Finding All Eigenvalues

- Directly from the Characteristic Equation: How to efficiently obtain the Characteristic Polynomial – Faddeev-Le Verrier
- Using similarity transformation: Reduce to diagonal or triangular form – QR decomposition
- The characteristic equation, det (A- λI)=0, may be written as

$$(-1)^n\left(\lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \ldots - a_1\lambda - a_0\right) = 0$$

- It may be seen that $a_{n-1} = \sum_{i=1,\ldots,n} a_{ii}$, i.e., trace(A)
- and $a_0 = \left(-1\right)^{n+1} \det(\mathrm{A})$

Finding All Eigenvalues

- Fedeev-Leverrier came up with an algorithm for obtaining all the coefficients of the polynomial.
- Set $A_{n-1}=A$ and, as seen, $a_{n-1}=trace(A_{n-1})$
- For i=n-2, n-3, …,1,0:
  - $A_i=A(A_{i+1}-a_{i+1} I)$; $a_i=trace(A_i)/(n-i)$
- Solve using any of the methods discussed earlier to get all the eigenvalues.
- Another side-benefit is that we get the inverse of A as $A^{-1}=[\, A_1-a_1 I \,]/a_0$

Faddeev-Le Verrier method: Example

- $A_i = A(A_{i+1} - a_{i+1} I)$; $a_i = \mathrm{trace}(A_i)/(n-i)$ $A^{-1} = [\,A_1 - a_1 I\,]/a_0$
- Example: $A = \begin{bmatrix} 3 & 4 & 1 \\ 3 & 5 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ $A_2 = \begin{bmatrix} 3 & 4 & 1 \\ 3 & 5 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ $a_2=9$;

  $A_1 = \begin{bmatrix} -4 & -2 & -1 \\ -1 & -6 & 0 \\ -4 & 2 & -4 \end{bmatrix}$ $a_1=-7$; $A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $a_0=1$

- Characteristic polynomial is

  $(-1)\,(\lambda^3 - 9\,\lambda^2 + 7\,\lambda - 1)$; Roots: 8.16, 0.66, 0.19

- Inverse is $\begin{bmatrix} 3 & -2 & -1 \\ -1 & 1 & 0 \\ -4 & 2 & 3 \end{bmatrix}$

Finding All Eigenvalues : Similarity Transform

- If $A=S^{-1}BS$, the eigenvalues of B will be same as those of A
- $S^{-1}BS\ x = \lambda x \Rightarrow By = \lambda y$, where $y=Sx \Rightarrow \det(B-\lambda I)=0$
- A and B are said to be similar

  (Note: Eigenvectors are NOT same –> x for A and Sx for B)
- If we could perform the similarity transformation in such a way that B is diagonal or triangular, the diagonal elements will give us the eigenvalues!

Finding All Eigenvalues : QR Method

- One of the methods is the QR method, in which Q is an **orthogonal matrix** and R is upper triangular. Since Q is orthogonal, its inverse is equal to its transpose.
- An iterative method is followed to achieve the transformation. Assumption: A has *n* linearly independent eigenvectors.
- Orthogonal matrix is generated by using the *Gram-Schmidt orthogonalization* technique

Orthogonalization: Gram-Schmidt Method

- If there are *n* linearly independent vectors, say, x_i, we can generate a set of *n* orthonormal vectors, say, y_i.
- Take the first orthogonal unit vector, y_1, in the the direction of any one of these, say, x_1.
- To generate the second unit vector, y_2, which is orthogonal to y_1, project x_2 on y_1 and take

$$y_2 = \frac{x_2 - \left(x_2^T y_1\right) y_1}{\left\| x_2 - \left(x_2^T y_1\right) y_1 \right\|}$$

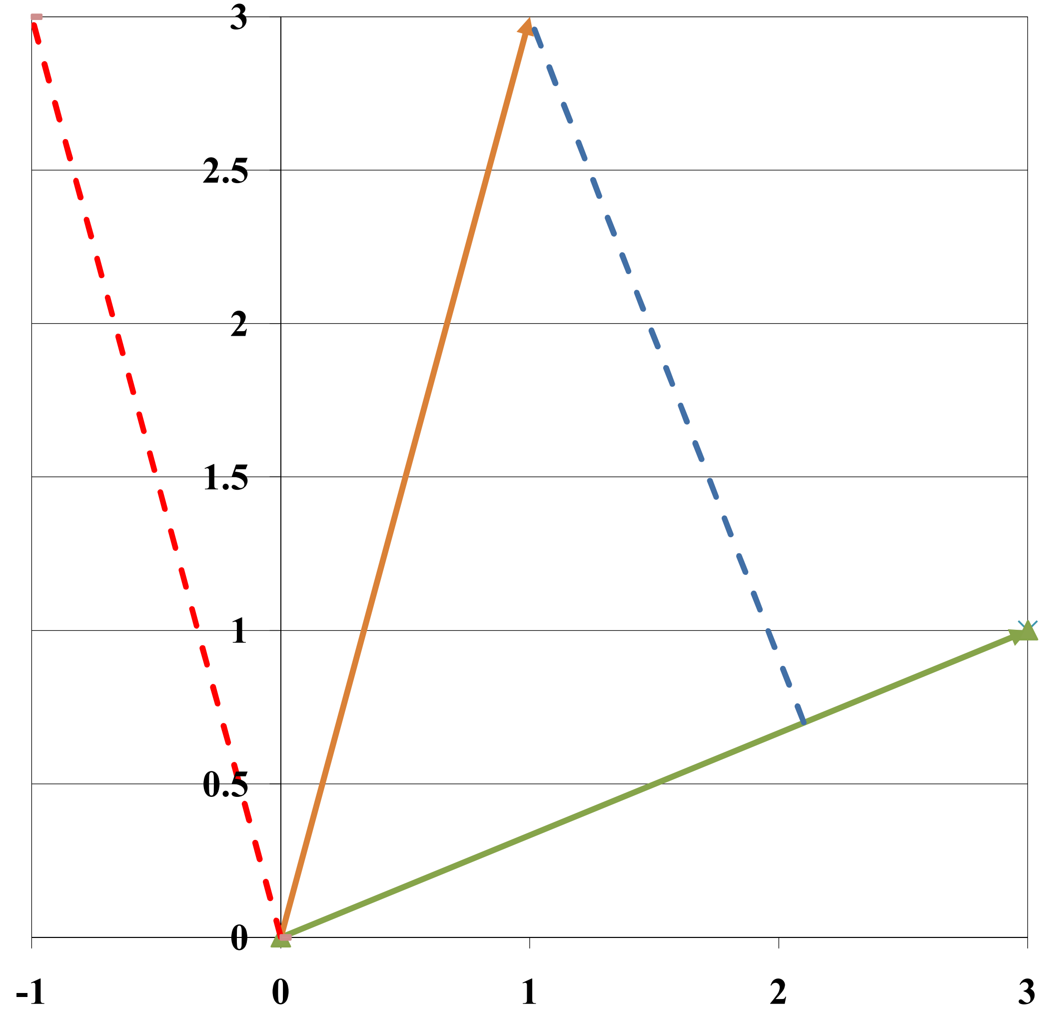
Take $x_1 = \begin{Bmatrix} 3 \\ 1 \end{Bmatrix}; x_2 = \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$

$$y_1 = \frac{1}{\sqrt{10}} \begin{Bmatrix} 3 \\ 1 \end{Bmatrix}$$

$$x_2^T y_1 = \frac{6}{\sqrt{10}}$$

$$x_2 - \left(x_2^T y_1\right) y_1 = \begin{Bmatrix} -4/5 \\ 12/5 \end{Bmatrix}$$

$$y_2 = \frac{1}{\sqrt{10}} \begin{Bmatrix} -1 \\ 3 \end{Bmatrix}$$



$$y_2 = \frac{x_2 - \left(x_2^T y_1\right) y_1}{\left\| x_2 - \left(x_2^T y_1\right) y_1 \right\|}$$

Orthogonalization: Gram-Schmidt Method

- It is easy to show that y_2 is orthogonal to y_1:

$$\left(x_2 - \left(x_2^T y_1\right) y_1\right)^T y_1 = x_2^T y_1 - \left(x_2^T y_1\right) y_1^T y_1 = 0$$

- Similar philosophy is used to generate the other vectors of the orthogonal set.
- The orthogonality may be proved by showing that if the y_i are orthogonal up to i=k, y_{k+1} is orthogonal to all y_i from i=1 to k. And we have already shown it to be true for k=2 (in fact, k=1 will work!)

Orthogonalization: Gram-Schmidt Method

- To generate the third unit vector, y_3, which is orthogonal to both y_1 and y_2,

$$y_3 = \frac{x_3 - \left(x_3^T y_1\right) y_1 - \left(x_3^T y_2\right) y_2}{\left\| x_3 - \left(x_3^T y_1\right) y_1 - \left(x_3^T y_2\right) y_2 \right\|}$$

- The general equation is

$$y_{k+1} = \frac{x_{k+1} - \sum_{i=1}^{k} \left(x_{k+1}^T y_i\right) y_i}{\left\| x_{k+1} - \sum_{i=1}^{k} \left(x_{k+1}^T y_i\right) y_i \right\|}$$

Finding All Eigenvalues : QR method

- We generate an orthogonal matrix Q
- We know that if $A=S^{-1}BS$, the eigenvalues of B will be same as those of A
- Also, if Q is orthogonal, its transpose is its inverse
- If $A=Q^TBQ$ for some Q and B, and if B is diagonal or triangular, we get the eigenvalues
- For example, since a symmetric matrix has orthogonal eigenvectors, we could construct Q by using the eigenvectors as its columns

Finding All Eigenvalues : QR method

$$Q = \left[\{x_1\} \quad \{x_2\} \quad . \quad . \quad \{x_n\} \right]$$

$$AQ = \left[\lambda_1 \{x_1\} \quad \lambda_2 \{x_2\} \quad . \quad . \quad \lambda_n \{x_n\} \right] = QD$$

where D is a diagonal matrix with eigenvalues on the diagonal

Consequently, A=QDQ^T

Since we don't know the eigenvectors, how to construct Q to obtain the triangular form of B? (It is easier to achieve a triangular B than a diagonal B!)

Finding All Eigenvalues : QR method

- An iterative technique is used
- The iterative sequence is written as:
  - A_0=A
  - For k=0,1,2,….till convergence
  - ➢ $A_k=Q_kR_k$: Perform a QR decomposition of A
  - ➢ $A_{k+1}=Q_k^T A_k Q_k = R_k Q_k$

Finding All Eigenvalues : QR method

- The QR decomposition of A is obtained by using the Gram-Schmidt orthogonalization with columns of A as the x vectors. $A=QR \Rightarrow R=Q^TA$. When would R be upper triangular?
  - 2^{nd} col of Q is orthogonal to 1^{st} col of A: $r_{21}=0$
  - 3^{rd} col of Q is orthogonal to 1^{st} and 2^{nd} columns of A: $r_{31}=r_{32}=0$. And so on.
- If we take the first col of Q as a unit vector along the first column vector of A, Q will be orthogonal
  - Take the first column of Q as a unit vector in the same direction as the first column of A: $\{q\}_1=\{a\}_1/|\{a\}_1|$
  - Follow the orthogonalization procedure described earlier, using subsequent columns of A

QR method: Example

- $A_k = Q_k R_k$; $R_k = Q_k^T A_k$; $A_{k+1} = R_k Q_k$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}; Q = \begin{bmatrix} 0.81650 & -0.49237 & -0.30151 \\ 0.40825 & 0.86164 & -0.30151 \\ 0.40825 & 0.12309 & 0.90453 \end{bmatrix}; R = \begin{bmatrix} 2.44949 & 2.04124 & 2.04124 \\ 0 & 1.35401 & 0.61546 \\ 0 & 0 & 1.20605 \end{bmatrix}$$

$$A = RQ = \begin{bmatrix} 3.6667 & 0.8040 & 0.4924 \\ 0.8040 & 1.2424 & 0.1485 \\ 0.4924 & 0.1485 & 1.0909 \end{bmatrix}; Q = \begin{bmatrix} 0.9685 & -0.2155 & -0.1249 \\ 0.2124 & 0.9765 & -0.0377 \\ 0.1301 & 0.0099 & 0.9915 \end{bmatrix}; R = \begin{bmatrix} 3.7859 & 1.0619 & 0.6503 \\ 0 & 1.0414 & 0.0497 \\ 0 & 0 & 1.0145 \end{bmatrix}$$

$$A = RQ = \begin{bmatrix} 3.977 & 0.2276 & 0.1319 \\ 0.2276 & 1.0174 & 0.0101 \\ 0.1319 & 0.0101 & 1.0058 \end{bmatrix}$$

- After 3 iterations A is nearly diagonal

Q

$$\begin{matrix} 0.9999 & -0.0143 & -0.0083 \\ 0.0143 & 0.9999 & -0.0002 \\ 0.0083 & 0.0000 & 1.0000 \end{matrix}$$

R

$$\begin{matrix} 3.9991 & 0.0717 & 0.0414 \\ 0.0000 & 1.0002 & 0.0002 \\ 0.0000 & 0.0000 & 1.0001 \end{matrix}$$

A

$$\begin{matrix} 3.9999 & 0.0144 & 0.0083 \\ 0.0144 & 1.0001 & 0.0000 \\ 0.0083 & 0.0000 & 1.0000 \end{matrix}$$

- Eigenvalues of 4,1, and 1

Finding Eigenvectors

- Once the eigenvalues are obtained, use

$$[A-\lambda I]\{x\}=\{0\}$$

  to solve for $\{x\}$
- Note that a unique solution does not exist
- It will only give the direction of the vector
- Example:

$$A=\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Eigenvalues: 2 and 4

  - For 2, $x_1+x_2=0$ => Eigenvector is $\{1,-1\}^T$
  - For 4, $-x_1+x_2=0$ => Eigenvector is $\{1,1\}^T$