Taylor's Series: Truncation Error

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\ldots+\frac{h^m}{m!}f^{[m]}(x_0)+\frac{h^{m+1}}{(m+1)!}f^{[m+1]}(\zeta)$$

$$\zeta\in(x_0, x_0+h)$$

- *E.g.:* Compute $e^{0.1}$ (T.V. = 1.1051709180756480)
- Use x_0=0, h=0.1.
  - First-order approx. (m=1), 1+h = 1.1, Error = 0.00517
    - From Taylor's series, residual = 0.005 e^{ζ} (range 0.005 to 0.0055)
  - Second-order approx. (m=2), 1+h+$h^2/2$ = 1.105, Error = 0.000171
    - From Taylor's series, residual = 0.000167 e^{ζ} (range 0.000167 to 0.000184)

Roots of Nonlinear Equations: Introduction and types of methods

- *f(x)*=0. "A root" is ξ, *i.e., f(ξ)*=0
- *Graphical method*: Plot and see the intersection with x-axis. Not very accurate. Zoom-in improves accuracy, less efficient.
- *Analytical*: E.g., for quadratic, cubic, and quartic equations. Not possible for general cases.
- *Numerical*: Works for most cases. Need to look at accuracy and efficiency.

Broadly classified: Bracketing and Open methods

Bracketing: First bracket the root and then refine the bracket

Open: Start from an initial guess (sometimes, more than one values) and try to move towards ξ.

Roots of Nonlinear Equations: Bracketing methods

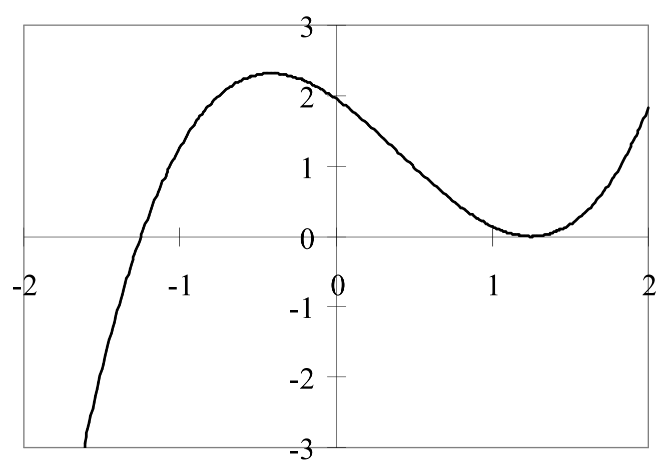
Initial Bracket: By finding a change of sign of *f(x)*

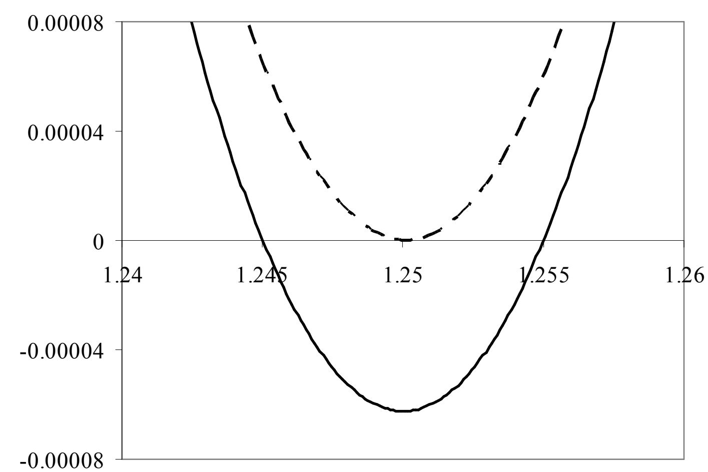
E.g.,

$$(a) \quad x^3 - 1.2500000x^2 - 1.5625250x + 1.9530938 = 0$$

$$(b) \quad x^3 - 1.2502000x^2 - 1.56249999x + 1.95343750 = 0$$

Solid Line: (a) Dashed Line: (b)





Roots of Nonlinear Equations: Bracketing methods

- Search technique is used to bracket the root
- Choice of increment is problem-specific
- Large increment may not be able to bracket a root
- Small increment increases the computational time of bracketing (but will generally reduce the effort in refining)

Once a bracket is obtained, how to refine the bracket? Leads to different methods:

Bisection, False Position, Modified False Position...

We assume that a bracket [x_l, x_u] has been obtained, i.e., $f_l \cdot f_u$ is negative.

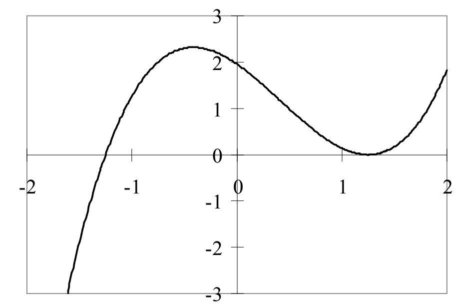
Roots of Nonlinear Equations: Bisection Method

Reduce the bracket to half at each step:

- Find f_m at the midpoint of the interval, i.e., $f_m = f\left(\dfrac{x_l + x_u}{2}\right)$
- The new bracket will be **[x_l, x_m] if $f_l \,.\, f_m$ is negative** or **[x_m, x_u] if $f_u \,.\, f_m$ is negative** (Typically not multiplied, just sign comparison)
- Repeat till desired accuracy is achieved

Example: Find the root with a 1% accuracy

$$x^3 - 1.2500000x^2 - 1.5625250x + 1.9530938 = 0$$



f(x)=−7.922 at x= − 2 and 1.265 at x= − 1

(To bracket the other two roots between 1.2 and 1.3, we need a fine increment of 0.01)

Bisection: Example

Iteration no.	x_l	x_u	f_l	f_u	$x_m = (x_l+x_u)/2$	ε_r (%)	f_m
1	-2	-1	-7.92186	1.265619	-1.5		-1.89062
2	-1.5	-1	-1.89062	1.265619	-1.25	20	5E-08
3	-1.5	-1.25	-1.89062	5E-08	-1.375	9.090909	-0.86132
4	-1.375	-1.25	-0.86132	5E-08	-1.3125	4.761905	-0.4104
5	-1.3125	-1.25	-0.4104	5E-08	-1.28125	2.439024	-0.20022
6	-1.28125	-1.25	-0.20022	5E-08	-1.26563	1.234568	-0.09888
7	-1.26563	-1.25	-0.09888	5E-08	**-1.25781**	0.621118	Not needed

Error: ***Approx. Error (ε) = Current Approx. – Previous Approx.***

Note: The table shows relative error.
Also note the small function value at the second iteration

At any iteration, ***ε = Half of bracket length (Always > True Error)***
Indicates that the error reduces by a factor of 2 at each iteration: Called ***"Linear Convergence"***

Roots of Nonlinear Equations: Order of Convergence

- Denote by $x^{(i)}$, the estimate of the root at the i^{th} iteration
- If an iterative sequence $x^{(0)}$, $x^{(1)}$, $x^{(2)}$ … converges to the root ξ, (which means the true error at iteration *i* is $e^{(i)} = \xi - x^{(i)}$) and if

$$\lim_{i \to \infty} \frac{\left|e^{(i+1)}\right|}{\left|e^{(i)}\right|^{p}} = C$$

  then *p* is the **order of convergence** and *C* the **asymptotic error constant**. The convergence is called linear if *p*=1, quadratic if *p*=2, and cubic if *p*=3 (*p* does not have to be an integer).
- For bisection method, using approx. error as a proxy for true error (as we approach the root, these tend to be the same)

$$\frac{\left|e^{(i+1)}\right|}{\left|e^{(i)}\right|} = \frac{1}{2}$$

  Implying linear convergence and error constant of 0.5

Roots of Nonlinear Equations: Order of Convergence

- **For linearly convergent methods**: *C* must be less than 1
- **For superlinearly convergent methods (*p*>1):** Not necessary
- Convergent methods: Approximate Error at any iteration is representative of True Error at the previous iteration

$$x^{(i+1)} - x^{(i)} \cong \xi - x^{(i)}$$

- For the bisection method, the approximate error at the n^th^ iteration is

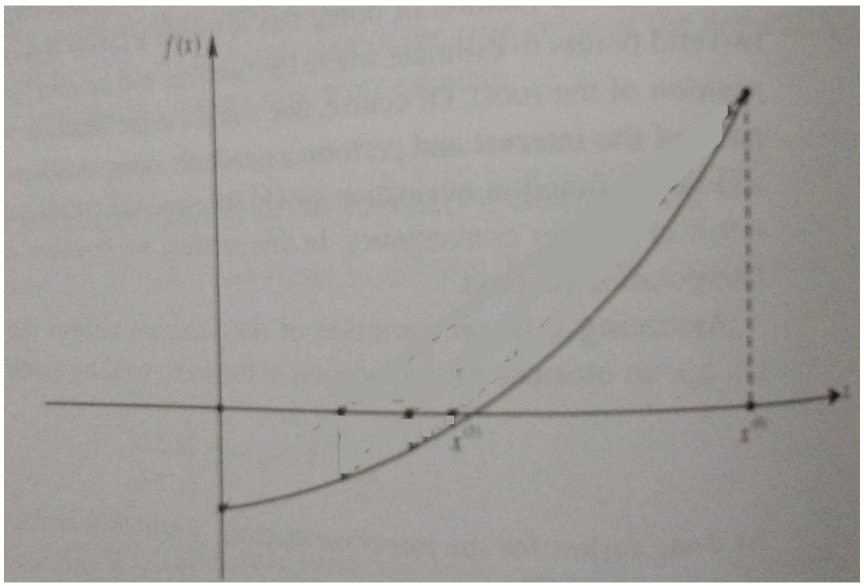
$$\frac{\Delta x^{(0)}}{2^n}$$

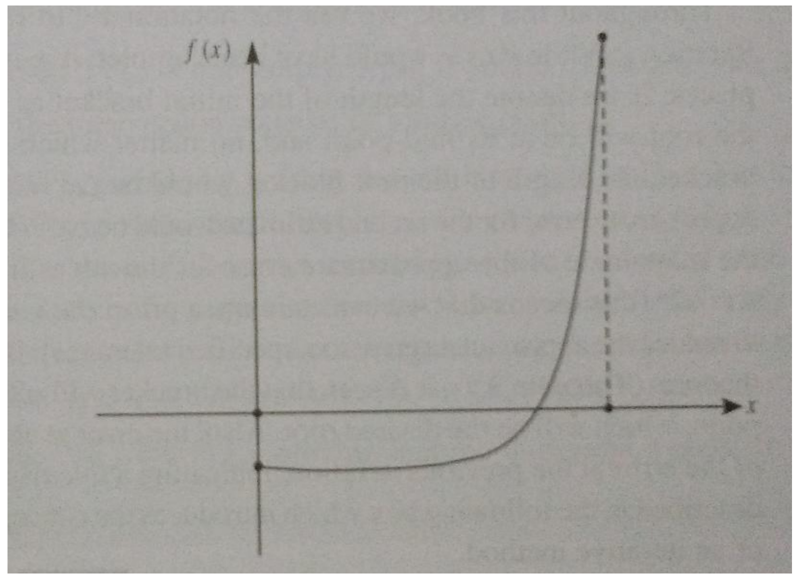
which means the number of iterations for a desired accuracy are known *a priori*.

Bisection Method: Comments

• Magnitude of *f(x)* is given no consideration in shrinking the bracket

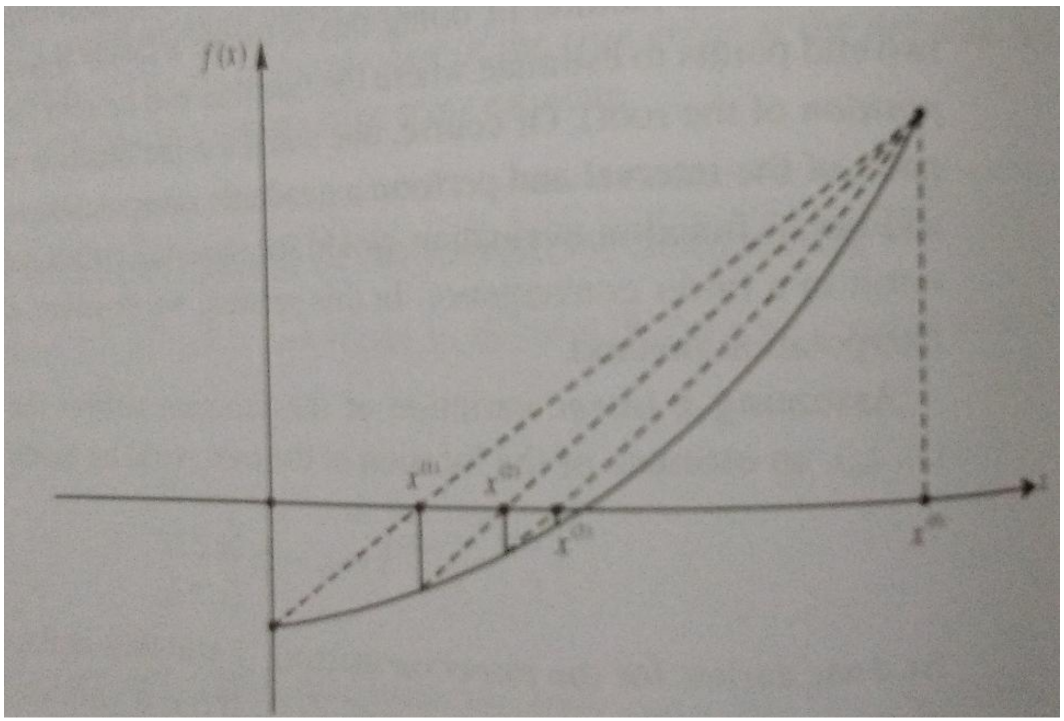
•Root SHOULD be closer to the end where the function magnitude is smaller





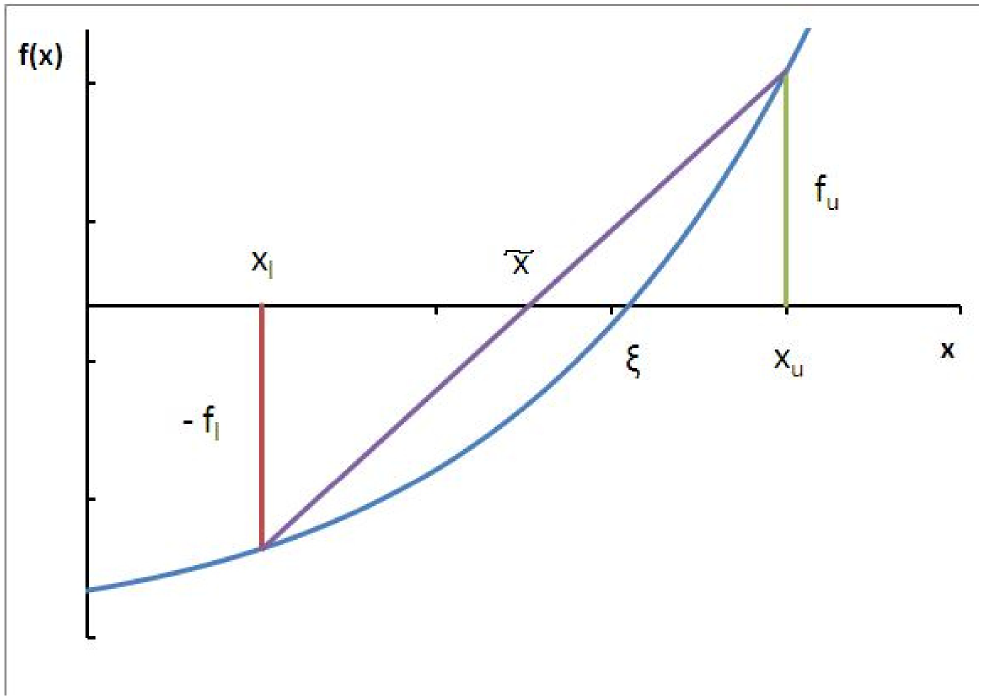
Linear Interpolation, False Position, or Regula Falsi Method

- Giving weight to f_l and f_u in choosing the bracket at the next iteration may work better
- The most common method: a **linear interpolation** between the two end points to estimate where the function will be zero (which indicates a **false position** of the root, Latin *Regula Falsi*)



Linear Interpolation: Algorithm

- Using similar triangles, denoting the "false position" as $\tilde{x}$,



$$\frac{\tilde{x}-x_l}{-f_l}=\frac{x_u-\tilde{x}}{f_u} \Rightarrow \tilde{x}=x_l-f_l\frac{x_u-x_l}{f_u-f_l}=x_u-f_u\frac{x_u-x_l}{f_u-f_l}$$

- Again, $\tilde{x}$ will replace either x_l or x_u.
- Same Example

Linear Interpolation: Example

Iteration no.	x_l	x_u	f_l	f_u	$\tilde{x}$	ε_r (%)	$\tilde{f}$
1	-2	-1	-7.92186	1.265619	-1.13775		0.639949
2	-2	-1.13775	-7.92186	0.639949	-1.2022	5.360842	0.287417
3	-2	-1.2022	-7.92186	0.287417	-1.23013	2.270632	0.122192
4	-2	-1.23013	-7.92186	0.122192	**-1.24183**	0.941715	-

Error reduces by a factor of about 0.4 at both steps, indicating a possibly *linear convergence*

Linear Interpolation: Error Analysis

- Similar to Taylor's series, equation for interpolation of data by a polynomial (details later in "Approximation")
- For linear interpolation between two points, x_l and x_u, it is written as

$$f(x) = f(x_u) + (x - x_u)\frac{f_u - f_l}{x_u - x_l} + (x - x_u)(x - x_l)\frac{f''(\zeta)}{2}; x, \zeta \in (x_l, x_u)$$

Taylor's: $$f(x) = f(x_u) + (x - x_u) f'(x_u) + \frac{(x - x_u)^2}{2!} f''(\zeta); \zeta \in (x, x_u)$$

- If we assume the function to be uniformly concave/convex, one end of the interval, x_u(or x_l), remains fixed, say, $x^{(0)}$, and

$$x^{(i+1)} = x^{(0)} - f_0 \frac{x^{(0)} - x^{(i)}}{f_0 - f_i}$$