ESO 208A: Computational Methods in Engineering
Problem Set 6

1. If cubic splines [$q(x)$] are fitted through (M+1) points $\{x_0, x_1, x_2 \ldots\ldots\ldots x_M\}$, one obtains the following for $i = 1, 2\ldots\ldots$ (M-1),

$$h_{i+1}k_{i-1} + 2(h_{i+1} + h_i)k_i + h_i k_{i+1} = 3(h_i g_{i+1} + h_{i+1} g_i)$$

where, k_i is the derivative at x_i, $h_i = x_i - x_{i-1}$, $g_i = (y_i - y_{i-1})/h_i$, and y_i is the functional value at x_i. The above gives (M-1) equations for (M+1) unknown k_i's. Two other equations are needed to fully determine the system. Assume the following:

- ✓ Functional values are known at the mid-points of two end intervals, i.e., at points $x_{1/2} = \frac{1}{2}(x_0 + x_1)$ and $x_{M-1/2} = \frac{1}{2}(x_{M-1} + x_M)$. Denote these values as $y_{1/2}$ and $y_{M-1/2}$, respectively.

- ✓ The triple derivative $q'''(x)$ is continuous at these two points $x_{1/2}$ and $x_{M-1/2}$

Using the above assumptions, derive the other two equations involving k_i's. Please note that number of unknown k_i's do not increase, *i.e.*, $k_{1/2}$ and $k_{M-1/2}$ do not appear in the equation.

2a). Derive finite difference approximations for f_j' and f_j'' in terms of f_j, f_{j+1} and f_{j+2} using Taylor's series. What are the orders of accuracies of these approximations?

b). Fit a piecewise Lagrange Polynomial through a set of N+1 equispaced (regular grid) discrete points by taking three points at a time. The grid points are denoted as $x_0, x_1, x_2, \ldots. x_n$ and the corresponding functional values as $f_0, f_1, f_2, \ldots. f_n$. Consider any three consecutive grid points x_i, x_{i+1} and x_{i+2} where corresponding functional values are f_i, f_{i+1} and f_{i+2}, respectively. Write the expression for the Lagrange Polynomial $\hat{f}(x)$ through these three points. Take the derivative of this polynomial to obtain the expressions for $\hat{f}_i'$ and $\hat{f}_i''$ at point x_i. Compute the order of the errors for these expressions using the residual of polynomials. Compare the expressions and the orders of the truncation error with the results of (a).

3. Consider the function $f(x) = \dfrac{Sinx}{x^3}$

(a) Obtain finite difference approximations of f' with first order backward difference, second order central difference and 4th order central difference. Evaluate f' by the three methods at 20 equally spaced points in the interval $[1,2\pi]$. Also evaluate the true value of f' at the same points. Plot f' vs. x and graphically compare the true values with the three approximations you have obtained, all in the same plot. Show them by different styles of lines.

(b) Start with $h = 1$ and do repeated interval halving for 10 times. For each h value, obtain the approximate derivative at $x = 4$. Also calculate the true derivative at $x = 4$. Now, compute the absolute error for each h-value. Plot ln[error] vs. ln[h] and obtain the slope of the line. Repeat this procedure for each of the three methods mentioned in (a). What are the slopes of these lines? Do they relate to the orders of truncation errors of these methods?

4. The following table is given for the functional values of e^x :

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
e^x	1.0000	1.2840	1.6487	2.1170	2.7183	3.4903	4.4817	5.7546	7.3891

a) Compute $\left.\dfrac{de^x}{dx}\right|_{x=1}$ using central difference scheme with h = 0.25, 0.50 and 1.00.

b) Using the values computed in (a), obtain most accurate estimate for the derivative by successive application of Richardson's extrapolation.

c) Compute absolute values of the true relative error for each computed value of the derivative.

5. The location of an object at various times was measured as follows:

Time (min)	0	1	2	3	4	5	6	7	8	9
Distance (cm)	0	3	14	39	84	155	258	399	584	819

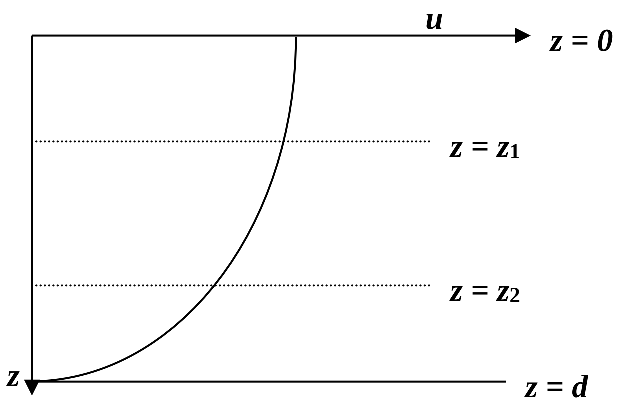
Estimate the speed and acceleration of the object at 5 minutes using (a) Forward difference, O(h^2); (b) Backward difference, O(h^2); (c) Central difference, O(h^2); and (d) Richardson extrapolation, O(h^6) using three central differences of O(h^2). (e) Perform an error analysis for the results obtained in the above problem, using the fact that the distance is given by $x = t + t^2 + t^3$.

6. Solve the integral $\int_1^8 \frac{\log x}{x}dx$ numerically using 5 points in the interval by,

a) Trapezoidal rule, b) Simpson's rule, c) Gaussian Quadrature, d) compute the %error in each of the three cases.

7. The flow rate through a circular pipe is given by $Q = \int_0^{r_0} 2\pi r\, v\, dr$, where v is the velocity at a distance of r from the centre of pipe and r_0 is the radius of the pipe. If the velocity is approximated by $v = 2\left(1 - \frac{r}{r_0}\right)^{1/7}$ (in m/s), and the pipe radius is 12 cm, compute Q using (a) trapezoidal rule with h = 2 cm; (b) Simpson's one-third rule with h = 3 cm; (c) Simpson's three-eighth rule with h = 4 cm; and (d) 3-point Gauss-Legendre quadrature. (e) Perform an error analysis for the results obtained in the above problem, using the true value of the flow rate as 0.0738902 m^3/s.

8. Velocity profile in an open channel flow is shown below.



The depth of the channel is d. The velocity at any depth is given by an arbitrary function $u(z)$, graphically shown above. The mean velocity of the channel $\bar{u}$ is given by $\bar{u} = \frac{1}{d}\int_0^d u(z)dz$.

a) Find two constants c_1 and c_2 such that $z_1 = c_1 d$, $z_2 = c_2 d$, and $\bar{u} = \frac{u(z_1) + u(z_2)}{2}$.

b) The velocities u_0, u_1, u_2 and u_3 have been measured at depths of $0.2d$, $0.4d$, $0.6d$ and $0.8d$, respectively. Compute ω_0, ω_1, ω_2 and ω_3 such that the velocity at $0.5d$ can be expressed as $(\omega_0 u_0 + \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3)$.

9. Show that the trapezoidal rule, with $h = \frac{2\pi}{n+1}$ is exact for **trigonometric polynomials** of period 2π, *i.e.*, for functions of the form $\sum_{k=-n}^{n} c_k e^{ikt}$; k integer, when it is used for integration over a whole period.

10. A common problem is that of solving the Fredholm integral equation :

$$f(x) = \phi(x) + \int_a^b K(x,t)\phi(t)dt$$

where the function $f(x)$ and $K(x,t)$ are given and the problem is to obtain $\phi(x)$. Given $f(x) = \pi x^2$, $K(x,t) = -3(0.5 \sin 3x - tx^2)$, $a = 0$ and $b = \pi$.

a) Obtain $\phi(x)$ at $x = 0$, $\pi/2$ and π and compare with the true solution given by $\phi(x) = \sin 3x$. Comment on the amount of error with reasoning. Use 3-point Simpson's rule approximation for the integral evaluation.

b) The above equation becomes the Volterra integral equation when $b = x$. For the same $f(x)$ and $K(x,t)$ and a as given above for the Fredholm equation, compute $\phi(x)$ at $x = 0$, $\pi/2$ and π for the Volterra equation. Use Trapezoidal rule for the evaluation of the integral with $h = \pi/2$.