MTL122 - Real and complex analysis

Assignment-3

Department of Mathematics
Indian Institute of Technology Delhi

Question 1

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Proof

- Here $A \cap B = \emptyset$ and A, B are closed sets of (X, d).

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- Here $A \cap B = \emptyset$ and A, B are closed sets of (X, d).
- Then $X - A$ and $X - B$ are open sets in (X, d) and $B \subset X - A$, $A \subset X - B$.

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- If $A = \emptyset$, take $U = \emptyset$ and $V = X$.
- If $B = \emptyset$, take $V = \emptyset$ and $U = X$.
- Now assume A and B both are non-empty disjoint closed subsets of X, consider $U = \{x \in X : d(x, A) < d(x, B)\}$ and $V = \{x \in X : d(x, B) < d(x, A)\}$.

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Claim: We need to show $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$.

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- Let $x \in A$. Then $d(x, A) = 0 < d(x, B)$, thus $x \in U$. Hence $A \subseteq U$.

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- Let us assume $U \cap V \neq \emptyset$, then there exist $y \in U \cap V$. Thus

$$d(y, A) < d(y, B) \text{ and } d(y, B) < d(y, A),$$

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$$\begin{aligned} U &= \{x \in X : d(x, A) < d(x, B)\} \\ &= \bigcup_{r>0} \{x \in X : d(x, A) < r < d(x, B)\} \end{aligned}$$

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- Hence $\{x \in X : d(x, A) < r\}$ is open in X.
- Simillarly, we can show $\{x \in X : d(x, B) > r\}$ is open in X. Hence U is open in X.
- Simillarly V can be open in X.

Question 2

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Let (X, d) be a metric space with $E \subset X$. Prove that $(E^\circ)^c = \overline{(E^c)}$.

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Proof

- We know $E^\circ \subset E$, then $E^c \subset (E^\circ)^c$.

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- Since $(E^\circ)^c$ is closed set

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- We know $E^\circ \subset E$, then $E^c \subset (E^\circ)^c$.
- Since $(E^\circ)^c$ is closed set and $\overline{(E^c)}$ is the smallest closed set containing E^c.

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- Let $x \in (E^\circ)^c$. Since $E^c \subset (E^\circ)^c$, then $x \in E^c$ or $x \notin E^c$.

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- If $x \in E^c$, then $x \in \overline{(E^c)}$. We are done.
- If $x \notin E^c$ and $x \in (E^\circ)^c$, we need to show $x \in (E^c)'$.

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- So $x \in (E^c)' \subset \overline{(E^c)}$.Thus $(E^\circ)^c \subset \overline{(E^c)}$.

Question 3

A point x not belonging to a closed set $M \subset (X, d)$ always has a nonzero distance from M. (Hint: To prove this, show that $x \in \overline{A}$ if and only if $D(x, A) = \text{dist}(x, A) = \inf_{y \in A} d(x, y) = 0$; here A is any nonempty subset of X).

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Let A and B be non-empty subsets of a metric space $(X; d)$. Prove that:

(i) $A \subset B \implies \mathrm{diam}(A) \leq \mathrm{diam}(B)$

(ii) $\mathrm{diam}(A) = 0$ if and only if for some $x \in X, A = \{x\}$.

(iii) If $a \in A$ and $b \in B$, then
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Proof(ii) Its proof is obvious. Do yourself.

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- By taking supremum bothside over $x, y \in A \cup B$, we have

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Let $X = (X, d)$ be a metric space and $CS(X)$ the collection of all Cauchy sequences in X. For (x_n) and (y_n) in $CS(X)$, define
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- Since $\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2}$. Thus for $\epsilon > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that $|\arctan(n) - \frac{\pi}{2}| < \frac{\epsilon}{2}$ for all $n > n_0$.
- Now consider $m, n > n_0$

$$\begin{aligned} d(x_m, x_n) &= |\arctan m - \arctan n| \\ &\leq |\arctan(m) - \frac{\pi}{2}| + |\arctan(n) - \frac{\pi}{2}| \end{aligned}$$

Question 10

Show that the set of all real numbers constitutes an incomplete metric space if we choose $d(x, y) = |\arctan x - \arctan y|$.

Solution:

- Take $\{x_n\} = n$.
- We will prove that $\{x_n\}$ is cauchy.
- Since $\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2}$. Thus for $\epsilon > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that $|\arctan(n) - \frac{\pi}{2}| < \frac{\epsilon}{2}$ for all $n > n_0$.
- Now consider $m, n > n_0$

$$\begin{aligned} d(x_m, x_n) &= |\arctan m - \arctan n| \\ &\leq |\arctan(m) - \frac{\pi}{2}| + |\arctan(n) - \frac{\pi}{2}| \\ &< \epsilon \end{aligned}$$

Question 10 Contd...

- Hence it is cauchy.

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- Suppose that $\{x_n\} \to x$ i.e $d(x_n, x) \to 0$ as $n \to \infty$.

Question 10 Contd...

- Hence it is cauchy.
- Suppose that $\{x_n\} \to x$ i.e $d(x_n, x) \to 0$ as $n \to \infty$.
- Let $\epsilon > 0$ and $N > n_0$,

$$d(x_n, x) \quad < \quad \frac{\epsilon}{2} \quad \text{for all} \quad n > n_0,$$

Question 10 Contd...

- Hence it is cauchy.
- Suppose that $\{x_n\} \to x$ i.e $d(x_n, x) \to 0$ as $n \to \infty$.
- Let $\epsilon > 0$ and $N > n_0$,

$$d(x_n, x) \quad < \quad \frac{\epsilon}{2} \quad \text{for all} \quad n > n_0,$$

- Consider

$$\left|\frac{\pi}{2} - \arctan(x)\right|$$

- Hence it is cauchy.
- Suppose that $\{x_n\} \to x$ i.e $d(x_n, x) \to 0$ as $n \to \infty$.
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- Consider

$$\left|\frac{\pi}{2} - \arctan(x)\right| \quad \leq \quad \left|\frac{\pi}{2} - \arctan(n)\right| + |\arctan(n) - \arctan(x)|$$

Question 10 Contd...

- Hence it is cauchy.
- Suppose that $\{x_n\} \to x$ i.e $d(x_n, x) \to 0$ as $n \to \infty$.
- Let $\epsilon > 0$ and $N > n_0$,

$$d(x_n, x) \quad < \quad \frac{\epsilon}{2} \quad \text{for all} \quad n > n_0,$$

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$$\begin{aligned} \left|\frac{\pi}{2} - \arctan(x)\right| \quad &\leq \quad \left|\frac{\pi}{2} - \arctan(n)\right| + |\arctan(n) - \arctan(x)| \\ &< \quad \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

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- Thus $\left|\frac{\pi}{2} - \arctan(x)\right| < \epsilon$ and $\epsilon > 0$ is arbitrary.

Question 10 Contd...

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- Suppose that $\{x_n\} \to x$ i.e $d(x_n, x) \to 0$ as $n \to \infty$.
- Let $\epsilon > 0$ and $N > n_0$,

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- Hence it is cauchy.
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- Thus $\left|\frac{\pi}{2} - \arctan(x)\right| < \epsilon$ and $\epsilon > 0$ is arbitrary.
- This gives $\arctan(x) = \frac{\pi}{2}$, but there doesn't exist any real number such that $\arctan(x) = \frac{\pi}{2}$.

Question 10 Contd...

- Hence it is cauchy.
- Suppose that $\{x_n\} \to x$ i.e $d(x_n, x) \to 0$ as $n \to \infty$.
- Let $\epsilon > 0$ and $N > n_0$,

$$d(x_n, x) \quad < \quad \frac{\epsilon}{2} \quad \text{for all} \quad n > n_0,$$

- Consider

$$\begin{aligned} \left|\frac{\pi}{2} - \arctan(x)\right| \quad &\leq \quad \left|\frac{\pi}{2} - \arctan(n)\right| + |\arctan(n) - \arctan(x)| \\ &< \quad \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

- Thus $\left|\frac{\pi}{2} - \arctan(x)\right| < \epsilon$ and $\epsilon > 0$ is arbitrary.
- This gives $\arctan(x) = \frac{\pi}{2}$, but there doesn't exist any real number such that $\arctan(x) = \frac{\pi}{2}$. contradiction.
- Hence it is not complete.