Lecture 24

Real and Complex Analysis

MTL- 122

$f(z) = u(x,y) + iv(x,y)$

$\hookrightarrow$ analytic $\Rightarrow$ CR eqns satisfied.

$\Rightarrow$ $u_x = v_y$

$u_y = -v_x$

$u_{xx} = v_{xy}$

$u_{yy} = -v_{xy}$

$$\left.\begin{aligned} & \cdot\ u_{xx} + u_{yy} = 0 \\ & \cdot\ v_{xx} + v_{yy} = 0 \end{aligned}\right\} \text{Laplace eqn}$$

Then $u, v \rightarrow$ harmonic fn.

$\underline{v}$ is a conjugate of u.

Exponential.

e^x, $\forall\, x \in \mathbb{R}$

Want to define e^z.

Expect.

- e^z is an entire.
- $e^{z_1} e^{z_2} = e^{z_1+z_2}$, $z_1, z_2 \in \mathbb{C}$
- $e^0 = 1$.

$z = x + iy \in \mathbb{C}$.

$$e^z = e^{x+iy}$$
$$= e^x e^{iy}$$
$$= e^x \left(c(y) + i s(y) \right)$$

$e^0 = 1 \Rightarrow c(0) + i s(0) = 1$

Since

$$c(0) + i s(0) = 1 + i0$$

$$\left.\begin{aligned} c(0) &= 1 \\ s(0) &= 0 \end{aligned}\right\} \longrightarrow (1)$$

$$e^z = \underbrace{e^x c(y)}_{u(x,y)} + i \underbrace{e^x s(y)}_{v(x,y)}$$

$$\left.\begin{aligned} u(x,y) &= e^x c(y) \\ v(x,y) &= e^x s(y) \end{aligned}\right| \quad \forall z \in \mathbb{C}.$$

Using C-R eqns.

$$u_x = e^x c(y)$$
$$\parallel$$
$$v_y = e^x s'(y)$$

$$\Rightarrow \quad s' = c \quad \} (2)$$

$$c' = -s \Big\}$$

$$\left.\begin{array}{l} s'' + s = 0 \\ s(0) = 0 \end{array}\right\} \text{ Solving.}$$

$$\left.\begin{array}{l} s(y) = \sin y \\ c(y) = \cos y. \end{array}\right| \; y \in \mathbb{R}.$$

Euler's identity

$$e^{iy} = \cos y + i \sin y, \; y \in \mathbb{R}$$

$$\boxed{e^{z} = e^{x}(\cos y + i \sin y)}$$

$$\frac{d}{dz} e^{z} = e^{z}.$$

Ex. Find the zeros of $e^z = 1$

Soln: $z = x + iy$

Defn:

$e^x \cos y = 1$ ✓

$e^x \sin y = 0$ —

2nd eqn $\Rightarrow$

$y = k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$

If <u>k is odd</u> ×

we get

$e^x \cos \underline{k}\pi = 1$

or $\quad -e^x = 1$ ↯ ✗

Not possible.

We have to look,
for even k.

$$y = 2m\pi, \quad m = 0, \pm 1, \pm 2, \ldots$$

We get $e^x = 1$

$$\Rightarrow \quad x = 0.$$

$$\therefore \quad z = iy = 2k\pi i, \quad k = 0, \pm 1, \pm 2, \ldots$$

Exercise.

Prove that $w = e^z, z \in \mathbb{C}$ is periodic with period $2\pi i$, ie, $e^{z+2\pi i} = e^z, z \in \mathbb{C}$

Trigonometric

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Hyperbolic fn.

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

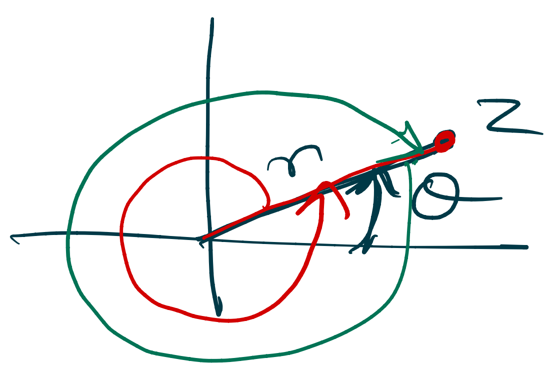
$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

Logarithms

$$z = x + iy$$

$$z \longrightarrow (r, \theta)$$

$$r = |z| = \sqrt{x^2 + y^2}$$



& , $x = r\cos\theta$

$y = r\sin\theta$.

If $z = (r, \theta)$

then $z = (r, \theta + 2k\pi)$,

$k \in \mathbb{Z}$.

We call

$\underbrace{\theta + 2k\pi}$ } → argument of z.

Demote it

argz.

$$\arg z = \theta + 2k\pi \quad k \in \mathbb{Z}.$$

Express, $z = x + iy$

as

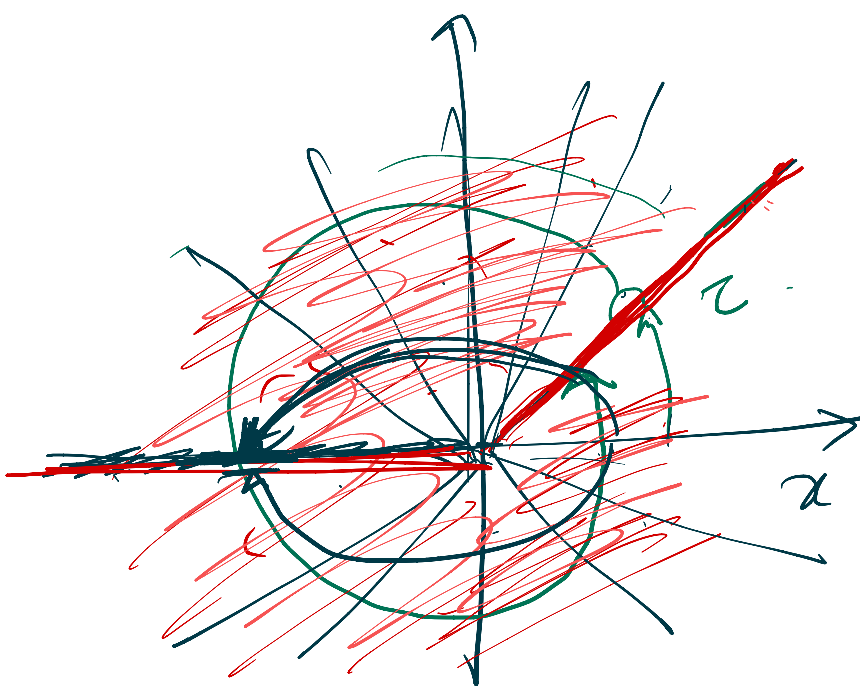
$$z = r(\cos\theta + i\sin\theta).$$

Cut planes.

Every $z \in \mathbb{C}$ if Θ is a value or a branch of $\arg z$

$$\boxed{\arg z = \Theta + 2k\pi, \quad k = 0, \pm 1, \pm 2 \ldots}$$

Let $\tau \in \mathbb{R}$.



Consider

$$C_\tau = \{ (r, \theta) \; ; \; r > 0, \; \tau < \theta \leq \tau + 2\pi \}.$$

This is called a cut plane along the branch cut

$\{(r, \tau) : r > 0\}$.

The branch of $\arg z$ that lies in

$(\tau, \tau + 2\pi] \rightarrow$ denoted by

$\arg_\tau z$.

Principal branch. $\rightarrow \operatorname{Arg} z$

$(-\pi, \pi]$

$\tau = -\pi$.

$\arg_{-\pi} z \in (-\pi, \pi + 2\pi]$

$= (-\pi, \pi]$

$$\boxed{\arg_{-\pi} z = \operatorname{Arg} z}$$

$$\boxed{C_z = C_{-\pi}}$$

Example

Find $z = 1 + i$

$\operatorname{Arg} z$ $\qquad$ $\arg_{\pi/2} z$.

Soln

$\arg z = \frac{\pi}{4} + 2k\pi$

$k = 0, \pm 1, \pm 2$ ....

$$\operatorname{Arg} z \in (-\pi, \pi]$$

$$\frac{\pi}{4} + 4\pi = \frac{17\pi}{4}$$

$$\arg_{\pi/2} z \in (\pi/2, \pi/2 + 2\pi]$$

$$= \left(\frac{\pi}{2}, \frac{5\pi}{2}\right]$$

$$\frac{10\pi}{4}$$

$$\operatorname{Arg} z = \pi/\cancel{2}4$$

$$\arg_{\pi/2} z = 9\pi/4$$

Logarithms

$$e^x = y \iff \ln y = x$$

$$w = \log z$$

$$\Updownarrow$$

$$z = e^w$$

We can write,

$w = u + iv$, $z = x + iy$.

$|z| = |e^w| = e^u \neq 0$.

$$\left[\therefore |z| = e^{u+i\theta} = |e^u (e^{i\theta})| = e^u \right]$$

$\Rightarrow z \neq 0$.

Let

$z = r e^{i\theta}$.

Then $r e^{i\theta} = z = e^w$.

$= e^u e^{iv}$

$\Rightarrow r = e^u = |z| \Rightarrow u = \ln r$.

$\Rightarrow v = \arg z = \theta + 2k\pi$.

$\Rightarrow$ Then

$$w = u + iv$$
$$= \ln r + i(\theta + 2k\pi)$$

$$\Rightarrow \boxed{\begin{aligned} \log z &= \ln r + i(\theta + 2k\pi) \\ &= \ln|z| + i(\theta + 2k\pi) \\ &= \ln|z| + i\arg z \end{aligned}}$$