Error Analysis of Newton Raphson Method:-

① By Taylor Series

$$F(x) = F(a) + \frac{F'(a)}{1!}(x-a) + \frac{F''(a)}{2!}(x-a)^2 + \ldots$$

$a = x_i$, $x = x_{i+1}$

$$F(x_{i+1}) = F(x_i) + F'(x_i)(x_{i+1} - x_i) + \frac{F''(x_i)(x_{i+1} - x_i)^2}{2!} + \ldots$$

$$\ldots + \frac{F^{(n)}(x_i)}{n!}(x_{i+1} - x_i) + R_n \text{ (Higher Order Terms)}$$

$$R_n = \frac{F^{(n+1)}(\xi)}{(n+1)!}(x_{i+1} - x_i)^{n+1}$$

(ξ is some const. b/w x_i & x_{i+1})

Step size , $h = x_{i+1} - x_i$

$$F(x_{i+1}) = F(x_i) + F'(x_i)h + \frac{F''(x_i)h^2}{2!} + \ldots + \frac{F^{(n)}(x_i)h^n}{n!} + R_n$$

$$R_n = \frac{F^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$

$$F(x_{i+1}) \cong F(x_i) + F'(x_i)(x_{i+1} - x_i)$$

x_{i+1} estimated to be the root with $F(x_{i+1}) = 0$

$$\Rightarrow 0 = F(x_i) + F'(x_i)(x_{i+1} - x_i) \quad \text{——(i)}$$

$$\boxed{x_{i+1} = x_i - \frac{F(x_i)}{F'(x_i)}} \quad \text{(Newton Raphson)}$$

$F(x_r) = 0$

$F(x_r) = 0 = F(x_i) + F'(x_i)(x_r - x_i) + \frac{F''(\xi)}{2!}(x_r - x_i)^2$ —— (ii)

By (ii) − (i),

$$0 = F'(x_i)(x_r - x_i) + \frac{F''(\xi)}{2!}(x_r - x_i)^2$$

$$E_{t,i+1} = x_r - x_{i+1}$$

$$0 = F'(x_i)\, E_{t,i+1} + \frac{F''(\xi)}{2}(E_{t,i})^2$$

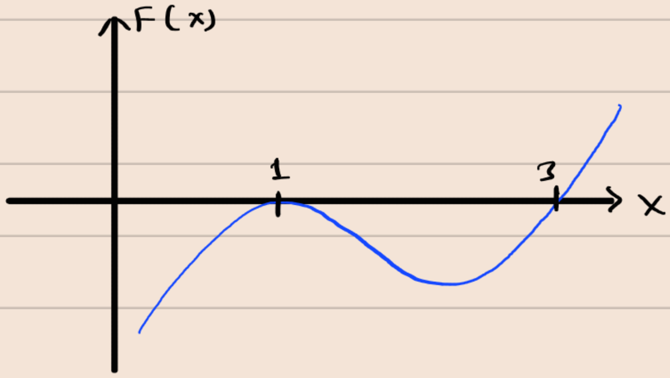
Assuming the method converges, both x_i & ξ should be approximated by x_r,

$$E_{t,i+1} \cong \frac{-F''(x_r)}{2F'(x_r)} E_{t,i}^2$$

$\Rightarrow$ quadratic convergence of newton – Raphson.

MULTIPLE ROOTS :-

$$F(x) = (x-3)(x-1)(x-1)$$
$$= x^3 - 5x^2 + 7x - 3$$



- Bracketing methods doesn't work as the function doesn't change sign
- Not just $F(x)$ but $F'(x)$ also goes to 0.

Convergence of Newton Raphson in case of multiple roots:-

$$e_{i+1} = \left(1 - \frac{1}{m}\right) e_i + \frac{1}{M^2(m+1)} e_i^2 \frac{F^{(m+1)}(x_r)}{F^{(m)}(x_r)} + O(e_i^3)$$

$m \Rightarrow$ multiplicity of the root

When $m = 1$, coefficient of $e_i = 0$; e_i^2 is dominant term.

when $m > 1$, e_i's coefficient $\neq 0$, e_i is dominant term.

$$e_{i+1} = O(e_i)$$ (Linear convergence)

Modified Newton - Raphson :-

multiplicity of the root

$$x_{i+1} = x_i - \frac{m F(x_i)}{F'(x_i)}$$

Newton Raphson

arbitrary

$$x_{i+1} = x_i - \frac{a F(x_i)}{F'(x_i)}$$

$$e_{i+1} = \left(1 - \frac{a}{m}\right) e_i + \frac{a^2 e_i}{m^2(m+1)} \frac{F^{m+1}(x_r)}{F^{(m)}(x_r)} + O(e_i^3)$$

goes to zero when $a = m$

Again we get quadratic convergence.

Note:- To check order of convergence, check if e_{i+1}/e_i is constant. Then linear convergence or if e_{i+1}/e_i^2 approaches a constant, then quadratic convergence, . . .

* modified Newton - Raphson requires us to know multiplicity of the root → Too much to ask for

$$\therefore u(x) = \frac{F(x)}{F'(x)}$$

$u(x)$ as same zeros as $F(x)$ but its roots have less multiplicity.

$$x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}$$

$$u'(x) = \frac{F'(x) \cdot F'(x) - F(x)\, F''(x)}{[F'(x)]^2}$$

$$\boxed{x_{i+1} = x_i - \frac{F(x_i)\, F'(x_i)}{(F'(x_i))^2 - F(x_i)\, F''(x_i)}}$$