Lecture 5

**Real and Complex Analysis**

MTL122/ MTL503/ MTL506

Lecturer: A. Dasgupta aparajita.dasgupta@gmail.com

## 1. Metric Spaces

**Definition 1.1.** *Let $X$ be a set. Define the Cartesian product* $X \times X = \{(x,y) \; : \; x, y \in X\}$.

**Definition 1.2.** *Let $d : X \times X \to \mathbb{R}$ be a mapping. The mapping $d$ is a* <u>*metric on $X$*</u> *if the following four conditions hold for all $x, y, z \in X$ :*

*i)* $d(x,y) = d(y,x)$,

*ii)* $d(x,y) \geq 0$,

*iii)* $d(x,y) = 0 \iff x = y$, *and*

*iv)* $d(x,z) \leq d(x,y) + d(y,z)$.

Given a metric $d$ on $X$, the pair $(X, d)$ is called a *metric space.*

Suppose $d$ is a metric on $X$ and that $Y \subseteq X$. Then there is an automatic metric $d_Y$ on $Y$ defined by restricting $d$ to the subspace $Y \times Y$,

$$d_Y = d|Y \times Y.$$

Together with $Y$, the metric $d_Y$ defines the automatic metric space $(Y, d_Y)$.

The elements of a metric space $(X, d)$ are usually referred to as **points**. If $x, y \in X$, then $d(x, y)$ is called the *distance* between $x$ and $y$. A set can have more than one metric defined on it.
If condition (3) is replaced by the condition

- $d(x,x) = 0,$ for all $x \in X$,

then $d$ is a pseudo-metric on $X$ and $(X, d)$ is a pseudo-metric space.

**Example 1.3.** *(1) Let $X = \mathbb{R}$ and for $x, y \in X$, define $d : X \times X \to \mathbb{R}$ by $d(x,y) = |x - y|$. Then $(\mathbb{R}, d)$ is a metric space. This metric is called the* ***usual metric*** *on* $\mathbb{R}$.

*(2) Let $X = \mathbb{C}$, the st of complex numbers. For $x, y \in X$, define $d : X \times X \to \mathbb{R}$ by $d(x,y) = |x - y|$. Then $(\mathbb{C}, d)$ is a metric space. This is metric called the usual metineric on* $\mathbb{C}$.

(3) *Let* $X = \mathbb{R}^n$, *where* $n$ *is a natural number. The elements of* $X$ *are ordered* $n$*-tuples* $x = (x_1, x_2, ..., x_n)$ *of real numbers. For* $x = (x_1, x_2, ..., x_n)$ *and* $y = (y_1, y_2, ..., )$ *in* $X$, *define*

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|,$$

$$d_2(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2},$$

$$d_\infty(x, y) = \max_{1 \le i \le n} |x_i - y_i|.$$

*Then each of* $d_1$, $d_2$ *and* $d_\infty$ *defines a metric on* $\mathbb{R}^n$.

*These metrics have special names attached to them:*

$d_1$ *is also called the* $1-$*metric.*

$d_2$ *is called the Euclidean metric or the usual metric on* $\mathbb{R}^n$.

$d_\infty$ *is called the supremum, maximum, or the infinity metric.*

i) *We leave it as an easy exercise to show that* $(\mathbb{R}^n, d_1)$ *is a metric space.*

ii) *We show that* $(\mathbb{R}^n, d_2)$ *is a metric space. Checking that* $d_2$ *satisfies properties* $(i)$, $(ii)$ *and* $(iii)$ *is straightforward. We prove property* $(iv)$. *To do that end, let* $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ *and* $z = (z_1, z_2, ..., z_n)$ *be elements of* $\mathbb{R}^n$. *We want to show that*

$$d_2(x, z) \le d_2(x, y) + d_2(y, z).$$

*This is equivalent to showing that*

$$\left[\sum_{i=1}^{n} (x_i - z_i)^2\right]^{1/2} \le \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} + \left[\sum_{i=1}^{n} (y_i - z_i)^2\right]^{1/2}. \tag{1.1}$$

*For each* $i = 1, 2, ..., n$, *let* $a_i = x_i - y_i$ *and* $b_i = y_i - z_i$. *Then* (1.1) *can be rewritten as*

$$\left[\sum_{i=1}^{n} (a_i + b_i)^2\right]^{1/2} \le \left[\sum_{i=1}^{n} a_i^2\right]^{1/2} + \left[\sum_{i=1}^{n} b_i^2\right]^{1/2}.$$

*Since both sides of the inequality are nonnegative, it suffices to show that inequality holds for the squares of the left and right hand sides of the inequality. That is, we have to show that*

$$\sum_{i=1}^{n} (a_i + b_i)^2 \le \sum_{i=1}^{n} a_i^2 + 2\left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2} + \sum_{i=1}^{n} b_i^2. \tag{1.2}$$

*It now follows that inequality (1.2) is equivalent to the inequality*

$$\sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2} \tag{1.3}$$

*Equation (1.3) is called the Cauchy-Schwarz Inequality. We now prove the Cauchy-Schwarz Inequality.*

***Cauchy-Schwarz Inequality:*** *If* $(a_1, a_2, ..., a_n) \in \mathbb{R}^n$ *and* $(b_1, b_2, ..., b_n) \in \mathbb{R}^n$*, then*

$$\sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2}.$$

*Proof.* If $a_i = 0$ for all $i = 1, 2, ..., n$ or $b_i = 0$ for all $i = 1, 2, ..., n$, then the inequality obviously holds. Assume that there is an $i \in \{1, 2, ..., n\}$ such that $a_i \neq 0$ and a $j \in \{1, 2, ..., n\}$ such that $b_j \neq 0$. For each $i = 1, 2, ..., n$, let

$$\alpha_i = \frac{a_i}{(\sum\limits_{i=1}^{n} a_i^2)^{1/2}} \text{ and } \beta_i = \frac{b_i}{(\sum\limits_{i=1}^{n} b_i^2)^{1/2}}.$$

Recall that if $a, b \in \mathbb{R}$, then $2ab \leq a^2 + b^2$. Therefore

$$\begin{aligned}
2\alpha_i\beta_i \leq \alpha_i^2 + \beta_i^2 \quad &\iff \quad \frac{2a_i b_i}{(\sum\limits_{i=1}^{n} a_i^2)^{1/2}(\sum\limits_{i=1}^{n} b_i^2)^{1/2}} \leq \frac{a_i^2}{\sum\limits_{i=1}^{n} a_i^2} + \frac{b_i^2}{\sum\limits_{i=1}^{n} b_i^2} \\
&\implies \quad \frac{2\sum_{i=1}^{n} a_i b_i}{(\sum\limits_{i=1}^{n} a_i^2)^{1/2}(\sum\limits_{i=1}^{n} b_i^2)^{1/2}} \leq \frac{\sum\limits_{i=1}^{n} a_i^2}{\sum\limits_{i=1}^{n} a_i^2} + \frac{\sum\limits_{i=1}^{n} b_i^2}{\sum\limits_{i=1}^{n} b_i^2} = 2 \\
&\implies \quad \frac{\sum_{i=1}^{n} a_i b_i}{(\sum\limits_{i=1}^{n} a_i^2)^{1/2}(\sum\limits_{i=1}^{n} b_i^2)^{1/2}} \leq 1 \\
&\implies \quad \sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2},
\end{aligned}$$

which proves the Cauchy-Schwarz Inequality. □

*iii) We show that* $(\mathbb{R}^n, d_\infty)$ *is a metric space. Let* $x = (x_1, x_2, ..., x_n)$*,* $y = (y_1, y_2, ..., y_n)$ *and* $z = (z_1, z_2, ..., z_n)$ *be elements of* $\mathbb{R}^n$*.*

*i) Since for each* $i = 1, 2, ..., n$*,* $|x_i - y_i| \geq 0$*, it follows that*

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i| \geq 0.$$

*ii)*

$$\begin{aligned} d_\infty(x,y)=0 &\iff \max_{1\le i\le n}|x_i-y_i|=0\\ &\iff |x_i-y_i|\le 0 \text{ for each } i=1,2,...,n\\ &\iff |x_i-y_i|=0 \text{ for each } i=1,2,...,n\\ &\iff x_i=y_i \text{ for each } i=1,2,...,n\\ &\iff x=y. \end{aligned}$$

*iii)* $d_\infty(x,y)=\max\limits_{1\le i\le n}|x_i-y_i|=\max\limits_{1\le i\le n}|y_i-x_i|=d_\infty(y,x)$.

*iv) Since, for each* $j=1,2,...,n$, $|x_j-z_j|\le|x_j-y_j|+|y_j-z_j|$, *it follows that*

$$|x_j-z_j|\le \max_{1\le i\le n}|x_i-y_i|+\max_{1\le i\le n}|y_i-z_i|=d_\infty(x,y)+d_\infty(y,z).$$

*Hence,*

$$d_\infty(x,z)=\max_{1\le j\le n}|x_j-z_j|\le d_\infty(x,y)+d_\infty(y,z).$$

*Hence,*

$$d_\infty(x,z)=\max_{1\le j\le n}|x_j-z_j|\le d_\infty(x,y)+d_\infty(y,z).$$

*iv) For* $1\le p<\infty$, *let* $X=l_p$ *be a set of sequences* $(x_i)_{i=1}^\infty$ *of real (or complex) numbers such that* $\sum\limits_{i=1}^{\infty}|x_i|^p<\infty$. *That is,*

$$l_p=\left\{x=(x_i)_{i=1}^\infty | \sum_{i=1}^{\infty}|x_i|^p<\infty\right\}.$$

*For* $x=(x_i)_{i=1}^\infty$ *and* $y=(y_i)_{i=1}^\infty$ *in* $l^p$, *define* $d_p: X\times X\to\mathbb{R}$ *by*

$$d_p(x,y)=\left(\sum_{i=1}^{\infty}|x_i-y_i|^p\right)^{1/p}.$$

*Then* $(l_p,d_p)$ *is a metric space.*

*Properties i), ii), iii) aare easy to prove. Property iv) requires Minkowski's Inequality:*
*If* $p>1$ *and* $(a_i)_{i=1}^\infty$ *and* $(b_i)_{i=1}^\infty$ *are in* $l_p$, *then*

$$\left(\sum_{i=1}^{\infty}|a_i+b_i|^p\right)^{\frac{1}{p}}\le\left(\sum_{i=1}^{\infty}|a_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{\infty}|a_i|^p\right)^{\frac{1}{p}}.$$

*v) Let* $X=l_\infty$ *be a set of bounded sequences of real (or complex) numbers. For* $x=(x_i)_{i=1}^\infty$ *and* $y=(y_i)_{i=1}^\infty$ *in* $X$, *define* $d: X\times X\to\mathbb{R}$ *by*

$$d(x,y)=\sum_{i=1}^{\infty}\frac{|x_i-y_i|}{2^i(1+|x_i-y_i|)}.$$

*Then* $(X,d)$ *is a metric space.*

*vi) Let $X$ be a set. For $x, y \in X$, define $d : X \times X \to \mathbb{R}$ by*

$$d(x,y) = \begin{cases} 0 & \textit{if } x = y \\ 1 & \textit{if } x \neq y \end{cases} \tag{1.4}$$

*Then $(X, d)$ is a metric space. This metric $d$ is called the **discrete metric** on $X$.*

**Proposition 1.4.** *Let $(X, d)$ be a metric space. Then for all $x, y, z \in X$,*

$$|d(x,z) - d(y,z)| \leq d(x,y).$$

By the triangle inequality we have that

$$d(x,z) \leq d(x,y) + d(y,z) \iff d(x,z) - d(y,z) \leq d(x,y) \tag{1.5}$$

Interchanging the roles of $x$ and $y$ in (1.5),

$$d(y,z) - d(x,z) \leq d(x,y). \tag{1.6}$$

It now follows from equations (1.5) and (1.6) that

$$|d(x,z) - d(y,z)| \leq d(x,y).$$