Mathematical Induction:-

Ex:-

Prove $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, $n \in \mathbb{N}$, $\theta \in \mathbb{R}$ } De Moivre's Theorem

Proof:-

For $n=1$, obvious

IH:- $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Then $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k (\cos\theta + i\sin\theta)$

$$= (\cos k\theta + i\sin k\theta)(\cos\theta + \sin\theta)$$

$$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\cos k\theta\sin\theta + \cos\theta\sin k\theta)$$

$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

Steps of MI:-

To prove a statement for every natural number.

Step 1: Prove for $n=1$

Step 2: Assume the statement for $n=k$ (this assumption is called induction hypothesis)

Step 3:- Prove the statement for $n=k+1$.

Conclusion:- statement is true for every n.

* Another forms of Induction:-

1) The IH is true for $1 \le j \le k$

2) Induction may be used to prove a statement to hold for $n \ge L \in \mathbb{Z}$

Let $\phi = \frac{\theta}{n}$

$$(\cos\phi + i\sin\phi)^n = \cos\theta + i\sin\theta$$

$\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)$ is a root of $x^n = \cos\theta + i\sin\theta$

$x^n = 0$ has n roots

0 is root with multiplicity n

Observe :-

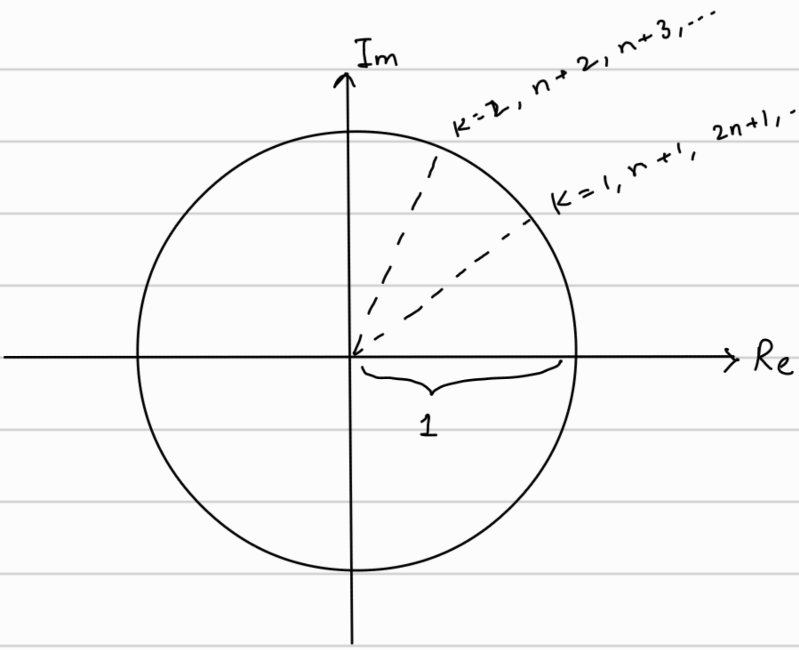
$$\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi) = \cos\theta + i\sin\theta$$

So $\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)$ is also a root of

$x^n = \cos\theta + i\sin\theta$ for $k \in \mathbb{Z}$

Observe, that there are n distinct root of this eqⁿ, namely :

$$\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right), \text{ for } k = 0, 1, \ldots, n-1$$



$(\cos\theta + i\sin\theta)^{1/n}$ is a set of complex numbers

Complex mapping

$\mathbb{C} \longrightarrow \mathbb{C}$

$z \longmapsto z^n$

Onto if $n \neq 0$

one-one if $n = 1$ (For $n = -1$, $\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$)

Eg: $z \mapsto z^2$

Non zero complex numbers has n inverse images

$A \xrightarrow{f} B$, $b \in B$

Then $f^{-1}(b) = \{ a \in A \mid f(a) = b \}$

$n = \frac{p}{q}$, $q \neq 0$, $(p, q) = 1$, HCF = GCD (greatest common divisor)

$(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{p\theta}{q}\right) + i\sin\left(\frac{p\theta}{q}\right)$?

Note:- $(\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$

$(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$\therefore$ $\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad n \in \mathbb{Z}}$

$$(\cos\theta + i\sin\theta)^{p/q} = (\cos p\theta + i\sin p\theta)^{1/q}$$

$$= \left\{ \cos\left(\frac{p\theta + 2k\pi}{q}\right) + i\sin\left(\frac{p\theta + 2k\pi}{q}\right) : 0 \leq k \leq q-1 \right\}$$

Non-terminating, non-recurring decimal :- $0.010010001000001\ldots$

the n^{th} 1 is followed by $n+1$ zeros.

Congruence modulo $n \in \mathbb{N}$:-

Let $x, y \in \mathbb{Z}$

Defⁿ: $x \equiv y \mod n$ if $n \mid x-y$

This is an equivalence relation

✵ Notation

Find $[2]_7 \longrightarrow$ equivalence class of 2 for the relation "congruence modulo 7"

$$[2]_7 = \{2 + 7t \mid t \in \mathbb{Z}\}$$

$$[x]_n = \{x + nt \mid t \in \mathbb{Z}\}$$

* Congruence classes are $[0]_n, [1]_n, \ldots, [n-1]_n$ & they are all.

Proof :- $x \in \mathbb{Z}$, By division algorithm

$$x = qn + r \quad , \quad 0 \leq r \leq n-1$$

$$[x]_n = [r]_n$$

Moreover, if $0 \leq r < s \leq n-1$

then $[r]_n = [s]_n$

(since $0 < s - r < n$, not divisible by n)

Assignment 3

Q) Prove if we have an equivalence rel^n on a set then

$[x] \cap [y] \neq \phi \Rightarrow [x] = [y]$, where $[x]$ is equivalence class of x.

$[x] = \{ t \in X \mid t \sim x \}$ related to

$$* \quad \underset{[x']_n}{\overset{\shortparallel}{[x]_n}} + \underset{[y']_n}{\overset{\shortparallel}{[y]_n}} \overset{\text{def}^n}{:=} \underset{[x'+y']_n}{\overset{?\,\shortparallel}{[x+y]_n}}$$

Well definess of binary operation.

Proof: Suppose $[x]_n = [x']_n$ & $[y]_n = [y']_n$

Then $n \mid x - x'$, $n \mid y - y'$

$\Rightarrow n \mid (x - x') + (y - y') = (x+y) - (x'+y')$

$\Rightarrow [x+y]_n = [x'+y']_n$

Try: $[x]_n [y]_n \overset{\text{def}^n}{:=} [xy]_n$

Proof: Suppose $[x]_n = [x']_n$ & $[y]_n = [y']_n$

Then $n \mid x - x'$, $n \mid y - y'$

$n \mid (x - x')(y - y') \Rightarrow n \mid xy - xy' - yx' + x'y'$ — ①

$n \mid x(y - y') \Rightarrow n \mid xy - xy'$ — ②

$n \mid y(x - x') \Rightarrow n \mid yx - yx'$ — ③

① − ② − ③

$n \mid x'y' - xy$

$\therefore [xy]_n = [x'y']_n$