Lecture 3

Real and Complex Analysis

MTL122/ MTL503/ MTL506

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1. Topology on $\mathbb{R}$

2. Open Sets and Closed Sets

2.1. Open Sets.

Definition 2.1. *Let $a \in \mathbb{R}$ and $\epsilon > 0$.*

*(1) An $\epsilon-$**neighbourhood** of a is the set*

$$N(a, \epsilon) = \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

*(2) A **deleted** ϵ**-neighbourhood** of a is the set*

$$N^*(a, \epsilon) = \{x \in \mathbb{R} : 0 < |x - a| < \epsilon\}.$$

It is clear that

$$N(a, \epsilon) = (a - \epsilon, a + \epsilon) \text{ and } N^*(a, \epsilon) = (a - \epsilon, a) \cup (a, a + \epsilon).$$

Definition 2.2. *A subset U of $\mathbb{R}$ is said to be open if for each $s \in U$ there is an $\epsilon > 0$ such that $(s - \epsilon, s + \epsilon) \subset U$.*

Example 2.3. *$I = (0, 1)$, the open interval, is open. If $x \in I$, then*

$$(x - \delta, x + \delta) \subset I, \ \delta = \min\left(\frac{x}{2}, \frac{1 - x}{2}\right) > 0$$

Similarly, every finite or infinite open interval (a, b), $(-\infty, b)$, or (a, ∞) is open.

Example 2.4. *The half-open interval $J = (0, 1]$ isn't open, since $1 \in J$ and $(1 - \delta, 1 + \delta)$ isn't a subset of J for any $\delta > 0$,however small.*

Proposition 2.5. *An arbitrary union of open sets is open, and a finite intersection of open sets is open.*

Example 2.6. *The interval $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ is open for every $n \in \mathbb{N}$, but*

$$\bigcap_{n=1}^{\infty} I_n = \{0\}$$

is not open.

Definition 2.7. *A set $G \subset \mathbb{R}$ is open if every $x \in G$ has a neighborhood U such that $U \subset G$.*

Definition 2.8. *Let S be a subset of $\mathbb{R}$. Then*

*a) $x \in S$ is called an **interior point** of s if there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset S$. The set of all the interior points of a set S is denoted by S° or $int(S)$.*

b) x *is called a* ***boundary point*** *of* S *if for every* $\epsilon > 0$ *the interval* $(x-\epsilon, x+\epsilon)$ *contains points of* S *as well as points of* $\mathbb{R} \setminus S$. *The set of boundary points of* S *is denoted by* δS *or* $bd(S)$.

c) $x \in S$ *is called an* ***isolated point*** *of* S *if there exists an* $\epsilon > 0$ *such that* $(x-\epsilon, x+\epsilon) \cap S = \{x\}$.

It is clear from the definition that each point of an open set S is an interior point of S. Also, every isolated point of a set S is a boundary point of S.

Example 2.9.

(1) $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. *Then each point of* S *is an isolated point of* S. *Therefore* $S \subset \delta S$.

(2) *The set* $\mathbb{N}$ *of natural numbers consists of isolated points only. Therefore, every point of* $\mathbb{N}$ *is a boundary point. Clearly,* $\mathbb{N}^\circ = \phi$

2.2. Closed sets.

Definition 2.10. *A set* $F \subset \mathbb{R}$ *is closed if* $F^c = \{x \in \mathbb{R} : x \notin F\}$ *is open.*

Example 2.11. *The closed interval* $I = [0, 1]$ *is closed since* $I^c = (-\infty, 0) \cup (1, \infty)$ *is a union of open intervals, and therefore it's open. Similarly, every finite or infinite closed interval* $[a, b]$, $(-\infty, b]$ *or* $[a, \infty)$ *is closed.*

The empty set ϕ and $\mathbb{R}$ are both open and closed. They are only such sets. Many subsets of $\mathbb{R}$ are neither open nor closed.

Example 2.12. $I = (0, 1]$ *isn't open because it doesn't contain any neighborhood of the end point* $1 \in I$. *Its complement*

$$I^c = (\infty, 0] \cup (1, \infty)$$

isn't open either, since it doesn't contain anu neighborhood of $0 \in I^c$. *Thus* I *isn't closed either.*

Example 2.13. *The set of rational numbers* $Q \subset \mathbb{R}$ *is neither open nor closed. It isn't open because every neighborhood of a rational number contains irrational numbers, and its complement isn't open because every neighborhood of an irrational number contains rational numbers.*

Proposition 2.14. *An arbitrary intersection of closed sets is closed, and a finite union of closed sets is closed.*

Exercise.

Example 2.15. *If* I_n *is the closed interval*

$$I_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right],$$

then the union of the I_n *is the open interval*

$$\bigcup_{n=1}^{\infty} I_n = (0, 1).$$

Proposition 2.16. *A set $F \subset \mathbb{R}$ is closed if and only if the limit of every convergent sequence in F belongs to F.*

Proof First suppose that F is closed and (x_n) is a convergent sequence of points $x_n \in F$ such that $x_n \to x$. Then every neighborhood of x contains points $x_n \in F$. It follows that $x \notin F^c$, since F^c is open and every $y \in F^c$ has a neighborhood $U \subset F^c$ that contains no points in F. Therefore, $x \in F$.

Conversely, suppose that the limit of every convergent sequence of points in F belongs to F. Let $x \in F^c$. Then x must have a neighborhood $U \subset F^c$; otherwise for every $n \in \mathbb{N}$ there exists $x_n \in F$ such that $x_n \in (x - 1/n, x + 1/n)$, so $x = \lim x_n$, and x is the limit of a sequence in F. Thus, F^c is open and F is closed. $\square$