Lecture 4

Real and Complex Analysis

MTL122/ MTL503/ MTL506

Lecturer: A. Dasgupta aparajita.dasgupta@gmail.com

Definition 0.1. *Let S be a subset of $\mathbb{R}$.*

(1) A point $x \in \mathbb{R}$ is an ***accumulation point*** *of S if for every $\epsilon > 0$ $N^*(x, \epsilon) \cap S \neq \phi$.*
The set of all accumulation points of S is called the ***derived set*** *of S and is denoted by S'.*

(2) S is said to be ***dense*** *in itself if $S \subset S'$.*

(3) S is called ***perfect*** *if $S = S'$.*

(4) The ***closure*** *of S is the set $\bar{S} = S \cup S'$.*

Remark 0.2.

- *An accumulation point of a set S need not be an element of S.*
- *A real number x is an accumulation point of a set $S \subset \mathbb{R}$ if for each $\epsilon > 0$ the interval $(x - \epsilon, x + \epsilon)$ contains infinitely many elements of S. Indeed, if x is an accumulation point of S then, for any $\epsilon > 0$, there exists an element $s_1 \in S$ with $s_1 \neq x$, such that $0 < |x - s_1| < \epsilon$. Taking $\epsilon_1 = |x - s_1|$, there exists an element $s_2 \in S$ with $s_2 \neq x$, such that $0 < |x - s_2| < \epsilon_1 < \epsilon$. Taking $\epsilon_2 = |x - s_2|$, there exists $s_3 \in S$ with $s_3 \neq x$ such that $0 < |x - s_3| < \epsilon_2 < \epsilon$. Continuing in this way we obtain a sequence (s_n) with the property that $s_n \neq x$ and $|s_n - x| < \epsilon$ for all n.*

Example 0.3.

(1) $S = \{x \in \mathbb{R} : 0 < x \leq 1\}$. Then $S' = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Therefore $\bar{S} = S \cup S' = S'$.

(2) $S = \{x \in \mathbb{R} : a \leq x \leq b\}$, then $S' = S$. Therefore $\bar{S} = S$.

(3) Every real number is an accumulation point of the set $\mathbb{Q}$, that is , $\mathbb{Q}' = \mathbb{R}$.

(4) $\mathbb{Z}' = \phi$. Indeed, for any $x \in \mathbb{R}$ we can find an $\epsilon > 0$ small enough such that $(x - \epsilon, x + \epsilon)$ contains no integer, except possibly when x is itself an integer. It thus follows that $\bar{\mathbb{Z}} = \mathbb{Z} \cup \mathbb{Z}' = \mathbb{Z}$

Theorem 0.4. *Let $S \subset \mathbb{R}$. Then S is closed if and only S contains all its accumulation points.*

Proof Suppose S is closed and let $x \in S'$. We want to show that $x \in S$. If $x \notin S$, then $x \in \mathbb{R} \setminus S$. Since S is closed so $\mathbb{R} \setminus S$ is open. So there exists $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset \mathbb{R} \setminus S$. This implies $(x-\epsilon, x+\epsilon) \cap S = \phi$. A contradiction. Thus $S' \subset S$.

To prove the converse assume, $S' \subset S$. We will show that $\mathbb{R} \setminus S$ is open. Let $x \in \mathbb{R} \setminus S$. Then $x \notin S'$, and so there is an $\epsilon > 0$ such that

$$N^*(x, \epsilon) \cap S = \phi.$$

Since $x \notin S$, we have that $(x-\epsilon, x+\epsilon) \cap S = \phi$. Thus $(x-\epsilon, x+\epsilon) \subset \mathbb{R} \setminus S$. So $\mathbb{R} \setminus S$ is open. □

1. Compact Sets

The significance of compact sets is not as immediately apparent as the significance of open sets, but the notion of compactness plays a central role in analysis. One indication of its importance already appears in the Bolzano-Weierstrass theorem.

Definition 1.1. *A set $K \subset \mathbb{R}$ is sequentially compact if every sequence in K has a convergent subsequence whose limit belongs to K.*

Example 1.2. *The open interval $I = (0, 1)$ is not compact. The sequence $(1/n)$ in I converges to 0, so every subsequence also converges to $0 \in I$. Therefore, $(1/n)$ has no convergent subsequence whose limit belongs to I.*

Example 1.3. *The set $\mathbb{N}$ is closed, but it is not compact. The sequence (n) in $\mathbb{N}$ has no convergent subsequence since every subsequence diverges to infinity.*

Theorem 1.4. *A subset of $\mathbb{R}$ is sequentially compact if and only if it is closed and bounded.*

Proof First, assume that $K \subset R$ is sequentially compact. Let (x_n) be a sequence in K that converges to $x \in \mathbb{R}$. Then every subsequence of K also converges to x, so the compactness of K implies that $x \in K$. It follows from Proposition that K is closed. Next, suppose for contradiction that K is unbounded. Then there is a sequence (x_n) in K such that $|x_n| \to \infty$ as $n \to \infty$. * * *

Hence K is bounded.

Conversely, assume that $K \subset R$ is closed and bounded. Let (x_n) be a sequence in K. Then (x_n) is bounded since K is bounded, and so (x_n) has a convergent subsequence. Since K is closed the limit of this subsequence belongs to K, so K is sequentially compact. □

Definition 1.5. *Let $A \subset \mathbb{R}$. A cover of A is a collection of sets $\{A_i \subset \mathbb{R} : i \in I\}$ whose union contains A,*

$$\bigcup_{i \in I} A_i \supset A.$$

An open cover of A is a cover that A_i is open for every $i \in I$.

[* * * Since K is sequentially compact so
$\exists\ (x_{n_k})$ s.t. $x_{n_k} \to x (\in K)$
$\Rightarrow$ (x_{n_k}) is bdd.

$\Rightarrow \exists M$ s.t $|x_{n_k}| \le M$

But $|x_{n_k}| > n_k > k \quad \forall k \geq 1$

$\Rightarrow k \le M \; \forall k \geq 1$ ↯]

Example 1.6. *Let $S = (0,1)$ and $\mathcal{U} = \{(\frac{1}{n}, 2) | n \in \mathbb{N}\}$. Then $\mathcal{U}$ is an open cover for S. Indeed, let $x \in (0,1)$. Then, by the Archimedean Property, there is a natural number m such that $0 < \frac{1}{m} < x$. Therefore $x \in (\frac{1}{m}, 2)$, whence, $(0,1) \subset \bigcup_{n\in\mathbb{N}} (\frac{1}{n}, 2)$.*

On the other hand, $\mathcal{U}$ is not a cover of $[0,1]$ since its union does not contain 0. If for any $\delta > 0$, we add the interval $B = (-\delta, \delta)$ to $\mathcal{U}$, then

$$\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2\right) \cup B = (-\delta, 2) \supset [0,1],$$

so $\mathcal{U}' = \mathcal{U} \cup \{B\}$ is an open cover of $[0,1]$.

Definition 1.7. *A set $K \subset \mathbb{R}$ is compact if every open cover of K has a finite subcover.*

Theorem 1.8. *Let S be a compact subset of $\mathbb{R}$. If F is a closed subset of S, then F is compact.*

Proof Let $\mathcal{U} = \{U_\alpha : \alpha \in \Omega\}$ be an open cover for F. Then $\mathcal{G} = \mathcal{U} \cup F^c$ is an open cover for S. Since S is compact, the cover $\mathcal{G}$ is reducible to a finite subcover. That is, there are indices $\alpha_1, \alpha_2, ..., \alpha_n$ such that

$$S \subset \bigcup_{i=1}^{n} U_{\alpha_i} \cup F^c.$$

Since $F \subset S$ and $F \cap F^c = \phi$, it follows that $F \subset \bigcup_{i=1}^{n} U_{\alpha_i}$. Hence F is compact. // $\square$

Theorem 1.9. *(Exercise) Let a and b be real numbers such that $-\infty < a < b < \infty$. Then the interval $[a,b]$ is compact.*

Theorem 1.10. *(Heine-Borel).*
A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Proof Let $K \subset \mathbb{R}$. Assume that K is compact. We show that K is closed and bounded.

Closedness of K: It suffices to show that the complement $\mathbb{R} \setminus K$, of K is open. To that end, let $x_0 \in \mathbb{R} \setminus K$ and for each $k \in \mathbb{N}$, let

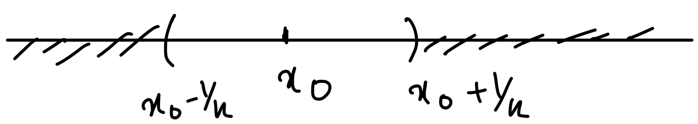
$$U_k = \{x \in \mathbb{R} : |x - x_0| > \frac{1}{k}\} = (-\infty, x_0 - \frac{1}{k}) \cup (x_0 + \frac{1}{k}, \infty).$$

$K \subset \mathbb{R} \setminus \{x_0\}$

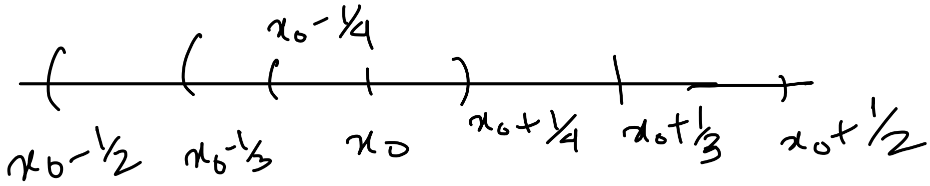
Then $\bigcup_{k=1}^{\infty} U_k = \mathbb{R} \setminus \{x_0\}$ and $\mathcal{U} = \{U_k : k \in \mathbb{N}\}$ is an open cover for K. Since K is compact, this cover of K is reducible to finite subcover. That is, there are indices $k_1, k_2, ..., k_n$ such that $K \subset \bigcup_{j=1}^{n} U_{k_j}$. Let $k_{\max} = \max\{k_1, k_2, ..., k_n\}$. Then

$$K \subset \bigcup_{j=1}^{n} U_{k_j} = (-\infty, x_0 - \frac{1}{k_{\max}}) \cup (x_0 + \frac{1}{k_{\max}}, \infty) = \{x \in \mathbb{R} : |x - x_0| > \frac{1}{k_{\max}}\}.$$

$K \subset \{x \in \mathbb{R} \mid |x - x_0| > \frac{1}{k_{max}}\}$



Example.



Hence,

$$\{x \in \mathbb{R} : |x - x_0| < \frac{1}{k_{\max}}\} \subset \{x \in \mathbb{R} : |x - x_0| \leq \frac{1}{k_{\max}}\} \subset \mathbb{R} \setminus K,$$

which implies $\mathbb{R} \setminus K$ is open and so K is closed.
Boundedness of K: Let $\mathcal{U} = \{(-k, k) : k \in \mathbb{N}\}$. Then $\mathcal{U}$ is an open cover for K. Indeed,

$$K \subset \mathbb{R} = \bigcup_{k \in \mathbb{N}} (-k, k).$$

Since K is compact, there are natural numbers $k_1, k_2, k_3, ..., k_n$ such that $K \subset \bigcup_{j=1}^{n} (-k_j, k_j)$.
Let $k_{\max} = \max\{k_1, k_2, ..., k_n\}$. Then

$$K \subset \bigcup_{j=1}^{n} (-k_j, k_j) = (-k_{\max}, k_{\max}).$$

It now follows that K is bounded since it is contained in the bounded interval $(-k_{\max}, k_{\max})$.

Conversely, assume that K is a closed and bounded subset of $\mathbb{R}$. Then there are real numbers a and b such that $K \subset [a, b]$. Then K being a closed subset of a compact set is compact.

$\square$

Corollary 1.11. *A subset of $\mathbb{R}$ is compact if and only if it is sequentially compact.*

Ex: $\bar{S}$ is closed.

- Prove sequentially compact $\iff$ compact.

Let $(x_n) \in \bar{S}$ s.t $x_n \to x$ as $n \to \infty$

to show $x \in \bar{S}$

Let $x \notin \bar{S} = S \cup S'$

$\Rightarrow x \notin S$ and $x \notin S'$

Since $x \notin S'$ $\exists$ $N_\epsilon^*(x) \cap S = \emptyset$.

and also since

$x \notin S \Rightarrow N_\epsilon(x) \cap S = \emptyset$

$\Rightarrow N_\epsilon(x) \subset S^c$ — (1)

Given $x_n \to x$ and $x_n \in \bar{S}$

$\Rightarrow$ for any $\epsilon > 0$ $\exists$ N s.t

$|x_n - x| < \epsilon/2$ $\forall n \geq N$.

if $x_n \in S$ then (1) is a $\Downarrow$.

if $x_n \in S'$ $\forall \delta > 0$ $N_\delta(x_n) \cap S \neq \emptyset$.

Let $y \in N_\delta(x_n) \cap S$.

Let $\delta = \epsilon/2$.

then, $|x_n - y| < \epsilon/2$

$\therefore$ $|y - x| < |x_n - x| + |x_n - y_n|$
$< \epsilon$

$\Rightarrow y \in N_\epsilon(x) \cap S$ $\forall y$ $\rightarrow$ ①

$\Rightarrow$ S is closed.

Hence,

$$\{x \in \mathbb{R} : |x - x_0| < \frac{1}{k_{\max}}\} \subset \{x \in \mathbb{R} : |x - x_0| \leq \frac{1}{k_{\max}}\} \subset \mathbb{R} \setminus K,$$

which implies $\mathbb{R} \setminus K$ is open and so K is closed.

Boundedness of K: Let $\mathcal{U} = \{(-k, k) : k \in \mathbb{N}\}$. Then $\mathcal{U}$ is an open cover for K. Indeed,

$$K \subset \mathbb{R} = \bigcup_{k \in \mathbb{N}} (-k, k).$$

Since K is compact, there are natural numbers $k_1, k_2, k_3, ..., k_n$ such that $K \subset \bigcup_{j=1}^{n} (-k_j, k_j)$.

Let $k_{\max} = \max\{k_1, k_2, ..., k_n\}$. Then

$$K \subset \bigcup_{j=1}^{n} (-k_j, k_j) = (-k_{\max}, k_{\max}).$$

It now follows that K is bounded since it is contained in the bounded interval $(-k_{\max}, k_{\max})$.

Conversely, assume that K is a closed and bounded subset of $\mathbb{R}$. Then there are real numbers a and b such that $K \subset [a, b]$. Then K being a closed subset of a compact set is compact.

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