Lecture 1

Real and Complex Analysis

MTL122/ MTL503/ MTL506

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1. Basic Ideas

1.1. **Sets.**

A set is a collection of objects called elements. Usually, sets are denoted by capital letters A, B,...,Z. A set can consist any type and number of elements. Even other sets can be elements of a set. The sets dealt with here usually have real numbers as their elements.

If a is an element of the set A, we write $a \in A$. If a is not an element of the set A we write $a \notin A$.

If all the elements of A are also elements of B, then A is a subset of B. In this case, we write $A \subseteq B$ or $B \subseteq A$. In particular, notice that whenever A is a set, then $A \subseteq A$.

Two sets A and B are equal, if they have the same elements. In this case we write $A = B$. It is easy to see that $A = B$ iff $A \subseteq B$ and $B \subseteq A$. Establishing that both of these containments are true is the most common way to show two sets are equal.

If $A \subseteq B$ and $A \neq B$, then A is a proper subset of B. In this cases it is written $A \subsetneq B$.

The union of A and B is the set containing all the elements in either A or B:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The intersection of A and B is the set containing the elements contained in both A and B:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The difference of A and B is the set of elements in A and not in B:

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

Another common set operation is complementation. The complement of a set A is usually thought of as the set consisting of all elements which are not in A.To make sense of the complement of a set, there must be a well-defined universal set U which contains all the sets in question. Then the complement of a set $A \subset U$ is $A^c = U \setminus A$. It is usually the case that the universal set U is evident from the context in which it is used.

Theorem 1.1. *(Exercise) Let A, B and C be sets.*

(a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

(b) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Corollary 1.2. *(De Morgan's Laws) Let A and B be sets.*

(a) $(A \cup B)^c = A^c \cap B^c$

(b) $(A \cap B)^c = A^c \cup B^c$.

Example 1.3.

(1) $\mathbb{N} = \{1, 2, 3, ...\}$, *set of natural numbers.*

(2) $\mathbb{Z} = \{0, 1 - 1, 2, -2, , ...\}$, *set of integers.*

(3) $\mathbb{Q} = \{\frac{m}{n}, m.n \in \mathbb{Z}, n \neq 0\}$, *set of rational numbers.*

(4) $\mathbb{R}$, *the set of all real numbers. The set $\mathbb{R}$ will include all rational numbers, all irrational numbers.*

1.2. **Tuples.**

When listing the elements of a set, the order in which they are listed is unimportant; e.g., $\{e, l, v, i, s\} = \{l, i, v, e, s\}$. If the order in which n items are listed is important, the list is called an *n-tuple.* We denote a *n-tuple* by enclosing the ordered list in parentheses. For example, if x_1, x_2, x_3, x_4 are four items, the $4-$tuple (x_1, x_2, x_3, x_4) is different form the $4-$tuple (x_2, x_1, x_3, x_4).

Because they are used so often, the cases when $n = 2$ and $n = 3$ have special names: 2-tuples are called ordered pairs and a 3-tuple is called an ordered triple.

Definition 1.4. *Let A and B be sets. The sets of all ordered pairs*

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

is called the Cartesian product of A and B.

Example 1.5.

If $A = \{a, b, c\}$ and $B = \{1, 2\}$ then

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\},$$

and

$$B \times A = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

Notice that $A \times B \neq B \times A$ because of the importance of order in the ordered pairs.

2. Relations.

Definition 2.1. *If A and B are sets, the any $R \subset A \times B$ is a relation from A to B. If $(a, b) \in R$, we write aRb.*

In the special case when $R \subset A \times A$, for some set A, there are some additional terminologies.

R is symmetric, if $aRb \iff bRa$.

R is reflexive, if aRa.

R is transitive. if aRb and $bRc \implies aRc$
R is an equivalence relation on A, if it is symmetric, reflexive and transitive.

Example 2.2. *Let R be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by $aRb \iff a \leq b$. Then R is reflexive and transitive, but not symmetric.*

Example 2.3. *Let R be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by $aRb \iff a < b$. Then R is transitive, but not reflexive and symmetric.*

Example 2.4. *Let R be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by $aRb \iff a^2 = b^2$. Then R is and equivalence relation.*

3. Functions.

Definition 3.1. *A relation $R \subset A \times B$ is a function if*

$$aRb_1 \text{ and } aRb_2 \implies b_1 = b_2.$$

If $f : A \to B$ is a function, the usual intuitive interpretation is to regard f as a rule that associates each element of A with a unique element of B. It's not necessarily the case that each element of B is associated with something from A; B may not be $ran(f)$. It's s also common for more than one element of A to be associated with the same element of B. Functions are also called maps, mappings, or transformations. The set A on which f is defined is called the domain of f and the set Y in which it takes its values is called the co domain. We write $f : a \to f(b)$ to indicate that f is the function that maps a to $f(b)$. If $f \subset A \times B$ is a function and $dom(f) = A$, then we usually write $f : A \to B$ and use the usual notation $f(a) = b$ instead of afb.

The **domain** of a function is $dom(f) = \{a \in A : f(a) \in B\}$ and $dom(f) \subseteq A$.
The **range** of a function is $ran(f) = \{b \in B : b = f(a) \text{ for some } a \in A\}$.

Example 3.2. *The identity function $id_A : A \to A$ on a set A is the function $id_A : a \to a$ that maps every element to itself.*

Example 3.3. *Let $A \subset X$. The characteristic function of A,*

$$\chi_A : X \to \{0, 1\}$$

is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A. \\ 0, & \text{if } x \notin A. \end{cases} \tag{3.1}$$

The successive application of mappings leads to the notion of the composition of functions.

Definition 3.4. *The composition of functions $f : A \to B$ and $g : B \to C$ is the function defined by $g \circ f : A \to C$ defined by*

$$g \circ f(a) = g(f(a)).$$

The order of application of the functions in a composition is crucial and is read from right to left. The composition $g \circ f$ can only be defined if the domain of g includes the range of f, and the existence of $g \circ f$ does not imply that $f \circ g$ even makes sense.

Example 3.5. *Define* $f : \mathbb{N} \to \mathbb{Z}$ *by* $f(n) = n^2$ *and* $g : \mathbb{Z} \to \mathbb{Z}$ *by* $g(n) = n^2$. *In this case* $ran(f) = \{n^2 : n \in \mathbb{N}\}$ *and* $ran(g) = ran(f) \cup \{0\}$. *Notice that even though* f *and* g *use the same formula, they are actually different functions.*

In the above Example,

$$g \circ f(n) = g(f(n)) = g(n^2) = (n^2)^2 = n^4$$

makes sense for all $n \in \mathbb{N}$, but $f \circ g$ is undefined at $n = 0$.

There are several important types of functions.

Definition 3.6. *A function* $f(A) \to B$ *is a constant function, if* $ran(f)$ *has a single element; i.e., there is a* $b \in B$ *such that* $f(a) = b$ *for all* $a \in A$. *The function* f *is surjective (or onto* B*), if* $ran(f) = B$.

In a sense, constant and surjective functions are the opposite extremes. A constant function has the smallest possible range and a surjective function has the largest possible range. Of course, a function $f : A \to B$ can be both constant and surjective, if B has only one element.

Definition 3.7. *A function* $f : A \to B$ *is injective(or one to one), if* $f(a) = f(b)$ *implies* $a = b$.

The terminology "one-to-one" is very descriptive because such a function uniquely pairs up the elements of its domain and range.

Definition 3.8. *A function* $f : A \to B$ *is bijective, if it is both surjective and injective.*

A bijective function can be visualized as uniquely pairing up all the elements of A and B. Some use the more descriptive terminology *one-to-one correspondence* instead of bijection. This pairing up of the elements from each set is like counting them and finding they have the same number of elements. Given any two sets, no matter how many elements they have, the intuitive idea is they have the same number of elements if, and only if, there is a bijection between them.

Example 3.9. *The function* $f : [0, \infty) \to [1, \infty)$ *defined by*

$$f(x) = 1 + x^2, \ x \geq 0$$

is a bijective function.

Theorem 3.10. *If* $f : A \to B$ *and* $g : B \to C$ *are bijections, then* $g \circ f : A \to C$ *is a bijection.*

Inverse Functions.

Definition 3.11. *If* $f : A \to B$, *then the image* $f(A) \subset B$ *of* A *is the set* $f(A) = \{f(a) : a \in A\}$.

The inverse image or pre-image of f *is the set* $f^{-1}(B) = \{a : f(a) \in Ran(f)\} \subset A$.

Definition 3.12. *If $f : A \to B$ is bijective, the inverse of f is the function $f^{-1} : B \to A$ with the property that $f^{-1} \circ f(a) = a$ for all $a \in A$ and $f \circ f^{-1}(b) = b$ for all $b \in B$.*

Example 3.13. *Let $A = \mathbb{N}$ and B be the even natural numbers. If $f : A \to B$ is $f(n) = 2n$ and $g : B \to A$ is $g(n) = n/2$,it is clear f is bijective. Since $f \circ g(n) = f(n/2) = 2n/2 = n$ and $g \circ f(n) = g(2n) = 2n/2 = n$, we see $g = f^{-1}$.*

The use of the notation f^{-1} to denote the inverse function should not be confused with its use to denote the reciprocal function; it should be clear from the context which meaning is intended.

Example 3.14. *If $f : \mathbb{R} \to \mathbb{R}$ is a function $f(x) = x^3$, which is one-to-one and onto, then the inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ is given by*

$$f^{-1}(x) = x^{1/3}.$$

On the other hand, the reciprocal function $g = 1/f$ is given by

$$g(x) = \frac{1}{x^3}, \ \ g : \mathbb{R} \setminus \{0\} \to \mathbb{R}.$$

The reciprocal function is not defined at $x = 0$ where $f(x) = 0$.

Note that $f^{-1}(B)$ makes sense as a set even if the inverse function $f^{-1} : Y \to X$ does not exist.

Example 3.15. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. If $A = (-2, -2)$, then $f(A) = [0, 4)$. If $B = (0, 4)$, then*

$$f^{-1}(B) = (-2, 0) \cup (0, 2).$$

If $C = (-4, 0)$, then $f^{-1}(C) = \phi$.

Theorem 3.16. *(Exercise) Let $\mathcal{S}$ be a collection of sets. The relation on $\mathcal{S}$ defined by*

$$A \sim B \iff \text{there is a bijection } f : A \to B$$

is an equivalence relation.

Finally, we introduce operations on a set.

Definition 3.17. *A binary operation on a set X is a function $f : X \times X \to X$.*

Example 3.18. *Addition, $a : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and multiplication $m : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ are binary operations where*

$$a(x, y) = x + y, \ \ m(x, y) = xy.$$

4. Indexed Sets

We often have occasion to work with large collections of sets. For example, we could have a sequence of sets $A_1, A_2, A_3, ...,$ where there is a set A_n associated with each $n \in \mathbb{N}$. In general, let Ω be a set and suppose for each $\lambda \in \Omega$ there is a set A_λ. The set $\{A_\lambda : \lambda \in \Omega\}$ is called a collection of sets indexed by Ω. In this case, Ω is called the indexing set for the collection.

In other words, we say that a set A is indexed by a set I if there is an onto function $f : I \to A$. We then write

$$A = \{a_i : i \in I\},$$

where $a_i = f(i)$.

Example 4.1. $\{1, 4, 9, 16, ..\} = \{n^2 : n \in \mathbb{N}\}$.

Example 4.2.
For each $n \in \mathbb{N}$, let $A_n = \{k \in \mathbb{Z} : k^2 \leq n\}$. Then

$$A_1 = A_2 = A_3 = \{-1, 0, 1\},\ A_4 = \{-2, -1, 0, 1, 2\}, ...,$$

is a collection of sets indexed by $\mathbb{N}$.

Two of the basic binary operations can be extended to work with indexed collections. In particular, using the indexed collection from the previous paragraph, we define

$$\bigcup_{\lambda \in \Omega} A_\lambda = \{x : x \in A_\lambda \text{ for some } \lambda \in \Omega\}$$

and

$$\bigcap_{\lambda \in \Omega} A_\lambda = \{x : x \in A_\lambda \text{ for all } \lambda \in \Omega\}.$$

Example 4.3. *For $n \in \mathbb{N}$, define the intervals*

$$A_n = [1/n, 1 - 1/n] = \{x \in \mathbb{R} : 1/n \leq x \leq 1 - 1/n\},$$

$$B_n = (-1/n, 1/n) = \{x \in \mathbb{R} : -1/n < x < 1/n\}.$$

Then

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n = (0, 1),\ \bigcap_{n \in \mathbb{N}} B_n = \bigcap_{n=1}^{\infty} B_n = \{0\}.$$

De Morgan's Laws can be generalized to indexed collections.

Theorem 4.4. *If $\{B_\lambda : \lambda \in \Omega\}$ is an indexed collection of sets and A is a set, then*

$$A \setminus \bigcup_{\lambda \in \Omega} B_\lambda = \bigcap_{\lambda \in \Omega} (A \setminus B_\lambda)$$

and

$$A \setminus \bigcap_{\lambda \in \Omega} B_\lambda = \bigcup_{\lambda \in \Omega} (A \setminus B_\lambda).$$

5. Cardinality

There is a way to use sets and functions to formalize and generalize how we count. For example, suppose we want to count how many elements are in the set $\{a, b, c\}$. The natural way to do this is to point at each element in succession and say "one, two, three." What we're doing is defining a bijective function between $\{a, b, c\}$ and the set $\{1, 2, 3\}$. This idea can be generalized.

Definition 5.1. *Given $n \in \mathbb{N}$. A set S has cardinality n, if there is a bijective function $f : S \to \{1, 2, 3, ..., n\}$. We write card(S) $= n$.*

Two sets A and B are said to have *same cardinality* if there exists a bijection from A to B. It is common to write $card(N) = \mathfrak{N}_0$.

Example 5.2. *Let $f : \mathbb{N} \to \mathbb{Z}$ be defined as*

$$f(x) = \begin{cases} \frac{n+1}{2}, & \textit{when } n \textit{ is odd.} \\ 1 - \frac{n}{2}, & \textit{when } n \textit{ is even.} \end{cases} \tag{5.1}$$

It's easy to show f is a bijection, so $card(\mathbb{N}) = card(\mathbb{Z}) = \mathfrak{N}_0$.

Finite and Infinite Sets: A set is called finite if it is empty or has the same cardinality as the set $\{1, 2, ..., n\}$ fro some $n \in \mathbb{N}$; it is **infinite** otherwise.

Countable Sets: A set is called **countable**(or **countably infinite**) if it has the same cardinality as $\mathbb{N}$. Equivalently, a set A is countable if all of its elements can be listed as a sequence $a_1, a_2, a_3, ...$ A set is called **uncountable** if it is infinite and not countable.

Some important results on countable and uncountable sets.

- An infinite subset of a countable set is countable; a superset of an uncountable set is uncountable.
- A finite or countable union of countable sets countable.
- The cartesian product of finitely many countable sets is countable.

So we have,
A set X is :

(1) Finite if it is the empty set or $A \approx \{1, 2, ..., n\}$ for some $n \in \mathbb{N}$;
(2) Countably infinite (or denumerable) if $X \approx \mathbb{N}$;
(3) Infinite if it is not finite;
(4) Countable if it is finite or countably infinite;
(5) Uncountable if it is not countable.

Example 5.3.
$\mathbb{N}, \mathbb{Q}$ *is countable.*
$\mathbb{R}$ *is uncountable.*