

Assignment 3

Real and Complex Analysis

MTL122/ MTL503/ MTL506

Lecturer: A. Dasgupta

aparajita.dasgupta@gmail.com

- (1) Let A and B be disjoint closed subsets of a metric spaces (X, d) . Prove that there are disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.
- (2) Let (X, d) be a metric space with $E \subset X$. Prove that $(E^\circ)^c = \overline{(E^c)}$.
- (3) A point x not belonging to a closed set $M \subset (X, d)$ always has a nonzero distance from M . (Hint: To prove this, show that $x \in \bar{A}$ if and only if $D(x, A) = \text{dist}(x, A) = \inf_{y \in A} d(x, y) = 0$; here A is any nonempty subset of X).
- (4) Let A and B be non-empty subsets of a metric space (X, d) . Prove that
 - (i) $A \subset B$ implies $\text{diam}(A) \leq \text{diam}(B)$.
 - (ii) $\text{diam}(A) = 0$ if and only if for some $x \in X$, $A = \{x\}$.
 - (iii) If $a \in A$ and $b \in B$, then
$$\text{diam}(A \cup B) \leq \text{diam}(A) + \text{diam}(B) + d(a, b).$$
- (5) Let (X, d) be a metric space with the property that every bounded sequence has a convergent subsequence. Prove that X is complete. Does Bolzano-Weierstrass theorem holds for any metric space? Give reasons/counterexamples.
- (6) If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d) , show that (a_n) , where $a_n = d(x_n, y_n)$, converges.
- (7) Let $X = (X, d)$ be a metric space and $CS(X)$ the collection of all Cauchy sequences in X . For (x_n) and (y_n) in $CS(X)$, define

$$(x_n) \sim (y_n) \text{ if and only if } \lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

Show that \sim is an equivalence relation on $CS(X)$.

- (8) Show that the set X of all integers, with metric d defined by $d(m, n) = |m - n|$, is a complete metric space.

- (9) Show that (l^∞, d_∞) , d_∞ (defined in Lecture) is a complete metric space.
- (10) Show that the set of all real numbers constitutes an incomplete metric space if we choose $d(x, y) = |\arctan x - \arctan y|$.