Lecture 9, 10, 11

Real and Complex Analysis

MTL122/ MTL503/ MTL506

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1. Convergence of Sequence in Metric Spaces

Definition 1.1. *A sequence (x_n) in a metric space (X, d) is said to converge to a point $x \in X$ if, given any $\epsilon > 0$, there is a natural number N (which depends on ϵ, in general) such that $d(x_n, x) < \epsilon$ for all $n \geq N$. In this case the point x is called the* ***limit*** *of the sequence (x_n) and we write $\lim\limits_{n\to\infty} x_n = x$.*
Equivalently, (x_n) converges to x if, given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $x_n \in B(x, \epsilon)$ for all $n \geq N$.

Example 1.2. *The sequence $x_n = \frac{1}{n}$ converges to 0.*

Example 1.3. $x_n = 1 + [\frac{(-1)^n}{n}]$ *converges to* 1.

Theorem 1.4. *(Limits of convergent sequences are unique). Let (x_n) be a sequence in a metric space (X, d). If $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} x_n = y$, then $x = y$.*

Proof. Assume that $x \neq y$ and let $0 < \epsilon < \frac{d(x,y)}{2}$. Then there are natural numbers N_1 and N_2 such that

$$d(x_n, x) < \epsilon \text{ for all } n \geq N_1$$

and

$$d(x_n, y) < \epsilon \; for \; all \; n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$,

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < 2\epsilon < d(x, y),$$

which is not possible. Hence $x = y$. $\square$

Theorem 1.5. *Suppose x_n, y_n are complex sequences with $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Then*

a) $\lim\limits_{n\to\infty} (s_n + t_n) = s + t$
b) $\lim\limits_{n\to\infty} c \cdot s_n = c \cdot s$ *and* $\lim\limits_{n\to\infty} c + ts_n = c + s$ *for any number c.*
c) $\lim\limits_{n\to\infty} (s_n t_n) = st$
d) $\lim\limits_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, *$s_n \neq 0$, and $s \neq 0$.*

Proposition 1.6. *Every convergent sequence is bounded.*

Exercise.
Converse is not true.

Proposition 1.7. *If a sequence (x_n) converges to x, then every subsequence of (x_n) also converges to x.*

Exercise.

1.1. **Sequential Characterization of Closed Sets.**

Theorem 1.8. *Let K be a nonempty subset of a metric space (X, d) and $x \in X$. Then*

(a) $x \in \overline{K}$ if and only if there is a sequence $(x_n) \subset K$ such that $x_n \to x$ as $n \to \infty$.

(b) K is closed if and only if K contains the limit of every convergent sequence in K.

Proof. (a) Assume that $x \in \overline{K}$. Then either $x \in K$ or $x \in K'$. If $x \in K$, then the constant sequence $(x, x, x, ...)$ in K converges to x. If $x \in K'$, then, for each $n \in \mathbb{N}$, the open ball $B(x, \frac{1}{n})$ contains a point $x_n \in K$ distinct from x. It now follows that $d(x_n, x) < \frac{1}{n}$. Clearly, $(x_n) \subset K$ and $x_n \to x$ as $n \to \infty$.

Conversely, assume that there is a sequence $(x_n) \subset K$ such that $x_n \to x$ as $n \to \infty$. Then, either $x \in K$ or every ϵ-ball centred at x contains a point $x_n \neq x$, in which case $x \in K'$. Thus $x \in \overline{K}$.

(b) By Corollary 0.2 (previous lecture), K is closed if and only if $K = \overline{K}$. Hence (b) follows from (a).

□

2. Completeness in Metric Spaces

Definition 2.1. *A sequence (x_n) in a metric space (X, d) is a **Cauchy** sequence if, given any $\epsilon > 0$, there is a natural number N (which depends on ϵ, in general) such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.*

Example 2.2. *In $\mathbb{R}$, $a_n = \frac{n+1}{n-3}$, $n \in \mathbb{N}$,.*

Proposition 2.3. *A convergent sequence in a metric space (X, d) is a Cauchy sequence .*

Proof. Let (x_n) be a sequence in X which converges to $x \in X$ and let $\epsilon > 0$. Then there is a natural number N such that $d(x_n, x) < \frac{\epsilon}{2}$ for all $n \geq N$. For all $n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, (x_n) is a Cauchy sequence in X. □

Converse is not true.

Example 2.4. *In $(\mathbb{Q}, d)$, $a_n = \left(1 + \frac{1}{n}\right)^n$ is a Cauchy seq. But not convergent in $\mathbb{Q}$ as the limit is e.*

Proposition 2.5. *A Cauchy sequence in a metric space (X, d) is bounded.*

Proof. Let (x_n) be a Cauchy sequence in X. Choose $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < 1 \text{ for all } n, m \geq N.$$

Let $r = \max\{d(x_1, x_N), d(x_2, x_N), ..., d(x_{N-1}, x_N), 1\}$. Clearly $d(x_n, x_N) \leq r$ for all $n = 1, 2, ..., N-1$. If $n \geq N$, then $d(x_n, x_N) \leq r$ for all $n \in \mathbb{N}$ and so (x_n) is bounded. □

Proposition 2.6. *Let (X, d) be a metric space. A cauchy sequnce in X which has a convergent subsequence is convergent.*

Proof. Let (x_n) be a Cauchy sequence in X and x_{n_k} its subsequence which converges to $x \in X$. Then, for any $\epsilon > 0$, there are positive integers N_1 and N_2 such that

$$d(x_n, x_m) < \frac{\epsilon}{2} \text{ for all } n, m \geq N_1$$

and

$$d(x_{n_k}, x) < \frac{\epsilon}{2} \text{ for all } k \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. If $k \geq N$, then since $n_k \geq k$,

$$d(x_k, x) \leq d(x_k, x_{n_k}) + d(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence e $x_n \to x$ as $n \to \infty$. □

Definition 2.7. *A metric space (X, d) is said to be complete if every Cauchy sequence in X converges (to a point in X).*

Theorem 2.8. *Every Cauchy sequence of real numbers converges.*

Proof. Let (s_n) be a Cauchy sequence of real numbers. By Proposition 2.5, (s_n) is bounded, and therefore, by the Bolzano-Weierstrass Theorem, (s_n) has a subsequence $\{s_{n_k}\}$ which converges to some real number l. Let $\epsilon > 0$ be given. Then there exist natural numbers N_1 and N_2 such that

$$|s_n - s_m| < \frac{\epsilon}{2} \text{ for all } n, m \geq N_1 \text{ and}$$

$$|s_{n_k} - l| < \frac{\epsilon}{2} \text{ for all } k \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$|s_n - l| \leq |s_n - s_{n_k}| + |s_{n_k} - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $\lim_{n\to\infty} s_n = l$. □

Example 2.9. $\mathbb{R}$, *with its usual metric, is a complete metric space.*

Proposition 2.10. *A closed subset F of a complete metric space (X, d) is complete.*

Proof. Let (x_n) be a Cauchy sequence in F. Then (x_n) is a Cauchy sequence in X. Since X is complete, this sequence converges to some x in X. Since F is closed, $x \in F$. Hence F is complete. □

Exercise 2.11. *$(l^p(\mathbb{R}), d_p)$ $1 \le p < \infty$, with $d_p(x,y) = \left(\sum\limits_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}$, $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}}$ is a complete metric space.*

Solution: Suppose that, $x_n = (x_n^j)_{j\in\mathbb{N}} \in l^p(\mathbb{R})$ is a Cauchy sequence. Then for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for $m, n > N$,

$$d(x_n, x_m) = \left(\sum_{j=1}^{\infty} |x_n^j - x_m^j|^p\right)^{1/p} < \epsilon^{1/p}.$$

This implies

$$|x_n^j - x_m^j| < \epsilon, \text{ for each j and } m, n > N.$$

So $(x_n^j)_{n\ge 1}$ for each i is a Cauchy sequence in $\mathbb{R}$. By completeness of $\mathbb{R}$ we get $x_n^j \to a_j$ as $n \to \infty$. Now let $a = (a_1, a_2, a_3, ...)$. Note that for any $M \ge 1$ and $m, n > N$,

$$\sum_{j=1}^{M} |x_n^j - x_m^j|^p \le \sum_{j=1}^{\infty} |x_n^j - x_m^j|^p < \epsilon.$$

Then taking limit $n \to \infty$ we get,

$$\sum_{j=1}^{M} |x_n^j - x_m^j|^p \le \epsilon, \ \forall M.$$

This holds for all M so we take limit $M \to \infty$ we get

$$\sum_{j=1}^{\infty} |x_n^j - x_m^j|^p \le \epsilon.$$

Then $x_m - a = (x_m^j - a_j)_{j\in\mathbb{N}} \in l^p$. Since $l^p(\mathbb{R})$ is a vector space and $x_m \in l^p$, so $a \in l^p$. Also for any $\epsilon' > 0$

$$d(x_m, a) < \epsilon', \ \forall m \ge N,$$

that is, $x_m \to a$ in l^p and $a \in l^p$. This implies l^p is complete.

Exercise 2.12. *$M \subset l^\infty(\mathbb{R})$, be a subspace consisting of all sequences $x = (\xi_j)$ with at most finitely many nonzero terms. Show that (M, d_∞) is not a complete metric space.($d_\infty(x,y) = \sup\limits_{n\in\mathbb{N}} |x_n - y_n|$, for $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in l^\infty(\mathbb{R})$).*

Solution: Let (x_n) be a sequence in $M \subset l^\infty(\mathbb{R})$, where $x_n = (\xi_j^n)_{j\in\mathbb{N}}$ and

$$\xi_j^n = \begin{cases} \frac{1}{j}, \ j \le n \\ 0. \ j > n \end{cases} \tag{2.1}$$

Then

$$
\begin{aligned}
&x_1 = (\xi_j^1) = (1, 0, 0, 0, ...)\\
&x_2 = (\xi_j^2) = (1, \frac{1}{2}, 0, 0, ...)\\
&x_3 = (\xi_j^2) = (1, \frac{1}{2}, \frac{1}{3}, 0, ...)\\
&.\\
&.\\
&.\\
&x_n = (\xi_j^2) = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...),\\
&.\\
&.\\
&.
\end{aligned}
\tag{2.2}
$$

and so on. Then for any $\epsilon > 0$ choose $N \in \mathbb{N}$ such that $N + 1 > 1/\epsilon$. Then for any $m > n > N$ we have

$$d(x_m, x_n) = \sup_j |\xi_j^m - \xi_j^n| = \frac{1}{n+1} < \frac{1}{N+1} < \epsilon.$$

This implies that (x_n) is a Cauchy sequence in M. By the construction it is obvious that

$$x_n \to x = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, \frac{1}{n+1}, ...).$$

But $x \notin M$ and M is not a complete space wrt the metric d_∞.

Exercise 2.13. *Show that $C[0, 2]$ with the metric, d_1, given by $d_1(f, g) = \int_0^2 |f(x) - g(x)|dx$ is not complete.*

Solution: Consider

$$f_n(t) = \begin{cases} t^n, \ 0 \le t \le 1 \\ 1, \ 1 \le t \le 2. \end{cases} \tag{2.3}$$

Then for $n < m$

$$
\begin{aligned}
d(f_n, f_m) &= \int_0^2 |f_n(x) - f_m(x)|dx \\
&= \int_0^1 |x^n - x^m|dx \\
&= \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n+1}
\end{aligned}
\tag{2.4}
$$

and consequently $m, n > N$ we have

$$d(f_n, f_m) < \frac{1}{N} = \epsilon.$$

Then (f_n) is a Cauchy sequence in $C[0,2]$. Suppose that $f_n \to f$ where $f \in C[0,2]$. Then

$$d(f_n, f) = \int_0^1 |t^n - f(t)|dt + \int_1^2 |1 - f(t)|dt \to 0$$

as $n \to \infty$. Now since

$$|f(t)| - t^n \leq |t^n - f(t)| \leq |f(t)| + t^n$$

we can show that

$$\int_0^1 |t^n - f(t)|dt \to \int_0^1 |f(t)|dt$$

as $n \to \infty$. This implies

$$\int_0^1 |f(t)|dt + \int_1^2 |1 - f(t)|dt = 0.$$

So then f must be,

$$f(t) = \begin{cases} 0, \ 0 \leq t \leq 1 \\ 1 \ 1 \leq t \leq 2. \end{cases} \tag{2.5}$$

This shows that $f \notin C[0,2]$. Hence $C[0,2]$ is not a complete space.

Exercise 2.14. *Show that $\mathbb{Z}_{>0}$ is an incomplete metric space with respect to the metric $d(m,n) = |m^{-1} - n^{-1}|$, $m, n \in \mathbb{Z}$.*

Solution: Let (x_n) be a sequence in $\mathbb{Z}_{>0}$ where $x_n = n$. Then $d(x_m, x_n) = |\frac{1}{m} - \frac{1}{n}|$. Given any $\epsilon > 0$ choose $N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$ and

$$d(x_m, x_n) = |\frac{1}{m} - \frac{1}{n}| < \epsilon.$$

This implies (x_n) is a Cauchy sequence in $\mathbb{Z}_{>0}$. Now if (x_n) was convergent to x positive then $d(x_n, x) \to 0$, as $n \to \infty$. This would imply that $\frac{1}{x} = 0$, a contradiction.