Lectures 20–30	12-31 March 2020

Real and Complex Analysis

MTL122/ MTL503/ MTL506

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1. Cauchy's Integral Formula

Theorem 1.1. *Let $w = f(z)$ be a analytic function on a simply connected domain D. Let Γ be a simple closed contour in D oriented once inthe counter clockwise direction. Then for every point z_0 inside Γ,*

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz.$$

Proof. We begin with the function $\frac{f(z)}{z-z_0}$. It is analytic everywhere on D except at z_0. Let C_r be the circle inside Γ with centre at z_0 and radius r. We suppose that C_r is oriented once in the counter clockwise direction. Since z_0 lies outside the annular region of Γ and C_r so $\frac{f(z)}{z-z_0}$ is analytic in the annular region with boundary Γ and C_r. Then by Corollary 2.10 (previous lecture) we have

$$\begin{aligned}
\int_\Gamma \frac{f(z)}{z-z_0}dz &= \int_{C_r} \frac{f(z)}{z-z_0}dz \\
&= \int_{C_r} \frac{f(z_0)}{z-z_0}dz + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz \\
&= f(z_0)\int_{C_r} \frac{1}{z-z_0}dz + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz \\
&= f(z_0)2\pi i + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz. \qquad (1.1)
\end{aligned}$$

To estimate $\left|\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz\right|$, we note that on C_r,

$$M_r = \max_{z\in C_r}|f(z)-f(z_0)|$$

and

$$|z-z_0| = r,$$

so we have

$$\left|\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz\right| \leq \frac{M_r}{r}2\pi r = 2\pi M_r.$$

Since f is continuous at z_0, it follows that $M_r \to 0$ as $r \to 0+$. Therefore

$$\int_\Gamma \frac{f(z)}{z-z_0}dz = f(z_0)2\pi i + \lim_{r\to 0+}\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz = f(z_0)2\pi i.$$

Hence

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz,$$

as required. □

Example 1.2. *Let Γ be the circle $\{z \in \mathbb{C} : |z-2| = 3\}$ traversed once in the counter clockwise direction. Compute $\int_\Gamma \frac{e^z+\sin z}{z} dz$.*

Solution 1.3. *Let $f(z) = e^z + \sin z$. Then f is holomorphic on and inside Γ. By Cauchy's integral formula, we get*

$$\int_\Gamma \frac{e^z + \sin z}{z} = \int_\Gamma \frac{f(z)}{z-0} dz = 2\pi i f(0) = 2\pi i.$$

Example 1.4. *Compute $\int_C \frac{z^2 e^z}{2z+i} dz$, where C is the unit circle centered at the origin and traversed once in the clockwise direction.*

Solution: We write

$$\frac{z^2 e^z}{2z+i} = \frac{\frac{z^2 e^z}{2}}{z + \frac{i}{2}}$$

and we let

$$f(z) = \frac{z^2 e^z}{2}.$$

Then

$$\int_C \frac{z^2 e^z}{2z+i} dz = \int_C \frac{f(z)}{z+\frac{1}{2}} dz = -2\pi i f\left(-\frac{i}{2}\right) = \frac{\pi i}{4} e^{-i/2}.$$

Theorem 1.5. *(Cauchy's Integral Formula) Let $w = f(z)$ be holomorphic on a simply connected domain D. Let Γ be a simple closed contour in D oriented once in the counterclockwise direction. Then for all z inside Γ,*

$$f^n(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\tau)}{(\tau - z)^{n+1}} d\tau.$$

Example 1.6. *Let Γ be the contour given by $\Gamma = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0 = \{|z| = 1\}$in counterclockwise and $\Gamma_1 = \{|z-1| = 1\}$ in clockwise (this is called the ∞-contour). Compute $\int_\Gamma \frac{2z+1}{z(z-1)^2} dz$.*

Solution: Obviously,

$$\int_\Gamma \frac{2z+1}{z(z-1)^2} dz = \int_{\Gamma_0} \frac{2z+1}{z(z-1)^2} dz + \int_{\Gamma_1} \frac{2z+1}{z(z-1)^2} dz.$$

In Γ_0, let $f(z) = \frac{2z+1}{(z-1)^2}$, which is analytic. By Cauchy's integral formula,

$$\int_{\Gamma_0} \frac{2z+1}{z(z-1)^2} dz = 2\pi i \frac{2z+1}{(z-1)^2}|_{z=0} = 2\pi i.$$

Similarly in Γ_1, let $f(z) = \frac{2z+1}{z}$, which is analytic. Then by Cauchy's integral formula for first derivative,

$$\int_{\Gamma_1} \frac{2z+1}{z(z-1)^2} dz = -2\pi i \frac{d}{dz}\left(\frac{2z+1}{z}\right)|_{z=1} = 2\pi i \left(\frac{1}{z^2}\right)|_{z=1} = 2\pi i.$$

Thus

$$\int_\Gamma \frac{2z+1}{z(z-1)^2} dz = 4\pi i.$$

Corollary 1.7. *Let $w = f(z)$ be a holomorphic function on a domain D. Then f', f'',..., $f^{(n)}$,... exist and are holomorphic on D.*

Example 1.8. *Let Γ be the contour self-intersecting at 2. Compute $\int_\Gamma \frac{\cos z}{z^2(z-3)}dz$.*

Solution: Let Γ_b be the big circle and Γ_s be the small circle. By Cauchy's integral theorem,

$$\int_{\Gamma_s} \frac{\cos z}{z^2(z-3)}dz = 0.$$

By Cauchy's integral formula,

$$\int_{\Gamma_b} \frac{\cos z}{z^2(z-3)}dz = 2\pi i f'(0),$$

where

$$f(z) = \frac{\cos z}{z-3}.$$

But

$$f'(z) = \frac{-(z-3)\sin z - \cos z}{(z-3)^2}$$

and hence

$$\int_\Gamma \frac{\cos z}{z^2(z-3)} = \int_{\Gamma_b} \frac{\cos z}{z^2(z-3)} = -\frac{2}{9}\pi i.$$

2. Liouville's Theorem and Fundamental Theorem of Algebra

Lemma 2.1. *Suppose that a function f is analytic inside and a positively oriented circle C_R, centered at z_0 and with radius R. If M_R denotes the maximum value of $|f(z)|$ on C_R, then*

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M_R}{R^n},\ (n = 1, 2, ...,).$$

The above inequality is known as Cauchy's Inequality. This lemma can be used to show that no entire function except a constant is bounded in the complex plane. The next theorem is know as the Liouville's theorem.

Theorem 2.2. *If f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.*

Proof Since f is entire and bounded so there exists a $M > 0$ such that $|f(z)| \leq M$, for all z. Then by Cauchy's integral formula we have for $n = 1$,

$$|f'(z_0)| \leq \frac{M}{R},\ \forall R > 0.$$

Then $f'(z) = 0$, for all z and this implies f is a constant function. □

The following theorem, known as the fundamental theorem of algebra and it follows readily from Liouville's theorem.

Theorem 2.3. *Any polynomial*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n,\ (a_n \neq 0)$$

of degree n ($n \geq 1$) has atleast one zero. That is, there exist at least one point z_0 such that $P(z_0) = 0$.

Proof. The proof here is by contradiction. Suppose that $P(z)$ is not zero for any value of z. Then

$$f(z) = \frac{1}{P(z)}$$

is entire. We claim that f is bounded. Let $w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + ... + \frac{a_{n-1}}{z}$, so that $P(z) = (a_n + w)z^n$. Now

$$\frac{a_i}{z^{n-i}} \to 0, \ i = 0, 1, ..., n-1$$

as $z \to \infty$. So for $\epsilon = \frac{|a_n|}{2n} > 0$ there exists $R > 0$ such that

$$|\frac{a_i}{z^{n-i}}| < \frac{|a_n|}{2n}, \ i = 0, 1, ..., n-1$$

for $|z| \geq R$. Then using triangle inequality we have, $|w| < \frac{|a_n|}{2}$. This gives when $|z| \geq R$,

$$|a_n + w| \geq |a_n| - |w| > \frac{|a_n|}{2}.$$

So then

$$|P(z)| = |a_n + w||z^n| > \frac{|a_n|}{2}|z|^n \geq \frac{|a_n|}{2}R^n$$

whenever $|z| \geq R$. Then $|f(z)| < \frac{2}{|a_n|R^n}$ for $|z| \geq R$. So f is bounded in the region exterior to the disk $|z| \leq R$. Now f is entire so it is continuous and bounded in $|z| \leq R$. Hence by the Liouvile's theorem f is constant. Hence $P(z)$ is also a constant function, which is a contradiction. □

3. Sequence and Series

An infinite sequence

$$z_1, z_2, ...z_n, ..$$

of complex numbers has a limit z if for each $\epsilon > 0$ there exists a positive integer n_0 such that

$$|z_n - z| < \epsilon$$

whenever $n > n_0$.
Suppose that $z_n = x_n + iy_n$ $(n = 1, 2, ...,)$ and $z = x + iy$. Then

$$\lim_{n\to\infty} z_n = z$$

if and only if

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

An infinite series

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + ... + z_n + ...$$

of complex numbers converges to the sum S if the sequence

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + z_3 + ... + z_N, \ N = 1, 2, 3, ...$$

of partial sums converge to S; we then write

$$\sum_{n=1}^{\infty} z_n = S.$$

Test for Convergence of Series

Comparison Test

If a series $\sum_{n=1}^{\infty} z_n$ is given and if we can find a converging series $\sum_{n=1}^{\infty} b_n$ of non-negative real numbers such that $|z_n| \leq b_n$, for all $n = 1, 2, ...$ then the given series $\sum_{n=1}^{\infty} z_n$ converges.

Ratio Test

Suppose $\sum_{n=1}^{\infty} w_n$ is a given series with $w_n \neq 0, n = 1, 2, 3, ...,$ and suppose $\lim\limits_{n\to\infty} \left|\frac{w_{n+1}}{w_n}\right| = L$ exists.
(1) If $L < 1$, then the series $\sum_{n=1}^{\infty} w_n$ is absolutely convergent.
(2) If $L > 1$, then the series $\sum_{n=1}^{\infty} w_n$ is divergent.
(3) If $L = 1$, the test fails.

Root Test

Suppose that the sequence $|w_n|^{1/n}$ is convergent with the limit L, that is, $\lim\limits_{n\to\infty} |w_n|^{1/n} = L$. Then
(1) If $L < 1$, then the series $\sum_{n=1}^{\infty} w_n$ is absolutely convergent.
(2) If $L > 1$, then the series $\sum_{n=1}^{\infty} w_n$ is divergent.
(3) If $L = 1$, the test fails.

4. Power Series

A power series in powers of $(z - z_0)$ is an infinite series of the form

$$\sum_{n=1}^{\infty} a_n (z - z_0)^n$$

where z_0 and a_n are complex constants and z_0 may be any point in a stated region containing z_0.

Theorem 4.1. *If a power series $\sum\limits_{n=1}^{\infty} a_n z^n$ converges when $z = z_1$ $(z_1 \neq 0)$, then it is absolutely convergent at each point z in a open disk $|z| < |z_1|$.*

The above theorem enable us to define the circle of convergence and the radius of convergence for a power series $\sum\limits_{n=1}^{\infty} a_n z^n$. The set of all points inside that circle about the origin is therefore a region of convergence for the power series $\sum\limits_{n=1}^{\infty} a_n z^n$. The greatest circle about the origin such that the series converges at each point inside is called the circle of convergence of the power series and the radius of this circle is called the radius of convergence.

Remark 4.2.

(1) *If the power series* $\sum_{n=1}^{\infty} a_n z^n$ *converges only at* $z = 0$, *then the radius of convergence is* 0.
(2) *If* $\sum_{n=1}^{\infty} a_n z^n$ *converges everywhere, then the radius of convergence is infinite.*

5. Taylor Series

Definition 5.1. *Let* $w = f(z)$ *be a analytic function at* z_0. *Then the series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

is called the Taylor series of f *at* z_0. *In special case when* $z_0 = 0$, *we call it the Maclaurin series of* f.

We can always write down the Taylor series. The real issue is whether or not it converges to the function in question.

Theorem 5.2. *Let* $w = f(z)$ *be a holomorphic function on the open disk* $\{z \in \mathbb{C} : |z-z_0| < R\}$. *Then the Taylor series* $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ *converges to* $f(z)$ *uniformly on every closed subdisk* $\{z \in \mathbb{C} : |z - z_0| \leq \rho\}$.

Example 5.3. *Find the Taylor series of Log*(z) *at* $z_0 = 1$. *State also the largest disk of convergence centered at 1.*

Solution:
Let $f(z)$ =Log(z). Then f is holomorphic at 1 and

$$\{z \in \mathbb{C} : |z - 1| < 1\}$$

is the largest disk centered at 1 on which f is holomorphic. Now,

$$\begin{aligned} f'(z) = \frac{1}{z} &\implies f'(1) = 1, \\ f''(z) = -\frac{1}{z^2} &\implies f''(1) = -1, \\ f'''(z) = \frac{2}{z^3} &\implies f'''(1) = 2, \\ &\cdots . \end{aligned}$$

Thus,

$$\begin{aligned} Log(z) &= f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \frac{f'''(1)}{3!}(z-1)^3 + ... \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n, \ |z-1| < 1. \end{aligned}$$

Example 5.4. *Find the Taylor series of* $\frac{1}{1-z}$ *at* $z_0 = 0$. *State also the largest disk of convergence at* 0.

Solution:
Let $f(z) = \frac{1}{1-z}$. Then f is holomorphic at 0 and

$$\{z \in \mathbb{C} : |z| < 1\}$$

is the largest disk centered at 0 on which f is holomorphic. Now,

$$\begin{aligned} f'(z) &= \frac{1}{(1-z)^2} \implies f'(0) = 1, \\ f''(z) &= \frac{2}{(1-z)^3} \implies f''(0) = 2, \\ f'''(z) &= \frac{3!}{(1-z)^4} \implies f'''(0) = 3!, \\ &\cdots . \end{aligned}$$

So,

$$\begin{aligned} \frac{1}{1-z} &= f(0) + f'(0)z + \frac{f''}{2!}z^2 + ... \\ &= 1 + z + z^2 + z^3 + ... \\ &= \sum_{n=0}^{\infty} z^n, \ |z| < 1, \end{aligned}$$

which is the well known geometric series.

Theorem 5.5. *Let* $w = f(z)$ *be holomorphic at* z_0. *Then the Taylor series of* f' *at* z_0 *can be obtained from that of* f *at* z_0 *by differentiating term by term, and it converges on the same disk as that of* f.

Example 5.6. *Find the Maclaurin series of* $\sin z$ *and* $\cos z$.

Solution:

Let $f(z) = \sin z$. Then

$$\begin{aligned} f'(z) = \cos z &\implies f'(0) = 1, \\ f''(z) = -\sin z &\implies f''(0) = 0, \\ f'''(z) = -\cos z &\implies f'''(0) = -1, \\ f^4(z) = \sin z &\implies f^4(0) = 0, \\ &\cdots . \end{aligned}$$

So,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ..., \ z \in \mathbb{C}.$$

By Theorem 5.5,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} - ..., \ z \in \mathbb{C}.$$

Now the following example illustrates the point that sometimes we need to use an indirect method to compute a Taylor series or Maclaurin series.

Example 5.7. *Find the first three terms of the Maclaurin series for* $\tan z$*.*

Solution:
First let us where $\tan z$ is holomorphic. Since

$$\tan z = \frac{\sin z}{\cos z}$$

we see that $\tan z$ is holomorphic at all points z unless $\cos z = 0$. But

$$\cos z = 0 \iff e^{iz} = -e^{-iz} \iff e^{2iz-\pi i} = 1 \iff 2iz - \pi i = 2k\pi i,$$

where $k \in \mathbb{Z}$. Therefore $\tan z$ is holomorphic everywhere except at

$$z = \left(k + \frac{1}{2}\right)\pi,\ k \in \mathbb{Z}.$$

Therefore $\{z \in \mathbb{C} : |z| < \frac{\pi}{2}\}$ is the largest disk on which $\tan z$ is holomorphic. Now, suppose that the Maclaurin series of $\tan z$ is given by

$$\tan z = \frac{\sin z}{\cos z},$$

we get

$$\cos z \tan z = \sin z$$

and hence

$$\begin{aligned} (1 \quad &- \quad \frac{z^2}{2!} + \frac{z^4}{4!} - ...)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ...+) \\ &= \quad a_0 + a_1 z + \left(a_2 - \frac{a_0}{2!}\right) z^2 + \left(a_3 - \frac{a_1}{2!}\right) z^3 \\ &+ \quad \left(a_4 - \frac{a_2}{2!} + \frac{a_0}{4!}\right) z^4 + \left(a_5 - \frac{a_3}{2!} + \frac{a_1}{4!}\right) z^5 + ... \end{aligned} \tag{5.1}$$

Therefore and this is the same as

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - ... \ .$$

Therefore

$$a_0 = 0, a_1 = 1, a_2 = 0, a_3 = \frac{1}{3}, a_4 = 0, a_5 = \frac{2}{15}, ...$$

Hence

$$\tan z = z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + ...,\ |z| < \frac{\pi}{2}.$$

We now look at series that are apparently more general than Taylor series. A series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is called power series at z_0. Obviously, it converges at z_0. The most basic property of power series is given in the following theorem.

Theorem 5.8. *Let* $\sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$ *be a power series. Then there exists a real* R *in* $[0, \infty]$ *such that*

(1) the series converges absolutely on $\{z \in \mathbb{C} : |z - z_0| < R\}$*,*
(2) the series diverges on $\{z \in \mathbb{C} : |z - z_0| > R\}$*,*

(3) the series converges uniformly on every closed subdisk of the disk of convergence $\{z \in \mathbb{C} : |z - z_0| < R\}$*.*

In fact,

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Remark 5.9. *We call* R *the radius of convergence of the power series. The formula for* R *in Theorem 5.8 is known as Hadamard's radius of convergence. If* $R = \infty$*, then the power series converges at every complex number* z*.*

Proof. Without loss of generality, we assume that $z_0 = 0$. Suppose that $R = 0$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \infty.$$

Let z be a nonzero complex number. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n z^n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \infty.$$

So by Cauchy's root test, the power series $\sum_{n=0}^{\infty} a_n z^n$ diverges for all nonzero complex numbers z. Next we suppose that $R = \infty$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0.$$

Let $z \in \mathbb{C}$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n z^n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0.$$

So by Cauchy's root test, the power series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z in $\mathbb{C}$. Let $\rho \in [0, \infty)$. Then for all z in the disk $\{z \in \mathbb{C} : |z| \le \rho\}$,

$$|a_n z^n| \le |a_n| \rho^n, \; n = 0, 1, 2, ...$$

Since $\sum_{n=0}^{\infty} |a_n| \rho^n < \infty$, it follows from Weierstrass' M-test that $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on $\{z \in \mathbb{C} : |z| \le \rho\}$. Finally, we suppose that $0 < R < \infty$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R}.$$

Let $z \in \mathbb{C}$. Then

$$\limsup_{n\to\infty} \sqrt[n]{a_n z^n} = |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} \begin{cases} < 1, & |z| < R \\ > 1, & |z| > R. \end{cases}$$

So by Cauchy's root test, the power series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely on $\{z \in \mathbb{C} : |z| < R\}$ and diverges on $\{z \in \mathbb{C} : |z| > R\}$. That the convergence is uniform on every closed subdisk is the same as before. $\square$

Theorem 5.10. *Suppose that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ \ |z - z_0| < R,$$

where R is the radius of convergence of the power series. Then f is holomorphic on the disk $\{z \in \mathbb{C} : |z - z_0| < R\}$.

Theorem 5.11. *Suppose that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ \ |z - z_0| < R,$$

where $R > 0$. Then

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \ \ n = 0, 1, 2, 3....$$

Thus, the power series is the Taylor series of f at z_0.

Remark 5.12. *It is interesting to note that a power series with positive radius of convergence is a holomorphic function inside the radius of convergence. Since a holomorphic function has a Talor series expansion, it turns out that a power series with positive radius of convergence is a Taylor series.*

6. Laurent Series

In the last lecture we know that a holomorphic function on an open disk can be represented by a power series. What happens if a function is holomorphic only on an annulus?

Theorem 6.1. *Let $w = f(z)$ be a holomorphic function on an annulus $D = \{z \in \mathbb{C} : r < |z - z_0| < R\}$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n},$$

where both series converge on the annulus and converge uniformly on every closed subannulus $\{z \in \mathbb{C} : \rho_1 \leq |z - z_0| \leq \rho_2\}$ of D. Moreover,

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \ \ n = 0, \pm 1, \pm 2, ...,$$

where C is any simple closed contour in the annulus D enclosing z_0 and oriented once in the counterclockwise direction.

(1) Suppose f is analytic throughout the disk $|z - z_0| < R$. Then the integrand for the expression a_{-n}, that is $f(\zeta)(\zeta - z_0)^{n-1}$ for all n is also analytic throughout the disk $|z - z_0| < R$. Consequently by Cauchy-Goursat Theorem $a_{-n} = 0$, $n = 1, 2, 3.....$
(2) Also by generalized form of Cauchy's Integral formula

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!}, \ \ n = 0, 1, 2, ...$$

Remark 6.2. *We allow $r = 0$ or $R = \infty$. The coefficient a_{-1} of $(z - z_0)^{-1}$ is a significant number in complex analysis. It is known as the residue of the function $w = f(z)$ at z_0.*

We call $\sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ the Laurent series of f at z_0. The series $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ is called the singular part of f at z_0. If f is holomorphic on the whole disk $\{z \in \mathbb{C} : |z - z_0| < R\}$, then the singular part disappears.

Theorem 6.3. *Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ and $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ be series such that*

(1) $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *converges on* $\{z \in \mathbb{C} : |z - z_0| < R\}$,
(2) $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ *converges on* $\{z \in \mathbb{C} : |z - z_0| < r\}$,
(3) $r < R$.

Then there exists a unique holomorphic function $w = f(z)$ on the annulus $D = \{z \in \mathbb{C} : r < |z - z_0| < R\}$ such that the Laurent series of f on D is

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}.$$

Example 6.4. *Find the Laurent series of*

$$f(z) = \frac{z^2 - 2z + 3}{z - 2}$$

on $\{z \in \mathbb{C} : |z - 1| > 1\}$.

Solution:
Note that $z_0 = 1$, $r = 1$ and $R = \infty$. So, f is holomorphic on the annulus $\{z \in \mathbb{C} : r < |z - 1| < R\}$. Now,

$$\frac{1}{z - 2} = \frac{1}{(z - 1) - 1} = \frac{1}{z - 1}\frac{1}{1 - \frac{1}{z-1}}.$$

Thus for $|z - 1| > 1$,

$$\frac{1}{z - 2} = \frac{1}{z - 1}\sum_{n=0}^{\infty} \frac{1}{(z - 1)^n} = \sum_{n=0}^{\infty} \frac{1}{(z - 1)^{n+1}}.$$

Note that

$$z^2 - 2z + 3 = z^2 - 2z + 1 + 2 = (z - 1)^2 + 2$$

and hence we get

$$\begin{aligned} f(z) &= \{(z-1)^2+2\}\left\{\frac{1}{z-1}+\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+...\right\} \\ &= \left\{(z-1)+1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+...\right\}+\left\{\frac{2}{z-1}+\frac{2}{(z-1)^2}+...\right\} \\ &= (z-1)+1+\sum_{n=1}^{\infty}\frac{3}{(z-1)^n},\ |z-1|>1. \end{aligned}$$

Example 6.5. *Find the Laurent series of*

$$f(z)=\frac{1}{(z-1)(z-2)}$$

on $\{z\in\mathbb{C}:1<|z|<2\}$ *and also on* $\{z\in\mathbb{C}:|z|>2\}$.

Solution We can write,

$$\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}.$$

For $1<|z|<2$

$$\frac{1}{z-2}=-\frac{1}{2}\frac{1}{1-\frac{z}{2}}=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

and

$$\frac{1}{z-1}=\frac{1}{z}\frac{1}{1-\frac{1}{z}}=\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Hence for $1<|z|<2$,

$$f(z)=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}+-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}.$$

For $|z|>2$, we still have

$$\frac{1}{z-1}=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}},$$

but

$$\frac{1}{z-2}=\frac{1}{z}\frac{1}{1-\frac{2}{z}}=\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n=\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}.$$

So for $|z|>2$,

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}.$$

Laurent series can be used to classify isolated singularities of complex valued variable.

A point z_0 is called a singular point of a function f if f fails to be analytic at z_0 but is analytic at some point in every neighbourhood of z_0.

Definition 6.6. *Let $w = f(z)$ be a complex-valued function. Let z_0 be a point in $\mathbb{C}$ such that*

(1) f is not holomorphic at z_0.
(2) f is holomorphic on some punctured disk $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$.

Then we say that z_0 is an isolated singularity of f.

Example 6.7. *(1) $f(z) = \frac{1}{z}$ is analytic everywhere except at $z = 0$, so the origin is an isolated singular point of f.*

(2) The function $f(z) = \frac{z+1}{z^3(z^2+1)}$ has the three isolated singular points $z = 0$ and $z = \pm i$.

(3) The function $f(z) = \frac{1}{\sin(\frac{\pi}{z})}$ has the singular points $z = 0$ and $z = \frac{1}{n}$, $(n = \pm 1, \pm 2, ...)$ all lying on the line segment $[-1, 1]$. Each $z = \frac{1}{n}$ is an isolated singular point but $z = 0$ is not an isolated singular point because every deleted neighbourhood of 0 contains other singular points of f.

Suppose that z_0 is an isolated singularity of f. Then f is holomorphic on $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}, \; 0 < |z - z_0| < R.$$

Then we introduce the following terminology,

(1) If $a_{-n} = 0, n = 1, 2, 3, ...$, we say that z_0 is removable singularity.
(2) If $a_{-m} \neq 0$ for some positive integer m, but $a_{-n} = 0$ for $n > m$, we say that z_0 is a pole of order m. A pole of order 1 is called a simple pole.
(3) If $a_{-n} \neq 0$ for infinitely many positive integers n, we say that z_0 is an essential singularity.

Example 6.8. *Classify the isolated singularities, if any, for each of the following functions.*

(1) $w = f(z) = e^{1/z}$
(2) $w = f(z) = \frac{\sin z}{z}$
(3) $w = f(z) = \frac{e^z}{z^2}$
(4) $w = f(z) = \frac{e^z - 1}{z}$

Solution:

(1) 0 is the isolated singularity of each of the given functions. Now note that

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}, \; |z| > 0.$$

So, there are infinitely many negative powers. Therefore 0 is an essential singularity.

(2) $\frac{\sin z}{z} = \frac{1}{z}\left\{z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...\right\} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ....$
There are no negative powers. So 0 is a removable singularity.

(3) $\frac{e^z}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{1}{3!}z + \frac{1}{4!}z^2 + ...$
So, 0 is a pole of order 2.

(4) For $0 < |z| < \infty$

$$\begin{aligned} \frac{e^z - 1}{z} &= \frac{1}{z}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + ...\right) \\ &= 1 + \frac{z}{2!} + \frac{z^2}{3!} + ... \qquad (6.1) \end{aligned}$$

The point $z = 0$ is a removable singularity and if we write $f(0) = 1$ the function becomes entire.

7. Residues

If $w = f(z)$ is a holomorphic function on and inside a simple closed contour Γ, then

$$\int_\Gamma f(z)dz = 0.$$

Question: What happens if f has an isolated singularity z_0 inside Γ?

It turns out that the answer is related to a number associated with the function f at z_0. The number is called the residue of f at z_0.

Definition 7.1. *Let z_0 be an isolated singularity of $w = f(z)$. Then there exists a positive number R such that f is holomorphic on the punctured disk $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. Let*

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=0}^{\infty} a_{-n}(z - z_0)^{-n}$$

be the Laurent series of f on $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. Then we call the coefficient a_{-1} of $(z - z_0)^{-1}$ the residue of f at z_0 and denote it by $Res(f, z_0)$.

Example 7.2. *Compute the residue $Res(f, z_0)$ of a function $w = f(z)$ at a removable singularity z_0.*

Solution: Since z_0 is a removable singularity, we get $a_{-1} = 0$. Therefore

$$Res(f, z_0) = 0.$$

Example 7.3. *Compute the residue $Res(f, z_0)$ of a function $w = f(z)$ at a simple pole z_0.*

Solution: There exists a positive number R such that

$$f(z) = \sum_{n=0}^{\infty} a_z(z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}, \ 0 < |z - z_0| < R.$$

Since z_0 is a simple pole, we get

$$a_{-1} \neq 0$$

and

$$a_{-n} = 0,\ n = 2, 3, \ldots$$

Therefore

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + a_{-1}(z - z_0)^{-1},\ 0 < |z - z_0| < R.$$

So

$$(z - z_0) f(z) = a_{-1} + (z - z_0) \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Therefore

$$\lim_{z \to z_0} (z - z_0) f(z) = a_{-1}$$

and we get

$$Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Example 7.4. *Compute the residue $Res(f, z_0)$ of a function f at a pole z_0 of order m.*

Solution: There exists a positive number R such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n},\ 0 < |z - z_0| < R.$$

Since z_0 is a pole of order m, we get

$$a_{-m} \neq 0$$

and

$$a_{-n} = 0,\ n > m.$$

Thus, for $0 < |z - z_0| < R$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + a_{-m}(z - z_0)^{-m} + \ldots + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1}$$

and we get

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + a_{-m} + \ldots + a_{-2}(z - z_0)^{m-2} + a_{-1}(z - z_0)^{m-1}.$$

Recall the formula

$$\frac{d^\beta}{dz^\beta} z^\alpha = \begin{cases} \binom{\alpha}{\beta} \beta! z^{\alpha - \beta},\ \beta \leq \alpha, \\ 0,\ \beta > \alpha. \end{cases}$$

So, for $0 < |z - z_0| < R$,

$$\frac{d^{m-1}}{dz^{m-1}} \{(z - z_0)^m f(z)\} \tag{7.1}$$

$$= \sum_{n=0}^{\infty} a_n \binom{m+n}{m-1} (m-1)! (z - z_0)^{n+1} + \binom{m-1}{m-1} ! (m-1)! a_{-1}.$$

Therefore

$$\lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\{(z-z_0)^m f(z)\} = (m-1)!a_{-1}$$

and we get

$$Res(f, z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\{(z-z_0)^m f(z)\}$$

Example 7.5.
$f(z) = \frac{z^2-2z+3}{z-2} = z + \frac{3}{z-2} = 2 + (z-2) + \frac{3}{z-2}$.
The function f has a simple pole at $z = 2$ and the residue of f at $z = 2$ is 3.

For $0 < |z| < \infty$,

$$\begin{aligned} f(z) = \frac{\sinh z}{z^4} &= \frac{1}{z^4}(z + \frac{z^3}{3!} + \frac{z^5}{5!} + ...) \\ &= \frac{1}{z^3} + \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}z + \frac{1}{7!}z^3 + ..., \end{aligned} \tag{7.2}$$

that is, $f(z)$ has a pole of order 3 *at $z = 0$ with residue* $\frac{1}{6}$.

The significance of residues is given by the following theorem, which is known as Cauchy's residue theorem.

Theorem 7.6. *Let Γ be a simple closed contour oriented once in the counterclockwise direction. Let $w = f(z)$ be a holomorphic function on and inside Γ except at the isolated singularities $z_1, z_2, ..., z_n$ inside Γ. Then*

$$\int_\Gamma f(z)dz = 2\pi i \sum_{j=1}^{n} Res(f, z_j).$$

Example 7.7. *Compute $\int_C \frac{1-2z}{(z-1)(z-3)}dz$, where C is the circle with center at* 0 *and radius* 2*, and oriented once in the counterclockwise direction.*

Solution: Let

$$f(z) = \frac{1-2z}{(z-1)(z-3)}.$$

Then f is holomorphic everywhere except at $z = 1$ and $z = 3$. They are simple poles by inspection(Checked in class). Only $z = 1$ is inside C. So by Cauchy's residue theorem,

$$\int_C \frac{1-2z}{(z-1)(z-3)}dz = 2\pi i Res(f, 1).$$

But

$$Res(f, 1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1} \frac{1-2z}{z-3} = \frac{1}{2}.$$

Hence

$$\int_C \frac{1-2z}{(z-1)(z-3)}dz = \pi i.$$

Example 7.8. *Compute $\int_C \left(ze^{1/z} + \frac{\cos z}{z^2}\right) dz$, where C is the unit circle centered at the origin and orientated once in the counterclockwise direction.*

Solution: Let I be the integral to be computed. Then

$$I = \int_C ze^{1/z}dz + \int_C \frac{\cos z}{z^2}dz.$$

Let $I_1 = \int_C ze^{1/z}dz$. Then

$$ze^{1/z} = z\sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^{n-1}}.$$

Therefore

$$Res(ze^{1/z}, 0) = \frac{1}{2}$$

and

$$I_1 = \pi i.$$

Let $I_2 = \int_C \frac{\cos z}{z^2}dz$. Then the integrand in I_2 has a pole of order 2 at the origin. So,

$$Res\left(\frac{\cos z}{z^2}, 0\right) = \lim_{z\to 0} \frac{d}{dz}(\cos z) = \lim_{z\to 0}(-\sin z) = 0.$$

Hence $I_2 = 0$ and

$$\int_C ze^{1/z}dz + \int_C \frac{\cos z}{z^2}dz = \pi i.$$

8. Trigonometric Integrals

Now lets check the trigonometric integrals of the form

$$I = \int_0^{2\pi} U(\cos\theta, \sin\theta)d\theta,$$

where $U(\cos\theta, \sin\theta)$ is a rational function with real coefficients of $\cos\theta$ and $\sin\theta$. To compute I, we use the unit circle C centered at the origin and oriented once in the counterclockwise direction. We parametrize it by the equation

$$z = e^{i\theta},\ 0 \le \theta \le 2\pi.$$

Since

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}2i = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

and

$$dz = ie^{i\theta}d\theta,$$

we get

$$\int_0^{2\pi} U(\cos\theta, \sin\theta)d\theta = \int_C U\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)\frac{1}{iz}dz$$

Example 8.1. *Compute* $I = \int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}d\theta$

Solution: Let C be the unit circle centered at the origin and oriented once in the counterclockwise direction. Then

$$I = \int_C F(z)dz,$$

where

$$F(z) = \frac{(\frac{1}{2i}(z - \frac{1}{z}))^2}{5 + 4(\frac{1}{2}(z + \frac{1}{z}))} \frac{1}{iz} = -\frac{1}{4i} \frac{(z^2 - 1)^2}{z^2(2z^2 + 5z + 2)}.$$

Therefore

$$I = -\frac{1}{4i} \int_C \frac{(z^2 - 1)^2}{z^2(2z^2 + 5z + 2)} dz.$$

Let

$$f(z) = \frac{(z^2 - 1)^2}{z^2(2z^2 + 5z + 2)}.$$

Then

$$f(z) = \frac{(z^2 - 1)^2}{2z^2(z + \frac{1}{2})(z + 2)}.$$

So, f has simple poles at $z = -\frac{1}{2}$ and $z = -2$, and a pole of order 2 at $z = 0$. Now,

$$Res\left(f, -\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z) = \lim_{z \to -\frac{1}{2}} \frac{(z^2 - 1)^2}{2z^2(z + 2)} = \frac{3}{4}$$

and

$$\begin{aligned} Res(f, 0) &= \lim_{z \to 0} \frac{d}{dz} \left(\frac{(z^2 - 1)^2}{z^2(2z^2 + 5z + 2)} \right) \\ &= \lim_{z \to 0} \frac{(2z^2 + 5z + 2)2(z^2 - 1)2z - (z^2 - 1)^2(4z + 5)}{(2z^2 + 5z + 2)^2} = -\frac{5}{4}. \end{aligned}$$

Thus,

$$I = -\frac{1}{4i} 2\pi i \left(Res\left(f, -\frac{1}{2}\right) + Res(f, 0) \right) = \frac{\pi}{4}.$$