MTL122 - Real and complex analysis

Assignment-4

Department of Mathematics
Indian Institute of Technology Delhi

Question 1

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Let (E, d) be a metric space, and let $f, g : (E, d) \to (\mathbb{R}, \text{usual metric})$ be bounded and uniformly continuous functions. Show that the product $f \cdot g : (E, d) \to (\mathbb{R}, \text{usual metric})$ is bounded and uniformly continuous.

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- Now for all $s, t \in E$ and $d(s,t) < \delta = \min(\delta_1, \delta_2)$, then

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- Hence $f \cdot g$ is uniformly continuous.

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- Show that if $f : (0, 1) \to \mathbb{R}$ is uniformly continuous, then it is bounded.
- Give an example of a function $f : (0, 1) \to \mathbb{R}$ that is continuous but unbounded.

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Solution: (a)

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- $\exists\ x_1, x_2, ..., x_n$ such that $(0, 1) \subset \bigcup\limits_{i=1}^{n} N_{\delta_\epsilon}(x_i)$ and $x_i \in (0, 1)$ for all $i = 1, 2, ..., n$. (Why?)

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- $\forall\ x \in (0,1), \exists\ x_j$ such that $x \in N_{\delta_\epsilon}(x_j)$.
- Thus for $M = max\{|f(x_1)|, |f(x_2)|, ..., |f(x_n)|\}$, we have

$$|f(x)| = |f(x) - f(x_j) + f(x_j)| \leq 1 + |f(x_j)| \leq 1 + M$$

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- (b) The obvious answer to this question is $f(x) = 1/x$ (Why?)

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If the metric space (X, d) is compact and an open cover of X is given, then there exists a number $\delta > 0$ such that every subset of X having diameter less than δ is contained in some member of the cover.

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$$d_1(x, y) < \eta \implies d_1(x, y) < \delta_a \implies d_2(f(x), f(y)) < \epsilon.$$

- Hence f is uniformly continuous.

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- Since $B \subset A$, we have $B \subset \bigcup_{i=1}^{N} B_{\epsilon/2}(a_i)$.
- Let $b_i \in B_{\epsilon/2}(a_i) \cap B$ whenever $B_{\epsilon/2}(a_i) \cap B$ is non empty.
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- Thus the needful is proved. (Why?)

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- Let $(x_n)_{n>0}$ be a sequence in $A \cap B$. Thus $(x_n)_{n>0} \subset A$ and $(x_n)_{n>0} \subset B$.
- Thus we have $(x_{n_k})_{k>0} \to a \in A$.
- Now since $(x_{n_k})_{k>0} \subset B$ we have $x_{n_{k_j}} \to b \in B$.

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- Since $x_{n_{k_j}} \subset A$ and sequentially compact implies closed we have $b \in A$ and thus $b \in A \cap B$.
- Thus we have proven every sequence of $A \cap B$ has a convergent subsequence and hence proved.

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- Take $X = (\mathbb{R}, d_{dis})$

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Question 9

Let (X, d) be a metric space.

(a) Show that if A is a totally bounded subset of (X, d), then $\bar{A}$ is also totally bounded.

(b) Use (a) to show that if (X, d) is complete and A is a totally bounded subset of (X, d), then $\bar{A}$ is compact.

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Question 9(b)

Theorem

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(2) *X is sequentially compact.*

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Let A be a non-empty subset of a metric space (X, d). Recall that the distance of a point $x \in X$ to a set A is defined by

$$d(x, A) := \inf\{d(x, y) : y \in A\}$$

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