ESO 208A: Computational Methods in Engineering
Problem Set 7

1. Solve the differential equation $\frac{dy}{dx} = x^2 y - 2y$ with $y(0) = 1$ and $h = 0.1$, over the interval $x \in (0,1)$ using the following methods:

(a) Euler Forward

(b) Euler Backward

(c) Trapezoidal

(d) 2nd Order Runge-Kutta

(e) Solve analytically and compare (i) the solutions graphically with the analytical solution and, (ii) numerically the true relative errors of the above methods at every time step.

2. Consider a *LR*-circuit with a resistance (R) and an inductance (L). A time varying potential of $E(t)$ is applied to the circuit. Application of Kirchoff's law leads to the following differential equation for the current (i):

$$L\frac{di}{dt} + Ri = E(t)$$

Compute the current at every 0.2 hours for 3 hours due to application of a square voltage

$$E(t) = \begin{cases} 0 & t < 0 \\ 24 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

Given are the values of inductance $L = 12\ H$ and the resistance $R = 18\ \Omega$. The initial condition is $i = 0$ at $t = 0$. Solve using 4th order Adam's Moulton, 4^{th} Order Adam's Bashforth and 4^{th} order BDF methods. Compare the solutions graphically with the analytical solution. Use progressively higher order implicit methods for the startup.

3. Dissolved oxygen deficit (y) in a stream due to pollutants is given by the *Streeter-Phelps* equation as follows:

$$u\frac{dy}{dx} + k_a y = k_d L_0 e^{-k_d \frac{x}{u}}$$

where, the stream velocity u = 10 km/day, k_d = 0.5 /day, k_a = 0.2 /day, and p_0 = 12 mg/L. The initial condition $y(0)$ = 3 mg/L. The x is measured along the length of the river. Using 4^{th} order Runge-Kutta method, compute y for a stretch of 10 km by taking h = 0.2 km. Graphically compare the approximate solution with the analytical solution.

4. Solve the differential equation $\frac{dy}{dt} + 100y = 99e^{-t}$ with the initial condition $y(0)=2$ using, (a) Euler forward (explicit) method, and (b) Euler backward (implicit) method, to obtain the values of y at t=1.0. Use time steps of 0.01, 0.02 and 0.025. Find the analytical solution and graphically compare the solutions of both the methods with varying time steps. Observe the behaviour of the errors with time step of the two methods.

5. The amount of lowering of water level s, in a well at a time t, due to pumping from groundwater is governed by an equation of the form $s=A\ W(u)$, where A is a constant (proportional to the discharge), W is called the *Well Function*, and u is inversely proportional to t. The well function is given by the equation $\frac{dW(u)}{du} = -\frac{e^{-u}}{u}$. If the value of $W(1)$ is 0.2194, find the value of $W(0.5)$ using the following methods:

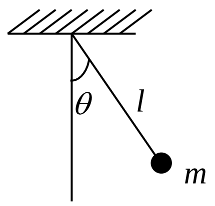
(a) Romberg integration with accuracy $O(h^6)$,

(b) Heun's method with h=-0.25.

(c) Fourth-order Runge-Kutta method with h=-0.25.

(d) Using a series expansion for e^{-u}, integrating term by term, and minimizing the error, an approximate expression for $W(u)$ for $u < 1$ is obtained as: $W(u)= -\ln u - 0.57722 + 0.99999\,u - 0.24991\,u^2 + 0.05519\,u^3 - 0.00976\,u^4 + 0.00108\,u^5$. Assuming that the true value is obtained from this expression, analyze the errors in solutions obtained by three methods above.

6. Consider the pendulum shown below:



Equating the forces, one obtains $ml\frac{d^2\theta}{dt^2} = -mg\sin\theta$. For small angles θ, $\sin\theta \approx \theta$ and the linearized equation of motion is the Newtons equation:

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \quad \text{where, } \omega = \sqrt{\frac{g}{l}} \text{ is the frequency of the pendulum}$$

The acceleration due to gravity is g = 9.81 m/sec^2 and l = 0.6 m. Assume that the pendulum starts from rest with $\theta(0) = 10^\circ$. Solve the linearized equation for $0 \le t \le 2.0$ using a time step h = 0.2 by the following methods:

a) True solution by analytical method.

b) First order explicit and implicit methods (*Euler*)

c) Second Order explicit and implicit methods (2^{nd} order *R-K* and *Trapezoidal*)

d) Fourth Order R-K method.

d) Graphically compare the numerical solutions with the true solution and comment on the stability and, amplitude and phase errors for each method. Would your comments be different if a finer time step (h = 0.05) was used?

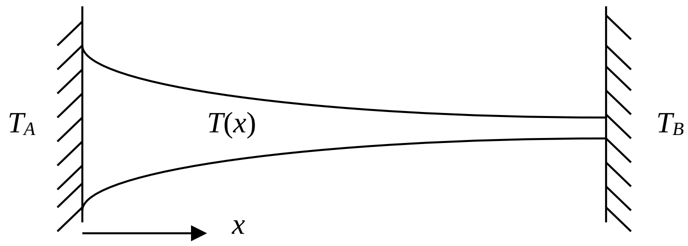
7. (41) Let us consider a LCR circuit where the current is governed by the following equation:

$$L\frac{d^2 i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=-220\sin t$$

Given are the values of inductance L = 12 H, resistance R = 18 Ω and conductance C = 0.75 F. The initial condition is i = 0 and $\frac{di}{dt}=0$ at t = 0. Use 4th order Runge Kutta method with a suitable time step for stability.

8. Solve the differential equation $d^2y/dx^2 - dy/dx - 2y + 2x = 3$ with the boundary conditions $y(0)=0$ and $y(0.5)=0.6967$ using (a) the shooting method with Ralston's method and (b) the direct method. For both cases, use $\Delta x = 0.25$.

9. The diagram shows a body of conical section fabricated from stainless steel immersed in air at zero temperature. It is of circular cross section that varies with x. The large end is located at $x = 0$ and is held at temperature $T_A = 5$. The small end is located at $x = L = 2$ and is insulated (i.e, the temperature gradient is zero).



Conservation of energy can be used to develop a heat balance equation at any cross section of the body. When the body is not insulated along its length and the system is at steady state, its temperature satisfies the following ODE:

$$\frac{d^2T}{dx^2}+a(x)\frac{dT}{dx}+b(x)T = f(x)$$

where, a(x), b(x), and f(x) are functions of the cross-sectional area, heat transfer coefficients, and the heat sinks inside the body. In the present case, they are given by

$$a(x)=-\frac{x+3}{x+1}, \qquad b(x)=\frac{x+3}{(x+1)^2}, \qquad \text{and} \qquad f(x)=2(x+1)+3b(x).$$

a)Discretize the above equation using 2nd order central difference approximation and formulate the set of linear simultaneous equations. Incorporate the boundary conditions such that the accuracy of the scheme is preserved. Use $\Delta x = 0.5$.

b)Solve the system of equations using Thomas Algorithm and draw the temperature profile indicating the values at the nodes.

10. The following scheme has been proposed for solving $y' = \frac{dy}{dt} = f(y)$:

$$y_{n+1} = y_n + \omega_1 k_1 + \omega_2 k_2$$

where, $k_1 = hf(y_n)$, $k_0 = hf(y_n + \beta_0 k_1)$, and $k_2 = hf(y_n + \beta_1 k_0)$, with h being the time step.

(a) Determine the coefficients ω_1, ω_2, β_0 and β_1 that would maximize the order of accuracy of this method.[Hint: Perform all the Taylor series expansions for y_{n+1}, k_0, k_1 and k_2 up to h^3 exactly and represent the residual as $O(h^4)$.]

(b) Applying this method to $y' = \alpha y$, what is the maximum step size h for purely imaginary α ?

(c) Applying this method to $y' = \alpha y$, what is the maximum step size h when α is real and negative?

11. The transport and decay of pollutants in river at steady state is governed by the following 1-*D* differential equation: $D\frac{d^2C}{dx^2} - u\frac{dC}{dx} - kC = 0$, where, C is the concentration of pollutant, u is the velocity of water, D is the dispersion parameter and k is the decay constant of pollutant. An industry discharges a pollutant at the rate of $\frac{dC}{dx} = 0.2$ at $x = 0$. Compute the concentration of that pollutant immediately before the point of discharge, *i.e.*, $C(0)$, such that the concentration at $x = 1$, *i.e.*, $C(1) = 1$. Also compute the concentration of the pollutant at three locations of x= (0.25, 0.50, 0.75). Use shooting method with trapezoidal method. All variables and parameters are dimensionless quantities. Given data, $u = 1$, D=1 and k=0.1. Assume two initial guesses of $C(0)$ as 0 and 1.