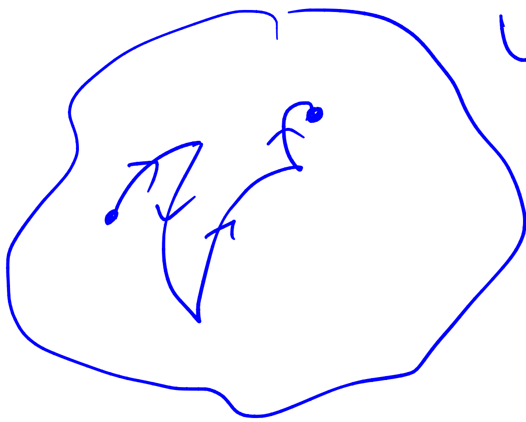
Lecture - 29 . MTL 122

Real and complex Analysis.

$U \subseteq \mathbb{C} \longrightarrow$ connected

open $\longrightarrow$ if every point in U is joined by a contour.



Connected open subset $=$ domain.

$D \to$ domain

$f : D \to \mathbb{C}$ — continuous complex valued fn on D

f has an "antiderivative" in D if $\exists\ F : D \to \mathbb{C}$

$$\boxed{F'(z) = \frac{dF(z)}{dz} = f(z)} \quad \forall z \in D$$

- F is analytic in D.
- f has an

Γ — contour in D. (z_0, z_1 are the end points)

If f has an antiderivative F on D,

$$\int_\Gamma f(z)\,dz = F(z_1) - F(z_0).$$

Γ:

- Smooth curve

Let Γ be parameterised by $z(t)$, $0 \le t \le 1$ (you can choose any end pts).

$$\int_\Gamma f(z)\,dz = \int_0^1 f(z(t))\,\dot{z}(t)\,dt$$

$$= \int_0^1 F'(z(t))\,\dot{z}(t)\,dt$$

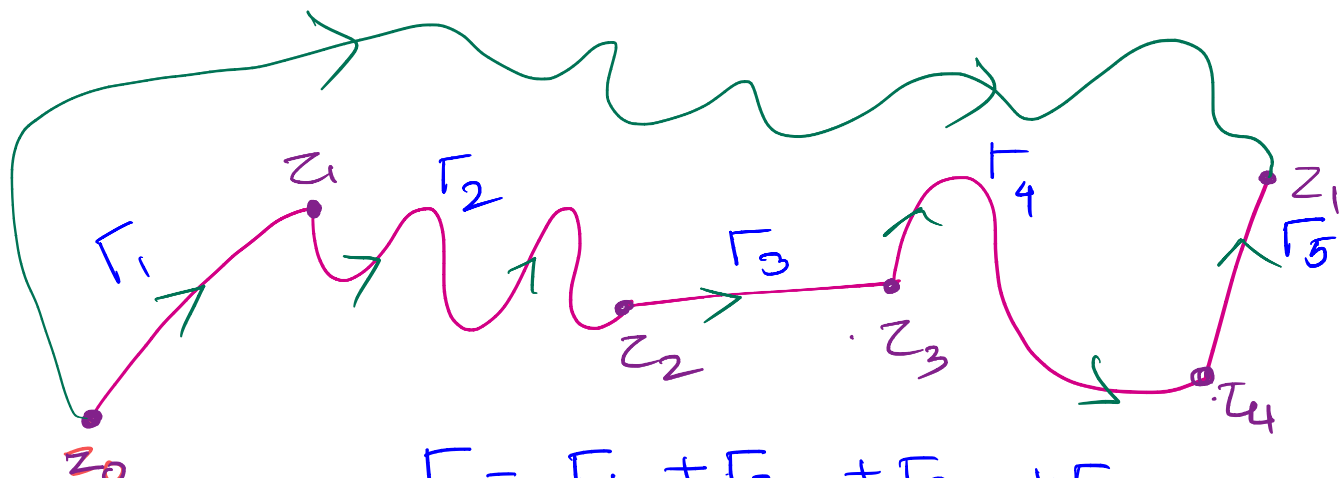
$$= \int_0^1 \frac{d}{dt} F(z(t))\, dt.$$

$$= F(z(1)) - F(z(0))$$

$$= F(z_1) - F(z_0)$$

General Case.

$\Gamma : \{\Gamma_j\}, \; j = 1, 2, \ldots, n, \; \Gamma_j$ smooth curves.



$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$
(notation)

$$\int_\Gamma f(z)\, dz = \sum_{j=1}^{n} \int_{\Gamma_j} f(z)\, dz$$

$$= F(\cancel{z_1}) - F(z_0) + \cancel{F(z_2)} - \cancel{F(z_1)}$$

$$\cdots\ \cancel{\cdot}\ \cdots + F(z_1) - \cancel{F(z_{n-1})}$$

$$= F(z_1) - F(z_0)$$

Theorem

Let f be a continuous fn defined on a domain D and $f(z)$ has an antiderivative $F(z)$ in D. Let $z_1, z_2 \in D$. Then for any contour C lying in D with initial and final point z_1, z_2 respectively the value of the integral

$\int_C f(z)\,dz$ is independent of path.

Corollary.

If a cont fn f has an antiderivative then

$$\oint_C f(z)\,dz = 0$$

$\oint_C$

$\Rightarrow C$ is closed cont.

Theo.

Let $f : D \to \mathbb{C}$ be a continuous function. Then the following statements are equivalent.

a) f has an antiderivative F in D

$\Downarrow$ $\Uparrow$

b) $\oint_\Gamma f(z)\,dz$ vanishes $\forall$ closed

↑ contours

c) Independent of path.

$$\int_{\Gamma} f(z)\,dz.$$

⟶

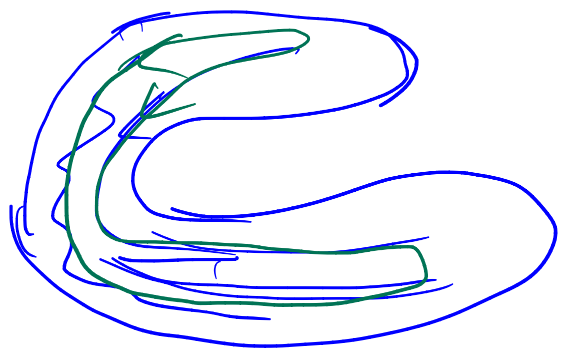
Questions:

1) Under what conditions on f we can guarantee the existence of F?

2) Under what assumptions on f we can get $\int_C f(z)dz = 0$, C closed?

Simple connected domain

A domain D is simply connected if every simple closed contour within it encloses points of D only.



No holes

Simple connected.

Multiple connected domain

Not simple connected.

One / Many holes.

Ex.

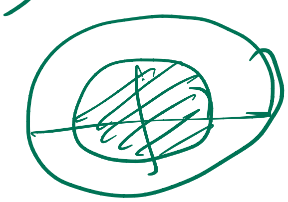
1) $\mathbb{C}$, $\mathbb{D}$ (unit disc)

$\Psi = \{z \,;\, \operatorname{Re} z > 0\}$

simple connected

2) $\mathbb{C}^*$, $\mathbb{D} \setminus \{0\}$

$A(a,b) = \{z \in \mathbb{C} \,;\, a < |z| < b\}$



multiple connected

Theo. (Cauchy's theorem)

If f is analytic in a simple connected domain D then

$$\oint_C f = 0 \quad , \quad \forall\, C - \text{closed}$$

Cor. For such f antiderivative exists

& $\displaystyle\int_{z_1}^{z_2} f(z)\,dz = F(z_1) - F(z_2)$, for z_1, z_2.

Mutiple connected domain

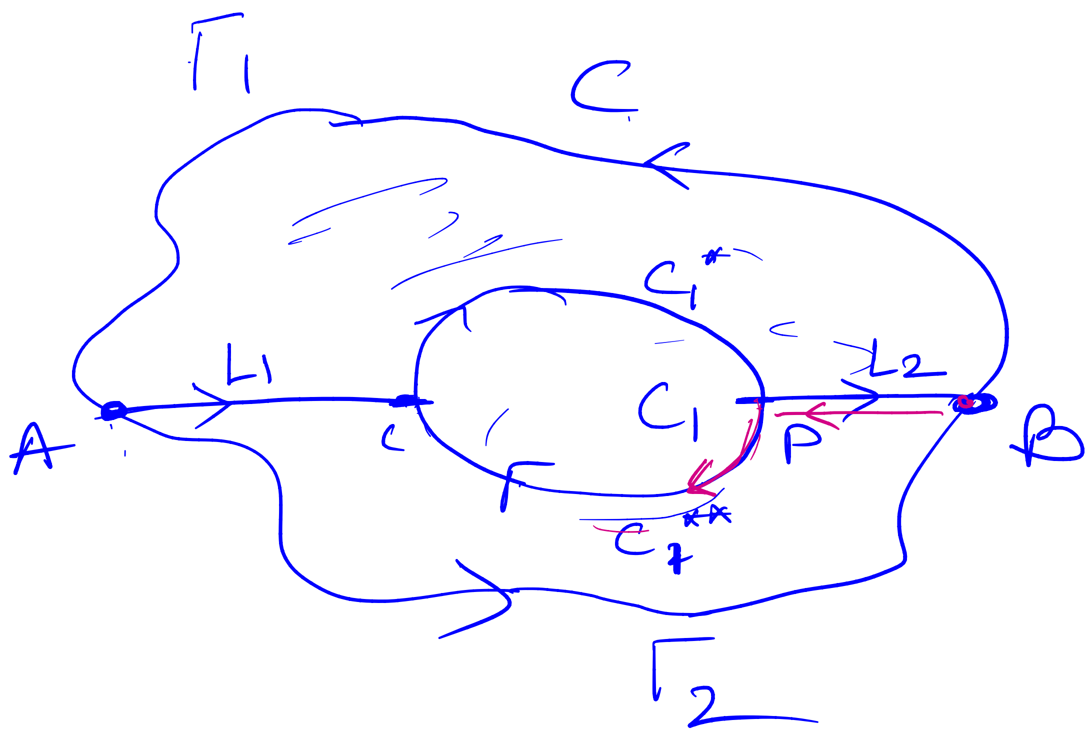
- $C \rightarrow$ simple closed contour.
- $C_k \rightarrow k = 1, 2, \ldots n$

  finite number of simple closed contours, s.t interiors has no points in common.

- f is analytic throughout the closed region consisting of points within and on C except for the pts interior of C_k's.

then,

$$\int_C f(z) + \sum_{k=1}^{n} \int_{C_k} f(z) = 0$$



$$K_1 = \Gamma_1 + L_1 + L_2 + C_1^*$$

$$K_2 = \Gamma_2 - L_2 + C_1^{**} - L_1$$

$$K_1 + K_2 = \Gamma_1 + \Gamma_2 + C_1^* + C_1^{**}$$

$$= C + C_1$$

By Cauchy's theorem

$$\oint_{K_1} f = 0 \qquad \oint_{K_2} f = 0$$

$$\Rightarrow \oint_{K_1+K_2} f = 0$$

$$\Rightarrow \int_C f + \int_{C_1} f = 0$$

Proceeding similarly we can show.

$$\int_C f + \sum_{k=1}^{n} \int_{C_k} f = 0 .$$

Cor: (Deformation of contour).

C_1, C_2 → 2 simple closed positively oriented contours s.t C_2 is in the interior of C_1

$K_1 = \Gamma_1 + L_1 - C_2^* + L_2$

$K_2 = \Gamma_2 - L_1 - L_2 - C_2^{**}$

$$\Rightarrow \int_{C_1} f = \int_{C_2} f .$$

Here $\underline{K_1 + k_2} = r_1 + r_2 - c_2^{*} - c_2^{**}$

$= \underline{c_1 - c_2}$

$$\int_C f(z)\,dz + \int_{C_1} f(z)\,dz = 0$$

<u>Corollary</u> C_1 positively oriented simple closed contour.

C_2 ! positive ~~d~~ oriented simple closed contour inside C_1

$$\int_{C_1} f = \int_{C_2} f .$$