MTL732: Financial Mathematics
Practice Sheet: Portfolio Optimization; Answers and Hints

1. Suppose there are three financial market scenarios $\Omega = \{w_1, w_2, w_3\}$ with different probabilities of occurrence. Consider the following table showing the returns on two different stocks in these three scenarios

scenario	prob	return k_1%	return k_2%
w_1	$0 \cdot 2$	-10	-30
w_2	$0 \cdot 5$	0	20
w_3	$0 \cdot 3$	20	15

   (a) What is the expected returns on the stocks?
   (b) Suppose 60% of the available fund is invested in stock 1 and the remaining is invested in stock 2, then what is the expected return of the portfolio?
   (c) Compute the weights if the expected return on a portfolio is 20%.

   Ans: 1%, 19%; 8.2%; (5%, 95%)

2. Consider the following data

scenario	prob	return k_1%	return k_2%
w_1	$0 \cdot 4$	-10	20
w_2	$0 \cdot 2$	0	20
w_3	$0 \cdot 4$	20	10

   Suppose a portfolio comprises of 40% of total investment in stock 1 and 60% in stock 2. Compare the risk of the portfolio with the risks of its individual components. What will be the risk situation if a portfolio is designed with investment of 80% in stock 1 and the remaining in stock 2.

   Ans: $\sigma_1^2 = 0.0184, \sigma_2^2 = 0.0024, \sigma_V^2 = 0.000736; \sigma_V^2 = 0.009824$

3. Prove that if short sales are not allowed then the risk of the portfolio can not exceed the greater of the risks of the individual components of the portfolio.

4. Let a portfolio be designed with investment of 50% in stock 1 and the remaining 50% in stock 2. Further let short sale be allowed in stock 1 and all the other data being the same as in Q 1. Does the conclusion of Q 3 hold?

   Ans: No. $\sigma_V^2 = 0.0196 > \sigma_1^2, \sigma_2^2$

5. Suppose the portfolios are constructed using three securities $a_1,\ a_2,\ a_3$ with expected returns, $\mu_1 = 20\%,\ \mu_2 = 13\%,\ \mu_3 = 4\%$, standard deviations of returns, $\sigma_1 = 25\%,\ \sigma_2 = 28\%,\ \sigma_3 = 20\%$, and the correlation between returns, $\rho_{12} = 0 \cdot 3,\ \rho_{13} = 0 \cdot 15$ and $\rho_{23} = 0 \cdot 4$. Among all the attainable portfolios, find the one with minimum variance. What are the weights of the three securities in this portfolio? Also compute the expected return and standard deviation of this portfolio.

   Ans: $w^T = \frac{e^T C^{-1}}{e^T C^{-1} e} = (0.314, 0.148, 0.538); \mu = m^T w = 0.173,\ \sigma = (w^T C w)^{0.5}$.

6. Among all attainable portfolios with expected return 20% constructed using the data provided in Exercise 5, find the portfolio with minimum variance. Compute the weights of individual assets in this portfolio.

   Ans: $w = (0.672, -0.246, 0.574)$.

7. Consider the following data

	μ	σ
asset 1	10%	5%
asset 2	8%	2%

   For each correlation coefficient $\rho = -1,\ -0 \cdot 5,\ 0,\ 0 \cdot 5,\ 1$, what is the combination of the two assets that yields the minimum standard deviation and what is the minimum value of the standard deviation?

   Hint: $\rho = 1 \Rightarrow w_1 = \frac{-\sigma_2}{\sigma_1 - \sigma_2},\ w_2 = \frac{\sigma_1}{\sigma_1 - \sigma_2}$. $\rho = -1 \Rightarrow w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2},\ w_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}$.

   $-1 < \rho < 1 \Rightarrow w_1 = 1 - w_2,\ w_2 = \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$.

8. Compute the minimum risk portfolio for the following rate return (%) data:

	Jan	Feb	Mar	Apr	May	June
asset 1	12	10	5	7	15	12
asset 2	7	12	10	10	12	15

   Also compute the expected return for the optimal portfolio.

   Hint: $\mu_1 = 10.167\%, \mu_2 = 11\%$. Find the two standard deviations and covariance (by formulas only) and write variance-covariance matrix C. Use same formulas as in Q7 to compute w and thereafter mu, σ of portfolio.

9. Consider three risky assets with the covariance matrix and expected returns (all data in %) as follows.

variance	- covariance	matrix(C)	return(M)
10	4	0	5
4	12	6	6
0	6	10	1

Find two portfolios yielding the minimum variance. Also, determine the expected returns from these two portfolios. Using the two fund theorem, construct the portfolio giving the return of $2 \cdot 8\%$ with minimum risk.

Hint: Solve $\sum_{j=1}^{3} \sigma_{ij} v_j = 1,\ i = 1, 2, 3$. Normalize the solution vector v to get w. Then solve the system $\sum_{j=1}^{3} \sigma_{ij} v_j = m_i,\ i = 1, 2, 3$. Normalize this to get say w^1. Then compute $\mu = m^T w, \mu_1 = m^T w^1$. And compute $\sigma = (w^T C w)^{0.5}, \sigma_1 = ((w^1)^T C w^1)^{0.5}$, to get two portfolios (σ, μ) and (σ_1, μ_1) on efficient frontier. After that use two fund theorem.

10. Suppose an investor is interested in constructing a portfolio with one risk-free asset a_1, with risk-free return 6%, and three risky assets a_2, a_3, a_4 with expected returns 10%, 12%, 18%, respectively. Given the covariance matrix of the three assets (data in %) as

$$C = \begin{pmatrix} 4 & 20 & 40 \\ 20 & 10 & 70 \\ 40 & 70 & 14 \end{pmatrix},$$

what is the optimum portfolio for the investor? What is the expected return of this portfolio?

Hint: Use $m = (0.1, 0.12, 0.18)^T$ and $w_M = \frac{C^{-1}(m - \mu_{rf} e)}{e^T C^{-1}(m - \mu_{rf} e)}$. Then find $\mu_M = m^T w_M$, $\sigma_M = (w_M^T C w_M)^{0.5}$. Find CML.

11. Consider the data of two risky assets a_1, a_2 with $\mu_1 = 12 \cdot 5\%$, $\mu_2 = 10 \cdot 5\%$, $\sigma_1 = 14 \cdot 9\%$, $\sigma_2 = 14\%$, $\rho = 0 \cdot 33$.

(a) Is it advisable to diversify the investment? If so then what composition of the assets will minimize the risk?

(b) What is the minimum value of the risk?

(c) If the risk-free rate of return is 5% then derive the equation of the capital market line?

Hint: Find matrix C. Compute $w = \frac{e^T C^{-1}}{e^T C^{-1} e}$, $\sigma = (w^T C w)^{0.5}$. For part (c), proceed as in Q10.

12. Given the following information about the one risk-free asset and three risky assets, find the expected return and standard deviation of the market portfolio. Also determine the equation of the capital market line.

$$\mu_{rf} = 5\%,\ \mu_1 = 14\%,\ \mu_2 = 8\%,\ \mu_3 = 20\%;$$
$$\sigma_1 = 6\%,\ \sigma_2 = 3\%,\ \sigma_3 = 15\%;\ \ \sigma_{12} = 0 \cdot 5,\ \sigma_{13} = 0 \cdot 2,\ \sigma_{23} = 0 \cdot 4.$$

Hint: Find C. Compute $w_M = \frac{C^{-1}(m - \mu_{rf} e)}{e^T C^{-1}(m - \mu_{rf} e)}$. Then find $\mu_M = m^T w_M$ and $\sigma_M = (w_M^T C w_M)^{0.5}$. Then substitute them in CML to get final equation.

13. Assume that the following assets are correctly priced according to the security market line. Derive the security market line. What is the expected return on an asset with $\beta = 2$?

$$\mu_1 = 6\%,\ \ \beta_1 = 0 \cdot 5;\quad \mu_2 = 12\%,\ \ \beta_2 = 1 \cdot 5.$$

Sol: $\mu_{rf} = 0.03, \mu_M = 0.09$; Write equation of SML. Use it to get, $\mu_{asset} = 0.15$.

14. If the following two assets are correctly priced according to the security market line, what is the return of the market portfolio? What is the risk-free return?

$$\mu_1 = 9 \cdot 5\%,\ \ \beta_1 = 0 \cdot 8;\quad \mu_2 = 13 \cdot 5\%,\ \ \beta_2 = 1 \cdot 3.$$

Hint: same approach as in Q13.

15. Let the expected rate of return on the market portfolio be 23% risk free return be 7%. Also let the standard deviation be 32% and let us assume that the market is efficient.

(a) What is the evaluation of capital market line?

(b) If Rs 300 is invested in the risk free asset, and Rs 700 in the market portfolio then what is the expected return at the end of the year?

(c) If an investor has Rs 1000 to invest and he/she desires a return of 39%, then what should be his/her portfolio.

Sol: $\mu = 0.07 + 0.5\sigma; 18.2\%$; Borrow, 1000 more and invest 2000 in market portfolio.