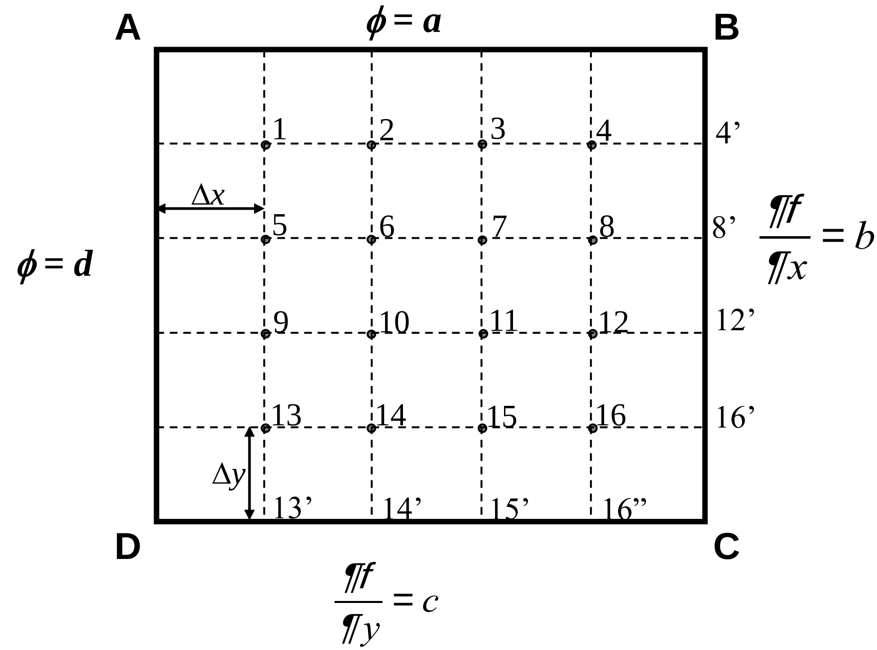
Backward Difference



Number of equations will remain at 16 and the size of the matrix ***A*** is 16×16

For Node 8, the 8th equation is:

$$\left(\frac{1}{\Delta y^2}\right)\phi_4 + \left(\frac{1}{\Delta x^2}\right)\phi_7 + \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)\phi_8 + \left(\frac{1}{\Delta x^2}\right)\phi_{8'} + \left(\frac{1}{\Delta y^2}\right)\phi_{12} = 0$$

$$\frac{\phi_7 - 4\phi_8 + 3\phi_{8'}}{2\Delta x} = b \quad \text{or} \quad \left(\frac{1}{2\Delta x}\right)\phi_7 + \left(-\frac{2}{\Delta x}\right)\phi_8 + \left(\frac{3}{2\Delta x}\right)\phi_{8'} = b$$

After obtaining the solutions for the 16 interior nodes, the values of phi at the boundary nodes are to be computed from the BC equations used for substitution!

Ghost Node

Number of equations is now 25 and the size of the matrix $\boldsymbol{A}$ is 25×25

For Node 5:

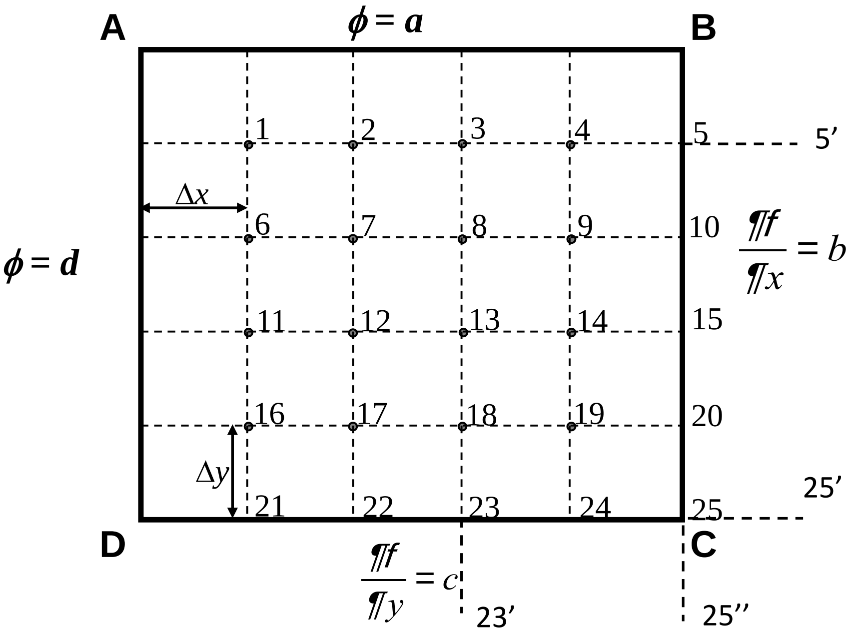
$$\left(\frac{1}{\Delta y^2}\right)a+\left(\frac{1}{\Delta x^2}\right)\phi_4+\left(-\frac{2}{\Delta x^2}-\frac{2}{\Delta y^2}\right)\phi_5+\left(\frac{1}{\Delta x^2}\right)\phi_{5'}+\left(\frac{1}{\Delta y^2}\right)\phi_{10}=0$$

For Node 23:

$$\left(\frac{1}{\Delta y^2}\right)\phi_{18}+\left(\frac{1}{\Delta x^2}\right)\phi_{22}+\left(-\frac{2}{\Delta x^2}-\frac{2}{\Delta y^2}\right)\phi_{23}+\left(\frac{1}{\Delta x^2}\right)\phi_{24}+\left(\frac{1}{\Delta y^2}\right)\phi_{23'}=0$$

$$\frac{\phi_{5'}-\phi_4}{2\Delta x}=b$$

$$\frac{\phi_{23'}-\phi_{18}}{2\Delta y}=c$$



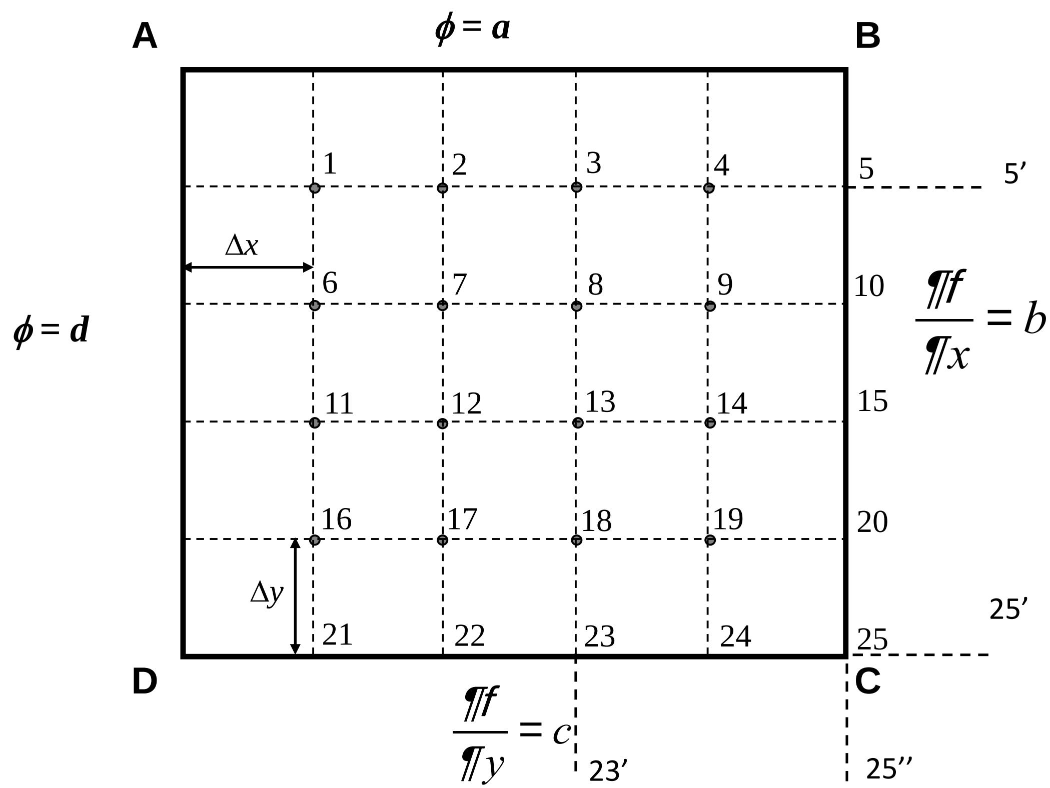
Ghost Node

Number of equations is now 25 and the size of the matrix $\boldsymbol{A}$ is 25×25
For Node 25:

$$\left(\frac{1}{\Delta y^2}\right)\phi_{20}+\left(\frac{1}{\Delta x^2}\right)\phi_{24}+\left(-\frac{2}{\Delta x^2}-\frac{2}{\Delta y^2}\right)\phi_{25}+\left(\frac{1}{\Delta x^2}\right)\phi_{25'}+\left(\frac{1}{\Delta y^2}\right)\phi_{25''}=0$$

$$\frac{\phi_{25'}-\phi_{24}}{2\Delta x}=b$$

$$\frac{\phi_{25''}-\phi_{20}}{2\Delta y}=c$$



ESO 208A: Computational Methods in Engineering

Partial Differential Equation: Hyperbolic Equation

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Wave Equation: 1st Order

$$\frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} = 0 \qquad x \in (-\infty, \infty), \qquad t > 0$$

$$\phi(x_0, t) = a(t), \qquad \phi(x, t_0) = b(x)$$

μ-CD scheme:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -\mu u_i^n \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} - (1-\mu)u_i^{n+1}\frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2\Delta x}$$

$$(1-\mu)u_i^{n+1}\frac{\Delta t}{2\Delta x}\phi_{i+1}^{n+1} + \phi_i^{n+1} - (1-\mu)u_i^{n+1}\frac{\Delta t}{2\Delta x}\phi_{i-1}^{n+1} = -\mu u_i^n\frac{\Delta t}{2\Delta x}\phi_{i+1}^n + \phi_i^n + \mu u_i^n\frac{\Delta t}{2\Delta x}\phi_{i-1}^n$$

If u is constant:

$$(1-\mu)\frac{C}{2}\phi_{i+1}^{n+1} + \phi_i^{n+1} - (1-\mu)\frac{C}{2}\phi_{i-1}^{n+1} = -\mu\frac{C}{2}\phi_{i+1}^n + \phi_i^n + \mu\frac{C}{2}\phi_{i-1}^n$$

Consistency

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0 \qquad x \in (-\infty, \infty), \qquad t > 0$$

$$\phi(x_0, t) = a(t), \qquad \phi(x, t_0) = b(x)$$

$$(1-\mu)u_i^{n+1}\frac{\Delta t}{2\Delta x}\phi_{i+1}^{n+1} + \phi_i^{n+1} - (1-\mu)u_i^{n+1}\frac{\Delta t}{2\Delta x}\phi_{i-1}^{n+1} = -\mu u_i^n \frac{\Delta t}{2\Delta x}\phi_{i+1}^n + \phi_i^n + \mu u_i^n \frac{\Delta t}{2\Delta x}\phi_{i-1}^n$$

$$\left.\frac{\partial \phi}{\partial t}\right|_i^n + u\left.\frac{\partial \phi}{\partial x}\right|_i^n = \boxed{-\left(\mu - \tfrac{1}{2}\right)u^2\Delta t \left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^n} + \left(\tfrac{\mu}{2} - \tfrac{1}{3}\right)u^3 \Delta t^2 \left.\frac{\partial^3 \phi}{\partial x^3}\right|_i^n - \frac{\Delta x^2}{6}\left.\frac{\partial^3 \phi}{\partial x^3}\right|_i^n + HOT$$

For $\mu \neq ½$:

- ✓ The method is $O(\Delta t, \Delta x^2)$
- ✓ Numerical Diffusion is present

For $\mu = ½$:

- ✓ The method is $O(\Delta t^2, \Delta x^2)$
- ✓ Numerical Diffusion is absent

Stability

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0$$

$$x \in (-\infty, \infty), \qquad t > 0$$

$$\phi(x_0, t) = a(t), \qquad \phi(x, t_0) = b(x)$$

$$\frac{d\phi_j}{dt} = -u\frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x}$$

$$\frac{\mathrm{d}\bar{\phi}}{\mathrm{dt}} = \mathbf{A}\bar{\phi} + \mathbf{b}$$

$$\mathbf{A} = -\frac{u}{2\Delta x}\mathbf{B}\left[-1,\ 0,\ 1\right]$$

$$\lambda_k = -i\frac{u}{\Delta x}\cos\frac{k\pi}{m} \qquad k=1,\ 2,\ \cdots\ m-1$$

For large *m*, largest eigenvalue (absolute) is:

$$\lambda_{m-1} \approx i\frac{u}{\Delta x}$$

Stability

- ✓ All eigenvalues are imaginary!
- ✓ Numerical time-stepping schemes, that are not stable for purely imaginary λ, cannot be applied for this problem with central difference approximation for spatial derivative
- ✓ Euler Forward ($\mu = 1$), multi-step methods up to 3rd order, 2nd Order Runge-Kutta method cannot be used
- ✓ Euler Backward ($\mu = 0$), Trapezoidal method, all BDFs and 4th order Runge-Kutta can be used.
- ✓ Some may have limitations on the time step, e.g., for 4th order R-K

$$\lambda_{m-1} \approx i\frac{u}{\Delta x} \qquad\qquad \left|\lambda_{n-1}\Delta t\right| \le 2.83 \text{ or } C \le 2.83$$

1st Order Wave Equation: Explicit Scheme

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0 \qquad x \in (-\infty, \infty), \qquad t > 0$$

$$\phi(x_0, t) = a(t), \qquad \phi(x, t_0) = b(x)$$

$$\left.\frac{\partial \phi}{\partial t}\right|_i^n + u\left.\frac{\partial \phi}{\partial x}\right|_i^n = -\left(\mu - \tfrac{1}{2}\right)u^2\Delta t\left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^n + \left(\tfrac{\mu}{2} - \tfrac{1}{3}\right)u^3\Delta t^2\left.\frac{\partial^3 \phi}{\partial x^3}\right|_i^n - \frac{\Delta x^2}{6}\left.\frac{\partial^3 \phi}{\partial x^3}\right|_i^n + HOT$$

The problem of numerical diffusion and stability with Euler Forward can be addressed by simulating the numerical diffusion term (Lax-Wendorf Scheme):

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -u\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} + \frac{u^2\Delta t}{2}\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}$$

$$\phi_i^{n+1} = \phi_i^n - \frac{C}{2}\left(\phi_{i+1}^n - \phi_{i-1}^n\right) + \frac{C^2}{2}\left(\phi_{i+1}^n - \phi_i^n + \phi_{i-1}^n\right)$$

1st Order Wave Equation: Upwind scheme

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0 \qquad x \in (-\infty, \infty), \qquad t > 0$$

$$\phi(x_0, t) = a(t), \qquad \phi(x, t_0) = b(x)$$

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + u_i^n \frac{\phi_i^n - \phi_{i-1}^n}{\Delta x} = 0 \qquad \phi_i^{n+1} = \phi_i^n - C\left(\phi_i^n - \phi_{i-1}^n\right)$$

Performing TE analysis:

$$\left.\frac{\partial \phi}{\partial t}\right|_i^n + u\left.\frac{\partial \phi}{\partial x}\right|_i^n = \left(\frac{u\Delta x}{2} - \frac{u^2\Delta t}{2}\right)\left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^n + \left(\frac{u\Delta x^2}{6} + \frac{u^3\Delta t^2}{6}\right)\left.\frac{\partial^3 \phi}{\partial x^3}\right|_i^n + HOT$$

Numerical Diffusion is eliminated by choosing $C = 1$.
The scheme is stable for

$$C \leq 1$$

1st Order Wave Equation: Upwind scheme

$$\frac{\partial \phi}{\partial t} - u\frac{\partial \phi}{\partial x} = 0$$

$$x \in (-\infty, \infty), \qquad t > 0$$
$$\phi(x_0, t) = a(t), \qquad \phi(x, t_0) = b(x)$$

$$\frac{\phi_i^{n+1} - \phi_i^{n}}{\Delta t} - u_i^n \frac{\phi_{i+1}^{n} - \phi_i^{n}}{\Delta x} = 0$$

1st Order Wave Equation Example: Upwind scheme

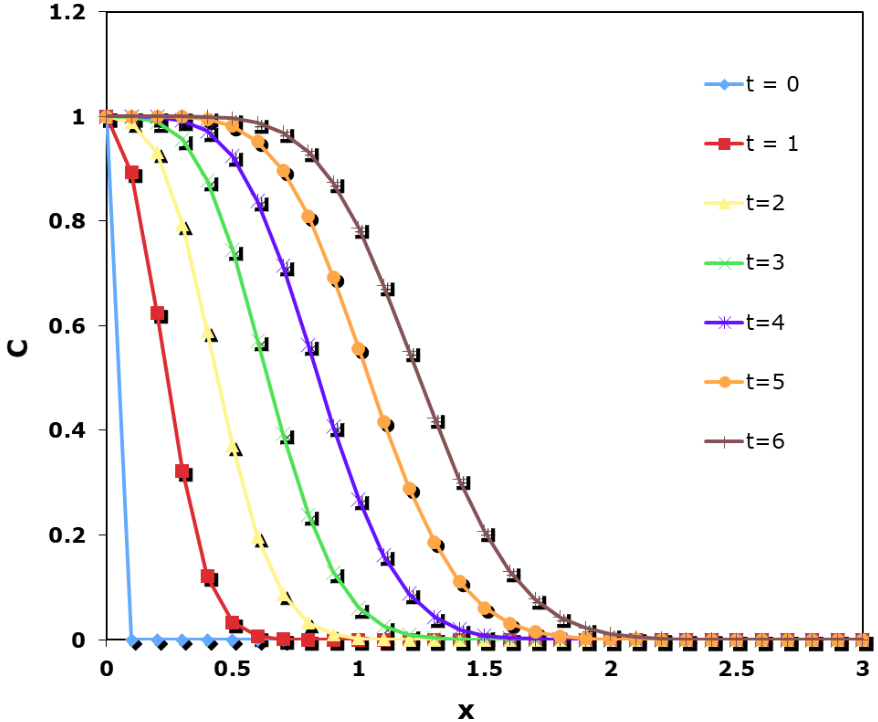
$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = 0 \qquad 0 \le x < \infty \qquad u = 0.2; \;\; C(0,t) = 1; \;\; C(x,0) = 0$$

$$C_i^{n+1} = C_i^n - \tfrac{u\Delta t}{\Delta x}\left(C_i^n - C_{i-1}^n\right)$$

$$\Delta t = 0.1;\; \Delta x = 0.1 \qquad \Longrightarrow \qquad C = \frac{u\Delta t}{\Delta x} = 0.2$$

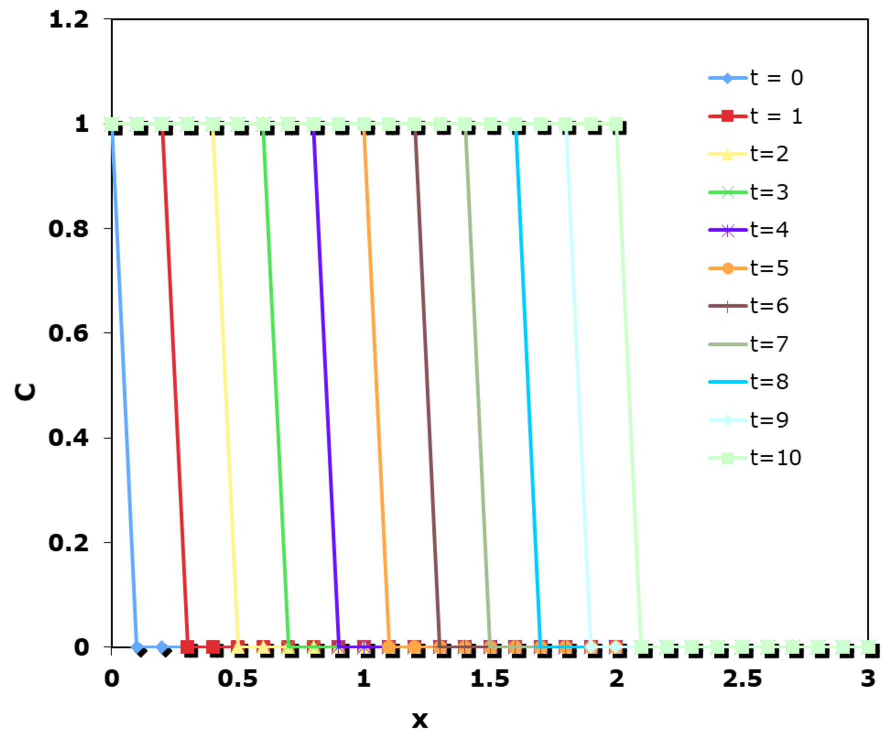
$$\Delta t = 0.5;\; \Delta x = 0.1 \qquad \Longrightarrow \qquad C = \frac{u\Delta t}{\Delta x} = 1.0$$

Let us see the solutions!



$$\Delta t = 0.1;\ \Delta x = 0.1$$
$$C = \frac{u\Delta t}{\Delta x} = 0.2$$
Numerical Diffusion Present, Stable

$$\Delta t = 0.5;\ \Delta x = 0.1$$
$$C = \frac{u\Delta t}{\Delta x} = 1.0$$
No Numerical Diffusion, Stable



2^{nd} Order Wave Equation: an implicit scheme

$$\frac{\partial^2 \phi}{\partial t^2} = u(x)^2 \frac{\partial^2 \phi}{\partial x^2}$$

$x \in (-\infty, \infty), \qquad t > 0$

$\phi(x_0, t) = a(t), \qquad \phi(x_1, t) = b(t) \qquad \phi(x,0) = a$ and $\left.\frac{\partial \phi}{\partial t}\right|_{(x,0)} = b$

$$\frac{\phi_j^{n+1} - 2\phi_j^{n} + \phi_j^{n-1}}{\Delta t^2} = u_j \left[\frac{1}{4} \frac{\phi_{j+1}^{n+1} - 2\phi_j^{n+1} + \phi_{j-1}^{n+1}}{\Delta x^2} + \frac{1}{2} \frac{\phi_{j+1}^{n} - 2\phi_j^{n} + \phi_{j-1}^{n}}{\Delta x^2} + \frac{1}{4} \frac{\phi_{j+1}^{n-1} - 2\phi_j^{n-1} + \phi_{j-1}^{n-1}}{\Delta x^2} \right]$$

2-D Unsteady Diffusion Equation

$$\frac{\P f}{\P t} = a\left(\frac{\P^2 f}{\P x^2} + \frac{\P^2 f}{\P y^2}\right)$$

$$\frac{df_{i,j}}{dt} = a\left(\frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{Dx^2} + \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{Dy^2}\right)$$

$$\frac{d\overline{f}}{dt} = \mathbf{A}\overline{f} + \mathbf{b}$$

Euler Forward:

$$\overline{f}^{n+1} = \overline{f}^{n} + Dt\left[\mathbf{A}\overline{f}^{n} + \mathbf{b}\right]$$

$$\frac{f^{n+1}_{i,j} - f^{n}_{i,j}}{Dt} = a\left(\frac{f^{n}_{i+1,j} - 2f^{n}_{i,j} + f^{n}_{i-1,j}}{Dx^2} + \frac{f^{n}_{i,j+1} - 2f^{n}_{i,j} + f^{n}_{i,j-1}}{Dy^2}\right)$$

2-D Unsteady Diffusion Equation

Euler Forward:

$$\frac{d\bar{f}}{dt} = \mathbf{A}\bar{f} + \mathbf{b} \qquad\qquad \bar{f}^{n+1} = \bar{f}^{n} + Dt\left[\mathbf{A}\bar{f}^{n} + \mathbf{b}\right]$$

$$\frac{f_{i,j}^{n+1} - f_{i,j}^{n}}{Dt} = a\left(\frac{f_{i+1,j}^{n} - 2f_{i,j}^{n} + f_{i-1,j}^{n}}{Dx^2} + \frac{f_{i,j+1}^{n} - 2f_{i,j}^{n} + f_{i,j-1}^{n}}{Dy^2}\right)$$

$$f_{i,j}^{n+1} = f_{i,j}^{n} + a\tfrac{Dt}{Dy^2}f_{i,j-1}^{n} + a\tfrac{Dt}{Dx^2}f_{i-1,j}^{n} - 2\left(a\tfrac{Dt}{Dx^2} + a\tfrac{Dt}{Dy^2}\right)f_{i,j}^{n} + a\tfrac{Dt}{Dx^2}f_{i+1,j}^{n} + a\tfrac{Dt}{Dy^2}f_{i,j+1}^{n}$$

Euler Backward: $\left[\mathbf{I} - Dt\mathbf{A}\right]\bar{f}^{n+1} = \bar{f}^{n} + Dt\mathbf{b}$

$$\frac{f_{i,j}^{n+1} - f_{i,j}^{n}}{Dt} = a\left(\frac{f_{i+1,j}^{n+1} - 2f_{i,j}^{n+1} + f_{i-1,j}^{n+1}}{Dx^2} + \frac{f_{i,j+1}^{n+1} - 2f_{i,j}^{n+1} + f_{i,j-1}^{n+1}}{Dy^2}\right)$$

$$-a\tfrac{Dt}{Dy^2}f_{i,j-1}^{n+1} - a\tfrac{Dt}{Dx^2}f_{i-1,j}^{n+1} + \left[1 + 2\left(a\tfrac{Dt}{Dx^2} + a\tfrac{Dt}{Dy^2}\right)\right]f_{i,j}^{n+1} - a\tfrac{Dt}{Dx^2}f_{i+1,j}^{n+1} - a\tfrac{Dt}{Dy^2}f_{i,j+1}^{n+1} = f_{i,j}^{n}$$

Need to solved by Gauss-Seidel at every time step!

2-D Unsteady Diffusion Equation: ADI

Alternating Direction Implicit or ADI Scheme:

$$\frac{\partial \phi}{\partial t} = \alpha\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)$$

1st Half Time-Step:

$$\phi_{i,j}^{n+\frac{1}{2}} = \phi_{i,j}^{n} + \alpha\frac{\Delta t}{2\Delta x^2}\left(\phi_{i+1,j}^{n+\frac{1}{2}} - 2\phi_{i,j}^{n+\frac{1}{2}} + \phi_{i-1,j}^{n+\frac{1}{2}}\right) + \alpha\frac{\Delta t}{2\Delta y^2}\left(\phi_{i,j+1}^{n} - 2\phi_{i,j}^{n} + \phi_{i,j-1}^{n}\right)$$

$$-\alpha\frac{\Delta t}{2\Delta x^2}\phi_{i+1,j}^{n+\frac{1}{2}} + \left(1+\alpha\frac{\Delta t}{\Delta x^2}\right)\phi_{i,j}^{n+\frac{1}{2}} - \alpha\frac{\Delta t}{2\Delta x^2}\phi_{i-1,j}^{n+\frac{1}{2}} = \alpha\frac{\Delta t}{2\Delta y^2}\phi_{i,j+1}^{n} + \left(1-\alpha\frac{\Delta t}{\Delta y^2}\right)\phi_{i,j}^{n} + \alpha\frac{\Delta t}{2\Delta y^2}\phi_{i,j-1}^{n}$$

2nd Half Time-Step:

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n+\frac{1}{2}} + \alpha\frac{\Delta t}{2\Delta x^2}\left(\phi_{i+1,j}^{n+\frac{1}{2}} - 2\phi_{i,j}^{n+\frac{1}{2}} + \phi_{i-1,j}^{n+\frac{1}{2}}\right) + \alpha\frac{\Delta t}{2\Delta y^2}\left(\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}\right)$$

$$-\alpha\frac{\Delta t}{2\Delta y^2}\phi_{i,j+1}^{n+1} + \left(1+\alpha\frac{\Delta t}{\Delta y^2}\right)\phi_{i,j}^{n+\frac{1}{2}} - \alpha\frac{\Delta t}{2\Delta y^2}\phi_{i,j-1}^{n+\frac{1}{2}} = \alpha\frac{\Delta t}{2\Delta x^2}\phi_{i+1,j}^{n+\frac{1}{2}} + \left(1-\alpha\frac{\Delta t}{\Delta x^2}\right)\phi_{i,j}^{n+\frac{1}{2}} + \alpha\frac{\Delta t}{2\Delta x^2}\phi_{i-1,j}^{n+\frac{1}{2}}$$