Lec - 12 - MTL 122

Real and complex Analysis

Sequential Characterization of closed sets.

(X, d)

Theo. $K \neq \emptyset$, $K \subseteq X$

$x \in X$.

a) $x \in \overline{K} \iff \exists$ a $(x_n) \subseteq K$ s.t $x_n \to x$ as $n \to \infty$

b) K is closed $\iff$ K contains the limits of every convergent seq.

Pf: $x \in \overline{K} = K \cup K'$

$\Rightarrow$ $x \in K$ or $x \in K'$.

If $x \in K$.

$(x_n) = (x) \subseteq K$ & $x_n \xrightarrow[n \to \infty]{} x$

$x \in K'$ (derived set).

For each $n \in \mathbb{N}$.

$\epsilon = 1/n$

$B(x, 1/n) \setminus \{x\} \cap K \neq \emptyset$.

(defn of limit pt).

$\Rightarrow \quad x_n \in K, \quad x_n \neq x$.

$x_n \in B(x, 1/n)$.

$d(x_n, x) < 1/n, \quad (x_n) \subset K$

as $n \to \infty$

$d(x_n, x) \to 0$

$\Rightarrow \quad x_n \to x \quad$ as $n \to \infty$.

Converse.

Assume $(x_n) \subset K$

$x_n \to x$ as $n \to \infty$.

$x \in K$

$x \notin K$.

Defn of conv. of seq

$\epsilon > 0$

$$B(x, \epsilon) \cap \{x_n\} \neq \emptyset \qquad x_n \neq x.$$

$\Rightarrow x \in K'$ $\Rightarrow x \in \overline{K}$

b) K is closed iff $K = \overline{K}$

(b) follows from (a).

— x —

Ex
$$\left((0, 1] , d_{Euc} \right)$$
$$\left\{ \frac{1}{n} \right\}$$

— x —

$1 \leq p < \infty$, l_p – complete.

$p = \infty$. ? $(\mathbb{N}, d_\infty)$

$(l^\infty(\mathbb{R}), d_\infty) \longrightarrow$ incomplete metric space.

$$d_\infty(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n|$$

Pf. $(x_n)_{n \geq 1} \in l^\infty(\mathbb{R})$.

$$\underline{\underline{x_n}} = \left(\xi_j^n\right)_{n\in\mathbb{N}}$$

$$\underline{\xi_j^n} = \begin{cases} \frac{1}{j}, & j \le n \\ 0, & \underline{j > n} \end{cases}$$

Then,

$$x_1 = \left(\xi_j^1\right)_{j\in\mathbb{N}}$$

$$= (1, 0, 0, 0, \ldots)$$

$$x_2 = \left(\xi_j^2\right)_{j\in\mathbb{N}}$$

$$= (1, 1/2, 0, 0, 0, \ldots)$$

$$x_3 = \left(\xi_j^3\right)_{j\in\mathbb{N}}$$

$$= (\underline{1, \tfrac{1}{2}, \tfrac{1}{3}}, 0, \ldots)$$

$$x_n = \left(\xi_j^n\right)_{j \in \mathbb{N}}$$

$$= \left(1, \underline{\tfrac{1}{2}, \tfrac{1}{3}}, \tfrac{1}{4}, \ldots, \tfrac{1}{n}, 0, 0, 0\right)$$

M = subspace consisting of $x = (\xi_j)$ with at finitely many non-zero terms

- $(M, d_\infty) \rightarrow$ incomplete.

$(x_n) \in M$.

- Show $(x_n)_{n \geq 1}$ is a C.S in M.

$\epsilon > 0 \quad N \in \mathbb{N},$

$N + 1 > 1/\epsilon$ (Archimedean prop).

Then $\quad m > n > N$

$d_\infty(x_m, x_n) = \sup_j |\xi_j^m - \xi_j^n|$

$= \frac{1}{n+1} < \frac{1}{N+1} < \epsilon$

$\Rightarrow (x_n)$ is a C.S.

$x_n \to x = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \ldots \frac{1}{n+1}, \frac{1}{n+2}, \ldots \ldots\right)$

x — non zero.

$x \notin M$?

$\Rightarrow$ M is incomplete.

ℓ^1

$\downarrow$

$L^1[0,1] \to \int_0^1 |f(x)|^2 dx < \infty$

$\mathbb{R}$

$\ell^2 \to d_2 = \sum |x_i - y_i|^2$

$L^2 \to d_2$

d_1

• $(C[0,2], d_1)$

$$d_1(f,g) = \int_0^2 |f(x) - g(x)| dx$$

is incomplete metric sp.

Seq $f_n \to f$.

Ex. $(\mathbb{Z}_{>0}, d_{inv})$ $\swarrow$

$d(m,n) = \left| \frac{1}{m} - \frac{1}{n} \right|$, $m, n \in \mathbb{Z}_{>0}$

— complete

→ incomplete ?

$(x_n)_{n \geq 1}$, $x_n = n$

$d(x_m, x_n) = \left| \frac{1}{m} - \frac{1}{n} \right|$

$\epsilon > 0$ $N > \frac{2}{\epsilon}$,

$d(x_m, x_n) < \epsilon$.

$\Rightarrow (x_n)$ in $\mathbb{Z}$.

Suppose $x_n \to x$ as $n \to \infty$

$\Leftrightarrow d(x_n, x) \to 0 \quad n \to \infty$

$\Rightarrow \frac{1}{x} = 0$ ↯.

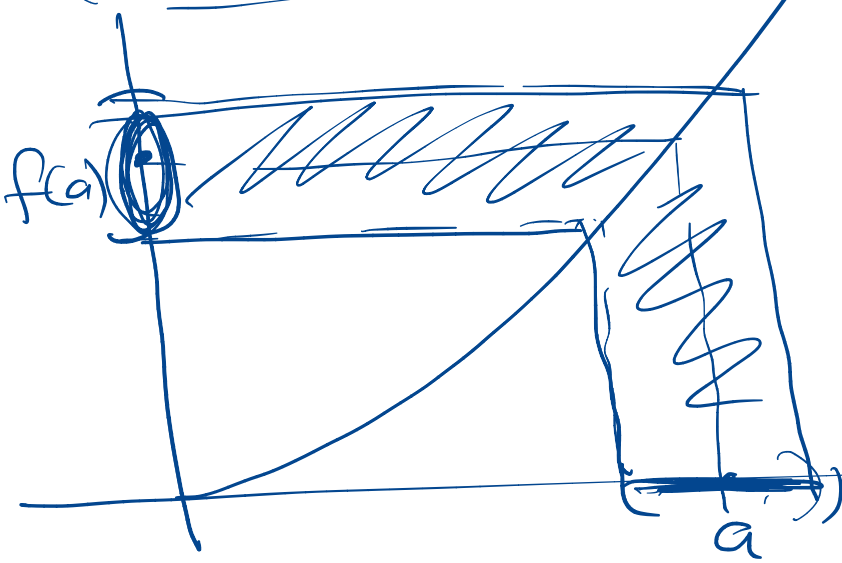
$\left|\frac{1}{n} - \frac{1}{x}\right| \to 0$

$\Rightarrow \frac{1}{x} = 0$ ↯

Continuity

$f : [0,1] \to \mathbb{R}$

Special



$f(U) \subseteq V$

$\epsilon \in \mathbb{R}_{>0}$ $\exists\, \delta \in \mathbb{R}_{>0}$

$x \in [0,1]$ $|x - a| < \delta$

$\Rightarrow |f(x) - f(a)| < \epsilon$

• open int. V centered at $f(a)$ $\exists$ U centered at a s.t

$$\boxed{f(U \cap [0,1]) \subseteq V}$$

Defn: (X, d_X) (Y, d_Y)

$f: X \to Y$ $f: \mathbb{R} \to \mathbb{R}$

continuous at a pt 'a' $\in X$

if for every $\epsilon > 0$

$\exists \delta > 0$ s.t

$d_X(x, a) < \delta \Rightarrow$

$d_Y(f(x), f(a)) < \epsilon$.

$$\Rightarrow \quad f(B_\delta(a)) \subseteq B_\epsilon(f(a))$$

for any ϵ $\exists\, \delta > 0$

Let $y \in f(B_\delta(a))$

$\exists\, x \in B_\delta(a)$

$f(x) = y$

$d_X(x, a) < \delta$

$\Rightarrow \quad d_Y(f(x), f(a)) < \epsilon$

$\Rightarrow \quad f(x) \in B_\epsilon(f(a))$

$\Rightarrow \quad y \in B_\epsilon(f(a))$

$\Leftrightarrow \quad f(B_\delta(a)) \subseteq B_\epsilon(f(a))$

$\Leftarrow$ One line

$\epsilon - \delta \iff f(B_\delta(a)) \subseteq B_\epsilon(f(a))$

'a'

$\epsilon > 0 \rightarrow \delta(\epsilon, a)$

$A \subseteq X$.

Defn. $f : A \to Y$ is cont. at $c \in A \subset X$. if $\forall\, V$ of $f(c)$ $\exists\, U$ of c s.t.

$$x \in U \cap A \Rightarrow f(x) \in V.$$

Note: Let 'c' be a isolated pt of A.

$\delta > 0 \quad x \in A.$

$d_X(x, c) < \delta$, is $x = c$.

$d_Y(f(x), f(c)) = 0 < \epsilon$

- A fn is continuous at every isolated pt of its domain.

- f is Lipschitz continuous. $C \in \mathbb{R}$

$$d_Y(f(x), f(y)) \leq C d_X(x, y)$$

$$\left[\ |f(x) - f(y)| \leq K |x - y|\ \right]$$

$\delta = \epsilon / K.$

- Lip cont $\Rightarrow$ continuity

Converse true?

$f : [0, \infty) \to \mathbb{R}$

$f(x) = \sqrt{x}$. is continuous

but not Lip.

- f continuity.

- $0 < c < \infty$.

$x = 0 \quad y > 0$.

$c\sqrt{y} < 1$. (A.P)

$c|y - x| = c|y - 0|$

$= cy < \sqrt{y} = |\sqrt{y} - \sqrt{x}|$

$= |f(y) - f(x)|$

- $\cancel{x}$ $f : X \to \mathbb{R}$.

  $f(x) = d(x, A)$ $A \neq \emptyset$, $A \subseteq X$.

- $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}$

  $\pi(x) = x_k$

  $|x_k - y_k| \leq d_{Euc}(x, y)$.

  $x, y \in \mathbb{R}^n$.

  Lip cont.

- (X, d_{dis}), (Y, d_{dis})

  $\left. f : X \to Y \right\}$ continuous.

  d

Ex: $X = \mathbb{R} = Y$

$f : \mathbb{R} \to \mathbb{R}$.

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q}. \end{cases}$$

d_{Euc}. d_{dis}.

- $f : (X, d_{Euc}) \to (Y, d_{Euc})$

  is not cont. (MTL100)

- $f : (X, d_{Euc}) \to (Y, d_{dis})$

  $\to$ not cont. (check)

- $f : (X, d_{dis}) \to (Y, d_{Euc})$

  $\to$ continuous.

- $f : (X, d_{dis}) \to (Y, d_{dis})$

$\to$ cont.

- $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$

$f+g$, $f \cdot g$, cf

- Sequential Criteria.

(X, d_X). (Y, d_Y)

$f : X \to Y$ $\to$ sequentially continuous

at $a \in X$.

$x_n \to a$ in X. $\implies$ $f(x_n) \to f(a)$ in Y