MTL122 - Real and complex analysis

Assignment-5

Department of Mathematics
Indian Institute of Technology Delhi

Question 1

Let $(X; d)$ be a metric space and Y, Z be subsets of X such that $Y \subset Z \subset \overline{Y}$. If Y is connected then Z is connected.

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- WLOG $Y \subset D$, then $\overline{Y} \subset \overline{D}$.
- Hence $Z \subset \overline{D}$, then $C = \emptyset$. Which is a contradiction, therefore Z is connected.

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- Now this is a contradiction to the fact that Y is connected (Why?)

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- Now suppose A and B are intervals then $A \cap B$ is an interval or ϕ which is connected. Similar reasoning when A or B or both are singletons. Hence Proved

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- Now $A \cap B = \{(1, 0)\} \cup \{(-1, 0)\}$ which is not connected.

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(a) Show that f is continuous if and only if whenever $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to 0, f_{a_n} converge pointwise to f.

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Suppose $\sum_{k=1}^{\infty} g_k$ converges uniformly to a function g on $\mathbb{R}$ and suppose that $h : \mathbb{R} \to \mathbb{R}$ is a bounded function on $\mathbb{R}$. Prove that $\sum_{k=1}^{\infty} hg_k$ converges uniformly to hg.

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- Hence $\sum_{k=1}^{\infty} hg_k$ converges uniformly to hg.