Lecture 2	
Real and Complex Analysis	
MTL122/ MTL503/ MTL506	
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1. Real Numbers

1.1. **Field.** These following first six axioms are called the *field axioms* because any object satifying them is called a *field.*
A *field* is a nonempty set $\mathbb{F}$ along with two binary operations, multiplication $\times : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and addition $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ satisfying the following axioms.

AXIOM 1(Associative Laws). If $a, b, c \in \mathbb{F}$, then $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.

AXIOM 2 (Commutative Laws). If $a, b, c \in \mathbb{F}$, then $a+b = b+a$ and $a\times b = b\times a$.

AXIOM 3 (Distributive Laws). $a, b, c \in \mathbb{F}$, then $a \times (b + c) = (a \times b) + (a \times c)$.

AXIOM 4 (Existence of identities). There are $0, 1 \in \mathbb{F}$ with $0 \neq 1$ such that $a + 0 = a$ and $a = a$, for all $a \in \mathbb{F}$.

AXIOM 5(Existence of an additive inverse). For each $a \in \mathbb{F}$ there is $-a \in \mathbb{F}$ such that $a + (-a) = 0$.

AXIOM 6(Existence of a multiplicative inverse). For each $a \in \mathbb{F} \setminus \{0\}$ there is $a^{-1} \in \mathbb{F}$ such that $a \times a^{-1} = 1$.

Definition 1.1. *(Ordered Fields)*
An ordered field is a field $\mathbb{F}$ with a relation, denoted $<$, obeying the

(a) For each pair $x, y \in \mathbb{F}$ precisely one of $x < y$, $x = y$, $y < x$ is true.
(b) $x < y$, $y < z \implies x < z$
(c) $y < z \implies x + y < x + z$
(d) $x > 0$, $y > 0 \implies xy > 0$

Example 1.2. $\mathbb{Q}$ *and* $\mathbb{R}$ *are ordered fields.*

Definition 1.3. *Let $x \in \mathbb{R}$. The **absolute value** of x is defined by*

$$|x| = \begin{cases} x, & \text{if } x \geq 0. \\ -x, & x < 0. \end{cases} \tag{1.1}$$

If we think of the real numbers as points on the real line, then $d(x, y) = |x - y|$ is just the distance between the real numbers x and y and it satisfies

i) $d(x, y) = d(y, x)$,

ii) $d(x, y) \geq 0$,

iii) $d(x, y) = 0 \iff x = y$, and

iv) $d(x, z) \leq d(x, y) + d(y, z)$.(Triangle Inequality)

This distance function is also called a **metric** of the space $\mathbb{R}$ and $(\mathbb{R}, d)$ is a **metric space**.

2. The supremum and infimum

Next, we use the ordering properties of $\mathbb{R}$ to define the supremum and infimum of a set of real numbers. These concepts are of central importance in analysis. In particular, in the next section we use them to state the completeness property of $\mathbb{R}$. First, we define upper and lower bounds.

Definition 2.1. *A set $A \subset \mathbb{R}$ of real numbers is bounded from above if there exists a real number $M \in \mathbb{R}$, called an upper bound of A, such that $x \leq M$ for every $x \in A$. Similarly, A is bounded from below if there exists $m \in \mathbb{R}$, called a lower bound of A, such that $x \geq m$ for every $x \in A$. A set is bounded if it is bounded both from above and below. Equivalently, a set A is bounded if $A \subset I$ for some bounded interval $I = [m, M]$*

- Equivalently, a set is $A \subset \mathbb{R}$ is bounded if and only if there exists a real number $M \geq 0$ such that

$$|x| \leq M \text{ for every } x \in A.$$

Definition 2.2. *Suppose that $A \subset \mathbb{R}$ is the set of real numbers. If $M \in \mathbb{R}$ is an upper bound of A such that $M \leq M'$ for every bound M' of A, then M is called the least upper bound or supremum of A, denoted by*

$$M = \sup A.$$

If $m \in \mathbb{R}$ is a lower bound or infimum of A, such that $m \geq m'$ for every lower bound m' of A, then m is called the greatest lower bound or infimum of A , denoted by

$$m = \inf A.$$

- Supremum or Infimum of a set is unique. (Exercise)

If $\sup A \in A$ then we denote it by $\max A$ and refer to it as the maximum of A; and if $\inf A \in A$, then we also denote it by $\min A$ and refer to it as the minimum of A. As the following examples illustrate, $\sup A$ and $\inf A$ may or may not belong to A, so the concepts of supremum and infimum must be clearly distinguished from those of maximum and minimum.

Example 2.3. *Every finite set of real numbers*

$$A = \{x_1, x_2, x_3, ..., x_n\}$$

is bounded. Its supremum is the greatest element, $\sup A = \max\{x_1, x_2, ..., x_n\}$ *and its infimum is the smallest element,* $\inf A = \min\{x_1, x_2, ..., x_n\}$.
Both the supremum and infimum of a finite set belong to the set.

Example 2.4. *If $A = (0,1)$, then every $M \geq 1$ is an upper bound of A. The lub is $M = 1$, so*

$$\sup(0,1) = 1.$$

Similarly, every $m \leq 0$ is a lower bound of A, so

$$\inf(0,1) = 0.$$

In this case neither $\sup A$ nor $\inf A$ belong to A.

Example 2.5. *Let*

$$A = \{\frac{1}{n} : n \in \mathbb{N}\}$$

be the set of reciprocals of the natural numbers. Then $\sup A = 1$, which belongs to A and $\inf A = 0$, which does not belong to A.

If a set $A \subset \mathbb{R}$ is not bounded from above then $\sup A = \infty$, and if $A \subset \mathbb{R}$ is not bounded from below then $\inf A = \infty$.

3. Completeness

The following axiomatic property of the real numbers is called Dedekind completeness. Dedekind (1872) showed that the real numbers are characterized by the condition that they are a complete ordered field.

Axiom . Every nonempty set of real numbers that is bounded from above has a supremum.

As a first application of this axiom, we prove that $\mathbb{R}$ has the Archimedean property, meaning that no real number is greater than every natural number.

Theorem 3.1. *If $x \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $x < n$.*

Proof Suppose, for contradiction, there exists a $x \in \mathbb{R}$ such that $x > n$ for every $n \in \mathbb{N}$. Then x is an upper bound of $\mathbb{N}$, so $\mathbb{N}$ has a supremum $M = \sup \mathbb{N} \in \mathbb{R}$. Since $n \leq M$ for every $n \in \mathbb{N}$, we have $n - 1 \leq M - 1$ for every $n \in \mathbb{N}$. This implies $n \leq M - 1$ for every $n \in \mathbb{N}$. But then $M - 1$ is an upperbound of $\mathbb{N}$. A contradiction. $\square$

Theorem 3.2. *Let S be a non empty subset of $\mathbb{R}$, and $M \in \mathbb{R}$. Then $M = \sup S$ if and only if*

i) M is an upper bound for S, and
ii) for any $\epsilon > 0$, there is an element $s \in S$ such that $M - \epsilon < s$.

Proof Assume that M is the supremum for S, i.e., $M = \sup S$. Then, by definition M is an upper bound for S. If there is an $\epsilon' > 0$ for which $M - \epsilon' \geq s$ for all $s \in S$, then $M - \epsilon'$ is an upper bound for S, which is smaller then M, a contradiction.

Assume now that $i), ii)$ hold. Since S is bounded above then by S has a least upper bound, say A. Since M is an upper bound for S so $A \leq M$. If $A < M$, then with $\epsilon = M - A$, there is an element $s \in S$ such that

$$M - (M - A) < s \leq A, \ i.e., \ A < A,$$

which is absurd. Therefore $A = M$, i.e., M is the supremum of S.

$\square$

4. Sequences

Definition 4.1. *A sequence (x_n) of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$, where $x_n = f(n)$.*

We write the sequence as $(x_n)_{n=1}^{\infty}$.

Definition 4.2. *A sequence (x_n) of real numbers converge to a limit $x \in \mathbb{R}$, written*

$$x = \lim_{n\to\infty} x_n, \text{ or } x_n \to x \text{ as} n \to \infty,$$

if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \epsilon \text{ for all } n > N.$$

A sequence converges if it converges to some limit $x \in \mathbb{R}$, otherwise it diverges. Note that $x_n \to x$ as $n \to \infty$ means the same thing as $|x_n - x| \to 0$ as $n \to \infty$.

Proposition 4.3. *(Exercise) If a sequence converges, then its limit is unique.*

Definition 4.4. *If (x_n) is a sequence then*

$$\lim_{n\to\infty} x_n = \infty,$$

or $x_n \to \infty$ as $n \to \infty$,if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that

$$x_n > M \text{ for all } n > N.$$

Also $\lim\limits_{n\to\infty} x_n = -\infty$, or $x_n \to -\infty$ as $n \to \infty$, if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that $x_n < M$ for all $n > N$.

Definition 4.5. *A sequence (x_n) of real numbers is bounded from above if there exists $M \in \mathbb{R}$ such that $x_n \leq M$ for all $n \in \mathbb{N}$, and bounded from below if there exists $m \in \mathbb{R}$ such that $x_n \geq m$ for all $n \in \mathbb{N}$. A sequence is bounded if it is bounded from above and below, otherwise it is unbounded.*

An equivalent condition for a sequence (x_n) to be bounded is that there exists $M \geq 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Example 4.6. *The sequence (n^3) is bounded from below but not from above, while the sequences $(1/n)$ and $((-1)^{n+1})$ are bounded. The sequence (x_n) where $x_n = (-1)^{n+1}n$ is not bounded from below or above.*

Proposition 4.7. *A convergent sequence is bounded.*

Proof. Let $(x)_n$ be a convergent sequence with limits x. There exists $N \in \mathbb{N}$ such that

$$|x_n - x| < 1 \text{ for all } n > N.$$

The triangle inequality implies that

$$|x_n| \leq |x_n - x| + |x| < 1 + |x|, \text{ for all } n > N.$$

Defining $M = \max\{|x_1|, |x_2|, ..., |x_N|, 1 + |x|\}$, we see that $|x_n| \leq M$ for all $n \in \mathbb{N}$, so (x_n) is bounded. $\square$

Thus, boundedness is a necessary condition for convergence. But boundedness is not a sufficient condition for convergence.

Example 4.8. $x_n = (-1)^{n+1}$. *(Check)*

Definition 4.9. *A sequence (x_n) of real numbers is a Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$|x_m - x_n| < \epsilon, \text{ for all } m, n > N.$$

Exercise 4.10. *Cauchy sequence is bounded.*

A subsequence of a sequence (x_n),

$$x_1, x_2, ...x_3, ..., x_n, ...$$

is the sequence (x_{n_k}) of the form

$$x_{n_1}, x_{n_2}, ..., x_{n_3}, ..., x_{n_k}, ...$$

where $n_1 < n_2 < n_3 ... < n_k < ...$

Example 4.11. *A subsequence of the sequence $(1/n)$,*

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$$

is the sequence $(1/k^2)$

$$1, \frac{1}{4}, \frac{1}{9}, \cdots$$

Here $n_k = k^2$. On the other hand, the sequence

$$1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}$$

aren't subsequences of $1/n$ since n_k is not a strictly increasing function of k.

Theorem 4.12. ***Nested Interval Theorem***
For each $n \in \mathbb{N}$, let $I_n = [a_n, b_n]$, where $-\infty < a_n < b_n < \infty$. If $I_{n+1} \subset I_n$ for each $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} (b_n - a_n) = 0$ then $\bigcap\limits_{n=1}^{\infty} I_n$ consists of exactly one point.

The Nested Intervals Theorem may fail for a decreasing sequence of open or half-open intervals. For example, if $I_n = (0, \frac{1}{n+1}]$ or $I_n = [n, \infty)$ for each $n \in \mathbb{N}$, then $\bigcap\limits_{n=1}^{\infty} I_n = \phi$.

Theorem 4.13. *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Suppose that (x_n) is a bounded infinite sequence of real numbers. Let

$$M = \sup_{n\in\mathbb{N}} x_n, \; m = \inf_{n\in\mathbb{N}} x_n,$$

and define the closed interval $I_0 = [m, M]$.
Divide $I_0 = L_0 \cup R_0$ in half into two closed intervals, where

$$L_0 = [m, (m + M)/2], \; R_0 = [(m + M)/2, M].$$

At least one of the interval L_0, R_0 contains infinitely many terms of the sequence, meaning that $x_n \in L_0$ or $x_n \in R_0$ for infinitely many $n \in \mathbb{N}$. Choose I_1 to be one of the intervals L_0, R_0 that contains infinitely many terms and choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in I_1$. Divide $I_1 = L_1 \cup R_1$ in half into two closed intervals. One or both of the intervals L_1, R_1 contains infinitely many terms of the sequence. Choose I_2 to be one of these intervals and choose $n_2 > n_1$ such that $x_{n_2} \in I_2$. This is always possible because I_2 contains infinitely many terms of the sequence. Divide I_2 in half, pick a closed half-interval I_3 that contains infinitely many terms, and choose $n_3 > n_2$ such that $x_{n_3} \in I_3$. Continuing in this way, we get a nested sequence of intervals $I_1 \supset I_2 \supset I_3 \ldots.$ of length $|I_k| = 2^{-k}(M - m)$, together with the subsequence (x_{n_k}) such that $x_{n_k} \in I_k$.

So $|I_k| \to 0$ as $k \to \infty$. So by Nested Interval Theorem we have $\bigcap\limits_{n=1}^{\infty} I_n$ consists of exactly one point, say l. Then,

$$|x_{n_k} - l| < 2^{-k}(M - m) \to 0$$

as $k \to \infty$. That is, $\lim\limits_{k \to \infty} x_{n_k} = l$.

□

Theorem 4.14. *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Proof. Let $\{a_n\}$ is a convergent sequence. Since $\{a_n\}$ converges to L, for every $\epsilon > 0$, there is an $N > 0$ so that when $j > N$, we have

$$|a_j - L| \leq \frac{\epsilon}{2}.$$

Now the for $j, k > N$ we have

$$|a_j - a_k| = |a_j - L + L - a_k| \leq |a_j - L| + |a_k - L| < \epsilon,$$

so that the sequence $\{a_j\}$ is a Cauchy sequence as desired.

Let $\{a_n\}$ be a Cauchy sequence. Then by a previous exercise we know it is bounded. By Bolzano Weierstrass Theorem (a_n) has a convergent subsequence $(a_{n_k}) \to l$,(say). Then

$$\exists N_1 \text{ such that } r \geq N_1 \implies |a_{n_r} - l| < \epsilon/2$$
$$\exists N_2 \text{ such that } m, n \geq N_2 \implies |a_m - a_n| < \epsilon/2.$$

We choose a $k > N_1$ such that $n_k > N_2$. Then for all $n \geq N_2$ we have

$$|a_n - l| = |a_n - a_{n_k} + a_{n_k} - l| < \epsilon.$$

Hence $\{a_n\}$ is convergent. □