Assignment 3

Real and Complex Analysis

MTL122/ MTL503/ MTL506

Lecturer: A. Dasgupta

(1) Let A and B be disjoint closed subsets of a metric spaces (X, d). Prove that there are disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

(2) Let (X, d) be a metric space with $E \subset X$. Prove that $(E^\circ)^c = \overline{(E^c)}$.

(3) A point x not belonging to a closed set $M \subset (X, d)$ always has a nonzero distance from M. (Hint: To prove this, show that $x \in \bar{A}$ if and only if $D(x, A) = dist(x, A) = \inf_{y \in A} d(x, y) = 0$; here A is any nonempty subset of X.

(4) Let A and B be non-empty subsets of a metric space (X, d). Prove that

(i) $A \subset B$ implies $diam(A) \leq diam(B)$.

(ii) $diam(A) = 0$ if and only if for some $x \in X$, $A = \{x\}$.

(iii) If $a \in A$ and $b \in B$, then

$$diam(A \cup B) \leq diam(A) + diam(B) + d(a, b).$$

(5) Let (X, d) be a metric space with the property that every bounded sequence has a convergent subsequence. Prove that X is complete.
Does Bolzano-Weierstrass theorem holds holds for any metric space? Give reasons/counterexamples.

(6) If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d), show that (a_n), where $a_n = d(x_n, y_n)$, converges.

(7) Let $X = (X, d)$ be a metric space and $CS(X)$ the collection of all Cauchy sequences in X. For (x_n) and (y_n) in $CS(X)$, define

$$(x_n) \sim (y_n) \text{ if and only if } \lim_{n \to \infty} d(x_n, y_n) = 0.$$

Show that $\sim$ is an equivalence relation on $CS(X)$.

(8) Show that the set X of all integers, with metric d defined by $d(m, n) = |m-n|$, is a complete metric space.

(9) Show that (l^∞, d_∞), d_∞ (defined in Lecture) is a complete metric space.

(10) Show that the set of all real numbers constitutes an incomplete metric space if we choose $d(x, y) = |\arctan x - \arctan y|$.