MTL122 - Real and complex analysis

Assignment-1

Department of Mathematics
Indian Institute of Technology Delhi

Question 1

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Let A, B and C be sets.

- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
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Let A, B, C be sets, $f : A \to B$ and $g : B \to C$ be functions, and let $h : A \to C$ be defined by $h(x) = g(f(x))$ for $x \in A$. State (give reasons/counterexamples) whether the following statements are true or false:

- If h is not injective, then at least one of the functions f and g is not injective.
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Let $f : A \to B$ be a function. Let $W \subseteq B$.

- Prove that $f(f^{-1}(W)) \subseteq W$.
- Prove that if f is surjective then $f(f^{-1}(W)) = W$.

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- Let $x \in f(f^{-1}(W))$

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- $\therefore \exists\ y \in A$ such that $x = f(y)$ and $y \in f^{-1}(W)$

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- Let $x \in f(f^{-1}(W))$
- $\therefore \exists\ y \in A$ such that $x = f(y)$ and $y \in f^{-1}(W)$
- Now since $y \in f^{-1}(W)$, we have $f(y) \in W$ and thus $x \in W$.

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- Now since $y \in f^{-1}(W)$, we have $f(y) \in W$ and thus $x \in W$.
- **Hence Proved**

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- Take $A = B = C = \mathbb{R}$. Let $f(x) = 2x$ and $g(x) = |x|$.
- We have $h(x) = |2x|$ and this is a sufficient counterexample.
- Thus this statement is **False**.

Question 3

Let $f : A \to B$ be a function. Let $W \subseteq B$.

- Prove that $f(f^{-1}(W)) \subseteq W$.
- Prove that if f is surjective then $f(f^{-1}(W)) = W$.

Solution:

a)To Prove: $f(f^{-1}(W)) \subseteq W$ for a function $f : A \to B$ and $W \subseteq B$

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Consider the formula $f(x) = 2 - \sqrt{x+4}$.

- What is the largest subset of $A \subseteq \mathbb{R}$ so that $f : A \to \mathbb{R}$ defined by $f(x) = 2 - \sqrt{x+4}$ is a function?
- Compute the image of $f : A \to \mathbb{R}$.
- Compute $f([5, 12])$.
- Compute $f^{-1}([0, 2])$.

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Hence $f^{-1}([0, 2]) = [-4, 0]$.

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Are the following sets finite, countable or uncountable? Explain or prove your answer in each case.

- $\{(x, y) \in \mathbb{N} \times \mathbb{R} : xy = 1\}$.
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- Similarly supremum case is obvious.

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(i) If $x > 0$, then there exist $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

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- Clearly $y - x > 0$, then there exist $n \in \mathbb{N}$ such that

$$\frac{1}{n} < y - x.$$

Then $nx + 1 < ny$.

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- Clearly $y - x > 0$, then there exist $n \in \mathbb{N}$ such that

$$\frac{1}{n} < y - x.$$

Then $nx + 1 < ny$.

Question 9 Contd...

- Since $x > 0$, then $nx > 0$. Now there exist $m \in \mathbb{N}$ s.t.

$$m - 1 \leq nx < m$$

Then $m \leq nx + 1 < m + 1$.

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- Since $x > 0$, then $nx > 0$. Now there exist $m \in \mathbb{N}$ s.t.

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Hence $x < \frac{m}{n} < y$, when $0 < x < y$.

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- Now replace x and y by $\sqrt{x}$ and $\sqrt{y}$ respectively. And take square both side we have given result.

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- If A is empty, then $\sup A = -\infty$ (we are done) and B is empty, then $\inf B = \infty$ (we are done).

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- Suppose that A and B are non-empty. Here $x \leq y$ for all $x \in A$.
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- Again for all $y \in B$, $\sup A$ is lower bound of B. Then

$$\sup A \leq \inf B.$$

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For each of the following sets S, find the sup S and inf S if they exist. You need to justify your answer.

(a) $S = \{x \in \mathbb{R} : x^2 < 5\}$.

(b) Let $A = \{1/n : n \in \mathbb{N} \text{ and } n \text{ is prime}\}$.

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Proof

(a) $S = (-\sqrt{5}, \sqrt{5})$, then $\sup S = \sqrt{5}, \quad \inf S = -\sqrt{5}$.

(b) $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{5} \cdots\}$, then $\sup A = \frac{1}{2}, \inf A = 0$.

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- So there is $\epsilon_0 > 0$ and for all $n_k \in \mathbb{N}$, $k = 1, 2, \cdots$, s.t.

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Show that the sequence converges.

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- Let $m > n$, Now

$$|a_n - a_m| = |a_n - a_{n+1} + a_{n+1} - a_{n+2} + a_{n+2} - \cdots + a_{m-1} - a_m|$$

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$$\begin{aligned} |a_n - a_m| &= |a_n - a_{n+1} + a_{n+1} - a_{n+2} + a_{n+2} - \cdots + a_{m-1} - a_m| \\ &\leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m| \\ &\leq \left(\frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}}\right)|a_1 - a_0| \end{aligned}$$

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|a_n - a_m| &= |a_n - a_{n+1} + a_{n+1} - a_{n+2} + a_{n+2} - \cdots + a_{m-1} - a_m| \\
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&\leq \left(\frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}}\right)|a_1 - a_0| \\
&= \frac{1}{2^n}\left(1 + 1/2 + \cdots + \frac{1}{2^{m-1-n}}\right)|a_1 - a_0|
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  Hence $\{a_n\}$ is Cauchy sequence, then its limit is unique.

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- Let us assume a_n converges to a and b. i.e., $a_n \to a$ and $a_n \to b$. We need to show $a = b$.

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- Let us assume a_n converges to a and b. i.e., $a_n \to a$ and $a_n \to b$. We need to show $a = b$.
- By definition of convergence of sequence
  For each $\epsilon > 0$ there exist a natural number $N_1 \in \mathbb{N}$ such that
$$|a_n - a| < \frac{\epsilon}{2} \quad \text{when } n \geq N_1.$$

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- Now, for $N > \max\{N_1, N_2\}$,
  $$|a - b| = |a - a_N + a_N - b| \leq |a_N - a| + |a_N - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Question 15 Contd...

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Question 15

Suppose that $0 < \alpha < 1$ and that (x_n) is a sequence which satisfies one of the following conditions:

1. $|x_{n+1} - x_n| \leq \alpha^n, n = 1, 2, 3, \ldots$
2. $|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|, n = 1, 2, 3, \ldots$

Then prove that (x_n) satisfies the Cauchy criterion.
Note: Whenever you use this result, you have to show that the number α that you get, satisfies $0 < \alpha < 1$. The condition
$|x_{n+2} - x_{n+1}| \leq |x_{n+1} - x_n|$ does not guarantee the convergence of (x_n).
Give examples.

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Question 15 Contd...

$$\leq \quad \alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^{m}$$

Question 15 Contd...

$$
\begin{aligned}
&\leq \alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^{m} \\
&= \alpha^{m}(1 + \alpha + \alpha^{2} + \cdots + \alpha^{n-m-1})
\end{aligned}
$$

Question 15 Contd...

$$
\begin{aligned}
&\leq \alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^{m} \\
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&= \alpha^{m}\frac{1-\alpha^{n-m}}{1-\alpha}
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Question 15 Contd...

Now consider

$$|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - \cdots + x_{m+1} - x_m|$$

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$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - \cdots + x_{m+1} - x_m| \\ &\leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \end{aligned}$$

Question 15 Contd...

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$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - \cdots + x_{m+1} - x_m| \\ &\leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \\ &\leq \left(\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m\right) |x_2 - x_1| \end{aligned}$$

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Now we will check that whether the condition $|x_{n+2} - x_{n+1}| \leq |x_{n+1} - x_n|$ does not guarantee the convergence of (x_n).

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- Take $x_n = n$.

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- Then clearly $|x_{n+2} - x_{n+1}| \leq |x_{n+1} - x_n|$ is satisfied but x_n is not cauchy.

Question 15 Contd...

- Therefore $|x_n - x_m| < \frac{\alpha^m}{1-\alpha}|x_2 - x_1| < \epsilon$ for all $n, m > n_0$.
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- Hence the condition $|x_{n+2} - x_{n+1}| \leq |x_{n+1} - x_n|$ does not guarantee the convergence of (x_n).

Question 16

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For two sets S_1 and S_2 in $\mathbb{R}^n$, prove or disprove:

1. $S_1 + S_2$ is open if both S_1 and S_2 are open.
2. $S_1 + S_2$ is closed if both S_1 and S_2 are closed.
3. $S_1 + S_2$ is bounded if both S_1 and S_2 are bounded. Are the converses of these statements true? Prove or disprove their converses.

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Solution: True.

- Assume that S_1 and S_2 are open.

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- Let $p \in S_2$. Since S_2 is open, there exists $r > 0$ such that $N_r(p) \subseteq S_2$.

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- Take $A = 1, -1$ and $B = \mathbb{R} \backslash \{0\}$, then $A + B = \mathbb{R}$.
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Solution: False.

- Take $A = \{n + \frac{1}{n} : n \in \mathbb{N}\}$ and $B = \{-n : n \in \mathbb{N}\}$.

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- Then $A + B = \{-m + n + \frac{1}{n} : n, m \in \mathbb{N}\}$

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- Let $p \in S_2$. Since S_2 is open, there exists $r > 0$ such that $N_r(p) \subseteq S_2$.
- Let $q \in S_1$. Then $N_r(q+p) = q + N_r(p) \subseteq q + S_2$. (why?)
- Therefore $q + S_2$ is open.
- Hence $S_1 + S_2$ is open.

What about converse? False

- Take $A = 1, -1$ and $B = \mathbb{R} \setminus \{0\}$, then $A + B = \mathbb{R}$.
- Cleary $A + B$ is open but A is not open.

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$S_1 + S_2$ is closed if both S_1 and S_2 are closed.

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$S_1 + S_2$ is bounded if both S_1 and S_2 are bounded.

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- Since S_1 and S_2 is bounded.
- There exists $M_1, M_2 > 0$ such that $|a| < M_1$ and $|b| < M_2$ for all $a \in S_1$, $b \in S_2$.

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- Hence S_2 is bounded. (why?)

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- Since $S_1 + S_2$ is bounded, this implies $a_0 + S_2$ is bounded.
- Hence S_2 is bounded. (why?)
- Similarily we can show that S_1 is also bounded.

Question 17 Contd...

Question 17

Show that the following sets are open in $\mathbb{R}$.

1. $A = \{x \in \mathbb{R} : x^3 > x\}$
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- Since, arbitrary union of open sets is open. Hence B is open.

Question 18

Decide whether the following statements are true or false. If they're true, prove them. If they are false, provide counter examples

1. An open set that contains every rational number must necessarily contain all of $\mathbb{R}$.
2. Every nonempty open set contains a rational number.

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18(a)

An open set that contains every rational number must necessarily contain all of $\mathbb{R}$.

Solution: False

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- Let α be any irrational number.
- Then $A = (-\infty, \alpha) \cup (\alpha, \infty)$ is a open set containing all the rational numbers.

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An open set that contains every rational number must necessarily contain all of $\mathbb{R}$.

Solution: False

- Let α be any irrational number.
- Then $A = (-\infty, \alpha) \cup (\alpha, \infty)$ is a open set containing all the rational numbers.
- But A doesn't contain all of $\mathbb{R}$

Question 18 Contd...

18(b)

Every nonempty open set contains a rational number.

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Solution:

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Solution:

- Let A be a nonempty open set.
- Let $x \in A$. Then there exists $r > 0$ such that $N_r(x) \subseteq A$.

Question 18 Contd...

18(b)

Every nonempty open set contains a rational number.

Solution:

- Let A be a nonempty open set.
- Let $x \in A$. Then there exists $r > 0$ such that $N_r(x) \subseteq A$.
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Question 20

Decide whether the following sets are open or closed. Determine the interior

1. $\mathbb{Z} \subseteq \mathbb{R}$.
2. $\{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$

Solution: 1) $\mathbb{Z} \subseteq \mathbb{R}$.

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- Then $(a - r, a + r) \subset (k, k + 1) \subset \mathbb{R} \backslash \mathbb{Z}$.

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- Then $(a - r, a + r) \subset (k, k + 1) \subset \mathbb{R} \backslash \mathbb{Z}$.
- So $\mathbb{Z}$ is closed.

Question 20 Contd...

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