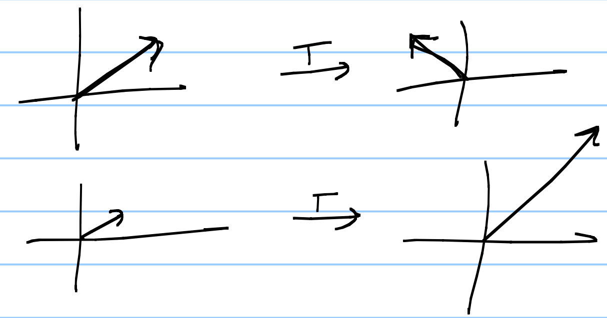
MTL104 Lec17

Note Title 14-02-2023

§ Invariant Subspaces.



$T : V \longrightarrow W \qquad \mathcal{M}(T)$

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$$

Restrict T to U_i & study the resulting transform

$T : V \longrightarrow V$

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$$

Restrict T to U_i

$T(U_i)$ may not be in any of the U_i's

If $T(U_i) \subseteq U_i$ for all i then we can study the restrictions of T to U_i & then put the data together.

$T : U_1 \oplus \cdots \oplus U_m \longrightarrow U_1 \oplus \cdots \oplus U_m \qquad \boxed{T(U_i) \subseteq U_i}$

the restriction of T to U_i is also a transform from U_i to U_i

$$\mathcal{M}(T) = \begin{bmatrix} \mathcal{M}(T|_{U_1}) & & & \\ & \mathcal{M}(T|_{U_2}) & & \\ & & \ddots & \\ & & & \mathcal{M}(T|_{U_m}) \end{bmatrix}$$

$T|_{U_i}$ is the restriction of T to U_i

Basis $\quad U_1 = \{u_{11}, \dots, u_{1n_1}\}$

$u_2 = \{u_{12}, \dots, u_{1n_2}\}$

$T(\overset{\in U_1}{u_{11}}) = \sum \alpha_i u_{1i}$

Def: Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called <u>invariant under T</u> if
<u>$u \in U$</u> $\Rightarrow$ <u>$T(u) \in U$</u>

<u>Ex:</u> ① $\{0\}$

② V

③ null(T). $\left[u \in \text{null}\, T \Rightarrow Tu = 0 \in \text{null}(T) \Rightarrow \text{null}(T) \text{ is invariant under } T \right]$

④ range T

Def: Eigenvalue: Suppose $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$ is called an eigenvalue if $\exists$ $v \in V$, $v \neq 0$ s.t. $Tv = \lambda v$.

[$U = \{\lambda v : \lambda \in \mathbb{F}\}$ & v is fixed. If U is invariant under T: $\boxed{T(v) = \lambda v} \in U$ $\forall v \in U$]

Eigenvector: $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$ is an eigenvalue of T. $v \in V$ is called an eigenvector of T corresponding to λ, if $v \neq 0$ & $T(v) = \lambda v$.

Theorem: Suppose V is f-d v.s. $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$. Then TFAE

(a) λ is an eigenvalue of T

(b) $T - \lambda I$ is not injective

(c) $T - \lambda I$ is not surjective

(d) $T - \lambda I$ is not invertible.

$T : V \to V$

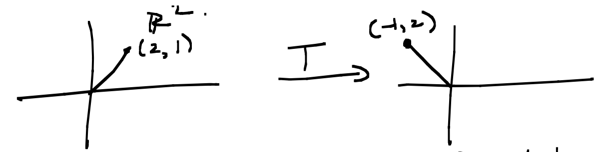
$T(v) = \lambda v$ $\quad v \neq 0$

$Tv - \lambda v = 0$

$(T - \lambda I)(v) = 0$

$v \in \ker(T - \lambda I)$

Ex: $T \in \mathcal{L}(\mathbb{F}^2)$



$$T(w, z) = (-z, w)$$

It has no eigenvalues!! as no vector is scaled!!

(a) If $\mathbb{F} = \mathbb{R}$ $\nexists$ no eigen values!

(b) If $\mathbb{F} = \mathbb{C}$, $\exists$ eigenvalues!

Proof: (a) No vector is scaled $\Rightarrow$ no eigenvalues

(b) Suppose $T(w, z) = (-z, w)$ $\quad w, z \in \mathbb{C}$

Suppose T has an eigenvalue $\quad T(w, z) = \lambda(w, z) = (\lambda w, \lambda z)$, $T(w,z) = (-z, w)$

$$(-z, w) = (\lambda w, \lambda z)$$

$$\Rightarrow \quad \lambda w = -z, \quad \underline{\lambda z = w}.$$

$$-z = \lambda^2 z$$

$$(\lambda^2 + 1) z = 0 \text{ in } \mathbb{C}$$

$$\Rightarrow z = 0 \quad \text{or} \quad \lambda^2 + 1 = 0$$

$\Downarrow$. $\quad$ $\Downarrow$.

$w = 0.$
$\Downarrow$
$(w, z) = 0$
$\Rightarrow$ the vector $(w, z) = 0$
$\rightarrow\leftarrow$

$\lambda = \pm i$ in $\mathbb{C}$
eigenvalues are $i, -i$

Theorem: $T \in \mathcal{L}(V)$ $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T & $v_1 \dots v_m$ are corresponding eigenvectors

Then $v_1 \dots v_m$ are linearly independent.

Proof: Suppose $v_1 \dots v_m$ are l.i $\Rightarrow$ one of the vectors v_i is written as a linear combination of $v_1 \dots v_{i-1}$.

Choose the smallest possible i & call it k.

$v_k \in \operatorname{span}(v_1 \dots v_{k-1})$.

$\Rightarrow \exists\ c_i$ s.t $\boxed{v_k = \sum_{i=1}^{k-1} c_i v_i}$

(1) $\lambda_k v_k = T(v_k) = T\left(\sum c_i v_i\right) = \sum c_i T(v_i) = \sum c_i \lambda_i v_i$

(2) $\lambda_k v_k = \sum c_i \lambda_k v_i$

$$① - ② \Rightarrow 0 = \sum_{i=1}^{k-1} c_i \underbrace{(\lambda_i - \lambda_k)}_{\neq 0} v_i \Rightarrow v_{k-1} \in \operatorname{Span}(v_1 \ldots v_{k-2})$$

$\rightarrow\leftarrow$ as k is least with such property.

$\Rightarrow v_1 \ldots v_m$ are l.i

Corollary: V is a f.d.v.s. $T \in \mathcal{L}(V)$. Then T can have atmost dim V eigenvalues.

Proof: $\lambda_1 \ldots \lambda_m$ are eigenvalue

$\Rightarrow v_1 \ldots v_m$ corresponding eigenvectors are linearly independent.

The largest linearly independent set contains only dim V elements.

$$\Rightarrow m \leq \dim V$$

§ Restriction & Quotient Operator.

Def: $T \in \mathcal{L}(V)$ U is a subspace of V which is <u>also invariant under T</u>

- The restriction operator $T|_U \in \mathcal{L}(U)$

$$T|_U(u) = T(u) \quad \forall u \in U$$

- The quotient Operator $T/U \in \mathcal{L}(V/U)$

$$(T/U)(v+U) = T(v) + U$$

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§ Restriction & Quotient Operators

Def: $T \in \mathcal{L}(V)$, U is a subspace of V that is invariant under T

$U \le V$

$T|_U \in \mathcal{L}(U, V)$

- The restriction operator $T|_U \in \mathcal{L}(U)$.

$$T|_U(u) = T(u) \quad \forall u \in U$$

- quotient operator $T/U \in \mathcal{L}(V/U)$

$$(T/U)(v+U) = Tv + U$$

$T/U\,(v+U) = 0.$

$\Rightarrow v+U = U$

$\Leftrightarrow v \in U$

$\left[v+U = w+U \Leftrightarrow v-w \in U \Rightarrow T(v-w) \in T(U) \subseteq U \right.$

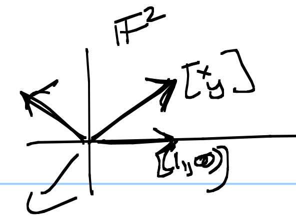
$T(v) - T(w) \in U$

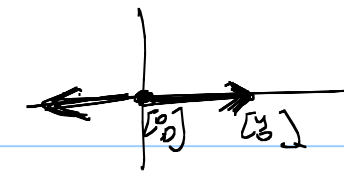
$\Leftrightarrow T(v) + U = T(w) + U$

$T/U(v+U) = T/U(w+U) \left.\right]$

Ex: $T \in \mathcal{L}(\mathbb{F}^2)$

$$\boxed{T(x,y) = (y,0)}$$





$\ker T = \boxed{U = \{(x,0) : x \in \mathbb{F}\}} \subseteq \mathbb{F}^2$

$T(v) = \lambda v$

a). U is invariant under T & $T|_U$ is the zero operator on U

$(x,0) \in U \Rightarrow T((x,0)) = (0,0) \in U \Rightarrow U$ is invariant under T

$T|_U \in \mathcal{L}(U)$ $\quad T|_U(u) = T(u)$

$T|_U(x,0) = T(x,0) = (0,0) \Rightarrow T|_U$ is the zero operator

b). there is NO subspace W that is invariant under T & $\mathbb{F}^2 = U \oplus W$

$U \leq V$; $W \leq V$. Suppose $\mathbb{F}^2 = V = U \oplus W$

U is 1-dim. $\Rightarrow W$ is also 1-dim.

$\Rightarrow$ each elt of W is an eigenvector for T.

$W = \langle v \rangle = \{\alpha v\}$ $\alpha \in \mathbb{F}$

$T(v) \in W$, $T(v) = \alpha v$

But observe that the only eigenvalue of T is 0.

The only eigenvector that has eigenvalue 0 is in U.

$T(u)=0 \Leftrightarrow$ eigenvalue 0
$\Leftrightarrow$ kernel elts.

$\Rightarrow$ W cannot be an invariant subspace of V

(b) T/U is the zero operator on $\mathbb{F}^2/U$

$$(T/U)(v+U) = (T/U)((x,y)+U)$$
$$= T(x,y) + U$$
$$= (y,0) + U = U \quad \text{as } (y,0) \in U$$

$\Rightarrow (T/U)(v+U) = U \leftarrow$ zero elt of V/U

Ex: (1) $U \subseteq \text{null } T$ is also an invariant subspace.

$u \in U \subseteq \text{null}(T) \Rightarrow T(u) = 0 \in U \Rightarrow U$ is invariant under T.

(2) $\text{range } T \subseteq U$ then U is invariant under T. (Check)

(3). $S, T \in \mathcal{L}(V)$ $ST = TS \Rightarrow \text{null}(S)$ is invariant under T.

- $u \in null(S)$ $\quad ST(u) = TS(u) = T(0) = 0$

$S(T(u)) = 0 \Rightarrow T(u) \in null(S)$

$\Rightarrow$ null (S) is invariant under T.

(4) Suppose $S, T \in \mathcal{L}(V)$, $ST = TS$, then range S is invariant under T.

(check)

Def: $T \in \underline{\mathcal{L}(V)}$, $m > 0$, then T^m is defined by $T^m = \underbrace{T \circ T \circ T \circ \cdots \circ T}_{m \text{ times}}$

[T^0 is defined to be identity operator]

If T is invertible then $T^{-m} = \underbrace{T^{-1} \circ T^{-1} \circ \cdots \circ T^{-1}}_{m \text{ times.}}$

Def: $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbb{F})$ i.e., $p(z) = a_0 + a_1 z + \cdots + a_n z^n$.

$p(T) = a_0 I + a_1 T + \cdots + a_n T^n \in \mathcal{L}(V)$

Theorem: Every operator on a finite dimensional, nonzero complex vector space has an eigenvalue.

Proof: V is a f.d.v.s over $\mathbb{C}$. $\dim V = n$

$V \neq 0 \Rightarrow v \neq 0 \quad v \in V$

$v, Tv, T^2v, \ldots, T^nv$ are $n+1$ vector

$\Rightarrow v, Tv, T^2v, \ldots, T^nv$ are linearly dependent.

$\Rightarrow a_0 v + a_1 Tv + \ldots + a_n T^n v = 0 \quad a_i \in \mathbb{C}$ not all $a_i = 0$

not all $a_1 \ldots a_n = 0$

$\Rightarrow p(z) = a_0 + a_1 z + \ldots + a_n z^n \in \mathcal{P}(\mathbb{C})$

By Fundamental theorem of Algebra, $p(z)$ has a root in $\mathbb{C}$.

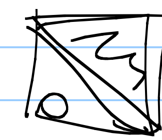
$p(z) = c(z - \lambda_1) \ldots (z - \lambda_n)$

$$p(T) = c(T-\lambda_1 I) \cdots \cdots (T-\lambda_m I)$$

$$\begin{aligned} 0 &= a_0 v + a_1 T v + \cdots + a_n T^n v \\ &= (a_0 I + a_1 T + \cdots + a_n T^n)(v) \\ &= c(T-\lambda_1 I)(T-\lambda_2 I) \cdots (T-\lambda_n I)(v) \end{aligned}$$

$\Rightarrow$ v is in the kernel of $(T-\lambda_j I)$ for some j $\Rightarrow$ T has an eigenvalue j & eigenvector v.

§ Upper triangular Matrices



Theorem: $T \in \mathcal{L}(V)$. $v_1 \ldots v_n$ is a basis for V. TFAE

(a) the matrix of T with respect to $v_1 \ldots v_n$ is upper triangular.

(b) $Tv_j \in \text{Span}(v_1, \ldots, v_j)$ $\forall$ $j = 1 \ldots n$.

(c) $\text{Span}(v_1 \ldots v_j)$ is invariant under T $\forall j = 1 \ldots n$.

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Theorem: $T \in \mathcal{L}(V)$. $v_1 \dots v_n$ be a basis for V. TFAE

(a) The matrix of T w.r.t $v_1 \dots v_n$ is upper triangular

(b) $Tv_j \in \text{span}\langle v_1 \dots v_j \rangle \ \forall j = 1 \dots n$.

(c) $\text{span}\langle v_1 \dots v_j \rangle$ is invariant under T.

$$\begin{bmatrix} a_{11} & & a_{ij} \\ & a_{22} & \\ 0 & & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Proof: (a) $\Rightarrow$ (b) ✓ $\mathcal{M}(T, v_1 \dots v_n) = A$ is upper triangular.

$T(e_1) = a_{11} e_1$

$T(v_j) = Av_j = \sum_{k \le j} a_{ik} v_k \in \text{span}\{v_1 \dots v_j\}$ check

$$\begin{bmatrix} a_{11} & & a_{ij} \\ & a_{22} & \\ 0 & & a_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ 0 \\ 0 \end{bmatrix}$$

e_2

$T(e_2) = a_{12} e_1 + a_{22} e_2$

$\vdots$

$T(e_i) = \sum_{j \le i} a_{ij} e_j$

(b) $\Rightarrow$ (a) $Tv_j \in \text{span}\langle v_1 \dots v_j \rangle$

$Tv_j = \sum_{i \le j} c_i v_i$

$\begin{bmatrix} c_1 \\ \vdots \\ c_j \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is the j^{th} column of $\mathcal{M}(T, v_1 \dots v_n)$

$$\begin{bmatrix} a_{11} & a_{12} & & * \\ 0 & a_{22} & & \\ \vdots & 0 & \ddots & \\ 0 & 0 & & \end{bmatrix} = \mathcal{M}(T, v_1 \dots v_n) \text{ is Upper Triangular.}$$

(b) $\Rightarrow$ (c) $\quad Tv_j \in \operatorname{Span}(v_1 \dots v_j) \quad \forall\, j = 1 \dots n.$

$v \in \operatorname{Span}(v_1 \dots v_j) \implies v = \sum_{i \le j} c_i v_i$

$$T(v) = T\Big(\sum_{i \le j} c_i v_i\Big) = \sum_{i \le j} c_i \underbrace{T(v_i)}_{\in \operatorname{Span}(v_1 \dots v_j)}$$

$T(v_1) \in \operatorname{Span}\langle v_1 \rangle \subseteq \operatorname{Span}(v_1 \dots v_j)$

$T(v_2) \in \operatorname{Span}(v_1, v_2) \subseteq \operatorname{Span}(v_1 \dots v_j)$

$\vdots$

$T(v_j) \in \operatorname{Span}(v_1 \dots v_j)$

$T(v) \in \operatorname{Span}(v_1 \dots v_j)$

$\Rightarrow \operatorname{Span}(v_1 \dots v_j)$ is invariant under T

(c) $\Rightarrow$ (b) Span$(v_1 \dots v_j)$ is invariant under T

$v_j \in$ Span$(v_1 \dots v_j)$ $\Rightarrow$ $T(v_j) \in$ Span$(v_1 \dots v_j)$ $j = 1 \dots n$.

Theorem: V is a f-d-v-s over $\mathbb{C}$ & $T \in \mathcal{L}(V)$. Then T has a upper triangular matrix with respect to some basis of V

Proof 1: Using induction on dim V.

Clearly this true if dim $V = 1$

Assume dim $V > 1$ & suppose that the result holds for all f-d-v-s over $\mathbb{C}$ whose dim is $< n = \dim V$.

Use the one of the previous theorem, to see that T has an eigenvalue λ.

$$U = \text{range}(T - \lambda I)$$

[clear U is invariant under $T - \lambda I$]

$T - \lambda I$ is not injective (λ is an eigenvalue $\Rightarrow$ v is an eigenvector corresponding to λ $\Rightarrow v \in \text{Ker}(T - \lambda I)$)

$\Rightarrow T - \lambda I$ is not surjective! $\Rightarrow$ $\underline{\dim U < \dim V}$.

$u \in U$, $T(u) = \underbrace{(T-\lambda I) u}_{} + \lambda u$.

U is invariant under $T - \lambda I \Rightarrow (T - \lambda I) u \in U$.

$\lambda u \in U$ (as $u \in U$)

$\Rightarrow T(u) \in U$ $\quad$ $\underline{U \text{ is invariant under } T}$

Use the restriction operator $T|_U \in \mathcal{L}(U)$ $\quad T|_U(u) = T(u) \in U \quad \forall u \in U$

$\dim U < \dim V \Rightarrow$ By induction hypothesis, $\exists$ a basis $\underline{u_1 \dots u_m \text{ of } U}$ s.t.

$\mathcal{M}\left(T|_U, u_1 \dots u_m\right)$ is $\underline{\text{upper triangular}} \Leftrightarrow \underbrace{T|_U(u_i)}_{= T(u_i)} \in \underline{\operatorname{Span}(u_1 \dots u_i)}$

Extend $u_1 \dots u_m$ to a basis $u_1 \dots u_m, \underline{v_1 \dots v_n}$ of V.

$$\left[\begin{array}{c|c} \begin{matrix} \diagdown \\ 0 \end{matrix} & \begin{matrix} \\ \end{matrix} \end{array}\right]$$

$\forall k$, $$T(v_k) = \underbrace{(T-\lambda I)v_k}_{\text{range of } T-\lambda I} + \lambda v_k.$$

$$(T-\lambda I)v_k \in \text{range}\,(T-\lambda I) = U \qquad (T-\lambda I)v_k = \sum_{i=1}^{m} c_i u_i$$

$$\underline{T(v_k)} = \sum c_i u_i + \lambda v_k \in \text{Span}(u_1 \dots u_m, v_k)$$

$$\subseteq \underline{\text{Span}(u_1 \dots u_m, v_1 \dots v_k)}$$

$\Rightarrow$ T is upper triangular (using the previous theorem).

Proof 2: Use induction on dim V.

Clearly the result holds for dim $V = 1$

Assume the result hold for all f-d v-s over $\mathbb{C}$ whose dim $< n = \dim V$.

Let λ be an eigenvalue of T (since V is f-d v-s over $\mathbb{C}$)

$\Rightarrow$ Let v_1 be an eigenvector of T w.r.t λ.

$\underline{U = \text{Span} \langle v_1 \rangle}$ $\quad$ v_1 is an eigenvector of $T \Rightarrow T(v_1) = \lambda v_1$

$T(\alpha v_1) = \lambda(\alpha v_1)$.

$\Rightarrow U$ is invariant under T

V/U is the quotient space.

$\dim(V/U) = \dim V - \dim U = n-1$.

By induction hypothesis $\exists$ a basis $v_2+U, \ldots, v_n+U$ of V/U s.t.

$(T/U) \in \mathcal{L}(V/U)$ & T/U w.r.t $v_2+U \ldots v_n+U$ is upper triangular.

$\Rightarrow$ By previous theorem $\underset{\substack{\| \\ T(v_j)+U}}{(T/U)(v_j+U)} \in \text{Span}\,\underbrace{(v_2+U, \ldots, v_j+U)}$

$$T(v_j)+\underset{\substack{\| \\ \alpha v_1}}{U} = \sum_{2 \le i \le j} c_i(v_i + \underset{\substack{\| \\ \alpha_i v_1}}{U})$$

$T(v_j) \in \text{Span}(v_1, v_2, \ldots, v_j)$

From Minor 1, $\{v_1 \ldots v_n\}$ is a basis for V. & $T(v_j) \in \text{Span}(v_1 \ldots v_j)$

$\Rightarrow T$ is upper triangular w.r.t $\{v_1 \ldots v_n)$

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Theorem: V is f.d. v.s over $\mathbb{C}$, $T \in \mathcal{L}(V)$, Then T has an upper triangular matrix w.r.t some basis of V.

Theorem: $T \in \mathcal{L}(V)$. has an upper triangular matrix w.r.t some basis of V.
Then T is invertible $\iff$ all entries on the diagonal of that upper triangular matrix are nonzero.

Proof: Suppose $v_1 \ldots v_n$ is a basis w.r.t which T has the form

$$\mathcal{M}(T, v_1 \ldots v_n) = \begin{bmatrix} \lambda_1 & a_{12} & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

T is invertible $\iff \lambda_i \neq 0 \ \forall i$

Suppose $\lambda_i \neq 0 \ \forall i$ $\qquad v_i \in \text{range}\, T \ \forall i$

$$T v_1 = \lambda_1 v_1 \Rightarrow T\left(\frac{1}{\lambda_1} v_1\right) = \frac{1}{\lambda_1} T(v_1) = \frac{1}{\lambda_1} \lambda_1 v_1 = v_1 \Rightarrow v_1 \in \text{range}\, T$$

$$T v_2 = a_{12} v_1 + \lambda_2 v_2 \Rightarrow T\left(\frac{1}{\lambda_2} v_2\right) = \frac{1}{\lambda_2} T(v_2) = \frac{1}{\lambda_2}(a_{12} v_1 + \lambda_2 v_2) = \frac{a_{12}}{\lambda_2} v_1 + v_2$$

$$v_2 = \underbrace{T\left(\frac{1}{\lambda_2} v_2\right)} - \underbrace{\frac{a_{12}}{\lambda_2} v_1} \in \text{range}\, T$$

$\vdots$

$$T v_i = \sum a_i v_i + \lambda_i v_i \Rightarrow T\left(\frac{1}{\lambda_i} v_i\right) = \sum \frac{a_i}{\lambda_i} v_i + v_i \Rightarrow v_i \in \text{range}\, T$$

$v_i \in \text{range}\, T \ \forall i \Rightarrow V = \text{range}\, T \Rightarrow T$ is surjective.

$\Rightarrow T$ is injective $(\because T \in \mathcal{L}(V))$

$\Rightarrow T$ is invertible

Conversely suppose T is invertible $\Rightarrow \lambda_1 \neq 0$. [If $\lambda_1 = 0 \Rightarrow T(v_1) = \lambda_1 v_1 = 0$
$v_1 \in \text{Null}(T) \Rightarrow\Leftarrow$]

Suppose $\lambda_i = 0$ for some i

$$\Rightarrow T(v_i) = \sum_{k=1}^{i-1} a_k v_k.$$

$U = \mathrm{Span}\langle v_1 \dots v_i \rangle$ $\quad$ U is invariant under T $\quad$ (T is upper triangular w.r.t $v_1 \dots v_n$

$\Leftrightarrow T v_j \in \mathrm{Span}\langle v_1 \dots v_j \rangle \ \forall j$

$\Leftrightarrow \mathrm{Span}\langle v_1 \dots v_j \rangle$ is invariant under T)

Consider restriction operator

$$T|_U \in \mathcal{L}(U)$$

$T|_U : \mathrm{Span}\langle v_1 \dots v_i \rangle \longrightarrow \mathrm{Span}\langle v_1 \dots v_i \rangle$

$T(v_1) \in \mathrm{Span}(v_1)$

$T(v_2) \in \mathrm{Span}(v_1, v_2)$

$\vdots$

$T(v_{i-1}) \in \mathrm{Span}(v_1 \dots v_{i-1})$

$T(v_i) \in \mathrm{Span}(v_1 \dots v_{i-1})$

$\mathrm{Span}\langle v_1 \dots v_{i-1} \rangle \supseteq \mathrm{Range}(T|U)$ has dim $i-1$.

$\Rightarrow \mathrm{Null}(T|U) \neq 0$.

$\Rightarrow v \in U$ s.t $T|_U(v) = 0$.

$\quad\quad \| $

$\quad\quad T(v)$.

$\Rightarrow \exists\, v \in U \subseteq V$ s.t $T(v) = 0$ $\longrightarrow\!\leftarrow$ to assumption that T is invertible

Theorem: $T \in \mathcal{L}(V)$ has upper triangular matrix w.r.t some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of the upper triangular matrix.

Proof: Suppose $v_1 \dots v_n$ is a basis of V s.t

$$\mathcal{M}(T, v_1 \dots v_n) = \begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{bmatrix}$$

$T(v_1) = \lambda_1 v_1$

$(T - \lambda_1 I)(v_1) = 0$

$v_1 \in \text{Null}(T - \lambda I)$

λ is an eigenvalue $\Leftrightarrow \text{Null}(T - \lambda I) \neq 0$

$$\mathcal{M}(T - \lambda I) = \begin{bmatrix} \lambda_1 - \lambda & & * \\ & \lambda_2 - \lambda & \\ & & \ddots \\ 0 & & & \lambda_n - \lambda \end{bmatrix}$$

$T - \lambda I$ is invertible $\Longleftrightarrow \lambda_i - \lambda \neq 0 \ \forall i$

If $\lambda = \lambda_i \Rightarrow T - \lambda I$ is not invertible $\Rightarrow T - \lambda I$ is not injective

$\Rightarrow \text{Null}(T - \lambda I) \neq 0$

λ is an eigenvalue of $T \Longleftrightarrow \lambda = \lambda_i$ for some i $\in$

Ex: $T \in \mathcal{L}(\mathbb{F}^3)$ $T(x,y,z) = (2x+y,\ 5y+3z,\ 8z)$.

w.r.t the standard basis
$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{bmatrix}$$ is upper triangular.

$\Rightarrow$ Eigenvalues are $2, 5, 8$.

§ Eigenspaces & Diagonal matrix.

Def: Diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal

Def: Eigenspace $E(\lambda, T)$

$T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$, the Eigenspace $E(\lambda, T)$ is defined by

$$E(\lambda, T) = \text{Null}(T - \lambda I).$$

[$E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ]

Theorem: V is f-d-v-s. $T \in \mathcal{L}(V)$. $\lambda_1 \ldots \lambda_m$ are distinct eigenvalues of T.
Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T) \text{ is a direct sum.}$$

Furthermore: $\dim E(\lambda_1, T) + \cdots + \dim(E(\lambda_m, T)) \leq \dim V$

Proof: WTS $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum.

$$\boxed{u_1 + \cdots + u_m = 0} \text{ where } u_i \in E(\lambda_i, T)$$

u_i's are eigenvectors corresponding to eigenvalues λ_i

λ_i's are distinct. $\lambda_i \neq \lambda_j$ $i \neq j$.

Eigenvectors corresponding to distinct eigenvalues are linearly independent.

$\Rightarrow$ $u_1 \dots u_m$ are linearly independent.

$\Rightarrow$ $u_i = 0$. $\Rightarrow$ $E(\lambda_1, T) + \dots + E(\lambda_m, T)$ is a direct sum.

$E(\lambda_1, T) + \dots + E(\lambda_m, T)$ is a subspace of V.

$\dim(E(\lambda_1, T) + \dots + E(\lambda_m, T)) \leq \dim V$.

$\quad\quad\quad\quad\quad\quad\quad \parallel$

$\dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$. as the sum is a direct sum.

$$\boxed{\begin{aligned}&\dim(V_1 + V_2)\\ &= \dim V_1 + \dim V_2\\ &\quad - \dim V_1 \cap V_2.\end{aligned}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Def: $T \in \mathcal{L}(V)$ is diagonalizable if T has a diagonal matrix w.r.t some basis of V.

<u>Theorem</u>: V is a f.d.v.s. $T \in \mathcal{L}(V)$. $\lambda_1 \dots \lambda_m$ are distinct eigenvalues of T.

TFAE.

(a) T is diagonalizable

(b) V has basis consisting of eigenvector of T

(c) $\exists$ 1-dim subspace $U_1 \dots U_n$ of V invariant under T s.t

$$V = U_1 \oplus \dots \oplus U_n.$$

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Note Title 22-02-2023

Theorem V is f.d v.s. $T \in \mathcal{L}(V)$. Let $\lambda_1 \dots \lambda_m$ be the distinct eigenvalues of T

TFAE

(a) T is diagonalizable

(b) V has a basis consisting of eigenvectors of T &

(c) $\exists$ 1-dim subspace $U_1 \dots U_n$ of V s.t U_i is invariant under T &

$$V = U_1 \oplus \dots \oplus U_n.$$

(d) $V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$

(e) $\dim V = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$.

Proof: (a) $\Leftrightarrow$ (b) T is diagonalizable iff $\exists$ a basis $v_1 \dots v_n$ s.t.

$$\Leftrightarrow \quad \mathcal{M}(T, v_1 \dots v_n) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$\Leftrightarrow$

$T(v_i) = \lambda_i v_i \quad \forall i = 1 \dots n. \Leftrightarrow v_i$'s are eigenvectors w.r.t eigenvalue λ_i.

(b) $\Rightarrow$ (c) If V has a basis $v_1 \ldots v_n$ consisting of eigenvectors of T

$T(v_i) = \lambda_i v_i \in U_i$

$U_i = \text{span}\langle v_i \rangle \Rightarrow V = U_1 + \cdots + U_n$.

$0 = a_1 v_1 + \cdots + a_n v_n \Rightarrow a_i = 0$

$0 = v_1 + \cdots + v_n$.

(c) $\Rightarrow$ (b) $\exists$ 1-dim invariant subspaces U_i s.t. $V = \underset{\underset{\langle v_1 \rangle}{\|}}{U_1} \oplus \cdots \oplus \underset{\underset{\langle v_n \rangle}{\|}}{U_n}$.

$v_i \in U_i$ invariant under T

$\Rightarrow T(v_i) = \lambda_i v_i \Rightarrow v_i$ is an eigenvector

$v_1 \ldots v_n$ form a basis of V s.t. each v_i is an eigenvector of T.

(b) $\Rightarrow$ (d). V has a basis consisting of eigenvectors of T.

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$$

(d) $\Rightarrow$ (e) ✓

(e) $\Rightarrow$ (b) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$

$\langle v_{11} \ldots v_{1n_1} \rangle$ $\quad\quad$ $\langle v_{m1}, \ldots, v_{mn_m} \rangle$

Consider $\langle v_1 \dots v_n \rangle = \langle v_{11} \dots v_{1n_1}, v_{21} \dots v_{2n_2}, \dots, v_{m1}, \dots, v_{mn_m} \rangle$ Check

$\dim V = n = n_1 + n_2 + \dots + n_m.$

$$\sum c_i v_i = 0 = \sum_{i=1}^{m} \sum_{j=1}^{n_i} c_i v_{ij}$$

$u_i = \sum c_{ij} v_{ij}$, $v_{ij} \in E(\lambda_i, T)$

$$\sum u_i = 0.$$ u_i is the linear combination of vectors from $\sum c_i v_i$ where $v_{ij} \in E(\lambda_i, T)$

$\Rightarrow$ Eigenvectors corresponding to distinct eigenvalues are linearly independent

u_i are eigenvectors of T w.r.t λ_i

$\Rightarrow u_i = 0$

$\Rightarrow \sum_{j=1}^{n} c_i v_{ij} = 0. \Rightarrow c_i = 0$ as v_{ij}'s form a basis for $E(\lambda_i, T)$

Ex: $T \in \mathcal{L}(\mathbb{C}^2)$ $\quad T(w, z) = (z, 0)$

From previous lecture we see that 0 is the only eigenvalue of T.

$E(0, T) = \{(w, 0) : w \in \mathbb{C}\}$. $\dim E(0,T) = 1 < \dim V$

$\Rightarrow T$ is not diagonalizable.

Theorem: If $T \in \mathcal{L}(V)$ has dim V eigenvalues then T is diagonalizable.

Proof: $\dim V = n$ & distinct eigenvalues are $\lambda_1 \dots \lambda_n$.

$$\begin{array}{lccc} & \lambda_1 & \dots & \lambda_n \\ & \downarrow & & \downarrow \\ \text{eigenvects} & v_1 & \dots & v_n \end{array}$$

Eigenvector corresponding to distinct eigenvalues are linearly independent

$\Rightarrow v_1 \dots v_n$ is l-i

$\Rightarrow v_1 \dots v_n$ forms a basis for V.

$\Rightarrow T$ is diagonalizable as v_i's are eigenvectors of T.

Ex: Converse is false. $T: \mathbb{R}^3 \to \mathbb{R}^3$. $\mathcal{M}(T) = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 2 \end{bmatrix}$ eigenvalues are 1, 2. but diagonalizable!

§ Change of Basis.

$\mathcal{B}_1$ is a basis for Vectorspace V $\quad v \in V \quad \mu(v, \mathcal{B}_1) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}. \quad v = \sum a_i v_i.$

$\mathcal{B}_1 = \{v_1 \dots v_n\}$

$\mathcal{B}_2$ is another basis for V. $\quad \mu(v, \mathcal{B}_2) = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}. \quad v = \sum b_i w_i$

$\mathcal{B}_2 = \{w_1 \dots w_n\}$

$$v_i \in V \qquad v_i = \sum_{j=1}^{n} s_{ji} w_j \qquad S = (s_{ij})$$

$$v = \sum a_i v_i = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} s_{ji} w_j = \sum_{j=1}^{n} \underbrace{\sum_{i=1}^{n} a_i s_{ji}} w_j$$

$$\mu(v, \mathcal{B}_1) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\mu(v, \mathcal{B}_2) = S\, \mu(v, \mathcal{B}_1)$$

$$\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = S \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Def: The S defined above is called the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$.

$$M(v, \mathcal{B}_2) = S\,M(v, \mathcal{B}_1) \qquad M(v, \mathcal{B}_1) = T\,M(v, \mathcal{B}_2).$$

$$M(v, \mathcal{B}_2) = ST\,M(v, \mathcal{B}_2) \qquad M(v, \mathcal{B}_1) = TS\,M(v, \mathcal{B}_1).$$

$$\Rightarrow ST = TS = I.$$

$\Rightarrow S$ is invertible.

Ex: $\mathcal{P}_1(\mathbb{R})$. $\mathcal{B}_1 = \{2x+5, x+3\}$ $\quad \mathcal{B}_2 = \{2x-1, x-1\}$.

$$M(v, \mathcal{B}_1) = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \qquad M(v, \mathcal{B}_2) = ??$$

$$2x+5 = s_{11}(2x-1) + s_{21}(x-1)$$
$$= (2s_{11} + s_{21})x + (-s_{11} - s_{21}).$$

$$2s_{11} + s_{21} = 2 \qquad s_{11} = 7$$
$$-s_{11} - s_{21} = 5 \qquad s_{21} = -12$$

$$x+3 = s_{12}(2x-1) + s_{22}(x-1) \Rightarrow s_{12} = 4, \quad s_{22} = -7$$

$$S = \begin{bmatrix} 7 & 4 \\ -12 & -7 \end{bmatrix}$$

$$\mathcal{M}(v, B_2) = S\,\mathcal{M}(v, B_1) = \begin{bmatrix} 7 & 4 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

Theorem. $T \in \mathcal{L}(V)$ $\quad \mathcal{M}(T, B_1) = A \quad \mathcal{M}(T, B_2) = B$

Let S be the change of Basis matrix from B_1 to B_2.

Then $A = S^{-1} B S$.

Proof: $S\,\mathcal{M}(v, B_1) = \mathcal{M}(v, B_2)$.

$$\mathcal{M}(T(v), B_2) = B(\mathcal{M}(v, B_2)) = B(S\,\mathcal{M}(v, B_1))$$

$$\mathcal{M}(T(v), B_1) = S^{-1}(\mathcal{M}(T(v), B_2)) = S^{-1} B S\,\mathcal{M}(v, B_1)) \Rightarrow \boxed{A = S^{-1} B S}$$

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Note Title 28-02-2023

§ Change of Basis

$B_1 = \{v_1 \dots v_n\}$ $B_2 = \{w_1 \dots w_n\}$.

$v_i = \sum_{j=1}^{n} s_{ji} w_j$ $S = (s_{ij})$ the change of Basis matrix from B_1 to B_2.

$M(v, B_2) = S\, M(v, B_1)$

· S is invertible and S^{-1} is the change of basis matrix from B_2 to B_1

Theorem: $T \in \mathcal{L}(V)$. B_1, B_2 are bases for a f-d-vs V, S is the change of basis matrix from B_1 to B_2

$M(T, B_1) = A$ $M(T, B_2) = B$.

Then $A = S^{-1} B S$.

Proof: $v \in V$, $T(v) \in V$ $M(T(v), B_1) = A\, M(v, B_1)$ $M(T(v), B_2) = B\, M(v, B_2)$.

$$B\,\mathcal{M}(v, \mathcal{B}_2) = B\,S\,\mathcal{M}(v, \mathcal{B}_1)$$
$$\parallel$$
$$\mathcal{M}(T(v), \mathcal{B}_2)$$

$$S^{-1}\mathcal{M}(T(v), \mathcal{B}_2) = \mathcal{M}(T(v), \mathcal{B}_1) = A\,\mathcal{M}(v, \mathcal{B}_1) \quad \forall v \in V.$$
$$\parallel$$
$$S^{-1}BS\,\mathcal{M}(v, \mathcal{B}_1) \qquad \Rightarrow \boxed{A = S^{-1}BS}$$

Def: A square matrix A is SIMILAR to a matrix B if $\exists$ an invertible matrix S s.t.
$A = S^{-1}BS$.

Proposition: Two matrices A & B are similar if and only if A & B are the matrix of a linear transform $T \in \mathcal{L}(V)$ w.r.t two bases $\mathcal{B}_1, \mathcal{B}_2$.

Proof: If $\exists$ a $T \in \mathcal{L}(V)$ s.t $\mathcal{M}(T, \mathcal{B}_1) = A$, $\mathcal{M}(T, \mathcal{B}_2) = B$, then A & B are similar by previous theorem.

If A & B are similar, $\exists$ invertible S s.t $A = S^{-1}BS$.

Let $T: V \to V$. $T(x) = Ax$. , $\exists$ a basis $\mathcal{B}_1$ of V s.t $\mathcal{M}(T, \mathcal{B}_1) = A$.

$\mathcal{B}_1 = \{v_1, \dots v_n\}$

$T \in \mathcal{L}(V)$

$\mathcal{B}_2 = \{Sv_1, \dots, Sv_n\}$.

$\mathcal{B}_2$ is also a basis for V. (Check: S is invertible!)

$Bx = \boxed{T(x) = Ax}$.

$\mathcal{M}(T, \mathcal{B}_2) = B$. (Check)

$\mathcal{B}'$

$\mathcal{M}(T, \mathcal{B}') = B$

$V = \mathbb{R}^n$

$\exists \mathcal{B}$ of V s.t

$\mathcal{M}(T, \mathcal{B}) = A$

Theorem: A & B are similar, then $\det(A - \lambda I) = \det(B - \lambda I)$ & they have same eigenvalues & $\det A = \det B$.

Proof: $\det(A - \lambda I) = \det(S^{-1}BS - \lambda I) = \det(S^{-1}BS - \lambda S^{-1}S)$

$= \det(S^{-1}(B - \lambda I)S) = \det S^{-1} \det(B - \lambda I) \det S$

$= \det B - \lambda I$.

$\Rightarrow$ Eigenvalues remain same!

$\det A = \det(S^{-1}BS) = \det S^{-1} \det B \det S = \det B$

§ Simultaneous triangulization.

$T, S \in \mathcal{L}(V)$

If V is f-d-v-s over $\mathbb{C}$, $\Rightarrow \exists\ \mathcal{B}_1, \mathcal{B}_2$ of V s.t $\mathcal{M}(T, \mathcal{B}_1)$ is upper triangular.
$\mathcal{M}(S, \mathcal{B}_2)$ is upper triangular

(revise)

Does $\exists$ a $\mathcal{B}$ of V s.t $\mathcal{M}(T, \mathcal{B})$ & $\mathcal{M}(S, \mathcal{B})$ is upper triangular??

Theorem: $T, S \in \mathcal{L}(V)$, V is f-d-v-s over $\mathbb{C}$

Suppose $TS = ST$ then $\exists$ a basis $\mathcal{B}$ of V s.t $\mathcal{M}(T, \mathcal{B})$, $\mathcal{M}(S, \mathcal{B})$ are upper triangular.

Proof: Let V be a f-d-v-s over $\mathbb{C}$, $T, S \in \mathcal{L}(V)$, $ST = TS$. $\qquad$ $TS = ST$

$\dim V = 1$ ✓

$T \in \mathcal{L}(V)$ $\qquad$ $U = E(\lambda, T)$ $\quad$ λ is an eigenvalue of T.

$T(v) = \lambda v \quad \forall\ v \in U \Rightarrow (T - \lambda I)v = 0 \quad \forall\ v \in U$.

$v \in \ker(T - \lambda I)$.

U is invariant under S!!!

$$(T-\lambda I)S = TS - \lambda S = ST - \lambda S = S(T-\lambda I)$$

$v \in U$ $\quad (T-\lambda I)Sv = S(T-\lambda I)v = S(0) = 0$

$$\Rightarrow Sv \in \operatorname{Ker}(T-\lambda I) = U$$

Consider $S|_U \in \mathcal{L}(U) \Rightarrow \exists\, \lambda'$ s.t. $U' \subseteq U$ & $U' = E(\lambda', S)|_U = E(\lambda', S|_U)$.

$w \in U'$, $\quad W = \langle w \rangle$.

$T(w) = \lambda w$ $\qquad\qquad w \in U' \subseteq U = E(\lambda, T)$

$S(w) = \lambda' w$ $\qquad\qquad w \in U' = E(\lambda', S|_U)$.

W is invariant under T & S !!!

Consider V/W $\quad T/W,\ S/W \in \mathcal{L}(V/W)$

$$\dim V/W = \dim V - \dim W = \dim V - 1$$

By induction $\exists$ $\mathcal{B}' = \{v_2 + W, \ldots, v_n + W\}$ of V/W s.t. $\mu(T/W, \mathcal{B}')$ &

$\mathcal{M}(S/W, \mathcal{B}')$ are upper triangular!!

$\Rightarrow$ $\mathcal{B} = \{w, v_2, \dots, v_n\}$ is a basis for V.

$\mathcal{M}(T, \mathcal{B})$, $\mathcal{M}(S, \mathcal{B})$ are upper triangular. (check). $\leftarrow$

$\mathcal{B} = \{w, v_2, \dots, v_n\}$

$T(v_i) \in \{w, v_2, \dots, v_i\}$

$S(v_i) \in \{w, v_2, \dots, v_i\}$.

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Note Title 01-03-2023

Def: $T, S \in \mathcal{L}(V)$, T, S are simultaneously diagonalizable if $\exists$ a basis $\mathcal{B}$ of V s.t. $\mathcal{M}(T, \mathcal{B})$ & $\mathcal{M}(S, \mathcal{B})$ are diagonal matrix.

Theorem: V is f.d.v.s over $\mathbb{C}$, T, S are diagonalizable

$TS = ST \overset{\Leftarrow}{\Rightarrow}$ T, S are simultaneously diagonalizable

Lemma: $W \subseteq V$ is an T-invariant subspace of V. If $T \in \mathcal{L}(V)$ is diagonalizable then $T|_W \in \mathcal{L}(W)$ is also diagonalizable.

Proof: Suppose T is diagonalizable

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$$

? $W = (W \cap E(\lambda_1, T)) \oplus \cdots \oplus (W \cap E(\lambda_m, T))$ if W is invariant under T.

$w \in W \subseteq V$ $\quad w = \underbrace{u_1 + \cdots + u_k}$. $\quad u_i \in E(\lambda_i, T)$ $\quad w - u_2 - \cdots u_k = u_1$

Claim: $u_i \in W$.

$k=1$ ✓. $w = u_1$. $w \in W$

$Tw = T(u_1 + \cdots + u_k) = T(u_1) + \cdots + T(u_k)$ $\quad u_i \in E(\lambda_i, T)$

$= \lambda_1 u_1 + \cdots + \lambda_k u_k$.

$\lambda_1 w = \lambda_1 u_1 + \cdots + \lambda_k u_k$.

W is invariant under T

$W \ni Tw - \lambda_1 w = (\lambda_2 - \lambda_1) u_2 + \cdots + (\lambda_k - \lambda_1) u_k$.

By induction hypothesis $(\lambda_i - \lambda_1) u_i \in W$ $\quad \Rightarrow u_i \in W$ $\quad i = 2 \cdots k$.

$\Rightarrow u_1 \in W$.

$u_i \in W$. $i = 1 \ldots k$

$u_i \in E(\lambda_i, T) \cap W$

$w = u_1 + \cdots + u_k$ $\quad u_i \in E(\lambda_i, T) \cap W$ $\quad \Rightarrow W = (W \cap E(\lambda_1, T)) \oplus \cdots \oplus (W \cap E(\lambda_m, T))$

Proof of Lemma: If $\dim V = 1$ ✓

Assume $\dim V > 1$.

If $T = \lambda I$ ✓ $V = E(\lambda, T)$ Check.

T is diagonalizable (& T is not a scalar multiple of I).

$$\Rightarrow V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$$
$$= W_1 \oplus \cdots \oplus W_m$$

As in the proof of yesterdays result W_i is invariant under S

$(T - \lambda_i I) S = S (T - \lambda_i I)$
$w \in W_i \quad (T - \lambda_i I) S w = 0$
$\Rightarrow Sw \in W_i$

$$T|_{W_i}, \; S|_{W_i} \in \mathcal{L}(W_i)$$

W_i are invariant under T & S!! By Lemma $T|_{W_i}, S|_{W_i}$ are diagonal.

$\Leftarrow$ $\dim W_i < \dim V$. By induction hypothesis $\exists$ a basis $\mathcal{B}_i$ of W_i s.t. $\mathcal{M}(T|_{W_i}, \mathcal{B}_i)$ & $\mathcal{M}(S|_{W_i}, \mathcal{B}_i)$ are diagonal.

$V = W_1 \oplus \cdots \oplus W_m \quad \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_m$

$$\mathcal{M}(T, \mathcal{B}) = \begin{pmatrix} T|_{W_1} & & & \\ & T|_{W_2} & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}$$

$\mathcal{M}(T, \mathcal{B})$ & $\mathcal{M}(S, \mathcal{B})$ are diagonal!!

$\Rightarrow T$ & S are simultaneously diagonalizable.

$$\left[\begin{array}{l} V = \underline{W_1} \oplus W_2 \\ \quad \{v_1, v_2\} \quad \{v_3, v_4\}. \end{array}\right.$$

W_i are invariant under T

$T(W_1) \subseteq W_1$
$T(W_2) \subseteq W_2$

$T \in \mathcal{L}(V)$ $\qquad \mathcal{M}(T, v_1, v_2, v_3, v_4)$.

$T(v_1) \in W_1 = \langle v_1, v_2 \rangle$.

$$T(v_1) = c_1 v_1 + c_2 v_2 + \underset{0}{c_3} v_3 + \underset{0}{c_4} v_4$$

$$T(v_2) = c_1' v_1 + c_2' v_2 + \underset{0}{c_3'} v_3 + \underset{0}{c_4'} v_4$$

$$T(v_3) = 0 v_1 + 0 v_2 + \cdot\, v_3 + \cdot\, v_4$$

$$\begin{bmatrix} \boxed{\begin{matrix} c_1 & c_1' \\ c_2 & c_2' \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} b & c \\ \cdot & \cdot \end{matrix}} \end{bmatrix}$$

$V = W_1 \oplus \cdots \oplus W_m$ where W_i are <u>generalized</u> eigenspaces

§ Inner Product Spaces.

Def: Dot Product: $x, y \in \mathbb{R}^n$, the dot product of x & y is $x \cdot y$

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n. \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n.$$

Def: Inner Product: An innerproduct on V is a function that takes an ordered pair to an number in $\mathbb{F}$.

$$\langle \, , \rangle : V \times V \longrightarrow \mathbb{F} \qquad (u, v) \longrightarrow \langle u, v \rangle$$

s.t it satisfies

1. $\langle v, v \rangle \geq 0$ for v.
2. $\langle v, v \rangle = 0 \iff v = 0$
3. $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
4. $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$
5. $\langle u, v \rangle = \overline{\langle v, u \rangle}$

Def: An innerproduct space is a vector space V along with an innerproduct on V

Ex: (1) $$\left\langle \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \right\rangle = w_1 \overline{z_1} + \cdots + w_n \overline{z_n}$$

$\in \mathbb{C}^n$.

(2) $c_1, \ldots, c_n$ are positive numbers.

$$\left\langle \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \right\rangle = c_1 w_1 \overline{z_1} + \cdots + c_n w_n \overline{z_n}.$$

(3) $V = \mathcal{C}([-1,1])$ = set of continuous functions on $[-1,1]$

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx.$$

Properties: ① $\forall u \in V$, the function that takes v to $\langle v, u\rangle$ is a linear map from V to $\mathbb{F}$

$$V \longrightarrow \mathbb{F}$$
$$v \longrightarrow \langle v, u\rangle \quad \text{for fixed } u \in V.$$

② $\langle 0, u\rangle = \langle u, 0\rangle = 0 \quad \forall u \in V.$

③ $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle \quad \forall u, v, w \in V.$

④ $\langle u, \lambda v\rangle = \bar{\lambda} \langle u, v\rangle$

Def: norm $\|u\|$

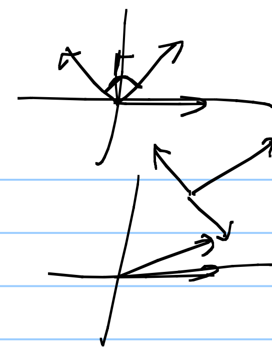
$v \in V$, the norm of v, denoted by $\|v\|$ is defined by

$$\|v\| = \sqrt{\langle v, v\rangle}$$

Properties: ① $\cdot\ v \in V$, $\quad \|v\| = 0 \iff v = 0$

② $\cdot\ \|\lambda v\| = |\lambda|\, \|v\| \quad \lambda \in \mathbb{F}$

Def: Two vectors $u, v \in V$ are orthogonal if $\langle u, v \rangle = 0$



Properties: ① 0 is orthogonal to every vector $v \in V$.

② 0 is the only vector that is orthogonal to itself.

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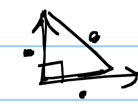
Note Title 03-03-2023

Theorem (Pythagoras theorem): If u & v are orthogonal vectors in V, Then
$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$



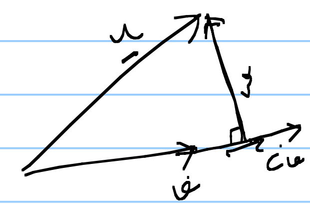
Proof: $\|u+v\|^2 = \langle u+v, u+v \rangle$ & use $\langle u,v\rangle = 0$

Theorem: (Orthogonal Decomposition): $u, v \in V$

$u = cv + w$ where $\langle w, v\rangle = 0$ ✓

$c = \dfrac{\langle u, v\rangle}{\|v\|^2}$ ✓

$w = u - cv$. ✓



Proof: $u = cv + (u - cv)$

$0 = \langle u - cv, v\rangle = \langle u, v\rangle - c\|v\|^2 \Rightarrow c = \dfrac{\langle u, v\rangle}{\|v\|^2}$

Theorem: <u>Cauchy-Schwarz Inequality</u>: $u, v \in V$,

$$|\langle u, v\rangle| \leq \|u\| \, \|v\|. \checkmark$$

Proof: Use the orthogonal decomposition

$$u = \frac{\langle u, v\rangle}{\|v\|^2} v + w.$$

$$\|u\|^2 = \left\| \frac{\langle u, v\rangle}{\|v\|^2} v \right\|^2 + \|w\|^2. \quad \text{(Pythagoras theorem)}$$

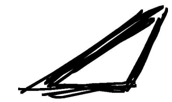
$$= \frac{|\langle u, v\rangle|^2}{\|v\|^2} + \|w\|^2$$

$$\geq \frac{|\langle u, v\rangle|^2}{\|v\|^2}.$$

Theorem (Triangular Inequality): $u, v \in V$.

$$\|u+v\| \le \|u\| + \|v\|.$$

[equality only when $\langle u, v\rangle = \|u\|\,\|v\|$]



Proof: Use $\|u+v\|^2 = \langle u+v, u+v\rangle$ & use $\langle u, v\rangle + \overline{\langle u, v\rangle} = 2\,\mathrm{Re}(\langle u, v\rangle)$.

Theorem (Parallelogram Equality): $u, v \in V$.

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

§ Orthonormal Bases

Def: Orthonormal

- A list of vectors is called orthonormal if each vector in the list has norm 1 & is orthogonal to all other vectors in the list.
- In other words, a list $e_1 \dots e_m \in V$ is orthonormal if $\langle e_j, e_k\rangle = \begin{cases} 1 & j = k. \\ 0 & j \ne k. \end{cases}$

Theorem: If $e_1 \dots e_m$ is an <u>orthonormal</u> list of vectors in V then

$$\|a_1e_1 + \dots + a_me_m\|^2 = |a_1|^2 + \dots + |a_m|^2. \qquad \forall a_1 \dots a_m \in \mathbb{F}$$

Proof: $\|a_1e_1 + a_2e_2\|^2 = \langle a_1e_1 + a_2e_2, a_1e_1 + a_2e_2\rangle$

$$= \langle a_1e_1, a_1e_1 + a_2e_2\rangle + \langle a_2e_2, a_1e_1 + a_2e_2\rangle$$

$$= \langle a_1e_1, a_1e_1\rangle + \underbrace{\langle a_1e_1, a_2e_2\rangle}_{=0} + \underbrace{\langle a_2e_2, a_1e_1\rangle}_{=0} + \langle a_2e_2, a_2e_2\rangle$$

$$= |a_1|^2 + |a_2|^2$$

Use induction & expand the proof (check!)

Theorem: Any orthonormal set of vectors are linearly independent!!

Proof: Suppose $e_1 \dots e_m$ are orthonormal.

$$c_1e_1 + \dots + c_me_m = 0$$

$$\langle c_1e_1 + \dots + c_me_m, e_i\rangle = \langle 0, e_i\rangle = 0$$

$$= c_1 \underbrace{\langle e_1, e_i \rangle}_{0} + c_2 \underbrace{\langle e_2, e_i \rangle}_{0} + \cdots + c_i \underbrace{\langle e_i, e_i \rangle}_{1} + \cdots + c_n \underbrace{\langle e_n, e_i \rangle}_{0} = 0$$

$$\Rightarrow c_i = 0 .$$

Def: Orthonormal Basis: An orthonormal basis of V is a list of orthonormal vectors which also forms a basis for V.

Theorem: Every orthonormal list of vectors in V with length $\dim V$ is an orthonormal basis for V.

Proof: Any linearly independent set with $\dim V$ vectors forms a basis for V.

Theorem: Suppose $e_1 \ldots e_n$ is an orthonormal basis for V, $v \in V$.

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n .$$

Furthermore

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2 .$$

Proof. $e_1 \dots e_n$ form a basis for V.

$$v = a_1 e_1 + \dots + a_n e_n.$$

$$\langle v, e_i \rangle = a_1 \underbrace{\langle e_1, e_i \rangle}_{0} + \dots + a_i \underbrace{\langle e_i, e_i \rangle}_{1} + \dots + a_n \underbrace{\langle e_n, e_i \rangle}_{0}$$

$$\langle v, e_i \rangle = a_i$$

Theorem (Gram-Schmidt): Let $v_1 \dots v_m$ is linearly independent list of vectors in V.

Set $e_1 = \dfrac{v_1}{\|v_1\|}$

$$e_j = \frac{v_j - \langle v_j, e_1 \rangle e_1 - \dots - \langle v_j, e_{j-1} \rangle e_{j-1}}{\| v_j - \langle v_j, e_1 \rangle e_1 - \dots - \langle v_j, e_{j-1} \rangle e_{j-1} \|}.$$

Then $e_1 \dots e_m$ form an orthonormal list of vectors in V. It also satisfies

$\operatorname{Span}\langle v_1 \dots v_j \rangle = \operatorname{Span}\langle e_1 \dots e_j \rangle$ $\forall j = 1 \dots m$.

Proof: $j=1$ ✓.

Assume by induction that $\text{span}\langle v_1 \dots v_{j-1}\rangle = \text{span}\langle e_1 \dots e_{j-1}\rangle$

$v_j \notin \text{span}\langle v_1 \dots v_{j-1}\rangle$

$v_j \notin \text{span}\langle e_1 \dots e_{j-1}\rangle$

$$\langle e_j, e_k\rangle = \left\langle \frac{v_j - \langle v_j, e_1\rangle e_1 - \dots - \langle v_j, e_{j-1}\rangle e_{j-1}}{\underbrace{\| v_j - \langle v_j, e_1\rangle e_1 - \dots - \langle v_j, e_{j-1}\rangle e_{j-1}\|}_{=c}},\ e_k \right\rangle. \qquad \underline{k = 1 \dots j-1}$$

$$= \frac{1}{c}\langle v_j - \langle v_j, e_1\rangle e_1 - \dots - \langle v_j, e_k\rangle e_k - \dots - \langle v_j, e_{j-1}\rangle e_{j-1}, e_k\rangle$$

$$= \frac{1}{c}\Big[\underline{\langle v_j, e_k\rangle} - \langle v_j, e_1\rangle \underbrace{\langle e_1, e_k\rangle}_{=0} - \dots - \underline{\langle v_j, e_k\rangle} \underbrace{\langle e_k, e_k\rangle}_{=1} - \dots - \langle v_j, e_{j-1}\rangle \underbrace{\langle e_{j-1}, e_k\rangle}_{=0}\Big].$$

$$= \frac{1}{c}\Big[\langle v_j, e_k\rangle - \langle v_j, e_k\rangle\Big] = 0$$

$\Rightarrow e_j$ is orthogonal to $e_1 \dots e_{j-1}$ $\Rightarrow \{e_1 \dots e_j\}$ forms an orthonormal set in V.

$v_j \in \text{Span}\langle e_1 \dots e_j \rangle$

$\text{Span}\langle v_1 \dots v_j \rangle \subseteq \text{Span}\langle e_1 \dots e_j \rangle$

$v_1 \dots v_j$ are linearly independent
$e_1 \dots e_j$ are linearly independent (orthonormal)
$\Big\} \Rightarrow \text{Span}\langle v_1 \dots v_j \rangle = \text{Span}\langle e_1 \dots e_j \rangle$

Theorem: Every finite dimensional inner product space has an orthonormal basis.

Proof: Every f.d.v.s has a basis $\mathcal{B}$, Use Gram-Schmidt process on $\mathcal{B}$ to construct orthonormal basis.

Theorem: V is f.d.v.s. Then every orthonormal ~~set~~ set of vectors extends to an orthonormal basis

Proof: $v_1 \dots v_j$ is orthonormal set $\Rightarrow v_1 \dots v_j$ is linearly independent.

$\Rightarrow$ Extend to a basis $\{v_1 \dots v_j, v_{j+1} \dots v_n\}$

$\Rightarrow$. Use Gram-Schmidt to construct orthonormal basis!!

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Note Title 14-03-2023

Gram-Schmidt.

$v_1 \dots v_m$ are linearly independent set of vectors in V.

$$e_1 = \frac{v_1}{\|v_1\|}$$

$$e_j = \frac{1}{c}\left(v_j - \langle v_j, e_1\rangle e_1 - \dots - \langle v_j, e_{j-1}\rangle e_{j-1}\right).$$

where $c = \| v_j - \langle v_j, e_1\rangle e_1 - \dots - \langle v_j, e_{j-1}\rangle e_{j-1} \|$

Then $e_1 \dots e_m$ is an orthonormal list of vectors s.t.

$\mathrm{Span}(v_1 \dots v_j) = \mathrm{Span}(e_1 \dots e_j)$ $\forall j$.

Theorem: Suppose $T \in \mathcal{L}(V)$. If T has an upper triangular matrix w.r.t some basis of V then T has an upper triangular matrix. w.r.t an orthonormal basis of V

Proof: T has a upper triangular matrix representation w.r.t some basis $v_1 \cdots v_n$ of V.

Apply Gram-Schmidt process to construct an orthonormal basis $e_1 \cdots e_n$.

[T is upper triangular w.r.t $v_1 \cdots v_n \iff$. Span $(v_1 \cdots v_j)$ is invariant under T $\forall j$]

By Gram Schmidt Span $(v_1 \cdots v_j) = \text{span}(e_1 \cdots e_j)$ $\forall j$

$\Rightarrow$ Span $(e_1 \cdots e_j)$ is invariant under T.

$\Rightarrow$ T is upper triangular w.r.t $e_1 \cdots e_j$.

Theorem (Schur's Theorem): If V is f.d. v.s over $\mathbb{C}$, then $T \in \mathcal{L}(V)$ has a upper triangular representation w.r.t some orthonormal basis of V.

§ Linear Functionals on Inner Product Spaces

Def: A linear functional on V is a linear map from V to $\mathbb{F}$. $\mathcal{L}(V, \mathbb{F})$

Ex: $\varphi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$

$$\varphi(p) = \int_{-1}^{1} p(t) \cos \pi t \, dt.$$

Theorem: V is a f-d v.s & φ is a linear functional on V. Then $\exists$ a Unique vector u s.t. $\varphi(v) = \langle v, u \rangle$

Proof: Let $e_1 \ldots e_n$ be an orthonormal basis for V.

$\varphi(v) \in \mathbb{F}$

$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$

$\varphi(\lambda v) = \lambda \varphi(v)$

$$\varphi(v) = \varphi\left(\langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n\right)$$

$$\langle v, \lambda u\rangle = \bar{\lambda}\langle v, u\rangle$$

$$= \langle v, e_1\rangle \varphi(e_1) + \cdots + \langle v, e_n\rangle \varphi(e_n)$$

$$= \langle v, \overline{\varphi(e_1)}\, e_1 + \cdots + \overline{\varphi(e_n)}\, e_n\rangle \quad \forall v \in V.$$

$$= \langle v, \overline{\varphi(e_1)}\, e_1\rangle + \cdots + \langle v, \overline{\varphi(e_n)}\, e_n\rangle$$

Set $u = \overline{\varphi(e_1)}\, e_1 + \cdots + \overline{\varphi(e_n)}\, e_n$. $\Rightarrow \varphi(v) = \langle v, u\rangle \ \forall v \in V.$

Suppose $\exists\, u_1, u_2 \in V$ s.t. $\varphi(v) = \langle v, u_1\rangle = \langle v, u_2\rangle \ \forall v \in V$

$$0 = \langle v, u_1\rangle - \langle v, u_2\rangle = \langle v, u_1 - u_2\rangle \quad \forall v \in V.$$

$$\Rightarrow u_1 - u_2 = 0$$

$$\Rightarrow u_1 = u_2$$

§ Orthogonal Complement.

Def: $U \subset V$, then the orthogonal complement of U, denoted by $U^{\perp}$ is the set of all vectors in V that are orthogonal to all vectors in U.

$$U^{\perp} = \{ v \in V : \langle v, u \rangle = 0 \ \forall u \in U \}$$

Basic Properties

1. If U is a subset of V then $U^{\perp}$ is a subspace of V.

$\langle 0, u \rangle = 0 \ \forall u \in U \Rightarrow 0 \in U^{\perp}$.

$u_1, u_2 \in U^{\perp} \Rightarrow \langle u_1, u \rangle = \langle u_2, u \rangle = 0 \ \forall u \in U$.

$\langle u_1 + u_2, u \rangle = \langle u_1, u \rangle + \langle u_2, u \rangle = 0 + 0 = 0 \Rightarrow u_1 + u_2 \in U^{\perp}$

$u \in U^{\perp}, \lambda \in \mathbb{F} \Rightarrow \langle \lambda u, v \rangle = 0 \ \forall v \in U$.

$= \lambda \langle u, w \rangle = 0 \Rightarrow \lambda u \in U^{\perp}$.

(2) $\{0\}^{\perp} = V$

(3) $V^{\perp} = \{0\}$

(4) If U is a subset of V then $U \cap U^{\perp} = \{0\}$.

(5) If U, W are subsets of V, $U \subseteq W$ then $W^{\perp} \subseteq U^{\perp}$

U subspace of V

$U \cap U^{\perp} = 0$

$U \oplus U^{\perp}$

Theorem: If U is a finite dimensional subspace of V. Then

$$V = U \oplus U^{\perp}$$

Proof: $v \in V$.

U is finite dimensional $\Rightarrow$ choose $e_1 \cdots e_m$ an orthonormal basis for U.

$$v = \underbrace{\langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m}_{u} + \underbrace{(v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_m\rangle e_m)}_{w}$$

$U = \langle e_1 \cdots e_m \rangle$

$\langle w, e_j \rangle = \langle v - \langle v, e_1 \rangle e_1 - \cdots - \langle v, e_m \rangle e_m, e_j \rangle$

$= \langle v, e_j \rangle - \langle v, e_1 \rangle \underbrace{\langle e_1, e_j \rangle}_{=0} - \cdots - \langle v, e_j \rangle \underbrace{\langle e_j, e_j \rangle}_{=1} - \cdots - \langle v, e_m \rangle \underbrace{\langle e_m, e_j \rangle}_{=0}$

$= \langle v, e_j \rangle - \langle v, e_j \rangle = 0$

$$V = U \oplus U^{\perp}$$ U is f-d. subspace of V.

Theorem: V is a f-d. and U is a subspace of V Then

$$\dim U^{\perp} = \dim V - \dim U.$$

Theorem: U is a finite dimensional subspace of V. Then

$$U = (U^{\perp})^{\perp}$$

Proof: ① $U \subseteq (U^\perp)^\perp$ $\qquad$ $U^\perp = \{v,\ \langle v, u\rangle = 0\ \forall u \in U\}$

$u \in U \Rightarrow \langle u, v\rangle = 0\ \forall v \in U^\perp$

$\Rightarrow$ u is orthogonal to every vector in $U^\perp$

$\Rightarrow u \in (U^\perp)^\perp$

② $v \in (U^\perp)^\perp \Rightarrow v = u + w \quad u \in U,\ w \in U^\perp$

$v - u = w \in U^\perp$

$v \in (U^\perp)^\perp \quad u \in U \subseteq (U^\perp)^\perp \Rightarrow v - u \in (U^\perp)^\perp$

$v - u \in U^\perp \cap (U^\perp)^\perp = \{0\}$

$\Rightarrow v = u \in U$

Def: Orthogonal Projection P_U.

Suppose U is a f.d.v.s of V. The orthogonal projection of V onto U is

$$P_U \in \mathcal{L}(V)$$

$$P_U(v) = u \quad \text{where} \quad v = u + w \text{ for } u \in U, w \in U^\perp.$$

Ex: $x \in V$, $x \neq 0$, $U = \operatorname{span}(x)$. $P_U(v) = \dfrac{\langle v, x\rangle}{\|x\|^2} x$. $\forall v \in V$.

Properties of Projection Operator

Suppose U is a f.d. subspace of V, $v \in V$.

(a) $P_U \in \mathcal{L}(V)$.

(b) $P_U(u) = u$, $u \in U$

(c) $P_U(w) = 0$, $w \in U^\perp$

$v_1 = u_1 + w_1$

$v_2 = u_2 + w_2$

$P_U(v_1 + v_2)$

$= (u_1 + u_2)$

$= P_U(u_1) + P_U(u_2)$

$\Leftarrow$

$$\boxed{\begin{aligned} &V = U \oplus U^\perp \\ &v = u + w \quad u \in U,\ w \in U^\perp \\ &P_U(v) = u \end{aligned}}$$

$u \in U \Rightarrow u = u + 0$

$w \in U^\perp \Rightarrow w = 0 + w$

(d) range $P_U = U$

(e) null $P_U = U^\perp$

(f) $v - P_U(v) \in U^\perp$

(g) $P_U^2 = P_U$

(h) $\|P_U v\| \le \|v\|$

(i) $\forall$ orthonormal basis $e_1 \cdots e_m$ of U $P_U(v) = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$.

$v = u + w$.

$P_U(v) =$

$v - P_U(v) = v - u = w \in U^\perp$

$P_u^2(v) = P_u P_u(v) = P_u(u) = u = P_u(v)$

$P_u(v) = u = \langle u, e_1\rangle e_1 + \cdots + \langle u, e_m\rangle e_m$.

$\langle \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m, e_1 \rangle$

$= \langle \langle u+w, e_1\rangle e_1 + \cdots + \langle u+w, e_m\rangle e_m \; e_1 \rangle$

$= \langle u+w, e_1\rangle e_1$

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Note Title 15-03-2023

Def: $T \in \mathcal{L}(V,W)$. The adjoint of T is $T^*: W \to V$

$$\langle Tv, w\rangle = \langle v, T^* w\rangle$$

Riez Representation theorem. $\varphi: V \to \mathbb{F}$

$v \to \langle Tv, w\rangle$.

$\varphi(v) = \langle v, \underline{u}\rangle$ u is unique

$u = T^* w$.

Ex: Fix $u \in V$, $x \in W$, Define $T \in \mathcal{L}(V,W)$.

$$\boxed{Tv = \langle v, u\rangle x}$$

Fix $w \in W$, $\forall v \in V$

$T^* = ??$

$$\langle v, T^* w\rangle = \langle Tv, w\rangle.$$

$$= \langle \langle v, u\rangle x, w\rangle.$$

$$= \langle v, u\rangle \langle x, w\rangle, \quad = \langle x, w\rangle \langle v, u\rangle$$

$$= \langle v, \langle w, x\rangle u\rangle.$$

$\overline{\langle x, w\rangle} = \langle w, x\rangle$

$$T^* w = \langle w, x\rangle u.$$

Theorem: $T^* \in \mathcal{L}(W, V)$

Proof: $T \in \mathcal{L}(V, W)$, $w_1, w_2 \in W$, $v \in V$.

$$\langle v, T^*(w_1 + w_2) \rangle = \langle Tv, w_1 + w_2 \rangle$$
$$= \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle.$$
$$= \langle v, T^* w_1 \rangle + \langle v, T^* w_2 \rangle$$
$$= \langle v, T^* w_1 + T^* w_2 \rangle$$
$$\Rightarrow T^*(w_1 + w_2) = T^* w_1 + T^* w_2.$$

Similarly $T^*(\lambda w) = \lambda T^* w$. (check)

Properties of Adjoint.

① $(S+T)^* = S^* + T^*$

② $(\lambda T)^* = \bar{\lambda} T^* \quad \forall \lambda \in \mathbb{F}$

③ $(T^*)^* = T$

④ $I^* = I$ where $I \in \mathcal{L}(V, V)$ is an identity operator

⑤ $(ST)^* = T^* S^* \quad \forall T \in \mathcal{L}(V, W) \; S \in \mathcal{L}(W, U)$.

Proof: ① $\langle v, (S+T)^* w \rangle = \langle (S+T) v, w \rangle = \langle Sv + Tv, w \rangle$

$= \langle Sv, w \rangle + \langle Tv, w \rangle = \langle v, S^* w \rangle + \langle v, T^* w \rangle$

$= \langle v, (S^* + T^*) w \rangle$

$(S+T)^* = S^* + T^*$

③ $T \in \mathcal{L}(V, W)$. $\langle v, (T^*)^* w \rangle = \langle T^* v, w \rangle = \overline{\langle w, T^* v \rangle} = \overline{\langle T w, v \rangle}$

$= \langle v, Tw \rangle$

(5) $T \in \mathcal{L}(V,W)$ $S \in \mathcal{L}(W,U)$. $v \in V$, $u \in U$.

$$\langle v, (ST)^* u\rangle = \langle (ST)v, u\rangle = \langle Tv, S^* u\rangle = \langle v, T^* S^* u\rangle.$$

$$(ST)^* = T^* S^*.$$

Theorem. $T \in \mathcal{L}(V,W)$

(1) $\operatorname{null} T^* = (\operatorname{range} T)^\perp$

(2) $\operatorname{range} T^* = (\operatorname{null} T)^\perp$

(3) $\operatorname{null} T = (\operatorname{range} T^*)^\perp$

(4) $\operatorname{range} T = (\operatorname{null} T^*)^\perp$

$$(U^\perp)^\perp = U$$

Proof: (1) $w \in \operatorname{null} T^* \iff T^* w = 0$

$\iff \langle v, T^* w\rangle = 0 \quad \forall v \in V.$

$\iff \langle Tv, w\rangle = 0 \quad \forall v \in V$

$\iff w \in (\operatorname{range} T)^\perp$

$\Rightarrow$ (3).

$$(\text{null } T)^{\perp} = \left((\text{range } T^*)^{\perp}\right)^{\perp} = \text{range } T^* \Rightarrow (2) \Rightarrow (4).$$

Theorem: $T \in \mathcal{L}(V, W)$. $e_1 \cdots e_n$ is an orthonormal basis for V
$f_1 \cdots f_m$ W

$\mathcal{M}(T^*, f_1 \cdots f_m, e_1 \cdots e_n)$ is the conjugate-transpose of $\mathcal{M}(T, e_1 \cdots e_n, f_1 \cdots f_m)$

Proof: $\mathcal{M}(T, e_1 \cdots e_n, f_1 \cdots f_m)$

k^{th} col: $T(e_k) = c_1 f_1 + \cdots + c_m f_m$

$$\begin{array}{c} k\text{th col} \\ \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \end{array}$$

$= \langle T(e_k), f_1 \rangle f_1 + \cdots + \langle T(e_k), f_m \rangle f_m$

k col of $\mathcal{M}(T, e_1 \cdots e_n, f_1 \cdots f_m)$ is $\begin{bmatrix} \langle T(e_k), f_1 \rangle \\ \vdots \\ \langle T(e_k), f_m \rangle \end{bmatrix}$

row j

$\langle T(e_k), f_j \rangle$

$= \langle e_k, T^*(f_k) \rangle$

$= \overline{\langle T^*(f_j), e_k \rangle}$.

j th col of $\mathcal{M}(T^*, f_1 \dots f_m, e_1 \dots e_n)$

$$T^*(f_j) = d_1 e_1 + \dots + d_n e_n.$$

$$= \langle T^*(f_j), e_1 \rangle e_1 + \dots + \langle T^*(f_j), e_n \rangle e_n$$

j^{th} col

$$\begin{bmatrix} \langle T^*(f_j), e_1 \rangle \\ \vdots \\ \langle T^*(f_j), e_n \rangle \end{bmatrix}$$

k^{th} row

$\langle T^*(f_j), e_k \rangle$.

(j, k) entry of $\mathcal{M}(T, e_1 \dots e_n, f_1 \dots f_m)$ is $\langle T^*(f_j), e_k \rangle$

(k, j) entry of $\mathcal{M}(T^*, f_1 \dots f_m, e_1 \dots e_n)$ is $\langle T^*(f_j), \cancel{e_k} \rangle$

Def: $T \in \mathcal{L}(V)$ is called self adjoint if $T = T^*$.

$$\langle Tv, w\rangle = \langle v, T^* w\rangle = \langle v, Tw\rangle$$

Ex: $T \in \mathcal{L}(\mathbb{F}^2)$. $\mathcal{M}(T) = \begin{bmatrix} 2 & b \\ 3 & 7 \end{bmatrix}$. If T is self adjoint, then $b = 3$.

Theorem: Every eigenvalue of a self adjoint operator is real!!

Proof: T is a self adjoint operator. λ is an eigenvalue of T, $v \neq 0$ $Tv = \lambda v$.

$$\lambda \|v\|^2 = \lambda \langle v, v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar{\lambda} \langle v, v\rangle = \bar{\lambda} \|v\|^2.$$

$$\Rightarrow \lambda = \bar{\lambda}$$

$\Rightarrow \lambda$ is real eigenvalue.

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Note Title 17-03-2023

Def: Self Adjoint Operator: An operator $T \in \mathcal{L}(V)$ is called self-adjoint if $T = T^*$.

$$\left[\Rightarrow \langle Tv, w\rangle = \langle v, T^* w\rangle = \langle v, Tw\rangle \quad \forall v, w \in V \right]$$

Remark: The entries of matrix of selfadjoint operators need not be real!

$$\mathcal{M}(T) = \begin{bmatrix} 2 & 2+i \\ 2-i & 5 \end{bmatrix} \text{ is selfadjoint } \left(\mathcal{M}(T^*) = \begin{bmatrix} 2 & 2+i \\ 2-i & 5 \end{bmatrix} \right)$$

Theorem: Every eigenvalue of selfadjoint operators are real!

Theorem: Suppose V is a Complex inner product space $T \in \mathcal{L}(V)$. Suppose

$$\langle Tv, v \rangle = 0 \quad \forall\, v \in V.$$

Then $T = 0$

Proof:

$$\langle Tu, w \rangle = \frac{\langle T(u+w), u+w \rangle - \langle T(u-w), u-w \rangle}{4} + \frac{\langle T(u+iw), u+iw \rangle - \langle T(u-iw), u-iw \rangle}{4}$$

$T(u+w) = T(u)+T(w)$, $T(u-w) = T(u)-T(w)$, $T(u+iw) = T(u)+iT(w)$, $T(u-iw) = T(u)-iT(w)$

$$= 0$$

$\langle Tu, w \rangle = 0 \;\; \forall\, u, w \in V \implies T = 0.$

$w = Tu.$

$\langle Tu, Tu \rangle = 0 \;\; \forall u \in V$

$\|Tu\|^2 = 0 \;\; \forall u \in V$

Theorem: Suppose V is a Complex inner product space & $T \in \mathcal{L}(V)$. Then T is self adjoint if & only if $\langle Tv, v \rangle \in \mathbb{R} \quad \forall\, v \in V.$

Proof: $v \in V.$

$$\langle Tv, v\rangle - \overline{\langle Tv, v\rangle} = \langle Tv, v\rangle - \langle v, Tv\rangle = \langle Tv, v\rangle - \langle T^*v, v\rangle = \langle (T-T^*)(v), v\rangle$$

Suppose $\langle Tv, v\rangle \in \mathbb{R}\ \forall\, v \in V \Rightarrow$ LHS $= 0 \Rightarrow T = T^* \Rightarrow$ self adjoint.

Conversely if T is self adjoint $\Rightarrow T = T^* \Rightarrow$ RHS $= 0 \Rightarrow \langle Tv, v\rangle = \overline{\langle Tv, v\rangle} \Rightarrow \langle Tv, v\rangle \in \mathbb{R}$

Theorem: Suppose T is a self adjoint operator on V, Then

$$\langle Tv, v\rangle = 0\ \ \forall\, v \in V \implies T = 0.$$

Def: Normal Operators: $T \in \mathcal{L}(V)$, T is said to be normal if $TT^* = T^*T$

(T commutes with T^*)

(self adjoint operator $\Rightarrow$ Normal)

But Normal Operators need not be selfadjoint.

Ex: $T = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ $\quad T^* = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ $\quad TT^* = T^*T$ but $T \neq T^*$.

Theorem: An operator $T \in \mathcal{L}(V)$ is normal if and only if

$$\|Tv\| = \|T^*v\| \quad \forall\, v \in V.$$

Proof: T is normal $\iff T^*T = TT^* \iff T^*T - TT^* = 0$

$\langle T^*Tv - TT^*v, v\rangle$

$$\iff \langle (T^*T - TT^*)v, v\rangle = 0 \quad \forall\, v \in V.$$

$$\iff \langle T^*Tv, v\rangle = \langle TT^*v, v\rangle \quad \forall\, v \in V$$

$$\iff \langle Tv, Tv\rangle = \langle T^*v, T^*v\rangle \quad \forall\, v \in V.$$

$$\iff \|Tv\| = \|T^*v\| \quad \forall\, v \in V.$$

Remark: If T is normal $\Rightarrow \operatorname{null} T = \operatorname{null} T^*$.

Theorem: Suppose $T \in \mathcal{L}(V)$. T is normal, $v \in V$ is an eigenvector of T with eigenvalue λ.
Then v is also a eigenvector for T^* with eigenvalue $\bar{\lambda}$.

Proof: T is normal $\Rightarrow TT^* = T^*T$

$\Rightarrow (T-\lambda I)$ is also normal

$$\left[(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)(T^* - (\lambda I)^*) = (T-\lambda I)(T^* - \bar{\lambda} I) \right.$$

$$= TT^* - \bar{\lambda} T - \lambda T^* + \lambda \bar{\lambda} I$$

$$= T^*T - \bar{\lambda} T - \lambda T^* + \lambda \bar{\lambda} I = (T^* - \bar{\lambda} I)(T - \lambda I)$$

$$\left. = (T-\lambda I)^*(T-\lambda I) \right]$$

$$0 = \|(T-\lambda I)v\| = \|(T-\lambda I)^* v\| = \|(T^* - \bar{\lambda} I)v\|$$

$\Rightarrow v$ is an eigenvector for T^* with eigenvalue $\bar{\lambda}$.

Theorem: $T \in \mathcal{L}(V)$ is normal. Then eigenvector of T corresponding to distinct eigenvalues are orthogonal.

Proof: Suppose α, β are eigenvalues for T with eigenvector u, v.

$$Tu = \alpha u, \quad Tv = \beta v.$$

$$\Rightarrow T^* u = \bar{\alpha} u \qquad T^* v = \bar{\beta} v$$

If $\alpha \neq \beta$.

$$(\alpha-\beta)\langle u, v\rangle = \langle \alpha u, v\rangle - \langle \beta u, v\rangle$$
$$= \langle \alpha u, v\rangle - \langle u, \bar{\beta} v\rangle$$
$$= \langle Tu, v\rangle - \langle u, T^* v\rangle = \langle Tu, v\rangle - \langle Tu, v\rangle$$
$$= 0$$

$\langle u, v\rangle = 0 \Rightarrow u$ is orthogonal to v.

§ Spectral Theorem.

Theorem: $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$. Then the following are equivalent.

(1) T is normal

(2) V has an orthonormal basis consisting of eigenvectors of T

(3) T has a diagonal matrix representation w.r.t some orthonormal basis of V.

Theorem: $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$. Then the following are equivalent.

1. T is selfadjoint
2. V has a orthonormal basis consisting of eigenvectors of T
3. T has a diagonal matrix representation w.r.t. some orthonormal basis of V

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Theorem (Spectral Theorem): $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$. The following are equivalent

① T is normal
② V has an orthonormal basis consisting of eigenvectors of T
③ T is a diagonal matrix w.r.t some orthonormal basis of V

Proof: ② $\Leftrightarrow$ ③ ✓

Suppose ① holds $\Rightarrow$ T is normal

By Schur's theorem $\exists$ an orthonormal basis $\mathcal{B} = \{e_1, \dots, e_n\}$ with respect to which $\mathcal{M}(T)$ is upper triangular.

$$\mathcal{M}(T, e_1, \dots, e_n) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{bmatrix}$$

$$\mathcal{M}(T^*, e_1 \cdots e_n) = \begin{bmatrix} \bar{a}_{11} & 0 & \cdots & 0 \\ \bar{a}_{12} & \bar{a}_{22} & & 0 \\ \vdots & & \ddots & 0 \\ \bar{a}_{1n} & & & \bar{a}_{nn} \end{bmatrix}$$

$T e_1 = a_{11} e_1 \qquad T^* e_1 = \bar{a}_{11} e_1 + \cdots + \bar{a}_{1n} e_n.$

T normal $\Rightarrow \|Tv\| = \|T^* v\|$

T is normal $\Rightarrow \|Te_1\| = \|T^* e_1\|$

$$\begin{array}{ccc} \|Te_1\| & = & \|T^* e_1\| \\ \| & & \| \\ |a_{11}|^2 & & |a_{11}|^2 + \cdots + |a_{1n}|^2 \end{array}$$

$\Rightarrow |a_{12}|^2 + \cdots + |a_{1n}|^2 = 0 \Rightarrow |a_{1i}|^2 = 0 \Rightarrow a_{1i} = 0 \quad \forall\, 1 \le i \le n$

$$\mathcal{M}(T) = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ & a_{22} & a_{2\cdot} \cdots & a_{2n} \\ 0 & & \ddots & \\ & & & a_{nn} \end{bmatrix} \qquad \mathcal{M}(T^*) = \begin{bmatrix} \bar{a}_{11} & & & \\ 0 & \bar{a}_{22} & & 0 \\ \vdots & & \ddots & \\ 0 & & & \bar{a}_{nn} \end{bmatrix}$$

$T(e_2) = a_{22} e_2 \qquad T^*(e_2) = \overline{a_{22}} e_2 + \overline{a_{23}} e_3 + \cdots + \overline{a_{2n}} e_n$

T is normal $\Rightarrow \| T(e_2) \| = \| T^*(e_2) \| \Rightarrow a_{2i} = 0 \quad 3 \le i \le n$. (same as above)

$$\underset{|a_{22}|^2}{\|T(e_2)\|} = \underset{|a_{22}|^2 + \cdots + |a_{2n}|^2}{\|T^*(e_2)\|}$$

Keep repeating this process to see

$\mathcal{M}(T)$ is a diagonal matrix. $\Rightarrow$ (3)

Suppose (3) is true: $\mathcal{M}(T)$ is diagonal w.r.t some orthonormal basis of V.

$\Rightarrow \mathcal{M}(T^*)$ is also diagonal w.r.t the same orthonormal basis.

$\mathcal{M}(T)$ & $\mathcal{M}(T^*)$ commute. $\Rightarrow$ (1)

Real Spectral Theorem.

Lemma: $T \in \mathcal{L}(V)$ & T is selfadjoint, $b, c \in \mathbb{R}$, $b^2 < 4c$. then

$T^2 + bT + cI$ is invertible.

Proof: $\forall v \in V$, $v \neq 0$

$$\langle (T^2 + bT + cI)v, v\rangle = \langle T^2 v, v\rangle + b\langle Tv, v\rangle + c\underbrace{\langle v, v\rangle}_{= \|v\|^2}$$

$$= \langle Tv, T^* v\rangle + b\langle Tv, v\rangle + c\|v\|^2. \quad (T^* = T)$$

$$= \langle Tv, Tv\rangle + b\langle Tv, v\rangle + c\|v\|^2$$

$$= \|Tv\|^2 + b\langle Tv, v\rangle + c\|v\|^2.$$

$$= \underbrace{\left(\|Tv\| - \frac{|b|\,\|v\|}{2}\right)^2}_{\geq 0} + \underbrace{\left(c - \frac{b^2}{4}\right)}_{>0}\|v\|^2$$

$$> 0$$

$(T^2+bT+cI)v \neq 0 \quad \forall \overset{\neq 0}{v} \in V. \Rightarrow T^2+bT+cI$ is injective.

$\Rightarrow T^2+bT+cI$ is invertible.

Lemma: $p \in \mathcal{P}(\mathbb{R})$ is a non constant polynomial. Then p has a Unique factorization (except for the order of the factors) of the form.

$$p(x) = c(x-\lambda_1)\cdots(x-\lambda_m)(x^2+b_1x+c_1)\cdots(x^2+b_Mx+c_M)$$

where $c, \lambda_1 \cdots \lambda_m, b_1 \cdots b_m, c_1 \cdots c_m \in \mathbb{R}$ & $b_j^2 < 4c_j \quad \forall \; 1 \leq j \leq M$.

Proof: Assume $p(x) \in \mathcal{P}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{C})$.

Suppose $\lambda \in \mathbb{C}$ is a root of $p(x)$ & $\lambda \notin \mathbb{R}$. $\Rightarrow \bar{\lambda}$ is a root of $p(x)$

$\Rightarrow p(x) = (x-\lambda)(x-\bar{\lambda}) \cdot q(x) \qquad q(x) \in \mathcal{P}(\mathbb{C})$.

$= (x^2 - 2(\mathrm{Re}\,\lambda)x + |\lambda|^2) \cdot q(x)$.

$\hookrightarrow$ a polynomial in $\mathcal{P}(\mathbb{R})$

$$q(x) = \frac{p(x)}{x^2 - 2(\operatorname{Re}\lambda)x + |\lambda|^2}$$

$\forall x \in \mathbb{R} \Rightarrow q(x) \in \mathbb{R}.$

$q(x) = a_0 + a_1 x + \cdots + a_{n-2} x^{n-2} \quad a_i \in \mathbb{C},$

$\forall x \in \mathbb{R} \Rightarrow 0 = \operatorname{Im} q(x) = \operatorname{Im}(a_0) + \operatorname{Im}(a_1)x + \cdots + \operatorname{Im}(a_{n-2}) x^{n-2}$

$\Rightarrow \operatorname{Im}(a_0) = \operatorname{Im}(a_1) = \cdots = \operatorname{Im}(a_{n-2}) = 0$

$\Rightarrow q(x) \in \mathcal{P}(\mathbb{R})$

Now $q(x) \in \mathcal{P}(x)$ & $\deg q(x) = n-2 < \deg p(x)$.

By induction we can write $q(x) = c(x-\lambda_1)\cdots(x-\lambda_m)(x^2+b_1x+c_1)\cdots(x^2+b_{M-1}x+c_{M-1})$

$$\Rightarrow p(x) = \underbrace{c(x-\lambda_1)\cdots(x-\lambda_m)(x^2+b_1x+c_1)\cdots(x^2+b_{M-1}x+c_{M-1})}_{q(x)}\underbrace{(x^2+b_Mx+c_M)}_{x^2-2\operatorname{Re}\lambda x+|\lambda|^2}.$$

Theorem: $V \neq 0$, $T \in \mathcal{L}(V)$ is a selfadjoint operator. Then T has an eigenvalue.

(Compare with an earlier theorem for vector spaces over $\mathbb{C}$).

Proof: Let V be a real inner product space.

$n = \dim V$. Then

$$v, Tv, \ldots, T^n v \quad \text{are linearly dependent}$$

$$\Rightarrow \exists\, a_0 \ldots a_n \in \mathbb{R} \text{ s.t. } \quad (\text{not all } a_i = 0)$$

$$a_0 v + a_1 Tv + \cdots + a_n T^n v = 0$$

$a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{P}(\mathbb{R})$

By previous theorem $\quad a_0 + a_1 x + \cdots + a_n x^n = c(x-\lambda_1)\cdots(x-\lambda_m)(x^2+b_1x+c_1)\cdots(x^2+b_M x+c_M)$.

$$b_j^2 < 4c_j \quad 1 \leq j \leq M.$$

$$0 = a_0 v + a_1 Tv + \cdots + a_n T^n v = \underbrace{(a_0 + a_1 T + \cdots + a_n T^n)}\, v \qquad b_j^2 < 4c_j$$

$$= c(T-\lambda_1)\cdots(T-\lambda_m)(T^2+b_1T+c_1)\cdots(T^2+b_M T+c_M)\, v.$$

$b_j^2 < 4c_j \quad 1 \le j \le M \Rightarrow$ By previous lemma $\Rightarrow$ $T^2 + b_j T + c_j$ is invertible

$\Rightarrow 0 = c(T-\lambda_1) \cdots (T-\lambda_m) v$

$\Rightarrow v$ is an eigenvector with eigenvalue $\lambda_i \quad 1 \le i \le m.$

Theorem: $T \in \mathcal{L}(V)$ is self adjoint. U is a subspace of V that is invariant under T. Then

1. $U^\perp$ is also invariant under T
2. $T|_U \in \mathcal{L}(U)$ is selfadjoint
3. $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is selfadjoint.

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Theorem: $T \in \mathcal{L}(V)$. self adjoint & U is a subspace invariant under T. Then

(1) $U^{\perp}$ is invariant under T

(2) $T|_U$ is self adjoint

(3) $T|_{U^{\perp}}$ is also selfadjoint.

Proof: (1) $v \in U^{\perp}$ & $\forall u \in U$

$$\langle Tv, u\rangle = \langle v, T^* u\rangle \underset{\substack{\uparrow \\ T \text{ self adjoint} \\ T = T^*}}{=} \langle v, Tu\rangle \underset{\substack{\uparrow \\ U \text{ is invariant under } T \\ \Downarrow \\ Tu \in U}}{=} 0$$

$v \in U^{\perp} \Rightarrow \langle v, Tu\rangle = 0$

$\Rightarrow Tv$ is orthogonal to $\forall u \in U$ $\Rightarrow Tv \in U^{\perp}$.

(2) T is self adjoint, U is invariant under $T \Rightarrow T|_U \in \mathcal{L}(U)$

$u, v \in U$

$$\langle (T|_U)(u), v \rangle = \langle Tu, v \rangle = \langle u, T^* v \rangle = \langle u, Tv \rangle = \langle u, (T|_U)(v) \rangle$$

$$\Rightarrow (T|_U)^* = T|_U .$$

$T|_U$ is also self adjoint.

(3) Replace U by $U^\perp$ in (2) & use (1) to see $U^\perp$ is also invariant under T.

Real Spectral Theorem : $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$. Then TFAE

(1) T is self adjoint

(2) V has a orthonormal basis consisting of eigenvectors of T

(3) T has diagonal matrix representation w.r.t some orthonormal basis of V.

Proof ① $\Rightarrow$ ②

Induct on $\dim V$.

$\dim V = 1$ ✓

$\dim V > 1$.

From previous lecture, $\exists$ an eigenvalue λ & eigenvector u (T is self adjoint)

$U = \langle u \rangle$. U is invariant under T. (one should normalize u to get a orthonormal basis for U)

From previous lemma $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ & $U^\perp$ is invariant under T.

$$\dim U^\perp = \dim V - \dim U < \dim V.$$

By induction $\exists$ an orthonormal basis $v_1 \dots v_n$ for $U^\perp$.

Now $u, v_1 \dots v_n$ forms an <u>ortho</u> <u>normal</u> <u>basis</u> for V. $\left(\begin{array}{l} V = U \oplus U^\perp \\ \langle v, w \rangle = 0 \ \forall v \in U, w \in U^\perp \end{array} \right)$

② $\Rightarrow$ ③ ✓

③ $\Rightarrow$ ① $\exists$ a basis consisting of orthonormal vectors $e_1, \dots e_n$ for V s.t.

$\mathcal{M}(T, e_1 \ldots e_n)$ is diagonal

$$\mathcal{M}(T) = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \quad a_i \in \mathbb{F} = \mathbb{R}$$

$$\mathcal{M}(T^*) = \begin{bmatrix} \bar{a}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{a}_n \end{bmatrix} \text{ is also diagonal}$$

$$= \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$$

$$= \mathcal{M}(T).$$

$\Rightarrow$ T is self adjoint.

Problem Sheet 7.

① Gram-schmidt.

② U is a subspace of V. with basis $u_1 \dots u_m$.
Extend to a basis $u_1 \dots u_m, w_1 \dots w_n$.
Gram-schmidt: $\underbrace{e_1 \dots e_m}_{U}, \underbrace{f_1 \dots f_m}_{U^\perp}$.

G-S.
$\{v_1 \dots v_n\}$
$\Rightarrow \{e_1 \dots e_n\}$.
$\mathrm{Span}(v_1 \dots v_j) = \mathrm{Span}(e_1 \dots e_j)$

$V = U \oplus U^\perp$

$\mathrm{Span}(u_1 \dots u_m) = \mathrm{Span}(e_1 \dots e_m)$ (G-S).

$\Rightarrow e_1 \dots e_m$ is a orthonormal basis for U.

$\Rightarrow \mathrm{Span}(f_1 \dots f_m) \subseteq U^\perp$

$v \in U^\perp$, $v = a_1 e_1 + \dots + a_m e_m + b_1 f_1 + \dots + b_m f_m$

$\langle v, u \rangle = 0$ $\forall u \in U$ $\quad \langle v, e_1 \rangle = a_1 = 0$

$\Rightarrow U^\perp = \mathrm{Span}(f_1 \dots f_m)$

$|||^{ly}$ $a_1 \dots a_m = 0$ $\Rightarrow v = b_1 f_1 + \dots + b_m f_m$.
$\Rightarrow v \in \mathrm{Span}(f_1 \dots f_m)$

③ V f.d.v.s. $P \in \mathcal{L}(V)$ s.t $P^2 = P$.

every vector in null P is orthogonal to every vector in range P.

? $U \leq V$ s.t $P = P_U$.

$P^2 = P \Rightarrow V = \text{null}\, T \oplus \text{range}\, T$. (check)

$v \in V \Rightarrow v = \underset{\in \text{null}\, T}{u} + \underset{\in \text{range}\, T}{w}$

$$
\begin{aligned}
Pv &= P(u+w) \\
&= Pu + Pw \\
&= 0 + Pw \\
&= Pw
\end{aligned}
$$

$U = \text{range}\, T$ $\quad P_U(v) = P_U(u+w) = P_U(w) = w$.

④ V = f.d.v.s. $T \in \mathcal{L}(V)$. U is a subspace of V.

P.T. U & $U^\perp$ are invariant under $T \iff P_U T P_U = T P_U$

Proof Suppose $P_U T P_U = T P_U$.

$V = U \oplus U^\perp$

$v = u + w$.

$P_U(v) = u$

$P_U(u) = u$

range $P_U = U$.

$u \in U$. $Tu = T P_U(u) = P_U T P_U(u)$

$P_U(u) = u$

$T P_U(u) = Tu \in V$

$P_U T P_U(u) = P_U(Tu) \in U$

U is invariant under T

$|||^{ly}$ show $U^\perp$ is invariant under T (check)

Conversely if U is invariant under T. $v \in V$. $P_U(v) \in U$

$T P_U(v) \in U$ (U is invariant under T).

$P_U T P_U(v) = T P_U(v)$. (P_U is an identity operator on U).

(5) ✓

(6) check.

(7) $\dim \operatorname{null} T^{\#} = \dim \operatorname{null} T + \dim W - \dim V.$

$$\|$$

$\dim (\operatorname{range} T)^{\perp} = \dim W - \dim \operatorname{range} T \qquad (\dim U + \dim U^{\perp} = \dim V)$

$= \dim W - \dim V + \dim \operatorname{null} T.$

$\dim \operatorname{range} T^{\#} = \dim \operatorname{range} T$ check

(9) check (3).

Theorem: V f.d. v.s. $T \in \mathcal{L}(V)$. P.T TFAE

(1) $V = \operatorname{null} \oplus \operatorname{range}$.

(2) $V = \operatorname{null} T + \operatorname{range} T$

(3) $\operatorname{null} T \cap \operatorname{range} T = \{0\}$

Proof: (1) $\Rightarrow$ (2) ✓

(2) $\Rightarrow$ (3) $\quad \underset{\dim V}{\overset{\shortparallel}{\dim}}\ \text{null}\,T + \text{range}\,T = \underbrace{\dim \text{null}\,T + \dim \text{range}\,T}_{\dim V} - \dim(\text{null} \cap \text{range}\,T)$.

$\Rightarrow \dim(\text{null}\,T \cap \text{range}\,T) = \{0\}$.

$\text{null}\,T \cap \text{range}\,T = \{0\}$

(3) $\Rightarrow$ (1) $\quad \dim(\text{null}\,T + \text{range}\,T) = \underbrace{\dim \text{null}\,T + \dim \text{range}\,T}_{\dim V} - \underset{0}{\dim(\text{null} \cap \text{range})}$.

$\text{null}\,T + \text{range}\,T \leq V$ & their dims are equal

$\Rightarrow V = \text{null}\,T + \text{range}\,T$

$\Rightarrow V = \text{null}\,T \oplus \text{range}\,T$ (intersection is zero)