DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI

MAJOR TEST 2022-2023 SECOND SEMESTER

MTL 107(NUMERICAL METHODS AND COMPUTATION)

Time: 2 hours **Max. Marks: 40**

** Answer to each question should begin on a new page. **

All notations are standard. Exhibit clearly all the steps to deserve full credit.

"As a student of IIT Delhi, I will not give or receive aid in examinations. I will do my share and take an active part in seeing to it that others as well as myself uphold the sprit and letter of the Honour Code"

1a. If x, y, and z are machine numbers in a 32−bit word length computer, what upper bound can be given for the relative roundoff error in computing $z(x+y)$? (2)

1b. Consider Newton's method for finding the positive square root of $a > 0$. Then derive the following, assuming $x_0 > 0, x_0 \neq \sqrt{a}$.

$$x_{n+1}^2 - a = \left[\frac{x_n^2 - a}{2x_n}\right]^2,$$

$n \geq 0$, and thus $x_n > \sqrt{a}$ for all $n > 0$. Also prove that the iterates $\{x_n\}$ are a strictly decreasing sequence for $n \geq 1$. (4)

2a. A function $f(x)$ is defined on [0,1] and $| f^{(m)}(x) | \leq m!$ for $m = 1, 2, \ldots\ldots$ Let $P_n(x)$ be the interpolating polynomial of $f(x)$ at the points $1, q, q^2, \ldots\ldots, q^n$ where $0 < q < 1$. Then prove or disprove that $\lim_{n\to\infty} P_n(0) = f(0)$. (3)

2b. Determine a polynomial $p(x)$ of degree as low as possible such that

$$\max_{-1\leq x\leq 1} \left|\frac{1}{3+x} - p(x)\right| \leq 0.01$$

(Use Lanczos Economization). (4)

3. By use of repeated Richardson extrapolation, find $f'(1)$ from the following values:

x	0.6	0.8	0.9	1.0	1.1	1.2	1.4
f(x)	0.707178	0.859892	0.925863	0.984007	1.033743	1.074575	1.127986

Apply the approximate formula $f'(x_0) \simeq \frac{f(x_0+h)-f(x_0-h)}{2h}$ with $h = 0.4, 0.2, 0.1$. (4)

P.T.O.

4a. Derive two point Gauss-Chebyshev quadrature formula and hence evaluate

$$\int_{\frac{1}{2}}^{1} \frac{dx}{1+x}.$$

(5)

4b. Find the error in the three point Gauss-Legendre quadrature formula. (3)

5. Let A and B be matrices of same order. Assume that A is non singular and suppose $\|A - B\| < \frac{1}{\|A^{-1}\|}$. Then find a bound on B^{-1} and hence Prove or disprove that

$$\|A^{-1} - B^{-1}\| \leq \frac{\|A^{-1}\|^2 \, \|A - B\|}{1 - \|A^{-1}\|.\, \|A - B\|}.$$

(4)

6a. Assume that A is a strictly diagonally dominant matrix. Then prove or disprove that Gauss-Seidel iterations converge to the solution of $Ax = b$ for any initial(starting) vector $x^{(0)}$. (4)

6b. Find the optimal relaxation factor ω_{opt} if the following linear system is solved by Relaxation method.

$$\begin{aligned} 4x + 0y + 2z &= 4 \\ 0x + 5y + 2z &= -3 \\ 5x + 4y + 10z &= 2 \end{aligned}$$

(4)

7. Find the solution at the end of the first step of the Fourth order Runge-Kutta method in finding an approximation to the solution to the initial value problem

$$y' = 2x - y, \quad y(0) = -1$$

at $x = 1$ with $N = 10$. Here N denotes number of subintervals. (3)

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI

MAJOR TEST 2020-2021 FIRST SEMESTER

MTL 107(NUMERICAL METHODS AND COMPUTATION)

Time: 1 hour 30 minutes **Max. Marks: 40**

$**$ **Answer to each question should begin on a new page** $**$

1a. If $x, y,$ and z are machine numbers in a $32-$bit word length computer, what upper bound can be given for the relative roundoff error in computing $z(x+y)$? (2)

1b. Consider Newton's method for finding the positive square root of $a > 0$. Then derive the following, assuming $x_0 > 0, x_0 \neq \sqrt{a}$.

$$x_{n+1}^2 - a = \left[\frac{x_n^2 - a}{2x_n}\right]^2,$$

$n \geq 0$, and thus $x_n > \sqrt{a}$ for all $n > 0$. Also prove that the iterates $\{x_n\}$ are a strictly decreasing sequence for $n \geq 1$. (4)

2a. If $f[x, x_0, \ldots\ldots, x_k]$ is a polynomial (in x) of degree $m > 0$, then prove or disprove that $f[x, x_0, \ldots\ldots, x_{k+1}]$ is a polynomial of degree $m-1$. (2)

2b. The interpolating polynomial for the function $f(x)$ on the set of distinct points $x_0, x_1, \ldots\ldots, x_n$ is given as $P_n(x) = \sum_{k=0}^{n} l_k(x) f(x_k)$. Find an explicit expression for $\sum_{k=0}^{n} l_k(0) x_k^{n+1}$. (2)

2c. A function $f(x)$ is defined on [0,1] and $\mid f^{(m)}(x) \mid \leq m!$ for $m = 1, 2, \ldots\ldots$. Let $P_n(x)$ be the interpolating polynomial of $f(x)$ at the points $1, q, q^2, \ldots\ldots, q^n$ where $0 < q < 1$. Then prove or disprove that $\lim_{n\to\infty} P_n(0) = f(0)$. (3)

3. Make a least squares fit for

x	1	2	3	4	5
y(x)	0.5	2	4.5	8	12.5

using the function ax^b where a and b are constants. (4)

4. Consider the linear system $Ax = b$ with $\mid A \mid \neq 0$. Let δA be perturbation of A and assume $\|\delta A\| < \frac{1}{\|A^{-1}\|}$. Then examine whether $A + \delta A$ is nonsingular or not. And if we define δx implicitly by $(A + \delta A)(x + \delta x) = b$ then prove or disprove

$$\frac{\|\delta x\|}{\|x\|} \leq Cond.(A) \frac{\|\delta A\|}{\|A\|}(1 + O(\|\delta A\|)).$$

(4)

P.T.O.

5. Consider the linear system $Ax = b$ with

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix},$$

Do the both Gauss-Jacobi and Gauss-Seidel iterative Methods converge/diverge?. Justify your answer. (6)

6. By use of repeated Richardson extrapolation, find $f'(1)$ from the following values:

x	0.6	0.8	0.9	1.0	1.1	1.2	1.4
f(x)	0.707178	0.859892	0.925863	0.984007	1.033743	1.074575	1.127986

Apply the approximate formula

$$f'(x_0) \simeq \frac{f(x_0+h) - f(x_0-h)}{2h}$$

with $h = 0.4, 0.2, 0.1$. (4)

7. Find a quadrature formula

$$\int_0^1 f(x) \frac{dx}{\sqrt{x(1-x)}} = \alpha_1 f(0) + \alpha_2 f(\frac{1}{2}) + \alpha_3 f(1)$$

which is exact for polynomials of highest possible degree. Use this formula to evaluate

$$\int_0^1 \frac{dx}{\sqrt{(x-x^3)}}.$$

(5)

8. Consider the initial value problem $y'(x) = xy,\ y(0) = 1$. Estimate the error at $x = 1$, when Euler's method is used, with step size $h = 0.01$

(Hint: $\mid e_n \mid = \mid y(x_n) - y_n \mid \leq [\frac{e^{L(x_n - a)} - 1}{L}] M \frac{h}{2}$ when Euler method is applied to the problem $y'(x) = f(x,y); y(a) = A$, in $a \leq x \leq b$ and $h = (b-a)/N,\ x_k = a + kh$ and $|\frac{\partial f}{\partial y}| \leq L,\ \ |y''(x)| \leq M$.) (4)

Department of Mathematics
MTL107:Numerical Methods and Computations

Major,

Max Marks: 50

Max Time: 2 Hours

Answer ALL questions

1. (a) Given $g \in C[a, b]$, $g(x) \in [a, b]$ $\forall x \in [a, b]$ and $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with $|g'(x)| \leq k$, $\forall x \in (a, b)$. Then prove the existence and uniqueness of fixed point for the function $g(x)$ in $[a, b]$. [4 Marks]

(b) Apply the Gauss elimination with partial pivoting and 4−digit rounding arithmetic to the following linear system

$$0.003x_1 + 59.14x_2 = 59.17$$

$$5.291x_1 - 6.13x_2 = 46.78$$

and compare the results to the exact solution $x_1 = 10$ and $x_2 = 1$ [4 Marks]

2. (a)Using the given data, determine the coefficient of x^2 in $P(x)$, a polynomial of unknown degree and assume that all third-order forward differences are 1. [3 Marks]

x	0	1	2
$P(x)$	2	-1	4

(b) Using the given data, construct the least squares approximation of the form $y(x) = be^{ax}$. [5 Marks]

x_i	4.0	4.5	5.1	5.9	6.8
y_i	102	130	167	224	299

3. (a) Using Lagrange interpolation, derive a forward difference formula along with the error term for the first derivative $y'(x_0)$. [3 Marks]
(b) Using the above forward difference formula for $y'(x_0)$ derive an inequality for the total error $E(h)$. [2 Marks]
(c) Using the above inequality for $E(h)$, compute the optimal h to minimize $E(h)$ for the given initial value problem

$$y' = -y + 2, \quad 0 \leq x \leq 1, \quad y(0) = 0.$$

(Assume that the maximum round-off error is bounded by $\epsilon = 10^{-2}$). [3 Marks]

(Please Turn Over for the rest of the Questions)

4. Let $T(a,b)$ and $T(a,\frac{a+b}{2})+T(\frac{a+b}{2},b)$ be the single and double applications of the Trapezoidal rule respectively to approximate $\int_a^b f(x)dx$. Derive a relationship between [6 Marks]

$$\left|T(a,b)-T(a,\frac{a+b}{2})-T(\frac{a+b}{2},b)\right|$$

and

$$\left|\int_a^b f(x)dx-T(a,\frac{a+b}{2})-T(\frac{a+b}{2},b)\right|.$$

5. (a) Suppose that $x_1, x_2, \cdots, x_n$ are the roots of the nth Legendre polynomial $P_n(x)$ and that for each $i=1,2,\cdots,n$, the numbers c_i are defined by

$$c_i=\int_{-1}^{1}\prod_{j=1, j\neq i}^{n}\frac{x-x_j}{x_i-x_j}dx.$$

If $P(x)$ is any polynomial of degree less than $2n$, then prove that [6 Marks]

$$\int_{-1}^{1}P(x)dx=\sum_{i=1}^{n}c_iP(x_i)$$

(b) Use Gaussian Quadrature with $n=3$ in both the dimensions to approximate the following double integral: [6 Marks]

$$\int_{1.4}^{2.0}\int_{1.0}^{1.5}\ln(x+2y)\ dy\ dx.$$

6. Compute the approximate solutions $y(0.5)$ and $y(1.0)$ using

(a) Taylor's Series Method of order two

and

(b) a Runge-Kutta Method of order two,

for the following initial value problem

$$y'=\cos x+e^x,\quad 0\leq x\leq 1,\quad y(0)=1$$

with $h=0.5$.

Tabulate the computed approximate solutions along with the exact solutions and the corresponding absolute errors. [8 Marks]

Department of Mathematics

MTL107: Numerical Methods and Computations

Major | Max Marks 50 | Max Time 2 Hours

Answer ALL questions ONLY by the methods indicated.
No marks for using Graphical Methods and No marks for using different methods.

1. (a) Complete the missing lines in the below MATLAB code for finding the solution accurate to within 10^{-5} of the equation $x - 2^{-x} = 0$ for $0 \leq x \leq 1$: [2 Marks]

```
clc; close all; clear all; a=0; b=1; if func(a)*func(b) > 0 break else n=1;
c=(a+b)/2; while abs(func(c))>epsilon if (func(a)*func(c) < 0) b=c else a=c end
n=n+1; c=(a+b)/2; end end
```

(b) Modify the below code for finding the solution accurate to within 10^{-4} of the equation $x - \cos x = 0$ for $0 \leq x \leq \frac{\pi}{2}$ using the False Position Method: [2 Marks]

```
clc;clear all;close all; x(1)=0; x(2)=1.57; eps=.0001; if new(x(1))*new(x(2))> 0
break end for i=2:100
x(i+1)=x(i)-(new(x(i))*(x(i)-x(i-1)))/(new(x(i))-new(x(i-1)))
if abs(new(x(i+1)))<eps break end end
```

(c) Add the lines of code concerning the back-substitution for implementing the Gaussian Elimination Method [3 Marks]

```
clc; clear all; close all; disp('Solution of N-equation "[A][X]=[r]"') n=input ('Enter number of Equations :'); A=input ('Enter Matrix [A]:'); r=input ('Enter Matrix [r]:'); D=A;d=r; s=0;
for j=1:n-1 for i=1+s:n-1 L=A(i+1,j)/A(j,j); A(i+1,:)=A(i+1,:)-L*A(j,:);
r(i+1)=r(i+1)-L*r(j); end s=s+1; end
```

2. Compute two iterations $\underline{x}^{(1)}, \underline{x}^{(2)}$ using Conjugate Gradient Method for the linear system:

$$3x_1 - x_2 + x_3 = 1$$
$$-x_1 + 6x_2 + 2x_3 = 0$$
$$x_1 + 2x_2 + 7x_3 = 4,$$

by taking initial guess $\underline{x}^{(0)} = (0,0,0)^t$. [7 Marks]

0.153

-0.169

$t = \langle v, r \rangle / \langle v, Av \rangle$

$x' = x + tv$

3. (a) Let $f(x) = \sqrt{x - x^2}$ and $P_2(x)$ be the Lagrange interpolation polynomial on $x_0 = 0$, x_1 and $x_2 = 1$. Find the largest value of x_1 in $(0, 1)$ for which $f(0.5) - P_2(0.5) = -0.25$. [4 Marks]

(b) Derive the formula (inequality) for the total error $E(h)$ (both round-off and truncation) for the numerical differentiation formula [3 Marks]

$$y'(x_0) = \frac{1}{2h}\Big[y(x_0 + h) - y(x_0 - h)\Big] - \frac{h^2}{6} y^{(3)}(\xi_0), \quad x_0 - h < \xi_0 < x_0 + h.$$

(c) By using the above error formula $E(h)$ compute the optimal h to minimize $E(h)$ for the given initial value problem [4 Marks]

$$y' = -y + 2, \qquad y(0) = 0, \qquad 0 \leq x \leq 1.$$

0.25

(Assuming that the maximum round-off error is bounded by $\epsilon = 10^{-2}$.)

4. (a) Prove that there exists a $\mu \in (a, b)$ for which the Composite Simpson's rule for n subintervals can be written with its error term as [4 Marks]

$$\int_a^b f(x)dx = \frac{h}{3}\Big[f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(b)\Big] - \frac{(b-a)}{180} h^4 f^{(4)}(\mu).$$

where $f \in C^4[a, b]$, n is even, $h = \frac{(b-a)}{n}$, and $x_j = a + jh$, for each $j = 0, 1, \cdots, n$.

(b) Write the statement of the theorem of Composite Trapezoidal rule for n subintervals along with the error term. Determine the values of n and h required to approximate using Composite Trapezoidal rule, the $\int_0^2 \frac{dx}{x+10}$ to within 10^{-6}. [7 Marks]

(c) Approximate the improper integral $\int_0^\infty \frac{dx}{1+x^4}$ using the Gaussian quadrature with $n = 2$ (given the roots 0.57735, -0.57735 and coefficients 1,1.) [5 Marks]

5. Compare the approximate solutions $y(0.5), y(1.0)$ with the exact solutions by tabulating the values and the corresponding absolute errors for the initial value problem [9 Marks]

$$y' = \sin x + e^{-x}, \quad 0 \leq x \leq 1, \quad y(0) = 1$$

with $h = 0.5$, using

(a) the Euler Method,

(b) the Taylor Series Method of order two,

(c) and any of the Runge-Kutta Methods.

MTL107: NUMERICAL METHODS AND COMPUTATION
MAJOR

Total Marks: 50 Time: Two Hour

1 **(5 Marks)** Assume that a smooth function f has root x^* which is m times repeated. Show that, in general, Newton method will converge only linearly to x^*.

2. (5 Marks) Consider the following clamped cubic spline on $[0,2]$

$$C(x) = \begin{cases} 1 + ax + 2x^2 - 2x^3, & \text{if } 0 \le x \le 1, \\ 1 + b(x-1) - 4(x-1)^2 + 7(x-1)^3 & \text{if } 1 \le x \le 2. \end{cases}$$

find $a, b, f'(0)$ and $f'(2)$.

3. (5 Marks) Perform two iterations of Steepest Decent method with initial guess $(0,0,0)^T$ for the following linear system of equations:

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 2, \\ -x_1 + 6x_2 + x_3 &= 1, \\ x_1 + 2x_2 + 5x_3 &= 3. \end{aligned}$$

4. (5 Marks) Find the best fit quadratic polynomial to the following data:

x	0	1	3	4	2
f	3	1	4	3	0

5. (5 Marks) Consider the following inner product:

$$(f,g) = \int_0^\infty w(x)f(x)g(x)dx$$

with $w(x) = e^{-ax}$. Find first three orthogonal polynomials (zero, first and second degree) with respect to this inner product.

6. (5 Marks) State and prove Chebyshev Min-Max property for Chebyshev monic polynomials.

7. (5 Marks) Consider the following central difference formula approximating the derivative:

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h}.$$

Assuming the roundoff error bound of ϵ find the optimal value of h for minimum error.

8. **(5 Marks)** Find the order of following quadrature rule on interval $[-1,1]$:

$$\int_{-1}^{1} f(x)dx \approx \frac{1}{4}\left(f(-1)+3f(-1/3)+3f(1/3)+f(1)\right)$$

9. **(5 Marks)** Consider the following on step method for first order ODE:

$$y_{j+1} = y_j + h\phi(t_j, y_j, h),$$

where ϕ is Lipschitz continuous w.r.t. y with Lipschitz contant L and consistent. Assume that local truncation error is $O(h^p)$, then show that:

$$\|e_h(t_n)\| \leq \frac{Mh^p}{L}(e^{L(t_n-t_0)}-1).$$

Assume that initial error is 0.

10. **(5 Marks)** Consider the following initial value problem:

$$y' = -y^2 + t \qquad y(0) = 1,$$

Compute one iteration using Classical Runge-Kutta 4th order method with $h = 0.1$.

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI

MAJOR TEST 2015-2016 FIRST SEMESTER

MTL 107/MAL 230 (NUMERICAL METHODS AND COMPUTATION)

Time: 2 hours Max. Marks: 50

(**) This question paper has two parts: Part-A (for 40 Marks) and Part-B (for 10 Marks). Part-B is objective type. Attach Part-B to the main answer book. (**)

PART-A

** **Answer to each question should begin on a new page** **

1. Find the interval of unit length in which the smallest root of the equation $x^5 - x + 1 = 0$ lies. Taking midpoint of that interval as initial approximation perform one iteration of the second order Birge-Vieta method. (4)

2. Find leat square approximation of degree 2 for the function

x	0	1	2	3	4
f(x)	-4	-1	4	11	20

(4)

3. By use of repeated Richardson extrapolation , find $f'(1)$ from the following values:

x	0.6	0.8	0.9	1.0	1.1	1.2	1.4
f(x)	0.707178	0.859892	0.925863	0.984007	1.033743	1.074575	1.127986

Apply the approximate formula

$$f'(x_0) \simeq \frac{f(x_0+h) - f(x_0-h)}{2h}$$

with $h = 0.4, 0.2, 0.1$. (4)

4a. Derive two point Gauss-Chebyshev quadrature formula and hence evaluate

$$\int_{\frac{1}{2}}^{1} \frac{dx}{1+x}.$$

$0, \pm\sqrt{\frac{3}{5}}$ (5)

4b. Find the error in the three point Gauss-Legendre quadrature formula. (3)

5a. Using Crout's decomposition, solve the system of linear equations

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ 3 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5.5 \end{bmatrix}.$$

3   −0.5   0.5

1   −0.5   −0.5

3 − 0.5 − 0.5 = 2

1 − 0.5

(4)

5b. Let A and B be matrices of same order. Assume that A is non singular and suppose $\|A - B\| < \frac{1}{\|A^{-1}\|}$. Then find a bound on B^{-1} and hence Prove or disprove that

$$\|A^{-1} - B^{-1}\| \leq \frac{\|A^{-1}\|^2 \, \|A - B\|}{1 - \|A^{-1}\| \, \|A - B\|}.$$

(4)

6a. Assume that A is a strictly diagonally dominant matrix. Then prove or disprove that Gauss-Seidel iterations converge to the solution of $Ax = b$ for any initial(starting) vector $x^{(0)}$. (4)

6b. Find the optimal relaxation factor ω_{opt} if the following linear system is solved by Relaxation method.

$$\begin{array}{rcrcrcr} 4x & + & 0y & + & 2z & = & 4 \\ 0x & + & 5y & + & 2z & = & -3 \\ 5x & + & 4y & + & 10z & = & 2 \end{array}$$

(4)

7. Find the solution at the end of the first step of the Fourth order Runge-Kutta method in finding an approximation to the solution to the initial value problem

$$y' = 2x - y, \quad y(0) = -1$$

at $x = 1$ with $N = 10$. Here N denotes number of subintervals. (4)