Complex Analysis-I

# Real and Complex Analysis

MTL122/ MTL503/ MTL506

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## 1. Complex numbers

The complex numbers can be defined as pairs of real numbers,

$$\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\},$$

equipped with the addition

$$(x, y) + (a, b) = (x + a, y + b)$$

and the multiplication

$$(x, y) \cdot (a, b) = (xa - yb, xb + ya).$$

at $\mathbb{C}$ is an extension of $\mathbb{R}$, in the sense that the complex numbers of the form $(x, 0)$ behave just like real numbers; that is, $(x, 0) + (y, 0) = (x + y, 0)$ and $(x, 0) \cdot (y, 0) = (x \cdot y, 0)$. So we can think of the real numbers being embedded in $\mathbb{C}$ as those complex numbers whose second coordinate is zero.
The definition of our multiplication implies

$$(0, 1) \cdot (0, 1) = (-1, 0). \tag{1.1}$$

This identity together with the fact that

$$(a, 0) \cdot (x, y) = (ax, ay)$$

allows an alternative notation for complex numbers. The latter implies that we can write

$$(x, y) = (x, 0) + (0, y) = (x, 0) \cdot (1, 0) + (y, 0) \cdot (0, 1).)$$

If we think—in the spirit of our remark on the embedding of $\mathbb{R}$ in $\mathbb{C}$—of $(x, 0)$ and $(y, 0)$ as the real numbers $x$ and $y$, then this means that we can write any complex number (x, y) as a linear combination of $(1, 0)$ and $(0, 1)$, with the real coefficients $x$ and $y$. $(1, 0)$, in turn, can be thought of as the real number 1. So if we give $(0, 1)$ a special name, say $i$, then the complex number that we used to call $(x, y)$ can be written as $x \cdot 1 + y \cdot i$, or in short,

$$x + iy.$$

The number $x$ is called the real part and y the imaginary part of the complex number $x + iy$, often denoted as $\mathrm{Re}(x + iy) = x$ and $\mathrm{Im}(x + iy) = y$. The identity (1.1) then reads

$$i^2 = -1.$$

Although we just introduced a new way of writing complex numbers, let's for a moment return to the $(x, y)$-notation. It suggests that one can think of a complex number as a two-dimensional real vector. When plotting these vectors in the plane $\mathbb{R}^2$ , we will call the $x$-axis the real axis and the $y$-axis the imaginary axis. The plane of two axes representing complex numbers as points is called the complex plane or

Argand Plane. The addition that we defined for complex numbers resembles vector addition. The analogy stops at multiplication: there is no "usual" multiplication of two vectors in $\mathbb{R}^2$ and certainly not one that agrees with our definition of the product of two complex numbers.

Any vector in $R^2$ is defined by its two coordinates. On the other hand, it is also determined by its length and the angle it encloses with, say, the positive real axis.

- The **absolute value** or **modulus** $r = |z| \in \mathbb{R}$ of $z = x + iy$ is

$$r = |z| = \sqrt{x^2 + y^2}.$$

- The **argument** of $z = x + iy$ is a number $\theta \in \mathbb{R}$ such that

$$x = r\cos\theta \text{ and } y = r\sin\theta,$$

that is

$$\text{arg}z = \arctan\left(\frac{y}{x}\right).$$

A given complex number $z = x+iy$ has infinitely many possible arguments. For instance, the number $1 = 1 + 0i$ lies on the $x$-axis, and so has argument 0, but we could just as well say it has argument $2\pi, 4\pi, -2\pi$, or $2\pi * k$ for any integer $k$. The number $0 = 0 + 0i$ has modulus 0, and every number $\phi$ is an argument. Aside from the exceptional case of 0, for any complex number $z$, the arguments of $z$ all differ by a multiple of $2\pi$, just as we saw for the example $z = 1$.

$\theta$ is not uniquely defined: If $\theta = \arctan(\frac{y}{x})$ then $\theta + 2k\pi$, $k = \pm 1 \pm 2...$ also fit. **Note: When calculating $\theta$ you must take account of the quadrant in which $z$ lies.**

- The **principal value of the argument** is denoted by Arg$z$, and is the unique value of arg$z$ such that $-\pi < \text{arg}z \leq \pi$. Arg$z$ is obtained by adding or subtracting integer multiples of $2\pi$ from arg$z$.
By arg$z$ we denote the set of all possible arguments, ie.

$$\text{arg}z = \{\text{Arg}z + 2\pi k : K \in \mathbb{Z}\}.$$

- Writing a complex numbers in terms of polar coordinates $r$ and $\theta$:

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = r^{i\theta}.$$

- For any two complex numbers $z_1$ and $z_2$

$$\text{arg}(z_1 z_2) = \text{arg}z_1 + \text{arg}z_2$$

and for $z_2 \neq 0$,

$$\text{arg}\left(\frac{z_1}{z_2}\right) = \text{arg}z_1 - \text{arg}z_2.$$

The 'metric' $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ in $\mathbb{C}$ is defined by ,

$$d(z_1, z_2) = |z_1 - z_2| = |z_2 - z_1|.$$

$d(z_2, z_1) =$ distance between the endpoints of the two vectors in $\mathbb{R}^2$.
That $|z_1 - z_2| = |z_2 - z_1|$ simply says that the vector from $z_1$ to $z_2$ has the same length as its inverse, the vector from $z_2$ to $z_1$.

**Euler's Identity**
We all know what $e^\theta$ is if $\theta$ is a real number. What is $e^{i\theta}$?

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\
&= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + .... \\
&= \left(1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + ...\right) + \left(\frac{(i\theta)}{1!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + ...\right) \\
&= \left(1 - \frac{(\theta)^2}{2!} + \frac{(\theta)^4}{4!} - ...\right) + i\left(\theta - \frac{(\theta)^3}{3!} + \frac{\theta^5}{5!} - ...\right) \\
&= \cos\theta + i\sin\theta. \qquad (1.2)
\end{aligned}$$

So we have the celebrated Euler's identity,

$$e^{i\theta} = cos\theta + i\sin\theta.$$

Recall that the polar form of a nonzero complex number $z$ is given by

$$z = r\left(\cos\theta + i\sin\theta\right).$$

thus, using Euler's identity, the polar form becomes

$$z = re^{i\theta}.$$

We have the following inequalities,

$$-|z| \leq \mathrm{Re}z \leq |z| \text{ and } -|z| \leq \mathrm{Im}z \leq |z|.$$

We know $|z| = |x + iy| = \sqrt{x^2 + y^2}$, that is,

$$|x + iy|^2 = x^2 + y^2 = (x + iy)(x - iy).$$

$x - iy$ is called the (complex) conjugate of $x + iy$. We denote the conjugate by

$$\overline{x + iy} = x - iy.$$

Geometrically, conjugating $z$ means reflecting the vector corresponding to $z$ with respect to the real axis. The following collects some basic properties of the conjugate. For any $z, z_1, z_2 \in \mathbb{C}$,

(a) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

(b) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

(c) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

(d) $\overline{\overline{z}} = z$

(e) $|\overline{z}| = |z|$

(f) $|z|^2 = z\overline{z}$

(g) $\text{Re}z = \frac{1}{2}(z + \overline{z})$

(h) $\text{Im}z = \frac{1}{2i}(z - \overline{z})$

(i) $\overline{e^{i\phi}} = e^{-i\phi}$.

From part (f) we have a neat formula for the inverse of a non-zero complex number:

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

A famous geometric inequality is the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

**Definition 1.1.** *Let $z = x + iy$ be a complex number. Then we define $e^z$ by*

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + \sin y).$$

Let $z = re^{i\theta}$ be a nonzero complex number. Then for every positive integer $n$, the $n^{th}$ power $z^n$ of $z$ is given by

$$z^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta).$$

Is there a formula for all the $n^{th}$ roots of $z$?
To solve this problem, let $\zeta = \sqrt[n]{z}$. Suppose that $\zeta = \rho e^{i\phi}$. Then

$$\zeta^n = z \implies \rho^n e^{in\phi} = re^{i\theta}.$$

So,

$$\rho^n = r$$

and $\phi = \frac{1}{n}(\theta + 2k\pi)$, where $k = 0, 1, 2, ..., n-1$. Thus, there are $n$ $n^{th}$ roots of $z$ given by

$$\zeta_k = \sqrt[n]{r} e^{i(\theta+2k\pi)/n}, \ k = 0, 1, 2, ..., n-1.$$

**Example 1.2.** *Find the fourth roots of unity, i.e., $\sqrt[4]{1}$.*

**Solution 1.3.** *Since $1 = 1e^0$, we get $r = 1$ and $\theta = 0$. Thus, the four fourth roots of unity are*

$$\zeta_k = e^{i(2k\pi)/4} = e^{ik\pi/2}, \ k = 01, 2, 3.$$

*So,*

$$\zeta_0 = 1, \zeta_1 = e^{i\pi/2} = i, \zeta_2 = e^{i\pi} = -1, \zeta_3 = e^{i3\pi/2} = -i.$$

## 2. Functions of Complex Variable

A function defined on a set $D$ is a rule that associates to each point $z$ of $D$ a complex number $w$. Set $D$ is called the domain and $w$ is called the image of $f$ at $z$ and is denoted by $f(z) = w$.

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

$$f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta).$$

**Example 2.1.** *Write $f(z) = 1/z^2$ in $u + iv$ form.*
$u(x, y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ *and* $v(x, y) = \frac{2xy}{(x^2+y^2)^2}$.
*Also* $u(r, \theta) = r^{-2}\cos 2\theta$ *and* $v(r, \theta) = -r^{-2}\sin 2\theta$.
*Domain of $f$ is* $\mathbb{C} \setminus \{0\}$.

**Example 2.2.** *(1) Polynomials*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$$

*where the coefficients are real.*

*(2) Exponential Function*

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$$

*converges for all $z$.*

*(3) Trigonometric Functions*

*Define*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\tan z = \frac{\sin z}{\cos z}$$

*(4) Hyperbolic Functions*

$$\cos hz = \frac{e^z + e^{-z}}{2}$$

$$\sin hz = \frac{e^z - e^{-z}}{2}$$

## 3. Limits

**Definition 3.1.** *The limit of the function $f(z)$ as $z \to z_0$ is a number $w_0$ if for any given $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|z - z_0| < \delta \implies |f(z) - w_0| < \epsilon.$$

**Example 3.2.** *Prove that* $\lim_{z\to 1+i}(2+i)z = \frac{5}{2}(1+3i)$. *According to the definition,* $\lim_{z\to 1+i}(2+i)z = \frac{5}{2}(1+3i)$. *if, for every* $\epsilon > 0$, *there is a* $\delta > 0$, *such that* $|(2+i)z - \frac{5}{2}(1+3i)| < \epsilon$ *whenever* $0 < |z-(1+i)| < \delta$. *We first factor* $(2+i)$ *out of the left-hand side:*

$$|2+i| \cdot \left| z - \frac{5}{2}\frac{1+3i}{2+i} \right| < \epsilon.$$

*Because* $|2+i| = \sqrt{5}$ *and* $\frac{1+3i}{2+i}$ = -$(1+i)$, *we get* $\sqrt{5}\cdot|z-(1+i)| < \epsilon$ *or* $|z-(1+i)| < \frac{\epsilon}{\sqrt{5}}$. *This indicate that we should take* $\delta = \frac{\epsilon}{\sqrt{5}}$.

**Theorem 3.3.** *Let* $f(z) = u(x,y) + iv(x,y)$ *and* $w_0 = u_0 + iv_0$. $\lim_{z\to z_0} f(z) = w_0$ *if and only if* $\lim_{(x,y)\to(x_0,y_0)} u = u_0$ *and* $\lim_{(x,y)\to(x_0.y_0)} v = v_0$.

**Example 3.4.** *Compute* $\lim_{z\to 1+i}(z^2+i)$.
*Since* $f(z) = z^2 + i = x^2 - y^2 + (2xy+1)i$, *we set*

$$u(x,y) = x^2 - y^2, v(x,y) = 2xy + 1$$

*and* $z_0 = 1+i$, *i.e.,* $x_0 = 1$ *and* $y_0 = 1$. *We next compute the two real limits:*

$$u_0 = \lim_{(x,y)\to(1,1)}(x^2-y^2) = 1^2 - 1^2 = 0,$$

$$v_0 = \lim_{(x,y)\to(1,1)}(2xy+1) = 2.1.1 + 1 = 3.$$

*Therefore,* $L = u_0 + iv_0 = 0 + 3i = 3i$, *that is,* $\lim_{z\to 1+i} = 3i$.

- There is an important difference between the two concepts of limit:
  - In a real limit, there are two directions from which $x$ can approach $x_0$ on the real line, from the left or from the right.
  - In a complex limit, there are infinitely many directions from which $z$ can approach $z_0$ in the complex plane. In order for a complex limit to exist, each way in which $z$ can approach $z_0$ must yield the same limiting value.

**Example 3.5.** ***Real One-Sided Limits***
*There is at least one very important difference between real and complex limits.*

- *For real functions* $\lim_{x\to x_0} f(x) = L$ *if and only if* $\lim_{x\to x_0^+} f(x) = L$ *and* $\lim_{x\to x_0^-} f(x) = L$. . *Since there are two directions from which* $x$ *can approach* $x_0$ *on the real line, the real limit exists if and only if these two one-sided limits have the same value.*

- ***Example***
  *Consider the real function*

$$f(x) == \begin{cases} x^2, & \text{if } x < 0. \\ x-1, & \text{if } x \geq 0. \end{cases} \tag{3.1}$$

  *The limit of* $f$ *as* $x$ *approaches* $0$ *does not exist:*

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^-} x^2 = 0,$$

*but*

$$\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^+} x - 1 = -1,$$

For limits of complex functions, $z$ is allowed to approach $z_0$ from any direction in the complex plane, i.e., along any curve or path through $z_0$.

For $\lim\limits_{z\to z_0} f(z)$ to exist and to equal $L$, we require that $f(z)$ approach the same complex number $L$ along every possible curve through $z_0$. $f(z)$ to exist and to equal $L$, we require that $f(z)$ approach the same complex number $L$ along every possible curve through $z_0$.

**Remark 3.6.** *If $f$ approaches two complex numbers $L_1 \neq L_2$ for two different curves or paths through $z_0$, then $\lim\limits_{z\to z_0} f(z)$ does not exist.*

**Example 3.7.** *Show that $\lim\limits_{z\to 0} \frac{z}{\bar{z}}$ does not exist.*

- *First, we let $z$ approach $0$ along the real axis. That is, we consider complex numbers of the form $z = x + 0i$, where the real number $x$ is approaching $0$. For these points we have:*
$$\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{x\to 0} \frac{x+0i}{x-0i} = \lim_{x\to 0} 1 = 1.$$
- *On the other hand, if we let $z$ approach $0$ along the imaginary axis, then $z = 0 + iy$, where the real number $y$ is approaching $0$. For this approach we have*
$$\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{y\to 0} \frac{0+iy}{0-iy} = \lim_{y\to 0}(-1) = -1.$$

*Since the two values are not the same, we conclude that $\lim\limits_{z\to 0} \frac{z}{\bar{z}}$ does not exist.*

**Properties of Complex Limits**

Suppose that $f$ and $g$ are complex functions. If $\lim\limits_{z\to z_0} f(z) = L$ and $\lim\limits_{z\to z_0} g(z) = M$, then:

(i) $\lim\limits_{z\to z_0} cf(z) = cL$, $c$ a complex constant;
(ii) $\lim\limits_{z\to z_0} (f(z) \pm g(z)) = L \pm M$;
(iii) $\lim\limits_{z\to z_0} f(z) \cdot g(z) = L \cdot M$, and
(iv) $\lim\limits_{z\to z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$, provided $M \neq 0$.

## 4. Continuity

A complex function $f$ is continuous at a point $z_0$ if $\lim\limits_{z\to z_0} f(z) = f(z_0)$.

A complex function $f$ is sontinuous at a point $z_0$ if each of the following three conditions hold:

- $\lim\limits_{z\to z_0} f(z)$ exists;
- $f$ is defined at $z_0$ :

- $\lim_{z \to z_0} f(z) = f(z_0)$.

If a complex function $f$ is not continuous at a point $z_0$, then we say that $f$ is discontinuous at $z_0$.

**Example 4.1.** *The function $f(z) = \frac{1}{1+z^2}$ is discontinuous at $z = i$ and $z = -i$.*

- Suppose that $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$. Then the complex function $f$ is continuous at the point $z_0$ if and only if both real functions $u$ and $v$ are continuous at the point $(x_0, y_0)$.

Suppose that the function $f(z) = u(x, y)+iv(x, y)$ is defined on a closed and bounded region $R$ in the complex plane. As with real functions, we say that the complex functions $f$ is bounded on $R$ if there exists a real constant $M > 0$, such that $|f(z)| \leq M$, for all $z$ in $R$.

## 5. Differentiability

Let $f : D \to \mathbb{C}$, $z_0$ and interior point in $D$. Then, $f$ is differentiable at $z_0$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The value of the limit is called the derivative of $f$ at the point $z_0$ and denoted by $f'(z_0)$.

If we write $h$ or $\Delta z$ as $z - z_0$, then we can write

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

**Chain Rule:**

Suppose $f$ is differentiable at $z_0$ and $g$ is differentiable at $f(z_0)$. Then the composite function $g \circ f$ is differentiable at $z_0$ and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

**Question 5.1.** *Is there any difference between the differentiability in $\mathbb{R}^2$ and $\mathbb{C}$?*

The function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = |z|^2$ is differentiable only at 0. We have (using $|z|^2 = z\overline{z}$)

$$\frac{|z_0 + h|^2 - |z_0|^2}{h} = \frac{z_0\overline{h} + \overline{z_0}h + h\overline{h}}{h} = z_0 \left(\frac{\overline{h}}{h}\right) + \overline{z_0} + \overline{h}.$$

Note that the limit of the above exists as $h \to 0$ iff $z_0 = 0$.
However, looking $f$ as $\mathbb{R}^2 \to \mathbb{R}$ we have $f(x, y) = x^2 + y^2$. Observe that $f$ is differentiable everywhere on $\mathbb{R}^2$.

**Properties:**

- $\frac{dc}{dz} = 0$, $c$ a complex constant.

- $\frac{dz}{dz} = 1$, $\frac{d}{dz}[cf(z)] = cf'(z)$.

- $\frac{d}{dz}[f(z)+g(z)] = f'(z)+F'(z)$
- $\frac{d}{dz}[f(z)F(z)] = f(z)F'(z)+F(z)f'(z)$.
- $F(z)\neq 0$, $\frac{d}{dz}\left(\frac{f(z)}{F(z)}\right) = \frac{F(z)f'(z)-f'(z)F(z)}{(F(z))^2}$
- Chain Rule: Suppose $f$ has derivative at $z_0$ and $g$ has derivative at $f(z_0)$. Then $F(z)=g[f(z)]$ has a derivative at $z_0$ and $F'(z_0)=g'[f(z_0)]f'(z_0)$.

**A necessary condition**

Suppose $f$ is differentiable at $z_0$. Assume $f(x+iy)=u(x,y)+iv(x,y)$, $z_0=x_0+iy_0$. Now, $\lim\limits_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ exists and equals $f'(z_0)$ whichever way $z$ approaches $z_0$.

(1) Horizontally: $z=x+iy_0$ approaches $z_0$ as $x$ approaches $x_0$. So,

$$f'(z_0)=\lim_{x\to x_0}\frac{u(x,y_0)-u(x_0,y_0)}{x-x_0}+i\lim_{x\to x_0}\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}, \tag{5.1}$$

i.e. $f'(z_0)=u_x(x_0,y_0)+iv_x(x_0,y_0)$.

(2) Vertically:$z=x_0+iy$ approaches $z_0$ as $y$ approaches $y_0$, and $z-z_0=i(y-y_0)$. So,

$$f'(z_0)=\lim_{y\to y_0}\frac{u(x_0,y)-u(x_0,y_0)}{i(y-y_0)}+i\lim_{y\to y_0}\frac{v(x_0,y)-v(x_0,y_0)}{i(y-y_0)}, \tag{5.2}$$

i.e., $f'(z_0)=-iu_y(x_0,y_0)+v_y(x_0,y_0)$.

From (5.1) and (5.2) we have at $z_0=(x_0,y_0)$,

$$u_x=v_y,\ v_x=-u_y.$$

**Theorem 5.2.** *Suppose that $f(z)=f(x+iy)=u(x,y)+iv(x,y)$ is differentiable at $z_0=x_0+iy_0$. Then, the partial derivatives of $u$ and $v$ exist at the point $z_0=(x_0,y_0)$ and*

$$u_x=v_y,\ u_y=-v_x,\ \textit{at } z_0=x_0+iy_0\ (\textbf{\textit{Cauchy-Riemann(CR) Equations}})$$

**Example 5.3.**

- CR equations are necessary, but not sufficient conditions. A function may not be differentiable, even if CR equations are satisfied.

**Example 5.4.** *Consider*

$$f(z)=\begin{cases}\frac{\overline{z}^2}{z}, & \text{if } z\neq 0\\ 0, & \text{if } z=0.\end{cases} \tag{5.3}$$

*Then*

$$\frac{f(z)-f(0)}{z-0}=\frac{\overline{z}^2/z}{z}=\frac{\frac{x^3-3xy^2}{x^2+y^2}+i\frac{y^3-3x^2y}{x^2+y^2}}{x+iy}$$

*If $z$ approaches $0$ along the $x$-axis, then $\frac{f(z)-f(0)}{z-0}=\frac{x-0}{x}\to 1$.*
*If $z$ approaches $0$ along the line $y=x$, then $\frac{f(z)-f(0)}{z-0}=\frac{-x-ix}{x+ix}\to -1$.*

*So, $f$ is not differentiable at $z = 0$. Here,*

$$u = \frac{x^3 - 3xy^2}{x^2 + y^2}, \ v = \frac{y^3 - 3x^2y}{x^2 + y^2}$$

*at nonzero $z$ and $0$ at $0$. They satisfy CR equations at $z = 0$.*
*At $(0,0)$, $u_x = v_y = 1$, $u_y = -v_x = 0$.*

**Theorem 5.5.** *Let $f = u + iv$, defined on $B(z_0, r)$, be such that*

- *$u_x, u_y, v_x, v_y$ exist on $B(z_0, r)$.*
- *$u_x, u_y, v_x, v_y$ are continuous at $z_0$.*
- *$u$ and $v$ satisfies CR equations: $u_x = v_y$, $v_y = -u_x$ Then $f$ is differentiable at $z_0$ and $f'(z_0) = u_x(z_0) + iv_x(z_0)$.*

**Proof** Proved in detail in class lecture. □

**Example 5.6.** *Show that the function $f(z) = \frac{1}{z}$, is differentiable everywhere except $z = 0$. Let us write the function as*

$$f(x+iy) = \frac{1}{x+iy} \tag{5.4}$$

$$= \frac{x - iy}{x^2 + y^2}. \tag{5.5}$$

*Hence the real and imaginary component functions are*

$$u(x,y) = \frac{x}{x^2 + y^2},$$

$$v(x,y) = -\frac{y}{x^2 + y^2}.$$

*Except at the point $x = y = 0$, these functions are differentiable, and their partial derivatives satisfy:*

$$\frac{\partial u}{\partial x} = \frac{-x^2 + y^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{x^2 + y^2)^2} = -\frac{\partial u}{\partial y}.$$

*Here $u_x, u_y, v_x, v_y$ exist and continuous at all points except at $z = 0$. So $f$ is differentiable everywhere except $z = 0$.*

**Cauchy-Riemann equation in polar form:**
Suppose differentiable $f$ is given in polar coordinates:

$$w = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta).$$

Then the Cauchy-Riemann equation in polar form is

$$ru_r = v_\theta, \ u_\theta = -rv_r.$$

Formulated in class lecture.

## 6. Analytic Functions/Regular/Holomorphic

**Definition 6.1.** *A function $f(z)$ is said to be analytic in an open set of the complex plane if $f(z)$ has a derivative at each point of that set.*

**Remark 6.2.**

- *If we speak of a function $f$ that is analytic in a set $S$ that is not open, we mean that $f$ is analytic in an open set that contains $S$.*
- *$f$ is analytic at a point $z_0$ if its is analytic in a neighborhood of $z_0$.*

  **Example 6.3.**

  *i)* *$f(z) = \frac{1}{z}$ $(z \neq 0)$ is analytic everywhere except at $z = 0$.*

  *ii)* *$f(z) = |z|^2$ is not analytic at any point as $f'(z)$ exists only at $z = 0$ but not at any other point other in any neighborhood of* $0$.

- If a function $f$ fails to be analytic at a point $z_0$ but is analytic at some point in every neighborhood of $z_0$ then $z_0$ is a singular point.
- If $f(z)$ is analytic everywhere in the complex plane, it is called entire.

**Example 6.4.**

*(1)* *$\frac{1}{z}$ is analytic except at $z = 0$, so the function is singular at the point.*

*(2)* *The functions $z^n$, $n$ a non negative integer are entire functions.*

## 7. Harmonic Functions

The real part and the imaginary part of a complex function $f(z) = u(x, y)+iv(x, y)$ that is analytic in a domain $D$ are solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

in $D$ and have continuous partial derivatives in $D$. We call $u$ and $v$ the harmonic functions.

If two give functions $u$ and $v$ are harmonic in a domain $D$ and their first derivatives satisfy the C-R equations, i.e., $u_x = v_y$, $u_y = -v_x$ throughout $D$, $v$ is said to be a harmonic conjugate of $u$.

**Remark 7.1.** *If $v$ is a harmonic conjugate of $u$ in some domain it is not in general true that $u$ is a harmonic conjugate of $u$.*

## 8. Exponential, Trigonometric and Hyperbolic Functions

We begin with the definition of the function $e^z$ for all $z \in \mathbb{C}$. Guided by the properties of the exponential function $e^x$ for all $x \in \mathbb{R}$, we expect the function $e^z$ to be an entire function such that $e^{z_1}e^{z_2} = e^{z_1+z_2},\ z_1, z_2 \in \mathbb{C}$, and $e^0 = 1$.
To do so let $z = x + iy \in \mathbb{C}$. Then

$$e^z = e^{x+iy} = e^x c(y) + ie^x s(y),$$

where $c(y)$ and $s(y)$ are to be determined. First we note that

$$1 = e^0 = c(0) + is(0).$$

Hence $c(0) = 1$ and $s(0) = 0$. Let

$$u(x, y) = e^x c(y)$$

and

$$v(x, y) = e^x s(y)$$

for all $z \in \mathbb{C}$. Now using C-R equations we get a system ordinary differential equations

$$\begin{cases} s' = c, \\ c' = -s. \end{cases} \tag{8.1}$$

Thus we get

$$\begin{cases} s'' + s = 0, \\ s(0) = 0. \end{cases} \tag{8.2}$$

Therefore $s(y) = \sin y$ and $c(y) = \cos y$ for all $y \in \mathbb{R}$. This gives Euler's identity to the effect that

$$e^{iy} = \cos y + i \sin y,\ \ y \in \mathbb{R},$$

and the very important formula

$$e^z = e^x(\cos y + i \sin y).$$

Therefore $\frac{d}{dz}e^z = e^z$.

**Example 8.1.** *Find all zeroes of* $e^z = 1$.

Solved in class.

- $e^{z+2\pi i} = e^z$, $z \in \mathbb{C}$.

**Definition 8.2.** *Foe every complex number* $z$*, we define* $\cos z$ *and* $\sin z$ *by*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

*and*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

For every complex number $z$ we define $\tan z, \sec z, \csc z$ and $\cot z$ as in calculus. The derivatives and the trigonometric identities for these six functions are also the same.

**Definition 8.3.** *For every complex number $z$, we define* $\cosh z$ *and* $\sinh z$ *by*

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

*and*

$$\sinh z = \frac{e^z - e^{-z}}{2}.$$

## 9. Logarithms, Branches and Cuts

Before talking about the Logarithms we mention an important notion related to the argument of acomplex number. For every nonzero complex number $z$, if $\theta$ is a value or branch of arg $z$, then

$$\arg z = \theta + 2k\pi,\ \ k = 0, \pm 1, \pm 2, \pm 3, ...$$

Let $\tau$ be a real number. The place $\mathbb{C}_\tau$ given by $\mathbb{C}_\tau = \{(r,\theta) : r > 0, \tau < \theta \leq \tau + 2\pi\}$ is known as the cut plane along the branch cut $\{(r,\tau) : r > 0\}$. The branch of arg $z$ that lies in $(\tau, \tau + 2\pi]$ is denoted by $\arg_\tau z$. The principal branch is then $\arg_{-\pi} z = \operatorname{Arg} z$.

A very interesting function in complex analysis is the logarithmic function

$$w = \log z,$$

which we will introduce. As in calculus we want to define

$$w = \log z \iff z = e^w.$$

As usual, we write $w = u + iv$ and $z = x + iy$. Since $|e^w| = e^u \neq 0$, we see that $z \neq 0$. So, we can write $z$ in polar form as

$$z = re^{i\theta}.$$

Therefore

$$re^{i\theta} = e^u e^{iv}.$$

We get $r = e^u$ or $u = \ln r$. We also get

$$v = \theta = \arg z.$$

So, we have the formula

$$\log z = \ln|z| + i\arg z,\ \ z \neq 0.$$

**Remark 9.1.** *For every nonzero complex number $z$,*

$$\log z = ln|z| + i(\theta + 2k\pi),\ \ k = 0, \pm 1, \pm 2, \pm 3, ...,$$

*where $\theta$ is any value or branch of arg $z$.*

**Definition 9.2.** *Let $\tau \in \mathbb{R}$. Then for every nonzero complex number $z$, we define the branch* $\log_\tau z = ln|z| + i arg_\tau z$. *If we let $\tau = -\pi$ then we write Log$z$ for* $\log_{-\pi} z$ *and we call Log$z$ the principal logarithm of $z$.*

Thus it is important to note that

$$\operatorname{Log} z = \ln|z| + i\arg_{-\pi} z = \ln|z| + i\operatorname{Arg} z,\ \ z \neq 0.$$

**Example 9.3.** *Compute* $\log(1+i)$, *Log*$(1+i)$ *and* $\log_\pi(1+i)$.

Solved in class.

**Theorem 9.4.** *Let $\tau \in \mathbb{R}$. Then $w = f(z) = \log_\tau z$ is holomorphic on the domain $\mathbb{C}_\tau^\circ$ given by*

$$\mathbb{C}_\tau^\circ = \{(r,\theta) : r > 0, \tau < \theta < \tau + 2\pi\},$$

*and*

$$\frac{d}{dz}(\log_\tau z) = \frac{1}{z},\ z \in \mathbb{C}_\tau^\circ.$$

**Proof** Let $z_0 \in \mathbb{C}_\tau^\circ$. Then we need to prove that $f$ is differentiable at $z_0$. Let $w_0 = \log_\tau z_0$. Then we need to prove that

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{w - w_0}{z - z_0} = \frac{1}{z_0}.$$

But $w = \log_\tau z \iff z = e^w$ and

$$e^{w_0} = \lim_{w\to w_0} \frac{e^w - e^{w_0}}{w - w_0} = \lim_{w\to w_0} \frac{z - z_0}{w - w_0}.$$

Note that

$$f(z) = \log_\tau z = \ln|z| + i\mathrm{arg}_\tau z \to \ln|z_0| + i\mathrm{arg}_\tau z_0 = \log_\tau z_0$$

as $z \to z_0$. Therefore $z \to z_0 \Rightarrow w \to w_0$. Moreover,

$$z \neq z_0 \Rightarrow w \neq w_0$$

because

$$w = w_0 \Rightarrow z = e^w = e^{w_0} = z_0.$$

Therefore

$$\lim_{z\to z_0} \frac{w - w_0}{z - z_0} = \lim_{z\to z_0} \frac{1}{\frac{z-z_0}{w-w_0}} = \frac{1}{e^{w_0}} = \frac{1}{z_0}$$

as required. □

**Example 9.5.** *Find the domain on which the function*

$$w = f(z) = Log(3z - i)$$

*is holomorphic. Compute $f'(z)$.*

Since $f$ is the composition of Log and $3z - i$, $f$ is holomorphic at all points $z$ unless $3z - i \in (-\infty, 0]$. But it is easy to check that

$$3z - i \in (-\infty, 0] \iff z = \frac{x}{3} + \frac{i}{3}, x \leq 0. \tag{9.1}$$

Thus, $f$ is holomorphic on the domain $\mathbb{C} - \{x + \frac{i}{3} : x \leq 0\}$.

**Example 9.6.** *Find a branch of $w = f(z) = \log(z^3 - 2)$ that is holomorphic at $z = 0$.*

Let $\tau \in \mathbb{R}$ and take the branch $\log_\tau$. Then $f$ is the composition of $\log_\tau$ and the function $z^3 - 2$. So $f$ is holomorphic at $z = 0$ only if $-2$ is not on the branch cut $\{(r,\tau) : r > 0\}$. So we can use any branch where the branch cut $\{(r,\tau) : r > 0\}$ is not equal to $(-\infty, 0)$.
From the properties of arg one can derive the following properties of logarithm functions.

For any $z_1, z_2 \in \mathbb{C}$, with $z_1, z_2 \neq 0$

i) $\log(z_1 z_2) = \log(z_1) + \log(z_2)$

ii) $\log(\frac{z_1}{z_2}) = \log(z_1) - \log(z_2)$.

**Example 9.7.** *Let* $z_1 = -2i$, $z_2 = -i$. *Then check* $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ *but* $Log(z_1 z_2) \neq Log(z_1) + Log(z_2)$.

## 10. Complex-valued Function

First consider derivatives and definite integrals of complex-valued functions $w$ of a real variable $t$. We can write it as $w(t) = u(t) + iv(t)$ where $u$ and $v$ are real valued functions of $t$. Then $w'(t) = u'(t) + iv'(t)$, provided $u'(t)$, $v'(t)$ exists at $t$.

- Rules of differentiation of sums and products holds.
- Mean value theorem for derivatives no longer holds. For example, $f(z) = e^z,\ z = x + iy$. Here $f'(c) \neq 0$ but $f(z_1) - f(z_2) = 0$ for $z_2 = z_1 + 2\pi i$.

**Definition 10.1.** $\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$ *when the individual integral exists.*

- $Re \int_a^b w(t)dt = \int_a^b Re(w(t))dt$
- $Im \int_a^b w(t)dt = \int_a^b Im(w(t))dt$

Improper integrals of $w(t)$ over unbounded intervals are defined in a similar manner.

**Definition 10.2.** *A real valued function of a real variable $t$ is said to be piecewise continuous on $[a,b]$ if it is continuous everywhere on $[a,b]$ except possibly for a finite number of points where only the right hand limit is required at $a$ and only the left hand limit is required at $b$.*
*If $w(t) = u(t) + iv(t),\ a \leq t \leq b$, then $w$ is said to be piecewise continuous if both $u$ and $v$ are piecewise continuous on $[a,b]$.*

If $w$ is piecewise continuous on $[a,b]$ then $\int_a^b w(t)dt$ exists.

- $\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt$.
- $z_0 \int_a^b w(t)dt = \int_a^b z_0 w(t)dt$, where $z_0$ is a complex constant.
- $\int_a^b w_1(t)dt + \int_a^b w_2(t)dt = \int_a^b (w_1(t) + w_2(t))dt$.
- $\int_a^b w(t)dt = -\int_b^a w(t)dt$.

Suppose that the functions $w(t) = u(t) + iv(t)$ and $W(t) = U(t) + iV(t)$ are continuous on $[a,b]$.

(1) If $W'(t) = w(t)$ then $U'(t) = u(t)$, $V'(t) = v(t)$.
(2) $\int_a^b w(t)dt = W(b) - W(a)$.

Suppose $a < b$ and $\int_a^b w(t)dt \neq 0$.
Let $\int_a^b w(t)dt = r_0 e^{i\theta_0}$. So $r_0 = e^{-i\theta_0}\int_a^b w(t)dt = \int_a^b e^{-i\theta_0} w(t)dt$. Since $r_0$ is real,

$$r_0 = \int_a^b e^{-i\theta_0} w(t)dt = Re\int_a^b e^{-i\theta_0} w(t)dt = \int_a^b Re(e^{-i\theta_0} w(t))dt.$$

$$Re(e^{-i\theta_0} w(t)) \leq |e^{-i\theta_0} w| = |w|.$$

Then $\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt$, $a < b$.
With minor modification of the above proof we also get

$$\left|\int_a^\infty w(t)dt\right| \leq \int_a^\infty w(t)dt$$

## 11. Contour Integration

**Contour integration** is a powerful technique, based on complex analysis, that allows us to calculate certain integrals that are otherwise difficult or impossible to do. Contour integrals have important applications in many areas of physics, particularly in the study of waves and oscillations

### 11.1. Contour Integrals.

You have previously studied what it means to take the integral of a real function. To recap: if $f(x)$ is a real function, the integral from $x = a$ to $x = b$ is defined by dividing the interval into $N$ segments, and evaluating the sum of $f(x)\Delta x$ on each segment, in the limit where $N$ goes to infinity:

$$\int_a^b f(x)dx = \lim_{N\to 0}\sum_{n=0}^{N} \Delta x f(x_n), \text{ where } x_n = a + n\Delta x, \Delta x = \frac{b-a}{N}. \tag{11.1}$$

Now consider the case where $f$ is a complex function of a complex variable. The straight-foward way to define the integral of $f(z)$ is by an analogous expression like this:

$$\lim_{N\to 0}\sum_{n=0}^{N} \Delta z f(z_n) \tag{11.2}$$

However, since $f$ takes complex inputs, the values of $z_n$ need not lie along the real line. In general, the complex numbers $z_n$ form a set of points in the two-dimensional complex plane. We can imagine chaining together a sequence of points $z_1, z_2, ..., z_N$ , which are separated by displacements $\Delta z_1, \Delta z_2, \Delta z_3, ..., \Delta z_{N-1}$ such that

$$\begin{aligned} z_2 &= z_1 + \Delta z_1 \\ z_3 &= z_2 + \Delta z_2, \\ z_4 &= z_3 + \Delta z_3, \\ . &= . \\ . &= . \\ . &= . \\ z_N &= z_{N-1} + \Delta z_{N-1}. \end{aligned} \tag{11.3}$$

Figure shown in class. Now, the sum we are interested in is

$$\sum_{n=1}^{N-1} \Delta z_n f(z_n) = \Delta z_1 f(z_1) + \Delta z_2 f(z_2) + ... + \Delta z_{N-1} f(z_{N-1}).$$

Suppose we fix the end-points $z_1$ and $z_N$ , and take the limit $N \to \infty$, so that each displacement $\Delta z_n$ becomes infinitesimal. Then the sequence of points $z_1, z_2, ..., z_N$ turns into a continuous trajectory in the complex plane, with a certain starting point and end-point. Such a trajectory is called a **contour**, and can be denoted by an abstract symbol, such as $\Gamma$. Now we can define a **contour integral** over $\Gamma$, like this:

$$\int_\Gamma f(z)dz = \lim_{N\to\infty} \sum_{n=1}^{N-1} \Delta z_n f(z_n).$$

The symbol $\Gamma$ in the subscript of the integral sign indicates that the integral is to take place over the contour $\Gamma$. It is always necessary, when defining a contour integral, to specify the contour to integrate over. This is roughly analogous to how we need to specify the ends of the integration range, in order to properly define a real definite integral. In the complex case, we must specify an entire trajectory. It is important to note that the contour $\Gamma$ specifies a direction. In fact, if we integrate along the curve in the opposite direction, the value of the contour integral switches sign.

11.1.1. *Contour Integral along parametric curve.*
Simple contour integrals can be calculated by **parameterizing the contour.** Suppose we have a contour integral

$$\int_\Gamma dz f(z),$$

where $f$ is a complex function of a complex variable and $\Gamma$ is a given contour. As previously discussed, we can describe a trajectory in the complex plane by a complex function of a real variable, $z(t)$ :

$$\Gamma = \{z(t) | t_1 < t < t_2\},$$

where $t \in \mathbb{R}, z(t) \in \mathbb{C}$. The real numbers $t_1$ and $t_2$ specify two complex numbers, $z(t_1)$ and $z(t_2)$, which correspond to the endpoints of the contour. The rest of the contour is given by the values of $z(t)$ between these two end-points. Assuming we are able to parameterize $\Gamma$ in this way, we can express the complex displacement $dz$ in the contour integral by

$$dz \to dt \frac{dz}{dt}.$$

This allows us to express the contour integral $\Gamma$ as a definite integral over $t$.

$$\int_\Gamma dz f(z) = \int_{t_1}^{t_2} dt \frac{dz}{dt} f\left(z(t)\right).$$

The resulting definite integral can then be evaluated using standard integration techniques.

**Example 11.1.** *Calculate the following contour integral:*

$$\int_{\Gamma[\theta_1,\theta_2]} dz z^n, \ n \in \mathbb{Z},$$

*where the trajectory* $\Gamma[\theta_1, \theta_2]$ *is shown in the figure in class. We can parameterize the contour by defining the function* $z(\theta)$ *as follows:*

$$\Gamma[\theta_1, \theta_2] = \{z(\theta) : \theta_1 \leq \theta \leq \theta_2\},$$

*where* $z(\theta) = Re^{i\theta}$. *Then the contour integral can be converted into an integral over the real parameter* $\theta$ :

$$\begin{aligned} \int_{\Gamma[\theta_1,\theta_2]} dz z^n &= \int_{\theta_1}^{\theta_2} d\theta z^n \frac{dz}{d\theta} \\ &= \int_{\theta_1}^{\theta_2} d\theta (R^n e^{in\theta})(iRe^{i\theta}) \\ &= iR^{n+1} \int_{\theta_1}^{\theta_2} d\theta e^{i(n+1)\theta} \end{aligned} \tag{11.4}$$

*To proceed, there are two distinct cases that we need to handle separately. Firstly, if* $n \neq -1$, *then we can evaluate the integral on the last line as follows*

$$\begin{aligned} \int_{\theta_1}^{\theta_2} d\theta e^{i(n+1)\theta} &= \left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_{\theta_1}^{theta_2} \\ &= \frac{e^{i(n+1)\theta_2} - e^{i(n+1)\theta_1}}{i(n+1)} \end{aligned} \tag{11.5}$$

*However, this doesn't apply if* $n = -1$, *since the factor of* $n+1$ *in the denominator would vanish. Instead, in this case, the integrand is identically one, so we do the integral in a different way:*

$$\begin{aligned} \int_{\theta_1}^{\theta_2} d\theta e^{i(n+1)\theta} &= \int_{\theta_1}^{\theta_2} d\theta \\ &= \theta_2 - \theta_1 \end{aligned} \tag{11.6}$$

*Putting these two cases together, we arrive at the result*

$$\int_{\Gamma[\theta_1,\theta_2]} dz z^n = \begin{cases} i(\theta_2 - \theta_1), & \text{if } n = -1 \\ iR^{n+1} \frac{e^{i(n+1)\theta_2} - e^{i(n+1)\theta_1}}{i(n+1)} & \text{if } n \neq 1. \end{cases} \tag{11.7}$$

*The case where* $\theta_2 = \theta_1 + 2\pi$ *is of particular interest. Here,* $\Gamma$ *forms a complete loop, and the above equation simplifies to*

$$\oint_\Gamma z^n \, dz = \begin{cases} 2\pi i, & \text{if } n = 1 \\ 0, & \text{if } n \neq -1. \end{cases} \tag{11.8}$$

*Here, the special integration symbol* $\oint$ *is used to indicate that the contour integral is taken over a loop. The value of this particular contour integral is independent of*$R$ *(so long as* $R > 0$*). It is zero for all values of* $n$ *except* $n = -1$. *For the special case* $n = -1$, *the contour integral takes the value* $2\pi i$. *This is an important result.*

So we have seen that the integrals of complex-valued functions are defined on curves in the complex plane rather than on just intervals of the real lines.

**Definition 11.2.** ***Arc*** *A set of points $\gamma : z = (x, y)$ in the complex plane is said to be an arc or curve if*

$$x = x(t),\ y = y(t)\ a \leq t \leq b$$

*where $x(t)$ and $y(t)$ are continuous functions of the real parameter $t$. The set $\gamma$is described by $z(t)$ where*

$$z(t) = x(t) + iy(t),\ a \leq t \leq b$$

*The point $z(a)$ is called the initial point and the point $z(b)$ is called the terminal point of $\gamma$*

**Example 11.3.** *The polygonal line:*

$$z = \begin{cases} x + ix & \text{when } 0 \leq x \leq 1 \\ x + i & \text{when } 1 \leq x \leq 2 \end{cases} \tag{11.9}$$

*Here $x = x$ and $y = y(x)$. Here $x$ is the parameter $t$ and $z = z(t) = z(x)$ is the simple arc.*

**Definition 11.4.** *An arc or curve $C$ is called a simple arc (or curve), if it does not cross itself, that is, $C$ is simple if $z(t_1) \neq z(t_2)$ whenever $t_1 \neq t_2$.*

**Example 11.5.** *The curve in Example 11.1 is a simple curve.*

**Definition 11.6.** *If an arc is simple except for the fact that $z(b) = z(a)$, we say that $C$ is a simple closed curve or Jordan curve.*

**Example 11.7.**
*i) The unit circle $z = e^{i\theta} = \cos\theta + i\sin\theta$, $0 \leq \theta \leq 2\pi$ is a simple closed curve as $z(0) = z(2\pi)$, oriented in the counter-clockwise direction.*
*ii) $z = z_0 + Re^{i\theta}$, $0 \leq \theta \leq 2\pi$ is similarly a simple closed curve a circle centred at $z_0$ with radius $R$.*
*iii) The arc $z = e^{-i\theta}$, $0 \leq \theta \leq 2\pi$ is a simple closed curve , a unit circle, but not same as the curve described in the above example. The set of points are the same, but the circle is traversed in the clockwise direction.*

**Definition 11.8.** ***Smooth Curves***
*A curve (or arc) is said to be smooth if it obeys the following three conditions*

*(1) $z(t)$ has continuous derivative on the interval $[a, b]$*
*(2) $z'(t)$ is never zero on $(a, b)$*
*(3) $z(t)$ is a one to one function on $[a, b]$*

*If the first two conditions are met but $z(a) = z(b)$, then it is called a smooth closed curve.*
*The example of the circles are simple closed contour.*
*The boundary of a triangle or a rectangle taken in a specific direction is a simple closed contour.*

**Definition 11.9.** ***Contour****:*
*A contour or a piecewise smooth arc. Its is an arc consisting of finite number of smooth arcs joined end to end. Hence if $z = z(t) = x(t) + iy(t)$ represents a contour, then $z(t)$ is continuous, whereas its derivative $z'(t)$ is piecewise continuous. Also*

*$z'(t) \neq 0$ where it is defined.*

*If $z(a) = z(b)$ then it is called a simple closed contour.*

**Example 11.10.** *(1) The polygonal line described earlier is a contour.*

**Positive Orientation**
The direction of increasing values in the real parameter $t$ corresponds to the positive direction on a contour $C$. If the contour is closed the positive direction corresponds to the counter-clockwise direction or the direction in which you would walk so that the interior of the closed contour is always on your left.

**Length of an Arc**
The length of an arc is given by

$$L = \int_a^b |z'(t)|dt = \int_a^b \sqrt{(x')^2 + (y')^2}dt,$$

where $x(t)$ and $y(t)$ can be thought of as parametric representations of the curve $\gamma$ which consists of a set of points in the cartesian $(x, y)$ plane.

**Theorem 11.11.** ***Jordan Curve Theorem***
*A simple closed curve or simple closed contour divides the complex plane into two sets, the interior which is bounded and the exterior which is unbounded.*

## 12. Contour Integral

As we have discussed in the previous class that,

- $z = z(t)$, $a \leq t \leq b$ represents a contour $C$ extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$;
- $f(z)$ be piecewise continuous on $C$, that is $f(z(t))$ is piecewise continuous on the interval $a \leq t \leq b$;

The contour integral of $f$ along $C$ is defined as follows:

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

**Remark 12.1.** *Since $C$ is a contour, $z'(t)$ is also piecewise continuous on the interval $a \leq t \leq b$ and so the above integral exists.*

Associated with the contour $C$ in the above integral is the contour $-C$, consisting of the same set of points but with the order preserved so that the new contour extends from the point $z_2$ to the point $z_1$.
The contour $-C$ has the parametric representation $z = z(-t)$, $-b \leq t \leq -a$.

### 12.1. **Properties of the contour integral.**

i) $\int_C f(z)dz = -\int_{-C} f(z)dz$

ii) Suppose that $C$ consists of a contour $C_1$ from $z_1$ to $z_2$ followed by a contour $C_2$ from $z_2$ to $z_3$, the initial point of $C_2$ being the final point of $C_1$.

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

Some times the contour $C$ is called the sum of its legs $C_1$ and $C_2$ and is denoted by $C_1 + C_2$. So

$$\int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

iii) $\int_C z_0 f(z)dz = z_0 \int_C f(z)dz$

iv) $\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$

Now

$$\begin{aligned} \left|\int_C f(z)dz\right| &= \left|\int_a^b f(z(t))z'(t)dt\right| \\ &\leq \int_a^b |f(z(t))z'(t)|\, dt. \end{aligned} \tag{12.1}$$

Suppose there exist an $M > 0$ such that $|f(z)| \leq M$ for all $z \in C$. So

$$\left|\int_C f(z)dz\right| \leq M \int_a^b |z'(t)|dt = ML$$

where $L$ is the length of the contour given by $\int_a^b |z'(t)|dt$.

**Remark 12.2.** *Suppose $f$ is a continuous function of a real variable $t$ defined on the closed bounded interval $[a, b]$. Then it is well known that $f$ is bounded on $[a, b]$ and there exists $t_0 \in [a, b]$ such that $f(t_0) = \sup_{a \leq t \leq b} |f(t)|$, that is $|f(t)| \leq |f(t_0)|$ for all $t \in [a, b]$. So if $f$ is continuous on the contour $C : z = z(t)$, then $f(z) = f(z(t))$ is continuous on $[a, b]$ and consequently there exists an $M > 0$ such that $|f(z(t))| \leq M$ for all $t \in [a, b]$. It now follows that such an $M$ also exists when $f$ is piecewise continuous.*

**Example 12.3.**
*Find the value of the integral*

$$I = \int_C \overline{z} dz$$

*when $C$ is the right-handed half $z = 2e^{i\theta}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$.*

*Proof.* $z = 2e^{i\theta}$ so $\overline{z} = 2e^{-i\theta}$ and $\frac{d}{d\theta} 2e^{i\theta} = 2ie^{i\theta}$. Then

$$\begin{aligned} \int_C \overline{z} dz &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2e^{-i\theta} 2ie^{i\theta} d\theta \\ &= 4i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = 4\pi i. \end{aligned}$$

□

- *When $z$ is on the circle $|z| = 2$, then $z\overline{z} = |z|^2 = 4$ and so $\overline{z} = \frac{4}{z}$. Hence*
$$\int_C \frac{dz}{z} = \frac{1}{4}\int_C \frac{4}{z}dz = \frac{1}{4}\int_C \overline{z}dz = \pi i.$$
- *Show that $\int_C \frac{dz}{z} = 2\pi i$ if $C$ represents the circle $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$ or $-\pi \leq \theta \leq \pi$.*

**Example 12.4.** *Let $C_1$ denotes the contour $OAB$, where $OA$ is the $y$-axis, $0 \leq y \leq 1$, $AB$ is the line segment $y = 1$ with $0 \leq x \leq 1$ and $C_2$ is the segment $OB$ of the line $y = x$. Evaluate the integral on $C_1$ and $C_2$ respectively of the function $f(z) = y - x - i3x^2$, where $z = x + iy$.*

*The leg $OA$ has the parametric representation $z = 0 + iy$, $0 \leq y \leq 1$ and hence on $OA$, $f(z) = y$, $0 \leq y \leq 1$.*
*So*
$$\int_{OA} f(z)dz = \int_0^1 yidy = i\int_0^1 ydy = \frac{i}{2}.$$
*On the leg $AB$, $z = x + i$, $0 \leq x \leq 1$ and so*
$$\int_{AB} f(z)dz = \int_0^1 (1 - x - i3x^2)dx = \frac{1}{2} - \frac{i}{2}..$$
*So*
$$\int_{C_1} f(z)dz = \frac{1-i}{2}.$$
*Now in $C_2$ we have $z = x + ix$, $0 \leq x \leq 1$. Hence $\int_{C_2} f(z)dz = \int_{OB} f(z)dz = \int_0^1 -i3x^2(1+i)dx = 1 - i$*

**Remark 12.5.**

- *The integrals of $f(z)$ along the two paths $C_1$ and $C_2$ have different values even though those are paths having the same initial and same final points.*
- *If $C = C_1 - C_2$, then*
$$\int_C f(z)dz = \int_{C_1 - C_2} f(z)dz = \int_{C_!} f(z)dz - \int_{C_2} f(z)dz = \frac{-1+i}{2} \neq 0.$$

Let $C$ denote an arbitrary smooth arc $z = z(t)$, $a \leq t \leq b$ from a fixed point $z_1$ to a fixed point $z_2$. Then
$$I = \int_C zdz = \int_a^b z(t)z'(t)dt = \left[\frac{[z(t)]^2}{2}\right]_a^b = \frac{[z(b)]^2 - [z(a)]^2}{2} = \frac{z_2^2 - z_1^2}{2}.$$
The above expression is also valid when $C$ is a contour that is not necessarily smooth since a contour consists of a finite number of smooth arcs $C_k$, $k = 1, 2, ..., n-1$, joined end to end. More precisely, suppose that each $C_k$ extends from $z_k$ to $z_{k+1}$. Then
$$\int_C zdz = \sum_{k=1}^{n-1}\int_{C_k} zdz = \sum_{k=1}^{n-1}\frac{z_{k+1}^2 - z_k^2}{2} = \frac{z_n^2 - z_1^2}{2},$$

$z_1$ being the initial point of C and $z_n$ its final point. So when $z_1 = z_n$ then $\int_C z dz = 0$.

## 13. Antiderivative

**Definition 13.1.** *Suppose a function $f$ is continuous on a domain $D$ and there exists a function $F$ such that $F'(z) = f(z)$ for all $z \in D$. Then $F$ is called an antiderivative of $f$ in the domain $D$.*

**Remark 13.2.**

*i) An antiderivative is necessarily analytic on the domain $D$.*

*ii) If $F$ and $G$ are two antiderivatives of a continuous function $f$ on a domain $D$, then $F(z) - G(z) =$constant for all $z \in D$, implies the analytic function $F(z) - G(z)$ has derivative $0$ on $D$. Hence $F(z) - G(z)$ is constant on $D$.*

**Theorem 13.3.** *Suppose that a function $f$ is continuous on a domain $D$, then the followings are equivalent:*

*a) $f$ has an antiderivative $F$ in $D$.*

*b) The integrals of $f(z)$ along the contours lying entirely in $D$ and extending from any fixed point $z_1$ to any other point $z_2$ all have the same value.(i.e the intergral is independent of the path in D.)*

*c) The integrals of $f(z)$ around closed contours lying entirely in $D$ all have value $0$.*

**Example 13.4.** *The continuous functions $f(z) = z^2$ has an antiderivative $F(z) = z^3/3$ throughout the plane.*
*Hence $\int_0^{1+i} z^2 dz = \frac{z^3}{3}]_0^{1+i} = \frac{2}{3}(-1+i)$ for every contour from $z = 0$ to $z = 1+i$.*

**Example 13.5.** *The function $1/z^2$ which is continuous every where except at the origin has an antiderivative $-1/z$ in the domain $|z| > 0$. Consequently*

$$\int_{z_1}^{z_2} \frac{dz}{z^2} = \frac{1}{z_1} - \frac{1}{z_2}, \ z_1 \neq 0, z_2 \neq 0$$

*for any contour from $z_1$ to $z_2$ that doesn't pass through the origin. In particular $\int_C \frac{dz}{z^2} = 0$ where $C$ is the circle $z = 2e^{i\theta}$, $-\pi \leq \theta \leq \pi$ about the origin.*

**Theorem 13.6.** *If a function is analytic at all points interior to and on a simple closed contour C, then $\int_C f(z)dz = 0$.*

**Definition 13.7.** *(Simply connected domain) A domain $D$ is called simply connected if every simple closed contour (within it) encloses points of $D$ only.*

A domain $D$ is called multiply connected if it is not simply connected.

**Example 13.8.** *(1) The set of points interior to a simple closed contour is a simple connected domain.*

(2) *The sets* $\mathbb{C}$ *and* $D = \{z : Rez > 0\}$ *are simply connected domains.*

(3) *The sets* $\mathbb{C}^*$ *and the annulus* $A(a, b) = \{z \in \mathbb{C} : a < |z| < b\}$ *are not simply connected domains.*

**Theorem 13.9.** *(An extension of Cauchy-Goursat)*
*If* $f$ *is analytic in a simply connected domain* $D$*, then*

$$\int_C f(z)dz = 0$$

*for every closed contour* $C$ *lying in* $D$*.*

**Corollary 13.10.** *A function* $f$ *which is analytic throughout a simple connected domain* $D$ *must have an anti derivative,* $F$ *say in* $D$*. In this case for all the paths in* $D$ *joining two points* $z_1$ *and* $z_2$ *in* $D$ *we have* $\int_{z_1}^{z_2} f(z)dz = F(z_1) - F(z_2)$*.*

**Example 13.11.** *If* $C_0$ *is a positively oriented circle of radius* $\epsilon_0$ *centered at the origin (for any* $\epsilon > 0$*), then* $\int_{C_0} \frac{1}{z} dz = 2\pi i$*.*

**Theorem 13.12.** *(Extension of Cauchy-Goursat Theorem to a multiply connected domain)*
*Suppose that*

(1) $C$ *is a simply closed contour in counter clockwise direction.*
(2) $C_k$*,* $k = 1, 2, 3, ..., n$ *denotes a finite number of simple closed contours all described in the clockwise direction that are interior to* $C$ *and whose interiors have no points in common.*

*If a function* $f$ *is analytic throughout the closed region consisting of all points within and on* $C$ *except for the points interior to each* $C_k$*, then*

$$\int_C f(z) + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0.$$

**Proof** Figure given in class.
By Cauchy Gourset Theorem,

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0 + 0 = 0,$$

which gives the result. □

**Example 13.13.**

$$\int_B \frac{dz}{z^2(z^2+16)} = 0,$$

*where* $B$ *consists of the annular region between the circle* $|z| = 2$ *in counter clockwise(positive) direction and the circle* $|z| = 1$ *in the negative direction. Then it can be seen that* $\frac{dz}{z^2(z^2+16)}$ *fails to be analytic at the points* $z = 0$ *and* $z = \pm 4i$*. These points lies outside our annular region. Hence from above theorem the integral is* 0*.*

**Corollary 13.14.** *Let* $C_1$ *be a positively oriented simple closed contour. Then,* $C_1$ *breaks the complex plane up into two regions: the interior of* $C_1$ *and the exterior of* $C_1$

*(by the Jordan curve theorem). Let $C_2$ be a positively oriented simple closed contour entirely inside the interior of $C_1$. If $f$ is analytic in between and on $C_1$ and $C_2$, then*

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$