Assignment 4

Real and Complex Analysis

MTL122/ MTL503/ MTL506

Lecturer: A. Dasgupta

(1) Let (E, d) be a metric space, and let $f, g : E \to \mathbb{R}$ be bounded, uniformly continuous functions, where $\mathbb{R}$ is equipped with the usual metric. Show that the product $f \cdot g : E \to \mathbb{R}$ is bounded and uniformly continuous.

(2) Equip the interval $(0, 1) \subset \mathbb{R}$ with the usual metric.
   a) Show that if $f : (0, 1) \to \mathbb{R}$ is uniformly continuous, then it is bounded.

   b) Give an example of a function $f : (0, 1) \to \mathbb{R}$ that is continuous but unbounded.

(3) Prove that a continuous function on a compact metric space is bounded and uniformly continuous.

(4) Prove or give a counterexample:
   i) The union of infinitely many compact sets is compact.

   ii) A non-empty subset S of real numbers which has both a largest and a smallest element is compact.

(5) Prove that every subset of a totally bounded subset A of a metric space X is totally bounded in X.

(6) Let (X, d) be a metric space, and let $A, B \subset X$. Show that, if A and B are sequentially compact, then so is $A \cap B$.

(7) Let $f : X \to Y$ be a uniformly continuous surjective mapping between two metric spaces. Assume X is totally bounded. Prove that Y is totally bounded as well.

(8) Let $f : X \to Y$ be a uniformly continuous surjective mapping between two metric spaces. Assume X is bounded. Does it necessarily mean that Y is bounded as well? If yes, give a proof; if no, create a counter-example.

(9) Let (X, d) be a metric space.
   a) Show that if A is a totally bounded subset of (X, d), then $\bar{A}$ is also totally bounded.

b) Use a) to show that if (X, d) is complete and A is a totally bounded subset of (X, d), then $\bar{A}$ is compact.

(10) Let A be a non-empty subset of a metric space (X, d). Recall that the distance of a point $x \in X$ to a set A is defined by

$$d(x, A) := \inf\{d(x, y) : y \in A\}.$$

Show that if A is compact subset of X, then there is $y \in A$ such that $d(x, A) = d(x, y)$.