Assignment 2

Real and Complex Analysis

MTL122/ MTL503/ MTL506

Lecturer: A. Dasgupta

(1) Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.
   i) Prove that $\text{Int}(A \cap B) = \text{Int}A \cap \text{Int}B$.
   ii) Prove that $\text{Int}A \cup \text{Int}B \subset \text{Int}(A \cup B)$
   iii) Give an example of two sets A and B with $\text{Int}(A \cup B) \neq \text{Int}A \cup \text{Int}B$.

(2) Prove that
   i) If A is bounded above then $\sup A \in \text{Bd}(A)$.

   ii) If $a < b < c$ and the two sets A and B has the property that $A \cap (a, c) = B \cap (a, c)$. Show that $b \in \text{Bd}(A)$ if and only if $b \in \text{Bd}(B)$.

(3) Prove or give a counterexample:
   i) The union of infinitely many compact sets is compact.
   ii) A non-empty subset S of real numbers which has both a largest and a smallest element is compact.

(4) For $A \subset \mathbb{R}$, $B \subset \mathbb{R}$, let

$$A + B = \{a + b : a \in A, b \in B\}.$$

Let A be closed set, B be a compact set. Show that $A + B$ is closed.

(5) Let (X, d) be a metric space. Define $\bar{d} : X \times X \to \mathbb{R}$ by $\bar{d}(x, y) = d(x, y)$ when $d(x, y) \leq 1$ and $\bar{d}(x, y) = 1$ when $d(x, y) \geq 1$.
Prove that $\bar{d}$ is a metric on X.

(6) Suppose that $\phi : [0, \infty) \to [0, \infty)$ satisfies $\phi(0) = 0, \phi(r) > 0$ for all $r > 0$ and for all $a, b \in [0, \infty)$:
   i) $\phi(a + b) \leq \phi(a) + \phi(b)$
   ii) if $a \leq b$ then $\phi(a) \leq \phi(b)$.
Let (X, d) be a metric space and let $D : X \times X \to \mathbb{R}$ be defined by $D(x, y) := \phi(d(x, y))$. Prove that D is a metric on X.

(7) Let (X_1, d_1), $(X_2, d_2), \ldots$ be a sequence of metric spaces. Let $X = \prod_{n \in \mathbb{N}} X_n$, i.e, X is the set of all sequences $x = (x_1, x_2, \ldots)$ with $x_n \in X_n$ for all $n \in \mathbb{N}$.

Prove that the function $d : X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

is a metric on X.

(8) Prove that the function $d(m,n) = |m^{-1} - n^{-1}|$ for any $m, n \in \mathbb{N}$ defines a metric on the set of natural numbers. Does this metric extend to $\mathbb{R}^+$.

(9) Let A be a subset of a metric space X with closure $\bar{A}$ and interior of A by A° and boundary of A by δA. Show that

i) Show that $\delta A = \bar{A} \setminus A^\circ$ and δA is closed.

ii) Prove that $X \setminus \bar{A} = (X \setminus A)^\circ$.

iii) Prove that A is closed if and only if $\delta A \subset A$, and A is open if and only if $\delta A \subset A^c$.

iv) If A is open, does it follow that $(\bar{A})^\circ = A$?

(10) Let $\mathbb{Q}$, the set of rational numbers, as a metric space with the Euclidean distance $d(p,q) = |p - q|$. Consider the set

$$E = \{p \in \mathbb{Q} | 2 < p^2 < 3\}.$$

Show that E is closed and bounded in $\mathbb{Q}$.