Pitfalls of Gaussian Elimination:-

1. Division by zero

$$x_2 + x_3 = 13$$
$$5x_1 + 3x_2 + 2x_3 = 4$$
$$6x_1 + 2x_2 + 8x_3 = 9$$

Solution : Pivoting → exchange the rows so that the pivot elements is non-zero

2. Round off errors

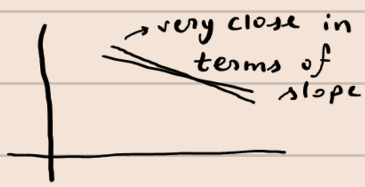
computers have a finite precision. Elimination has many divisions → has to be rounded off
In substitution → round off errors can propagate

Solution (sort of) :- use more significant figures.
Always check the solution by substituting in original equations.

3. 122 ? - conditioned systems

$$\left.\begin{aligned} x_1 + 2x_2 &= 10 \\ 1.1x_1 + 2x_2 &= 10.4 \end{aligned}\right\}$$ solution $x_1 = 4$, $x_2 = 3$

$$1.05x_1 + 2x_2 = 10.4$$ Solution



If we change coefficient slightly, solution changes a lot $x_1 = 8$, $x_2 = 1$

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_2 = -\frac{a_{11}}{a_{12}}x_1 + \frac{b_1}{a_{12}}$$

$$x_2 = -\frac{a_{21}}{a_{22}}x_1 + \frac{b_2}{a_{22}}$$

$$-\frac{a_{11}}{a_{12}} \cong -\frac{a_{21}}{a_{22}}$$

$$\underbrace{a_{11}a_{22} - a_{12}a_{21}}_{\text{Determinant}} \cong 0$$

Multiply by constant

→ system still remains ill-conditioned but determinant changes.

4. Singular Systems Rows are LI

determinant is zero

Output of Gaussian elimination - upper triangular matrix

Determinant of upper triangular matrix - product of diagonals

If during the elimination process, any diagonal entry is 0 → output that system is singular.

Improving Solutions :-

① Use more significant figures

② Pivoting we don't want to divide by numbers close to zero entries

look for the largest element in the column →
switch that row with the current one to make the largest element the pivot

partial pivoting

Full pivoting

Look for the largest element in the matrix
make that the pivot

not used in practice

③ Scaling Required when some equations have much larger coefficients than others

$$2x_1 + 100000\,x_2 = 100002$$
$$x_1 + x_2 = 2$$

? Correct : $x_1 = 1.00002$ $\quad x_1 = x_2 = 1$

$x_2 = 0.99998$

Scaling → look at a column, normalise all the entries to maximum of 1.

Gauss - Jordan :

Instead of eliminating unknown from all subsequent equations, eliminate from all (including previous)
Also normalise so that leading coefficients is 1.

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ & & \cdots & \\ \cdots & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1' \\ b_2' \\ \vdots \\ b_n' \end{bmatrix}$$

$$a_{ii} = 1 \quad \forall i \in [n]$$

operations $\longrightarrow n^3 + O(n^2)$

$n = m + 1$

n = no. of columns

m = no. of rows

```
Gauss-Jordan (arg, n, m, x)
for  k = 1 to m
        d = arg(k,k)
        for j=1 to n
            arg(k,j) = arg(k,j) / d
        end for
        for i=1 to m
            if  i ≠ k
                d = arg(i,k)
                for j= k to n
                    arg(i,j) = arg(i,j) - d * arg(k,j)
                end for
            end if
        end for
end for
```

Exactly as gaussian

?

```
for k=1 to m
      (k,k) = arg(k,n)
end for
```