Lecture 7 - MTL 122 -
Real and Complex Analysis:

- $K \subseteq \mathbb{R}$ is compact if every open cover of K has <u>finite subcover</u>.

$$K \subseteq \bigcup_{i \in I} U_i$$

$\underline{\mathcal{U}} = \{U_i, \ i \in I\}$

any index.

finite subcollection of ~~$\mathcal{U}$~~ U $\longrightarrow$ $J \subseteq I$

finite.

$V = \{\underline{U_j} \mid j \in J, \ J \subseteq I$

<u>J finite</u>$\}$

$$K \subseteq \bigcup_{j \in J} U_j$$

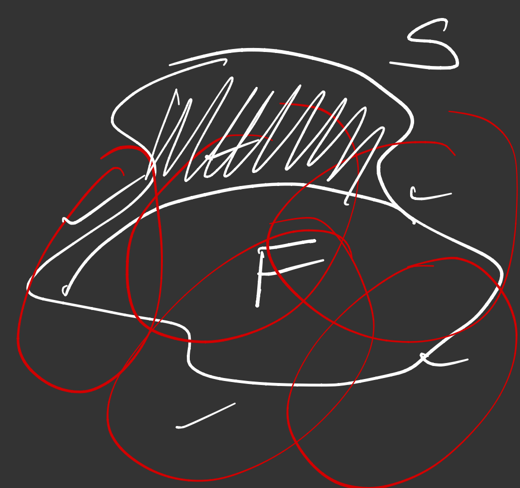
Theo. $S \subseteq \mathbb{R}$ compact.

If F is closed subset of S then F is compact.

Pf. $\mathcal{U} = \{U_\alpha, \alpha \in \Omega\}$ be an open cover of F.

$\mathcal{G} = \mathcal{U} \cup F^c$

$S \subseteq \bigcup_\alpha U_\alpha \cup F^c$.



$\mathcal{U} \cup F^c$ is an open cover of S.

and S is compact

$\Rightarrow \exists$ a finite sub cover of S.

$$S \subset \bigcup_{i=1}^{n} U_i \cup F^c$$

Since

$$F \subset S$$

$$\& \quad F \cap F^c = \emptyset$$

$$\Rightarrow \quad F \subseteq \bigcup_{i=1}^{n} U_i$$

$\Rightarrow$ F is compact.

$$S \subseteq \bigcup_{i=1}^{n} U_i \cup F^c.$$

$$F \subseteq \bigcup_{i=1}^{n} U_i \; \boxed{\cup F^c}$$

$$F \cap F^c = \emptyset$$

$$\Downarrow$$

$$F \subseteq \bigcup_{i=1}^{n} U_i$$

<u>Ex.</u> Let a, b be real number such that, $-\infty < a < b < \infty$. Then the $[a, b]$ is compact

(NIP)

Theo. (Heine-Borel)

A subset of $\mathbb{R}$ is compact iff it is closed & bounded.

[Note: Compactness $\Longleftrightarrow$ seq compact]

Pf. $K \subseteq \mathbb{R}$, K is compact.

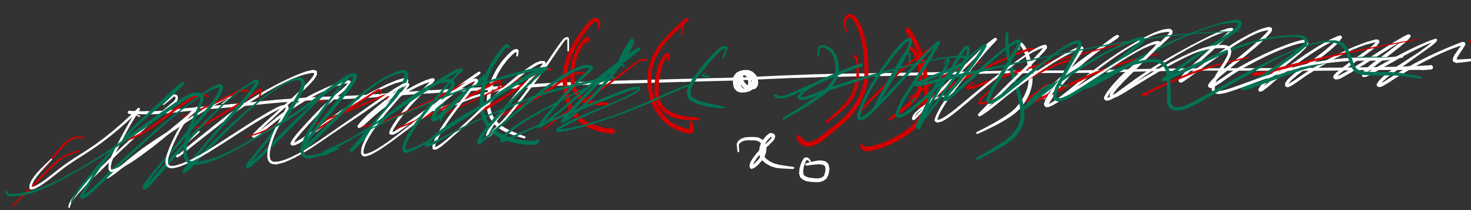
i) Closedness of K.

Show. $\mathbb{R} \setminus K$ is open.

Let $x_0 \in \mathbb{R} \setminus K$.

For each $k \in \mathbb{N}$

$$U_k = \left(-\infty, x_0 - \frac{1}{k}\right) \cup \left(x_0 + \frac{1}{k}, \infty\right)$$



$$= \left\{ x \in \mathbb{R} \;\middle|\; |x - x_0| > \frac{1}{k} \right\}$$

$$\mathbb{R} \setminus \{x_0\} = \bigcup_{k=1}^{\infty} U_k$$

$$\mathcal{U} = \{ U_k : k \in \mathbb{N} \}$$

$\downarrow$

open cover of K.

$K \subseteq \mathbb{R} \setminus \{x_0\}$

$x_0 \in \mathbb{R} \setminus K.$

$k_1, k_2, \ldots k_n$

$K \subset \bigcup_{i=1}^{n} U_{k_j}$ (This is because K is compact subset of $\mathbb{R}$)

Let

$k_{max} = \max\{k_1, \ldots k_n\}$

$$K \subset \bigcup_{i=1}^{n} U_{k_i} = \left(-\infty, x_0 - \frac{1}{k_{max}}\right) \cup \left(x_0 + \frac{1}{k_{max}}, \infty\right)$$

$$= \left\{ x \in \mathbb{R} \;\middle|\; |x - x_0| > \frac{1}{k_{max}} \right\}$$

$$K \subset \left\{ x \in \mathbb{R} \;\middle|\; |x - x_0| > \frac{1}{k_{max}} \right\}$$

Hence,

$$\left\{ x \in \mathbb{R} \;\middle|\; |x - x_0| < \frac{1}{k_{max}} \right\} \subseteq \mathbb{R} \setminus K.$$

$\Rightarrow$ x_0 is an int point

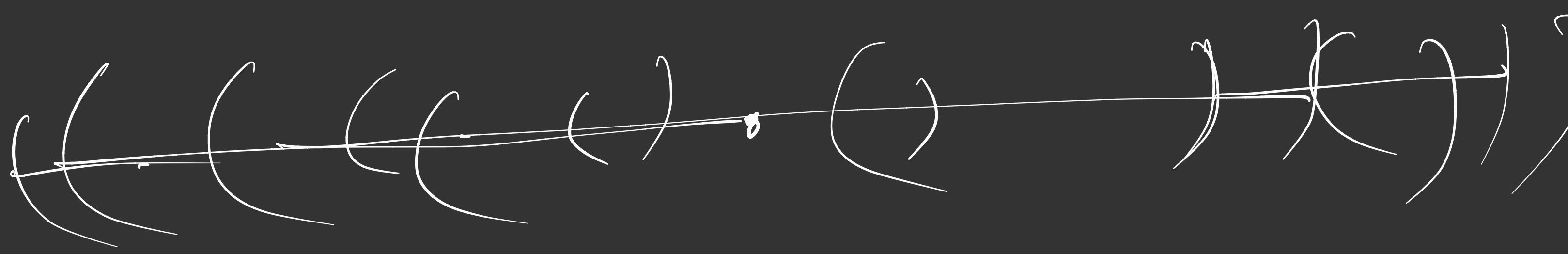
of $\mathbb{R}\setminus K$

$\Rightarrow$ $\mathbb{R}\setminus K$ is open

$\Rightarrow$ K is closed.

Boundedness.

$$\mathcal{U} = \{(-n, n) : n \in \mathbb{N}\}$$



$\mathcal{U}$ is an open cover of K.

$$K \subseteq \mathbb{R} = \bigcup_{n\in\mathbb{N}} (-n, n)$$

Then,

$$n_1, n_2 \ldots n_m$$

s.t.,

$$K \subseteq \bigcup_{i=1}^{m} (-n_i, n_i)$$

$$n_{max} = \max\{n_1, \ldots n_m\}$$

$$K \subseteq (-n_{max}, n_{max})$$

$\Rightarrow$ K is bdd.

m, M

$m \leq x \leq M$

$M \geq 0$

$|x| < M$

$\forall x \in K$

$x \in (m, M)$

Conversely, K is closed & bdd.

$$K \subseteq (m, M)$$

$K \subseteq [m, M] \leftarrow$ compact

$\Rightarrow K$ is compact.

Metric spaces

$d(x, y) = |x - y|$ $\quad \mathbb{R}$

X — set.

$X \times X = \{(x, y) : x, y \in X\}$

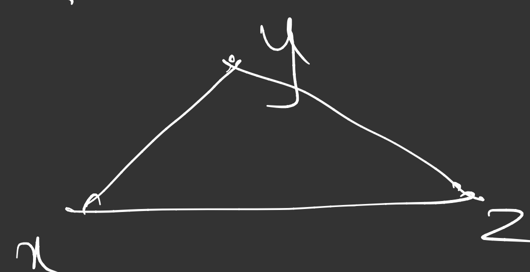
$d : X \times X \longrightarrow \mathbb{R}$

i) $d(x, y) = d(y, x)$

ii) $d(x,y) \geq 0$

iii) $d(x,y) = 0 \iff x = y$

iv) $d(x,z) \leq d(x,z) + d(y,z)$



$d \to$ dist fn on

metric

$(X, d) \to$ metric space

$Y \subseteq X$

$d_Y = d\big|_{Y \times Y} \to$

$(Y, d_Y) \to$ metric space.

$d(x, x) = 0$ instead.

↳ pseudo-metric.

Ex. 1. $X = \mathbb{R}$

$$d(x, y) = |x - y|$$

$(\mathbb{R}, d) \to$ metric space.

2) $X = \mathbb{C}$ → complex numbers.

$(\mathbb{C}, d)$, $d(x,y) = |x-y|$

3) $X = \mathbb{R}^n$. n = any

$x = (x_1, x_2 \ldots x_n)$

1 meteric

$\leftarrow d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

Euclidean ~~...~~ meteric

$\leftarrow d_2(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$

supremum.

$\uparrow d_\infty(x,y) = \max_{1 \le i \le n} |x_i - y_i|$

ii) $(\mathbb{R}^n, d_2)$

$$\left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2}$$

$$\leq \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{1/2}$$

$a_i = x_i - y_i$

$b_i = y_i - z_i$

$$\left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}$$

Calculation: -

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}$$

Cauchy - Schwarz

Ineq

Quiz syllabus.

upto Metric.
spaces.