# COL778: Principles of Autonomous Systems
# Semester II, 2023-24

## Sate Estimation - I

**Rohan Paul**

# Today's lecture

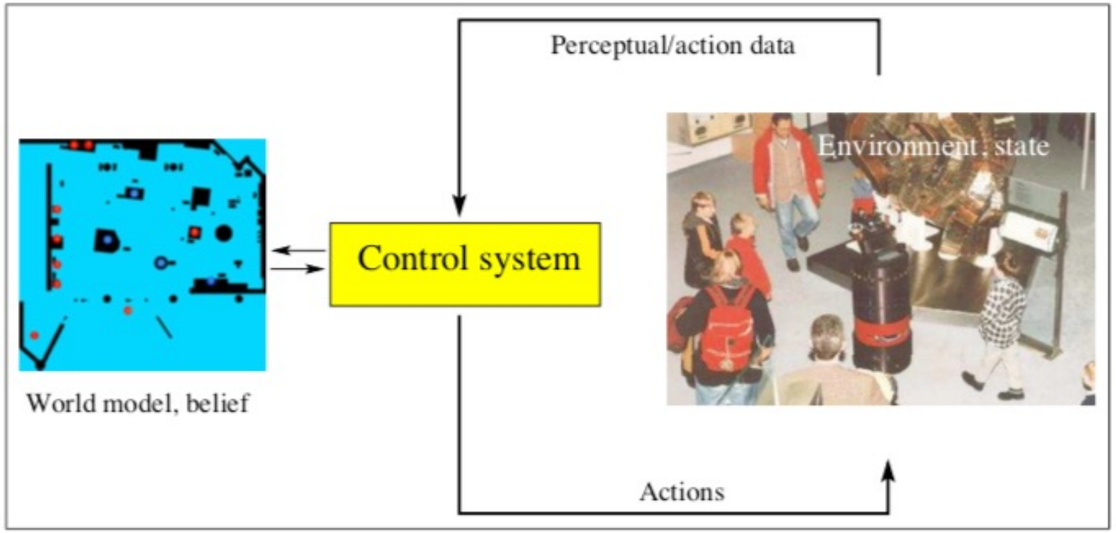
- Last Class
  - Agent Representation II (Sensing)
- This Class
  - State Estimation
    - Recursive State Estimation
    - Bayes Filter
  - References
    - Probabilistic Robotics Ch 1 & 2
    - AIMA Ch 15 (till sec 15.3)

# Acknowledgements

**These slides are intended for teaching purposes only. Some material has been used/adapted from web sources and from slides by Nicholas Roy, Wolfram Burgard, Dieter Fox, Sebastian Thrun, Siddharth Srinivasa, Dan Klein, Pieter Abbeel and others.**

# Robot Environment Interaction

- Environment or world
    - Objects, robot, people, interactions
    - Environment possesses a true internal state
- Observations
    - The agent cannot directly access the true environment state.
    - Takes observations via its sensors which are error prone.
- Belief
    - Agent maintains a belief or an estimate with respect to the state of the environment derived from observations.
    - The belief is used for decision making
- Actions
    - Agent can influence the environment through its physical interactions (actuations, motions, language interaction etc. )
    - The effect of actions may be stochastic.
    - Taking actions affects the world state and the robot's internal belief with regard to this state.

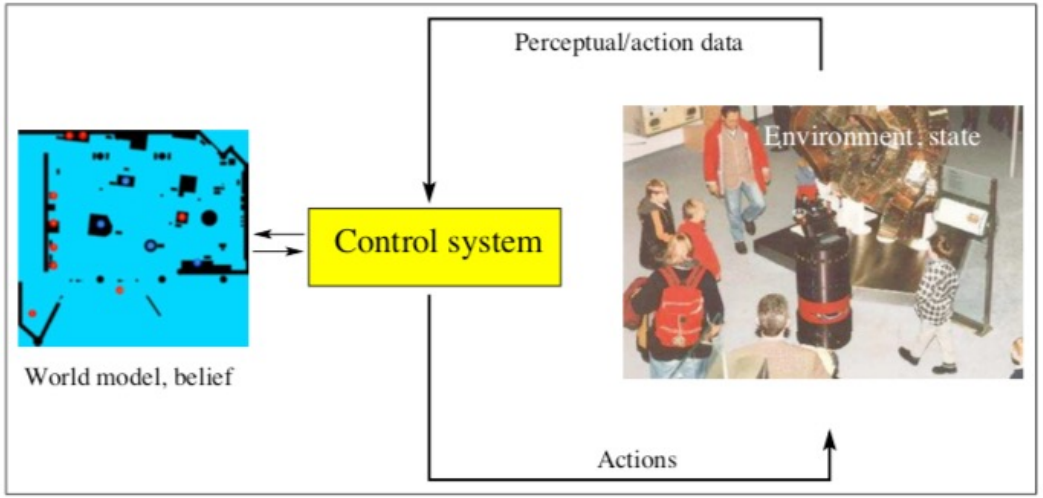


# State Estimation

- Framework for estimating the state from sensor data.
- Estimating quantities that are *not directly observable.*
  - *But* can be inferred if certain quantities are available to the agent.
- State estimation algorithms
  - Compute *belief distributions* over *possible states* of the world.

# State

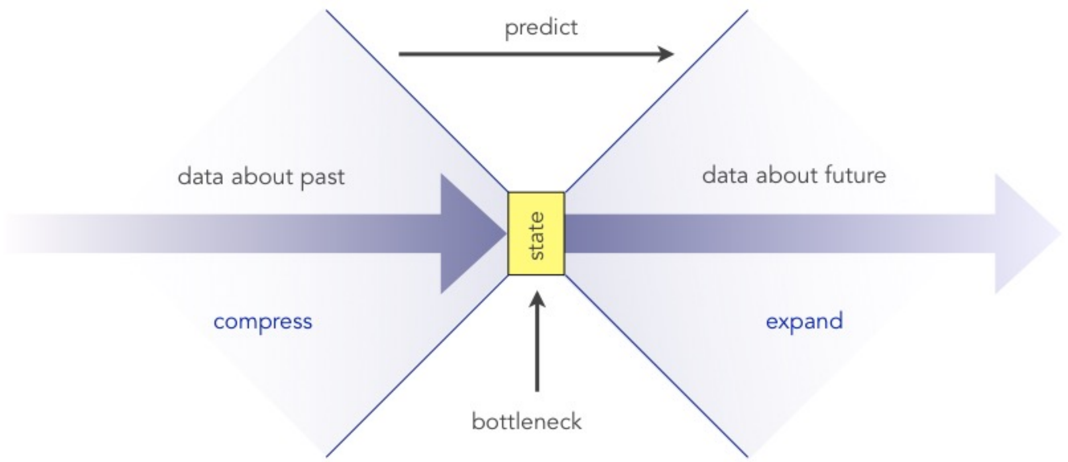
- What is typically part of the state, x?
  - Robot pose: position and orientation or kinematic state
  - Velocities: of the robot and other objects like people.
  - Location and features of surrounding objects in the environment.
  - Semantic states: is the door open or closed?
  - ……..
- What is put in the state is influenced by which task we seek to perform
  - Navigation
  - More complex example (e.g., delivery of hospital supplies)



# State

- Environment is characterized by the state.
  - *“A collection of all aspects of the agent and its environment that can impact the future”*
- A sufficient statistic of the past observations and interactions required for future tasks.
- State plays and important role for decision making.

Figure courtesy Byron Boots




State: statistic of history sufficient to predict the future

**Markovian** assumption:

Future is independent of past given present

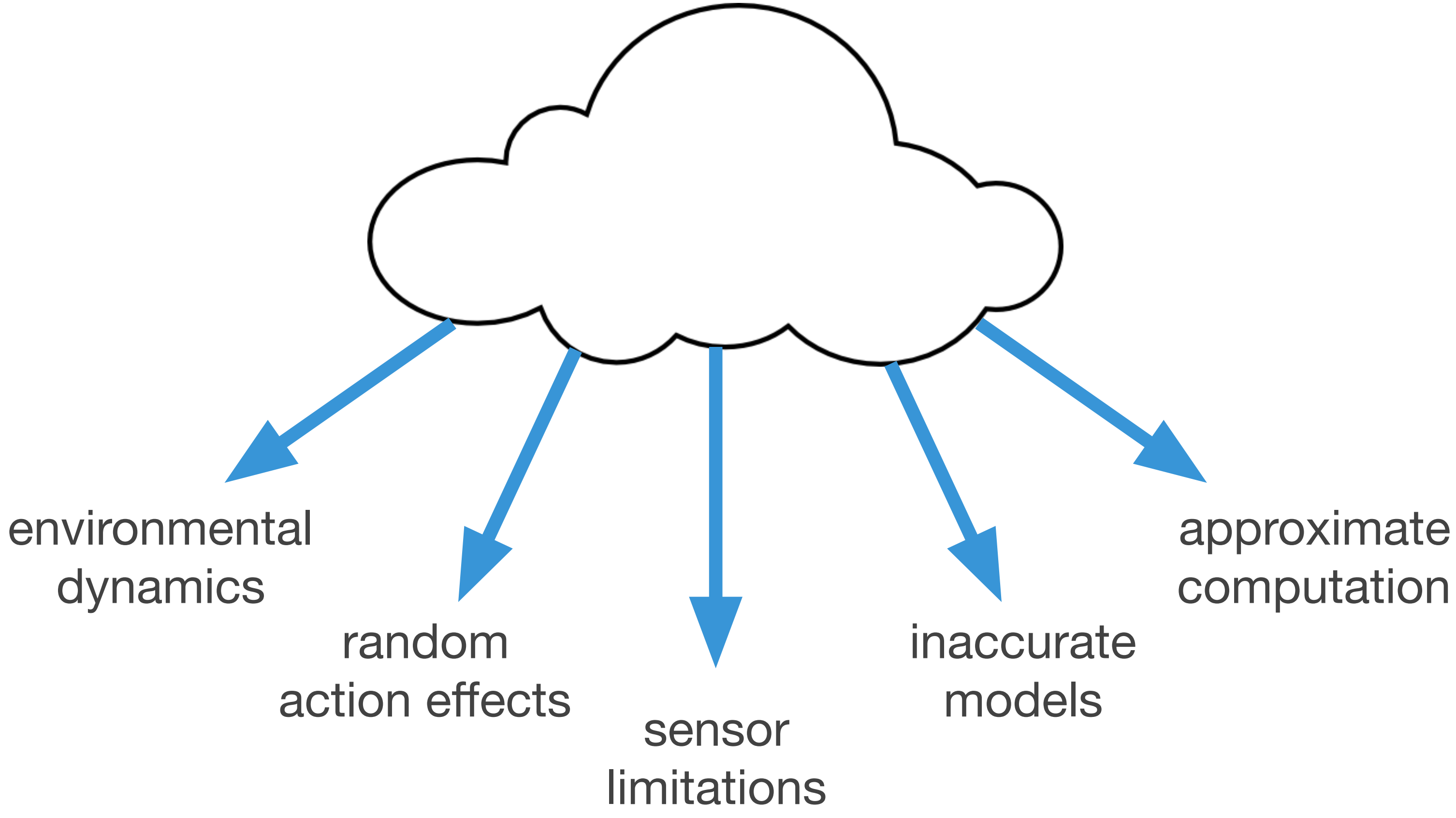
# Two aspects: Sensing and Taking Actions

- **Taking Sensor Measurements**
  - Camera, range, tactile, language query etc.
  - Denote measurement data as $z_t$
  - Noisy observations of the true state.
  - Measurements typically add information, *decrease uncertainty.*

- **Taking Actions (or Controls)**
  - Physical interaction: robot motion, manipulation of objects, *NO_OP* etc.
  - Carry information about the change of state.
  - Source of control data: odometers or wheel encoders.
  - Denote control data as $u_t$
  - Actions are never carried out with absolute certainty.
  - In contrast to measurements, actions generally *increase uncertainty.*

# Uncertainty

Explicitly represent uncertainty using probability theory.



# Probability Recap

## Independence

- X and Y are independent iff

  $P(x,y) = P(x)\,P(y)$

- $P(x \mid y)$ is the probability of $x$ given $y$

  $P(x \mid y) = P(x,y) / P(y)$

  $P(x,y) = P(x \mid y)\,P(y)$

- If X and Y are independent then

  $P(x \mid y) = P(x)$

## Marginalization

**Discrete case**

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x,y)$$

$$P(x) = \sum_y P(x \mid y) P(y)$$

**Continuous case**

$$\int p(x)\, dx = 1$$

$$p(x) = \int p(x,y)\, dy$$

$$p(x) = \int p(x \mid y) p(y)\, dy$$

# Conditioning

- Law of total probability:

Marginalize out z.

$$P(x) = \int P(x, z) dz$$

$$P(x) = \int P(x \mid z) P(z) dz$$

Conditioning on the extra variables.

$$P(x \mid y) = \int P(x \mid y, z)\, P(z \mid y)\, dz$$

# Conditional Independence

- X and Y are conditionally independent given Z.
- Given Z , X does not add information about Y and vice versa.

$$P(x,y|z)=P(x|z)P(y|z)$$

$$P(x|z)=P(x|z,y)$$

$$P(y|z)=P(y|z,x)$$

# Bayes Rule

$$P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\Rightarrow$$

$$\boxed{P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}}$$

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \eta\ P(y \mid x)\ P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x)P(x)}$$

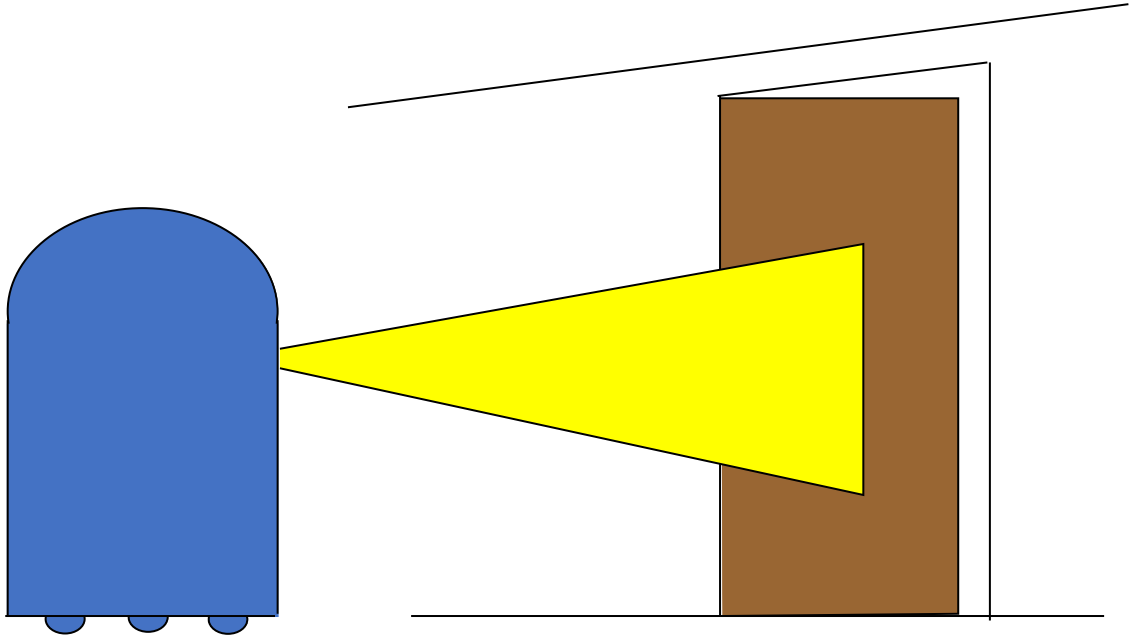
# Bayes Rule with Background Knowledge

When extra information is available. Incorporate that knowledge as observations of extra variables.

$$P(x \mid y, z) = \frac{P(y \mid x, z)\ P(x \mid z)}{P(y \mid z)}$$

# Example of State Estimation

- The robot wants to estimate the state of the door as closed or open
  - Has a noisy sensor that produces measurement, z
- Estimate: P(open|z)?
  - Likelihood that the true state of the door is open given that z was measured.



$$P(z \mid open) = 0.6 \qquad P(z \mid \neg open) = 0.3$$

$$P(open) = P(\neg open) = 0.5$$

The observation z is correlated with the true state as open or not-open.
E.g., measuring a particular distance or classifying an image.

# Causal vs. Diagnostic Reasoning

- P(open|z) is diagnostic reasoning
  - *Given that I observe z what is the likelihood that the door state is actually open?*
- P(z|open) is causal reasoning (can estimate by counting frequencies)
  - *Given that the door state is open what is the likelihood of getting measurement z?*
- Often causal knowledge is easier to obtain
  - Given the underlying state collect the data.
- Bayes rule enables the use of causal knowledge:

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z)}$$

# Incorporating a single measurement

- **Higher** likelihood of observation z when the door **is open** compared to when the door is closed.
- The incorporation of the measurement *z* **raises** the probability that the door is open.

$$P(z \mid open) = 0.6 \qquad P(z \mid \neg open) = 0.3$$

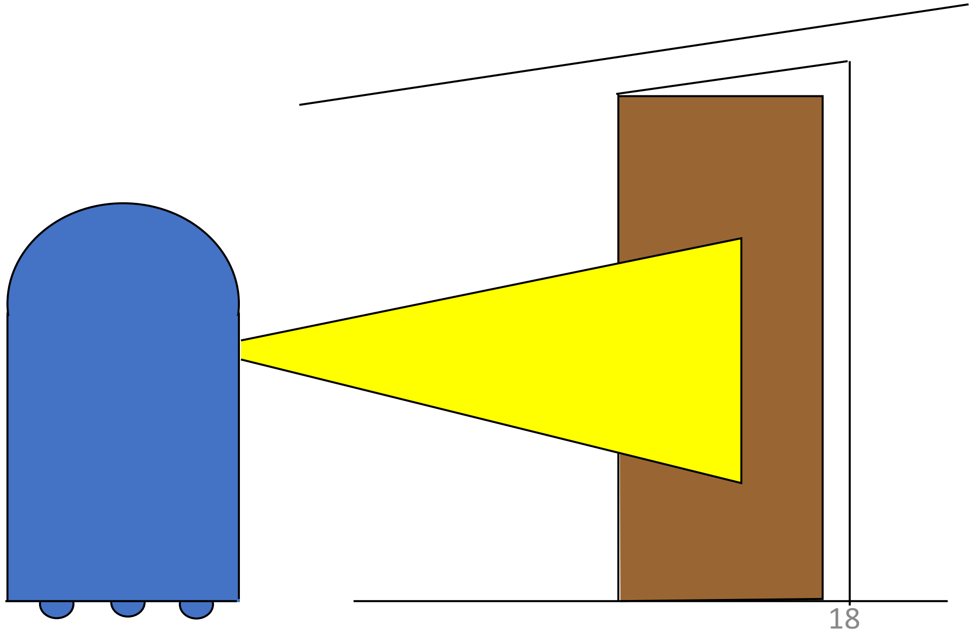
$$P(open) = P(\neg open) = 0.5$$

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)\,p(open) + P(z \mid \neg open)\,p(\neg open)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67$$

# Incorporating multiple measurements

- Suppose the robot has another sensor that produces a second observation $z_2$
- How can we combine the measurement of the second sensor
- What is $P(open|z_1, z_2)$?
- In general, how to estimate $P(x|z_1 \ldots z_n)$



# Recursive Bayesian Updating

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1})\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

Markov assumption: $z_n$ is conditionally independent of $z_1, \ldots, z_{n-1}$ given *x*.

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

$$= \eta\, P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})$$

$$= \eta_{1\ldots n} \prod_{i=1\ldots n} P(z_i \mid x)\, P(x)$$

In our causal modeling view, the world state is *causing* all the observations.

# Incorporating second sensor measurement

- Higher likelihood of observation z when the door is **not** open compared to when the door is open.
- The inclusion of the second measurement $z_2$ **lowers** the probability for the door to be open.

$$P(z_2 \mid open) = 0.5 \qquad P(z_2 \mid \neg open) = 0.6$$

$$P(open \mid z_1) = 2/3 \qquad P(\neg open \mid z_1) = 1/3$$

$$P(open \mid z_2, z_1) = \frac{P(z_2 \mid open)\, P(open \mid z_1)}{P(z_2 \mid open)\, P(open \mid z_1) + P(z_2 \mid \neg open)\, P(\neg open \mid z_1)}$$

$$= \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{2} \cdot \frac{2}{3} + \frac{3}{5} \cdot \frac{1}{3}} = \frac{5}{8} = 0.625$$

# Sensor Model

- Sensor model
  - What is the likelihood of obtaining this sensor measurement given the true state?
  - A conditional distribution over observations given the true state. Generative Model.
  - Observations or measurements can be considered as the noisy projection of the state

$$p(z_t|x_t)$$

# Action Model

- Action or Motion model
  - How the actions or controls change the state of the world?
  - Incorporate the outcome of an action u into the current “belief”, we use the conditional distribution.
  - Specifies how does the state change by application of the action (from the state, $x_{t-1}$ to the state, $x_t$ by executing the action, $u_t$).

$$p(x_t | x_{t-1}, u_t)$$

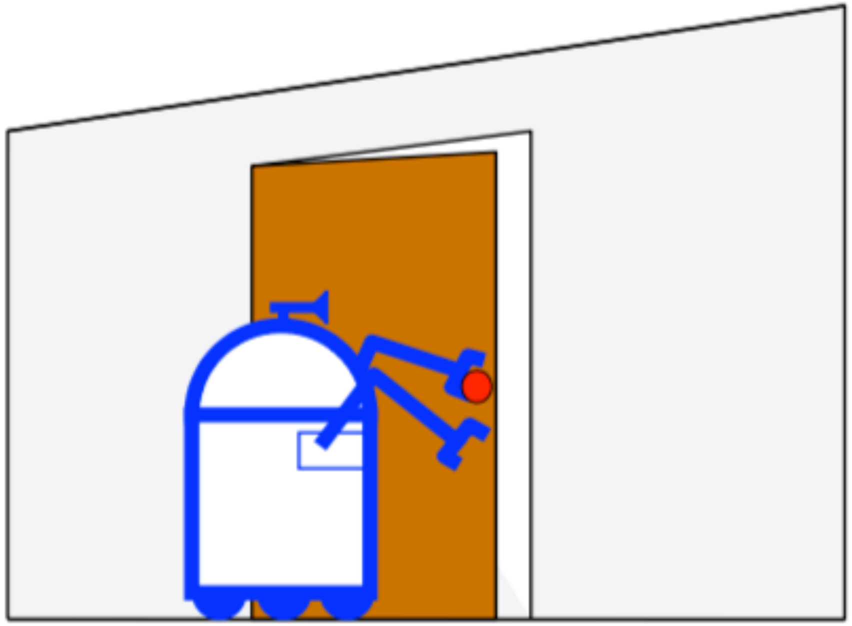
# Belief over the world state

- Belief
  - Expresses the agent's internal knowledge about the state of an aspect of the world.
  - Note: we do not know the true state.
  - The belief is estimated from the sensor measurement data and the actions taken till now.

$$Bel(x_t) = p(x_t | z_{1:t}, u_{1:t})$$

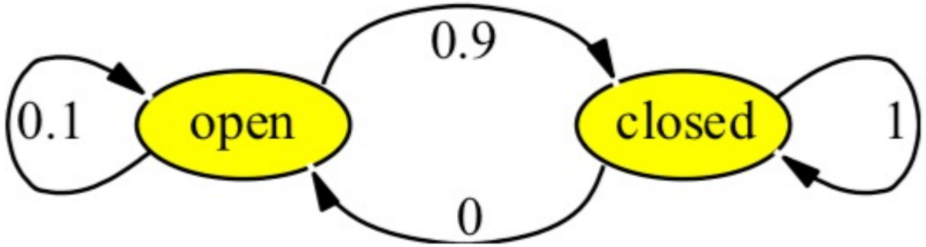
# Example: Incorporating action effects

**Example: Closing the door**



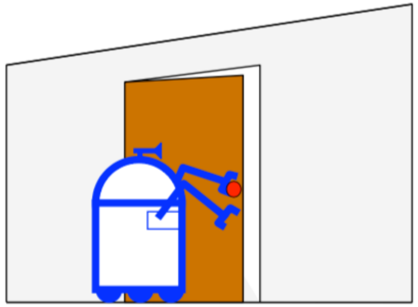
**Probabilistic effects**

$P(x|u,x')$ for $u$ = "close door":




If the door is open, the action "close door" succeeds in 90% of all cases.

# Example: The Resulting Belief



**Marginalizing out the outcome of actions using the distribution over the state at the last time step.**

Continuous case:

$$P(x \mid u) = \int P(x \mid u, x')P(x')dx'$$

Discrete case:

$$P(x \mid u) = \sum P(x \mid u, x')P(x')$$

$$\begin{aligned}
P(closed \mid u) &= \sum P(closed \mid u, x')P(x') \\
&= P(closed \mid u, open)P(open) \\
&\quad + P(closed \mid u, closed)P(closed) \\
&= \frac{9}{10} * \frac{5}{8} + \frac{1}{1} * \frac{3}{8} = \frac{15}{16} \\
P(open \mid u) &= \sum P(open \mid u, x')P(x') \\
&= P(open \mid u, open)P(open) \\
&\quad + P(open \mid u, closed)P(closed) \\
&= \frac{1}{10} * \frac{5}{8} + \frac{0}{1} * \frac{3}{8} = \frac{1}{16} \\
&= 1 - P(closed \mid u)
\end{aligned}$$

# Incorporating Measurements

- Bayes rule

$$P(x \mid z) = \frac{\boxed{P(z \mid x)}\, P(x)}{P(z)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

# Bayes Filter

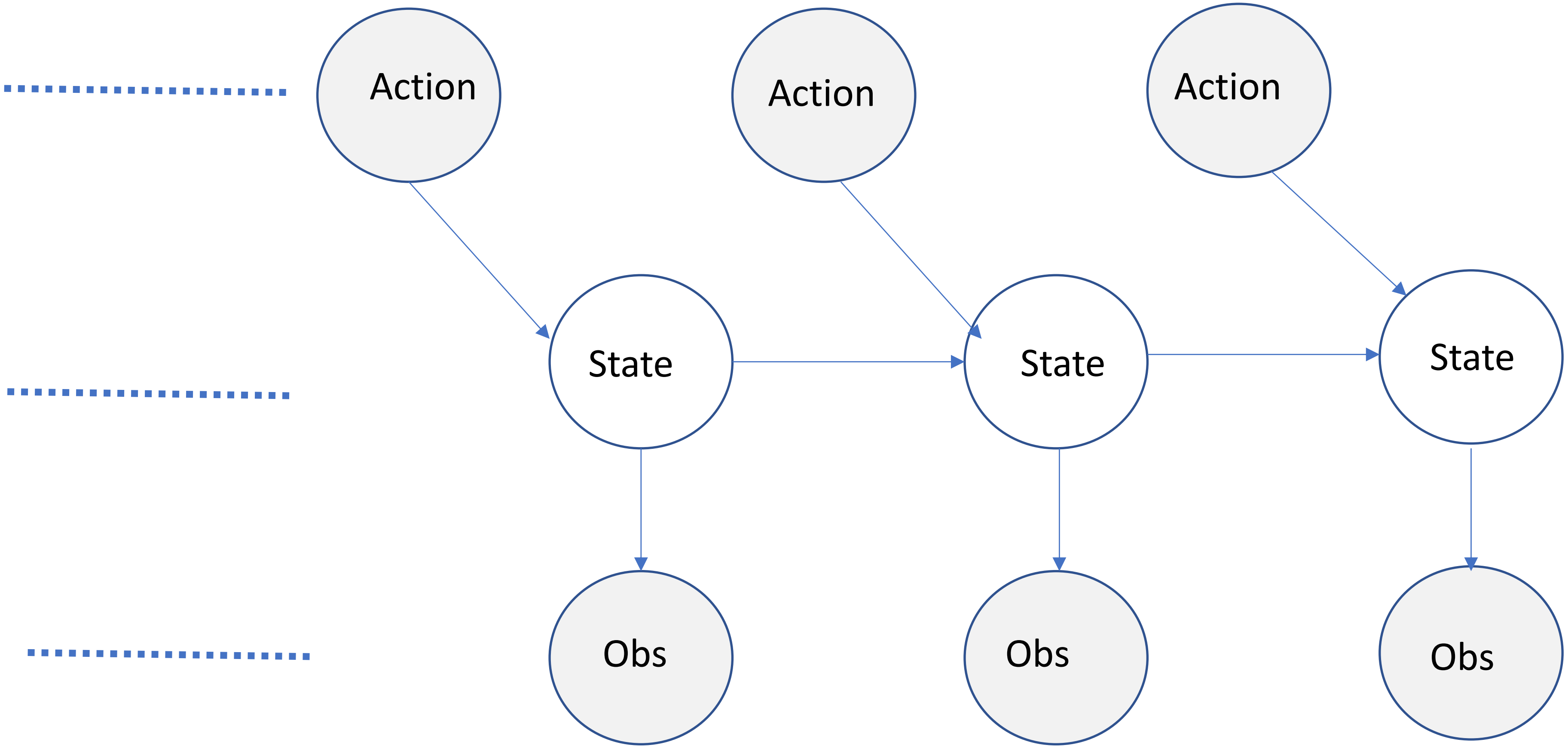
- **Given:**
  - Stream of observations *z* and action data *u:* $d_t = \{u_1, z_2 \ldots, u_{t-1}, z_t\}$
  - Sensor model $p(z_t | x_t)$
  - Action model $p(x_t | x_{t-1}, u_t)$
  - Prior probability of the system state *P(x).*

- **What we want to estimate?**
  - The state at time t
  - A belief or the posterior over the state: $\boxed{Bel(x_t) = P(x_t \mid u_1, z_2 \ldots, u_{t-1}, z_t)}$

***Intuitively: Given all observations collected by the agent till time t and all the actions taken by the agent till time t, what is our estimate over its state?***

# In essence

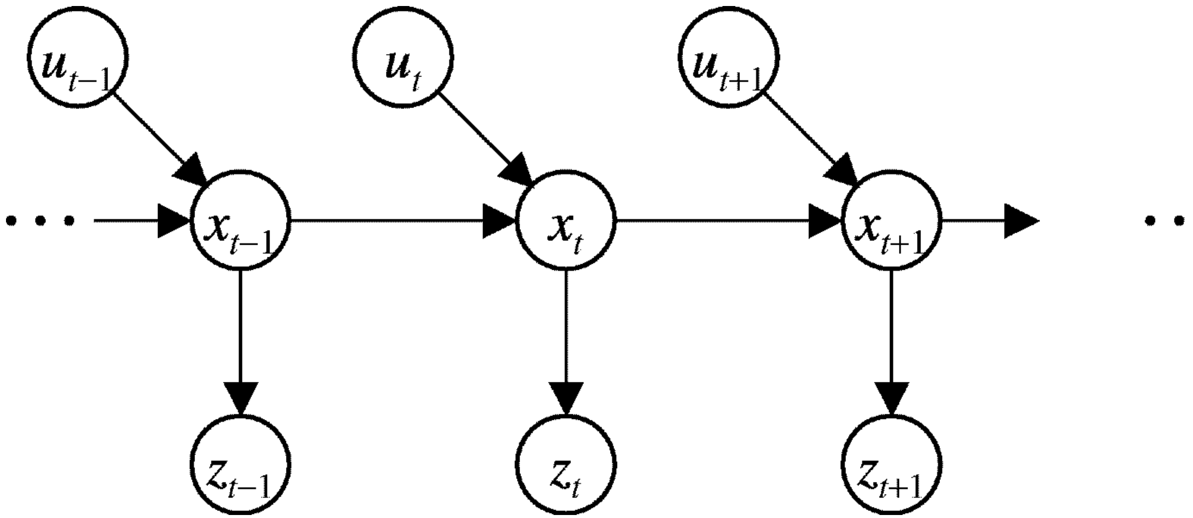
We estimate the state of the agent via measurements and knowledge of what actions were taken. Bayes Filter provides a recursive way to estimate the likelihood given the conditional independence assumptions.



# Formally, a Generative Model

**Assumptions**

- Static world
- Independent noise
- Perfect model, no approximation errors
- Markov assumption (once you know the state the past actions and observations do not affect the future).



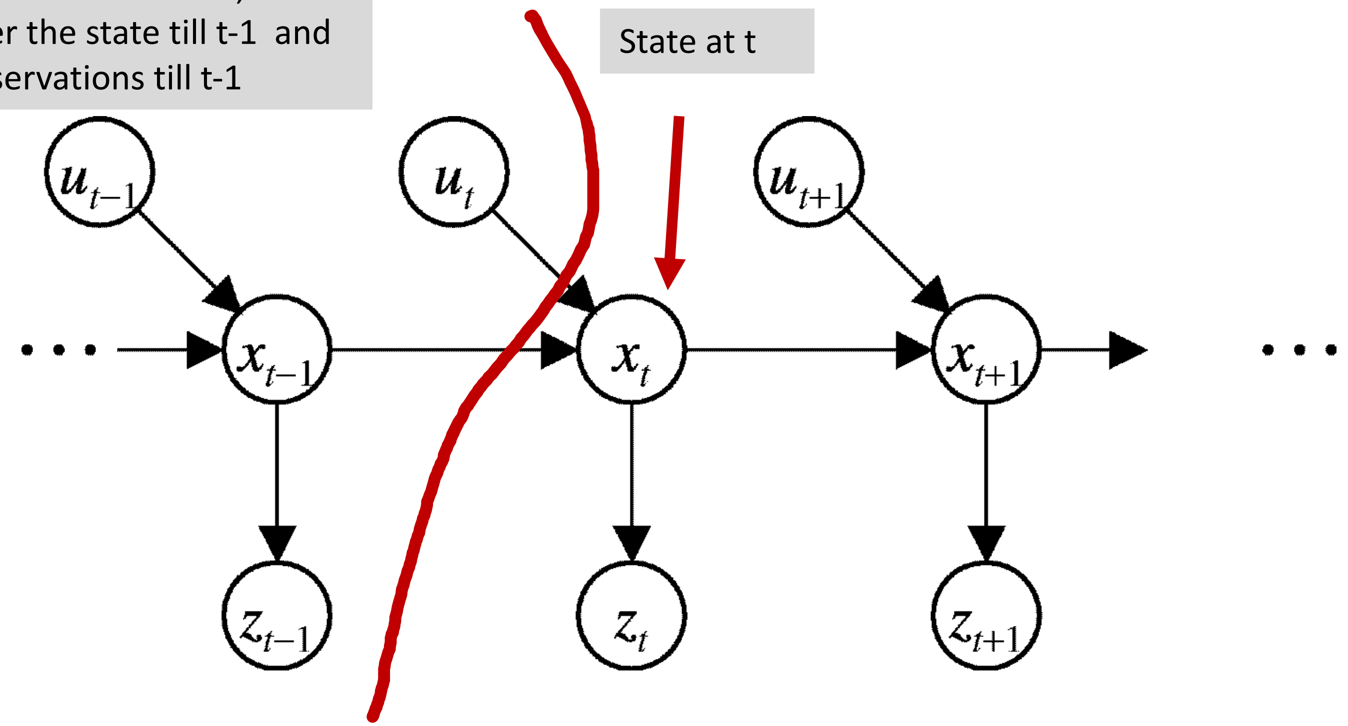

$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$

$$p(x_t \mid x_{1:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

# Generative Model

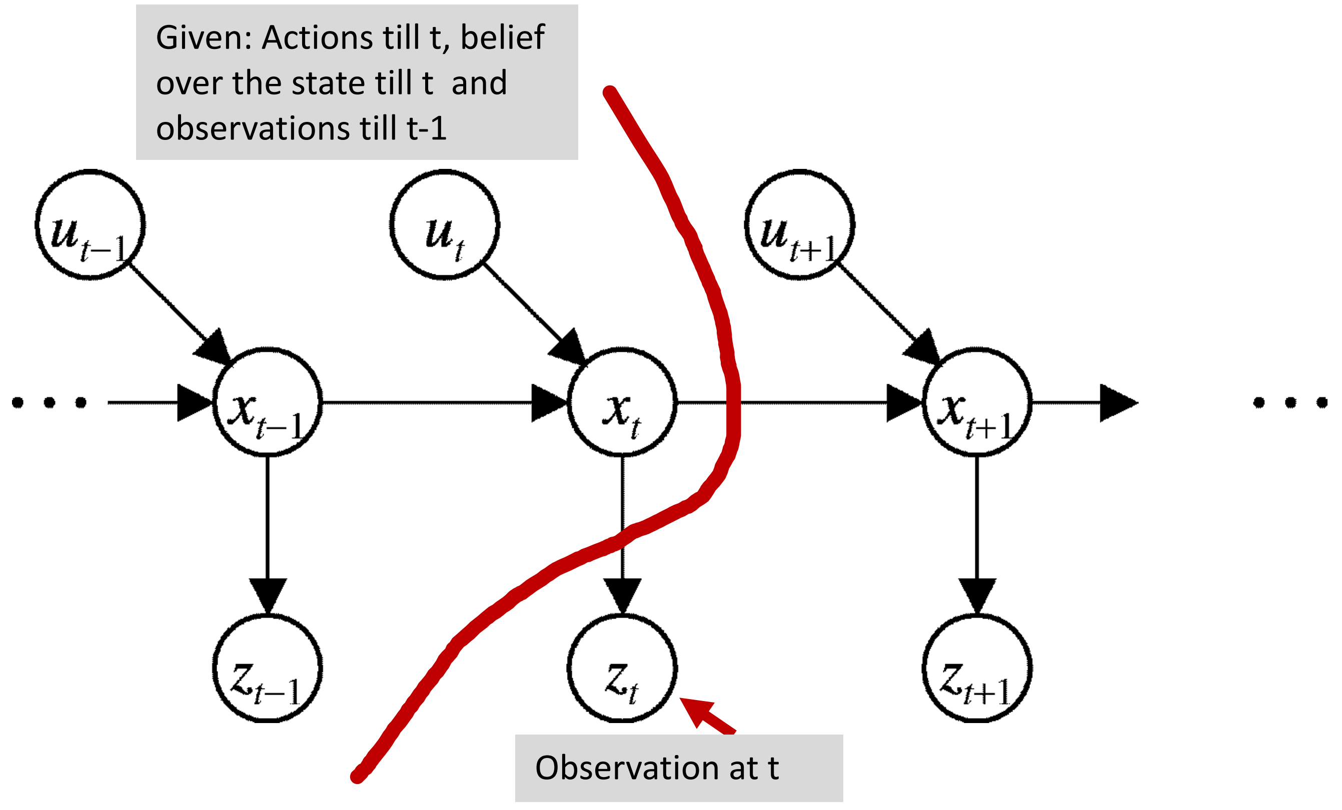
Given: Actions till t, belief over the state till t-1  and observations till t-1

State at t




$$p(x_t \mid x_{1:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

# Generative Model




$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$

# Bayes Filters

$z$ = observation
$u$ = action
$x$ = state

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

**Bayes**
$$= \eta \; P(z_t \mid x_t, u_1, z_1, \ldots, u_t) \; P(x_t \mid u_1, z_1, \ldots, u_t)$$

**Markov**
$$= \eta \; P(z_t \mid x_t) \; P(x_t \mid u_1, z_1, \ldots, u_t)$$

**Total prob.**
$$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \, dx_{t-1}$$

**Markov**
$$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \, dx_{t-1}$$

$$\boxed{= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; Bel(x_{t-1}) \, dx_{t-1}}$$

# Bayes Filters Algorithm

1. **Algorithm Bayes_filter** ( *Bel(x), d*):
2. $n=0$
3. **If *d* is a perceptual data item *z* then**
4. For all $x$ do
5. $$Bel'(x) = P(z \mid x) Bel(x)$$
6. $$\eta = \eta + Bel'(x)$$
7. For all $x$ do
8. $$Bel'(x) = \eta^{-1} Bel'(x)$$
9. **Else if *d* is an action data item *u* then**
10. For all $x$ do
11. $$Bel'(x) = \int P(x \mid u, x')\, Bel(x')\, dx'$$
12. **Return *Bel'(x)***

$$\boxed{Bel(x_t) = \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\, Bel(x_{t-1})\, dx_{t-1}}$$

# Bayes Filter: Takeaways

- Bayes filters are a probabilistic tool for estimating the state of with observations acquired over time.
  - Target is to obtain the belief over the current state given past actions and observations.
  - Estimate this distribution in a recursive manner.
  - Update using actions
  - Update using measurements.
- Bayes rule allows us to compute probabilities that are difficult to determine otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.