Sayan Karmakar
22MS172.

PH2201

Assignment-05

Q1. Show that in region ①, the square integrability of the wave function implies $A_2 = 0$: $\boxed{\psi_I = A_1 e^{kx}}$

Solution in region (I):-

$$\psi_I(x) = A_1 e^{+kx} + A_2 e^{-kx}$$

$$k = \frac{\sqrt{-2mE}}{\hbar}$$

We know that wave-functions to be square-integrable,

$$\int_{-\infty}^{-a} \psi_I^* \psi_I \, dx < \infty$$

$$= \int_{-\infty}^{-a} (A_1^* e^{kx} + A_2^* e^{-kx})(A_1 e^{kx} + A_2 e^{-kx}) < \infty$$

$$\boxed{\begin{aligned} A_1^* A_1 &= A_1^2 \\ A_2^* A_2 &= A_2^2 \end{aligned}}$$

$$= \int_{-\infty}^{-a} A_1^2 e^{2kx} dx + \int_{-\infty}^{-a} A_2^2 e^{-2kx} dx + (A_1^* A_2 + A_2^* A_1) \int_{-\infty}^{-a} dx < \infty$$

$$= \frac{A_1^2}{2K} e^{2Kx}\Big|_{-\infty}^{-a} + \frac{A_2^2}{-2K} e^{-2kx}\Big|_{-\infty}^{-a} + (A_1^* A_2 + A_2^* A_1)\, x\Big|_{-\infty}^{-a} < \infty$$

$$= \frac{A_1^2}{2K}\left[e^{-2Ka} - e^{-\infty}\right] + \frac{A_2^2}{-2K}\left(e^{2Ka} - e^{\infty}\right) + (A_1^* A_2 + A_2^* A_1)\left[-a + \infty\right] < \infty$$

$\therefore$ e^{-2Kx} blows up at $x \to -\infty$, so for the above statement to be true ; $A_2 = 0$.

so, $\psi_I(x) = A_1 e^{+kx}$; $k = \frac{\sqrt{-2mE}}{\hbar}$

Q2. For $B_1 = -B$ show that the odd-wave functions in the infinite square well limit lead to the energy eigen value

$$\left[E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}\right] \text{ with } n = 2,4,6,8,\ldots$$

Sol^n:-

For the finite Square Well:-

$$\psi(x) = \begin{cases} Ae^{+Kx} & -\infty < x < -a \\ B_1 e^{ilx} + B_2 e^{-ilx} & -a < x < a \\ De^{-Kx} & a < x < \infty \end{cases}$$

$$K^2 = -\frac{2mE}{\hbar^2}$$

$$l^2 = \frac{2m(E+V_0)}{\hbar^2}$$

for $B_1 = B_2$] $\Rightarrow$ At infinite squarewell limit.

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad [n = 1, 3, 5, \ldots]$$

$\Rightarrow \underline{B_1 = -B_2}$

- We know that $\psi(x)$ must be continuous at $x = a$.

$$\psi_I(x=-a) = \psi_{II}(x=-a)$$

$$\Rightarrow Ae^{-Kx} = B_1(e^{ilx} - e^{-ilx}) = 2iB_1 \sin(lx) \quad \text{--- (1)}$$

- $\psi'(x)$ must also be continuous at $x = -a$.

$$\Rightarrow KAe^{-Kx} = ilB_1(e^{ilx} + e^{-ilx}) = 2ilB_1 \cos(lx) \quad \text{--- (2)}$$

(2) ÷ (1)

$$\Rightarrow K = l\cot(lx)$$

at $x = -a$; $K = -l\cot(la)$

$$\Rightarrow K^2 = l^2\cot^2(la)$$

$$\Rightarrow -\frac{2mE}{\hbar^2} = \frac{2m(E+V_0)}{\hbar^2}\cot^2\left(\frac{\sqrt{2m(E+V_0)a^2}}{\hbar}\right)$$

$$\Rightarrow \left[E = -(E+V_0)\cot^2\left(\frac{\sqrt{2m(E+V_0)a^2}}{\hbar}\right)\right]$$

from finite to infinite.

- width : $a \to \frac{a}{2}$
- Shift the origin of energy $(E+V_0) \longrightarrow E$
- limit : $V_0 \longrightarrow \infty$

- $a \to \frac{a}{2}$

$$E = -(E+V_0)\cot^2\left(\frac{\sqrt{m(E+V_0)a^2/2}}{\hbar}\right)$$

- $(E+V_0) \longrightarrow E$

$$\Rightarrow \frac{V_0}{E_0} - 1 = \cot^2\left(\frac{\sqrt{mEa^2/2}}{\hbar}\right)$$

- $V_0 \longrightarrow \infty$

$$\Rightarrow \left(\frac{V_0}{E_0} - 1\right) \longrightarrow \infty$$

$$\cot\theta \longrightarrow \infty \quad , \theta = \frac{n\pi}{2} \; [n = 0, 2, 4, 6, \ldots]$$

$$\therefore \frac{\sqrt{mEa^2/2}}{\hbar} = \frac{n\pi}{2}$$

$$\Rightarrow \frac{mEa^2}{2} = \frac{n^2\pi^2\hbar}{4}$$

$$\Rightarrow \boxed{E = \frac{n^2\pi^2\hbar^2}{2ma^2}} \quad (n = 2, 4, 6, 8, \ldots)$$