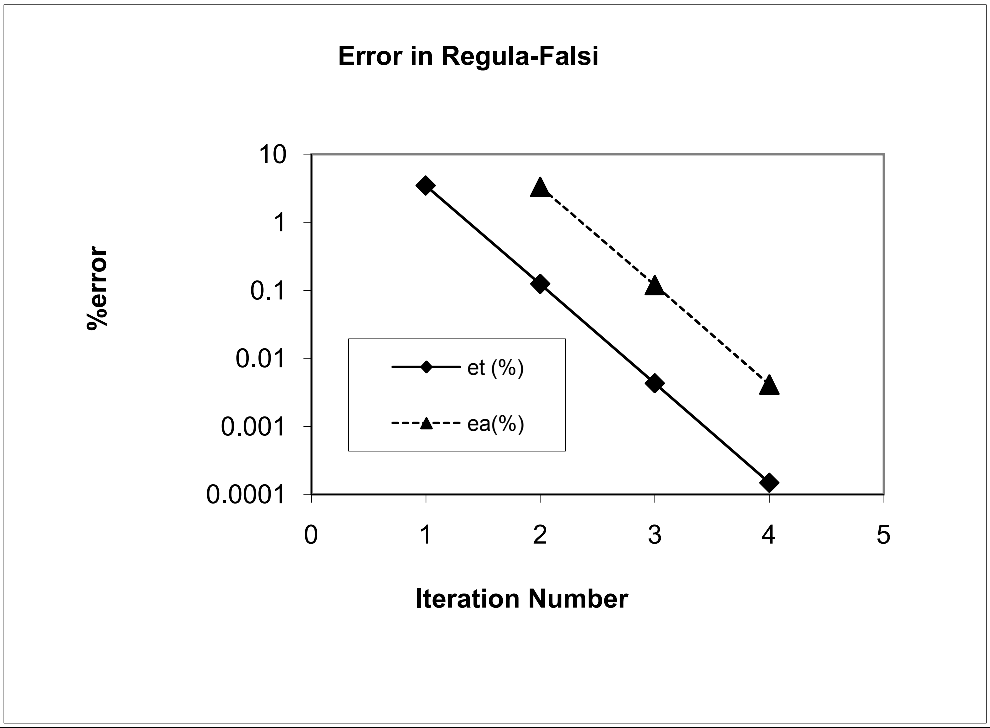


False Position

Iteration 1: $x^{(3)}=2-((1-2)/(0.591470985-(-0.090702573)))*(-0.090702573)=1.867039$
Iteration 2: $x^{(4)}=2-((1.867039-2)/(0.084981622-(-0.090702573)))*(-0.090702573)=1.931355$ and so on.

Iteration	x	sinx-(x/2)^2	e_r (%)	ε_r (%)
	1	0.591470985		
	2	-0.090702573		
1	1.867039	0.084981622	3.450020524	
2	1.931355	0.003167407	0.124069284	3.330083

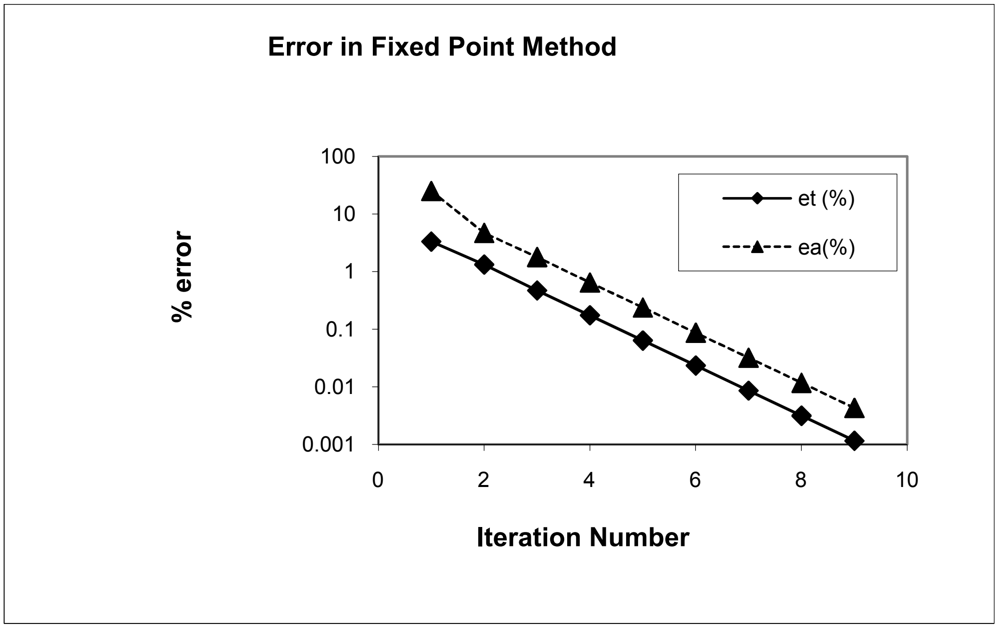
3	1.933671	0.000109618	0.004288389	0.119786
4	1.933751	3.78371E-06	0.000148017	0.00414



<u>Fixed Point</u>

<u>Iteration1:</u> f(x)=2*sqrt(sin(x))=2*sqrt(sin(1.5))= 1.997493416 and so on…

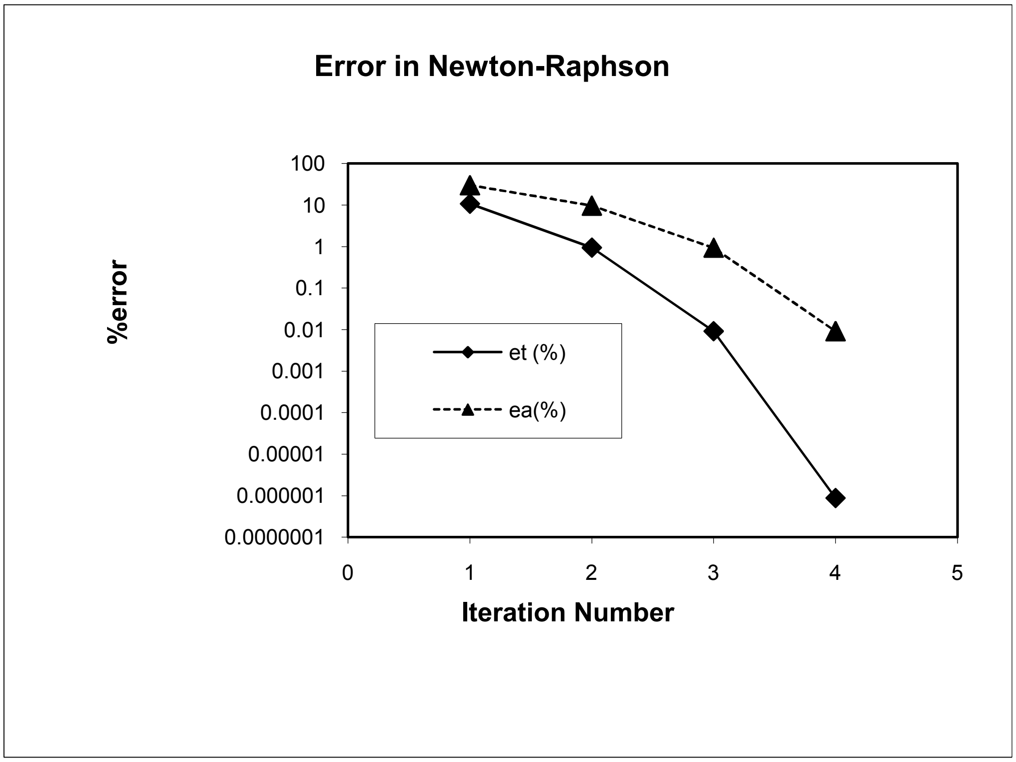
Iteration	x	2sqrt(sinx)	e_r (%)	ε_r (%)
	1.5	1.997493416		
1	1.997493	1.908232351	3.296162	24.90589
2	1.908232	1.942788325	1.319786	4.677683
3	1.942788	1.930393907	0.467203	1.778679
4	1.930394	1.934981664	0.173748	0.642067
5	1.934982	1.933302092	0.063498	0.237096
6	1.933302	1.933919512	0.023357	0.086876
7	1.93392	1.933692885	0.008571	0.031926
8	1.933693	1.933776116	0.003148	0.01172
9	1.933776	1.933745555	0.001156	0.004304



Newton Raphson

f(x)= **sinx-(x/2)^2, f'(x)=cos(x)-x/2**
Iteration 1: $x^{(2)}=x^{(1)}-(f_1)/(\cos(x^{(1)})-x^{(1)}/2)=1.5$-(0.434994987)/(cos(1.5)-1.5/2)= 2.140393 and so on..

Iteration	x	sinx-(x/2)^2	e_r (%)	ε_r (%)
	1.5	0.434994987		
1	2.140393	-0.303201628	10.68590084	29.9194
2	1.952009	-0.024370564	0.944028342	9.650767
3	1.933931	-0.000233752	0.009143414	0.934799
4	1.933754	-2.24233E-08	8.77191E-07	0.009143



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2. Find *a root* of the following equation using Mueller's method to an approximate error of $\varepsilon_r \leq$ 0.1%:

$$x^4 - 2x^3 - 53x^2 + 54x + 504 = 0$$

Take the three starting values as 1, 2, and 3.

Solution:

504	c0
54	c1
-53	c2
-2	c3
1	c4

$$a = \frac{\dfrac{f_i - f_{i-1}}{x^{(i)} - x^{(i-1)}} - \dfrac{f_{i-1} - f_{i-2}}{x^{(i-1)} - x^{(i-2)}}}{x^{(i)} - x^{(i-2)}}$$

$$b = \frac{f_i - f_{i-1}}{x^{(i)} - x^{(i-1)}} + a\left(x^{(i)} - x^{(i-1)}\right)$$

$$c = f_i$$

$$\Delta x^{(i)} = -\frac{2c}{b + Sign(b)\sqrt{b^2 - 4ac}}$$

Note: Δx2 in the table below shows the other root of the quadratic equation, i.e., with a negative sign in denominator.

i	$x^{(i)}$	f	a	b	c	Δx	Δx2	$x^{(i+1)}$	ε_r (%)
	1	504							
	2	400							
0	3	216	-40	-224	216	0.8387	-6.439	3.83868	
1	3.83868	34.3133	-17.75	-231.5	34.313	0.1466	-13.19	3.98524	3.67764
2	3.98524	3.10289	3.7396	-212.4	3.1029	0.0146	56.783	3.99986	0.36532
3	3.99986	0.0302	16.562	-210	0.0302	0.0001	12.682	4	0.00359

Therefore, one of the roots is 4.

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3. Find *all the roots* of the above polynomial using Bairstow's method with $\varepsilon_r \leq 0.1\%$. Use the starting guess as α_0=2 and α_1=2.

Solution:

Initial $\alpha_0=2, \alpha_1=2$

$d_n=c_n$, $d_{n-1}=c_{n-1}+ \alpha_1*d_n$, $d_j=c_j+ \alpha_1*d_{j+1}+ \alpha_0*d_{j+2}$ for j=n-2 to 0

$\delta_{n-1}=d_n$, $\delta_{n-2}=d_{n-1}+ \alpha_1*\delta_{n-1}$, $\delta_j=d_{j+1}+ \alpha_1*\delta_{j+1}+ \alpha_0*\delta_{j+2}$ for j=n-3 to 0

$\Delta\alpha_0$ and $\Delta\alpha_1$ can be obtained by solving simultaneous linear equations:

$$\delta_1^{(i)}\Delta\alpha_0^{(i)} + \delta_0^{(i)}\Delta\alpha_1^{(i)} = -d_0^{(i)}$$
$$\delta_2^{(i)}\Delta\alpha_0^{(i)} + \delta_1^{(i)}\Delta\alpha_1^{(i)} = -d_1^{(i)}$$

Finally, roots are obtained by: $r_{1,2} = 0.5\left(\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_0}\right)$

The relative error is computed as the maximum of the relative errors in α_0 and α_1 and is highlighted.
(For iteration 1: Max(8.81/10.81,0.6751/1.3249)*100% = 81.5%)

	c	d	δ	
0	504	306	-134	
1	54	-48	-45	
2	-53	-51	2	
3	-2	0	1	
4	1	1		
$\Delta\alpha_0$	8.810292	$\Delta\alpha_1$	-0.6751	ε_r (%)
$\alpha_0$new	10.81029	$\alpha_1$new	1.324902	81.50
	c	d	δ	
0	504	24.494941	-44.9748	

$$-45\Delta\alpha_0 - 134\Delta\alpha_1 = -306$$
$$2\Delta\alpha_0 - 45\Delta\alpha_1 = 48$$
$$\Rightarrow \Delta\alpha_0 = 8.8103, \Delta\alpha_1 = -0.6751$$

1	54	-10.38027	-31.4129	
2	-53	-43.08415	0.649804	
3	-2	-0.675098	1	
4	1	1		
$\Delta\alpha_0$	1.216843	$\Delta\alpha_1$	-0.30527	ε_r (%)
$\alpha_0$new	12.02713	$\alpha_1$new	1.019627	29.94
	c	d	δ	
0	504	-1.407564	-30.6075	
1	54	-0.587363	-29.9053	
2	-53	-41.97248	0.039255	
3	-2	-0.980373	1	
4	1	1		
$\Delta\alpha_0$	-0.026929	$\Delta\alpha_1$	-0.01968	ε_r (%)
$\alpha_0$new	12.00021	$\alpha_1$new	0.999951	1.97
	c	d	δ	
0	504	-0.004703	-29.9979	
1	54	0.0014621	-29.9997	
2	-53	-41.999794	-9.7E-05	
3	-2	-1.000049	1	
4	1	1		
$\Delta\alpha_0$	-0.000206	$\Delta\alpha_1$	4.87E-05	ε_r (%)
$\alpha_0$new	12.000000	$\alpha_1$new	1.000000	0.0049
Roots =	-3		4	

Reduced Polynomial: x^2 - 1.000049x - 41.999794

Roots = 7.000010 -5.999962

Tutorial 3

1. Solve the following system of equations by Gauss Elimination, Doolittle method, Crout method and Cholesky decomposition:

$$\begin{bmatrix} 9.3746 & 3.0416 & -2.4371 \\ 3.0416 & 6.1832 & 1.2163 \\ -2.4371 & 1.2163 & 8.4429 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9.2333 \\ 8.2049 \\ 3.9339 \end{bmatrix}$$

Solution:

Gauss Elimination				
Step 1	9.3746	3.0416	-2.4371	9.2333
	3.0416	6.1832	1.2163	8.2049
	-2.4371	1.2163	8.4429	3.9339
Step 2	9.3746	3.0416	-2.4371	9.2333
	0	5.196349	2.00702	5.209145
	0	2.00702	7.809331	6.334266
Step 3	9.3746	3.0416	-2.4371	9.2333
	0	5.196349	2.00702	5.209145
	0	0	7.034147	4.322304
	x=	**0.896424**	**0.76513**	**0.614475**

Doolittle method				
A=	9.3746	3.0416	-2.4371	
	3.0416	6.1832	1.2163	
	-2.4371	1.2163	8.4429	
				b
L=	1	0	0	9.2333
	0.324451	1	0	8.2049
	-0.25997	0.386237	1	3.9339
				y
U=	9.3746	3.0416	-2.4371	9.2333
	0	5.196349	2.00702	5.209145
	0	0	7.034147	4.322304
	x=	**0.896424**	**0.76513**	**0.614475**

Crout Method				
A=	9.3746	3.0416	-2.4371	
	3.0416	6.1832	1.2163	
	-2.4371	1.2163	8.4429	
				b
L=	9.3746	0	0	9.2333
	3.0416	5.196349	0	8.2049
	-2.4371	2.00702	7.034147	3.9339
				y
U=	1	0.324451	-0.25997	0.984927
	0	1	0.386237	1.002462

	0	0	1	0.614475
	x=	**0.896424**	**0.76513**	**0.614475**

Cholesky Decomposition:						
					b	
A=	9.3746	3.0416	-2.4371	=	9.2333	
	3.0416	6.1832	1.2163	=	8.2049	
	-2.4371	1.2163	8.4429	=	3.9339	
Cholesky: L11[=sqrt(A11)],L21(=A21/L11),L22[=sqrt(A22-L21^2)],L31(=A31/L11),L32[=(A32-L31*L21)/L22],L33[=sqrt(A33-L31^2-L32^2)]						
L	3.061797	0	0	y1=	3.015647	b1/L11
	0.993404	2.27955	0	y2=	2.285163	(b2-L21*y1)/L22
	-0.79597	0.880446	2.652197	y3=	1.629707	(b3-L31*y1-L32*y2)/L33
U	3.061797	0.993404	-0.79597	**x1=**	**0.896424**	(y1-U12*x2-U13*x3)/U11
	0	2.27955	0.880446	**x2=**	**0.76513**	(y2-U23*x3)/U22
	0	0	2.652197	**x3=**	**0.614475**	y3/U33

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2. Solve the following system of equations using Thomas algorithm:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Solution:

Thomas Algorithm							
index	**l**	**d**	**u**	**b**	**alpha**	**beta**	**x**
1		1	-1	0	1	0	5
2	-1	2	-1	1	1	1	5
3	-1	2	-1	2	1	3	4
4	0	1		1	1	1	1

3. Consider the following set of equations:

$$\begin{bmatrix} 0.123 & 0.345 & 2.00 \\ -2.34 & 0.789 & 1.98 \\ 12.3 & -5.67 & -0.678 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6.81 \\ 5.17 \\ -1.08 \end{bmatrix}$$

a) Solve the system using Gaussian elimination, without pivoting, using 3-digit floating-point arithmetic with round-off. Perform calculations more precisely but round-off to 3 significant digits when storing a result, and use this rounded-off value for further calculations.

b) Perform partial pivoting and carry out Gaussian elimination steps once again using 3-digit floating-point arithmetic with round-off. Comment on the results.

Solution:

Gauss Elimination				
Step 1	0.123	0.345	2.00	6.81
	-2.34	0.789	1.98	5.17
	12.3	-5.67	-0.678	-1.08
Step 2	0.123	0.345	2.00	6.81
	0	7.35	40.0	135
	0	-40.2	-201	-682
Step 3	0.123	0.345	2.00	6.81
	0	7.35	40.0	135
	0	0	17.8	56.4
	x=	**0.680**	**1.12**	**3.17**

Pivoting				
Step 1	12.3	-5.67	-0.678	-1.08
	-2.34	0.789	1.98	5.17
	0.123	0.345	2.00	6.81
Step 2	12.3	-5.67	-0.678	-1.08
	0	-0.290	1.85	4.96
	0	0.402	2.01	6.82
Step 3	12.3	-5.67	-0.678	-1.08
	0	0.402	2.01	6.82
	0	-0.290	1.85	4.96
Step 4	12.3	-5.67	-0.678	-1.08
	0	0.402	2.01	6.82
	0	0	3.3	9.88
	x=	**1.01**	**2.02**	**2.99**

Gauss Elimination: Exact Soln				
Step 1	0.123	0.345	2.00	6.81
	-2.34	0.789	1.98	5.17
	12.3	-5.67	-0.678	-1.08
Step 2	0.123	0.345	2	6.81
	0	7.352415	40.02878	134.7261
	0	-40.17	-200.678	-682.08
Step 3	0.123	0.345	2	6.81
	0	7.352415	40.02878	134.7261
	0	0	18.01969	53.99755
	x=	**1.003813**	**2.009739**	**2.996586**

Comment: Pivoting improves the solution significantly

Tutorial 4

1. Solve the following system of equations by Gauss Jacobi and Gauss Seidel methods, with $\varepsilon_r \leq 0.1\%$. Use starting guess of (0,0,0) for both the methods.

$$\begin{bmatrix} 9.3746 & 3.0416 & -2.4371 \\ 3.0416 & 6.1832 & 1.2163 \\ -2.4371 & 1.2163 & 8.4429 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9.2333 \\ 8.2049 \\ 3.9339 \end{bmatrix}$$

Solution:

	A		*x*		b
9.3746	**3.0416**	**-2.4371**	x_1	**=**	**9.2333**
3.0416	**6.1832**	**1.2163**	x_2		**8.2049**
-2.4371	**1.2163**	**8.4429**	x_3		**3.9339**

Jacobi

i	x_1	x_2	x_3	ε_r (%)
0	0	0	0	
1	0.984927	1.326967	0.465942	100
2	0.675522	0.750812	0.559082	76.73757
3	0.886669	0.884691	0.552772	23.81358
4	0.841592	0.782066	0.594435	13.12232
5	0.885719	0.796045	0.596207	4.982136
6	0.881645	0.773989	0.606931	2.849612
7	0.891589	0.773884	0.608932	1.115299
8	0.892143	0.768599	0.611818	0.687639
9	0.894608	0.767758	0.612739	0.275531
10	0.89512	0.766365	0.613572	0.181869
11	0.895789	0.765949	0.61392	0.074644

Gauss Seidel

i	x_1	x_2	x_3	ε_r (%)
0	0	0	0	
1	0.984927	0.842467	0.62888	100
2	0.875077	0.772797	0.607208	12.55325
3	0.892047	0.768712	0.612695	1.902424
4	0.894799	0.766279	0.61384	0.317513
5	0.895886	0.765519	0.614263	0.121335
6	0.896243	0.765261	0.614403	0.039787

2. Solve the following equations using (a) fixed-point iteration and (b) Newton-Raphson method, starting with an initial guess of x=1 and y=1 and $\varepsilon_r \leq 0.1\%$.

$$x^2 - x + y - 0.5 = 0$$

$$x^2 - 5xy - y = 0$$

Solution:

Fixed Point Iteration

Note: $x=\phi_1(x,y)=\sqrt{(x-y+0.5)}$ and $y=\phi_2(x,y)=(x^2-y)/5x$

Iteration					ε_r (%)
1	x	1	$\phi_1(x,y)$	0.707107	
	y	1	$\phi_2(x,y)$	-0.141421	
2	x	0.707107	$\phi_1(x,y)$	1.161261	
	y	-0.141421	$\phi_2(x,y)$	0.256609	807.107
3	x	1.161261	$\phi_1(x,y)$	1.18518	
	y	0.256609	$\phi_2(x,y)$	0.193733	155.112
4	x	1.18518	$\phi_1(x,y)$	1.221248	
	y	0.193733	$\phi_2(x,y)$	0.212523	32.455
5	x	1.221248	$\phi_1(x,y)$	1.228302	
	y	0.212523	$\phi_2(x,y)$	0.211056	8.841
6	x	1.228302	$\phi_1(x,y)$	1.231765	
	y	0.211056	$\phi_2(x,y)$	0.212084	0.695
7	x	1.231765	$\phi_1(x,y)$	1.232753	
	y	0.212084	$\phi_2(x,y)$	0.212142	0.485
8	x	1.232753	$\phi_1(x,y)$	1.233131	
	y	0.212142	$\phi_2(x,y)$	0.212219	0.080
9	x	1.233131			
	y	0.212219			

Newton Raphson

Note: $f_1=x^2-x+y-0.5$ and $f_2=x^2-5xy-y$. Derivatives are $f_1(x,y)$: $(2x-1,1)$; $f_2(x,y)$: $(2x-5y, -5x-1)$

Iteration	x	y	f_1	f_2	$f_1'x$	$f_1'y$	$f_2'x$	$f_2'y$
1	1.000000	1.000000	0.500000	-5.000000	1.000000	1.000000	-3.000000	-6.000000
2	1.666667	-0.166667	0.444444	4.333333	2.333333	1.000000	4.166667	-9.333333
3	1.339757	0.151677	0.106870	0.627218	1.679515	1.000000	1.921128	-7.698787
4	1.242124	0.208784	0.009532	0.037410	1.484248	1.000000	1.440328	-7.210621
5	1.233383	0.212226	0.000076	0.000227	1.466766	1.000000	1.405635	-7.166914

xnew	ynew	ε_r (%)
1.666667	-0.166667	700.000
1.339757	0.151677	209.882
1.242124	0.208784	27.352
1.233383	0.212226	1.622
1.233318	0.212245	0.009

3. Consider the following Matrix:

$$\begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix}$$

a) Find the largest eigenvalue and the corresponding eigenvector using the Power method with $\varepsilon_r \leq 0.1\%$. Take the starting z vector as $\{1,0,0\}^T$.

b) Obtain the equation of the characteristic polynomial using Fadeev-Leverrier Method.

c) Perform two iterations of the QR algorithm and compute the approximate eigenvalues of the matrix after this iteration.

Solution:

Part (a)

	A	
7	**-2**	**1**
-2	**10**	**-2**
1	**-2**	**7**

Power Method: Using L_2 norm for normalization (Error computed in λ)

k	Normalized $z^{(k)}$	$Az^{(k)}$	λ	ε_r (%)
0	1	7	7.348469	
	0	-2		
	0	1		
1	0.952579344	7.348469	9.165151	24.72191
	-0.272165527	-4.89898		
	0.136082763	2.44949		
2	0.801783726	6.948792	10.87592	18.66606
	-0.534522484	-7.48331		
	0.267261242	3.741657		
3	0.638915143	6.192562	11.66936	7.295328
	-0.688062462	-8.84652		
	0.344031231	4.423259		
4	0.530668631	5.609926	11.91348	2.092003
	-0.758098044	-9.40042		
	0.379049022	4.700208		
5	0.470888855	5.268864	11.97811	0.542519
	-0.789057	-9.6214		
	0.3945285	4.810702		

6	0.439874292	5.087242	11.99451	0.136901
	-0.803248707	-9.71548		
	0.401624354	4.857742		
7	0.424130777	4.993901	11.99863	0.034305
	-0.809994116	-9.7582		
	0.404997058	4.879098		

<u>Power Method: Using L_∞ norm for normalization</u>

k	Normalized z^k	Az^k	λ	ε_r (%)
0	1	7	7	
	0	-2		
	0	1		
1	1	7.714286	7.714286	10.20408
	-0.285714286	-5.14286		
	0.142857143	2.571429		
2	1	8.666667	9.333333	20.98765
	-0.666666667	-9.33333		
	0.333333333	4.666667		
3	0.928571429	9	12.85714	37.7551
	-1	-12.8571		
	0.5	6.428571		
4	0.7	7.4	12.4	3.555556
	-1	-12.4		
	0.5	6.2		
5	0.596774194	6.677419	12.19355	1.664932
	-1	-12.1935		
	0.5	6.096774		
6	0.547619048	6.333333	12.09524	0.806248
	-1	-12.0952		
	0.5	6.047619		
7	0.523622047	6.165354	12.04724	0.396801
	-1	-12.0472		
	0.5	6.023622		
8	0.511764706	6.082353	12.02353	0.196847
	-1	-12.0235		

	0.5	6.011765		
9	0.505870841	6.041096	12.01174	0.098039
	-1	-12.0117		
	0.5	6.005871		

Part (b)

Faddeev Le Verrier Method- **Characteristic Eqn:** $(-1)(\lambda^3 - a_2\lambda^2 - a_1\lambda^1 - a_0) = 0$

Step 0: $A_2 = A$; $a_2 = \text{trace}(A_2) = 24$ *(where trace of a matrix is sum of its diagonal elements)*
Iteration Step: $A_i = A(A_{i+1} - a_{i+1}I)$ and $a_i = \text{trace}(A_i)/(n-i)$ where i = 1, 0 (n = 3, I is a 3x3 identity matrix)

i	**$(A_{i+1} - a_{i+1}I)$**			**$A_i = A(A_{i+1} - a_{i+1}I)$**			**a_i**
1	-17	-2	1	-114	12	-6	-180
	-2	-14	-2	12	-132	12	
	1	-2	-17	-6	12	-114	
0	66	12	-6	432	0	0	432
	12	48	12	0	432	0	
	-6	12	66	0	0	432	

Characteristic Eqn: $-(\lambda^3 - 24.\lambda^2 + 180.\lambda - 432) = 0$ (**Eigenvalues: 6, 6, 12)**

Part (c) :

Note: After 2 iterations, the values are highlighted. The complete solution is shown here but not needed. The Eigenvalue estimates could be assumed to be on the diagonals of either A or R. Ultimately both will converge to the true values. Here, the errors are computed based on the diagonals of R and the maximum out of the three errors is chosen. However, at the final iteration, the diagonals of A are assumed to be the eigenvalues. Strictly speaking, diagonals of A will not be the eigenvalues till A becomes diagonal or triangular.

A		
7	**-2**	**1**
-2	**10**	**-2**
1	**-2**	**7**

Iteration	**$A_k = Q_k.R_k$ and $A_{k+1} = R_k.Q_k$**			**Q_k**			**R_k**			**ε_r (%)**
1	7.0000	-2.0000	1.0000	0.9526	0.2910	-0.0891	7.3485	-4.8990	2.4495	
	-2.0000	10.0000	-2.0000	-0.2722	0.9456	0.1782	0.0000	9.1652	-2.6186	
	1.0000	-2.0000	7.0000	0.1361	-0.1455	0.9800	0.0000	0.0000	6.4143	

2	8.6667	-2.8508	0.8729	0.9456	0.3168	-0.0741	9.1652	-5.5988	1.7143	19.82
	-2.8508	9.0476	-0.9331	-0.3110	0.9471	0.0792	0.0000	7.7143	-0.9331	-18.81
	0.8729	-0.9331	6.2857	0.0952	-0.0518	0.9941	0.0000	0.0000	6.1101	-4.98
										19.82
3	10.5714	-2.4884	0.5819	0.9720	0.2299	-0.0487	10.8759	-4.1183	0.9631	15.73
	-2.4884	7.3545	-0.3168	-0.2288	0.9731	0.0265	0.0000	6.5894	-0.2633	-17.07
	0.5819	-0.3168	6.0741	0.0535	-0.0146	0.9985	0.0000	0.0000	6.0280	-1.36
										17.07
4	11.5652	-1.5217	0.3225	0.9911	0.1306	-0.0269	11.6694	-2.3473	0.4975	6.80
	-1.5217	6.4161	-0.0882	-0.1304	0.9914	0.0074	0.0000	6.1628	-0.0681	-6.92
	0.3225	-0.0882	6.0187	0.0276	-0.0038	0.9996	0.0000	0.0000	6.0070	-0.35
										6.92
5	11.8851	-0.8055	0.1660	0.9976	0.0676	-0.0138	11.9135	-1.2171	0.2508	2.05
	-0.8055	6.1103	-0.0227	-0.0676	0.9977	0.0019	0.0000	6.0418	-0.0172	-2.00
	0.1660	-0.0227	6.0047	0.0139	-0.0010	0.9999	0.0000	0.0000	6.0018	-0.09
										2.05
6	11.9708	-0.4088	0.0836	0.9994	0.0341	-0.0070	11.9781	-0.6143	0.1257	0.54
	-0.4088	6.0280	-0.0057	-0.0341	0.9994	0.0005	0.0000	6.0105	-0.0043	-0.52
	0.0836	-0.0057	6.0012	0.0070	-0.0002	1.0000	0.0000	0.0000	6.0004	-0.02
										0.54
7	11.9927	-0.2051	0.0419	0.9998	0.0171	-0.0035	11.9945	-0.3079	0.0629	0.14
	-0.2051	6.0070	-0.0014	-0.0171	0.9999	0.0001	0.0000	6.0026	-0.0011	-0.13
	0.0419	-0.0014	6.0003	0.0035	-0.0001	1.0000	0.0000	0.0000	6.0001	-0.01
										0.14
8	11.9982	-0.1027	0.0210	1.0000	0.0086	-0.0017	11.9986	-0.1540	0.0314	0.03
	-0.1027	6.0018	-0.0004	-0.0086	1.0000	0.0000	0.0000	6.0007	-0.0003	-0.03
	0.0210	-0.0004	6.0001	0.0017	0.0000	1.0000	0.0000	0.0000	6.0000	0.00
										0.03
9	**11.9995**	-0.0513	0.0105							
	-0.0513	**6.0004**	-0.0001							
	0.0105	-0.0001	**6.0000**							

(**Eigenvalues: 6.0000, 6.0004, 11.9995)**

Tutorial 5

1. Approximate the function, $f(t) = e^t$ in the interval [−1, 3].
   (a) Use a second-degree polynomial using both the conventional form of polynomials and the Legendre polynomials. Then, use a third-degree polynomial and comment on the additional computational effort required in both the methods.
   (b) Obtain the second-degree Tchebycheff fit and compare the error with second-degree Legendre fit.

The Legendre Polynomials are: $P_0(x)=1$; $P_1(x) = x$; $P_2(x) = (-1+3x^2)/2$; $P_3(x) = (-3x+5x^3)/2$

The Tchebycheff Polynomials are: $T_0(x)=1$; $T_1(x) = x$; $T_2(x) = -1+2x^2$

Following integrals are useful:

$$\int xe^{2x+1}dx = e^{2x+1}(2x-1)/4; \int x^2e^{2x+1}dx = e^{2x+1}(2x^2-2x+1)/4;$$

$$\int x^3e^x dx = e^{2x+1}(4x^3-6x^2+6x-3)/8$$

$$\int_{-1}^{1}\frac{e^{2x+1}}{\sqrt{1-x^2}}dx = 19.4671; \int_{-1}^{1}\frac{xe^{2x+1}}{\sqrt{1-x^2}}dx = 13.5836; \int_{-1}^{1}\frac{x^2e^{2x+1}}{\sqrt{1-x^2}}dx = 12.6752$$

Solution:

(a)

Conventional Method:

Second-degree polynomial: $f_2(t) = c_0 + c_1\,t + c_2t^2$

$$\begin{bmatrix} \int_{-1}^{3}dt & \int_{-1}^{3}tdt & \int_{-1}^{3}t^2dt \\ \int_{-1}^{3}tdt & \int_{-1}^{3}t^2dt & \int_{-1}^{3}t^3dt \\ \int_{-1}^{3}t^2dt & \int_{-1}^{3}t^3dt & \int_{-1}^{3}t^4dt \end{bmatrix}\begin{Bmatrix} c_0 \\ c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} \int_{-1}^{3}e^t dt \\ \int_{-1}^{3}te^t dt \\ \int_{-1}^{3}t^2e^t dt \end{Bmatrix} \Rightarrow \begin{bmatrix} 4 & 4 & 28/3 \\ 4 & 28/3 & 20 \\ 28/3 & 41/2 & 244/5 \end{bmatrix}\begin{Bmatrix} c_0 \\ c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 19.7177 \\ 40.9068 \\ 98.5883 \end{Bmatrix}$$

$$f_2(t) = 0.358687 + 0.386269\,t + 1.79334\,t^2$$

Third-degree polynomial: $f_3(t) = c_0 + c_1\,t + c_2t^2 + c_3t^3$

$$\begin{bmatrix} 4 & 4 & 28/3 & 20 \\ 4 & 28/3 & 20 & 244/5 \\ 28/3 & 20 & 244/5 & 364/3 \\ 20 & 244/5 & 364/3 & 2188/7 \end{bmatrix}\begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 19.7177 \\ 40.9068 \\ 98.5883 \\ 246.913 \end{Bmatrix}$$

$$f_3(t) = 1.14751 + 0.724336\,t + 0.103008\,t^2 + 0.563445\,t^3$$

Legendre Polynomials:

Note: $x = (t-1)/2$

Using the integral expressions given in the question, we get

$$b_0 = \int_{-1}^{1} e^{2x+1} dx = 9.85883; b_1 = \int_{-1}^{1} x e^{2x+1} dx = 5.29729; b_2 = \int_{-1}^{1} \frac{(-1+3x^2)}{2} e^{2x+1} dx = 1.91289$$

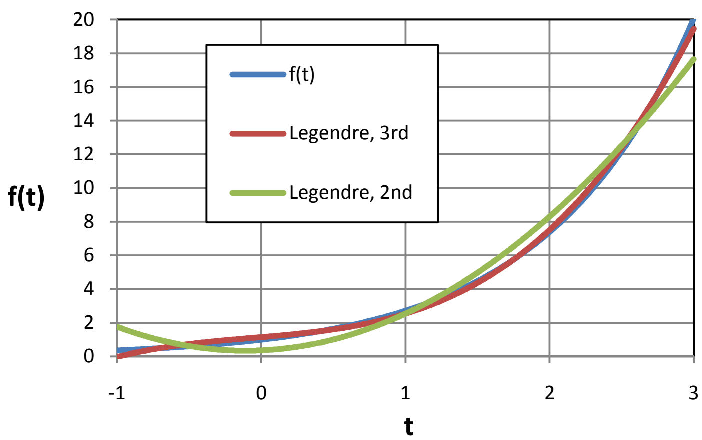
$$b_3 = \int_{-1}^{1} \frac{(-3x+5x^3)}{2} e^{2x+1} dx = 0.515074$$

From which, $c_0 = b_0/2 = 4.92941$; $c_1 = 3b_1/2 = 7.94594$; $c_2 = 5b_2/2 = 4.78222$; $c_3 = 7b_3/2 = 1.80276$

$$f_2(x) = 4.92941 + 7.94594\,x + \frac{4.78222(-1+3x^2)}{2}$$

$$f_3(x) = 4.92941 + 7.94594\,x + \frac{4.78222(-1+3x^2)}{2} + 1.80276(-3x+5x^3)/2$$

Very little additional effort required. Only c_3 needs to be computed. In conventional polynomial, the entire set of linear equations is to be solved again.



(b)

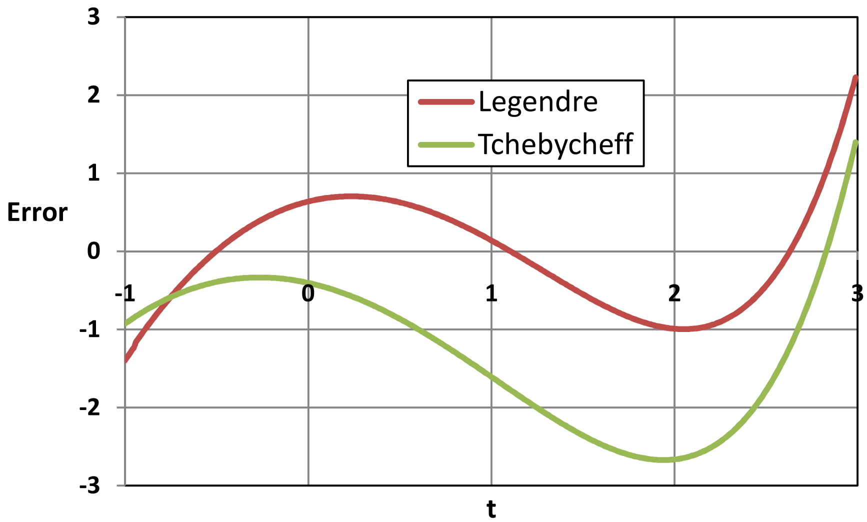
Tchebycheff Polynomials:

Using the integral expressions given in the question, we get ($c_0 = b_0/\pi$; others divided by $\pi/2$)

$$b_0 = \int_{-1}^{1} \frac{e^{2x+1}}{\sqrt{1-x^2}} dx = 19.4671;\ b_1 = \int_{-1}^{1} \frac{xe^{2x+1}}{\sqrt{1-x^2}} dx = 13.5836$$

$$b_2 = \int_{-1}^{1} \frac{(2x^2-1)e^{2x+1}}{\sqrt{1-x^2}} dx = 2 \times 12.6752 - 19.4671 = 5.88330$$

$$f_2(x) = 6.19657 + 8.64759\,x + 3.74543(-1 + 2x^2)$$



Error is reduced near the ends in Tchebycheff method. However, for this case, it is not minimax approximation! It can be shown that Tchebycheff is minimax when approximating a (d+1)-degree polynomial by a d-degree polynomial.

2. Estimate the value of the function at x = 4 from the table of data given below, using, (a) Lagrange interpolating polynomial of 2nd degree using the points x=2,3,5; (b) Newton's interpolating polynomial of 4th degree.

x	***f(x)***
1	1
2	12
3	54
5	375
6	756

Solution:

<u>**Lagrange Polynomials**</u>

Using x_0=2, x_1=3, x_2=5; we get $L_0 = \frac{(x-3)(x-5)}{3}$; $L_1 = \frac{(x-2)(x-5)}{-2}$; $L_2 = \frac{(x-2)(x-3)}{6}$. The values at x=4 are −1/3, 1, and 1/3, respectively. Hence, $f(4)$=12.(−1/3)+54.(1)+375.(1/3)=**175.**

Newton's Divided Difference

x	$f(x)$	$f[x_i,x_j]$	$f[x_i,x_j,x_k]$	…	…
1	1				
		11			
2	12		15.5		
		42		6.0	
3	54		39.5		0.5
		160.5		8.5	
5	375		73.5		
		381			
6	756				

Interpolating polynomial: $1+11(x-1)+15.5(x-1)(x-2)+6(x-1)(x-2)(x-3)+0.5(x-1)(x-2)(x-3)(x-5)$
Therefore, $f(4)=1+33+93+36-3=$**160**

Tutorial 6

1. The mass of a radioactive substance is measured at 2-day intervals till 8 days. Unfortunately, the reading could not be taken at 6 days due to equipment malfunction. The following table shows the other readings:

Time (d)	Mass (g)
0	1.0000
2	0.7937
4	0.6300
8	0.3968

(a) Estimate the mass at 6 days using cubic spline

(b) Using the table and the value obtained in (a), estimate the half-life of the substance using least squares regression after linearising the exponential decay equation.

Solution:

(a) Using natural spline, S''(0)=S''(8)=0. From the tridiagonal system, equations for S''(2) and S''(4) are written as below, and solved to get their values.

$$(x_i - x_{i-1})S''_{i-1} + 2(x_{i+1} - x_{i-1})S''_i + (x_{i+1} - x_i)S''_{i+1}$$
$$= 6\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} - 6\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

		Equations For S" at x=2 and 4			Gauss Elimination:			S" at x=2 and x=4
x	f(x)							
0	1.0000	8	2	0.1278	8	2	0.12780	0.0136
2	0.7937	2	12	0.1413	0	11.5	0.10935	0.00951
4	0.6300							
8	0.3968							

From the spline equation between x_i=4 and x_{i+1}=8:

$$S_i(x) = \frac{(x_{i+1}-x)^3 S_i''(x_i) + (x-x_i)^3 S_i''(x_{i+1})}{6(x_{i+1}-x_i)} + \left[\frac{f(x_i)}{x_{i+1}-x_i} - \frac{(x_{i+1}-x_i)S_i''(x_i)}{6}\right](x_{i+1}-x) + \left[\frac{f(x_{i+1})}{x_{i+1}-x_i} - \frac{(x_{i+1}-x_i)S_i''(x_{i+1})}{6}\right](x-x_i)$$

$f(x=6) = 0.5039$

(b) Decay equation $m = m_0 e^{-\lambda t}$. Linearizing: $\ln(m) = \ln(m_0) - \lambda t$

	x (=t)	m	y=[ln(m)]	x^2	xy
	0	1.0000	0.0000	0	0
	2	0.7937	-0.2310	4	-0.4621
	4	0.6300	-0.4620	16	-1.848
	6	0.5039	-0.6854	36	-4.112
	8	0.3968	-0.9243	64	-7.395
Sum	**20**		**-2.303**	**120**	**-13.82**

$$\begin{bmatrix} \sum_{k=0}^{4} 1 & \sum_{k=0}^{4} x_k \\ \sum_{k=0}^{4} x_k & \sum_{k=0}^{4} x_k^2 \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \end{Bmatrix} = \begin{Bmatrix} \sum_{k=0}^{4} y_k \\ \sum_{k=0}^{4} x_k y_k \end{Bmatrix} \qquad \begin{bmatrix} 5 & 20 \\ 20 & 120 \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \end{Bmatrix} = \begin{Bmatrix} -2.303 \\ -13.82 \end{Bmatrix}$$

Note: Computations are done with more significant digits!

$\lambda = -c_1 = 0.1151$ $\quad c_0 = 3.8 \times 10^{-5}$ $\quad m_0 = e^{c_0} = 1.00004$ $\quad$ Half Life $= \frac{\ln 2}{\lambda} = \mathbf{6.02}$ **days**

2. The velocity of an object, travelling along a straight line, was measured at various times as follows:

Time (min)	0	1	2	3	4	5	6	7	8	9	10
Velocity (cm/min)	0.00	0.65	1.72	3.48	6.39	11.18	19.09	32.12	53.60	89.02	147.41

Estimate the acceleration at 5 minutes using (i) forward difference, O(h^2), with h=1 min, (ii) backward difference, O(h^2), with h=1 min, and (iii) central difference O(h^2) with h=1, 2, and 3 min. Use Richardson extrapolation to obtain an O(h^6) estimate from the three central difference estimates.

Solution:

$$f_i' = \frac{-3f_i + 4f_{i+1} - f_{i+2}}{2h} \qquad f_i' = \frac{3f_i - 4f_{i-1} + f_{i-2}}{2h} \qquad f_i' = \frac{f_{i+1} - f_{i-1}}{2h}$$

t	v	Acceleration					
0	0	Forward	Backward	Central		Richardson	
1	0.65						
2	1.72						
3	3.48						
4	6.39						
5	11.18	5.35	5.73	6.35	(for h=1)	6.08	O(h^4)
6	19.09			7.16	(for h=2)	5.86	O(h^4)
7	32.12			11.05	(for h=4)	6.09	O(h^6)
8	53.6						
9	89.02						
10	147.41						