Lec 10 - MTL 122 :

Real and Complex Analysis.

$A \subseteq \mathbb{R}$.

A - is bdd. $\Updownarrow$

$\forall x \in A$

$|x| \leq M$.

$\Leftrightarrow \quad x \in (-M, M)$

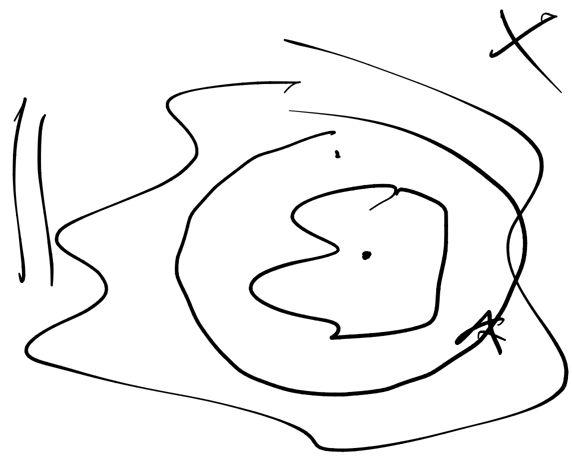
$\Rightarrow \quad A \subseteq (-M, M)$ //

(X, d)

$A \subseteq X$ is bounded

if $A \subseteq B(x, r)$,

$x \in X$, $r > 0$



- A is bdd $\Leftrightarrow M > 0$

$d(x, y) \leq M$, $\forall x, y \in A$

$B(a, M)$

$(\mathbb{R}, d)$

$\left(1 + \frac{1}{n}\right)^n$

$\operatorname{diam}(A) = \sup\{ \underline{d(x,y) \mid x, y \in A} \}$ $A \subseteq X$,

= diameter of A.

A is bdd $\iff \operatorname{diam}(A) < \infty$.

- $(\underline{X}, \underline{d_{dis}})$
- Any subset of a discrete metric is bdd.

$d(x,y) \in \mathbb{R}^+ \cup \{0\}$

<u>Prop</u> Finite union of bdd sets is bdd.

<u>Pf.</u> $\{ \underline{U_i} \mid i \in \{1, 2, \ldots, n\} \}$

of bdd. subsets of X.

$\Rightarrow$ For each 'i' $\exists\, r_i$

$$d(x,y) \le r_i \quad \forall\, x, y \in U_i$$

$$r = \max\{r_1, \ldots, r_n\}$$

$$U = \bigcup_{i=1} U_i$$

[U_i bdd $\Rightarrow \exists\, R_i$

$U_i \subseteq B(x, R_i)$

$x \in X$]

$\Updownarrow$

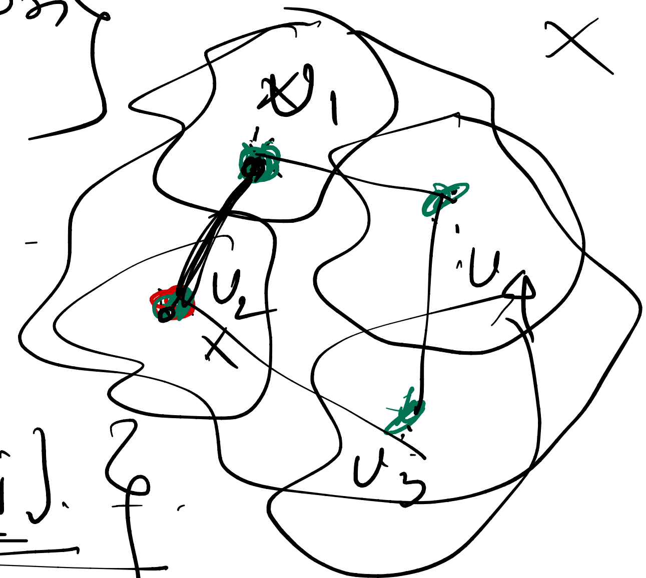
[$\exists\, r_i$, $d(x,y) < r_i$

$\forall\, x, y \in U_i$]

collection

$U_i \to \{\text{finite}\}$

$x_i \in U_i$, $i = 1, 2, \dots n$



$S = \max\{d(x_i, x_j)\}$

$x, y \in U$

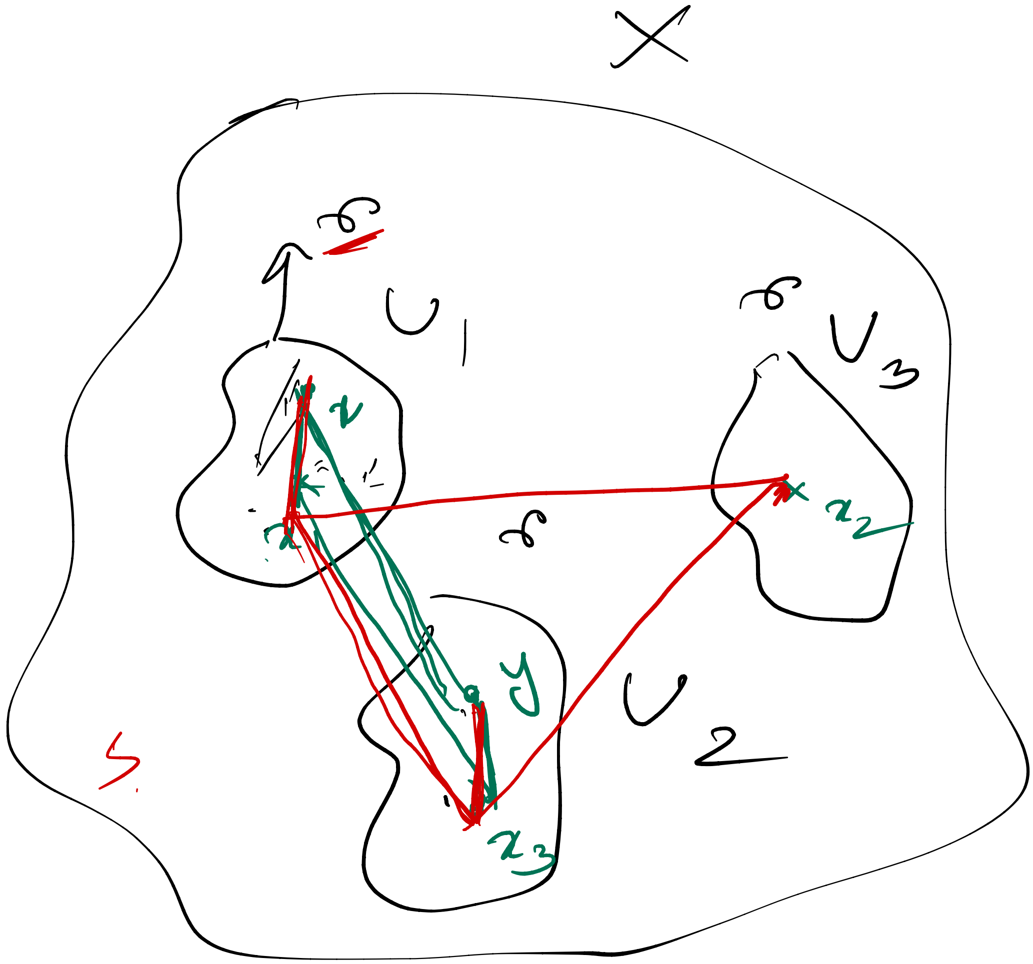
$x \in U_i$, $y \in U_j$

$$d(x, y) \le d(x, x_i) + d(x_i, x_j) + d(x_j, y)$$

$$\le 2r + S = M$$

$\Rightarrow$ $x, y \in U$, $d(x, y) \le M$.

$n = 3$



$$B(a, r) \subseteq (\mathbb{R}^2, d)$$

$$2r$$

$$\boxed{|x - a_1|^2 + (y - a_2)^2 \leq r^2}$$

$\underline{X} = [0, 1] \cup [7, 9]$,

$\underline{d \to \text{Eucl.}}$

$B(8, \underline{7})$

$a = 8 \quad \underline{r = 7}$

$= [7, 9]$

$\cancel{[0, 1]}$

$\operatorname{diam}\{B(8, 7)\}$

$= \underline{2 < 14}$

In general metric space diameter of a ball may not be twice its radius!!

Theo. (X, d) metric space

$r \in \mathbb{R}^+$, $a \in X$.

i) $\operatorname{diam}(B[a, r]) \leq 2r$

Proof Triangle ineq.

Theo. Suppose $S \subseteq \mathbb{R}$ bdd.

Then $\operatorname{diam}(S) = \sup(S) - \inf(S)$

Pf. Let $\epsilon > 0$.

$a, b \in S$

$$a - \epsilon/2 < \inf S \leq a \leq b \leq \sup S \leq b + \epsilon/2$$

[Sup, inf prop]

$\sup S - \inf S$

$< \underline{b-a} + \epsilon$

$< \underline{\mathrm{diam}(S)} + \epsilon$

$\Rightarrow \sup S - \underline{\inf \leq \mathrm{diam}(S)}$

$x, y \in S$

$\inf S \leq x \leq \sup S$

$\underline{|x-y|} \leq \underline{\sup S - \inf S}$

$\mathrm{diam}(S) \leq \sup S - \inf S$

$\Rightarrow \mathrm{diam}(S) = \sup S - \inf S.$

$(\mathbb{R}, d)$

$(x_n)_{n\geq 1}$

$x_n \to x$

$\epsilon > 0 \ \exists N \in \mathbb{N}$

$|x_n - x| < \epsilon$

$\forall n \geq N$

(X, d)

$(x_n)_{n \geq 1}$

$x_n \to x$

$\epsilon > 0 \ \exists N \in \mathbb{N}$

$d(x_n, x) < \epsilon$

$\forall n \geq N$

$\Updownarrow$

$x_n \in B(x, \epsilon), \forall n \geq N$

$(\mathbb{R}, d)$

$(x_n + y_n)$

$x_n \cdot y_n$

$\frac{x_n}{y_n}, \ y_n \neq 0$

$x_n \overset{?}{+} y_n \quad x_n \in X$

$y_n \in X$

$d : X \times X \to \mathbb{R}^+ \cup \{0\}$

(X, d)

- Limit of seq is unique.

- $(s_n) \in \mathbb{R} / \mathbb{C}$, $(y_n) \in \mathbb{R}$

- $\lim_{n \to \infty} (s_n + t_n) = s + t$.

- $\lim_{n \to \infty} (c s_n) = c s$.

- $\lim_{n \to \infty} (s_n \cdot t_n) = s t$

- $\lim_{n \to \infty} \left(\frac{1}{s_n} \right) = \frac{1}{s}$, $s_n \neq 0$
  $s \neq 0$.

$$\begin{bmatrix} s_n \to s \\ t_n \to t \end{bmatrix}$$

- Every convergent seq is bdd.

$(-1)^n$

$|x_n - x|$

$d(x_n, x)$

Seq Characterization of closed sets.

Theo. $K \neq \emptyset$ (X, d)

$x \in X$.

a) $x \in \overline{K}$ iff $\exists\ (x_n) \subset K$ s.t $x_n \to x$ as $n \to \infty$

b) K is closed iff the limits of every convergent seq is in K.

$(X, d_{dis}) \to (\{\frac{1}{n}\}, d)$