Lecture-5 : MTL 122

Real & Complex Analysis.

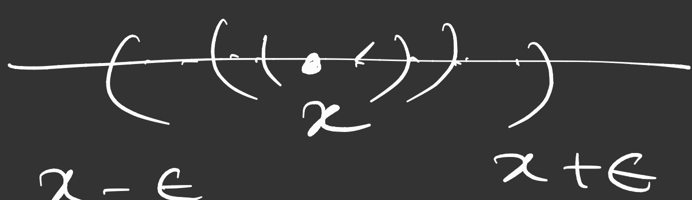
Limit points / accumulation pts.

$\mathbb{R}$.

$S \subseteq \mathbb{R}$ $(\neq \emptyset)$

$\boxed{x \in \mathbb{R}}$ is an accumulation point if for any $\epsilon > 0$

$$\boxed{N^*_\epsilon(x) \cap S \neq \emptyset}$$



Note: Defn of limit point does not say anything about whether $x \in S$ or

$x \notin S$.

Ex.

1) $c < d$, $[c, d]$

(c, d), c, d are limit points.

2) $\mathbb{Q} \to$ Set of limit points is $\mathbb{R}$.

$x \in \mathbb{R}$, $\epsilon > 0$ $\exists$ r

$r \neq x$, $x < r < x + \epsilon$.

$r \in N_\epsilon^*(x) \cap \mathbb{Q}$

$(\neq \phi)$

$S \subseteq \mathbb{R}$

Suppose $\boxed{x \text{ is a l.p of } S}$.

$n \in \mathbb{N}$, $\frac{1}{n} > 0$

$(\epsilon = 1/n)$

$N^*_{1/n}(x) \cap S \neq \emptyset$ (Defn of lp)

$n=1$, $\underline{N^*_1(x)} \cap S \neq \emptyset$

$|\underline{\underline{a_1 \neq x}}$, $a_1 \in N_1(x) \cap S \neq \emptyset$

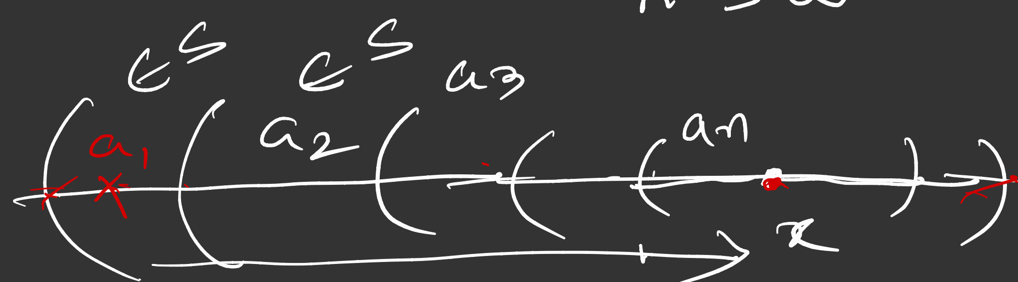
$n=2$, $a_2 \neq x$, $a_2 \in N_{1/2}(x) \cap S \neq \emptyset$

$\vdots$

$(a_n)_{n \geq 1}$, $a_n \in S$, $\underline{a_n \neq x}$ $\forall n \in \mathbb{N}$.

Claim: $a_n \to x$ as $n \to \infty$.

$|b_n - a_n| \to 0$



$\epsilon > 0$ arbitrary,

$N \in \mathbb{N}$, $\frac{1}{N} < \epsilon$. (A.P)

Then if $n \geq N$.

$a_n \in N_{1/n}(x)$

$\Rightarrow \quad |x - a_n| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$

Hence $(a_n) \to x$ as $n \to \infty$.

$a_n \in S$

$a_n \neq x$.

Conclusion:

A point x is a limit point of a set S

iff

$\Leftrightarrow \exists$ a seq (a_n) of S

such that,

$a_n \neq x$, $a_n \to x$.

as $n \to \infty$.

$\Leftarrow$

$a_n \in S$ & $a_n \neq x$ $\forall n$

Suppose, $(a_n)_{n \geq 1}$

$$\lim_{n \to \infty} a_n = x. \quad (a_n \neq x)$$

$\forall \epsilon > 0 \quad \exists N \in \mathbb{N}$ s.t

$$|a_n - x| < \epsilon, \quad \forall n \geq \underline{N}$$

In particular,

$$|\underline{a_N} - x| < \epsilon$$

$$\Rightarrow \underline{a_N} \in N^*_{\underline{\epsilon}}(x) \cap S$$

$\Rightarrow$ x is a limit point of S.

S' = set of limit points of S.
= derived set.

- $S \subseteq \mathbb{R}$ dense in itself

  $S \subset S'$

- S is perfect. $S = S'$

- Closure of set S

  $\bar{S} = S \cup S'$

Ex. 1 $S = \{x \in \mathbb{R} : 0 < x \leq 1\}$

$S' = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$

$\bar{S} = S \cup S' = S'$.

2) $S = [a, b]$, $\boxed{S' = S}$

$\overline{S} = S$. (*)

3) $\mathbb{Q}' = \mathbb{R}$,

$\mathbb{Q} \subseteq \mathbb{R} = \mathbb{Q}'$

$\mathbb{Q}$ dense in $\mathbb{R}$.

$\overline{\mathbb{Q}} = \mathbb{R}$.

4) $\boxed{\mathbb{Z}' = \emptyset}$ (Same argument

$\overline{\mathbb{Z}} = \mathbb{Z}$ (*) closed set.

Theo: $S \subseteq \mathbb{R}$.

Then S is closed iff $\Longleftrightarrow$

$\boxed{S' \subseteq S.}$

Pf: Suppose S is closed.

Let $x \in S'$ $\longrightarrow$ lp.

Show: $x \in S$.

- If, $x \notin S$

$\Rightarrow$ $x \in \mathbb{R} \setminus S$.

Now $\mathbb{R} \setminus S$ is open:

$\exists\ \epsilon > 0$ such that,

$(x-\epsilon, x+\epsilon) \subset \mathbb{R} \setminus S$.

$\Rightarrow$ $(x-\epsilon, x+\epsilon) \cap S = \emptyset$

$\Rightarrow$ $(x-\epsilon, x+\epsilon) \setminus \{x\} \cap S = \emptyset$

↯

Then $x \in S$.

$\Rightarrow$ $S' \subseteq S$.

[x is a lp of S.

$\Downarrow$

$\forall \epsilon$, $N_\epsilon^*(x) \cap S \neq \emptyset$]'

$\equiv$ $\exists \epsilon$ s.t $N_\epsilon^*(x) \cap S = \emptyset$

$\Downarrow$

x is not a lp

Converse. $S' \subseteq S$.

Show

- S is closed.
- $\mathbb{R} \setminus S$ is open.

$(x \in \mathbb{R} \setminus S) \Rightarrow x \notin S$ & $S' \subseteq S$ (here)

$\Rightarrow x \notin S'$

$\exists\, \epsilon > 0$ s.t

$N^*_\epsilon(x) \cap S = \emptyset$

$\Downarrow$

$(x-\epsilon, x+\epsilon) \cap S = \emptyset$

$\Rightarrow (x-\epsilon, x+\epsilon) \subset \mathbb{R} \setminus S$

$\Rightarrow$ ~~x~~ $\mathbb{R} \setminus S$ is an open set.

$\Rightarrow S$ is closed.

Ex. 1) $\underline{\mathbb{Q}' = \mathbb{R}}$.

$\underline{\mathbb{R} = \mathbb{Q}' \not\subseteq \mathbb{Q}}$

$\Rightarrow$ $\mathbb{Q}$ is not closed

(From the above).

2) $\underline{(c, d)'} = [c, d]$

3) $A = \{\frac{1}{n}\}$, '0', $\underline{0 \notin A}$.

$A' \not\subseteq A \longrightarrow$ A is not closed

Theo. A set $S \subseteq \mathbb{R}$ is closed iff every (convergent) Cauchy seq in S has a limit that is also in S. (Ex).

(Cauchy $\Leftrightarrow$ convergent) in $\mathbb{R}$.

[What happens in Metric Spaces??]

$S \subseteq \mathbb{R}$ $S \neq \emptyset$

- $\underline{\text{int}(S)}$, S', $\bar{S}$, $\underline{bd(S)}$. $\}$

What are the connections between these sets?

$\rightarrow$ $$\boxed{\mathbb{R} = \text{int}(S) \cup bd(S) \cup \text{int}(\mathbb{R} \setminus S)}$$