Lecture 4, 5, 6

Real and Complex Analysis

MTL122/ MTL503/ MTL506

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Definition 0.1. *Let S be a subset of $\mathbb{R}$.*

(1) *A point $x \in \mathbb{R}$ is an **accumulation point** of S if for every $\epsilon > 0$ $N^*(x, \epsilon) \cap S \neq \phi$.*
*The set of all accumulation points of S is called the **derived set** of S and is denoted by S'.*

(2) *S is said to be **dense** in itself if $S \subset S'$.*

(3) *S is called **perfect** if $S = S'$.*

(4) *The **closure** of S is the set $\bar{S} = S \cup S'$.*

Remark 0.2.

- *An accumulation point of a set S need not be an element of S.*
- *A real number x is an accumulation point of a set $S \subset \mathbb{R}$ if for each $\epsilon > 0$ the interval $(x - \epsilon, x + \epsilon)$ contains infinitely many elements of S. Indeed, if x is an accumulation point of S then, for any $\epsilon > 0$, there exists an element $s_1 \in S$ with $s_1 \neq x$, such that $0 < |x - s_1| < \epsilon$. Taking $\epsilon_1 = |x - s_1|$, there exists an element $s_2 \in S$ with $s_2 \neq x$, such that $0 < |x - s_2| < \epsilon_1 < \epsilon$. Taking $\epsilon_2 = |x - s_2|$, there exists $s_3 \in S$ with $s_3 \neq x$ such that $0 < |x - s_3| < \epsilon_2 < \epsilon$. Continuing in this way we obtain a sequence (s_n) with the property that $s_n \neq x$ and $|s_n - x| < \epsilon$ for all n.*

Example 0.3.

(1) *$S = \{x \in \mathbb{R} : 0 < x \leq 1\}$. Then $S' = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Therefore $\bar{S} = S \cup S' = S'$.*

(2) *$S = \{x \in \mathbb{R} : a \leq x \leq b\}$, then $S' = S$. Therefore $\bar{S} = S$.*

(3) *Every real number is an accumulation point of the set $\mathbb{Q}$, that is , $\mathbb{Q}' = \mathbb{R}$.*

(4) *$\mathbb{Z}' = \phi$. Indeed, for any $x \in \mathbb{R}$ we can find an $\epsilon > 0$ small enough such that $(x - \epsilon, x + \epsilon)$ contains no integer, except possibly when x is itself an integer. It thus follows that $\bar{\mathbb{Z}} = \mathbb{Z} \cup \mathbb{Z}' = \mathbb{Z}$*

Theorem 0.4. *Let $S \subset \mathbb{R}$. Then S is closed if and only if S contains all its accumulation points.*

Proof Suppose S is closed and let $x \in S'$. We want to show that $x \in S$. If $x \notin S$, then $x \in \mathbb{R} \setminus S$. Since S is closed so $\mathbb{R} \setminus S$ is open. So there exists $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset \mathbb{R} \setminus S$. This implies $(x-\epsilon, x+\epsilon) \cap S = \phi$. A contradiction. Thus $S' \subset S$.

To prove the converse assume, $S' \subset S$. We will show that $\mathbb{R} \setminus S$ is open. Let $x \in \mathbb{R} \setminus S$. Then $x \notin S'$, and so there is an $\epsilon > 0$ such that

$$N^*(x, \epsilon) \cap S = \phi.$$

Since $x \notin S$, we have that $(x-\epsilon, x+\epsilon) \cap S = \phi$. Thus $(x-\epsilon, x+\epsilon) \subset \mathbb{R} \setminus S$. So $\mathbb{R} \setminus S$ is open. □

1. Compact Sets

The significance of compact sets is not as immediately apparent as the significance of open sets, but the notion of compactness plays a central role in analysis. One indication of its importance already appears in the Bolzano-Weierstrass theorem.

Definition 1.1. *A set $K \subset \mathbb{R}$ is sequentially compact if every sequence in K has a convergent subsequence whose limit belongs to K.*

Example 1.2. *The open interval $I = (0, 1)$ is not compact. The sequence $(1/n)$ in I converges to 0, so every subsequence also converges to $0 \in I$. Therefore, $(1/n)$ has no convergent subsequence whose limit belongs to I.*

Example 1.3. *The set $\mathbb{N}$ is closed, but it is not compact. The sequence (n) in $\mathbb{N}$ has no convergent subsequence since every subsequence diverges to infinity.*

Theorem 1.4. *A subset of $\mathbb{R}$ is sequentially compact if and only if it is closed and bounded.*

Proof First, assume that $K \subset R$ is sequentially compact. Let (x_n) be a sequence in K that converges to $x \in \mathbb{R}$. Then every subsequence of K also converges to x, so the compactness of K implies that $x \in K$. It follows from Proposition that K is closed. Next, suppose for contradiction that K is unbounded. Then there is a sequence (x_n) in K such that $|x_n| \to \infty$ as $n \to \infty$. Every subsequence of (x_n) is also unbounded and therefore diverges, so (x_n) has no convergent subsequence. This contradicts the assumption that K is sequentially compact, so K is bounded.

Conversely, assume that $K \subset R$ is closed and bounded. Let (x_n) be a sequence in K. Then (x_n) is bounded since K is bounded, and so (x_n) has a convergent subsequence. Since K is closed the limit of this subsequence belongs to K, so K is sequentially compact. □

Definition 1.5. *Let $A \subset \mathbb{R}$. A cover of A is a collection of sets $\{A_i \subset \mathbb{R} : i \in I\}$ whose union contains A,*

$$\bigcup_{i \in I} A_i \supset A.$$

An open cover of A is a cover that A_i is open for every $i \in I$.

Example 1.6. *Let $S = (0,1)$ and $\mathcal{U} = \{(\frac{1}{n}, 2) | n \in \mathbb{N}\}$. Then $\mathcal{U}$ is an open cover for S. Indeed, let $x \in (0,1)$. Then, by the Archimedean Property, there is a natural number m such that $0 < \frac{1}{m} < x$. Therefore $x \in (\frac{1}{m}, 2)$, whence, $(0,1) \subset \bigcup_{n\in\mathbb{N}} (\frac{1}{n}, 2)$.*

On the other hand, $\mathcal{U}$ is not a cover of $[0,1]$ since its union does not contain 0. If for any $\delta > 0$, we add the interval $B = (-\delta, \delta)$ to $\mathcal{U}$, then

$$\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2\right) \cup B = (-\delta, 2) \supset [0,1],$$

so $\mathcal{U}' = \mathcal{U} \cup \{B\}$ is an open cover of $[0,1]$.

Definition 1.7. *A set $K \subset \mathbb{R}$ is compact if every open cover of K has a finite subcover.*

Theorem 1.8. *Let S be a compact subset of $\mathbb{R}$. If F is a closed subset of S, then F is compact.*

Proof Let $\mathcal{U} = \{U_\alpha : \alpha \in \Omega\}$ be an open cover for F. Then $\mathcal{G} = \mathcal{U} \cup F^c$ is an open cover for S. Since S is compact, the cover $\mathcal{G}$ is reducible to a finite subcover. That is, there are indices $\alpha_1, \alpha_2, ..., \alpha_n$ such that

$$S \subset \bigcup_{i=1}^{n} U_{\alpha_i} \cup F^c.$$

Since $F \subset S$ and $F \cap F^c = \phi$, it follows that $F \subset \bigcup_{i=1}^{n} U_{\alpha_i}$. Hence F is compact.

□

Theorem 1.9. *(Exercise) Let a and b be real numbers such that $-\infty < a < b < \infty$. Then the interval $[a,b]$ is compact.*

Theorem 1.10. *(Heine-Borel).*
A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Proof Let $K \subset \mathbb{R}$. Assume that K is compact. We show that K is closed and bounded.

Closedness of K: It suffices to show that the complement $\mathbb{R} \setminus K$, of K is open. To that end, let $x_0 \in \mathbb{R} \setminus K$ and for each $k \in \mathbb{N}$, let

$$U_k = \{x \in \mathbb{R} : |x - x_0| > \frac{1}{k}\} = (-\infty, x_0 - \frac{1}{k}) \cup (x_0 + \frac{1}{k}, \infty).$$

Then $\bigcup_{k=1}^{\infty} U_k = \mathbb{R} \setminus \{x_0\}$ and $\mathcal{U} = \{U_k : k \in \mathbb{N}\}$ is an open cover for K. Since K is compact, this cover of K is reducible to finite subcover. That is, there are indices $k_1, k_2, ..., k_n$ such that $K \subset \bigcup_{j=1}^{n} U_{k_j}$. Let $k_{\max} = \max\{k_1, k_2, ..., k_n\}$. Then

$$K \subset \bigcup_{j=1}^{n} U_{k_j} = (-\infty, x_0 - \frac{1}{k_{\max}}) \cup (x_0 + \frac{1}{k_{\max}}, \infty) = \{x \in \mathbb{R} : |x - x_0| > \frac{1}{k_{\max}}\}.$$

Hence,

$$\{x \in \mathbb{R} : |x - x_0| < \frac{1}{k_{\max}}\} \subset \{x \in \mathbb{R} : |x - x_0| \leq \frac{1}{k_{\max}}\} \subset \mathbb{R} \setminus K,$$

which implies $\mathbb{R} \setminus K$ is open and so K is closed.
Boundedness of K: Let $\mathcal{U} = \{(-k, k) : k \in \mathbb{N}\}$. Then $\mathcal{U}$ is an open cover for K. Indeed,

$$K \subset \mathbb{R} = \bigcup_{k \in \mathbb{N}} (-k, k).$$

Since K is compact, there are natural numbers $k_1, k_2, k_3, ..., k_n$ such that $K \subset \bigcup_{j=1}^{n} (-k_j, k_j)$. Let $k_{\max} = \max\{k_1, k_2, ..., k_n\}$. Then

$$K \subset \bigcup_{j=1}^{n} (-k_j, k_j) = (-k_{\max}, k_{\max}).$$

It now follows that K is bounded since it is contained in the bounded interval $(-k_{\max}, k_{\max})$.

Conversely, assume that K is a closed and bounded subset of $\mathbb{R}$. Then there are real numbers a and b such that $K \subset [a, b]$. Then K being a closed subset of a compact set is compact.

□

Corollary 1.11. *A subset of $\mathbb{R}$ is compact if and only if it is sequentially compact.*

2. Metric Spaces

Definition 2.1. *Let X be a set. Define the Cartesian product $X \times X = \{(x, y) \ : \ x, y \in X\}$.*

Definition 2.2. *Let $d : X \times X \to \mathbb{R}$ be a mapping. The mapping d is a metric on X if the following four conditions hold for all $x, y, z \in X$:*

i) $d(x, y) = d(y, x)$,

ii) $d(x, y) \geq 0$,

iii) $d(x, y) = 0 \iff x = y$, *and*

iv) $d(x, z) \leq d(x, y) + d(y, z)$.

Given a metric d on X, the pair (X, d) is called a *metric space.*

Suppose d is a metric on X and that $Y \subseteq X$. Then there is an automatic metric d_Y on Y defined by restricting d to the subspace $Y \times Y$,

$$d_Y = d|Y \times Y.$$

Together with Y, the metric d_Y defines the automatic metric space (Y, d_Y).

The elements of a metric space (X,d) are usually referred to as **points**. If $x,y \in X$, then $d(x,y)$ is called the *distance* between x and y. A set can have more than one metric defined on it.
If condition (3) is replaced by the condition

- $d(x,x) = 0,$ for all $x \in X$,

then d is a pseudo-metric on X and (X,d) is a pseudo-metric space.

Example 2.3. *(1) Let $X = \mathbb{R}$ and for $x,y \in X$, define $d : X \times X \to \mathbb{R}$ by $d(x,y) = |x-y|$. Then $(\mathbb{R},d)$ is a metric space. This metric is called the* ***usual metric*** *on $\mathbb{R}$.*

(2) Let $X = \mathbb{C}$, the st of complex numbers. For $x,y \in X$, define $d : X \times X \to \mathbb{R}$ by $d(x,y) = |x-y|$. Then $(\mathbb{C},d)$ is a metric space. This is metric called the usual metineric on $\mathbb{C}$.

(3) Let $X = \mathbb{R}^n$, where n is a natural number. The elements of X are ordered n-tuples $x = (x_1, x_2, ..., x_n)$ of real numbers. For $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ...,)$ in X, define

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|,$$

$$d_2(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2},$$

$$d_\infty(x,y) = \max_{1 \le i \le n} |x_i - y_i|.$$

Then each of d_1, d_2 and d_∞ defines a metric on $\mathbb{R}^n$.

These metrics have special names attached to them:

d_1 is also called the $1-$metric.

d_2 is called the Euclidean metric or the usual metric on $\mathbb{R}^n$.

d_∞ is called the supremum, maximum, or the infinity metric.

i) We leave it as an easy exercise to show that $(\mathbb{R}^n, d_1)$ is a metric space.

ii) We show that $(\mathbb{R}^n, d_2)$ is a metric space. Checking that d_2 satisfies properties (i), (ii) and (iii) is straightforward. We prove property (iv). To do that end, let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ and $z = (z_1, z_2, ..., z_n)$ be elements of $\mathbb{R}^n$. We want to show that

$$d_2(x,z) \le d_2(x,y) + d_2(y,z).$$

This is equivalent to showing that

$$\left[\sum_{i=1}^{n}(x_i - z_i)^2\right]^{1/2} \leq \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{1/2} + \left[\sum_{i=1}^{n}(y_i - z_i)^2\right]^{1/2}. \tag{2.1}$$

For each $i = 1, 2, ..., n$, *let* $a_i = x_i - y_i$ *and* $b_i = y_i - z_i$. *Then* (2.1) *can be rewritten as*

$$\left[\sum_{i=1}^{n}(a_i + b_i)^2\right]^{1/2} \leq \left[\sum_{i=1}^{n}a_i^2\right]^{1/2} + \left[\sum_{i=1}^{n}b_i^2\right]^{1/2}.$$

Since both sides of the inequality are nonnegative, it suffices to show that inequality holds for the squares of the left and right hand sides of the inequality. That is, we have to show that

$$\sum_{i=1}^{n}(a_i + b_i)^2 \leq \sum_{i=1}^{n}a_i^2 + 2\left(\sum_{i=1}^{n}a_i^2\right)^{1/2}\left(\sum_{i=1}^{n}b_i^2\right)^{1/2} + \sum_{i=1}^{n}b_i^2. \tag{2.2}$$

It now follows that inequality (2.2) *is equivalent to the inequality*

$$\sum_{i=1}^{n}a_ib_i \leq \left(\sum_{i=1}^{n}a_i^2\right)^{1/2}\left(\sum_{i=1}^{n}b_i^2\right)^{1/2} \tag{2.3}$$

Equation (2.3) *is called the Cauchy-Schwarz Inequality. We now prove the Cauchy-Schwarz Inequality.*

Cauchy-Schwarz Inequality: *If* $(a_1, a_2, ..., a_n) \in \mathbb{R}^n$ *and* $(b_1, b_2, ..., b_n) \in \mathbb{R}^n$, *then*

$$\sum_{i=1}^{n}a_ib_i \leq \left(\sum_{i=1}^{n}a_i^2\right)^{1/2}\left(\sum_{i=1}^{n}b_i^2\right)^{1/2}.$$

Proof. If $a_i = 0$ for all $i = 1, 2, ..., n$ or $b_i = 0$ for all $i = 1, 2, ..., n$, then the inequality obviously holds. Assume that there is an $i \in \{1, 2, ..., n\}$ such that $a_i \neq 0$ and a $j \in \{1, 2, ..., n\}$ such that $b_j \neq 0$. For each $i = 1, 2, ..., n$, let

$$\alpha_i = \frac{a_i}{(\sum\limits_{i=1}^{n}a_i^2)^{1/2}} \text{ and } \beta_i = \frac{b_i}{(\sum\limits_{i=1}^{n}b_i^2)^{1/2}}.$$

Recall that if $a, b \in \mathbb{R}$, then $2ab \le a^2 + b^2$. Therefore

$$\begin{aligned}
2\alpha_i\beta_i \le \alpha_i^2 + \beta_i^2 \quad &\Longleftrightarrow \quad \frac{2a_ib_i}{(\sum\limits_{i=1}^{n} a_i^2)^{1/2}(\sum\limits_{i=1}^{n} b_i^2)^{1/2}} \le \frac{a_i^2}{\sum\limits_{i=1}^{n} a_i^2} + \frac{b_i^2}{\sum\limits_{i=1}^{n} b_i^2} \\
&\Longrightarrow \quad \frac{2\sum_{i=1}^{n} a_ib_i}{(\sum\limits_{i=1}^{n} a_i^2)^{1/2}(\sum\limits_{i=1}^{n} b_i^2)^{1/2}} \le \frac{\sum\limits_{i=1}^{n} a_i^2}{\sum\limits_{i=1}^{n} a_i^2} + \frac{\sum\limits_{i=1}^{n} b_i^2}{\sum\limits_{i=1}^{n} b_i^2} = 2 \\
&\Longrightarrow \quad \frac{\sum_{i=1}^{n} a_ib_i}{(\sum\limits_{i=1}^{n} a_i^2)^{1/2}(\sum\limits_{i=1}^{n} b_i^2)^{1/2}} \le 1 \\
&\Longrightarrow \quad \sum_{i=1}^{n} a_ib_i \le \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2},
\end{aligned}$$

which proves the Cauchy-Schwarz Inequality. □

iii) We show that $(\mathbb{R}^n, d_\infty)$ *is a metric space. Let* $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ *and* $z = (z_1, z_2, ..., z_n)$ *be elements of* $\mathbb{R}^n$.

i) Since for each $i = 1, 2, ..., n$, $|x_i - y_i| \ge 0$, *it follows that*

$$d_\infty(x, y) = \max_{1\le i\le n} |x_i - y_i| \ge 0.$$

ii)

$$\begin{aligned}
d_\infty(x, y) = 0 \quad &\Longleftrightarrow \quad \max_{1\le i\le n} |x_i - y_i| = 0 \\
&\Longleftrightarrow \quad |x_i - y_i| \le 0 \textit{ for each } i = 1, 2, ..., n \\
&\Longleftrightarrow \quad |x_i - y_i| = 0 \textit{ for each } i = 1, 2, ..., n \\
&\Longleftrightarrow \quad x_i = y_i \textit{ for each } i = 1, 2, ..., n \\
&\Longleftrightarrow \quad x = y.
\end{aligned}$$

iii) $d_\infty(x, y) = \max\limits_{1\le i\le n} |x_i - y_i| = \max\limits_{1\le i\le n} |y_i - x_i| = d_\infty(y, x)$.

iv) Since, for each $j = 1, 2, ..., n$, $|x_j - z_j| \le |x_j - y_j| + |y_j - z_j|$, *it follows that*

$$|x_j - z_j| \le \max_{1\le i\le n} |x_i - y_i| + \max_{1\le i\le n} |y_i - z_i| = d_\infty(x, y) + d_\infty(y, z).$$

Hence,

$$d_\infty(x, z) = \max_{1\le j\le n} |x_j - z_j| \le d_\infty(x, y) + d_\infty(y, z).$$

Hence,

$$d_\infty(x, z) = \max_{1\le j\le n} |x_j - z_j| \le d_\infty(x, y) + d_\infty(y, z).$$

iv) For $1 \leq p < \infty$, *let* $X = l_p$ *be a set of sequences* $(x_i)_{i=1}^{\infty}$ *of real (or complex) numbers such that* $\sum_{i=1}^{\infty} |x_i|^p < \infty$. *That is,*

$$l_p = \left\{ x = (x_i)_{i=1}^{\infty} | \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}.$$

For $x = (x_i)_{i=1}^{\infty}$ *and* $y = (y_i)_{i=1}^{\infty}$ *in* l^p, *define* $d_p : X \times X \to \mathbb{R}$ *by*

$$d_p(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}.$$

Then (l_p, d_p) *is a metric space.*

Properties i), ii), iii) *aare easy to prove. Property* iv) *requires Minkowski's Inequality:*
If $p > 1$ *and* $(a_i)_{i=1}^{\infty}$ *and* $(b_i)_{i=1}^{\infty}$ *are in* l_p, *then*

$$\left(\sum_{i=1}^{\infty} |a_i + b_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{\frac{1}{p}}.$$

v) Let $X = l_\infty$ *be a set of bounded sequences of real (or complex) numbers. For* $x = (x_i)_{i=1}^{\infty}$ *and* $y = (y_i)_{i=1}^{\infty}$ *in* X, *define* $d : X \times X \to \mathbb{R}$ *by*

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i (1 + |x_i - y_i|)}.$$

Then (X, d) *is a metric space.*

vi) Let X *be a set. For* $x, y \in X$, *define* $d : X \times X \to \mathbb{R}$ *by*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \tag{2.4}$$

Then (X, d) *is a metric space. This metric* d *is called the* ***discrete metric*** *on* X.

Proposition 2.4. *Let* (X, d) *be a metric space. Then for all* $x, y, z \in X$,

$$|d(x, z) - d(y, z)| \leq d(x, y).$$

By the triangle inequality we have that

$$d(x, z) \leq d(x, y) + d(y, z) \iff d(x, z) - d(y, z) \leq d(x, y) \tag{2.5}$$

Interchanging the roles of x and y in (2.5),

$$d(y, z) - d(x, z) \leq d(x, y). \tag{2.6}$$

It now follows from equations (2.5) and (2.6) that

$$|d(x, z) - d(y, z)| \leq d(x, y).$$