Assignment 1

Real and Complex Analysis

MTL122/ MTL503/ MTL506

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(1) Prove Theorem 1.1 in Lecture 1.

(2) Let A, B, C be sets, $f : A \to B$ and $g : B \to C$ be functions, and let $h : A \to C$ be defined by $h(x) = g(f(x))$ for $x \in A$.
State (give reasons/counterexamples) whether the following statements are true or false:
  a) If h is not injective, then at least one of the functions f and g is not injective.
  b) If h is not injective then both the function f and g is not injective.

(3) Let $f : A \to B$ be a function. Let $W \subseteq B$.
  a) Prove that $f(f^{-1}(W)) \subseteq W$.
  b) Prove that if f is surjective then $f(f^{-1}(W)) = W$.

(4) Consider the formula $f(x) = 2 - \sqrt{x + 4}$.
  a) What is the largest subset of $A \subseteq \mathbb{R}$ so that $f : A \to \mathbb{R}$ defined by $f(x) = 2 - \sqrt{x + 4}$ is a function?
  b) Compute the image of $f : A \to \mathbb{R}$.
  c) Compute $f([5, 12])$.
  d) Compute $f^{-1}([0, 2])$.

(5) Theorem 3.16 in Lecture 1.

(6) Are the following sets finite, countable or uncountable? Explain or prove your answer in each case.
  a) $\{(x, y) \in \mathbb{N} \times \mathbb{R} : xy = 1\}$
  b) $(\frac{1}{4}, \frac{3}{4})$

(7) Let $\mathbb{N}$ be the set of natural numbers. Prove that $\mathbb{N} \times \mathbb{N}$ is countable.

(8) Prove that supremum and infimum of a set is unique.

(9) Prove that for any two number $x, y \in \mathbb{R}$ such that $0 < x < y$, there are positive integers m, n such that $x < \frac{m^2}{n^2} < y$.

(10) Suppose that A, B are nonempty sets of real numbers such that $x \leq y$ for all $x \in A$ and $y \in B$. Then $\sup A \leq \inf B$.

(11) For each of the following sets S find $\sup\{S\}$ and $\inf\{S\}$ if they exist. You need to justify your answer.
a) $S = \{x \in \mathbb{R} : x^2 < 5\}$.
b) Let $A = \{1/n : n \in \mathbb{N} \text{ and } n \textit{ is prime}\}$.

(12) Let $\{a_n\}$ be a bounded sequence with the property that every convergent subsequence converges to the same limit a. Show that the entire sequence $\{a_n\}$ converges and $\lim_{n\to\infty} a_n = a$.

(13) Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers satisfying

$$|a_{n+1} - a_n| \leq \frac{1}{2}|a_n - a_{n-1}|.$$

Show that the sequence converges.

(14) If a sequence converges, then its limit is unique.

(15) Suppose that $0 < \alpha < 1$ and that (x_n) is a sequence which satisfies one of the following conditions
a) $|x_{n+1} - x_n| \leq \alpha^n$, $n = 1, 2, 3, ...$
b) $|x_{n+2} - x_{n+1}| \leq \alpha|x_{n+1} - x_n|$, $n = 1, 2, 3, ...$
Then prove that (x_n) satisfies the Cauchy criterion.
Note: Whenever you use this result, you have to show that the number α that you get, satisfies $0 < \alpha < 1$. The condition $|x_{n+2} - x_{n+1}| \leq |x_{n+1} - x_n|$ does not guarantee the convergence of (x_n). Give examples.

(16) For two sets S_1 and S_2 in $\mathbb{R}^n$, prove or disprove
a) $S_1 + S_2$ is open if both S_1 and S_2 are open;
b) $S_1 + S_2$ is closed if both S_1 and S_1 are closed;
c) $S_1 + S_2$ is bounded if both S_1 and S_2 are bounded.
Are the converses of these statements true? Prove or disprove their converses.

(17) Show that the following sets are open in $\mathbb{R}$.

$$A = \{x \in \mathbb{R} : x^3 > x\}, \ B = \{x \in \mathbb{R} : 0 < x < 1, \frac{1}{x} \notin \mathbb{Z}\}.$$

(18) Decide whether the following statements are true or false. If they're true, prove them. If they are false, provide counter examples.

a) An open set that contains every rational number must necessarily contain all of $\mathbb{R}$.
b) Every nonempty open set contains a rational number.

(19) If $A \subseteq R$ is a closed set bounded from above (below), show that A has a maximum(minimum).

(20) Decide whether the following sets are open or closed. Determine the interior
a) $\mathbb{Z} \in \mathbb{R}$
b) $\{(-1)^n + 1/n : n \in \mathbb{N} \setminus \{0\}\} \subset \mathbb{R}$