Lec - 26

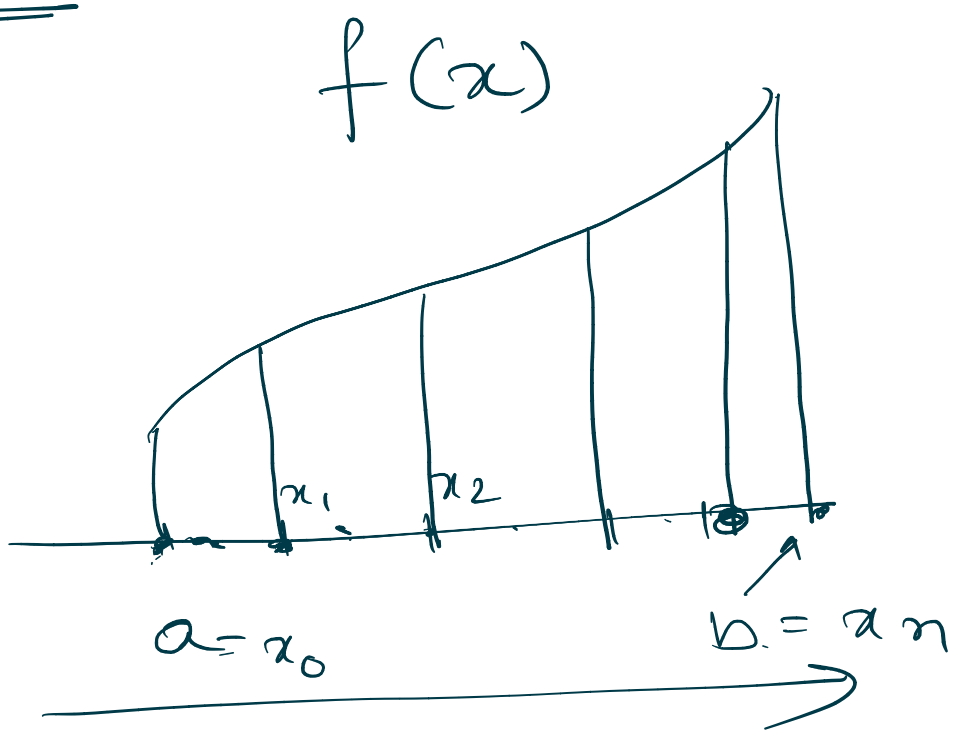
MTL - 122

Real and Complex Analysis

Contour Integration

$\mathbb{R}$

Recall — Riemann Int.



$$\Delta x = \frac{b-a}{n}.$$

$$\lim_{N \to \infty} \sum_{n=0}^{N} f(x) \Delta x = \int_a^b f(x)\,dx.$$

$$x_n = a + n \Delta x.$$

$$\Delta x = \frac{b-a}{n}$$

$\mathbb{C}$

$f(z)$

$$\boxed{\lim_{N\to\infty} \sum_{n=0}^{N} \Delta z \, f(z_n)}$$

$z_1, \; z_2, \; \ldots \; z_N \; \ldots$

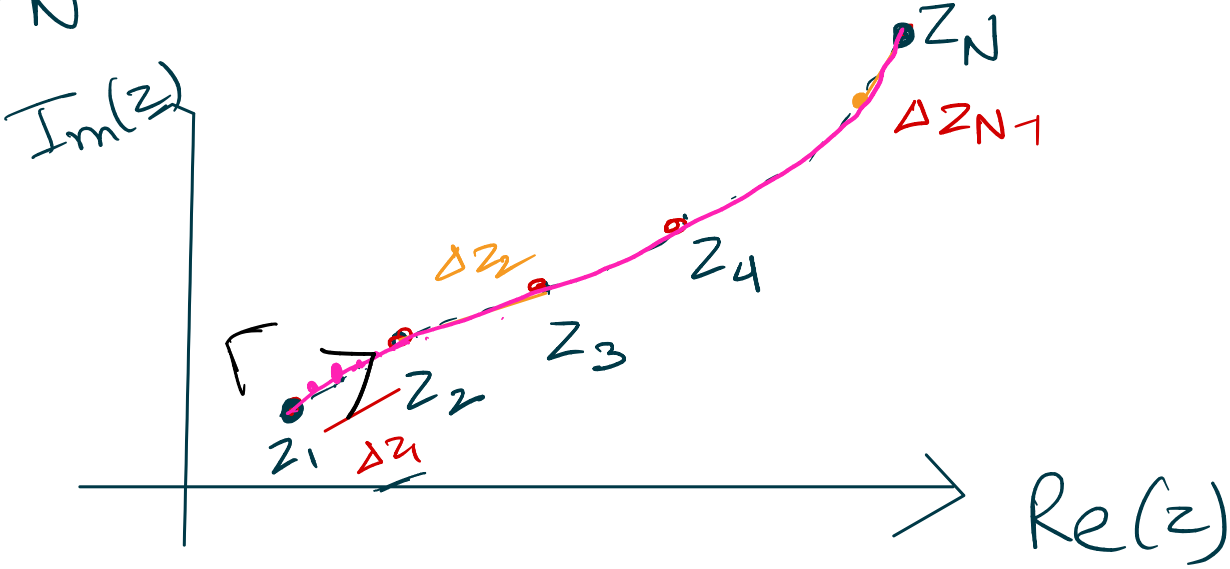
$\Delta z_1 \quad \Delta z_2 \quad \Delta z_{N-1}$

$$z_2 = z_1 + \Delta z_1$$

$$z_3 = z_2 + \Delta z_2$$

$$\vdots$$

$$z_N = z_{N-1} + \Delta z_{N-1}$$



$$\sum_{n=1}^{N-1} \Delta z_n \, f(z_n)$$

$a \longrightarrow b$

$\int_a^b$

$$= \Delta z_1 f(z_1) + \Delta z_2 f(z_2) + \cdots$$

$$+ \Delta z_{N-1} f(z_{N-1})$$

$N \to \infty$

$\Delta z_n \longrightarrow$ infinitesimal small

$\Gamma \to$ contour.

$$\boxed{\int_\Gamma f(z)\,dz = \lim_{N\to\infty} \sum_{n=1}^{N-1} \Delta z_n f(z_n)}$$

Integral along parametric curve.

$$\int_{\Gamma} f(z)\,dz$$

Trajectory

$z(t)$

$$\Gamma \equiv \{ z(t) \mid t_1 < t < t_2 \}$$

$t \in \mathbb{R}$, $z(t) \in \mathbb{C}$.

$$\begin{array}{ccc} t_1 & & t_2 \\ \downarrow & & \downarrow \\ z(t_1) & & z(t_2) \end{array}$$

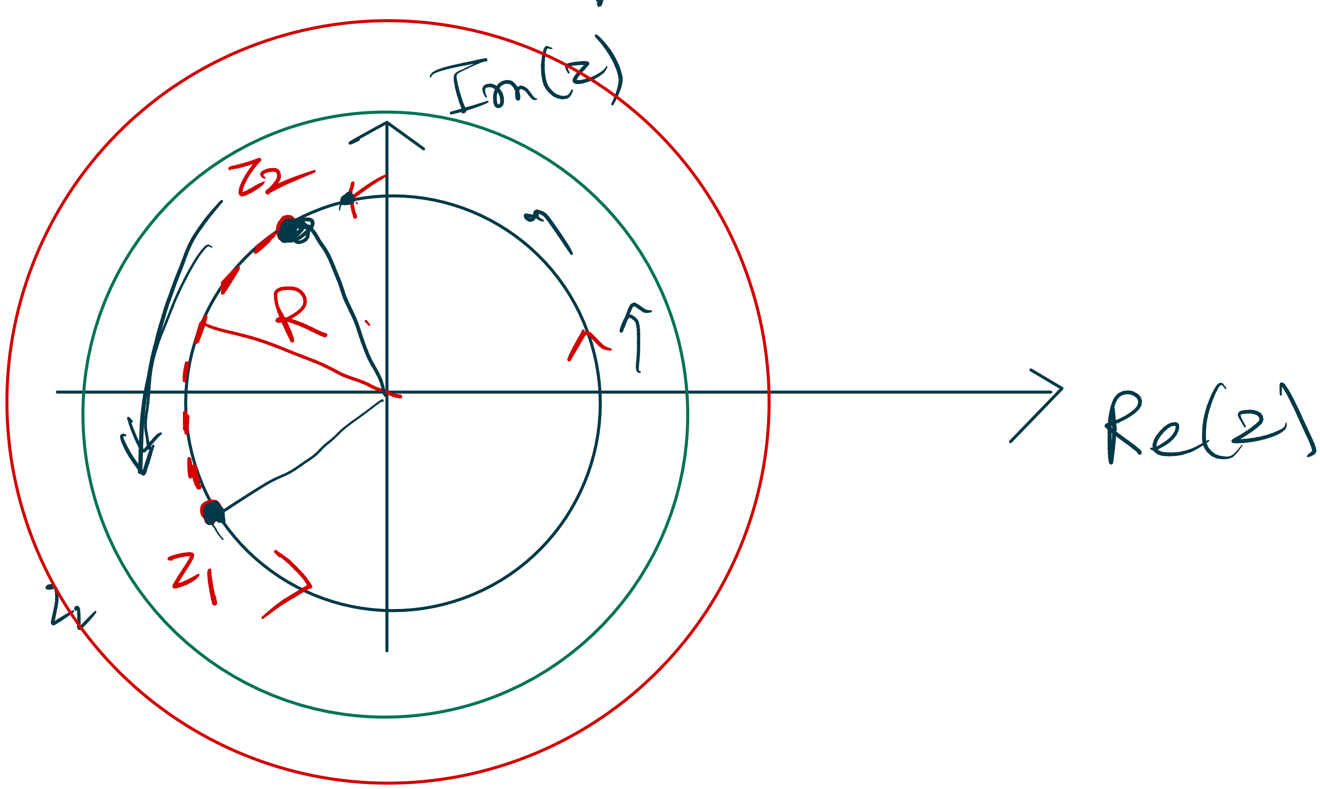
endpts of contour.

$$dz \to \frac{dz}{dt}\,dt.$$

$$\int_\Gamma f(z)\,dz = \int_{t_1}^{t_2} f(z(t))\,\frac{dz}{dt}\,dt$$

__Ex.__

$$\int_{\Gamma(\theta_1,\theta_2)} z^n\,dz \quad , \quad n \in \mathbb{Z}.$$



$\Gamma(\theta_1, \theta_2)$: counter clockwise arc of $R > 0$

$$z_1 = Re^{i\theta_1} \longrightarrow z_2 = Re^{i\theta_2}.$$

$z(\theta)$

$$\Gamma(\theta_1, \theta_2) = \{ z(\theta) \mid \theta_1 \leq \theta \leq \theta_2 \}$$

$$z(\theta) = Re^{i\theta}.$$

$$\int_{\Gamma(\theta_1, \theta_2)} z^n \, dz = \int_{\theta_1}^{\theta_2} z^n \frac{dz}{d\theta} \, d\theta$$

$$= \int_{\theta_1}^{\theta_2} R^n e^{in\theta} \left(iRe^{i\theta} \right) d\theta$$

$$= \int_{\theta_1}^{\theta_2} i R^{n+1} e^{i(n+1)\theta} \, d\theta .$$

$$= i R^{n+1} \int_{\theta_1}^{\theta_2} e^{i(n+1)\theta} \, d\theta \quad ⓐ .$$

$n \in \mathbb{Z}$

$n = -1$, $n \neq -1$.

$n = -1$.

$$\int_{\theta_1}^{\theta_2} d\theta = \theta_2 - \theta_1 \quad (1) .$$

$n \neq -1$

$$\int_{\theta_1}^{\theta_2} e^{i(n+1)\theta} \, d\theta = \frac{e^{i(n+1)\theta_2} - e^{i(n+1)\theta_1}}{i(n+1)} .$$

✓

$$\int_{\Gamma(\theta_1,\theta_2)} z^n \, dz = \begin{cases} i(\theta_2-\theta_1), & n=-1 \\ R^{n+1}\dfrac{e^{i(n+1)\theta_2}-e^{i(n+1)\theta_1}}{(n+1)} & n\neq -1 \end{cases}$$

$$\theta_2 = \theta_1 + 2\pi$$

$$\oint_{\Gamma} z^n \, dz = \begin{cases} 2\pi i, & n=-1 \\ 0, & n\neq -1. \end{cases}$$

$w(t) \in \mathbb{C}$; $t \in \mathbb{R}$.

$w(t) = u(t) + i\,v(t)$,

$w'(t) = u'(t) + i\,v'(t)$

- Mean value theorem for derivatives does not hold.

$f(z) = e^{z}$, $z = x + iy$.

$f'(c) \neq 0$

$z_2 = z_1 + 2\pi i$

$f(z_1) - f(z_2)$
$= 0$

Defn

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

- $$\operatorname{Re}\int_a^b w(t)\,dt = \int_a^b \operatorname{Re} w(t)\,dt$$

- $$\operatorname{Im}\int_a^b w(t)\,dt = \int_a^b \operatorname{Im}(w(t))\,dt.$$

- $w(t)$ piecewise continuous

  if both $u(t)$ & $v(t)$ are piecewise cont.

- $\int_a^b w(t)\,dt$ exists

1) $$\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt.$$

2) $$z_0 \int_a^b w(t) = \int_a^b z_0\, w(t)$$

3) $$\int_a^b (w_1 + w_2)(t)\, dt = \int_a^b w_1 + \int_a^b w_2$$

4) $$\int_a^b w(t)\, dt = -\int_b^a w(t)\, dt$$

5) $$w(t) = u(t) + i v(t)$$
$$W(t) = U(t) + i V(t)$$

a) $$W'(t) = w(t)$$
then $$U'(t) = u(t)$$
$$V'(t) = v(t)$$

b) $$\int_a^b w(t)\, dt = W(b) - W(a)$$

6) $$\left| \int_a^b w(t)\,dt \right| \leq \int_a^b |w(t)|\,dt$$

$a < b$ $$\int_a^b w(t)\,dt \neq 0.$$

$$\int_a^b w(t)\,dt = r_0 e^{i\theta_0}$$

$$r_0 = e^{-i\theta_0} \int_a^b w(t)\,dt$$

$$= \int_a^b e^{-i\theta_0} w(t)\,dt$$

$$= \operatorname{Re} \int_a^b e^{-i\theta_0} w(t)\,dt$$

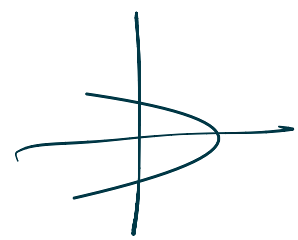
$$= \int_a^b \operatorname{Re}\left(e^{-i\theta_0} w(t)\right) dt$$

$$\operatorname{Re}\left(e^{-i\theta_0} w(t)\right) \leq \left|e^{-i\theta_0} w\right| = \underline{\underline{|w|}}$$

$$\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt$$

$a < b$.

Arc. / Curve



$\gamma : z = (x, y)$

$x = x(t), \quad y = y(t)$

$a \leq t \leq b.$

$x(t), y(t)$ are continuous fns of 't'

$$z(t) = x(t) + i\,y(t).$$

$$a \le t \le b.$$

$z(a) \to$ initial pt

$z(b) \to$ final pt / terminal pt.

Ex.

$$z = \begin{cases} x + ix, & 0 \le x \le 1 \\ x + i, & 1 \le x \le 2. \end{cases}$$

y

$x = x. \quad y = y(x)$

$z(x)$

Simple arc.

is a curve that is not crossing itself.

$$\boxed{\begin{array}{l} t_1 \neq t_2 \\ z(t_1) \neq z(t_2) \end{array}}$$

$$(x(t_1), y(t_1)) \neq (x(t_2), y(t_2))$$

$y = x^2$ ← simple curve?

$z(x) = (x, x^2)$

$y = 1$ $\quad x = 1$ $\quad x = -1$

$$(1, 1) \neq (-1, 1)$$