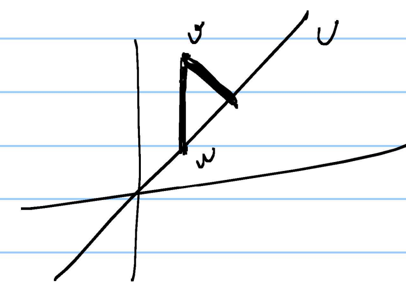
MTL 104 Lec 30

Note Title 28-03-2023

Least Square Approximation/Regression

Theorem: Suppose U is a finite dimensional subspace of V, $v \in V$, $u \in U$. Then

$$\|v - P_U(v)\| \le \|v - u\|$$



The inequality is an equality $\Leftrightarrow$ $u = P_U(v)$.

Proof: $\|v - P_U(v)\|^2 \le \|v - P_U(v)\|^2 + \|P_U(v) - u\|^2$

$= \|(v - P_U(v)) + (P_U(v) - u)\|^2$ Pythagoras.

$= \|v - u\|^2$

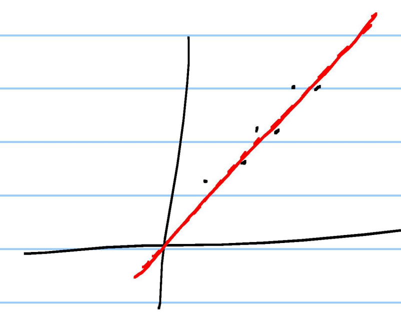
The first inequality = an equality when $\|P_U(v) - u\|^2 = 0$ or $u = P_U(v)$.

Ex: The boiling point of water is known to vary depending on the barometric pressure. Experiments show the following data

Barometric Pressure	20.2	22.1	24.5	27.3	30.1
Boiling Point	195	197	202	209	212

Find a linear equation of the form $T = c_0 + c_1 P$ that will make accurate predictions of the boiling point given the pressure.

$$c_0 + 20.2\, c_1 = 195$$
$$c_0 + 22.1\, c_1 = 197$$
$$\vdots$$
$$c_0 + 30.1\, c_1 = 212$$



This system has no solution!

So we try to get an approximate solution!

To find "approximate" solution to a system $Ax=y$ that has no solution

We find an approximate solution by replacing y by a suitable vector $\hat{y}$ in range of A.

Choose $\hat{y}$ s.t $\|y-\hat{y}\|$ is minimal (or $\hat{y}$ is closest to y)

Remark: The previous theorem says that the $\hat{y}$ closest to y is nothing but $P_U(y)$ where $U=$ range T_A.

Def: Finding such a solution is called the method of Least Square regression. The solution $\hat{x}$ for the system $Ax=\hat{y}$ is called a least square solution

Ex: Find the least square solution to $Ax=y$

$$A=\begin{bmatrix} 1 & 4 \\ -3 & 3 \\ 5 & 1 \end{bmatrix} \qquad y=\begin{bmatrix} -16 \\ 28 \\ 6 \end{bmatrix}.$$

$[a_1 \quad a_2]$

$V=U\oplus U^{\perp}$

$y=u+w$

$P_U(y)=u$

$U = \text{range } T_A$. $\quad \hat{y} = P_U(y) = \langle a_1, y\rangle \cdot \frac{a_1}{\|a_1\|} + \langle a_2, y\rangle \frac{a_2}{\|a_2\|}$ $\quad a_1 \cdot a_2 = 0$

$$= \begin{bmatrix} 2 \\ 9 \\ -9 \end{bmatrix}$$

$$Ax = \hat{y} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 4 & 2 \\ -3 & 3 & 9 \\ 5 & 1 & -9 \end{array}\right] \rightsquigarrow \hat{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Definition: If A is a $n \times m$ matrix & $y \in \mathbb{R}^n$, then a least squares solution to $Ax = y$ is a vector $\hat{x}$ s.t.

$$\|A\hat{x} - y\| \leq \|Ax - y\|. \qquad \forall\, x \in \mathbb{R}^n.$$

If $Ax = y$ has a solution x_0, then $\hat{x} = x_0$. If A has linearly independent columns, then $\hat{x}$ is unique. If not, $\exists$ infinite many choices of $\hat{x}$.

Theorem: The set of least square solution to $Ax=y$ is equal to the set of solution to

$$\boxed{A^T A x = A^T y}.$$

The solution to above system are called normal equation/solutions for $Ax=y$.

If A has linearly independent columns, then $\exists$ a unique solution given by

$$\hat{x} = (A^T A)^{-1} A^T y.$$

Proof: Suppose $\hat{x}$ is a least square solution to $Ax=y$.

$\Leftrightarrow$ $\hat{x}$ is a solution to $Ax = \hat{y}$ where $\hat{y} = P_U(y)$ $U = \text{range}\, T_A$.

$\Leftrightarrow$ $y - \hat{y} = y - P_U(y) \in U^\perp = (\text{range}\, T_A)^\perp = \text{null}\,(A^T)$.

$\Leftrightarrow$ $A^T(y - \hat{y}) = 0$

$A^T(y - A\hat{x}) = 0$. $\Rightarrow$ $A^T A \hat{x} = A^T y$.

(One can proceed the proof backwards to set viceversa)

A^TA is invertible $\iff$ A has linearly independent columns

$\Rightarrow$ the least square solution is unique if A has linearly independent columns

Ex: back to barometric pressure/Boiling pt example

$$A = \begin{pmatrix} 1 & 20.2 \\ 1 & 22.1 \\ 1 & 24.5 \\ 1 & 27.3 \\ 1 & 30.1 \end{pmatrix} \qquad x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \qquad y = \begin{bmatrix} 195 \\ 197 \\ 202 \\ 209 \\ 212 \end{bmatrix}$$

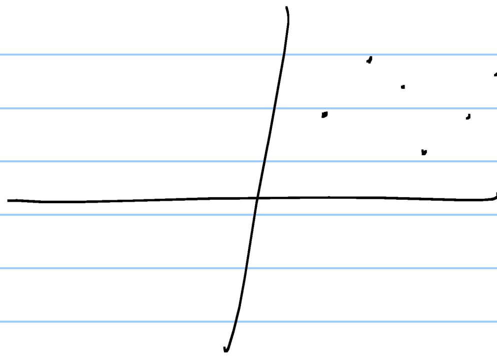
U = range T_A = space spanned by cols of A.

The colum of A are linearly independent!

$\Rightarrow$ solution is given by $(A^TA)^{-1}A^Ty = \begin{bmatrix} 157.17 \\ 1.845 \end{bmatrix}$

Ex: A data is given by an economist to correlate consumer confidence.

Quarter	1	2	3	4	5	6
confidence index	5	9	8	4	6	8



Find a cubic polynomial that approximates the data.

$$g(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 .$$

$$c_0 + c_1 + c_2 + c_3 = 5$$
$$c_0 + c_1 \cdot 2 + c_2 \cdot 2^2 + c_3 \cdot 2^3 = 9$$
$$\vdots$$
$$c_0 + 6c_1 + 6^2 c_2 + 6^3 c_3 = 8$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ \vdots & & & \\ 1 & 6 & 6^2 & 6^3 \end{bmatrix}}_{A} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 9 \\ 8 \\ 4 \\ 6 \\ 8 \end{bmatrix}}_{y}$$

In this case the columns are linearly independent.

$\Rightarrow$ the least square solution $(A^T A)^{-1} A^T y$

Ex: The time from sunrise to sunset in a certain place is given by the following data -

Date	Mar 20	June 1	Aug 12	Oct 25
Day length.	12.17	16.23	12.12	8.18
	0	0.2	0.4	0.6

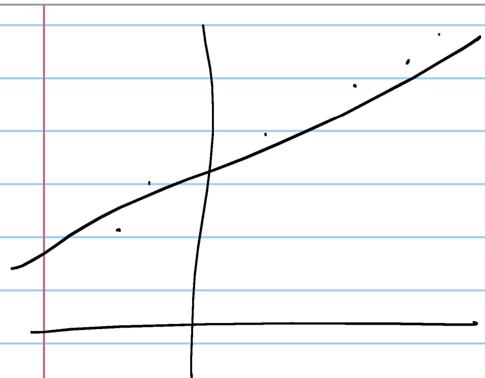
The length of the day is modelled on the function $L(t) = c_1 + c_2 \sin(2\pi t)$ where t = time in years.

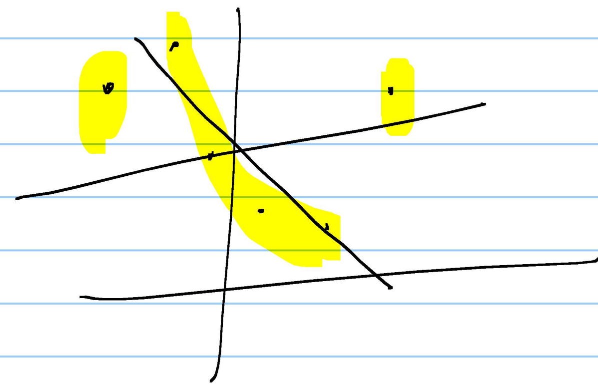
$t=0$ is evaluated as March 20. then rest of the data occur in $t = 0.2, 0.4, 0.6$

Check.

MTL 104 Lec 31

Note Title 31-03-2023





Def: $V = \mathbb{R}^n$ $t_1, \dots, t_n$ are positive numbers in $\mathbb{F}$.

$u, v \in V$ $\langle u, v \rangle = t_1 u_1 v_1 + t_2 u_2 v_2 + \dots + t_n u_n v_n$.

is an inner product on $\mathbb{R}^n$. (check!)

$Ax = y \qquad A x = \hat{y} \qquad \hat{y} = P_U(y) \qquad U = \text{range } A. \qquad \|y - \hat{y}\|$ is least.

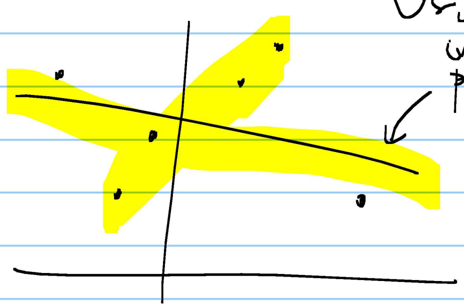
$$\| \; t_1(y_1 - \hat{y}_1)^2 + \cdots + t_n(y_n - \hat{y}_n)^2$$

Ex.: Use the least square regression to fit a line to the data set

$(-6, 2.9), (-3, 1.5), (-2, 2), (2, 2.7), (3, 3.3), (6, 1.1)$

Then use the weighted inner product to solve this problem s.t. middle four pts are more emphasized than the external pts.

Sol: $f(t) = c_0 + c_1 t$.



Using usual inner product

$$c_0 + c_1(-6) = 2.9$$
$$\vdots$$
$$c_0 + c_1(6) = 1.9.$$

$$\begin{array}{cc} a_1 & a_2 \end{array}$$
$$\begin{bmatrix} 1 & -6 \\ 1 & -3 \\ 1 & -2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 2.9 \\ \vdots \\ 1.1 \end{bmatrix}$$

$$\hat{y} = P_U(y) = \langle a_1, y \rangle \frac{a_1}{\|a_1\|} + \langle a_2, y \rangle \frac{a_2}{\|a_2\|}$$

$U = \text{range } U = \langle a_1, a_2 \rangle$

$$= 2.25\, a_1 - 0.0408\, a_2$$

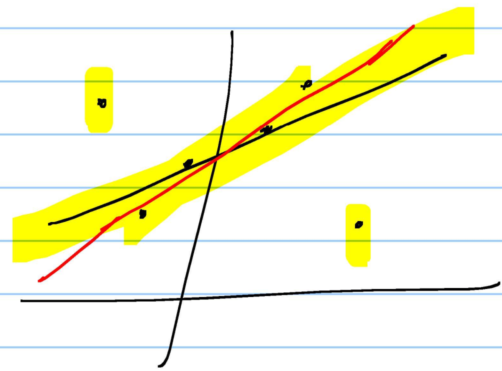
$Ax = \hat{y}$ $\quad\quad f(t) = 2.25 - 0.0408\,x.$

weighted inner product. $t_1 = t_6 = 1$ $\quad t_2 = t_3 = \cancel{t_4} = \cancel{t_5} = 5$

$(t_1, \ldots t_6) = (1, 5, \ldots 5, 1)$

Using this inner product $\hat{y} = P_V(y) = \langle a_1, y\rangle \cdot \frac{a_1}{\|a_1\|} + \langle a_2, y\rangle \cdot \frac{a_2}{\|a_2\|}.$

$= 2.34 a_1 + 0.1159 a_2.$



§ Fourier Approximation

$V = \mathcal{C}([-\pi, \pi])$ $\qquad\qquad$ $\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) g(x)\, dx$

Theorem: $\forall n \geq 1$. the set

$\{1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x \ldots \sin nx\}$.

is orthogonal on V.

(Check): $\langle 1, \underset{\sin nx}{\cos nx} \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} 1 \cdot \underset{\sin nx}{\cos nx}\, dx = 0$

$\langle \sin jx, \cos kx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos kx \cdot \sin jx\, dx = 0$ $\qquad$. $\cos kx \cdot \sin jx$ is odd!

For any $f(x) \in V$.

$f_n(x) = P_{F_n}(f(x)) = a_0 + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx$.

F_n = subspace spanned by $\{1, \cos x \ldots \cos nx, \sin x \ldots \sin nx\}$.

$f_n(x)$ is called the n^{th} Fourier approximation to $f(x)$.

$$a_k = \frac{\langle f, \cos kx\rangle}{\langle \cos kx, \cos kx\rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \qquad b_k = \frac{\langle f, \sin kx\rangle}{\langle \sin kx, \sin kx\rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx$$

$$\langle \cos kx, \cos kx\rangle = \pi \qquad \langle \sin kx, \sin kx\rangle = \pi$$

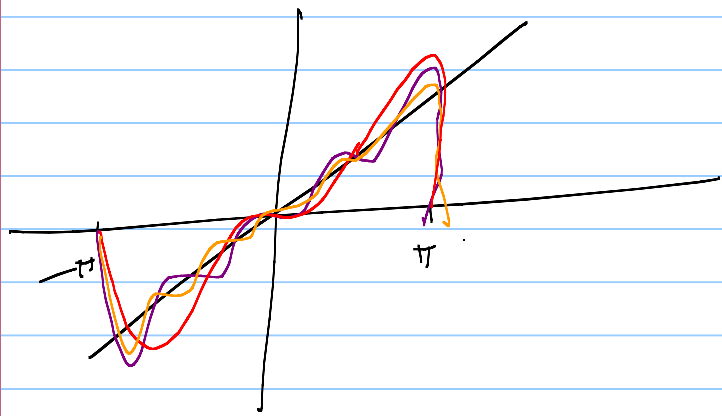
a_k, b_k's are called Fourier coefficients of $f(x)$.

Ex: $f(x) = x$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot 1\cdot dx = 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\cos kx\,dx = 0 \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\sin kx\,dx = \frac{2}{k}(-1)^{k+1}$$

$$f_n(x) = \sum_{k=1}^{n} \frac{2}{k}(-1)^{k+1} \sin kx.$$



$f_2(x)$ $f_3(x)$ $f_4(x$

If the Fourier coefficients decrease in size quickly then we can extend the n^{th} Fourier approximation to

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

is called a Fourier Series.

§ Positive Operators

Def: $T \in \mathcal{L}(V)$ is called positive if T is selfadjoint &

$$\langle Tv, v\rangle \geq 0.$$

Remark: If V is a complex vector space then T being self adjoint can be dropped.

Def: An operator R is called a square root of an operator T if $R^2 = T$.

Ex: ① If U is a subspace of V. Then P_U is a positive operator.

Theorem: $T \in \mathcal{L}(V)$. Then TFAE

1. T is positive
2. T is selfadjoint & all the eigenvalues of T are nonnegative.
3. T has a positive square root.
4. T has a selfadjoint square root.
5. $\exists$ an operator $R \in \mathcal{L}(V)$ s.t. $T = R^* R$.

Proof: 1 $\Rightarrow$ 2 T is already selfadjoint.

Let λ be an eigenvalue of $T \Rightarrow \exists\, v \in V$ s.t. $Tv = \lambda v$.

$$0 \le \langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda \langle v, v\rangle = \lambda \|v\|^2 \Rightarrow \lambda \ge 0.$$

2 $\Rightarrow$ 3 By spectral theorem $\exists$ orthonormal basis $e_1 \dots e_n$ (eigenvectors) of V with eigenvalues $\lambda_1 \dots \lambda_n \ge 0$.

$$T = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad R = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}$$

Define $R e_i = \sqrt{\lambda_i}\, e_i$ $R \in \mathcal{L}(V)$ & $R^2 = T$.

(3) $\Rightarrow$ (4) ✓

(4) $\Rightarrow$ (5) T has a selfadjoint squareroot $R \Rightarrow R^2 = T$

$$R^* R = RR = R^2 = T$$

(5) $\Rightarrow$ (1) $T = R^* R$.

$$T^* = (R^* R)^* = R^* (R^*)^* = R^* R = T \Rightarrow \text{selfadjoint}$$

$$\langle Tv, v \rangle = \langle R^* R v, v \rangle = \langle Rv, Rv \rangle \geq 0 \quad \forall v \in V. \Rightarrow \text{positive}$$

Theorem: Every positive operator on V has unique positive square root.

Proof: Suppose $T \in \mathcal{L}(V)$ is positive. Suppose v is an eigenvector for $T \Rightarrow \exists\ \lambda \geq 0$ s.t $Tv = \lambda v$.

Let R be a positive square root of T. (claim: $Rv = \sqrt{\lambda}\, v$).

By spectral theorem $\exists$ orthonormal basis $e_1 \ldots e_n$ of V s.t $e_1 \ldots e_n$ are eigenvectors of T.
$\lambda_1 \ldots \lambda_n$ are eigenvalues ≥ 0.

$e_1 \ldots e_n$ is a basis $\Rightarrow$ $v = a_1 e_1 + \cdots + a_n e_n$.

$$Rv = R(a_1 e_1 + \cdots + a_n e_n) = a_1 R e_1 + \cdots + a_n R e_n)$$
$$= a_1 \sqrt{\lambda_1} e_1 + \cdots + a_n \sqrt{\lambda_n} e_n.$$

$$R^2 v = R(Rv) = R(a_1 \sqrt{\lambda_1} e_1 + \cdots + a_n \sqrt{\lambda_n} e_n)$$
$$= a_1 \sqrt{\lambda_1} R e_1 + \cdots + a_n \sqrt{\lambda_n} R e_n.$$

$= a_1 \lambda_1 e_1 + \cdots + a_n \lambda_n e_n$.

But $R^2 v = Tv$ & $Tv = \lambda v$. $\Rightarrow$

$$a_1 \lambda e_1 + \cdots + a_n \lambda e_n = a_1 \lambda_1 e_1 + \cdots + a_n \lambda_n e_n.$$

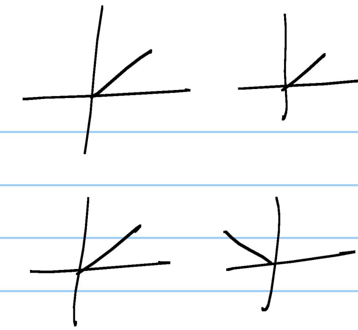
$$a_1 (\lambda - \lambda_1) e_1 + \cdots + a_n (\lambda - \lambda_n) e_n = 0.$$

$\Rightarrow a_i (\lambda - \lambda_i) = 0 \ \forall i$ $e_1 \ldots e_n$ are l.i.

$$\Rightarrow v = \sum_{(j : \lambda_j = \lambda)} a_j e_j$$

$$\Rightarrow Rv = \sum_{\{j : \lambda_j = \lambda\}} a_j \sqrt{\lambda}\, e_j = \sqrt{\lambda}\, v$$

§ Isometries.



Def: An operator $S \in \mathcal{L}(V)$ is called an isometry if

$$\|Sv\| = \|v\| \quad \forall v \in V.$$

Ex: ① λI is an isometry when $|\lambda| = 1$

② $\lambda_1 \dots \lambda_n$ are scalars. & $|\lambda_i| = 1 \ \forall i$ $S \in \mathcal{L}(V)$ $Se_j = \lambda_j e_j$ for some orthonormal basis $e_1 \dots e_n$.

$\|\sum a_i e_i\|^2 = |a_1|^2 + \dots + |a_n|^2$

$v \in V$. $\quad v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n$.

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_n \rangle|^2$$

$$Sv = S(\langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n)$$

$$= \langle v, e_1 \rangle Se_1 + \dots + \langle v, e_n \rangle Se_n.$$

$$= \langle v, e_1 \rangle \lambda_1 e_1 + \dots + \langle v, e_n \rangle \lambda_n e_n.$$

$$\|Sv\|^2 = |\langle v, e_1\rangle \lambda_1|^2 + \cdots + |\langle v, e_n\rangle \lambda_n|^2$$

$$= |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2$$

$$\Rightarrow \|Sv\| = \|v\| \quad \forall v \in V$$

$\Rightarrow$ S is an isometry.

Theorem: $S \in \mathcal{L}(V)$ TFAE

1. S is an isometry
2. $\langle Su, Sv\rangle = \langle u, v\rangle \quad \forall u, v \in V$.
3. $Se_1 \dots Se_n$ is orthonormal $\forall$ orthonormal list of vectors $e_1 \dots e_n$ in V
4. $\exists$ an orthonormal basis $e_1 \dots e_n$ of V s.t. $Se_1 \dots Se_n$ is orthonormal.
5. $S^* S = I$
6. $S S^* = I$

⑦ S^* is also an isometry.

⑧ S is invertible & $S^{-1} = S^*$.

Proof: ① $\Rightarrow$ ② - Suppose S is an isometry: $\|Sv\| = \|v\|$ $\forall v \in V$.

Suppose V is an inner product space over $\mathbb{R}$:

$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$

$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}$

$$\langle Su, Sv\rangle = \frac{\|Su+Sv\|^2 - \|Su-Sv\|^2}{4}$$

$$= \frac{\|S(u+v)\|^2 - \|S(u-v)\|^2}{4}$$

$$= \frac{\|u+v\|^2 - \|u-v\|^2}{4} = \langle u, v\rangle$$

Suppose V is an inner product space over $\mathbb{C}$

Check use the formula

(2) $\Rightarrow$ (3) Suppose $e_1 \dots e_n$ forms an orthonormal list in V

$$\langle S e_k, S e_j \rangle = \langle e_k, e_j \rangle \quad \text{(by assumption (2))}$$

$$= 0 \Rightarrow S e_k, S e_j \text{ are orthogonal.}$$

$\Rightarrow S e_1 \dots e_n$ forms an orthonormal list $\quad \|S e_i\| = \|e_i\| = 1$

(3) $\Rightarrow$ (4) ✓

(4) $\Rightarrow$ (5) Suppose $e_1 \dots e_n$ is an orthonormal basis s.t. $Se_1 \dots Se_n$ forms orthonormal basis.

$$\langle S^* S e_j, e_k \rangle$$

$$\|$$

$$\langle S e_j, S e_k \rangle = \langle e_j, e_k \rangle$$

$\forall$ vector $u, v \in V$. they can be written as a linear combination of vector $e_1 \dots e_n$

$$\Rightarrow \langle S^* S u, v \rangle = \langle u, v \rangle. \quad \forall u, v \in V.$$

$$\Rightarrow S^* S = I$$

(5) $\Rightarrow$ (6) ✓

(6) $\Rightarrow$ (7) Suppose $SS^* = I$. $v \in V$. $(S^*)^* = S$

$$\|S^* v\|^2 = \langle S^* v, S^* v\rangle = \langle SS^* v, v\rangle = \langle v, v\rangle = \|v\|^2.$$

$\Rightarrow S^*$ is an isometry.

(7) $\Rightarrow$ (8) Suppose S^* is an isometry. (use the implication (1) $\Rightarrow$ (5) $\Rightarrow$ (6) to see that

$$SS^* = S^*S = I$$

$$\Rightarrow S^{-1} = S^*.$$

(8) $\Rightarrow$ (1) S is invertible & $S^{-1} = S^*$

$v \in V$.

$$\|Sv\|^2 = \langle Sv, Sv\rangle = \langle S^*S v, v\rangle = \langle v, v\rangle = \|v\|^2$$

$\Rightarrow \|Sv\| = \|v\| \Rightarrow S$ is an isometry.

Theorem: Suppose V is a complex inner product space $S \in \mathcal{L}(V)$. Then TFAE

1. S is an isometry
2. $\exists$ an orthonormal basis of V consisting of eigenvectors of S which corresponding eigenvalues all have absolute value 1.

Proof: 1 $\Rightarrow$ 2

Suppose S is an isometry.

By the spectral theorem $\exists$ an orthonormal basis $e_1 \dots e_n$ which are all eigenvectors.

$\lambda_1 \dots \lambda_n \to$ eigenvalues.

$$|\lambda_j| = \|\lambda_j e_j\| = \|S e_j\| = \|e_j\| = 1$$

2 $\Rightarrow$ 1 Ex 2 above.

MTL104 Lec 33

Note Title 05-04-2023

§ Polar & Singular value Decomposition.

$T \in \mathcal{L}(V)$
$\sqrt{T} \in \mathcal{L}(V)$

Remark: $T \in \mathcal{L}(V) \Rightarrow T^*T$ is self adjoint & positive $\quad \langle T^*T v, v\rangle = \langle Tv, Tv\rangle \geq 0.$

$\Rightarrow \sqrt{T^*T}$ exists!

Theorem: Polar Decomposition: $T \in \mathcal{L}(V)$ Then $\exists$ isometry $S \in \mathcal{L}(V)$ s.t

$$T = S\sqrt{T^*T}$$

Proof: $v \in V$

$$\|Tv\|^2 = \langle Tv, Tv\rangle = \langle T^*Tv, v\rangle = \langle \sqrt{T^*T}\sqrt{T^*T}\, v, v\rangle$$

$$= \langle \sqrt{T^*T}\, v, \sqrt{T^*T}\, v\rangle$$

$$= \|\sqrt{T^*T}\, v\|^2$$

$\sqrt{T^*T}$ is self adjoint (infact positive)

$$\|Tv\|^2 = \|\sqrt{T^*T}\, v\|^2$$

$$S_1 : \text{range} \sqrt{T^*T} \longrightarrow \text{range}\, T$$
$$\sqrt{T^*T}\, v \longrightarrow Tv$$

check S_1 is linear.

$v_1 \neq v_2 \in V$

$\sqrt{T^*T}\, v_1 = \sqrt{T^*T}\, v_2$

$\Rightarrow T v_1 = T v_2$?

Suppose $v_1 \neq v_2 \in V$ & $\sqrt{T^*T}\, v_1 = \sqrt{T^*T}\, v_2$.

$$\|Tv_1 - Tv_2\|^2 = \|T(v_1 - v_2)\|^2 = \|\sqrt{T^*T}\,(v_1 - v_2)\|^2$$
$$= \|\sqrt{T^*T}\, v_1 - \sqrt{T^*T}\, v_2\|^2 = 0$$

$$Tv_1 = Tv_2$$

S_1 is clearly onto (surjective)

S_1 is also injective : $\|S_1 v\| = \|v\|$ $v \in \text{range} \sqrt{T^*T}$ = Domain S_1

By rank-Nullity theorem $\dim \text{range} \sqrt{T^*T} = \dim \text{range}\, T$

V $T \in \mathcal{L}(V)$

$\| \quad U \oplus U^\perp$

$\Downarrow$

$\dim (\text{range} \sqrt{T^*T})^\perp = \dim (\text{range}\, T)^\perp$

$S_2 \in \mathcal{L}((\text{range}\sqrt{T^*T})^\perp)$. Choose an orthonormal bas $e_1 \dots e_m$ of $(\text{range}\sqrt{T^*T})^\perp$

$f_1 \dots f_m$ of $(\text{range}\, T)^\perp$

$$S_2\left(\sum c_i e_i\right) = \sum c_i f_i$$

$$\forall w \in (\text{range}\sqrt{T^*T})^\perp \Rightarrow \|S_2 w\| = \|w\| = \sqrt{|c_1|^2 + \dots + |c_m|^2}$$

$$\|S_2 w\| = \left\|\sum c_i f_i\right\| = \sqrt{|c_1|^2 + \dots + |c_m|^2}.$$

$S : V \longrightarrow V.$

$$S(v) = \begin{cases} S_1(v) & \text{if } v \in \text{range}\sqrt{T^*T} \\ S_2(v) & \text{if } v \in (\text{range}\sqrt{T^*T})^\perp \end{cases}$$

Let $v \in V = \text{range}\sqrt{T^*T} \oplus (\text{range}\sqrt{T^*T})^\perp$

$$v = u + w$$

$$Sv = S(u+w) = Su + Sw = S_1 u + S_2 w.$$

$$S\left(\sqrt{T^*T}\,v\right) = S_1\left(\sqrt{T^*T}\,v\right) = Tv \quad \forall v \in V.$$

$$\Rightarrow \quad T = S\sqrt{T^*T}$$

S is an isometry!: $\|Sv\|^2 = \|S_1u + S_2w\|^2 = \|S_1u\|^2 + \|S_2w\|^2 = \|u\|^2 + \|w\|^2$

$$= \|u+w\|^2 = \|v\|^2.$$

Remark: S_1 is an isometry $\Rightarrow \exists$ orthogonal basis $\mathcal{B}_1$ s.t. $\mathcal{M}(S_1, \mathcal{B}_1)$ is diagonal

$\sqrt{T^*T}$ is an positive $\Rightarrow$ " $\mathcal{B}_2$ s.t $\mathcal{M}(\sqrt{T^*T}, \mathcal{B}_2)$ is diagonal.

But $\mathcal{B}_1$ may not be the same as $\mathcal{B}_2$.

Def: $T \in \mathcal{L}(V)$. The singular values of T are eigenvalues of $\sqrt{T^*T}$ with each eigenvalue λ repeated $\dim E(\lambda, \sqrt{T^*T})$ times.

Ex:

$$T\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 3z_1 \\ 2z_2 \\ 0 \\ 3z_4 \end{bmatrix} \qquad T^*T\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 9z_1 \\ 4z_2 \\ 0 \\ 9z_4 \end{bmatrix} \qquad \sqrt{T^*T}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 3z_1 \\ 2z_2 \\ 0 \\ 3z_4 \end{bmatrix}$$

Eigenvalues of $\sqrt{T^*T}$ is $3, 2, 0$.

$\dim E(3, \sqrt{T^*T}) = 2$

$\dim E(2, \sqrt{T^*T}) = 1$

$\dim E(0, \sqrt{T^*T}) = 1$

Theorem: $T \in \mathcal{L}(V)$. have singular values $s_1 \cdots s_n$. Then $\exists$ orthonormal basis $e_1 \cdots e_n$, & $f_1, \cdots, f_n$ of V s.t.

$$Tv = s_1 \langle v, e_1\rangle f_1 + \cdots + s_n \langle v, e_n\rangle f_n.$$

Proof: $\sqrt{T^*T}$ is a positive operator (& hence self adjoint)

By Spectral Theorem $\exists$ a basis $e_1 \cdots e_n$ of V s.t $\sqrt{T^*T}\, e_i = s_i e_i$

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n.$$

$$\sqrt{T^*T}\, v = \langle v, e_1\rangle \sqrt{T^*T} e_1 + \cdots + \langle v, e_n\rangle \sqrt{T^*T}\, e_n.$$

$$= \langle v, e_1\rangle s_1 e_1 + \cdots + \langle v, e_n\rangle s_n e_n$$

Using Polar decomposition $\exists$ isometry $S \in \mathcal{L}(V)$ s.t $T = S\sqrt{T^*T}$.

$$S\sqrt{T^*T}\, v = \langle v, e_1\rangle s_1 S e_1 + \cdots + \langle v, e_n\rangle s_n S e_n.$$

S is an isometry $\Rightarrow$ $Se_1 \cdots Se_n$ is an orthonormal basis. Set $Se_i = f_i$

$$Tv = S\sqrt{T^*T}\,v = \langle v, e_1\rangle s_1 f_1 + \cdots + \langle v, e_n\rangle s_n f_n .$$

T is diagonalizable $\Rightarrow \exists$ a basis $e_1 \dots e_n$ s.t. $\mathcal{M}(T, e_1 \dots e_n)$ is diagonal.

Now let $T \in \mathcal{L}(V)$ then by the singular value decomposition.

$$\mathcal{M}(T; e_1 \dots e_n; f_1 \dots f_n) \text{ diagonal!!}$$

$$= \begin{bmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{bmatrix}$$

$v = \sum c_i e_i$

$Tv = \sum d_i f_i$

$T(e_i) = \sum d_i f_i$

PD : $T \in \mathcal{L}(V) \Rightarrow T = S\sqrt{T^*T}$ where S is an isometry

SVD : $T \in \mathcal{L}(V)$, $\underbrace{s_1 \dots s_n \text{ are eigenvalues of } \sqrt{T^*T}}_{\text{Singular values of } T}$ $\Rightarrow \exists$ orthonormal bases $e_1 \cdots e_n$, & $f_1 \cdots f_n$ s.t.

$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$.

T is diagonalizable if $\exists$ basis $e_1 \dots e_n$ s.t. $\mathcal{M}(T; e_1 \dots e_n; e_1 \dots e_n) =$ diagonal

$$\mathcal{M}(T; e_1 \dots e_n; f_1 \dots f_n) = \begin{bmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{bmatrix}$$

Recall that T^*T is a positive operator

Theorem: $T \in \mathcal{L}(V)$. Then the singular values of T are nonnegative square roots of the eigenvalues of T^*T & each eigenvalue is repeated $\dim E(\lambda, T^*T)$ times.

§ Generalized Eigenvectors & Nilpotent Operators.

Lemma: $T \in \mathcal{L}(V)$. Then

$$\{0\} = \text{null}\, T^0 \subseteq \text{null}\, T^1 \subseteq \text{null}\, T^2 \subseteq \cdots$$

Proof: k is nonnegative integer. $v \in \text{null}\, T^k \overset{?}{\subseteq} \text{null}\, T^{k+1}$.

$$\Downarrow$$

$$T^k v = 0$$

$$\Downarrow$$

$$T^{k+1} v = T(T^k v) = T(0) = 0$$

Lemma: $T \in \mathcal{L}(V)$. Suppose m is positive integer s.t. $\text{null}\, T^m = \text{null}\, T^{m+1}$

Then $\text{null}\, T^m = \text{null}\, T^{m+1} = \text{null}\, T^{m+2} = \cdots = \text{null}\, T^{m+k+1} = \cdots$

Proof: WTS $\text{null}\, T^{m+k} = \text{null}\, T^{m+k+1}$.

$\subseteq$ ✓ (from previous lemma)

$v \in \text{null } T^{m+k+1} \Rightarrow T^{m+k+1}(v) = 0$

$$\begin{aligned} & T^{m+k+1}(v) = 0 \\ & \quad \| \\ & T^{m+1}(T^k v) = 0. \\ & \quad \Downarrow \\ & T^k v \in \text{null } T^{m+1} = \text{null } T^m \Rightarrow T^m(T^k v) = 0. \end{aligned}$$

$$T^m(T^k v) = T^{m+k} v = 0$$

$v \in \text{null } T^{m+k}$.

$\text{null } T^{m+k+1} \subseteq \text{null } T^{m+k}$.

Theorem: $T \in \mathcal{L}(V)$. $n = \dim V$ Then

$$\text{null } T^n = \text{null } T^{n+1} = \text{null } T^{n+2} = \cdots$$

Proof: Suppose $\text{null } T^0 \subsetneq \text{null } T^1 \subsetneq \text{null } T^2 \subsetneq \cdots$ (all are subspaces of V)

$\dim \text{null } T^0 < \dim \text{null } T^1 < \dim \text{null } T^2 < \cdots$

$\Rightarrow \dim \text{null } T^{n+1} \geq n+1. \rightarrow\leftarrow.$

$\Rightarrow \text{null } T^n = \text{null } T^{n+1} = \text{null } T^{n+2} = \cdots$

Compare with minor problem.

Theorem: $T \in \mathcal{L}(V)$ $n = \dim V$ Then

$$V = \operatorname{null} T^n \oplus \operatorname{range} T^n.$$

$V = \operatorname{null} T^n + \operatorname{range} T^n.$

$\operatorname{null} T^n \cap \operatorname{range} T^n = 0$.

Proof: $v \in \operatorname{null} T^n \cap \operatorname{range} T^n$.

$v \in \operatorname{null} T^n \Rightarrow T^n v = 0$

$v \in \operatorname{range} T^n \Rightarrow \exists\, u \in V$ s.t $T^n u = v$

$\Downarrow$

$T^n(T^n u) = T^n v = 0$

$\Rightarrow u \in \operatorname{null} T^{2n} = \operatorname{null} T^n$ (from previous result)

$\Rightarrow \underbrace{T^n u}_{= v} = 0$

$0 = \operatorname{null} T^n \cap \operatorname{range} T^n$

$\dim(\operatorname{null} T^n \oplus \operatorname{range} T^n) = \dim \operatorname{null} T^n + \dim \operatorname{range} T^n = \dim V.$ $T^n \in \mathcal{L}(V)$.

$\Rightarrow \operatorname{null} T^n \oplus \operatorname{range} T^n = V.$

Ex: $T \in \mathcal{L}(\mathbb{F}^3)$

$$T\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4z_2 \\ 0 \\ 5z_3 \end{bmatrix}$$

null T + range T is not a direct sum.

$$\operatorname{null} T = \left\{ \begin{bmatrix} z_1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\operatorname{range} T = \left\{ \begin{bmatrix} z_1 \\ 0 \\ z_3 \end{bmatrix} \right\}.$$

$$T^3\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 125 z_3 \end{bmatrix}$$

$$\operatorname{null} T^3 = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ 0 \end{bmatrix} \right\}$$

$$\operatorname{range} T^3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ z_3 \end{bmatrix} \right\}$$

$V = \operatorname{null} T^3 \oplus \operatorname{range} T^3$.

T is diagonalizable $\iff \exists$ basis $v_1 \dots v_n$ $\mathcal{M}(T, (v_1 \dots v_n))$ is diagonal

$\iff V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$

Def: $T \in \mathcal{L}(V)$. λ is an eigenvalue of T

$v \in V$ is called a generalized eigenvector of T corresponding to λ. if $v \neq 0$ &

$$(T - \lambda I)^j v = 0. \quad \text{for some positive integer } j.$$

Def: $T \in \mathcal{L}(V)$. $\lambda \in \mathbb{F}$. The generalized eigenspace of T corresponding to λ, denoted by $G(\lambda, T)$ is the set of all generalized eigenvectors of T corresponding to λ, along with 0 vector.

Every eigenvector is a generalized eigenvector

$$\Rightarrow E(\lambda, T) \subseteq G(\lambda, T).$$

$$G(\lambda, T) = \bigcup_{j \geq 0} \operatorname{null}(T - \lambda I)^j \qquad \operatorname{null}(T-\lambda I) \subseteq \operatorname{null}(T-\lambda I)^2 \subseteq \operatorname{null}(T-\lambda I)^3 \subseteq \cdots$$

$n = \dim V$.

$$\operatorname{null}(T-\lambda I)^n = \operatorname{null}(T-\lambda I)^{n+1} = \cdots$$

Theorem: $T \in \mathcal{L}(V)$ $\lambda \in \mathbb{F}$, $G(\lambda, T) = \text{null}\,(T - \lambda I)^{\dim V}$

Proof: $v \in \text{null}\,(T-\lambda I)^n$ $\quad n = \dim V$.

$\Rightarrow v \in G(\lambda, T)$

Conversely if $v \in G(\lambda,T) \Rightarrow v \in \text{null}\,(T-\lambda I)^j$ for some j

$\subseteq \text{null}\,(T-\lambda I)^n$ $\quad n = \dim V$.

Ex: $T \in \mathcal{L}(\mathbb{C}^3)$

Eigenvalues are 0 & 5.

$$T\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4z_2 \\ 0 \\ 5z_3 \end{bmatrix}$$

$E(0,T) = \left\{ \begin{bmatrix} z_1 \\ 0 \\ 0 \end{bmatrix} \right\}$ $\quad V \neq E(0,T) \oplus E(5,T)$

$E(5,T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ z_3 \end{bmatrix} \right\}$

$$T^3\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 125z_3 \end{bmatrix}.$$

$G(0,T) = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ 0 \end{bmatrix} \right\}$. $\quad \underset{E(5,T)}{\overset{\shortparallel}{G(5,T)}} = \left\{ \begin{bmatrix} 0 \\ 0 \\ z_3 \end{bmatrix} \right\}$

$V = G(0,T) \oplus G(5,T)$

Theorem: $T \in \mathcal{L}(V)$. $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T.

$v_1, \ldots, v_m$ are generalized eigenvectors of T

Then $v_1, \ldots, v_m$ are linearly independent.

MTL104 Lec35

Note Title 12-04-2023

Generalized Eigenvector: $v \in V$, $T \in \mathcal{L}(V)$, λ is an eigenvalue of T, $(T-\lambda I)^j v = 0$ for some j

$\Leftrightarrow v \in \text{null}(T-\lambda I)^j$ for some j

Generalized Eigenspace: $G(\lambda, T)$ = collection all generalised eigenvectors $= \bigcup_j \text{null}(T-\lambda I)^j$

$G(\lambda, T) = \text{null}(T-\lambda I)^n \quad n = \dim V$

$E(\lambda, T) = \text{null}(T-\lambda I) \subseteq G(\lambda, T)$

$T \in \mathcal{L}(V)$

$E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T) \neq V$

$G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T) = V$

Theorem: $T \in \mathcal{L}(V)$. $\lambda_1 \ldots \lambda_m$ are distinct eigenvalues of T.

$v_1 \ldots v_m$ are generalized eigenvectors of T

$\Rightarrow v_1 \ldots v_m$ are linearly independent

Proof: $a_1 v_1 + \cdots + a_m v_m = 0$ (*)

v_i's are generalized eigenvectors. $v_i \in \text{null}\,(T-\lambda_i I)^n$ "$G(\lambda_i, T)$" $n = \dim V$.

v_1 is a generalized eigenvector $\Rightarrow \exists$ some k s.t. $(T-\lambda_1 I)^k v_1 = 0$

$\Rightarrow$ choose k to be the smallest integer s.t. $(T-\lambda I)^{k-1} v_1 \neq 0$

$(\Rightarrow (T-\lambda I)^k v_1 = 0)$

$$w = (T-\lambda I)^{k-1} v_1.$$

$$(T-\lambda I) w = (T-\lambda I)(T-\lambda I)^{k-1} v_1 = (T-\lambda I)^k v_1 = 0$$

$$\Rightarrow \quad Tw = \lambda w.$$

$$(T-\lambda I) w = Tw - \lambda w = \lambda_1 w - \lambda w = (\lambda_1 - \lambda) w.$$

Apply $(T-\lambda_1 I)^{k-1} (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n$ to (*).

$$a_1 (T-\lambda_1 I)^{k-1} (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n v_1 = 0$$

(v_i is are generalized eigenvector $v_i \in \text{null}\,(T-\lambda_i I)^n$.)

$$a_1 (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n w = 0$$

$$a_1 (\lambda_1 - \lambda_2)^n \cdots (\lambda_1 - \lambda_m)^n w = 0$$

λ's are distinct $\Rightarrow$ $\lambda_1 - \lambda_2 \neq 0 \; \cdots \; (\lambda_1 - \lambda_m) \neq 0$ $\quad w \neq 0$ $\quad \Rightarrow a_1 = 0$.

Keep repeating this process to see that $a_2 = a_3 = \cdots = a_m = 0$.

$\Rightarrow$ $v_1 \ldots v_m$ are linearly independent

§ Nilpotent Operator.

Def: An operator $T \in \mathcal{L}(V)$ is nilpotent if $T^j = 0$ for some j

Ex: (1) $V = P_4(\mathbb{R})$ $\quad D \in \mathcal{L}(V)$ is the differential operator $D(f) = f'$ $\quad D^5 = 0$

(2) $V = \mathbb{R}^4$ $\quad T \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} z_3 \\ z_4 \\ 0 \\ 0 \end{bmatrix}$ $\quad T^2 = 0$

$G(\lambda, T) =$ null $(T-\lambda I)^n$.

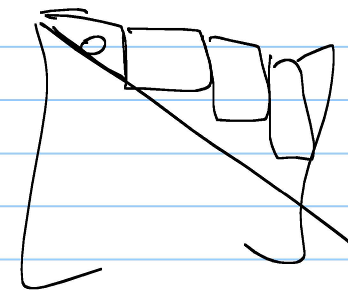
Theorem: $N \in \mathcal{L}(V)$ is nilpotent. Then $N^n = 0$, $n = \dim V$.

Proof: N is nilpotent $\Rightarrow$ $G(0, N) = V$ $\Rightarrow$ null $N^n = V$.

$$G(0,N) = \text{null}\,(N - 0I)^n = \text{null}\, N^n.$$

Theorem: N is an nilpotent operator on V. Then $\exists$ a basis of V with respect to which the matrix of N has the form

$$\begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$$



Proof: $V =$ null N^n for $n = \dim V$.

Basis for V is

$$\underbrace{\underbrace{\underbrace{v_{11}, \dots, v_{1n_1}}_{\text{null } N}, v_{21}, \dots, v_{2n_2}}_{\text{null } N^2}, \dots, v_{m1}, \dots, v_{mn_m}}_{\text{null } N^n = V} = \mathcal{B}$$

$N(v_{1i}) = \sum a_i v_{ij}$, $v_{ij} \in \mathcal{B}$

$v_{2i} \in \operatorname{null} N^2$

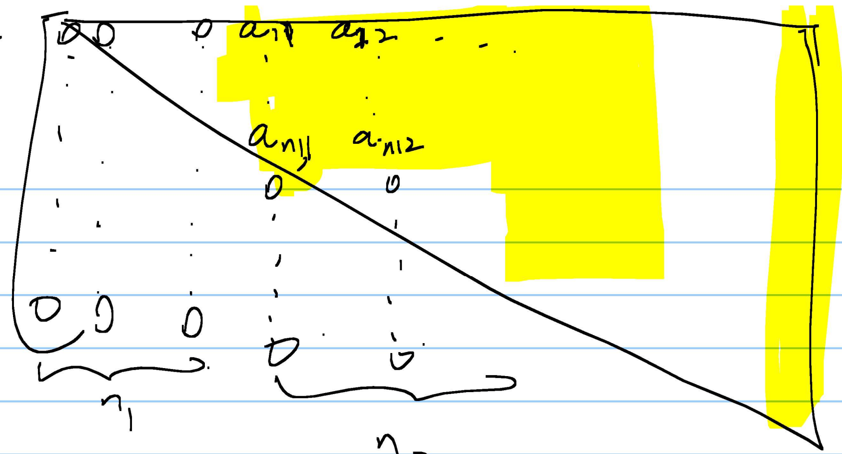
$N^2(v_{2i}) = 0$

$N(N(v_{2i})) = 0$

$N(v_{2i}) \in \operatorname{null} N$

$N(v_{2i}) = a_{11} v_{11} + \dots + a_{n_1} v_{n_1}$

$\mathcal{M}(T;\mathcal{B}) =$



$v \in \operatorname{null} N^i$

$N(v) \in \operatorname{null} N^{i-1}$

$x\,p(x) = x^3 + 3x^2 + x$

$p(x)\,x$ ✓

$p(x) = x^2 + 2x + 1$

$p(T) = T^2 + 2T + 1$

§ Decomposition of Operator

Theorem: $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbb{F})$. Then null $p(T)$ & range $p(T)$ are invariant under T.

Proof: $v \in \operatorname{null}(p(T))$ $\qquad p(T)v = 0$

$$p(T)(Tv) = (p(T)\cdot T)\,v = (T\cdot p(T))\,v = T(p(T)v) = T(0) = 0$$

$v \in \text{range}(p(T)) \Rightarrow v = p(T)u$

$Tv = T(p(T)u) = p(T)(Tu) \Rightarrow Tu \in \text{range } p(T).$

Theorem: V is a complex vector space. $T \in \mathcal{L}(V)$. Let $\lambda_1 \cdots \lambda_m$ are distinct eigenvalues of T. Then

1. $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$
2. each $G(\lambda_i, T)$ are invariant under T
3. each $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent.

Proof: (2) $G(\lambda_i, T) = \text{null } (T - \lambda_i I)^n$ $n = \dim V$. invariant under T by previous theorem

(3) $G(\lambda_j, T) = \text{null } (T - \lambda_j I)^n$ $n = \dim V$.

$\forall v \in G(\lambda_j, T) \Rightarrow (T - \lambda_j I)^n v = 0 \Rightarrow (T - \lambda_j I)^n = 0$ on $G(\lambda_j, T)$

$\Rightarrow (T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent.

$T \in \mathcal{L}(V)$

$V = \operatorname{null} T^n \oplus \operatorname{range} T^n$

① Induct on $n = \dim V$.

if $n=1$ T is already diagonalizable $\Rightarrow V = E(\lambda_1, T) = G(\lambda_1, T)$ ✓

Suppose $n>1$ & assume by induction hypothesis that it is true $\forall$ vector spaces of dim $< n$.

V is a complex vector space $\Rightarrow \exists\, \lambda_1$ eigenvalue of T.

$$(T - \lambda_1 I) \in \mathcal{L}(V)$$

$$V = \underbrace{\operatorname{null}(T - \lambda_1 I)^n}_{= G(\lambda_1, T)} \oplus \underbrace{\operatorname{range}(T - \lambda_1 I)^n}_{U}.$$

$\dim U < \dim V$ By induction hypothesis on $T|_U$

$$U = G(\lambda_2, T|_U) \oplus \cdots \oplus G(\lambda_m, T|_U)$$

MTL 104 Lec 36

Note Title 18-04-2023

$E(\lambda, T) = \{v : Tv = \lambda v\} = \text{null}(T - \lambda I)$

$G(\lambda, T) = \{v : (T-\lambda I)^j v = 0 \text{ for some } j\} = \text{null}(T-\lambda I)^{\dim V}$

$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$

$p(t) \in P(\mathbb{F})$

$\text{null}\, p(T)$, $\text{range}\, p(T)$ invariant under T

Theorem: V is a complex vector space. $T \in \mathcal{L}(V)$. $\lambda_1 \ldots \lambda_m$ are eigenvalues

1. $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$
2. $G(\lambda_j, T)$ is invariant under T
3. each $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent.

on $G(\lambda_j, T)$

$(T - \lambda I)^{\dim V} v = 0$

$\forall v \in G(\lambda_j, T)$

$\Rightarrow (T - \lambda I)^{\dim V} = 0$

$\Rightarrow T - \lambda I$ is nilpotent.

Proof: (1) Induct on $\dim V = n$.

$n = 1$ ✓

Suppose result is true for all vector spaces of dim $< n$.

V is complex vector space $\Rightarrow \exists \lambda_1 \in \mathbb{C}$ eigenvalue

$$V = \underset{\text{null } (T-\lambda I)^n}{\underset{\shortparallel}{G(\lambda_1, T)}} \oplus \underset{\text{range } (T-\lambda I)^n}{\underset{\shortparallel}{U}}$$

(result. $V = \text{null } T^n \oplus \text{range } T^n$)

$\dim U = \dim \text{range}(T-\lambda I)^n < \dim V$

$\Rightarrow$ By induction hypothesis $\lambda_2 \ldots \lambda_m$ s.t. applied to $T|_U \in \mathcal{L}(U)$

$$U = G(\lambda_2, T|_U) \oplus \cdots \oplus G(\lambda_m, T|_U)$$

$$G(\lambda_j, T|_U) = \{u \in U : (T-\lambda I)^k u = 0 \text{ for some } k\}$$

$$\cap|$$

$$G(\lambda_j, T) = \{v \in V : (T-\lambda I)^k v = 0 \text{ for some } k\}$$

Any generalized eigenvector of $T|_U$ is also a generalized eigenvector of T

$v \in G(\lambda_j, T)$. $v = v_1 + u$ where $v_1 \in G(\lambda_1, T)$, $u \in U$

$\Rightarrow u = v_2 + \cdots + v_m$ where $v_k \in G(\lambda_k, T|_U)$.

$$v = v_1 + v_2 + \cdots + v_m$$

generalized eigenvectors corresponding to distinct eigenvalues are linearly independent

all $v_i = 0$ except possibly v_j. $\Rightarrow$ $v = v_j \in G(\lambda_j, T|_U)$.

$$G(\lambda_j, T) = G(\lambda_j, T|_U)$$

$$\Rightarrow \quad V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$$

Theorem: Suppose V is a complex vector space. $T \in \mathcal{L}(V)$. Then $\exists$ a basis of V consisting of generalized eigenvectors of T

Proof: ✓

Def: $T \in \mathcal{L}(V)$. The multiplicity of an eigenvalue λ of T is the dim of $G(\lambda, T)$

Algebraic multiplicity = $\dim G(\lambda, T)$.

Geometric multiplicity = $\dim E(\lambda, T)$

Theorem: Suppose V is a complex vector space $T \in \mathcal{L}(V)$. Then the sum of the alg multiplicities of all eigenvalues of T equals $\dim V$.

Def: Block diagonal matrix.

A block diagonal matrix is a square matrix of the form

$$\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix}$$

where $A_1 \dots A_m$ are square matrices

$$\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Theorem: V is a complex vector space $T \in \mathcal{L}(V)$. Let $\lambda_1 \dots \lambda_m$ be distinct eigenvalues of T with multiplicities $d_1 \dots d_m$. Then $\exists$ a basis of V with respect to which T has a block diagonal form

$$\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix}$$

where A_j is $d_j \times d_j$ matrix

$$\begin{bmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{bmatrix}$$

Proof: $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$

$G(\lambda_j, T)$ are T-invariant

$T\big|_{G(\lambda_j, T)}$

$(T - \lambda_j I)\big|_{G(\lambda_j, T)}$ is nilpotent.

The matrix of nilpotent operator are $\begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$ Uppertriangular with diagonal entries 0!

$v' \in U_2$
$T(v') \in U_2$

$V = U_1 \oplus U_2$
$T \in \mathcal{L}(V)$
U_1, U_2 are T-invariant

$v \in U_1$
$T(v) \in U_1$
$T(v_2) \in U_1$

$u \in U_{1,2}$ $T(u) \in U_{1,2}$

$$\mathcal{M}(T) = \left[\begin{array}{cc|c} * & * & 0 \\ \vdots & \vdots & \vdots \\ * & * & 0 \\ \hline 0 & 0 & * \\ \vdots & \vdots & \vdots \\ 0 & 0 & * \end{array}\right]$$

restricted to $G(\lambda_j, T)$

$$\mathcal{M}(T - \lambda_j T) = \begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$$

$$\mathcal{M}(T) = \begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix} + \lambda I = \begin{bmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{bmatrix}$$

$V = U_1 \oplus U_2$

$\{v_1, v_2\}$ $\{v_3, v_4\}$

$T \in \mathcal{L}(V)$. U_1, U_2 are T-invariant

$$\mu(T) = \begin{bmatrix} \mu(T|_{U_1}) & 0 \\ 0 & \mu(T|_{U_2}) \end{bmatrix}$$

$$\mu(T) = \begin{bmatrix} \mu(T|_{G(\lambda_1, T)}) & & & \\ & \mu(T|_{G(\lambda_2, T)}) & & \\ & & \ddots & \\ & & & \mu(T|_{G(\lambda_m, T)}) \end{bmatrix}$$

$$= \begin{bmatrix} A_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & A_m \end{bmatrix}$$

$$A_m = \begin{bmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{bmatrix}$$

Ex: $T\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 6z_1 + 3z_2 + 4z_3 \\ 6z_2 + 2z_3 \\ 7z_3 \end{bmatrix}$

$\mathcal{M}(T) = \begin{bmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{bmatrix}$

$G(6,T) = \text{Span}\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \right\}$

$G(7,T) = \text{span}\left\{ \underset{v_3}{\begin{bmatrix} 10 \\ 2 \\ 1 \end{bmatrix}} \right\}.$

$\mathcal{M}(T, v_1, v_2, v_3) = \begin{bmatrix} \boxed{\begin{matrix} 6 & 3 \\ 0 & 6 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{7} \end{bmatrix}$

$G(\lambda, T) = \text{null}\,(T - \lambda I)^2.$

$E(\lambda, T) = \text{null}\,(T - \lambda I)$

$\subseteq \text{null}\,(T - \lambda I)^2$

Def: V is a complex vector space. $T \in \mathcal{L}(V)$

$\lambda_1 \dots \lambda_m$ are eigenvalue of T.

$d_1 \dots d_m$ are alg. multiplicities of T.

Then $p(z) = (z - \lambda_1)^{d_1} \dots (z - \lambda_m)^{d_m}$ is called

characteristic polynomial of T

Theorem: V is a complex vector space. $T \in \mathcal{L}(V)$.

(a) The characteristic polynomial of T has degree $\dim V$.

(b) The zeros of the characteristic polynomial of T are the eigenvalues of T

MTL 104 Lec 37

Note Title 19-04-2023

Characteristic polynomial: $T \in \mathcal{L}(V)$, $\lambda_1 \dots \lambda_m$ are eigenvalues
$d_1 \dots d_m$ are algebraic multiplicities

$$V = \underbrace{G(\lambda_1, T)}_{d_1} \oplus \cdots \oplus \underbrace{G(\lambda_m, T)}_{d_m}.$$

$$p(z) = (z-\lambda_1)^{d_1} \cdots (z-\lambda_m)^{d_m}.$$

① $\deg p = \dim V$.

② zeros of $p(z)$ are eigenvalues

Theorem (Cayley-Hamilton Theorem): Suppose V is a complex vector space, $T \in \mathcal{L}(V)$. Let $q(z)$ be the characteristic polynomial, then $q(T) = 0$

Proof: Let $\lambda_1 \dots \lambda_m$ be eigenvalues of T
$d_1 \dots d_m$ be algebraic multiplicities of T

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$$

$q(T) = 0$.

$q(T)v = 0 \ \forall v \in V$

$(T-\lambda_1)^{d_1} \cdots (T-\lambda_m)^{d_m} v = 0 \ \forall v \in V$

$v = v_1 + \cdots + v_m$

$(T-\lambda_i I)\big|_{G(\lambda_i,T)}$ is nilpotent.

$(T-\lambda_1)^{d_1}\cdots(T-\lambda_m)^{d_m}(v_1+\cdots+v_m)=0$

$N \in \mathcal{L}(V)$

$N^{\dim V}=0$

$(T-\lambda_i I)^{d_i}=0$ on $G(\lambda_i,T)$ $\quad d_i=\dim G(\lambda_i,T)$

$(T-\lambda_i I)^{d_i}u=0 \quad \forall u\in G(\lambda_i,T)$

$(T-\lambda_1)^{d_1}\cdots(T-\lambda_m)^{d_m}(v_1+\cdots+v_m)=0 \qquad v_i\in G(\lambda_i,T)$

$(T-\lambda_1)^{d_1}\cdots(T-\lambda_m)^{d_m}v_i=0 \quad \forall i$

$\parallel$

$(T-\lambda_1)^{d_1}\cdots(T-\lambda_{i-1})^{d_{i-1}}(T-\lambda_{i+1})^{d_{i+1}}\cdots(T-\lambda_m)^{d_m}(T-\lambda_i)^{d_i}v_i=0$

$(T-\lambda_1)(T-\lambda_2)$
$=T^2-(\lambda_1+\lambda_2)T+\lambda_1\lambda_2$
$=T^2-(\lambda_2+\lambda_1)T+\lambda_2\lambda_1$
$=(T-\lambda_2)(T-\lambda_1)$

Def: $T\in\mathcal{L}(V)$. A monic polynomial is a polynomial whose highest degree coefficient equals 1.

Theorem: $T \in \mathcal{L}(V)$. Then $\exists$ a unique monic polynomial $p(z)$ of smallest degree s.t. $p(T) = 0$

Proof: $n = \dim V$. Consider the list $1, T, T^2, \ldots, T^{n^2}$ is n^2+1 in number $\dim \mathcal{L}(V) = n^2$.

$\Rightarrow 1, T, T^2, \ldots, T^{n^2}$ is linearly dependent.

$\Rightarrow \exists\ a_0 \ldots a_{n^2}$ s.t. $a_0 + a_1 T + a_2 T^2 + \cdots + a_{n^2} T^{n^2} = 0$ not all $a_i = 0$.

Choose m to be the smallest integer s.t.

$1, T, T^2, \ldots, T^m$ is linearly dependent.

$\Rightarrow \exists\ a_0 \ldots a_m$ not all zero s.t.

$a_0 + a_1 T + \cdots + a_m T^m = 0$.

Define the monic polynomial $p(z)$ to be $\frac{a_0}{a_m} + \frac{a_1}{a_m} T + \cdots + \frac{a_{m-1}}{a_m} T^{m-1} + T^m = 0$

($a_m \neq 0$ as m is smallest s.t. $1, T, \ldots, T^m$ is l.d).

Suppose $\exists\, q(z)$, a monic polynomial of degree m s.t $q(T)=0$.

Consider $p(T)-q(T)$ has degree $m-1$.

$\left(p(T)-q(T)\right)v=0$ as $p(T)v=q(T)v=0$.

contradiction to minimality of m.

Def: $T\in\mathcal{L}(V)$. Then the minimal polynomial of T is the unique monic polynomial p of smallest degree s.t $p(T)=0$.

$T\in\mathcal{L}(V)$ $\mathcal{M}(T)$ For each m search $\mathcal{M}(T^2)$

solve $\mathcal{M}(T^2)=a_0+a_1\mathcal{M}(T)$

if not solve $\mathcal{M}(T^3)=a_0+a_1\mathcal{M}(T)+a_2\mathcal{M}(T^2)$

$\vdots$

keep going on.

Theorem: $T \in \mathcal{L}(V)$. $q(z)$ is a polynomial. Then $q(T)=0 \iff q$ is a polynomial multiple of the minimal polynomial

Proof: Suppose $q = f \cdot p$ where $p(z)$ is the minimal polynomial.

$q(T) = f(T)\,p(T)$ $\quad$ $p(T)=0$ $\quad \Rightarrow q(T)=0$.

Conversely suppose $q(z)$ is a polynomial s.t. $q(T)=0$.

By division algorithm. $\quad \underset{\substack{\| \\ 0}}{q(T)} = f(T)\,\underset{\substack{\| \\ 0}}{p(T)} + r(T)$ $\quad$ where $\deg r(T) < \deg p(T)$ $*$

$\Rightarrow r(T)=0$ & $\deg r(T) < \deg p(T)$

$\Rightarrow r \equiv 0$.

$\Rightarrow q(T) = f(T)\,p(T)$

Theorem: Suppose V is a complex vector space. $T \in \mathcal{L}(V)$. Then the characteristic polynomial of T is a polynomial multiple of the minimal polynomial.

$F = \mathbb{R}$

$(z^2+1)(z-\lambda)$

Theorem: $T \in \mathcal{L}(V)$. Then the zeros of minimal polynomial of T are precisely the eigenvalues of T.

Proof: $p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m$.

λ is a zero for $p(z)$.

$$\Rightarrow \quad p(z) = (z-\lambda)\, q(z)$$

$$p(T) = 0 \quad \Rightarrow \quad p(T) = (T-\lambda)\; q(T)$$

$\forall v \in V$. $\quad 0 = p(T)v = (T-\lambda)\; \underbrace{q(T)\, v}_{\neq 0}$. $\qquad \deg q(T) < \deg p(T)$

$$= q(T)\;(T-\lambda)\, v$$

$\deg q(T) < \deg p(T) \Rightarrow q(T) \neq 0 \Rightarrow q(T)\, v \neq 0$ for some $v \in V$.

$\Rightarrow$ For this v, $\quad 0 = q(T)(T-\lambda)v.$

$\Rightarrow \quad Tv = \lambda v.$

$\Rightarrow$ λ is an eigenvalue & v is its eigenvector.

Suppose $\lambda \in F$ is an eigenvalue of T. $\Rightarrow$. $Tv = \lambda v$. for some v.

$T^i v = \lambda^i v \quad \forall i$

$$0 = P(T)v = (T^m + a_{m-1}T^{m-1} + \cdots + a_1 T + a_0)v$$

$$= (\lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_1\lambda + a_0)v$$

$$\Rightarrow \quad \underbrace{\lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_1\lambda + a_0}_{p(\lambda)} = 0$$

$\Rightarrow$ λ is a root of the minimal polynomial $p(z)$.

Ex: $V = \mathbb{C}^3$. $T \in \mathcal{L}(V)$. Suppose $\det(T - \lambda I) = 0$ & solve to get eigenvalues of T.

$\lambda_1, \lambda_2, \lambda_3$.

① characteristic poly is $(T-\lambda_1)(T-\lambda_2)(T-\lambda_3)$

minimal poly $(T-\lambda_1)(T-\lambda_3)$

② $\lambda_1 = \lambda_2$ $\quad (T-\lambda_1)^2 (T-\lambda_3)$

$(T-\lambda_1)^2 (T-\lambda_3)$

③ $\lambda_1 = \lambda_2 = \lambda_3$ $\quad (T-\lambda_1)^3$.

$(T-\lambda_1)(T-\lambda_2) \neq 0$

Ex: $V = \mathbb{R}^3$. $\quad q(T) = 0 \quad q(z) = (z^2+1)(z-1)$

$z - 1$

$(z^2+1)(z-1)$

Ex: $V = \mathbb{C}^5$

$q(z) = z^2+1$

§ Jordan Canonical Form.

Ex. $N \in \mathbb{F}^4$. $N \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$ $N^4 = 0$.

$v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $Nv = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $N^2 v = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $N^3 v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\{v, Nv, N^2 v, N^3 v\}$ form a basis for V.

$Nv = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$= 0v + 1Nv +$

$$\mathcal{M}(N) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}^T$$

Ex: $N \in \mathcal{L}(\mathbb{C}^6)$

$$N \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ z_6 \end{bmatrix} = \begin{pmatrix} 0 \\ z_1 \\ z_2 \\ 0 \\ z_4 \\ 0 \end{pmatrix}. \qquad N^6 = 0.$$

$\exists$ no v s.t $v, Nv, N^2v, N^3v, \ldots, N^5v$ is a basis for V.

MTL101 Lec 38

Note Title 21-04-2023

§ Jordan Form

Ex: $N \in \mathcal{L}(\mathbb{C}^4)$

$$N\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \qquad N^4 = 0$$

$$v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad Nv = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad N^2 v = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad N^3 v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\mathcal{B} = \{N^3 v, N^2 v, Nv, v\}$. $\quad \mathcal{M}(N, \mathcal{B}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Ex: $N \in \mathcal{L}(\mathbb{F}^6)$

$\to N \in \mathcal{L}(T)$

$$\begin{bmatrix} 0 & & & * \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix}$$

$\to V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$

$$N \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} = \begin{bmatrix} 0 \\ z_1 \\ z_2 \\ 0 \\ z_4 \\ 0 \end{bmatrix}$$

There is no v s.t $N^5 v, N^4 v, N^3 v, N^2 v, v$ is a basis for V !!!

Chose $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\mathcal{B} \{ N^2 v_1, N v_1, v_1, N v_2, v_2, v_3 \}$ form a basis for V. !

$$\mathcal{M}(T, \mathcal{B}) = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & & 0 \\ & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \\ 0 & & [0] \end{bmatrix}$$

m_i's can be zero.

Theorem: $N \in \mathcal{L}(V)$ is nilpotent. Then $\exists\ v_1 \dots v_n$ of V & nonnegative integers $m_1 \dots m_n$ s.t.

(1) $\underbrace{N^{m_1}v_1, N^{m_1-1}v_1, \dots, Nv_1, v_1}, \underbrace{N^{m_2}v_2, \dots, v_2}, \dots\dots, \underbrace{N^{m_n}v_n, \dots, Nv_n, v_n}$ forms a basis of V.

(2) $N^{m_1+1}v_1 = \dots = N^{m_n+1}v_n = 0$

Proof: By induction on $\dim V$.

If $\dim V = 1 \Rightarrow N = 0 \Rightarrow$ choose any basis for V & it will satisfy the conclusion.

Assume by induction hypothesis the result is true $\forall$ vector spaces of $\dim < n$.

N is nilpotent $\Rightarrow N$ is not injective $\Rightarrow \operatorname{Ker} N \neq 0 \Rightarrow \dim \operatorname{Ker} N > 0$.

$\Rightarrow \dim V = \dim \operatorname{Ker} N + \dim \operatorname{range} N$ rank-nullity

$\Rightarrow \dim \operatorname{range} N < \dim V$.

range N is invariant subspace of V & $\dim \operatorname{range} N < \dim V$.

By induction hypothesis $\exists$ $v_1 \ldots v_n \in \text{range } N$ & $m_1 \ldots m_n \geq 0$ s.t.

(*) $N^{m_1} v_1, \ldots, N v_1, v_1, N^{m_2} v_2, \ldots, v_2, \ldots, N^{m_n} v_n, \ldots, v_n$ form a basis for range N.

& $N^{m_1+1} v_1 = N^{m_2+1} v_2 = \cdots = N^{m_n+1} v_n = 0$.

$v_1 \ldots v_n \in \text{range } N \Rightarrow \exists\, u_i \in V$ s.t. $N u_i = v_i$ $1 \leq i \leq n$.

$N^{k+1} u_i = N^k v_i$.

(**) Claim: $N^{m_1+1} u_1, \ldots, N u_1, u_1, N^{m_2+1} u_2, \ldots, N u_2, u_2, \ldots, N^{m_n+1} u_n, \ldots, N u_n, u_n$ is l.i in V.

Lift #

$\Rightarrow \alpha_{1 m_1+1} N^{m_1+1} u_1 + \cdots + \alpha_{11} u_1 + \alpha_{2 m_2+1} N^{m_2+1} u_2 + \cdots + \alpha_{21} u_2 + \cdots + \alpha_{n m_n+1} N^{m_n+1} u_n + \cdots + \alpha_{n1} u_n = 0$

Apply N to above.

$\underbrace{\alpha_{1 m_1+1} N^{m_1+2} u_1}_{=0} + \cdots + \alpha_{11} N u_1 + \underbrace{\alpha_{2 m_2+1} N^{m_2+2} u_2}_{=0} + \cdots + \alpha_{21} N u_2 + \cdots + \underbrace{\alpha_{n m_n+1} N^{m_n+2} u_n}_{=0} + \cdots + \alpha_{n1} N u_n = 0$

$\Rightarrow \alpha_{1 m_1} N^{m_1+1} u_1 + \cdots + \alpha_{11} N u_1 + \alpha_{2 m_2} N^{m_2+1} u_2 + \cdots + \alpha_{21} N u_2 + \cdots + \alpha_{n m_n} N^{m_n+1} u_n + \cdots + \alpha_{n1} N u_n = 0$

$\Rightarrow \alpha_{1m_1} = \alpha_{1m_1-1} = \cdots = \alpha_{11} = \alpha_{2m_2} = \cdots = \alpha_{21} = \cdots = \alpha_{nm_n} = \cdots = \alpha_{n1} = 0.$

by $(*)$

& u_i $Nu_i = v_i$

$\Rightarrow \alpha_{1m_1+1} N^{m_1+1} u_1 + \cdots + \alpha_{nm_n+1} N^{m_n+1} u_n = 0.$

$\alpha_{1m_1+1} N^{m_1} v_1 + \cdots + \alpha_{nm_n+1} N^{m_n} v_n = 0.$

$\Rightarrow \alpha_{ij} = 0$ by $(*)$

Extend $(+)$ to a basis

$(***)$ $N^{m_1}u_1, \ldots, u_1, N^{m_2}u_2, \ldots, u_2, \ldots, N^{m_n}u_n, \ldots, u_n, w_1, \ldots, w_p$ of V.

$Nw_j \in \text{range } N$ $\Rightarrow$ Nw_j is written as a linear combination of vectors in $(*)$

$\Rightarrow \exists$ some $x_j \in \text{range } N$ s.t. $N x_j = N w_j$

$\Rightarrow N(w_j - x_j) = 0.$

Set $w_j - x_j = u_{n+j}$

Check!!

$N^{m_1}u_1, \ldots, u_1, N^{m_2}u_2, \ldots, u_2, \ldots, N^{m_n}u_n, \ldots, Nu_n, u_n, \underset{w_1 - x_1}{u_{n+1}}, \ldots, \underset{w_p - x_p}{u_{n+p}}.$

is l.i in V & satisfies (1) & (2)

Def: Jordan Basis

$T \in \mathcal{L}(V)$. A basis of V is called a Jordan basis for T if w.r.t this basis the matrix of T is a block diagonal form

$$\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{bmatrix} \qquad A_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

Theorem: V is a complex vector space. $T \in \mathcal{L}(V)$. then $\exists$ a Jordan basis for T.

Proof: $N \in \mathcal{L}(V)$ be nilpotent.

$\Rightarrow \exists\, v_1 \dots v_n \in V$ s.t $N^{m_1} v_1, \dots, Nv_1, v_1, N^{m_2} v_2, \dots, v_2, \dots, N^{m_n} v_n, \dots, v_n$ forms basis for V

$$\mathcal{M}(N) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & 0 & 1 & & & & \vdots \\ \vdots & & 0 & \ddots & 0 & & \vdots \\ \vdots & & & \ddots & 1 & \ddots & \vdots \\ \vdots & & & & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

$$
= \begin{pmatrix}
\overbrace{\begin{matrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ & & & 0 \end{matrix}}^{m_1} & \overbrace{\begin{matrix} 0 \end{matrix}}^{m_2} & & \overbrace{}^{m_n} \\
0 & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{matrix} & & \\
0 & 0 & \ddots & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{matrix}
\end{pmatrix}
$$

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T).$$

$G(\lambda_i, T)$ are invariant

$(T - \lambda_i)\big|_{G(\lambda_i, T)}$ is nilpotent.

From what we just proved for nilpotent Operators

$\Rightarrow \exists$ a basis w.r.t which

$\mathcal{M}(T - \lambda_i)$ is a block diagonal matrix with 0's on diagonal

w.r.t to this basis

$\mathcal{M}(T\big|_{G(\lambda_i, T)})$ is a block diagonal matrix with λ_i on diagonal.

$$\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{bmatrix}. \qquad A_i = \begin{bmatrix} \lambda_i & & 0 \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}.$$

Put such basis of $G(\lambda_i, T)$ together to form a basis for V. w.r.t which the

$M(T)$ has the required form.

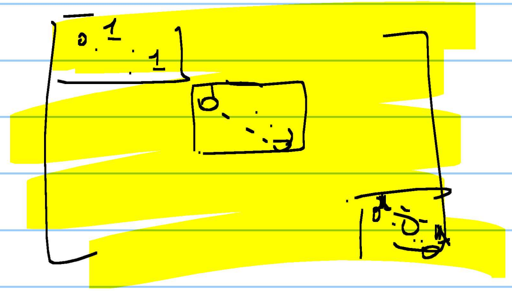
MTL104 Lec 39

Note Title 25-04-2023

① $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$ $T \in \mathcal{L}(V)$ V is a complex vectorspace

② $G(\lambda_i, T)$ are T invariant

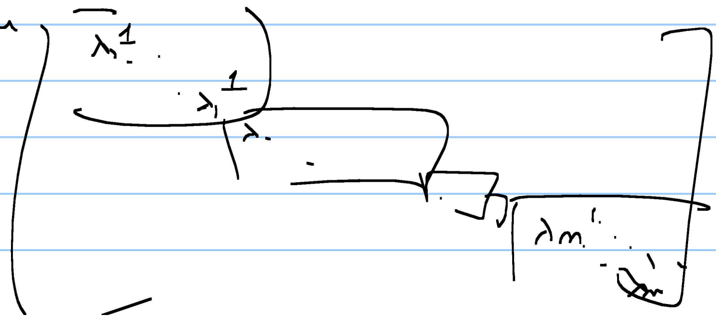
③ $(T - \lambda_i I)\big|_{G(\lambda_i, T)}$ are nilpotent. ④ $\Rightarrow \exists$ a basis $N^{m_1}v_1, \ldots, Nv_1, v_1, N^{m_2}v_2, \ldots, v_2, \ldots, N^{m_n}v_n, \ldots, v_n$ s.t.

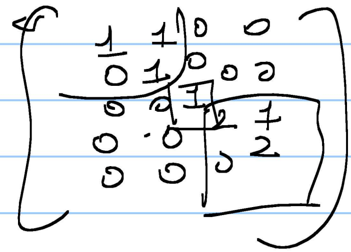


$N = T - \lambda_i I$.

$N^{m_i+1} v_i = 0$.

⑤ Jordan form, $T \in \mathcal{L}(V)$.





Ex.: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ $T: \mathbb{C}^3 \to \mathbb{C}^3$. $\lambda = 1$: null$(T - \lambda I)$

null$(T - \lambda I)^2 \overset{?}{=} 0$

null$(T - \lambda I)^3 = 0$

$\det A - \lambda I = (\lambda - 1)^2 (\lambda - 3) = 0 \Rightarrow \lambda = 1, 3.$

algebraic multiplicity

$\lambda = 1$ is $2 = \dim G(1, T)$

$\lambda = 3$ 1

null$(\cancel{A} - \lambda I)$

$\lambda = 1$: null$(A - I)$

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$\lambda = 3$: null$(A - 3I)$.

$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$G(1, T) = \text{span} \left\langle \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, ? \right\rangle$

No more generalized eigenvalues

Compute null$(A - I)^2$

Can get one by solving $(A - \lambda I)^2 v_3 = 0$ $(A - \lambda I) v_3 \neq 0$.

$v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$V = \underbrace{\text{Span} \left\langle \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle}_{G(1, T)} \oplus \underbrace{\text{Span} \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle}_{G(3, T)}$

Ex: $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$

$\det(A - \lambda I)$. $\lambda = 1$ is the only eigenvalue.

$\mathrm{null}(A - I)$ $v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ eigenvalue.

$(A - I)^2 = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}$. $\mathrm{null}(A - I)^2 = \langle \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{v_2}, v_1 \rangle$.

$(A - I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\mathrm{null}(A - I)^3 = V$.

Expand v_1, v_2 to a a basis of V to get v_3.

v_3 will be a generalized eigenvector as $\mathrm{null}(A - I)^3 = V$.

Ex: $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$.

$\lambda = 1$ is eigenvalue: $v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ is eigenvector. $\Rightarrow (A - I)v_1 = 0$

Soln $(A - I)\underline{x} = v_1$

If it has a solution, v_2, $\Rightarrow (A-I)v_2 = v_1$

$(A - I)^2 v_2 = (A-I)v_1 = 0$

$\Rightarrow v_2$ is a generalized eigenvector.

Soln $(A - I)\underline{x} = v_2$

If it has a solution v_3, $(A-I)v_3 = v_2$

$(A - I)^3 v_3 = (A-I)^2 v_2 = 0$

v_3 is a generalized eigenvector

A is a $n \times n$ matrix. v is a generalized eigenvector of A.

$$\underbrace{(A-\lambda I)^j}\, v = 0$$

$i < j$

$$(A-\lambda I)^{j-i}\,(A-\lambda I)^i v = 0 \Rightarrow (A-\lambda I)^i v \text{ is also an generalized eigenvector.}$$

If we choose j to be the smallest integer s.t. $(A-\lambda I)^j v = 0$

$1 \le i < j$ $\quad (A-\lambda I)^i v$ is a generalized eigenvector.

Ex: For the previous example

$$A - \lambda I = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \qquad (A-\lambda I) v = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \qquad (A-\lambda I)^2 v = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

$\Rightarrow (A-I)^3 v = 0$

$(A-I)^2 v \neq 0$

Ex: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$\det(A - \lambda I)$ $(\lambda - 2)(\lambda - 1)^2 = 0$. $\Rightarrow \lambda = 1, 2$.

$\lambda = 2$: Eigenvalue $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = v_1$.

$\lambda = 1$ eigenvector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = v_2$.

generalized eigenvector.

$(A - \lambda I)\underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = v_2$.

Solve to get $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = v_3$.

Ex:

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -3 & 0 \\ -1 & 1 & -1 \end{bmatrix} \qquad \det(A-\lambda I) = (\lambda+1)^3 = 0 \qquad \lambda = -1.$$

$$E(-1, I) = \operatorname{span}\left\langle \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{= v_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{= v_2} \right\rangle$$

null$(A+I)^2 = \cancel{\emptyset}$:

null$(A+I)^2 = V$.

$(A+I)\underline{x} = v_1$ $\qquad$ $(A+I)\underline{x} = v_2$

inconsistent!!

$$(A+I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Expand $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to a basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

third generalized eigenvector.

Ex: $A = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 1 & -2 & -2 \end{bmatrix}$.

The only eigenvalue is $\lambda = 0$ of algebraic multiplicity 4.

$$E(0,T) = \text{Span}\left\langle \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{v_2} \right\rangle = \text{eigenspace of } \lambda = 0. = \text{null}(A)$$

$A\underline{x} = v_1$
solve

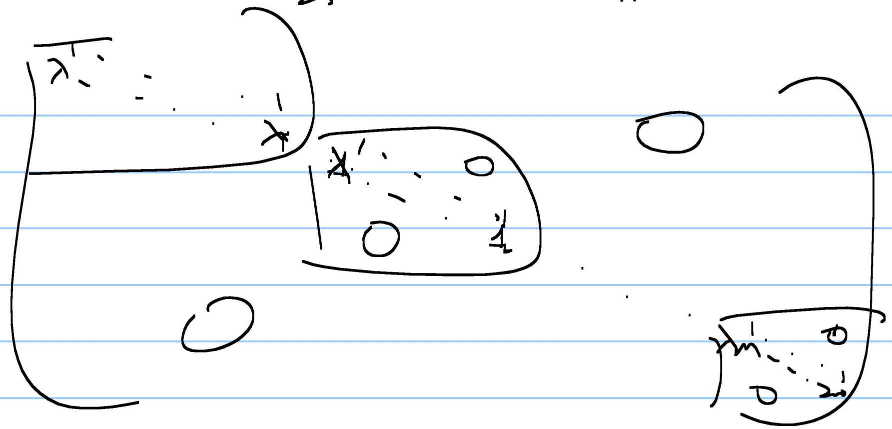
$$v_3 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}.$$

$A\underline{x} = v_2$
solve.

$$v_4 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

v_1, v_2, v_3, v_4 form a basis for $G(0,T) = V$.

∃ a basis $N^{m_1}v_1, \dots, v_1, N^{m_2}v_2, \dots, v_2, \dots, N^{m_n}v_n, \dots, v_n$ form a basis for V.



$$\begin{pmatrix} 1 & 1 & 0 & \\ 0 & 1 & & \\ & & 2 & 1 \\ & & 0 & 2 \end{pmatrix}$$

superdiagonal is zero!

Jordan Canonical Form

Ex: $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$

eigenvalues $\lambda = 3, 5$

$\lambda = 5$ $E(5, T) = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$

$\lambda = 3$ $E(3, T) = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

$\exists$ a basis $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ consisting of eigenvectors of A.

$A \sim \begin{bmatrix} \boxed{5} & 0 & 0 \\ 0 & \boxed{3} & 0 \\ 0 & 0 & \boxed{3} \end{bmatrix}$ = Jordan form. Diagonalizable

Ex: $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$N = A - I$. $N^{m_1} v_1 \ldots v_1 \ldots, N^{m_n} v_n \ldots v_n$.

$\lambda = 1$ is the only value

$(A-I)x = v_1$ or v_2

$E(1, A) = \text{Span}\left\langle \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{v_2} \right\rangle$.

$\lambda = 1$

$(A - I)^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 = 0$

Choose v_3 s.t v_1, v_2, v_3 form a basis

$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

generalized eigenvectors

$\text{Span}\{v_1, v_2, v_3\} = G(1, T)$

Choose $v_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$(A - I)x = v_2$. $\Rightarrow v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Jordan form $= \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Ex: $A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$

eigenvalue $= -1$.

$E(-1, A) = \text{span}\left\langle \overset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \right\rangle$

$(A+I)\underline{x} = v_1$ solve to get $v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

$G(1, T) = \text{span}\langle v_1, v_2, v_3 \rangle$

$(A+I)\underline{x} = v_2$ solve to get $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$

Jordan form $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

§ Real vectorspace.

Def: V is a real vector space

① The complexification of V, denoted by $V_{\mathbb{C}}$ is $V \times V$ $\{(u,v);\ u,v \in V\}$. The elements of $V \times V$ is denoted by $u+iv$ $u,v \in V$.

② Addition on $V_{\mathbb{C}}$ is

$$(u_1+iv_1)+(u_2+iv_2) = (u_1+u_2)+i(v_1+v_2)$$

③ Complex Scalar multiplication on $V_{\mathbb{C}}$

$$\underbrace{(a+ib)}_{\substack{\in \mathbb{C} \\ a,b \in \mathbb{R}}} \cdot \underbrace{(u+iv)}_{u,v \in V} = (au-bv)+i(av+bu) \in V \times V$$

Remark: $V_{\mathbb{C}}$ is a complex vector space. (Check)

Theorem: V is a real vector space.

① $v_1 \dots v_n$ is a basis for V. $\Rightarrow$ $v_1 \dots v_n$ is a basis for $V_{\mathbb{C}}$.

② $\dim_{\mathbb{C}} V_{\mathbb{C}} = \dim_{\mathbb{R}} V$.

Proof. $u+iv \in V_{\mathbb{C}}$ $\qquad u, v \in V = \operatorname{Span}\langle v_1 \dots v_n \rangle$

$$u = \sum c_j v_j \qquad v = \sum d_j v_j$$

$$u+iv = \sum c_j v_j + i \sum d_j v_j = \sum_j (c_j + i d_j) v_j \qquad V_{\mathbb{C}} = \operatorname{Span}_{\mathbb{C}} \langle v_1 \dots v_n \rangle$$

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0 \qquad \lambda_i \in \mathbb{C}$$

$$\lambda_j = \operatorname{Re}\lambda_j + i \operatorname{Im}\lambda_j$$

$$(\operatorname{Re}\lambda_1 v_1 + \dots + \operatorname{Re}\lambda_n v_n) + i(\operatorname{Im}\lambda_1 v_1 + \dots + \operatorname{Im}\lambda_n v_n) = 0$$

$\operatorname{Re}\lambda_j \in \mathbb{R}$ $\qquad \underset{0}{\shortparallel} \qquad \underset{0}{\shortparallel} \qquad \operatorname{Im}\lambda_j \in \mathbb{R}$

$\Rightarrow \operatorname{Re}\lambda_j = 0$ $\qquad$ $\operatorname{Im}\lambda_j = 0$

$\Rightarrow \lambda_j = 0$

$\Rightarrow v_1 \ldots v_n$ is l.i. in $V_{\mathbb{C}}$.

Def: Complexification of $T \in \mathcal{L}(V)$, V is a real vector space.

$T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}})$ $\qquad$ $T_{\mathbb{C}}(u+iv) = Tu + iTv \qquad u, v \in V.$

Remark: $T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}})$ addition ✓.

scalar mult.

$$T_{\mathbb{C}}(\lambda(u+iv)) = \lambda T_{\mathbb{C}}(u+iv). \qquad \lambda \in \mathbb{C}$$

$\lambda = a+ib.$

$$T_{\mathbb{C}}((a+ib)(u+iv)) = (a+ib) T_{\mathbb{C}}(u+iv).$$

Ex: $\mathbb{R}^n \rightsquigarrow \mathbb{C}^n$.
Complexification

$\mathcal{L}(\mathbb{C}^n)$.
$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$.
$\notin \mathcal{L}(\mathbb{R}^n)$

$T \in \mathcal{L}(\mathbb{R}^n)$ $\mathcal{M}(T)$ is $n \times n$ matrix with real entries

$T_{\mathbb{C}} \in \mathcal{L}(\mathbb{C}^n)$ $\mathcal{M}(T_{\mathbb{C}})$ is also $n \times n$ matrix with complex entries

Theorem: V is a real vector space. $v_1 \dots v_n$ is a basis for V. $T \in \mathcal{L}(V)$.

$$\mathcal{M}(T) = \mathcal{M}(T_{\mathbb{C}})$$

Theorem: Every operator on a nonzero f.d. v.s has an invariant subspace of dim 1 or 2.

Proof: If V is a complex vector space, use the eigenspace ✓

If V is a real vector space. $T \in \mathcal{L}(V)$.

$T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}}) \Rightarrow T_{\mathbb{C}}$ has an eigenvalue $\lambda = a + ib$

$\Rightarrow$ $\exists$ an eigenvector $u + iv$ (not both u, v are zero) s.t $T_{\mathbb{C}}((u+iv)) = \lambda(u+iv)$

$$\parallel$$

$$T(u) + i\,T(v) = (au - bv) + i\,(av + bu)$$

$$T(u) = au - bv.$$

$$Tv = av + bu.$$

$U = \operatorname{Span}\langle u, v\rangle$ $\quad$ $T(U) \subseteq U.$

$\Downarrow$

invariant subspace of dim 1 or 2.

Theorem: V is a real vector space. Then the minimum polynomial of T is the same as the minimum poly of $T_{\mathbb{C}}$.

Proof: PE

MTL104 Lec41

Note Title 28-04-2023

V is a real vector space $\rightsquigarrow$ $V_{\mathbb{C}}$ is a complex vectorspace

$V_{\mathbb{C}} = V \times V = \{u + iv \mid u, v \in V\}$

$T \in \mathcal{L}(V) \rightsquigarrow T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}})$ $T_{\mathbb{C}}(u+iv) = T(u) + iT(v)$

§ Minimum Polynomial

Theorem: V is a real vector space & $T \in \mathcal{L}(V)$. Then the minimum polynomial of $T_{\mathbb{C}}$ equals the minimum polynomial of T.

Proof: $p(x) \in \mathcal{P}(\mathbb{R})$ is a minimum polynomial of T.

$T_{\mathbb{C}}^n(u+iv) = T^n u + i T^n v$ (check!)

$T_{\mathbb{C}}(u+iv) = T(u) + iT(v)$

$T_{\mathbb{C}}^2(u+iv) = T_{\mathbb{C}}(T(u) + iT(v))$

$= T^2 u + i T^2(v)$

$p(T) = 0 \Rightarrow p(T_{\mathbb{C}}) = 0.$

$p(T)(u) = 0 \quad \forall u \in V$. $\qquad p(T_{\mathbb{C}})(u+iv) = \underset{0}{\overset{\shortparallel}{p(T)(u)}} + i\, \underset{0}{\overset{\shortparallel}{p(T)(v)}} = 0$

Choose $q(x) \in \mathcal{P}(\mathbb{C})$ s.t. $q(T_{\mathbb{C}}) = 0 \qquad T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}})$

$$\overset{\shortparallel}{T_{\mathbb{C}}^n + a_{n-1} T_{\mathbb{C}}^{n-1} + \cdots + a_0} = 0 \quad a_i \in \mathbb{C}$$

$$a_i = Re(a_i) + i\, Im(a_i)$$

$$\underset{\in \mathcal{P}(\mathbb{R})}{\underbrace{\left(T_{\mathbb{C}}^n + Re(a_i) T_{\mathbb{C}}^{n-1} + \cdots + Re(a_0)\right)}_{\overset{\shortparallel}{0}}} + i\left(Im\, a_{n-1} T^{n-1} + \cdots + Im\, a_0\right) = 0$$

$$\Rightarrow T^n + Re(a_i) T^{n-1} + \cdots + Re\, a_0 = 0$$

$$n \geq \deg p(x)$$

$\Rightarrow p(x)$ is the minimum polynomial for $T_{\mathbb{C}}$.

§ Eigenvalues of $T_{\mathbb{C}}$

V is a real vector space. $T \in \mathcal{L}(V)$. $\lambda \in \mathbb{R}$. λ is a eigenvalue for $T_{\mathbb{C}} \Leftrightarrow \lambda$ is an eigenvalue for T

Proof: Suppose λ is an eigenvalue for T. $\Rightarrow \exists\, v \overset{\neq 0}{\in} V$ s.t. $T(v) = \lambda v$.

$\in V_{\mathbb{C}}$

$\Downarrow$

$T_{\mathbb{C}}(v) = \lambda v$.

$\Downarrow$

λ is an eigenvalue for $T_{\mathbb{C}}$.

$V_{\mathbb{C}} = \{u + iv : u, v \in V\}$

$V \subseteq V_{\mathbb{C}}$

Suppose $\lambda \in \mathbb{R}$, λ is an eigenvalue for $T_{\mathbb{C}} \Rightarrow \exists\, u + iv \in V_{\mathbb{C}}$ s.t.

$T_{\mathbb{C}}(u + iv) = \lambda(u + iv)$.

$\lambda \in \mathbb{R}$

$T(u) + iT(v) \qquad \lambda u + i\lambda v$.

$\Rightarrow Tu = \lambda u \quad T(v) = \lambda v$.

Theorem: Suppose V is a real vectorspace. $T \in \mathcal{L}(V)$. $\lambda \in \mathbb{C}$. j is an integer $u, v \in V$. then

$$(T_{\mathbb{C}} - \lambda I)^j (u+iv) = 0 \iff (T_{\mathbb{C}} - \bar{\lambda} I)^j (u-iv) = 0$$

Proof: $j=0$ ✓

$j>0$ If $(T_{\mathbb{C}} - \lambda I)^j (u+iv) = 0$.

$T_{\mathbb{C}}(u+iv) = Tu + iTv$

$$(T_{\mathbb{C}} - \lambda I)^{j-1} (T_{\mathbb{C}} - \lambda I)(u+iv) = 0$$

$\lambda = a+ib$.

$$(T_{\mathbb{C}} - \lambda I)^{j-1} \left((Tu - au + bv) + i(Tv - av - bu) \right) = 0$$

By induction $(T_{\mathbb{C}} - \bar{\lambda} I)^{j-1} \left((Tu - au + bv) - i(Tv - av - bu) \right) = 0$

$$\Rightarrow (T_{\mathbb{C}} - \bar{\lambda} I)^j (u-iv) = 0$$

$T_{\mathbb{C}}(u-iv)$
$= Tu - iTv$.

Theorem: V is a real vector space. $T \in \mathcal{L}(V)$. $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of $T_{\mathbb{C}} \iff$ $\bar{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$

Theorem: V is a real vector space $T \in \mathcal{L}(V)$. $\lambda \in \mathbb{C}$ is an eigenvalue of $T_{\mathbb{C}}$. Then multiplicity of λ as an eigenvalue of $\mathbb{C}$ = multiplicity of $\bar{\lambda}$ as an eigenvalue of $T_{\mathbb{C}}$.

Theorem: Every operator on odd-dimensional real vector space has an eigenvalue.

Proof: Every odd degree polynomial has a real ~~ei~~ root.

§ Characteristic Polynomial

Theorem: V is a real vector space. $T \in \mathcal{L}(V)$. Then the coefficients of the characteristic polynomial of $T_{\mathbb{C}}$ are all real.

Proof: λ is a non real eigenvalue of $T_{\mathbb{C}}$ then $\bar{\lambda}$ is also an eigenvalue for $T_{\mathbb{C}}$.

Characteristic polynomial for $T_{\mathbb{C}}$ = $(z-\lambda_1)^{m_1}\cdots(z-\lambda_m)^{m_m}(z-\alpha_1)^{a_1}(z-\bar{\alpha}_1)^{a_1}\cdots(z-\alpha_b)^{a_b}(z-\bar{\alpha}_b)^{a_b}$

λ_i real. $\quad\alpha_i \in \mathbb{C}\setminus\mathbb{R}$.

$$(z-\alpha_1)^{a_1}(z-\bar{\alpha}_1)^{a_1} = (z^2-\underbrace{2\operatorname{Re}\alpha}_{\in\mathbb{R}}\, z+\underbrace{|\alpha|^2}_{\in\mathbb{R}})^{a_1} \in \mathcal{P}(\mathbb{R})$$

$\Rightarrow$ characteristic polynomial of $T_{\mathbb{C}}$ has real coefficients.

Def: V is a real vector space. The characteristic polynomial of T is the characteristic polynomial of $T_{\mathbb{C}}$.

Theorem: V is a real vector space. $T\in\mathcal{L}(V)$.

(a) The coefficients for the characteristic polynomial are all real

(b) The characteristic polynomial has degree $\dim V$ $\quad(\dim V_{\mathbb{C}} = \dim V)$.

(c) The eigenvalues of T are precisely the real eigenvalues of $T_{\mathbb{C}}$.

" the real roots of the characteristic polynomial.

Theorem (Cayley Hamilton). $T \in \mathcal{L}(V)$. Let q denote the characteristic polynomial of T. Then $q(T) = 0$

Theorem: $T \in \mathcal{L}(V)$. Then

1. degree of minimum polynomial of T is atmost $\dim V$.
2. the characteristic poly of T is a polynomial multiple of the minimum polynomial

V is $\mathbb{C}$-v.s.

char $(z-\lambda_1)^{n_1} \cdots (z-\lambda_m)^{n_m}$.

λ_i are eigenval of T

V is $\mathbb{R}$-v.s.

$\to (z-\lambda_1)^{n_1} \cdots (z-\lambda_m)^{n_m}$

λ_i is an eigenvalue

Spectral Theorem for normal operators on real vector spaces.

Theorem: V is a real vector space. $T \in \mathcal{L}(V)$. Then the following are equivalent

1. T is normal
2. $\exists$ an orthonormal basis for V. w.r.t which T has a block diagonal matrix s.t each block is a 1×1 matrix or a 2×2 matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $b>0$