Open Methods :-

1. Fixed - point Iteration :-

Aim :- Find root of $f(x) = 0$

rewrite as $x = g(x)$

Eg :- (1) $f(x) = x^2 - 2x + 3 = 0$

$$x = \frac{x^2 + 3}{2}$$

(2) $f(x) = \sin(x) = 0$

$$\sin x + x - x = 0$$

$$x = \sin x + x$$

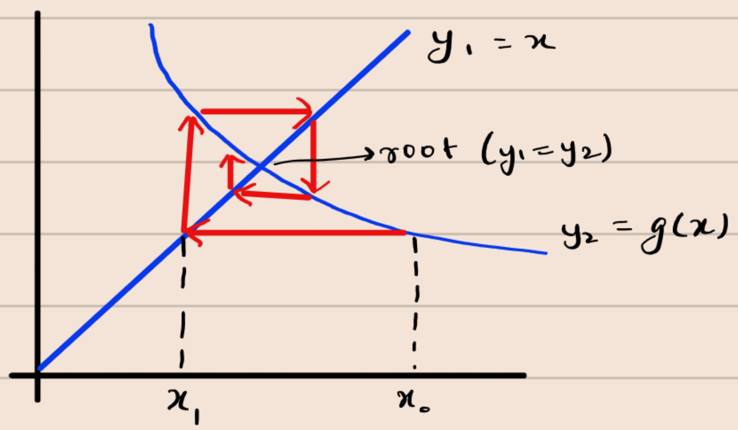
$$x_{i+1} = g(x_i)$$

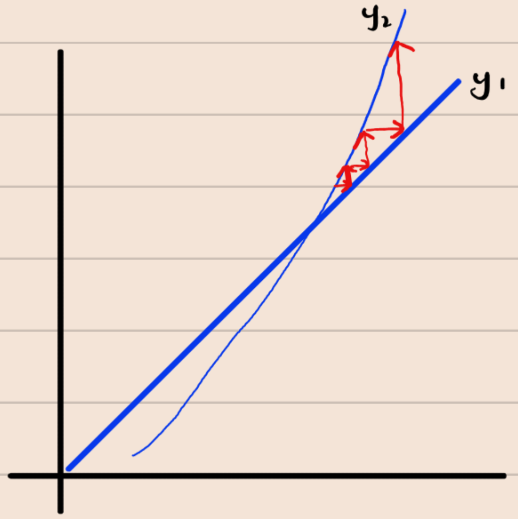
$$e_r = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\%$$

$x = g(x)$

$y_1 \qquad y_2$

$y_1 = x \qquad y_2 = g(x)$





does not converge!

→ Convergence happens when the slope of $y_2 = g(x)$ is less $y_1 = x$

i.e., $|g'(x)| < 1$

$x_{i+1} = g(x_i)$

$x_r = g(x_r)$ is the true solution

$x_r - x_{i+1} = g(x_r) - g(x_i)$

Theorem:– If $g(x)$ and $g'(x)$ are continuous over $[a,b]$ then $\exists\ \xi \in [a,b]$ such that

$$g'(\xi) = \frac{g(b) - g(a)}{b - a}$$

put $a = x_i$, $b = x_r$

$$g(x_r) - g(x_i) = (x_r - x_i)\, g'(\xi)$$

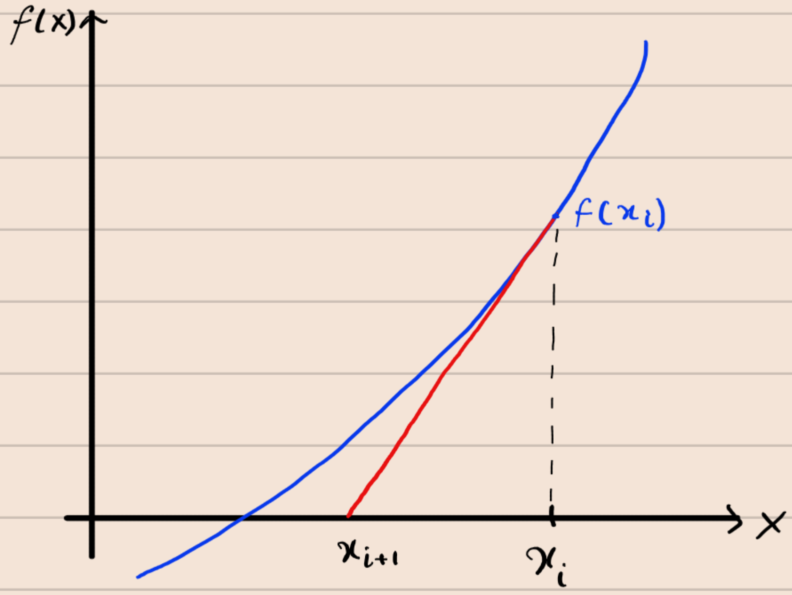
$$(x_r - x_{i+1}) = \underbrace{(x_r - x_i)}_{\text{True error } E_{t,i}}\, g'(\xi)$$

$$\therefore \quad E_{t,i+1} = E_{t,i}\, g'(\xi)$$

→ If $g'(\xi) < 1$, then error decreases otherwise grows.
converging ⇑ (decreases), diverging ⇑ (grows)

→ Linearly convergent ⟶ error in every iteration is proportional to (and less than) previous iteration's error.

Newton - Raphson



$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

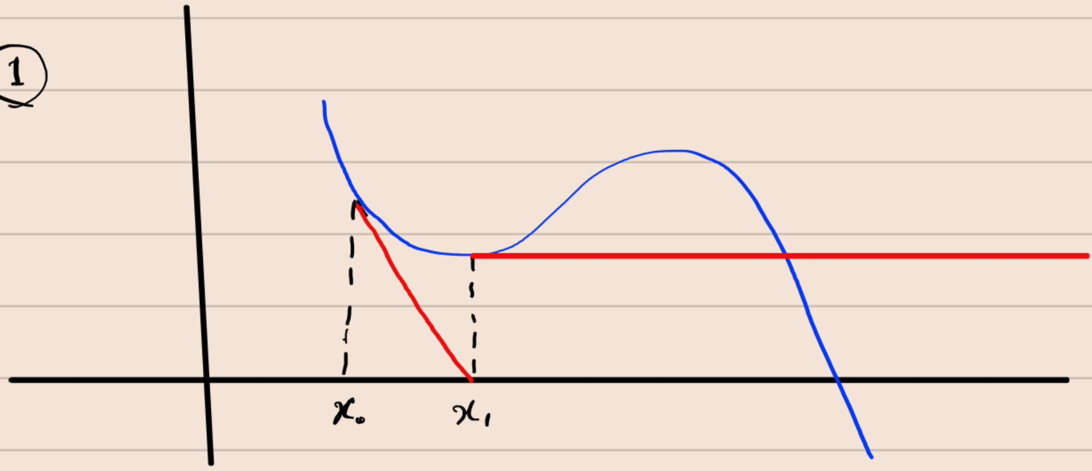
Newton Raphson Formula

Relative Error:- $E_r = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\%$

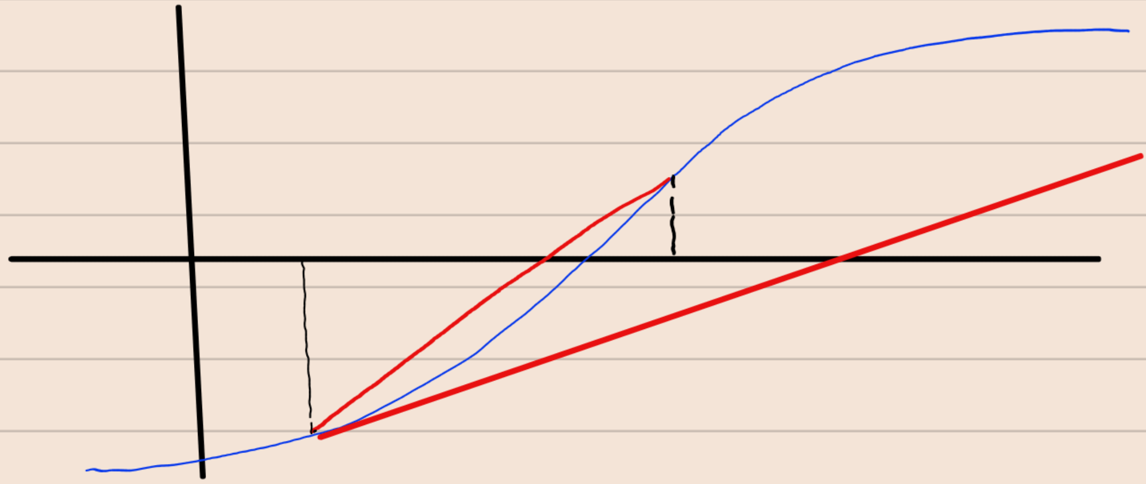
$$\boxed{E_{i+1} = O(E_i^2)}$$

Pitfalls of Newton-Raphson :-

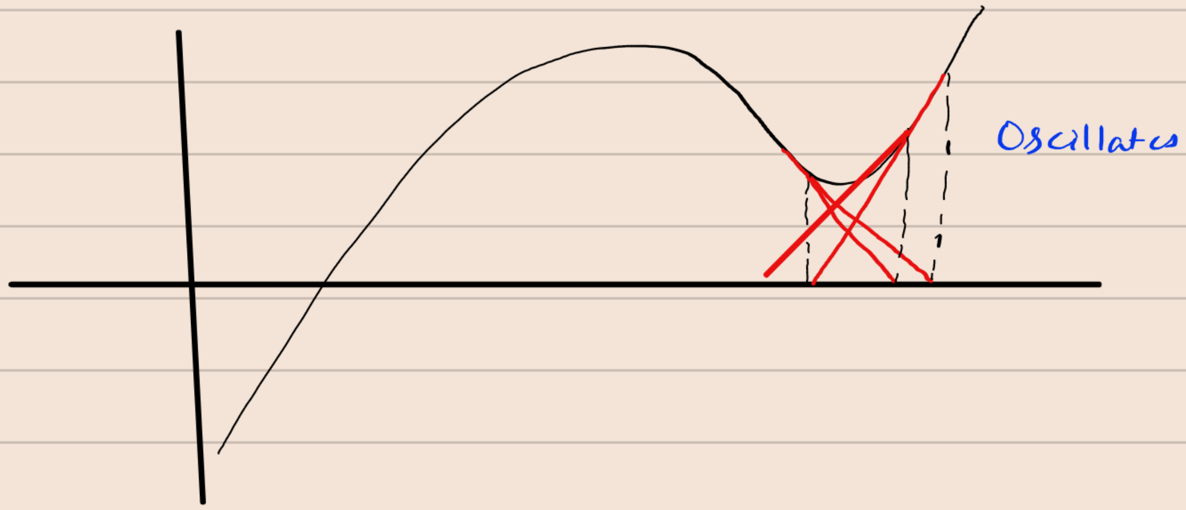
(1)



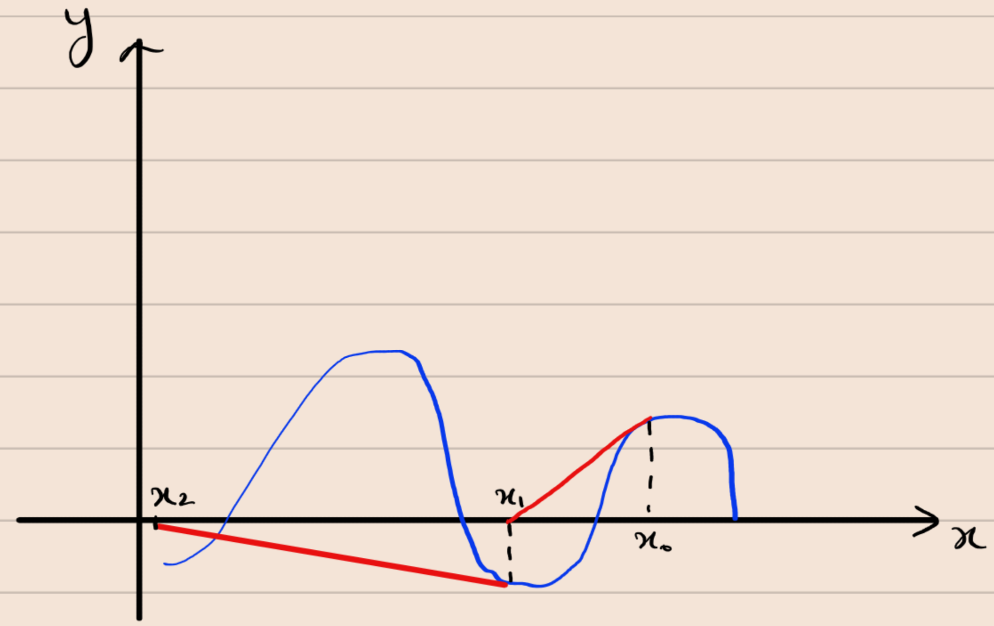
(2)



(3)



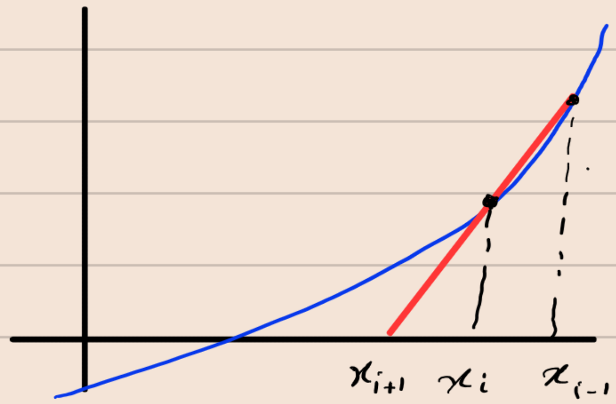
4



Secant Method :-

$$f'(x_i) \simeq \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)\,(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}}$$



Secant method → might not converge as bracketing property does not hold.