INDIAN INSTITUTE OF TECHNOLOGY DELHI
DEPARTMENT OF MATHEMATICS
SEMESTER II, 2020−21
MTL 122 (REAL AND COMPLEX ANALYSIS)
Major Examination(Open book)

DATE: 15/05/2021(5-00 pm) Total Marks: 40 Time: 24 hours
DEADLINE: 16/05/2021, 5-00 pm.

MARKS WILL BE AWARDED ONLY FOR THOSE ANSWERS WITH PROPER JUSTIFICATION

Question 1a: Let U be the plane with negative real axis deleted. Let $y > 0$. Find the limit

$$\lim_{y\to 0}\left[\log(a+iy) - \log(a-iy)\right],$$

where $a > 0$, and also when $a < 0$. [2+2=4]

Question 1b: Let f be analytic on a closed disc $\overline{D}$ of radius $b > 0$, centered at z_0. Show that

$$\frac{1}{\pi b^2}\int\int_D f(x+iy)dydx = f(z_0).$$

[6]

Question 2: Show that the function

$$f(z) = \sum_{n=1}^{\infty}\frac{z^2}{n^2z^2+8}$$

is defined and continuous for the real values of z. Determine the region of the complex plane in which this function is analytic. Determine its pole.

[2+3+5=10]

Question 3: Suppose $\mathbb{K}(\mathbb{R}) = \{A \subseteq \mathbb{R} : A \neq \phi \text{ and compact}\}$ and the mapping $D : \mathbb{K}(\mathbb{R}) \times \mathbb{K}(\mathbb{R}) \to \mathbb{R}$ is defined by

$$D(A,B) = \inf\{\delta > 0 : A \subseteq B_\delta \text{ and } B \subseteq A_\delta, A, B \in \mathbb{K}(\mathbb{R})\},$$

where $A_\delta = \{x \in \mathbb{R} : d(x,A) < \delta\}$ and $d(x,A) = \inf\{|x-a| : a \in A\}$.

a) Prove that $(\mathbb{K}(\mathbb{R}), D)$ is a metric space.

b) Assume that $(\mathbb{K}(\mathbb{R}), D)$ is complete. Let f_1 and f_2 be two contraction maps. Then prove that there exists a unique set $E \in \mathbb{K}(\mathbb{R})$ such that $E = f_1(E) \cup f_2(E)$

[4+6=10]

Question 4: Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial such that

$$f(z) = a_nz^n + a_{n-1}z^{n-1} + ...a_1z + a_0,$$

where $n \geq 2$ and $a_i \in \mathbb{C}$, $a_n \neq 0$. Define

$$K(f) = \left\{z \in \mathbb{C} | \lim_{k\to\infty}|f^k(z)| < \infty\right\}$$

and

$$J(f) = \text{bd}\left(K(f)\right),$$

where $f^k = f \circ f \circ f.... \circ f(k \text{ times})$.

a) Prove that $|f(z)| > 2|z|$, whenever $|z| > R$ for some $R > 0$.

b) Prove that $K(f)$ and $J(f)$ are compact subsets of $\mathbb{C}$.

c) Show that $(J(f))^{\circ} = \phi$.

[3+5+2=10]

—ALL THE BEST—