Lec - 22

MTL 122

Complex Analysis

field

$\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\}$ $\mathbb{R}^2$

+.

$(x, y) + (a, b) = (x + a, y + b)$

$(x, y) \cdot (a \cdot b) = (xa - yb, xb + ya)$

$(0, 1) \cdot (0, 1) = (-1, 0)$

$i^2 = -1$

$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$

$(x, 0)$

$(a, 0) \cdot (x, y) = (ax, ay)$

$(x, y) = (x, 0) + (0, y)$

$$= (x,0)\cdot(1,0) + \underbrace{(y,0)\cdot}_{(0,1)}$$

$$= x\cdot 1 + iy.$$

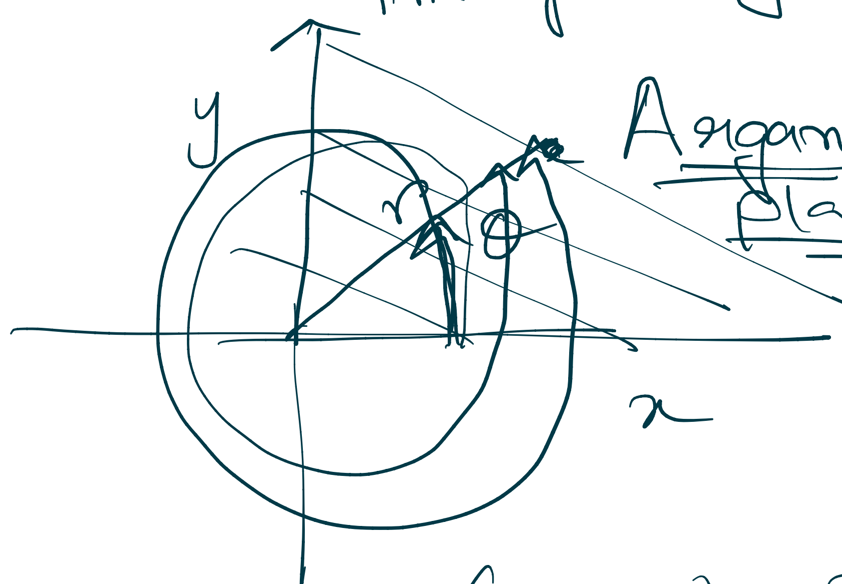
$$(1,0) \longrightarrow 1$$

$$(0,1) \longrightarrow i$$

$$i^2 = -1$$

$$x+iy \longrightarrow (x,y)$$

imaginary.



Argand plane.

"Usual"

$$(x,y) \in \mathbb{R}^2$$

$$\neq \; x+iy \in \mathbb{C}.$$

- $z = x + iy$

  $r = \sqrt{x^2 + y^2} = |z| \leftarrow$ Modulus.

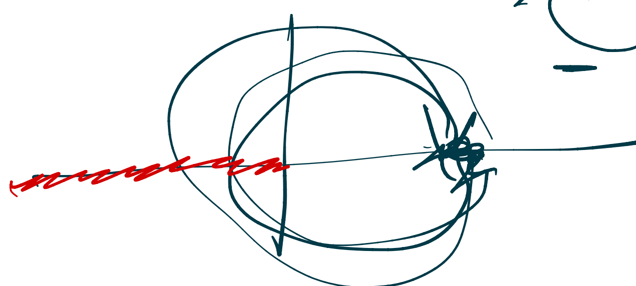
- Argument $\quad z = x + iy$

  $x = r\cos\theta \qquad \theta \in \mathbb{R}$

  $y = r\sin\theta$

$\arg z = \arctan\left(\frac{y}{x}\right)$

$1 = 1 + 0i$



$2\pi \qquad 4\pi \qquad -2\pi$

$2\pi * k, \qquad k \in \mathbb{Z}$

Questio

'0' argument.

$$\theta = \arctan\left(\frac{y}{x}\right)$$

not uniquely.

$$\theta + 2k\pi, \quad k = \pm 1, \pm 2.$$

Note: "quadrant in which z lies"

'Arg z' → principal value.

$$-\pi < \arg \leq \pi$$

$$\arg z = \{ \operatorname{Arg} z + 2\pi k,\ k \in \mathbb{Z} \}$$

$$z = x + iy$$

$$= r\cos\theta + ir\sin\theta$$

$$= r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Euler const.

- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- $z_2 \neq 0$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

$$\operatorname{Arg}(z_1 z_2)$$

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right)$$

?

$\mathbb{C} \longrightarrow$ field. $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$d : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$

$\underline{d(z_1, z_2)} = |z_1 - z_2|$

$= |z_2 - z_1|$

$e^\theta \longrightarrow \theta$ real.

$i^2 = -1$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} \cdots\right)$$

$= \cos\theta + i \, \cancel{\cos\theta} \, \sin\theta.$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

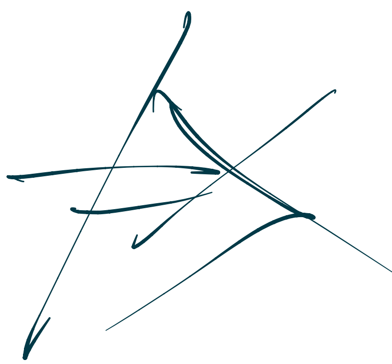
$$z = re^{i\theta}$$

- $-|z| \le \operatorname{Re} z \le |z|$
  $(\operatorname{Im} z)$.

- No order relation in Complex number

$x_1 \le x_2$

$y_1 \le y_2$



$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$\cancel{z_1 \le z_2}$

- Conjugate: $\overline{x+iy} = x - iy$

- $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
- $\overline{\overline{z}} = z$
- $|\overline{z}| = |z|$
- $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$

- $|z_1 + z_2| \leq |z_1| + |z_2|$

$e^x \quad e^y$

$$\underline{\underline{e^z}} = e^{x+iy} = e^x \underline{\underline{e}}^{iy}$$
$$= \underline{e}^x \underline{(\cos y + i \sin y)}$$

$$z = re^{i\theta}$$

$$z^n = r^n e^{in\theta}$$

$$= r^n(\cos n\theta + i\sin n\theta)$$

'n^{th} roots of z'?

$$\zeta = \sqrt[n]{z}$$

$\uparrow$

$\mathbb{C}$

$$\zeta = \rho e^{i\phi}$$

$$\zeta^n = z = re^{i\theta}$$

$$\rho^n e^{in\phi} = re^{i(\theta + 2k\pi)}$$

$$\rho^n = r \qquad \phi = \frac{1}{n}(\theta + 2k\pi)$$

$k = 0, 1, 2, \ldots, n-1$

$n \to n+1, n+2$

- $\sqrt[4]{1} \to$

Soln. :

$$1 = 1 \cdot e^{0}$$

$$r = 1 \quad \theta = 0$$

$$\zeta_k = e^{i(2k\pi)/4}$$

$$k = 0, 1, 2, 3$$

$$\zeta_0 = 1, \quad \zeta_1 = e^{i\pi/2}, \quad \zeta_2 = e^{i\pi}, \quad \zeta_3 = e^{i3\pi/2}$$

Functions of Complex variable.

$D \subseteq \mathbb{C}$ $\quad z \xrightarrow{f} w.$

z of D

$f(z) = w \in \mathbb{C}$

$f(z) = f(x+iy) = u(x,y) + iv(x,y)$

$f(z) = f(re^{i\theta})$

$= u(r,\theta) + iv(r,\theta)$

Ex. $\quad f(z) = \frac{1}{z^2}, \quad z \neq 0$

$u + iv$

$u(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2} \qquad v(x,y) = \frac{2xy}{(x^2+y^2)^2}.$

$u(r, \theta) = r^{-2} \cos 2\theta$ $\qquad v(r, \theta) = -r^{-2} \sin 2\theta$

$\mathbb{C} \setminus \{0\}$ . .

- $P(z) = a_0 + a_1 z + \cdots a_n z^n$.

  $a_i \in \mathbb{R}$.

- $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

  $\sin z$

- $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$

  $\sinh z = \dfrac{e^{z} - e^{-z}}{2}$.

Limit

- $f(z)$ $\quad z \to z_0$ $\quad \sqrt{x^2+y^2}$

w_0

$\epsilon > 0 \quad \exists\, \delta > 0$ s.t

$|z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon$

- $f(z) = u(x,y) + iv(x,y)$

$$\underset{(x,y)}{z} \to \underset{(x_0,y_0)}{z_0} \Rightarrow (x,y) \to (x_0,y_0)$$

$$f(z) \to w_0 = u_0 + iv_0$$

$\Downarrow \qquad \qquad \Uparrow$

$$u(x,y) \to u_0$$
$$v(x,y) \to v_0$$

Compute

$$\lim_{z \to 1+i} \left(z^2 + i \right)$$

$$f(z) = z^2 + i = \underbrace{\left(x^2 - y^2\right)}_{u(x,y)} + \underbrace{(2xy+1)}_{v(x,y)} i$$

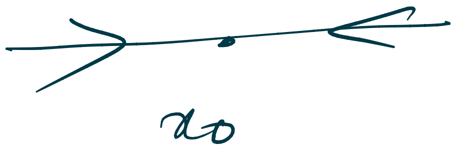
$$z_0 = 1 + i \qquad x_0 = 1 \qquad y_0 = 1$$

$$u_0 = \lim_{(x,y) \to (1,1)} u(x,y) = 0$$

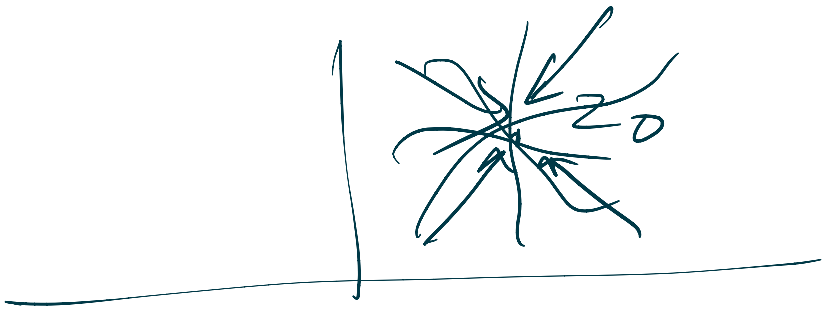
$$v_0 = 3$$

$$L = u_0 + i v_0 = 3i$$

$\mathbb{R}$



$\mathbb{C}$



$$\lim_{z \to z_0} f(z) = L$$

$$L_1 \neq L_2$$

Ex.

$$\lim_{z \to 0} \frac{z}{\bar{z}} = f(z)$$

$$z \to 0$$

$$z = x + 0i$$

$$\lim_{z \to 0} = \lim \frac{x + 0i}{x - 0i} = 1$$

$$z = 0 + iy$$

$$\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{y\to 0} \frac{0+iy}{0-iy}$$

$$= -1$$

$z \to z_0$

- $c f(z) \to cL$

$f(z) \to L$ as $z \to z_0$

- $f(z) \pm g(z) \longrightarrow L \pm M$

$g(z) \to M$ as $z \to z_0$

- $f(z) \cdot g(z) = L \cdot M$
- $\dfrac{f(z)}{g(z)} \longrightarrow \dfrac{L}{M}$ $\quad M \neq 0$

Continuity

$$\lim_{z \to z_0} f(z) = f(z_0)$$

- $\lim_{z \to z_0} f(z)$ exists
- f is defined z_0
- $\lim_{z \to z_0} f(z) = f(z_0)$

continuous

Ex. $f(z) = \dfrac{1}{1+z^2}$

discontinuous as $z = \pm i$

Quiz Till here.

$N_\epsilon \in \mathbb{N}$

$$|f_m(x) - f_n(x)| < \epsilon/2$$

$\forall m, n > N_\epsilon$

$\forall x \in$

- $n > N_\epsilon$

$m > N_\epsilon$

$$|f_m(x) - f(x)| < \epsilon/2 + |f_m(x) - f(x)| < \epsilon/2$$

$f_m(x) \to f(x)$ as $m \to \infty$

$m > N_\epsilon \to N(\epsilon, x)$

$m > N(\epsilon, x)$

$m_1 > m$