Lec 8 - MTL 122

Real and Complex Analysis.

$\cdot\ \underline{(X, d)}$, $X \neq \emptyset$ $\longrightarrow$ metric space

$d : X \times X \longrightarrow \mathbb{R}$ 'd' : metric

i) $d(x, y) \geq 0$

ii) $d(x, y) = d(y, x)$

iii) $d(x, y) = 0 \Leftrightarrow x = y$

iv) $d(x, y) \leq d(x, z) + d(y, z)$.

Ex: 1) $(\mathbb{R}, d)$, $d = |x - y|$.

$x, y \in \mathbb{R}$.

2) $\underline{1 \leq p < \infty}$

$X = l_p =$ set of all seq $(x_i)_{i=1}^{\infty}$, $x_i \in \mathbb{R}$, $\sum_{i=1}^{\infty} |x_i|^p < \infty$

$x \in \underline{l_p} \Rightarrow x = (x_i)_{i=1}^{\infty}$, $\sum_{i=1}^{\infty} |x_i|^p < \infty$

$$l_p = \left\{ x = (x_i)_{i=1}^{\infty} \;\middle|\; \sum_{i \in \mathbb{N}} |x_i|^p < \infty \right\}$$

$$d_p : l_p \times l_p \longrightarrow \mathbb{R}.$$

$$d_p(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}.$$

- Check $(l_p, d_p) \rightarrow$ metric space.

Minkowski's Ineq.

$p > 1$, $(a_i)_{i=1}^{n}$, $(b_i)_{i=1}^{n} \in l^p$

$$\left(\sum_i |a_i + b_i|^p \right)^{1/p} \le \left(\sum_i |a_i|^p \right)^{1/p} + \left(\sum_i |b_i|^p \right)^{1/p}$$

3) $X = l_\infty$ = set of all bdd seq.

- $d_{sup}(x, y) = \sup |x_i - y_i|$
- $d(x, y) = \sum \dfrac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$.

$$\left.\begin{array}{l}(X, d_{sup}) \\ (X, d)\end{array}\right\}$$ metric space.

4) $X \neq \emptyset$

$x, y \in X$

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

Is (X, d) ~~an~~ a metric space?

'd' $\rightarrow$ discrete metric

(X, d) is a discrete metric space.

Ex. (X, d) be a metric space.

- $|d(x,z) - d(y,z)| \leq d(x,y)$

How?

By triangle Ineq.

$$d(x,z) \leq d(x,y) + d(y,z)$$

$$\Leftrightarrow \quad d(x,z) - d(y,z) \leq d(x,y) \quad \text{--- } (1)$$

$$d(y,z) - d(x,z) \leq d(y,x)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$$

$$d(y,z) - d(x,z) \leq d(x,y) \quad \text{--- } (2)$$

We are getting.

$$|d(x,z) - d(y,z)| \leq d(x,y).$$

Ex (to try)

1) $X = \{P_1, P_2, P_3\}$

$d(P_1, P_2) = d(P_2, P_1) = 1$

$d(P_1, P_3) = d(P_3, P_1) = 2$

& $d(P_2, P_3) = d(P_3, P_2) = 3$

Can you find a triangle (P_1, P_2, P_3) in the plane with these distances?

2) <u>Chicago suburb metric</u>

$X = \mathbb{R}^2$

$x = (x_1, x_2)$
$y = (y_1, y_2)$

$x_0 = (0,0)$

$$d(x,y) = \begin{cases} d_2(x,y), & \exists\, t \in \mathbb{R} \;\; x_i = t y_i, \; i=1,2 \\ d_2(x, x_0) + d_2(x_0, y) & \text{else} \end{cases}$$

'd' satisfies triangle ineq.

3) 'n' be a prime number.

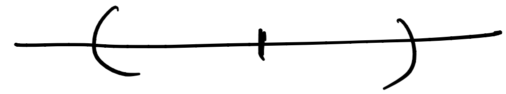
$\mathbb{Z}$.

$$d(x,y) = n^{-\max\{m \in \mathbb{N} : n^m | x-y\}}$$

$(\mathbb{Z}, d) \rightarrow$ metric space?

Topology

$(\mathbb{R}, d)$ $N(x,\epsilon)$



x_0

$|x-y|<\epsilon$

$N(x,\epsilon)=\{y\in\mathbb{R} \mid d(x,y)<\epsilon\}$

(X,d), $x\in X$, $r>0$

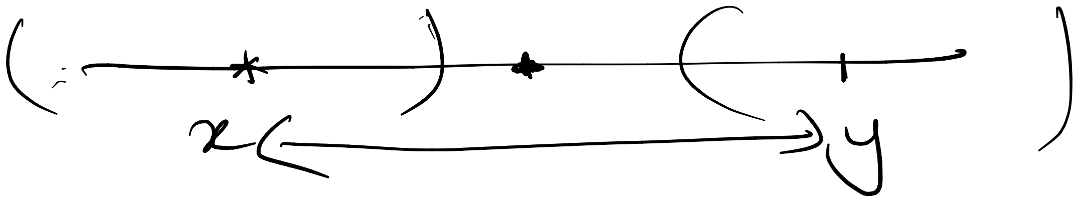
$$B(x,r)=\{y\in X \mid d(x,y)<r\}$$

$\hookrightarrow$ open ball with centre x and radius 'r'.

$$\overline{B(x,r)} = \{y\in X \mid d(x,y)\le r\}$$

Closed ball.

$(\mathbb{R}, d)$



$$\epsilon < \frac{|x-y| = d(x,y)}{2} \qquad N(x,\epsilon) \cap N(y,\epsilon) = \emptyset$$

Lemma.

Let x, y be distinct points in (X, d). Then there is $\epsilon > 0$ s.t

$$B(x,\epsilon) \cap B(y,\epsilon) = \emptyset$$

Pf. $x \neq y \Rightarrow d(x,y) > 0$

$\epsilon > 0$ s.t

$$0 < \epsilon < \frac{d(x,y)}{2}$$

Then $B(x,\epsilon) \cap B(y,\epsilon) = \emptyset$

If

$z \in B(x, \epsilon) \cap B(y, \epsilon)$

Then,

$d(x, z) < \epsilon$ & $d(y, z) < \epsilon$

$\therefore \quad 0 < d(x, y) \leq d(x, z) + d(y, z)$

$< \epsilon + \epsilon = 2\epsilon$

$\leq d(x, y)$

↯

So then,

$B(x, \epsilon) \cap B(y, \epsilon) = \emptyset.$

Open sets

$G \subseteq X$ open if $\exists \epsilon > 0$

s.t $B(x, \epsilon) \subset G$

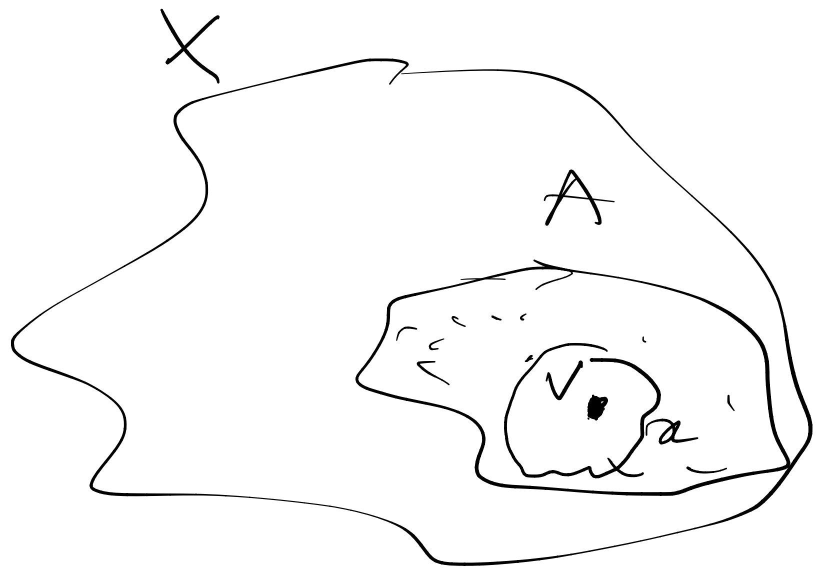
$\left[N(x,\epsilon) \subset G \quad (G \subseteq \mathbb{R}) \right]$

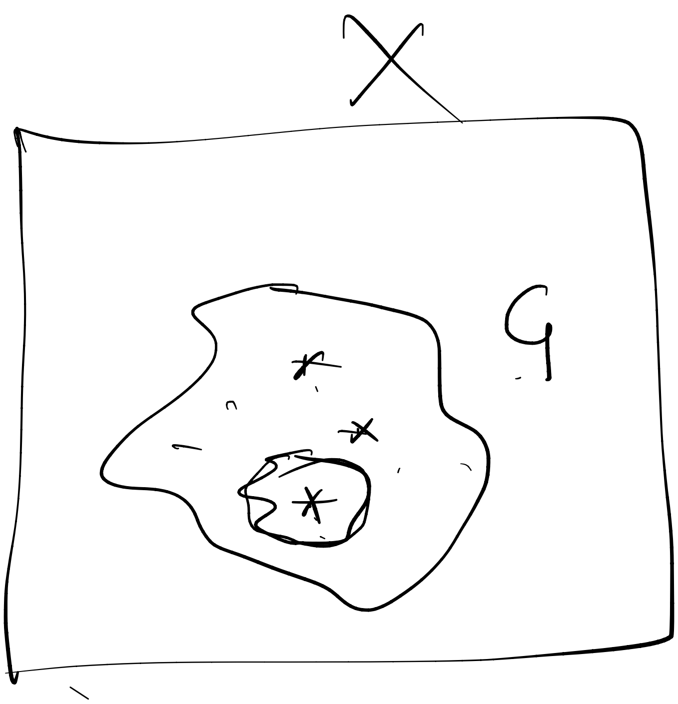
(X, d).

$A \subseteq X$ nqb of $x \in X$

if $\exists$ an open set

$V \subset X$ s.t $x \in V \subset A$





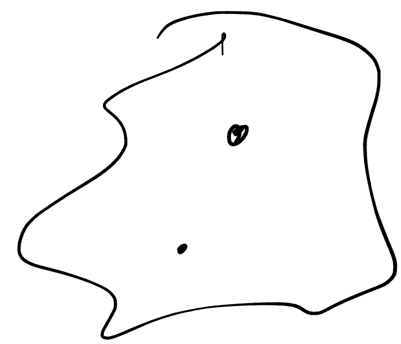
$B(x, \epsilon) \rightarrow$ open set

- Any open ball in a metric is an open set.

$x \in X$, $r > 0$.

$B(x,r) \longrightarrow$ show this an open set.

Let $y \in B(x,r)$



Then, $d(x,y) < r$.

$\epsilon = r - d(x,y)$

Show $B(y,\epsilon) \subset B(x,r)$.

$z \in B(y\ \epsilon)$

$\Rightarrow d(y,z) < \epsilon$.

By triang ineq.

$d(x,z) \leq d(x,y) + d(y,z)$

$$< d(x,y) + \epsilon$$

$$= r.$$

$\Rightarrow \quad d(x,z) < r.$

$\Rightarrow \quad z \in B(x,r)$

$\Rightarrow \quad B(\underline{y},\epsilon) \subset B(x,r)$

$\Rightarrow \quad B(x,r)$ is an open set.

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2) $(X,d) \to$ discrete metric space.

Claim : every subset of X is open.

$G \subset X$

$x \in G$, $\qquad 0 < \epsilon < 1$

$$B(x,\epsilon) = \{x\} \subset G.$$

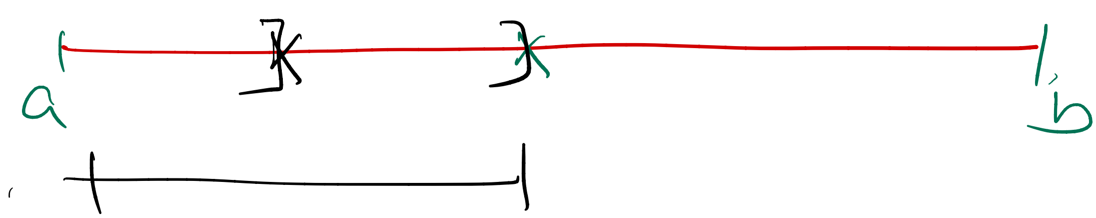
$(\mathbb{R}, d_{dis}) \longrightarrow \{x\}$ open.

$(\mathbb{R}, d_{Euc}) \rightarrow \{x\} \rightarrow$ closed

$[a, b] \rightarrow$ closed.

Proof: Idea is to keep dividing. $a \leq x \leq b$.

Let $\{G_\alpha\}_{\alpha \in \Omega}$ be an open cover of $[a, b]$ that does not have finite subcover.



- Divide $[a, b]$ in half.
- Then one subinterval (or both) are closed sets with an open cover without finite subcover. (Why?)
- Keep repeating.
- n-step
  We will have I_n, closed of length $\frac{b-a}{2^n}$, with an open cover without finite subcover of $\{G_\alpha\}_{\alpha \in \Omega}$.
- These are nested intervals.
  So by NIP, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ & has exactly one point.

Let, $\bigcap_{n=1}^{\infty} I_n = \{p\}$.

p is contained in atleast one of $\{G_\alpha\}_{\alpha \in \Omega}$, say G_β, $\beta \in \Omega$.

So $\exists\ \epsilon > 0$, $N(p, \epsilon) \subseteq G_\beta$.

$\Rightarrow$ for N sufficiently large,

$I_N \subset N(p, \epsilon)$. (Why?).

& hence

$I_n \subseteq G_\beta$, $\forall n \geq N$

— a contradiction.

(construction of I_n)

$\Rightarrow$ $\exists$ an open cover of $[a, b]$ that has finite subcover.