ESO 208A: Computational Methods in Engineering
Problem Set 5

1. Determine straight lines which approximate the curve $y = e^x$ such that, (i) the Euclidean semi-norm of the error function on the net (-1, -0.5, 0, 0.5, 1) is as small as possible; (ii) the Euclidean norm of the error function on the interval [-1,1] is as small as possible.

2. Compute a least square quadratic polynomial approximation of $f(x) = \frac{1}{1+x^2}; \quad x \in (-1,1)$ using Legendre polynomials as basis. Also expand $f(x)$ in Taylor series in the neighbourhood of $x = 0$ up to the quadratic term. Compare the second order polynomial approximation with the Taylor series. Using true absolute error, determine which polynomial approximates the function better at $x = -1$, 0 and 1? Justify your answer with analytical reasoning.

3. Consider the approximation of the function $f(t) = e^{-\frac{t^2}{24}}$ in the interval $[-2\pi, 2\pi]$. First map the t-domain to the x-domain in such a way that $[-2\pi, 2\pi]$ in t-domain maps into [-1,1] in x-domain. Approximate the function by employing a (i) Legendre basis $\{P_j(x)\}_{j=0}^{j=4}$; (ii) Tchebycheff basis $\{T_j(x)\}_{j=0}^{j=4}$; and (iii) Simple Polynomial Basis $\{x^j\}_{j=0}^{j=4}$. Graphically compare the function with the resulting approximations in the entire domain. Compute the maximum norm of the error in each case.

4. It is known that bacteria swim against the concentration gradient of the food. The following data were measured for the concentration gradient of the food and the corresponding speed of swim :

Conc. Grad., x, (μmol/cm)	0.1	0.2	0.5	1.3	4.2	9.5
Speed, y, (μm/min)	2.85	4.00	6.00	10.00	15.00	20.00

It was decided to fit the equation, $y = \frac{a\sqrt{x}}{1+b\sqrt{x}}$ with the data. a) Obtain least square estimates of a and b; b) Calculate r^2 for the fit; c) Graphically compare the data values with the fitted curve.

5. The cost of fuel consumed by a truck was assumed to be linearly related to the travel distance and the load carried. Over a certain period, the following data was recorded by the driver. Obtain the underlying relationship (add the constraint that there is no cost when both the distance and the load are zero). How good is the assumption that the relationship is linear?

Distance (km)	88	210	320	88	210	320	245	65
Load Factor	0.33	0.42	0.50	0.17	0.28	0.67	0.32	1.00
Cost (Rs.)	140	270	400	110	250	450	280	225

6. A new model of a sports car is being tested on the test-track for performance. The car started from the rest and the following times were recorded by an automatic data recorder at the fixed distance markers on the track:

Distance from the start (s), km	0.2	0.4	0.6	0.8	1.0
Time elapsed from the start (t), sec	2	3	3.8	4.4	4.8

a) Compute the least square estimate of the acceleration (a) for this model. The fixed distance markers have a measurement error of ±1m and the time-data recorder has a precision of ±0.01 sec. Compute an upper bound of the error in the value of acceleration. (Given: $s = \frac{1}{2} at^2$ and $v = at$ where, v is the speed at time t)

b) Compute the time and the distance required for this model to increase the speed from 0 to 100 km/hr. Compute an upper bound of the error in the estimate of the time required for 0-100 km/hr.

7. The following data have been measured in an experiment:

k	x_k	y_k	z_k
1	23.000	22.000	10800.000
2	35.000	21.999	162010.797
3	71.000	22.012	831492.000
4	103.000	22.078	2234520.000
5	111.000	22.622	4062960.000
6	109.000	25.536	5918854.000
7	100.000	36.094	7510450.000
8	86.000	57.113	8512614.000
9	71.000	76.565	8764492.000
10	59.000	85.632	8416764.000
11	47.000	86.572	7701761.000
12	39.000	82.884	6800436.000
13	32.000	76.928	5841266.500
14	28.000	70.121	4901137.000
15	24.000	63.270	4022114.000
16	22.000	56.796	3222201.250
17	22.000	50.913	2534144.000
18	22.000	45.663	1966323.250
19	22.000	41.076	1504742.000
20	22.000	37.144	1135166.000

It is proposed to approximate the data by an expression of the form

$$\hat{z}_k = A\left[Bx_k + (1-B)y_k\right] + C$$

where (^) denotes "estimated". Determine a least-squares approximation of the form given by the above equation. Set $\alpha = AB$, $\beta = A(1\text{-}B)$ and $\gamma = C$. Find the values of α, β and γ. Then compute A, B and C. Is the solution optimal (in the least-squares sense) with respect to A, B, and C? Provide a mathematical justification for your answer.

8. (a) Fit a second order polynomial of the form $f(x) = ax^2 + b$ through the following data points. Compute least square estimates of a and b, and estimate y at x=2.5. How good is the fit (compute r^2)?

x	1	2	3	4	5
$f(x)$	7.7	16	27.2	40.9	61.1

(b) For the data of (a), estimate the value of y at x=2.5 using a third order Lagrange interpolating polynomial.

9. The water level in the North Sea is mainly determined by the so-called M_2-tide, whose period is about 12 hours and thus has the form $H(t) = h_0 + a_1 \sin\frac{2\pi t}{12} + a_2 \cos\frac{2\pi t}{12}$, where t is in hours. One has made the following measurements:

***t*, hours**	0	2	4	6	8	10
***H*(*t*), meters**	1.0	1.6	1.4	0.6	0.2	0.8

Fit $H(t)$ to the series of measurements using the method of least squares and determine h_0, a_1 and a_2.

10. The mass of a radioactive substance is measured at 2-day intervals till 8 days. Unfortunately, the reading could not be taken at 6 days due to equipment malfunction. The following table shows the other readings:

Time (d)	Mass (g)
0	1.000
2	0.7937
4	0.6300
8	0.3968

Estimate the mass at 6 days using: (a) Newton's divided difference, (b) Lagrange polynomials, (c) Natural cubic spline.

11. Estimate the value of the function at x = 4 from the table of data given below, using, (a) Lagrange interpolating polynomial of 2nd order; (b) Newton's interpolating polynomial.

x	***f(x)***
1	1
2	12
3	54
5	375
6	756

Practice Problems (Derivation and Analysis): These will not be solved in Tutorials

12. Consider the following set of polynomials $\{p_n(x)\}$:

$$p_n(x) = \frac{\sin(n+1)\phi}{\sin\phi} \quad \text{where } x = \cos\varphi$$

a) Derive the recursion formula for this polynomial.

b) What are the first three polynomials, $p_0(x)$, $p_1(x)$ and $p_2(x)$?

c) Show that $\{p_n(x)\}$ forms an orthogonal system of polynomials with the weight function $w(x)$ in $x \in [-1, 1]$. What is $w(x)$ for this set of polynomials?

13. Show that if a matrix A has a zero eigenvalue, then A must be a singular matrix.

14. Consider the following set of linear equations:

$$\begin{aligned} x + y + z &= 0 \\ x + 2y - z &= 0 \\ x - y + az &= 0 \end{aligned}$$

Using Gauss-Elimination, find the value of a for which the above system have non-trivial solutions. How many solutions does this system have for the value of a computed above? Compute one non-trivial solution.

15. In a rectangular open channel flow of depth h, the velocities u_0, u_1, u_2 and u_3 have been measured at the depths (measured from the surface) of 0, 0.3h, 0.6h and 0.9h, respectively. Derive expressions for ω_0, ω_1, ω_2 and ω_3 such that the velocity at a depth of αh , $0 < \alpha < 1$, can be expressed as $(\omega_0 u_0 + \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3)$.

16. Consider the system of equation $\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ where, a is a real constant. For which values of a, the Jacobi and Gauss-Seidel methods converge?

17. Derive the following identity for Tchebycheff polynomials, show that $T_m(T_n(x)) = T_{mn}(x)$

18. Suppose a similarity transformation with X diagonalizes matrix A, i.e., $A = X\Lambda X^{-1}$ where, Λ is the diagonal matrix. Show that a similarity transformation $Y = \begin{bmatrix} X & X \\ X & -X \end{bmatrix}$ diagonalizes any matrix $B = \begin{bmatrix} P(A) & Q(A) \\ Q(A) & P(A) \end{bmatrix}$, where P and Q are arbitrary polynomials of A. Express the eigenvalues of B in terms of eigenvalues of A.

19. Derive the following identity for the divided difference:

$$f[x_0, x_1, x_2 \ldots\ldots x_m] = \sum_{i=0}^{m} \frac{f(x_i)}{\Phi'(x_i)} \quad \text{where,} \quad \Phi(x) = \prod_{j=0}^{m} (x - x_j)$$

20. Suppose that $f(x)$ possesses two zeros α_1 and α_2 which are nearly coincident, so that $f'(x)$ vanishes at a point β between α_1 and α_2.

a) Show that if β is determined first, the initial approximations to the nearby zeros are given by,

$$\alpha_{1,2} \approx \beta \pm \left(-\frac{2f(\beta)}{f''(\beta)} \right)^{1/2}$$

if $f''(\beta) \neq 0$ and both the roots are real if $f(\beta)$ and $f''(\beta)$ are of opposite signs.

b) Use the result of (a) to find the initial guesses of two real roots near $\beta = 1$ for the polynomial $3x^4 + 8x^3 - 6x^2 - 25x + 19 = 0$. Subsequently, refine the solution using Bairstow's method such that, both $\frac{|\Delta r|}{|r|}$ and $\frac{|\Delta s|}{|s|} \leq 0.05\%$. Are the other two roots real?