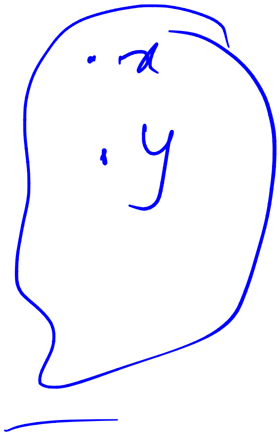
Lecture - 18

Real and Complex Analysis - MTL 122

• X — 

$\Rightarrow$ X is connected

• $x \in X$

$$C_x = \bigcup \{ C \subseteq X : C \text{ is connected}, \ x \in C \}$$

$\downarrow$

Components.

Ex.

1 Discrete metric space.

$$x \in (X, d_{dis})$$

$\{x\}$ $\quad$ $\{x, y\}$

$$C_x = \{x\}$$

2.) $X = \mathbb{R} \setminus \{0\}$ $d(x,y) = |x-y|$

$\underline{(0, \infty)}$ $\underline{(-\infty, 0)}$

$X = (0, \infty) \cup (-\infty, 0)$

- <u>Proposotition</u>

C_x — connected comp of X.

a) $x \in X$ — C_x connected & closed.

<u>Pf.</u>

$$C_x = \bigcup \{ C \subseteq X \mid C \text{ is connected } \& \ x \in C \}$$

$x \in \bigcap \{ C \subseteq X : C \text{ is connected } x \in C \} \neq \emptyset$

$\Rightarrow$ C_x is connected.

$\Rightarrow$ $\overline{C_x}$ is connected

$\Rightarrow$ $\overline{C_x} \subseteq C_x$

$\Rightarrow$ $C_x = \overline{C_x}$

$\Rightarrow$ C_x is closed.

$x \in \overline{C_x}$, $\overline{C_x}$ is connected

b) $x, y \in X$

$C_x = C_y$ or $C_x \cap C_y = \emptyset$

$C_x \cap C_y \neq \emptyset$ (Suppose)

$C_x \cup C_y \to$ connected

& $x, y \in C_x \cup C_y$

$C_x \cup C_y \subseteq C_x$

$C_x \cup C_y \subseteq C_y$

$\Rightarrow C_x = C_y$

Ex: subset of $\mathbb{R}$ is connected iff it is an interval.

~~Theo.~~

Ex:

$f : X \to \mathbb{R}$ cont.

X: connected space.

$x, y \in X \quad r \in \mathbb{R}$

s.t $f(x) \leq r \leq f(y)$

$\exists\ c \in X$ s.t $f(c) = r$

Pf

$X \to$ connected.

$f \to$ continuous.

$\Rightarrow f(X) \subseteq \mathbb{R}$ is connected.

$\Rightarrow \underline{f(X)} \subseteq \mathbb{R}$ is an interval.

$r \in \mathbb{R}$. $\quad f(x) \leq r \leq f(y)$

$\Rightarrow r \in f(X)$ $\quad [f(x), f(y)] \subseteq f(X)$

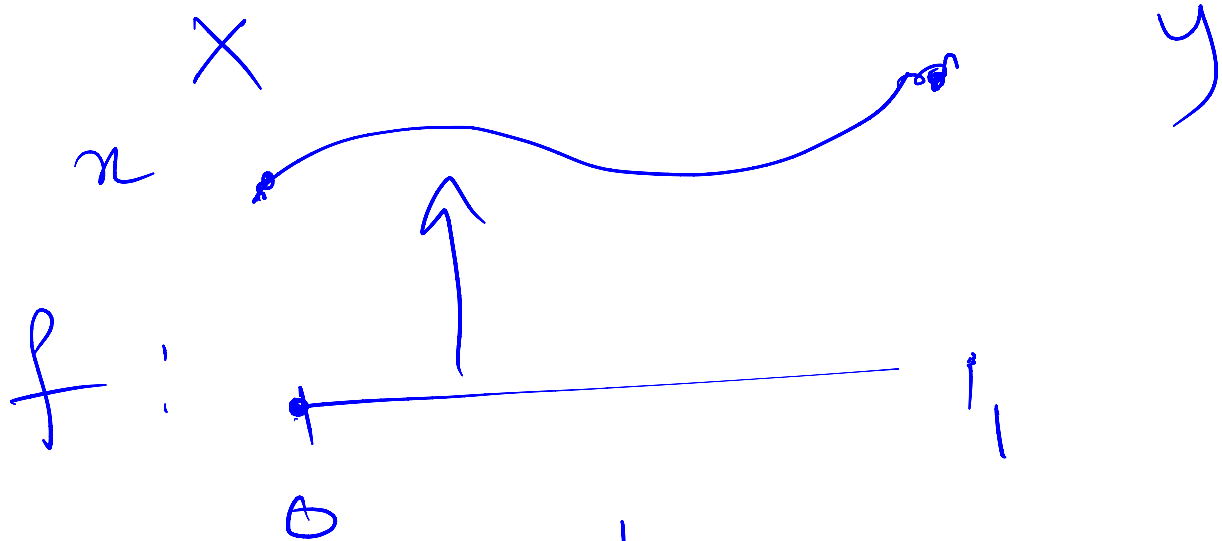
$\Rightarrow \exists\, c \in X$ s.t.

$f(c) = r$.

Path-connected Space

Defn.

$x, y \in X$.



$$f : [0,1] \xrightarrow{\text{cont.}} X$$

$$\left.\begin{array}{l} f(0) = x \qquad f(1) = y \end{array}\right\} \text{path.}$$

$X \to$ path connected if for every pair of points $\exists$ a path in X.

<u>Prop.</u> X path connected $\Rightarrow X$ is connected.

$\Leftarrow$?

$$S = \{(x, \underline{\overset{f(x)}{\sin 1/x}}) : 0 < x \leq 1\} \subseteq \mathbb{R}^2.$$

$\underline{(0, 1]}$ $\quad f : (0, 1] \to \mathbb{R}.$

$f(x) = \sin 1/x.$ cont.

$$\overline{S} = (\{0\} \times [-1, 1]) \cup S.$$

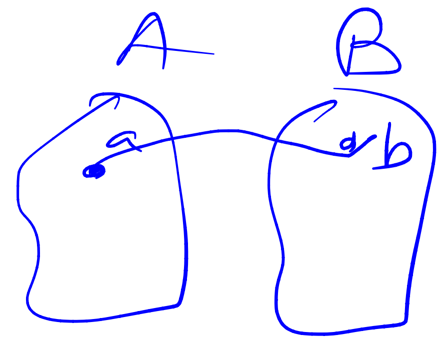
$\boxed{\text{Ex}}$ $\Rightarrow$ not path-connected.

$\boxed{\text{Why?}}$

Pf. X is not connected.

A, B

$a \in A$ $\qquad$ $b \in B$



$f : [0,1] \xrightarrow{\text{cont.}} X$.

Let this be a path

$f(0) = a$ & $f(1) = b$

connected $\Rightarrow$ $f([0,1])$ connected.

$$\Rightarrow f([0,1]) \begin{cases} \subseteq A \\ \text{or } \subseteq B \end{cases}$$

$f(0) = a$ & $f(1) = b$?

$\Rightarrow X$ has to be connected.

— x

Prop.

$$f : X \xrightarrow{\text{cont.}} Y$$

$\downarrow$

path-connected.

$f(X) = ?$

$x_1, x_2 \in X$. $\quad \left. \begin{array}{l} f(x_1) = y_1 \\ f(x_2) = y_2 \end{array} \right\} \in f(\boxed{X}) \subseteq Y$

$\swarrow$

$[0, 1] \longrightarrow X$