Partial Differential Equations: Numerical Solution

- Should be consistent with the physics of the problem. E.g., value of the dependent variable at a point should not depend on a point outside its domain of dependence
- Finite difference approximations of the derivatives are used
- We will use full-discretization, use a spatial grid with spacing of Δx (Δy, etc.), and temporal grid, with spacing Δt

Laplace Equation

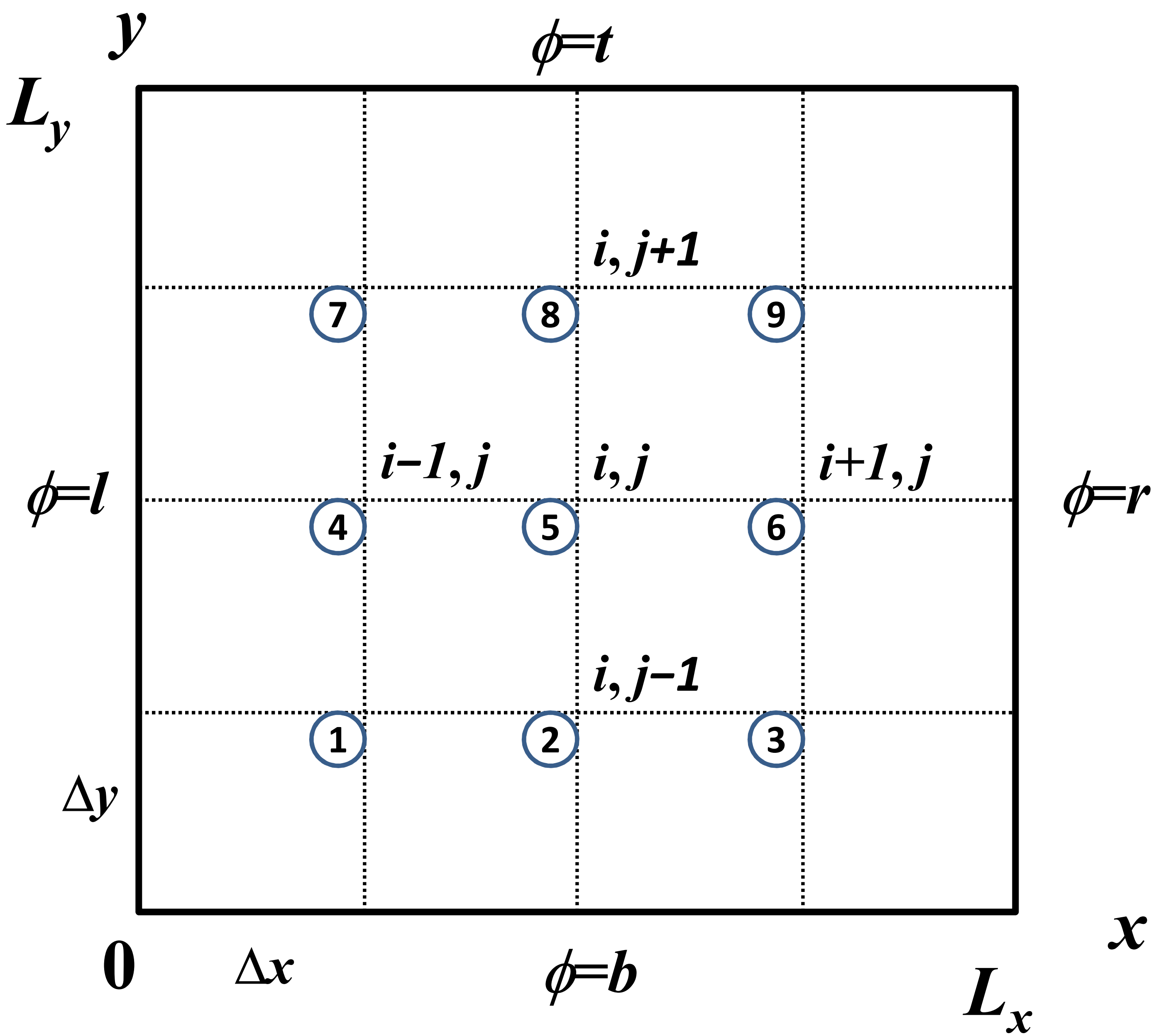
- Most commonly encountered
- Steady-state temperature, potential, stress
- We consider only 2-D, homogeneous, isotropic case

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

- For a general case, $K_x(x,y)$ and $K_y(x,y)$ need to be used in the two terms
- The objective is to find the value of ϕ at all points within the solution domain

Laplace Equation

- Rectangular domain, lengths L_x and L_y
- Elliptic equation: No characteristic lines, need boundary conditions on ALL boundaries
- Assume Dirichlet B.C. on all boundaries, $\phi(0,y)= l$; $\phi(L_x,y)= r$; $\phi(x,0)= b$; $\phi(x,L_y)= t$; at the left, right, bottom, and top
- Use a uniform grid, spacing Δx and Δy, which may not be equal
- How to find ϕ at each "grid point (node)"



Although there are 25 nodes, only 9 unknowns, shown by circles
(Also note the discontinuity at the corner nodes)

Laplace Equation

- At node (i,j), the discretized form is

$$\frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{\Delta x^2} + \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{\Delta y^2} = 0$$

- To simplify presentation, assume $\Delta x = \Delta y$
- The grid-point equation becomes

$$\phi_{i-1,j} + \phi_{i,j-1} - 4\phi_{i,j} + \phi_{i+1,j} + \phi_{i,j+1} = 0$$

- Applicable at all nodes, but needs modifications at nodes next to boundary

Laplace Equation: Applying B.C.

- Node 5

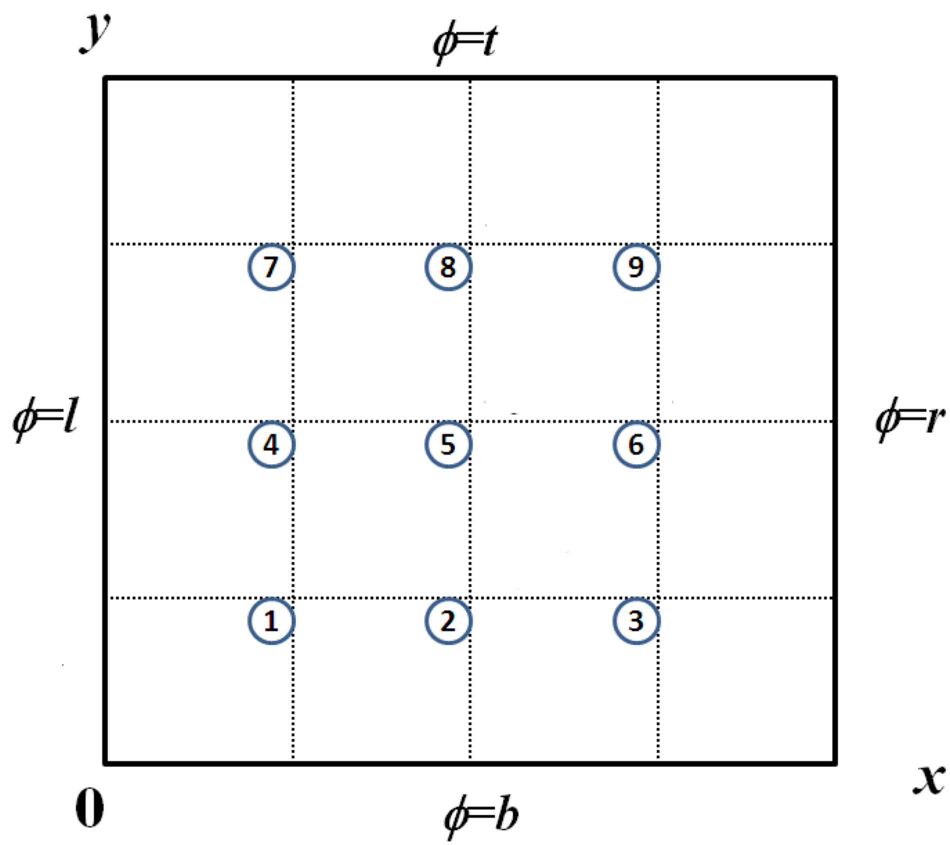
$$\phi_2 + \phi_4 - 4\phi_5 + \phi_6 + \phi_8 = 0$$

- Node 2

$$b + \phi_1 - 4\phi_2 + \phi_3 + \phi_5 = 0$$

- Node 1

$$b + l - 4\phi_1 + \phi_2 + \phi_4 = 0$$



Laplace Equation

- We get nine linear equations in nine unknowns. Five non-zero diagonals, but two are "away from" the main diagonal

$$
\begin{bmatrix}
-4 & 1 & 0 & 1 & & & & & \\
1 & -4 & 1 & 0 & 1 & & & & \\
0 & 1 & -4 & 0 & 0 & 1 & & & \\
1 & 0 & 0 & -4 & 1 & 0 & 1 & & \\
 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & \\
 & & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
 & & & 1 & 0 & 0 & -4 & 1 & 0 \\
 & & & & 1 & 0 & 1 & -4 & 1 \\
 & & & & & 1 & 0 & 1 & -4
\end{bmatrix}
\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{Bmatrix}
=
\begin{Bmatrix} -b-l \\ -b \\ -b-r \\ -l \\ 0 \\ -r \\ -l-t \\ -t \\ -r-t \end{Bmatrix}
$$

Laplace Equation

- The system of equations could be solved by using direct methods
- For sparse matrices, iterative solution is more efficient, specially for large systems
- Let us consider the bottom boundary at zero potential and the other three at 100
- The RHS vector is

$$\{-100,0,-100,-100,0,-100,-200,-100,-200\}^T$$

- And the solution is

$$\{57.1,47.3,57.1,81.3,75.0,81.3,92.9,90.2,92.9\}^T$$

Laplace Equation: Iterative solution

- The Gauss-Seidel iterations are written as

$$\phi_{i,j} = \frac{1}{4}\left(\phi_{i-1,j} + \phi_{i,j-1} + \phi_{i+1,j} + \phi_{i,j+1}\right)$$

- With appropriate values for the nodes near the boundaries
- Use the starting guess as 100 at all nodes
- In the first iteration:

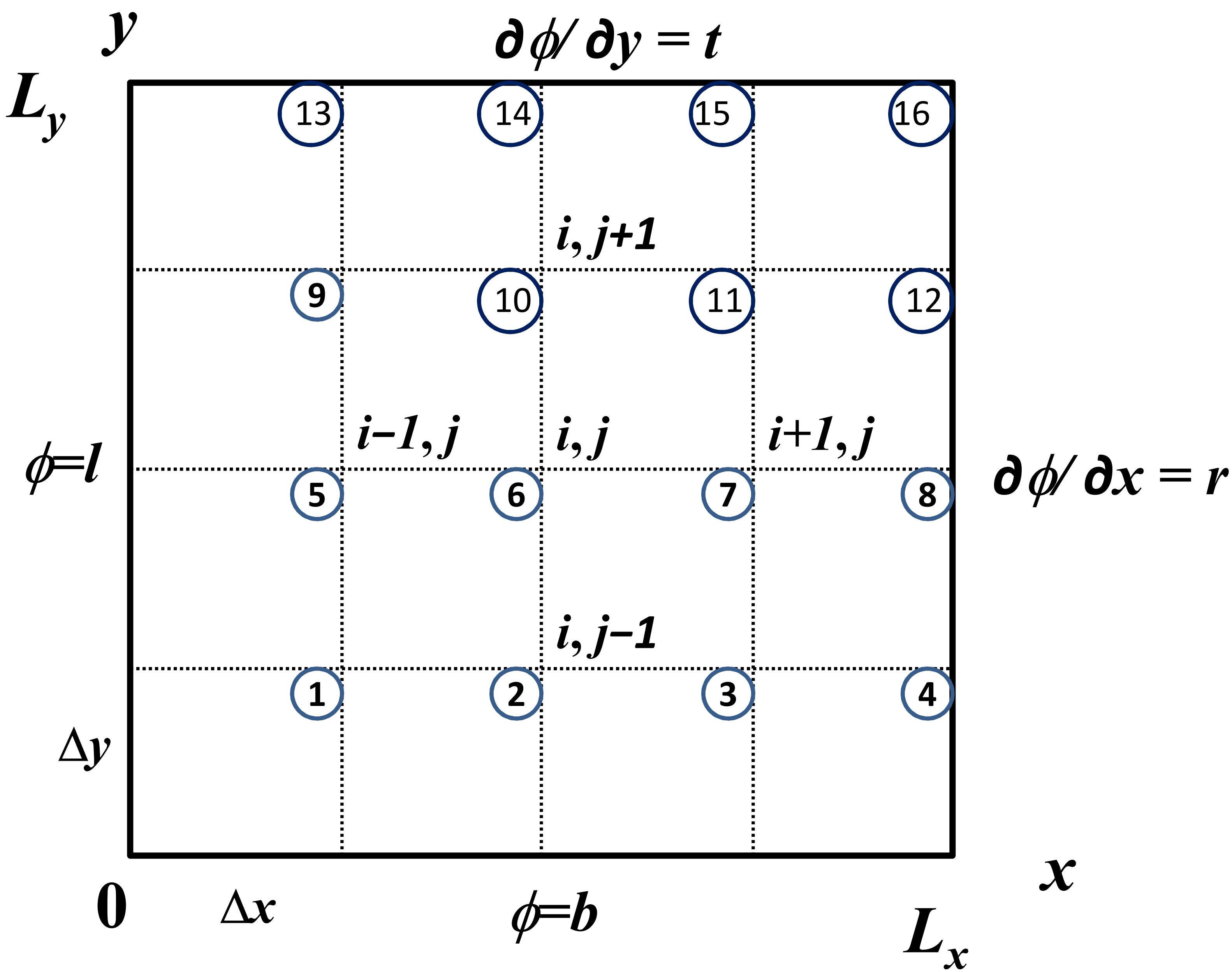
$$\phi_1 = \frac{1}{4}(0+100+100+100) = 75$$

$$\phi_2 = \frac{1}{4}(0+75+100+100) = 68.75......$$

Node ->	1	2	3	4	5	6	7	8	9
Iter 0	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Iter 1	75.00	68.75	67.19	93.75	90.63	89.45	98.44	97.27	96.68
Iter 2	65.63	55.86	61.33	88.67	82.81	85.21	96.48	93.99	94.80
Iter 3	61.13	51.32	59.13	85.11	78.91	83.21	94.78	92.12	93.83
Iter 4	59.11	49.29	58.12	83.20	76.95	82.23	93.83	91.15	93.35
Iter 5	58.12	48.30	57.63	82.23	75.98	81.74	93.34	90.67	93.10
Iter 6	57.63	47.81	57.39	81.74	75.49	81.49	93.10	90.42	92.98
Iter 7	57.39	47.57	57.26	81.49	75.24	81.37	92.98	90.30	92.92
Iter 8	57.26	47.44	57.20	81.37	75.12	81.31	92.92	90.24	92.89
Iter 9	57.20	47.38	57.17	81.31	75.06	81.28	92.89	90.21	92.87
Iter 10	57.17	47.35	57.16	81.28	75.03	81.27	92.87	90.19	92.86
Iter 11	57.16	47.34	57.15	81.27	75.02	81.26	92.86	90.19	92.86
Iter 12	57.15	47.33	57.15	81.26	75.01	81.25	92.86	90.18	92.86
Iter 13	57.15	47.33	57.14	81.25	75.00	81.25	92.86	90.18	92.86
Iter 14	57.14	47.32	57.14	81.25	75.00	81.25	92.86	90.18	92.86
Iter 15	57.14	47.32	57.14	81.25	75.00	81.25	92.86	90.18	92.86

Laplace Equation

- What if Neumann B.C.?
- Assume Dirichlet B.C. on bottom and left boundaries: $\phi(x,0)= b$; $\phi(0,y)= l$;
  and Neumann at the right and top:
  $\partial\phi/\partial x\,(L_x,y)= r$; $\partial\phi/\partial y(x,L_y)= t$
- We have already seen how to modify the equation at the boundary nodes for Dirichlet B.C.
- For Neumann boundary, we could use Ghost node or backward difference



Number of unknowns has increased from 9 to 16

Laplace Equation: Neumann B.C.

- Using backward difference, $O(h^2)$:

- At the right boundary,

$$\frac{\phi_{i-2,j} - 4\phi_{i-1,j} + 3\phi_{i,j}}{2\Delta x} = r$$

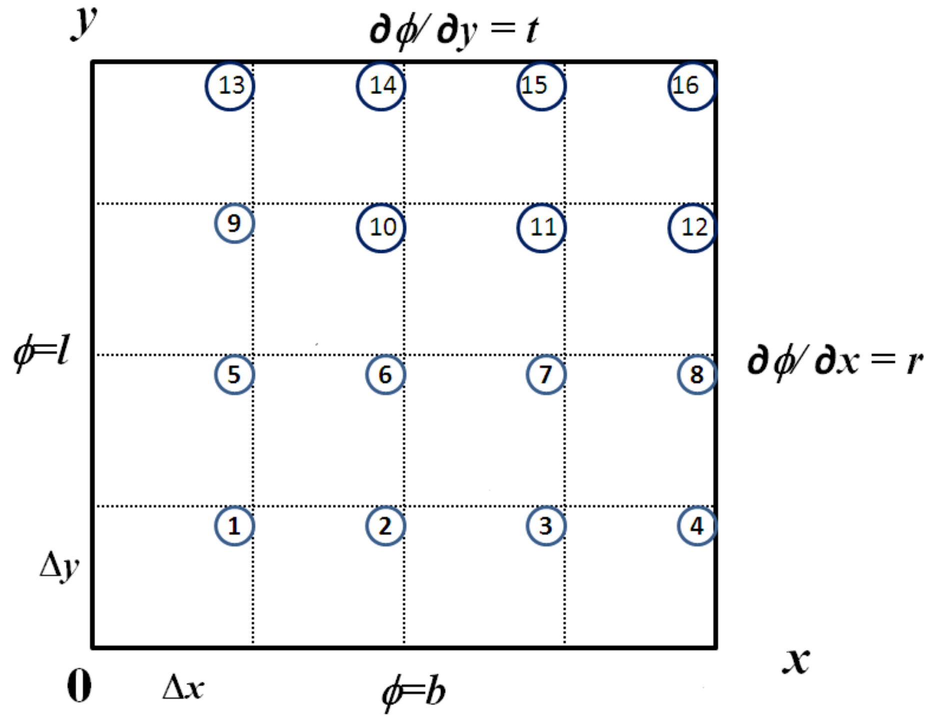
- And, at the top,

$$\frac{\phi_{i,j-2} - 4\phi_{i,j-1} + 3\phi_{i,j}}{2\Delta y} = t$$

Laplace Equation: Applying B.C.

- Node 6

$$\phi_2 + \phi_5 - 4\phi_6 + \phi_7 + \phi_{10} = 0$$



- Node 8

$$\phi_6 - 4\phi_7 + 3\phi_8 = 2\Delta x r$$

- Node 14

$$\phi_6 - 4\phi_{10} + 3\phi_{14} = 2\Delta y t$$

- Node 16? Does not occur in other Eqns. Could be computed separately!

Laplace Equation: Example of derivative B.C.

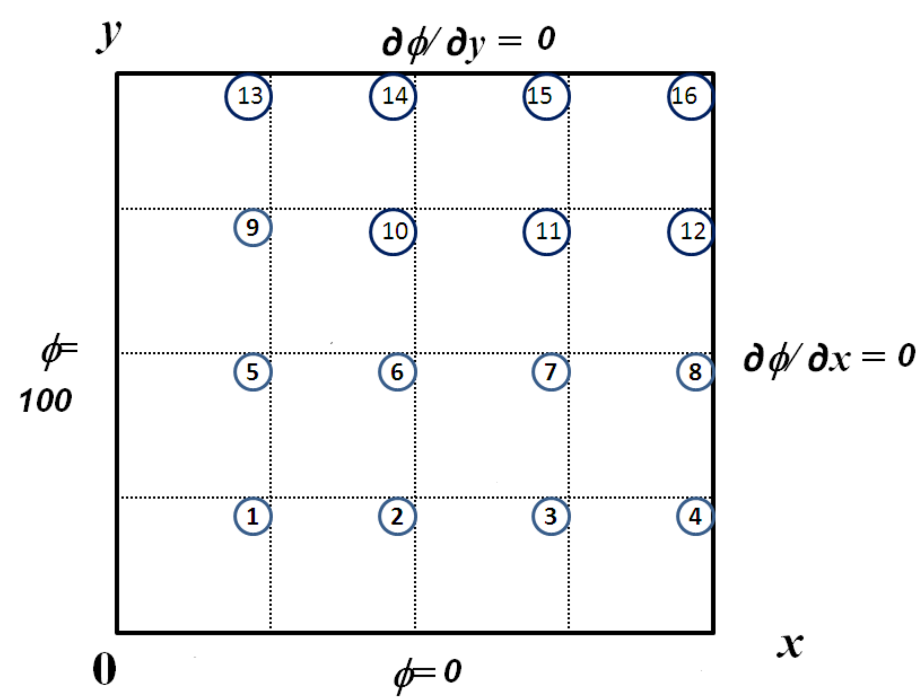
- Let us consider the bottom boundary at zero potential, left at 100, and the right and top to be insulated (zero gradient)
- Use the Gauss-Seidel method
- Some equations are:

$$\phi_1 = \left(0 + 100 + \phi_2 + \phi_5\right)/4$$

$$\phi_4 = \left(-\phi_2 + 4\phi_3\right)/3$$

$$\phi_6 = \left(\phi_2 + \phi_5 + \phi_7 + \phi_{10}\right)/4$$

$$\phi_{15} = \left(-\phi_7 + 4\phi_{11}\right)/3$$



Node ->	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Iter 0	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Iter 1	75.00	68.75	67.19	66.67	93.75	90.63	89.45	89.06	98.44	97.27	96.68	96.48	100.00	99.48	99.09
Iter 2	65.63	55.86	52.99	52.04	88.67	82.81	80.39	79.58	96.48	93.86	92.46	91.99	99.09	97.55	96.48
Iter 3	61.13	49.24	45.42	44.14	85.11	77.15	73.65	72.48	94.51	90.42	88.13	87.37	97.65	94.84	92.96
Iter 4	58.59	45.29	40.77	39.26	82.56	72.98	68.59	67.13	92.66	87.15	84.02	82.97	96.02	91.88	89.16
Iter 5	56.96	42.68	37.63	35.95	80.65	69.77	64.64	62.93	90.96	84.15	80.23	78.92	94.39	88.95	85.43
Iter 6	55.83	40.81	35.35	33.53	79.14	67.18	61.42	59.50	89.42	81.45	76.81	75.26	92.85	86.20	81.93
Iter 7	54.99	39.38	33.58	31.65	77.90	65.04	58.73	56.63	88.05	79.02	73.74	71.98	91.43	83.69	78.74
Iter 8	54.32	38.24	32.15	30.13	76.85	63.21	56.43	54.17	86.83	76.87	71.00	69.05	90.15	81.42	75.86
Iter 9	53.77	37.28	30.96	28.85	75.95	61.63	54.44	52.05	85.74	74.95	68.58	66.45	89.01	79.39	73.29
Iter 10	53.31	36.48	29.94	27.77	75.17	60.26	52.71	50.19	84.78	73.25	66.42	64.15	87.98	77.58	71.00
Iter 50	50.02	30.68	22.60	19.91	69.40	50.07	39.80	36.37	77.50	60.39	50.12	46.69	80.20	63.82	53.56
Iter 60	50.01	30.65	22.56	19.87	69.37	50.02	39.73	36.30	77.47	60.32	50.03	46.61	80.16	63.75	53.47
Iter 70	50.00	30.64	22.55	19.86	69.37	50.01	39.71	36.28	77.46	60.30	50.01	46.58	80.15	63.73	53.44
Iter 80	50.00	30.64	22.55	19.85	69.36	50.00	39.71	36.28	77.45	60.30	50.00	46.57	80.15	63.73	53.43

Node 16 value is obtained as 50, from both top boundary and right boundary conditions

Slow convergence. Neumann conditions make the equation “not diagonally dominant”

Laplace Equation: Mixed B.C.

- Sometimes the boundary condition is specified as a linear combination of the dependent variable and its derivative
- Known as Third Type, Mixed, or Robin
- E.g., convective heat transfer $\partial\phi/\partial x = k\,(\phi - \phi_0)$
- Similar procedure as Neumann. E.g.,

$$\frac{\phi_{i-2,j} - 4\phi_{i-1,j} + 3\phi_{i,j}}{2\Delta x} = k\left(\phi_{i-1,j} - \phi_0\right)$$

$$\phi_{i-2,j} - 4\phi_{i-1,j} + \left(3 - 2\Delta x k\right)\phi_{i,j} = -k\phi_0$$