COL778: Principles of Autonomous Systems
Semester II, 2023-24

Policy Gradients

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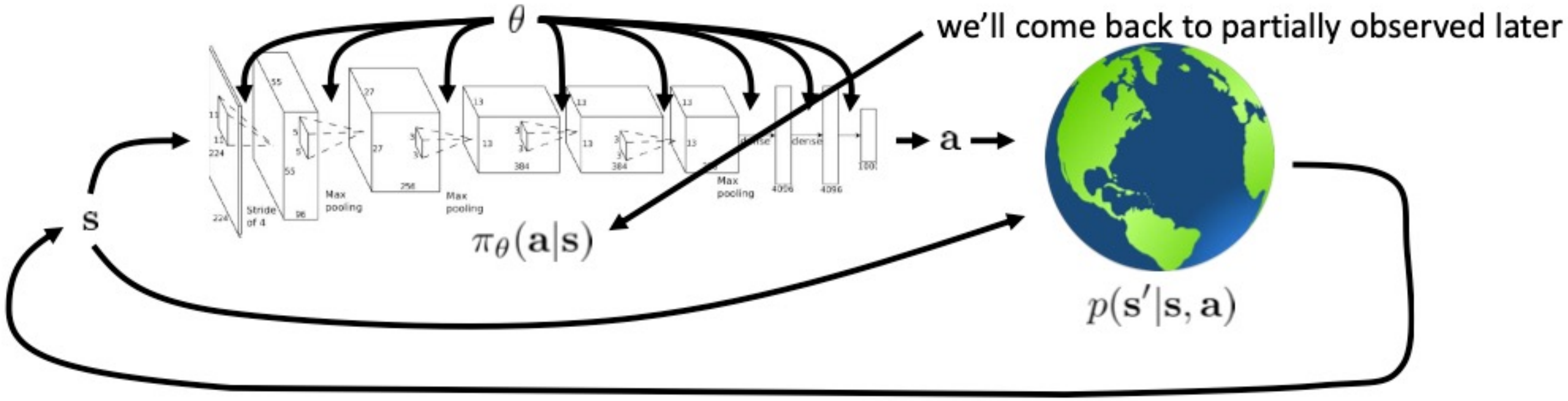
Outline

- Last Class
  - Imitation Learning
- This Class
  - Policy Gradients
- Reference Material
  - Please follow the notes as the primary reference on this topic.

Acknowledgements

These slides are intended for teaching purposes only. Some material has been used/adapted from web sources and from slides by Nicholas Roy, Wolfram Burgard, Dieter Fox, Sebastian Thrun, Siddharth Srinivasa, Dan Klein, Pieter Abbeel, Max Likhachev, Alexander Amini (MIT Introduction to Deep Learning) and others. This lecture builds on material from Sergey Levine's course on Deep RL.

Reinforcement Learning Objective



$$\underbrace{p_\theta(\mathbf{s}_1,\mathbf{a}_1,\ldots,\mathbf{s}_T,\mathbf{a}_T)}_{p_\theta(\tau)} = p(\mathbf{s}_1)\prod_{t=1}^{T}\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)p(\mathbf{s}_{t+1}|\mathbf{s}_t,\mathbf{a}_t)$$

$$\theta^\star = \arg\max_\theta E_{\tau\sim p_\theta(\tau)}\left[\sum_t r(\mathbf{s}_t,\mathbf{a}_t)\right]$$

The objective function: Sample a trajectory given the policy π_θ and estimate its cumulative reward.

Reinforcement Learning Objective

$$\theta^\star = \arg\max_\theta E_{\tau \sim p_\theta(\tau)}\left[\sum_t r(\mathbf{s}_t, \mathbf{a}_t)\right]$$

$$\theta^\star = \arg\max_\theta E_{(\mathbf{s},\mathbf{a}) \sim p_\theta(\mathbf{s},\mathbf{a})}[r(\mathbf{s}, \mathbf{a})]$$

infinite horizon case

$$\theta^\star = \arg\max_\theta \sum_{t=1}^{T} E_{(\mathbf{s}_t,\mathbf{a}_t) \sim p_\theta(\mathbf{s}_t,\mathbf{a}_t)}[r(\mathbf{s}_t, \mathbf{a}_t)]$$

finite horizon case

This lecture is about directly optimizing the policy parameters to output actions that lead to high long term rewards. The family of techniques is called ”Policy Gradients” as we directly differentiate the policy parameters w.r.t. objective.

PG Examples: Learning Locomotion for Humanoids (Sim)

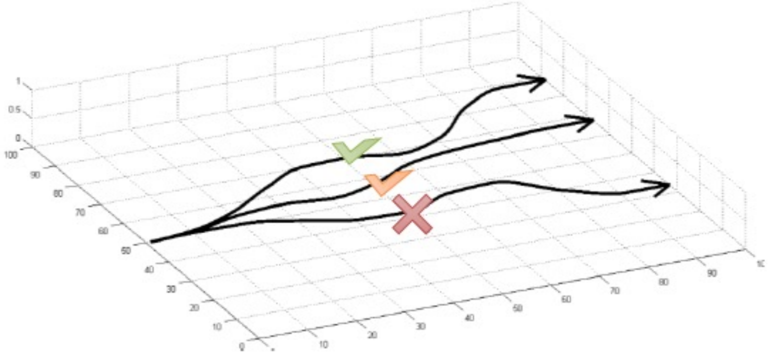
Heess et al., "Emergence of Locomotion Behaviours in Rich Environments"
https://www.youtube.com/watch?v=hx_bgoTF7bs

PG Examples: Learning Manipulation Skills

Levine et al., "End-to-End Training of Deep Visuomotor Policies"
https://sites.google.com/site/visuomotorpolicy/

Evaluating the Objective

$$\theta^\star = \arg\max_\theta \underbrace{E_{\tau\sim p_\theta(\tau)}\left[\sum_t r(\mathbf{s}_t, \mathbf{a}_t)\right]}_{J(\theta)}$$



$$J(\theta) = E_{\tau\sim p_\theta(\tau)}\left[\sum_t r(\mathbf{s}_t, \mathbf{a}_t)\right] \approx \frac{1}{N}\sum_i \sum_t r(\mathbf{s}_{i,t}, \mathbf{a}_{i,t})$$

sum over samples from π_θ

Expectation is via sampling state-action trajectories from the current policy.

Deriving Gradient of the Objective

$$\theta^\star = \arg\max_\theta \underbrace{E_{\tau\sim p_\theta(\tau)}\left[\sum_t r(\mathbf{s}_t,\mathbf{a}_t)\right]}_{J(\theta)}$$

a convenient identity

$$\underline{p_\theta(\tau)\nabla_\theta \log p_\theta(\tau)} = p_\theta(\tau)\frac{\nabla_\theta p_\theta(\tau)}{p_\theta(\tau)} = \underline{\nabla_\theta p_\theta(\tau)}$$

$$J(\theta) = E_{\tau\sim p_\theta(\tau)}[\underbrace{r(\tau)}_{\sum_{t=1}^{T} r(\mathbf{s}_t,\mathbf{a}_t)}] = \int p_\theta(\tau) r(\tau) d\tau$$

$$\nabla_\theta J(\theta) = \int \underline{\nabla_\theta p_\theta(\tau)} r(\tau) d\tau = \int \underline{p_\theta(\tau)\nabla_\theta \log p_\theta(\tau)} r(\tau) d\tau = E_{\tau\sim p_\theta(\tau)}[\nabla_\theta \log p_\theta(\tau) r(\tau)]$$

Direct Policy Differentiation

$$\theta^\star = \arg\max_\theta J(\theta)$$

$$J(\theta) = E_{\tau \sim p_\theta(\tau)}[r(\tau)]$$

$$\underbrace{p_\theta(\mathbf{s}_1, \mathbf{a}_1, \ldots, \mathbf{s}_T, \mathbf{a}_T)}_{p_\theta(\tau)} = p(\mathbf{s}_1) \prod_{t=1}^{T} \pi_\theta(\mathbf{a}_t|\mathbf{s}_t) p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)$$

log of both sides

$$\log p_\theta(\tau) = \log p(\mathbf{s}_1) + \sum_{t=1}^{T} \log \pi_\theta(\mathbf{a}_t|\mathbf{s}_t) + \log p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)$$

$$\nabla_\theta J(\theta) = E_{\tau \sim p_\theta(\tau)}[\nabla_\theta \underbrace{\log p_\theta(\tau)}_{\nabla_\theta \left[\cancel{\log p(\mathbf{s}_1)} + \sum_{t=1}^{T} \log \pi_\theta(\mathbf{a}_t|\mathbf{s}_t) + \cancel{\log p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)}\right]} r(\tau)]$$

$$\nabla_\theta J(\theta) = E_{\tau \sim p_\theta(\tau)}\left[\left(\sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)\right)\left(\sum_{t=1}^{T} r(\mathbf{s}_t, \mathbf{a}_t)\right)\right]$$

Evaluating the Policy Gradient

recall: $J(\theta) = E_{\tau \sim p_\theta(\tau)}\left[\sum_t r(\mathbf{s}_t, \mathbf{a}_t)\right] \approx \frac{1}{N}\sum_i \sum_t r(\mathbf{s}_{i,t}, \mathbf{a}_{i,t})$

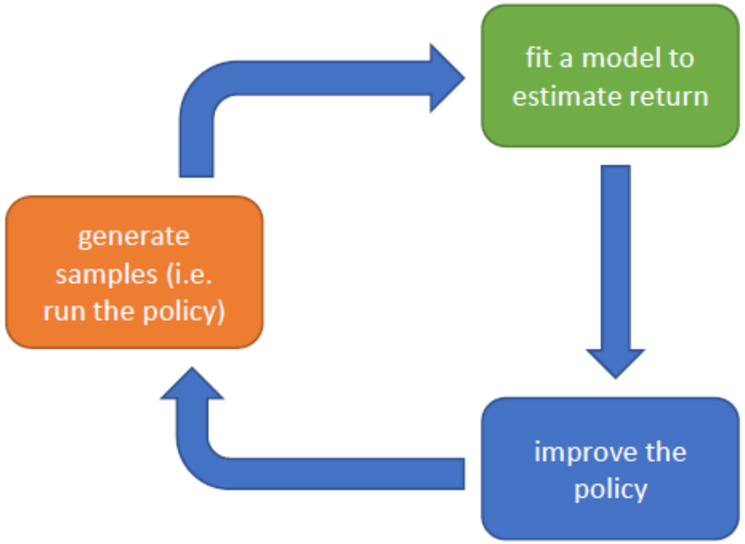
$$\nabla_\theta J(\theta) = E_{\tau \sim p_\theta(\tau)}\left[\left(\sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)\right)\left(\sum_{t=1}^{T} r(\mathbf{s}_t, \mathbf{a}_t)\right)\right]$$

$$\nabla_\theta J(\theta) \approx \boxed{\frac{1}{N}\sum_{i=1}^{N}}\left(\sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t})\right)\underline{\left(\sum_{t=1}^{T} r(\mathbf{s}_{i,t}, \mathbf{a}_{i,t})\right)}$$

$\underline{\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)}$

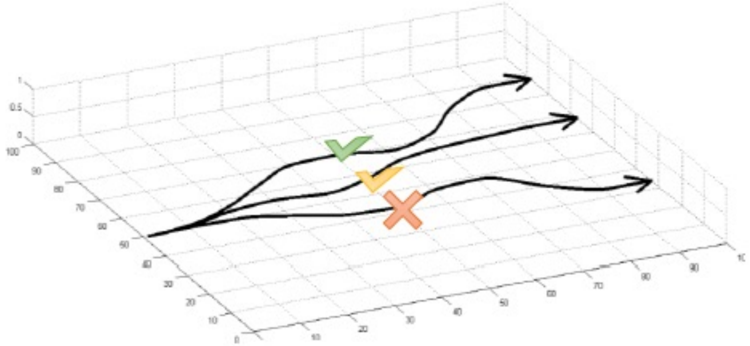
REINFORCE algorithm:

1. sample $\{\tau^i\}$ from $\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)$ (run the policy)
2. $\nabla_\theta J(\theta) \approx \sum_i \left(\sum_t \nabla_\theta \log \pi_\theta(\mathbf{a}_t^i|\mathbf{s}_t^i)\right)\left(\sum_t r(\mathbf{s}_t^i, \mathbf{a}_t^i)\right)$
3. $\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$



Evaluating the Policy Gradient

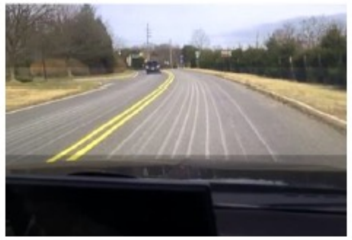
$$\text{recall: } J(\theta) = E_{\tau \sim p_\theta(\tau)}\left[\sum_t r(\mathbf{s}_t, \mathbf{a}_t)\right] \approx \frac{1}{N}\sum_i \sum_t r(\mathbf{s}_{i,t}, \mathbf{a}_{i,t})$$



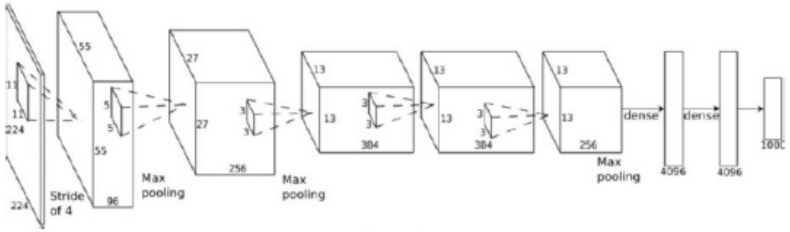
$$\nabla_\theta J(\theta) = E_{\tau \sim p_\theta(\tau)}\left[\left(\sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)\right)\left(\sum_{t=1}^{T} r(\mathbf{s}_t, \mathbf{a}_t)\right)\right]$$

$$\nabla_\theta J(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\left(\boxed{\sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t})}\right)\left(\sum_{t=1}^{T} r(\mathbf{s}_{i,t}, \mathbf{a}_{i,t})\right)$$

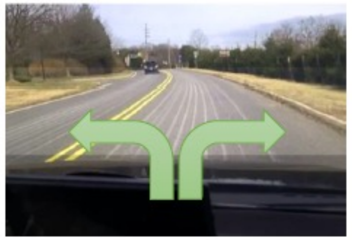
Predictive Likelihood of an action given a state. Can be continuous or discrete. We want the predictor to output actions that lead to higher rewards in expectation.



$\mathbf{s}_t$



$\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)$

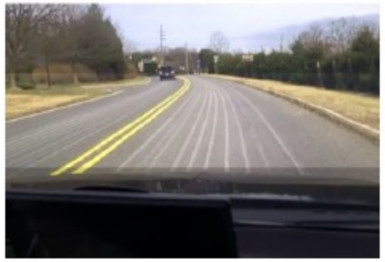


$\mathbf{a}_t$

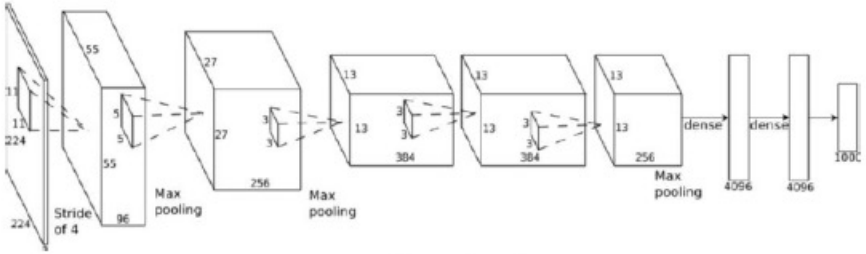
Connection to Maximum Likelihood Estimation

policy gradient: $\nabla_\theta J(\theta) \approx \boxed{\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t})\right)}\left(\sum_{t=1}^{T} r(\mathbf{s}_{i,t},\mathbf{a}_{i,t})\right)$

maximum likelihood: $\boxed{\nabla_\theta J_{\mathrm{ML}}(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t})\right)}$



$\mathbf{s}_t$



$\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)$

$\mathbf{a}_t$

$\mathbf{s}_t$ $\mathbf{a}_t$ → training data → supervised learning → $\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)$

Connection to Maximum Likelihood Estimation

We want the policy to predict those actions that lead to high long term rewards.
The objective is like the ML estimate which is to be optimized.

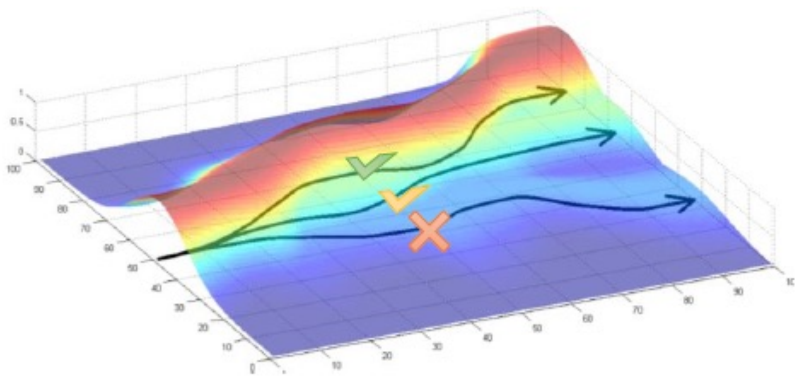
$$\nabla_\theta J(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t})\right)\left(\sum_{t=1}^{T} r(\mathbf{s}_{i,t},\mathbf{a}_{i,t})\right)$$

$$\nabla_\theta J(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\underbrace{\nabla_\theta \log \pi_\theta(\tau_i)}_{\sum_{t=1}^{T}\nabla_\theta \log_\theta \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t})} r(\tau_i)$$

maximum likelihood: $\nabla_\theta J_{\mathrm{ML}}(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\nabla_\theta \log \pi_\theta(\tau_i)$

REINFORCE algorithm:

1. sample $\{\tau^i\}$ from $\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)$ (run it on the robot)
2. $\nabla_\theta J(\theta) \approx \sum_i \left(\sum_t \nabla_\theta \log \pi_\theta(\mathbf{a}_t^i|\mathbf{s}_t^i)\right)\left(\sum_t r(\mathbf{s}_t^i,\mathbf{a}_t^i)\right)$
3. $\theta \leftarrow \theta + \alpha\nabla_\theta J(\theta)$



Actions that lead to higher rewards are made more likely
Actions that lead to lower rewards are made less likely under the policy.

Policy Gradients for Policies Parameterized as a Gaussian Likelihood

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t}) \right) \left(\sum_{t=1}^{T} r(\mathbf{s}_{i,t}, \mathbf{a}_{i,t}) \right)$$

example: $\pi_\theta(\mathbf{a}_t|\mathbf{s}_t) = \mathcal{N}(f_{\text{neural network}}(\mathbf{s}_t); \Sigma)$

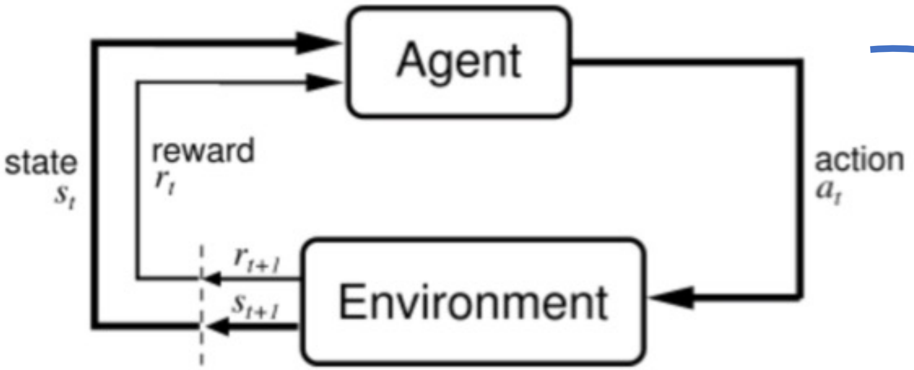
$$\log \pi_\theta(\mathbf{a}_t|\mathbf{s}_t) = -\frac{1}{2} \|f(\mathbf{s}_t) - \mathbf{a}_t\|_\Sigma^2 + \text{const}$$

$$\nabla_\theta \log \pi_\theta(\mathbf{a}_t|\mathbf{s}_t) = -\frac{1}{2} \Sigma^{-1}(f(\mathbf{s}_t) - \mathbf{a}_t) \frac{df}{d\theta}$$

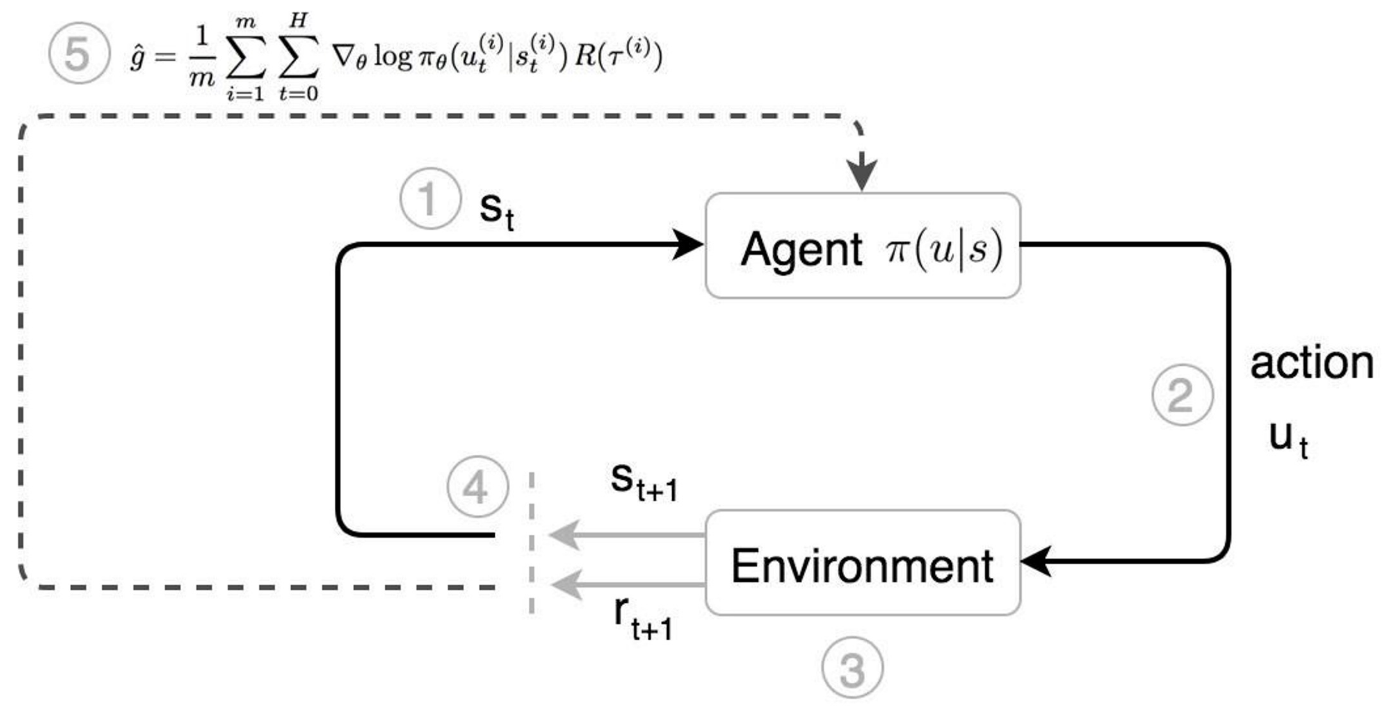
REINFORCE algorithm:

1. sample $\{\tau^i\}$ from $\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)$ (run it on the robot)
2. $\nabla_\theta J(\theta) \approx \sum_i \left(\sum_t \nabla_\theta \log \pi_\theta(\mathbf{a}_t^i|\mathbf{s}_t^i) \right) \left(\sum_t r(\mathbf{s}_t^i, \mathbf{a}_t^i) \right)$
3. $\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$

Policy Gradients: In Summary

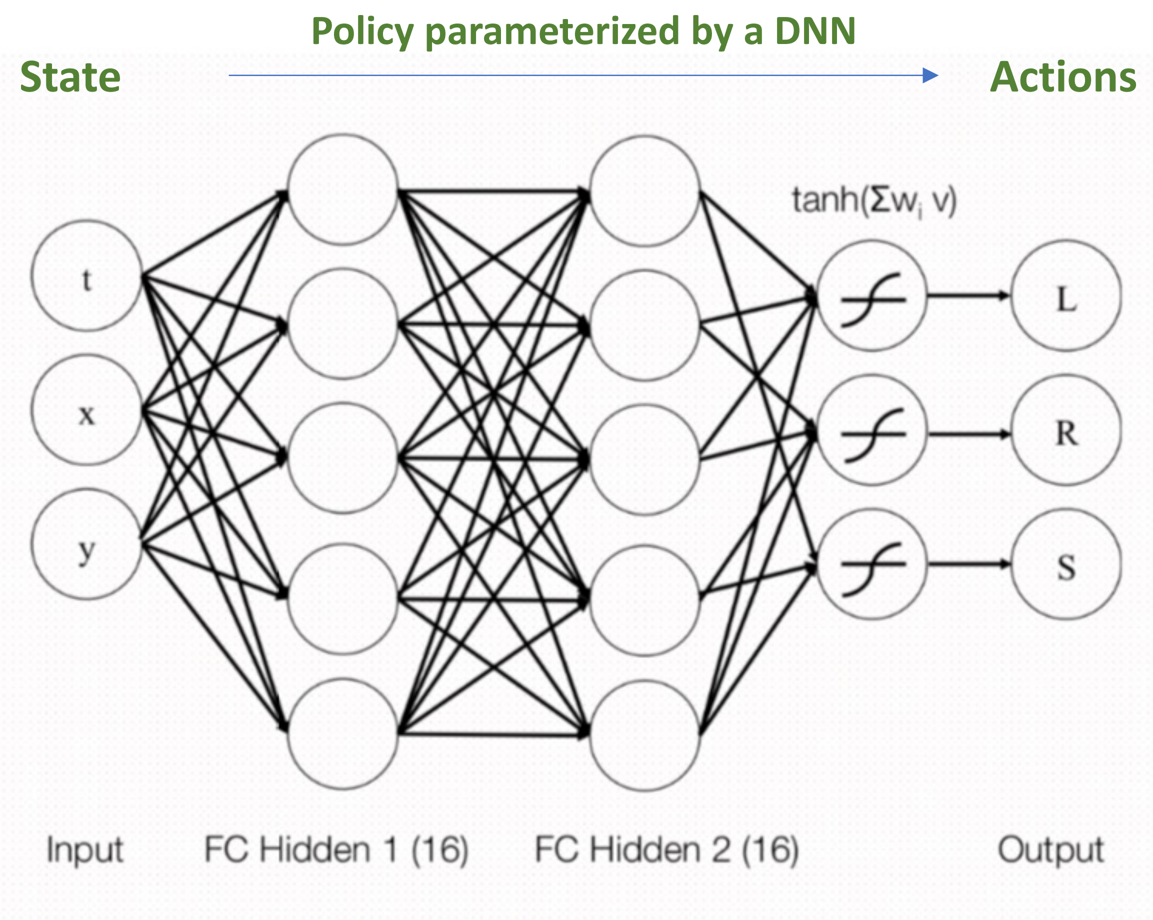


The agent learns to optimize the policy to best explain its experience.



"Deep" Policy Gradients

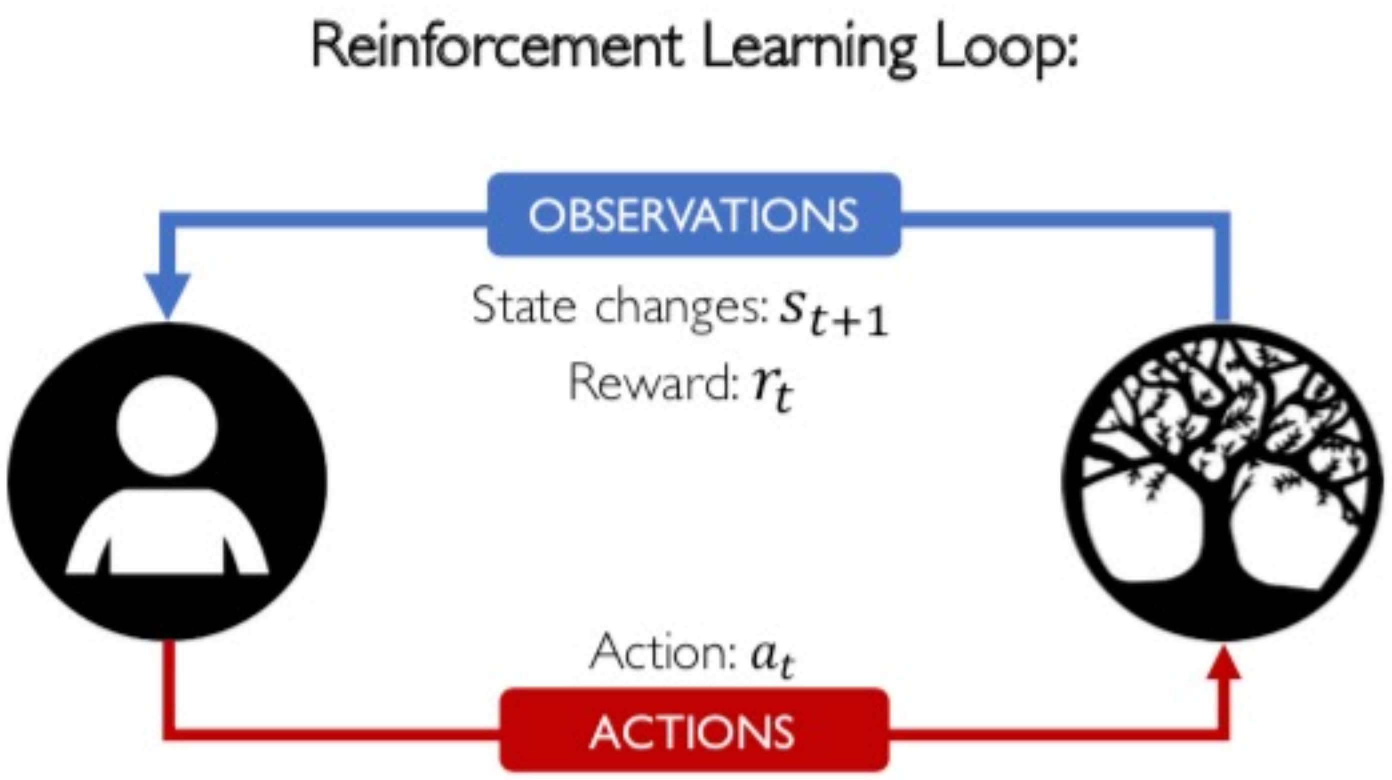
- Deep Policy Gradient:
  - Parameterize policy as deep neural network
  - Policy can act on high dimensional input, e.g. directly from visual feedback
  - Note that our derivation needed a policy parameterized by a set of parameters. We can plug in a neural network there.



PG Examples: Learning to Steer on the Road

Variational End-to-End Navigation and Localization. Alexander Amini, Guy Rosman, Sertac Karaman, Daniela Rus

Case Study: Learning to Drive



Case Study – Self-Driving Cars

Agent: vehicle

State: camera, lidar, etc

Action: steering wheel angle

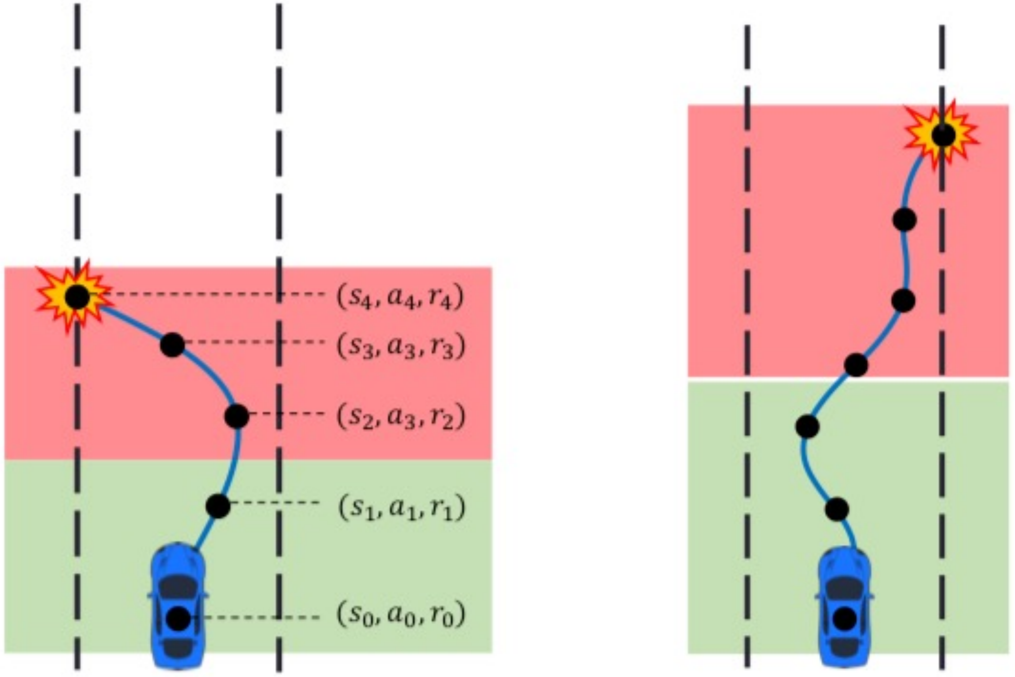
Reward: distance traveled without collisions

Example from Alexandar Amini (Introduction to Deep Learning)

Training Policy Gradients

Training Algorithm

1. Initialize the agent
2. Run a policy until termination
3. Record all states, actions, rewards
4. Decrease probability of actions that resulted in low reward
5. Increase probability of actions that resulted in high reward



Run the policy to obtain different episodes.

Training Policy Gradients

Training Algorithm

1. Initialize the agent
2. Run a policy until termination
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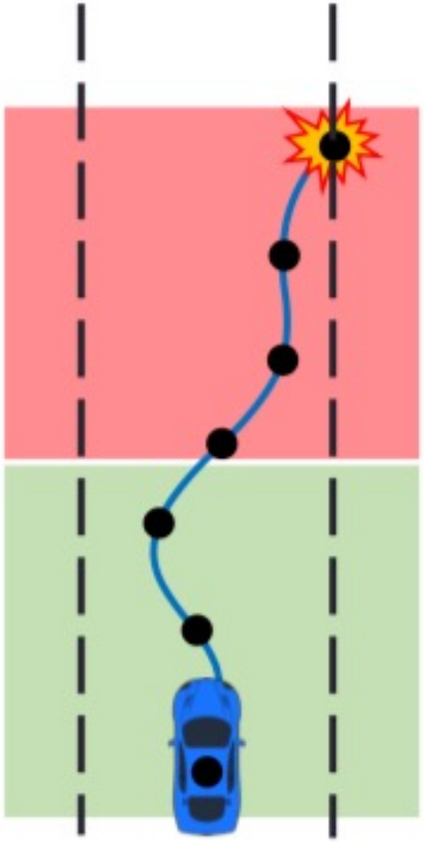

Training Policy Gradients

Training Algorithm

1. Initialize the agent
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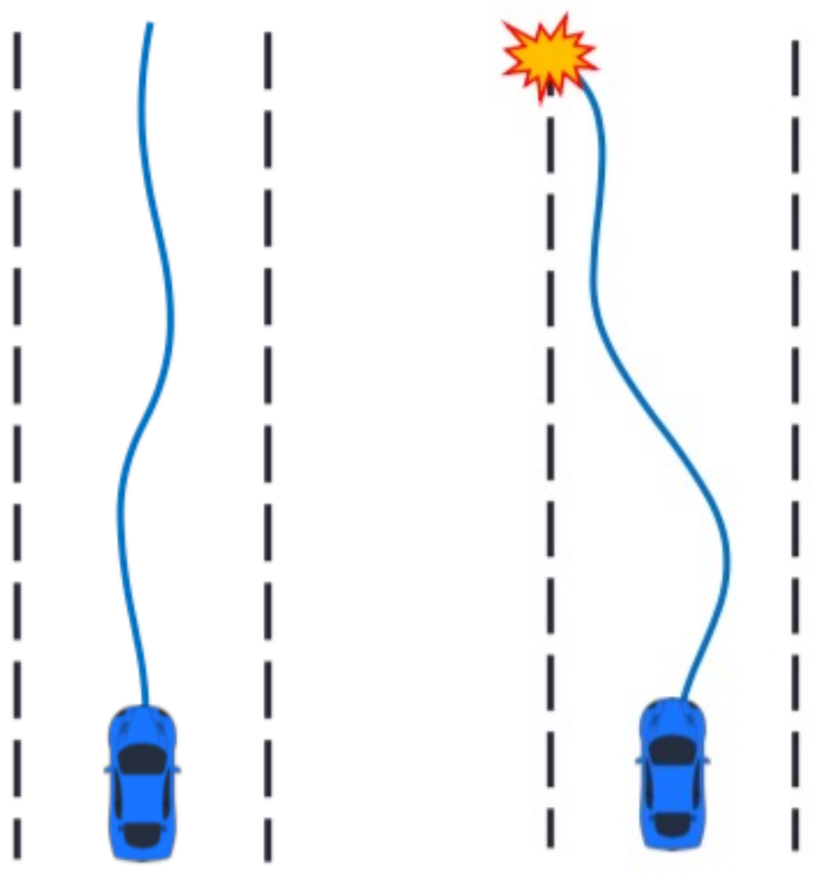


Increase probabilities of these actions

Decrease probabilities of these actions

Training Policy Gradients

Training Algorithm

1. Initialize the agent
2. Run a policy until termination
3. Record all states, actions, rewards
4. Decrease probability of actions that resulted in low reward
5. Increase probability of actions that resulted in high reward

$$\mathbf{loss} = -\underbrace{\overbrace{\log \mathrm{P}(a_t|s_t)}^{\text{log-likelihood of action}}\ R_t}_{}$$

log-likelihood of action

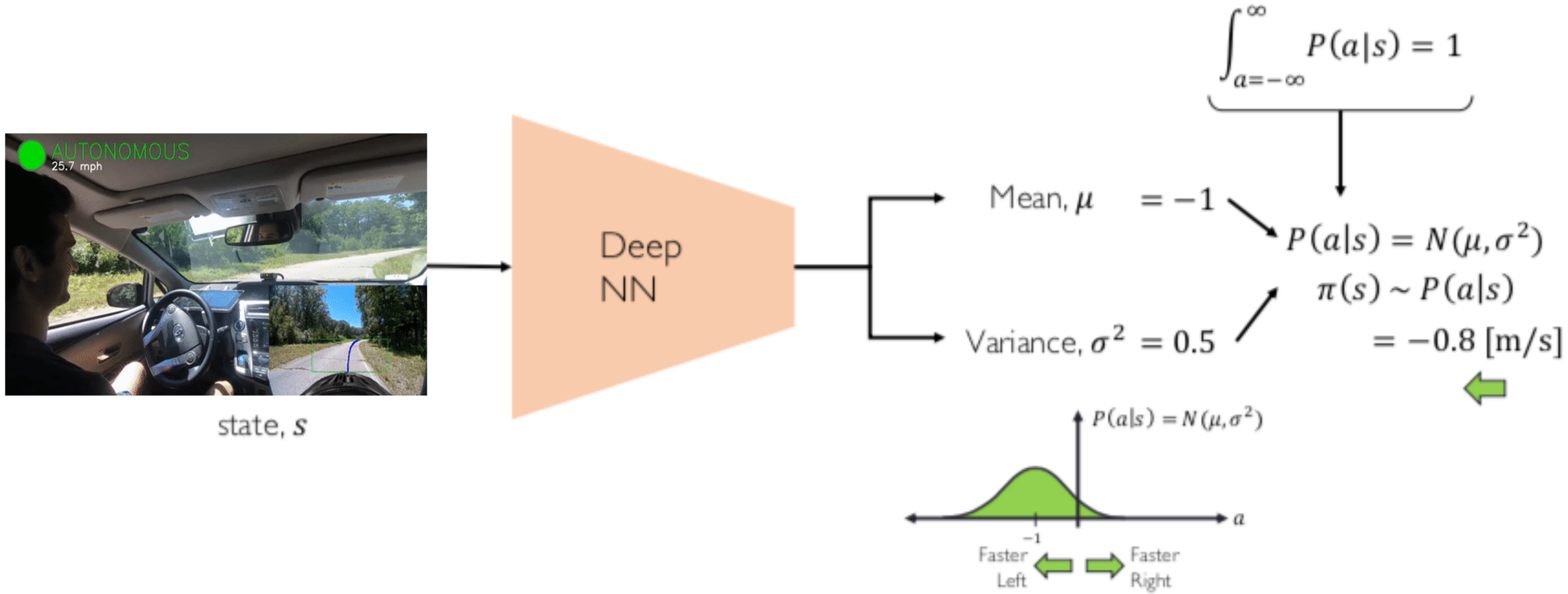
$$\mathbf{loss} = -\log \mathrm{P}(a_t|s_t)\ R_t$$

reward

$$w' = w - \nabla\,\mathbf{loss}$$
$$w' = w + \boxed{\nabla \log \mathrm{P}(a_t|s_t)\ R_t}$$

Policy gradient!

Handling continuous action spaces



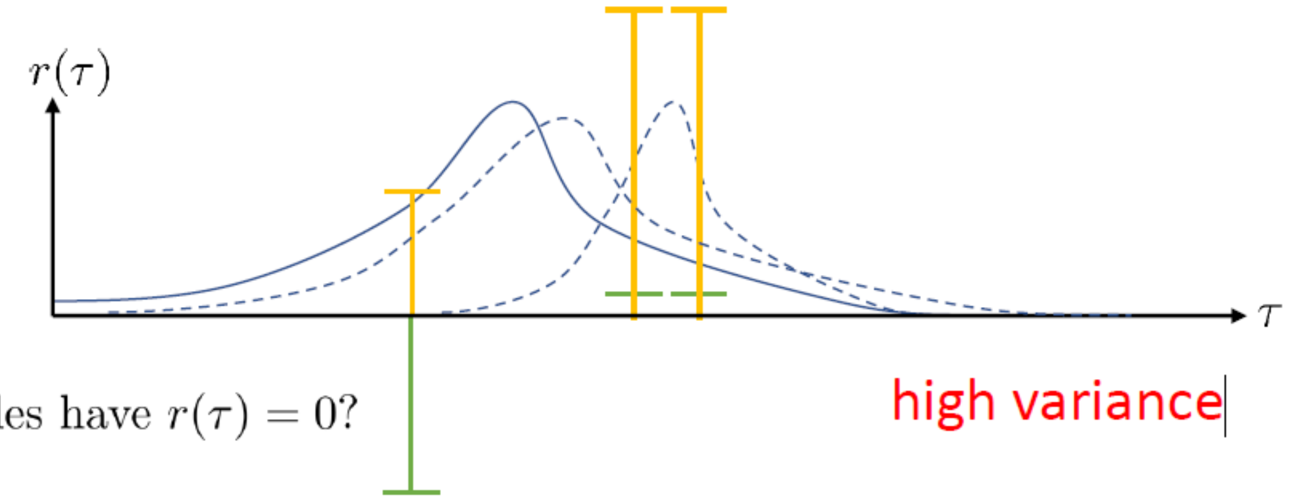
Robotics: Real world experiments are not easy

Real experiments may be dangerous

Perform roll outs in simulation.

Gradient Computation is Impacted by Variance during Averaging

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \nabla_\theta \log \pi_\theta(\tau) r(\tau)$$



even worse: what if the two "good" samples have $r(\tau) = 0$?

high variance

- Computation of the gradient of the PG objective is the crucial part of the method.
- This gradient is affected by high variance and those samples with zero rewards.
- This motivates research into approaches for reducing variance in the gradient steps.

Examining the Policy Gradient Objective

$$\nabla_\theta J(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t})\right)\left(\sum_{t=1}^{T} r(\mathbf{s}_{i,t},\mathbf{a}_{i,t})\right)$$

Causality: policy at time t' cannot affect reward at time t when $t < t'$

$$\nabla_\theta J(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t}) \underbrace{\left(\sum_{t'=t}^{T} r(\mathbf{s}_{i,t'},\mathbf{a}_{i,t'})\right)}_{\text{“reward to go”}}$$

$$\hat{Q}_{i,t}$$

Note:
- Since only future rewards matter from a time step, view it as a Q() function that measures reward to go after taking the current action.

Policy Gradient with Automatic Differentiation

$$\nabla_\theta J(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{\nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t})}\hat{Q}_{i,t}$$

pretty inefficient to compute these explicitly!

How can we compute policy gradients with automatic differentiation?

We need a graph such that its gradient is the policy gradient!

maximum likelihood: $\nabla_\theta J_{\mathrm{ML}}(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t})$ $\quad J_{\mathrm{ML}}(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t})$

Just implement "pseudo-loss" as a weighted maximum likelihood:

$$\tilde{J}(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t})\hat{Q}_{i,t}$$

cross entropy (discrete) or squared error (Gaussian)

Key Idea:

- Setup a computation graph that reflects the objective to be optimized.
- Use automatic differentiation to optimize. Invoke a package like PyTorch.

Policy gradient with automatic differentiation

Pseudocode example (policy with discrete actions):

Maximum likelihood:

```
# Given:
# actions -(N*T) x Da tensor of actions
# states -(N*T) x Ds tensor of states
# Build the graph:
logits = policy.predictions(states) # This should return (N*T) x Da tensor of action logits
negative_likelihoods= tf.nn.softmax_cross_entropy_with_logits(labels=actions, logits=logits)
loss = tf.reduce_mean(negative_likelihoods)
gradients = loss.gradients(loss, variables)
```

Note:
- If the output is a discrete label (not a Gaussian likelihood) then use a cross-entropy loss.

Policy gradient with automatic differentiation

Pseudocode example (with discrete actions):

Policy gradient:

```
# Given:
# actions -(N*T) x Da tensor of actions
# states -(N*T) x Ds tensor of states
# q_values–(N*T) x 1 tensor of estimated state-action values
# Build the graph:
logits = policy.predictions(states) # This should return (N*T) x Da tensor of action logits
negative_likelihoods= tf.nn.softmax_cross_entropy_with_logits(labels=actions, logits=logits)
weighted_negative_likelihoods= tf.multiply(negative_likelihoods, q_values)
loss = tf.reduce_mean(weighted_negative_likelihoods)
gradients = loss.gradients(loss, variables)
```

$$\tilde{J}(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t}) \hat{Q}_{i,t}$$

q_values

Reducing Variance by Subtracting Baseline

$$\nabla_\theta J(\theta) \approx \frac{1}{N}\sum_{i=1}^{N} \nabla_\theta \log p_\theta(\tau)[r(\tau) - b]$$

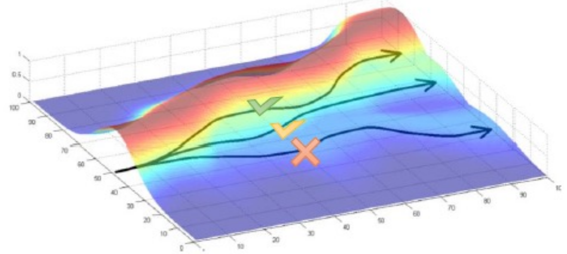
a convenient identity

$$p_\theta(\tau)\nabla_\theta \log p_\theta(\tau) = \nabla_\theta p_\theta(\tau)$$

$$b = \frac{1}{N}\sum_{i=1}^{N} r(\tau)$$

but... are we *allowed* to do that??

In a sense, does subtracting the baseline introduce a bias in the estimate?



$$E[\nabla_\theta \log p_\theta(\tau) b] = \int p_\theta(\tau)\nabla_\theta \log p_\theta(\tau) b\, d\tau = \int \nabla_\theta p_\theta(\tau) b\, d\tau = b\nabla_\theta \int p_\theta(\tau) d\tau = b\nabla_\theta 1 = 0$$

subtracting a baseline is *unbiased* in expectation!

average reward is *not* the best baseline, but it's pretty good!

The estimator is still unbiased, so fine to subtract the mean as baseline.

We will examine more methods possibly later.

Analyzing variance

Can we write down the variance?

$$\text{Var}[x] = E[x^2] - E[x]^2$$

$$\nabla_\theta J(\theta) = E_{\tau \sim p_\theta(\tau)}[\nabla_\theta \log p_\theta(\tau)(r(\tau) - b)]$$

$$\text{Var} = E_{\tau \sim p_\theta(\tau)}[(\nabla_\theta \log p_\theta(\tau)(r(\tau) - b))^2] - \underbrace{E_{\tau \sim p_\theta(\tau)}[\nabla_\theta \log p_\theta(\tau)(r(\tau) - b)]^2}_{\text{this bit is just } E_{\tau \sim p_\theta(\tau)}[\nabla_\theta \log p_\theta(\tau) r(\tau)]}$$

(baselines are unbiased in expectation)

$$\frac{d\text{Var}}{db} = \frac{d}{db} E[g(\tau)^2 (r(\tau) - b)^2] = \frac{d}{db}\left(\cancel{E[g(\tau)^2 r(\tau)^2]} - 2E[g(\tau)^2 r(\tau) b] + b^2 E[g(\tau)^2]\right)$$

$$= -2E[g(\tau)^2 r(\tau)] + 2bE[g(\tau)^2] = 0$$

$$b = \frac{E[g(\tau)^2 r(\tau)]}{E[g(\tau)^2]} \longleftarrow$$

This is just expected reward, but weighted by gradient magnitudes!