Lecture - 6 : MTL 122

Real and Complex Analysis.

- Closed set & Cauchy seq
- Closure is closed
- Closure & set relation using ngb
- Closure, int, boundary relation.
- Compact sets.

Ex.

$(\mathbb{R}, d) \quad d(x,y) = |x-y|$

Theo.

$F \subseteq \mathbb{R}$ is closed $\Leftrightarrow$ every ~~Cauchy seq~~ convergent seq in F has a limit in F.

Proof: $\Rightarrow$ $F \subseteq \mathbb{R}$ is closed.

& $(a_n)_{n \geq 1}$ be a C. Seq in F. ($a_n \in F \ \forall n$)

((a_n) is not a const)

$\Rightarrow$ We know

not true general metric space [Cauchy seq $\Leftrightarrow$ convergent seq (in $\mathbb{R}$)]

$a_n \to \underline{x}$ as $n \to \infty$. ($a_n \neq x$ $\forall n$.)

- $F' \subseteq F$
- limit pt $\Leftrightarrow$ $\exists$ seq converging to that pt.

$\Rightarrow x \in F$.

$\Leftarrow$ Suppose that <u>every</u> C.S in F has a limit in F also.

- Let $\underline{x \in \mathbb{R}}$ be a limit pt of F.

$\Rightarrow \exists \ \underline{(a_n)}_{n \geq 1}$, $a_n \neq x$

$a_n \to x$ as $n \to \infty$.

$a_n \in F$.

$\Rightarrow \underline{x \in F}$

$\Rightarrow F' \subseteq F \Rightarrow F$ is closed.

- $A \subseteq \mathbb{R}$.

$\overline{A} = A \cup \underline{A'} \leftarrow$

Claim: $\overline{A}$ is also a

closed set.

Note:

$\Rightarrow$ A is a closed set

- $A = \overline{A}$
- $\overline{\overline{A}} = \overline{A}$

gen M.sp.

$\overline{A}$ is a closed set. * $((\overline{A})' \subseteq \overline{A})$

$\overline{A} = A \cup A'$

Let x be a limit point of $\overline{A}$.

$\Rightarrow \quad x \in (\overline{A})'$

We will show

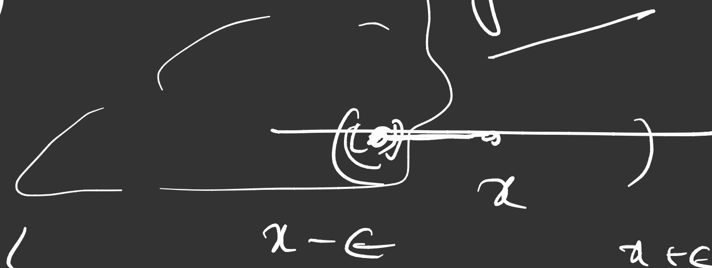
$x \in A'$. (Why?)

Let $\epsilon > 0$.

Since $\boxed{x \in (\bar{A})'}$ so there exists $y \neq x$ s.t

$y \in N_\epsilon(x)$ & $y \in \bar{A}$

$$y \in N_\epsilon(x) \cap (\bar{A})$$



$$\Rightarrow y \in \bar{A} = A \cup A'$$

$y \in A$ (trivial)

$y \in A'$ — (1)

Let $y \in A'$. $\Rightarrow$ y is a lp of A.

- $N_\epsilon(x)$ is an open set of 'y'

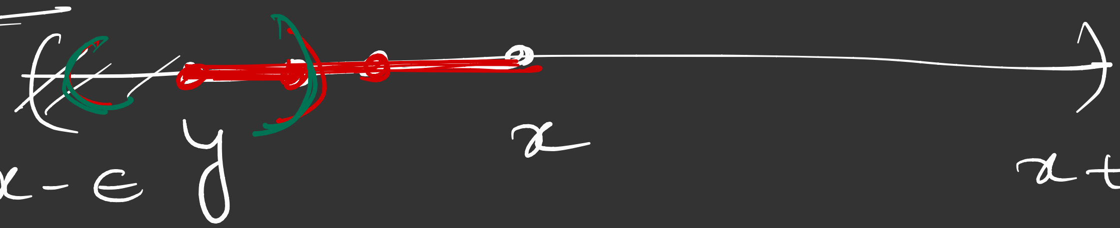
$\epsilon_1 < \epsilon$

By the defn of open sets.

$\exists\ r > 0 \qquad 0 < r < |x - y|$

then, $\boxed{N_r(y) \subset N_\epsilon(x)}$

$N_\epsilon(x) = (x - \epsilon,\ x + \epsilon)$



$y \in N_\epsilon(x)$

Now from defn of lp we get,

$\exists\ a \neq y$ s.t

$a \in N_r(y) \cap A$

(Because 'y' is a lp of A)

$\Rightarrow a \in N_\epsilon(x)$, also

& $a \in A$.

Hence if

$\underline{y \in A'}$ $\exists$ $a \in A$

in $\underline{N_\epsilon(x)}$, $\underline{a \neq x}$.

$\underline{\forall \epsilon > 0}$.

$\Rightarrow$ $\boxed{\underline{x \in A'}}$

If $y \in A$ always then

$x \in A'$ (trivial)

$\Rightarrow$ $\underline{y \in A \text{ or } y \in A'}$

then if $\underline{x \in (\bar{A})'}$

$\Rightarrow$ $\underline{x \in A'}$

$\Rightarrow$ $\underline{x \in \bar{A}} = A \cup A'$

$\Rightarrow (\bar{A})' \subseteq \bar{A}$

$\Rightarrow \bar{A}$ is a closed. ✓

Ex: $\bar{A}$ is the minimal closed set containing A.

$[A \subset B$ (closed) $\Rightarrow \bar{A} \subseteq B]$

Ex. 1 $x \in \bar{A}$ iff for each $\epsilon > 0$, $N_\epsilon(x) \cap A \neq \phi$.

<u>Ex.2)</u> $\overline{A} = \text{int}(A) \cup \text{bd}(A)$.

[Try].

<u>Ex 3</u> $\mathbb{R} = \text{int}(A) \cup \text{bd}(A)$
$\cup \, \text{int}(\mathbb{R} \setminus A)$.

Compact Set.

<u>Defn.</u> $K (\neq \emptyset) \subseteq \mathbb{R}$ is sequentially compact if every seq in K has a convergent subseq whose limit ~~bel~~ belongs

to $\underline{K}$.

- K is bdd ✓

$\underline{K=(0,1)}$, $\left\{\frac{1}{n}\right\}$

$\frac{1}{n} \to 0$, $0 \notin (0,1)$

- K is closed ✓

$\mathbb{N}$ is closed.

$\underline{(n)}$ in $\mathbb{N}$.

$\underline{\text{Theo}}$. A subset of $\mathbb{R}$ is seq compact iff it is closed & bdd.

Pf:- Assume,

$K \subseteq \mathbb{R}$ is <u>seq compact</u>.

Let $(x_n)_{n \geq 1}$, $x_n \in K$

& $x_n \to \underline{x}$ as $n \to \infty$

$\Rightarrow$ $x \in K$.

$\Rightarrow$ K is closed.

Assume K is not bdd

(x_n) in K s.t

$\underline{|x_n| \to \infty}$ as $n \to \infty$

(how)

$*$ [$n = 1$

$|x_1| > 1$ $\quad 2$

$x_2 > x_1$

$|x_2| > 2 \ldots$

$|x_n| > n.$

$n_1 < n_2 < n_3 \ldots$

$\{x_{n_k}\} \longrightarrow$ unbdd

$x_{n_k} \longrightarrow \infty.$

$\Rightarrow$ (x_n) has no convergent subseq.

$\Downarrow$

$\Rightarrow$ K has to be bdd.

$\Longleftarrow$ $K \subset \mathbb{R}$, closed & bdd.

Let $(x_n)_{n\geq 1}$, $x_n \in K$.

Now K is bdd

$\Rightarrow (x_n)_{n\geq 1}$ is bdd

$\Rightarrow (x_n)_{n\geq 1}$ has a convergent subseq.

(Bolzano - Weierstrass theo).

Say, (x_{n_k}), $x_{n_k} \to x$

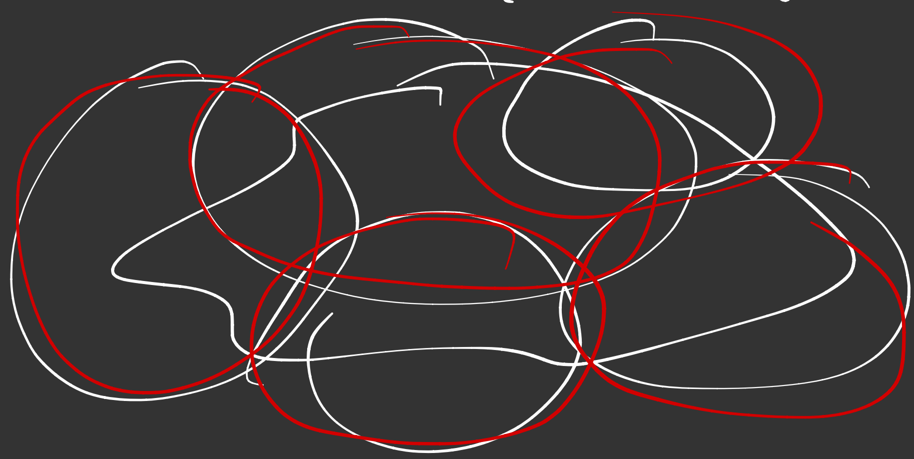
~~x_n~~ $x_{n_k} \in K$.

$\Rightarrow x \in K'$ (Limit pt characterization)

$\Rightarrow \underline{x \in K} \quad (\because K' \subset K)$

$\Rightarrow K$ is seq compact.

Defn. $A \subset \mathbb{R} \quad (A \neq \emptyset)$.



A <u>cover</u> of A is a collection of sets $\underbrace{\{A_i \subset \mathbb{R},\ i \in I\}}$

$$A \subset \bigcup_{i \in I} A_i$$

open cover if

A_i's are open sets.

Ex:

$$S = (0,1)$$

$$\mathcal{U} = \{ (\tfrac{1}{n}, 2) \mid n \in \mathbb{N} \}$$

Let $x \in (0,1)$

By A.P., $\exists\ m$ s.t

$$0 < \frac{1}{m} < x.$$

$$\Rightarrow x \in \left(\frac{1}{m}, 2\right)$$

$$\Rightarrow x \in \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n}, 2\right)$$

$$\Rightarrow \quad (0,1) \subset \bigcup_{n\in\mathbb{N}} \left(\frac{1}{n}, 2\right)$$

Is this an open cover of $[0,1]$?

$\delta > 0$. $B = (-\delta, \delta)$

$\underline{\mathcal{U} \cup B}$ is an open cover of $[0,1]$

$$\bigcup_{n=1}^{\infty} \underline{\left(\frac{1}{n}, 2\right) \cup (-\delta, \delta)}$$

$$= \underline{(-\delta, 2)} \supset \underline{\boxed{[0,1]}}$$

Defn. A set $K \subset \mathbb{R}$ is compact iff every open cover of K has a finite subcover

Ex: Show $[a,b]$ is compact.

(Use open cover defn of compact)

(Nested interval theorem).

Theo: S is a compact subset of $\mathbb{R}$

$\iff$ closed & bdd.

$\implies$ Seq compact

$\Updownarrow$

compact.