Lecture - 1

MTL-122 : Real and Complex Analysis.

Sets

- $A, B, \ldots, Z$
- $a \in A$

  $A \subseteq B$
- A is a set, $A \subseteq A$.
- $A = B$ iff $A \subseteq B$, $B \subseteq A$.
- $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- $A \cap B = \{x : x \in A \ \& \ x \in B\}$
- $A \setminus B = \{x : x \in A \ \& \ x \notin B\}$
- Universal set : U

  $A \subset U$, $A^c = U \setminus A$.

- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
- De-Morgan Law

$A, B,$

a) $(A \cup B)^c = A^c \cap B^c$

b) $(A \cap B)^c = A^c \cup B^c$

Example

1) $\mathbb{N} = \{1, 2, 3, \ldots\}$

2) $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$

3) $\mathbb{Q} = \{\frac{m}{n}, \; m, n \in \mathbb{Z}, n \neq 0\}$

4) $\mathbb{R}$ = set of real numbers.

$r \in \mathbb{R}$ $\quad \{r_1, r_2, \ldots r_n\}$

Tuples

$$\frac{(x, y)}{(y, x)} ?$$

$$\{e, l, v, i, s\}$$
$$= \{l, i, v, e, s\}$$

n-tuples

x_1, x_2, x_3, x_4

4-tuple (x_1, x_2, x_3, x_4)

diff from

(x_2, x_1, x_3, x_4).

- $A \times B = \{(a, b) : a \in A \text{ \& } b \in B\}$

$\neq B \times A$

ordering is imp.

Ex.

$A = \{a, b, c\}$

$B = \{1, 2\}$

$A \times B = \{(a, 1), (a, 2), \ldots \ldots (c, 2)\}$

$B \times A = \{(1, a), (1, b), \ldots (2, c)\}$

— × —

Relations

Defn.

$$A \times B$$
$$\cup\!|$$
$$R$$

$$R \subseteq A \times B.$$

$(a, b) \in R$, aRb.

$R \subset A \times A$.

- R is $\boxed{\text{symmetric}}$

$(a,b) \in R \iff (b,a) \in R$

$aRb \iff bRa$

- R is $\boxed{\text{reflexive}}$

$(a,a) \in R$.

aRa.

- R is $\boxed{\text{transitive.}}$

$(a,b) \in R \quad (b,c) \in R$

$\Rightarrow (a,c) \in R$.

$aRb \;\&\; bRc \Rightarrow aRc$.

Equivalence

Example

1. $R \subset \mathbb{Z} \times \mathbb{Z}$.

$(a, b) \in R \Leftrightarrow a \leq b$.

(aRb)

R: reflexive, transitive,
not symmetric.

2) $(a, b) \in R \Leftrightarrow a < b$

R: transitive
not reflexive
not symmetric.

3) $(a, b) \in R \Leftrightarrow a^2 = b^2$

- $R \rightarrow$ equivalence reln.

Functions

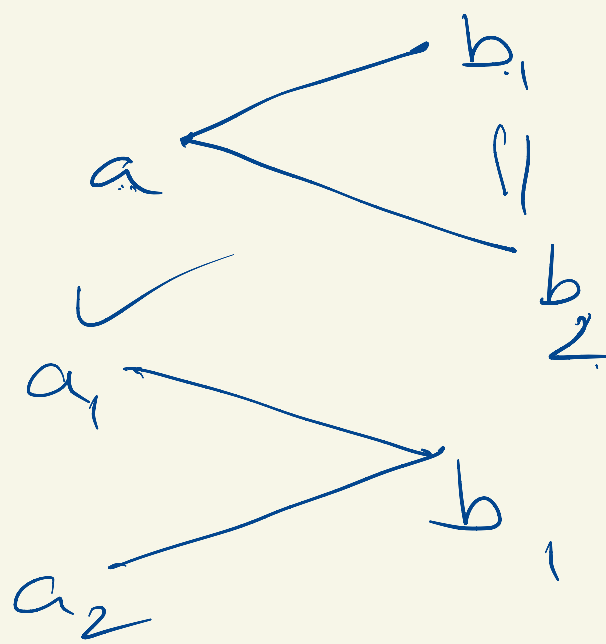
$R \subset A \times B$ is a function:

$$\left.\begin{array}{l} aRb_1 \ \& \ aRb_2 \\ \quad \Rightarrow b_1 = b_2 \end{array}\right\}$$

$f(a) = b_1$

$f : A \to B$

Maps, mapping, transformations.



$f : a \to f(b)$

$(a, f(a))$ $\quad$ $\underline{a f b}$

$\cancel{f(x)}$

$\mathrm{Dom}(f) = \{a \in A : f(a) \in B\}$

$\mathrm{Ran}(f) = \{b \in \underline{B} : b = f(a)$ for some $a \in A\}$.

- $id_A : A \to A$.

(a, a)

- $\chi_A : X \to \{0, 1\}$

$A \subset X$.

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Def. $f : A \to B$

$g : B \to C$.

$$\boxed{\begin{array}{c} \dfrac{f}{g} \\ fg \end{array}}$$

$g \circ f : A \to C$

$g \circ f(a) = g(\underline{f(a)})$.

Order is very crucial.

Example:

- $f : \underline{\mathbb{N}} \to \mathbb{Z}$

  $f(n) = \underline{n}^2$

- $g : \underline{\mathbb{Z}} \to \mathbb{Z}$

  $g(n) = \underline{n^2}$

$\text{ran} f = \{ n^2 : n \in \mathbb{Z} \}$

$\text{ran} g = \underline{\text{ran} f} \cup \{0\}$

$g \circ f(n) = n^4$. (Check)

$\forall n \in \mathbb{N}$.

$f \circ g = ?$ ~~X~~

$n = 0$

- $f : A \to B$ $\boxed{\text{const.}}$

ran(f) has single element.

$b \in B$ s.t $f(a) = b$

$\forall a \in A$.

- f is $\boxed{\text{surjective}}$

$\text{ran}(f) = B$

B is singleton.

- $f: A \to B$ is injective

  $f(a) = f(b)$.

  $\Longrightarrow \quad a = b$.

- Bijective $\begin{cases} \text{injective} \\ \& \\ \text{surjective.} \end{cases}$

Ex. $\quad f: [0, \infty) \longrightarrow [1, \infty)$

$$f(x) = 1 + x^2, \quad x \geq 0.$$

Theo. $\quad \left.\begin{array}{l} f: A \to B. \\ g: B \to C \end{array}\right\}$ bijection

$g \circ f: A \to C \longrightarrow$ bijection.

Inverse

$f : A \to B$.

$f(A) \subseteq B$

$\{ f(a) \mid a \in A \} = \mathrm{Ran}(f)$

$\boxed{f^{-1}(B)} = \{ a : f(a) \in \mathrm{Ran} f \}$
$\subset A$

$f^{-1} : B \to A$

$f^{-1} \circ f = I_d$.

Ex.

$A = \mathbb{N}$. $B = 2\mathbb{N}$

$f : A \to B$,

$f(n) = 2n$. $g : B \to A$

$g(n) = \frac{n}{2}$

- f is bijective.

$f \circ g(n) = f\left(\frac{n}{2}\right) = \frac{2n}{2} = n$

$g \circ f(n) = n$. , $\boxed{g = f^{-1}}$

- $f^{-1} \to$ inverse fn.

Ex. $f : \mathbb{R} \to \mathbb{R}$,

$f(x) = x^3 \to$ one-to-one & onto.

$\boxed{f^{-1}(x) = x^{1/3}} \to$ inverse.

$g = \frac{1}{f}$. $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$.

$$\boxed{g(x) = \frac{1}{x^2}}$$ ✓ Reciprocal.

<u>Ex.</u> $f : \mathbb{R} \to \mathbb{R}$.

$f(x) = x^2$

$A = (-2, 2)$

$f(A) = [0, 4)$

$B = (0, 4)$

$f^{-1}(B) = (-2, 0) \cup (0, 2)$

$C = (-4, 0)$

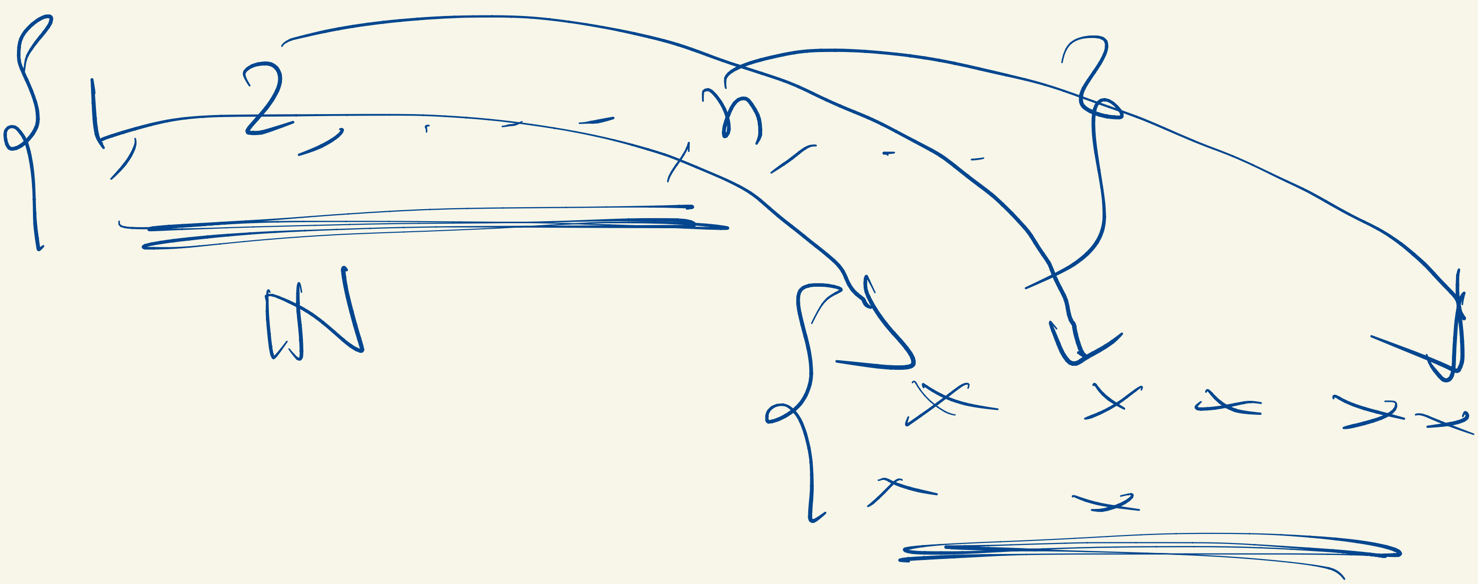
$f^{-1}(C) = \emptyset$.

Index.

Ω

$\lambda \in \Omega \longrightarrow A_\lambda$.

$\{A_\lambda : \lambda \in \Omega\} \to$ indexed by Ω.



A $\qquad\qquad$ I

$f : I \longrightarrow A$.

$A = \{a_i : i \in I\}$.

$$a_i = f(i).$$

Ex.

$$\{1, 4, 9, \ldots 16, \ldots\}$$

$$= \{n^2 : n \in \mathbb{N}\}$$