Lecture 11 - MTL 122

Real and Complex Analysis

Warm up.

Ex

(X, d)

$(x_n)_{n \geq 1}$, $x_n \in X$.

$(X, d) \to =$ discrete metric space.

$(x_n)_{n \geq 1} \in X$ $\quad x_n \to a$, $a \in X$.

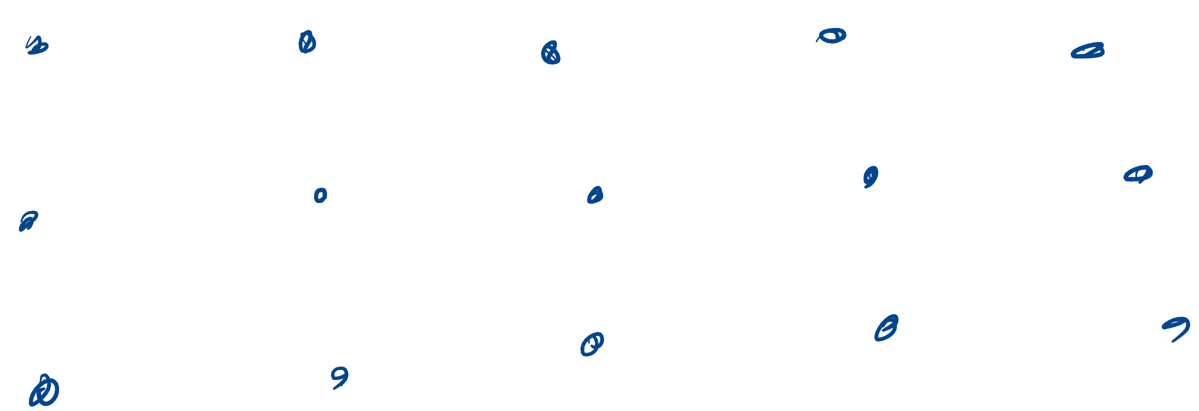
(X, d_{dis}) ,

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

(X, d)

Every open set is a closed set.

$\mathbb{Z}$



$\mathrm{Tail}_m(x_n) = \{ x_n \mid n \geq m \}$

$\boxed{x_n \to x}$

$d(x_n, x) < \epsilon \quad \forall n \geq m.$

$\forall \epsilon \; \exists m \geq 0.$

$x_n \in B(x, \epsilon) \quad \forall n \geq m$

$\{x\} \to$ open set.

$\cdot \; \epsilon_1 \quad \cdot \; B(x, \epsilon_1) \subseteq \{x\}$

$m_1 > 0$

$x_n \in B(x, \epsilon_1) \quad \forall n \geq m_1$

$x_n = x \quad \forall n \geq m_1$

Cauchy seq.

$(\mathbb{R}, d)$

$(x_n)_{n\geq 1}$.

$\epsilon > 0$ $\exists N$

$|x_m - x_n| < \epsilon$ $\forall m, n \geq N$

(X, d)

$(x_n)_{n\geq 1}$

$\epsilon > 0$ $\exists N \in \mathbb{N}$ s.t.

$d(x_m, x_n) < \epsilon$

$\forall m, n \geq N$

Prop. Any convergent seq in a metric space (X, d) is a Cauchy seq.

Pf $(x_n)_{n\geq 1}$ be a convergent seq in X.

$x_n \to x$ as $n \to \infty$.

- $\epsilon > 0$ $\exists N \in \mathbb{N}$ s.t

$d(x_n, x) < \epsilon/2$ $\forall n \geq N$.

$m, n \geq N$

- $d(x_m, x_n) \leq d(x_m, x) + d(x_n, x)$

$< \epsilon$

$\Rightarrow (x_n)_{n \geq 1}$ is a cauchy seq.

(X, d)

Convergent $\Rightarrow$ Cauchy.

$(\mathbb{R}, d)$:

Convergent $\overset{?}{\Leftarrow}$ Cauchy

$X = \mathbb{Q}$, d.

$\underline{(\mathbb{Q}, d)}$

$a_n = \left(1 + \frac{1}{n}\right)^n \longrightarrow$ Cauchy seq.

$\downarrow$

e

$e \notin \mathbb{Q}$

Cauchy $\not\Rightarrow$ convergent in general.

• Cauchy seq is bdd.

$\underline{\epsilon = 1}$ $\cdot$ (x_n)

$$\underline{d(x_m, x_n)} < 1$$

$\circledast$ $r = \max\{\underline{d(x_1, x_N)} \dots d(x_{N-1}, x_N), 1\}$

$d(x_n, x_N) \leq r$. $\forall$ $n = 1, 2, \dots, N-1$

$n \geq N$.

$d(x_n, x_N) \leq r$ $\forall n$.

$\Rightarrow$ $\underline{(x_n) \rightarrow \text{bdd.}}$

• (X, d)

~~Theo~~ A cauchy seq in X which has a convergent $\boxed{\text{subseq}}$ is convergent.

Ex. $\rightarrow$ Bolzano Weierstrass theo is not true in general metric.
(Exercise to find the example)

Cauchy seq $\Rightarrow$ bdd.

In $(\mathbb{R}, d)$

Every bdd seq has a convergent subseq

Pf: • (x_n) be a Cauchy seq in X.

• $(x_{n_k}) \subset (x_n)$

and $x_{n_k} \to x$, $k \to \infty$

$\epsilon > 0$, $\exists N_1$ s.t

$$d(x_n, x_m) < \epsilon/2 \quad \forall n, m \geq N_1$$

$\exists N_2$ s.t

$$d(x_{n_k}, x) < \epsilon/2 \quad \forall k \geq N_2$$

Let $N = \max\{N_1, N_2\}$

If $k \geq N$, $n_k \geq k$ (?)
(defn of subseq)
$n_1 < n_2 < \ldots$

$$d(x_k, x) \leq d(x_k, x_{n_k}) + d(x_{n_k}, x)$$

$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

$$\forall k \geq N.$$

$\Rightarrow (x_n)_{n\geq 1}$, $x_n \to x$ as $n \to \infty$.

$(\mathbb{R}, d)$

- Cauchy seq
$\Updownarrow$
Convergent seq

- B-W theo holds

(X, d)

$\Uparrow$ ✗ ✓ $\Downarrow$ ✗

✗

Complete metric space.

(X, d) is said to be complete if every Cauchy seq in X converge in X.

Ex. $(\mathbb{R}, d)$

- $F \subseteq X \to$ complete.
  closed.

  (F, d)

- $(\ell^p(\mathbb{R}), d_p) \to$ seq spaces.

  $\left(x = (x_n) : \sum_{n=1}^{\infty} |x_n|^p < \infty \right)$

  $$d_p(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}.$$

  Complete metric space.

- $(x_n)_{n \geq 1}$, $\quad x_n \in \ell^p(\mathbb{R})$

  $y = (y_1, y_2, \dots y_3 \dots)$

  $\boxed{x_n = (x_n^j)}_{j \in \mathbb{N}} \to$ Cauchy seq

  $x_n = x_n^1 \; x_n^2.$

  $\epsilon > 0 \;\; \exists \; N \in \mathbb{N}$ s.t

  $m, n > N$

$$d(x_n, x_m) = \left(\sum_{j=1}^{\infty} \left|x_n^j - x_m^j\right|^p\right)^{1/p}$$

$$< \epsilon$$

$$\Rightarrow \quad |x_n^j - x_m^j| < \epsilon, \quad \forall j$$

$$m, n > N.$$

$$\Rightarrow \quad (x_n^j)_{n \geq 1} \text{ is a Cauchy seq in } \mathbb{R} \rightarrow \text{C.S in } \mathbb{R}$$

$x_n = (x_n^j)$

$$\begin{array}{c|cccccc}
\ell^p(\mathbb{R}) & & \in\mathbb{R} & \in\mathbb{R} & & & \\
x_1 & \longrightarrow & x_1^1 & x_1^2 & x_1^3 & - & - \\
x_2 & \longrightarrow & x_2^1 & x_2^2 & x_2^3 & - & - \\
x_3 & \longrightarrow & x_3^1 & x_3^2 & x_3^3 & - & - \\
\vdots & & & & & & \\
x_n & \longrightarrow & x_n^1 & x_n^2 & x_n^3 & - & -
\end{array}$$

$x_1 = (x_1^j)_{j\in\mathbb{N}}$

$x_2 = (x_2^j)$

$a \Longrightarrow a_1 \quad a_2 \quad a_3$

By the completeness of $\mathbb{R}$

$$x_n^j \to a_j, \quad n \to \infty$$

$$a = (a_1, a_2, a_3 \ldots)$$

Note. $M \geq 1$, $\boxed{m, n} > N$.

$$\sum_{j=1}^{M} |x_n^j - x_m^j|^p \leq \sum_{j=1}^{\infty} |x_n^j - x_m^j|^p < \epsilon^p.$$

m.

Take $\boxed{n \to \infty}$

$$\sum_{j=1}^{M} |a_j - x_m^j|^p \leq \epsilon^p = \epsilon'$$

$M \to \infty$,

$$\sum_{j=1}^{\infty} |a_j - x_m^j|^p < \epsilon.$$

$$\underline{x_m} - a = (x_m^j - a_j)_{j \in \mathbb{N}} \in \ell^p.$$

$\ell^p(\mathbb{R}) \rightarrow$ vector space.

$$\Rightarrow \underline{\underline{a \in \ell^p}}.$$

$\epsilon' > 0$ $d(x_m, a) < \epsilon$ $\forall m \geq N$.

$\Rightarrow$ $x_m \rightarrow a$ as $m \rightarrow \infty$

$\underline{\underline{a \in \ell^p}}.$

$\Rightarrow$ ℓ^p is a complete metric space.

$$\boxed{1 \leq p < \infty}$$

$$\underline{\underline{p = \infty \quad ?}}$$