Problem Sheet 1, Math 104

Topics to revise from MTL101/MTL104

- Fields of the form $\mathbb{Z}/p\mathbb{Z}$ where p is a prime.
- Vector spaces
- Subspaces
- Span and Linear Independence
- Bases and Dimension of vector spaces

Exercises

1. Determine whether the following assertion are true or false giving brief justifications:
   (a) The empty set is a subspace of every vector space.
   (b) Any non-zero vector space over $\mathbb{R}$ has infinitely many distinct vectors.
   (c) The set of all non-invertible matrices in $M_n(\mathbb{F})$ is a subspace.
   (d) Let a be a fixed real number. For any $x, y, \alpha \in \mathbb{R}$ define $x \oplus y = x + y - a$ and $\alpha \odot x = \alpha x + a(1-\alpha)$. Then, $\mathbb{R}$ is a vector space over itself with respect to $\oplus$ and $\odot$.
   (e) Consider the set of real numbers in the open interval $(-1,1)$. For any $x, y \in (-1,1)$ and for any $\alpha \in \mathbb{R}$, define $x \oplus y = \dfrac{x+y}{1+xy}$ and $\alpha \odot x = \dfrac{(1+x)^\alpha - (1-x)^\alpha}{(1+x)^\alpha + (1-x)^\alpha}$. Then, $(-1,1)$ is a vector space over $\mathbb{R}$ with respect to $\oplus$ and $\odot$.
2. Prove or give a counterexample: if U_1, U_2 and W are subspaces of V such that
$$U_1 + W = U_2 + W,$$
then $U_1 = U_2$. What about the case $U_1 \oplus W = U_2 \oplus W$?
3. Let V be a vector space. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.
4. Determine whether or not the set
$$\left\{ \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$
is linearly independent over $\mathbb{Z}_5$.
5. Suppose $u, v \in V$, a vector space over a field $\mathbb{F}$. If $\mathbb{F} = \mathbb{R}$, then show that $\{u, v\}$ is linearly independent if and only if $\{u+v, u-v\}$ is linearly independent. What happens when $\mathbb{F} = \mathbb{Z}_2$?
6. If $\{u, v, w\}$ is a linearly independent subset of a vector space, show that $\{u, u+v, u+v+w\}$ is also linearly independent.
7. Let V be the vector space of all 2×2 matrices over the field $\mathbb{F}$. Let W_1 be the set of matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and let W_2 be the set of matrices of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$.
   (a) Prove that W_1 and W_2 are subspaces of V.
   (b) Is $W_1 \cup W_2$ subspace of V?
8. Let $P = \{(a, b, c) \mid a, b, c \in \mathbb{R}, a = 2b + 3c\}$. Prove that P is a subspace of $\mathbb{R}^3$. Find a basis for P. Give a geometric description of P.
9. If V is a vector space of dimension n over the field $\mathbb{Z}_p$, how many elements are in V?
10. Prove that the real vector space of all continuous real-valued functions on the interval $[0,1]$ is infinite-dimensional.

Problem Sheet 2, Math 104

Topics to revise from MTL101/MTL104

- row reduced echeolon forms, free variables, pivot rows/columns
- Bases and Dimension of vector spaces
- Linear Transformations
- Range, nullspace of a linear transformation, rank-nullity theorem

Exercises

1. In exercise 7 of problem sheet 1, find the dimensions of W_1, W_2, $W_1 + W_2$, and $W_1 \cap W_2$.
2. Suppose $V = \mathbb{R}^4$ and U is a subspace spanned by the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \\ 2 \end{bmatrix}$$

   Find the basis of U.

3. Let V is a vector space over $\mathbb{F}$ of dimension 5. Suppose U and W are subspaces of V of dimension 3, prove that $U \cap W \neq 0$. Generalize.

   space V over a field $\mathbb{F}$, then $m = n$.

4. (a) Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

   Find a basis of U.

   (b) Extend the basis in part (a) to a basis of $\mathbb{R}^5$.

   (c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

5. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(6) = 0\}$. Find a basis of U.
   Here, $\mathcal{P}_4(\mathbb{F})$ denotes the set of all polynomials with coefficients in $\mathbb{F}$ and degree at most 4.

   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{F})$.

   (c) Find a subspace W of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus V$.

6. Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.
7. Recall: If W_1 and W_2 are finite-dimensional subspace of a vector space V, then $W_1 + W_2$ is also finite-dimensional and

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2).$$

   You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned} \dim(U_1 + U_2 + U_3) = {} & \dim U_1 + \dim U_2 + \dim U_3 \\ & - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ & + \dim(U_1 \cap U_2 \cap U_3). \end{aligned}$$

   Prove this or give a counterexample.

8. Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\dim U = 3, \dim W = 5$, and $\mathbb{R}^8 = U + W$. Prove that $\mathbb{R}^8 = U \oplus W$.
9. (a) Give an example of a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\phi(av) = a\phi(v)$$

   for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but ϕ is not linear.

   (b) Give an example of a function $\phi : \mathbb{C} \to \mathbb{C}$ such that $\phi(w + z) = \phi(w) + \phi(z)$ for all $w, z \in \mathbb{C}$ but ϕ is not linear.

(c) Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$. Show that T is linear if and only if $b = c = 0$.

(d) Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq \{0\}$ (which means that $Su \neq 0$ for some $u \in U$). Define $T : V \longrightarrow W$ by

$$Tv = \begin{cases} Sv & \text{if} \quad v \in U, \\ 0 & \text{if} \quad v \in V \quad \text{and} \quad v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

(e) Let V be the vector space of all $n \times n$ matrices over the field $\mathbb{F}$, and let B be a fixed $n \times n$ matrix. If

$$T(A) = AB - BA$$

verify that T is a linear transformation from V into V.

10. (a) Give an example of a linear map $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ such that

$$\text{range } T = \text{null } T.$$

(b) Prove that there does not exist a linear map $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ such that

$$\text{null } T = \text{range } T.$$

(c) Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

(d) Describe the range and the null space for the differentiation transformation on $\mathcal{P}(\mathbb{R})$. Do the same for the integration transformation on $\mathcal{P}(\mathbb{R})$.

Problem Sheet 3, Math 104

Topics to revise from MTL101/MTL104

- Linear transformations, invertibility and isomorphic vector spaces
- Rank-Nullity theorem
- Matrix representation of linear maps and vectors
- Operators and its invertibility
- Product of vector spaces

Exercises

1. Suppose V is a finite dimensional vector space and $T \in \mathcal{L}(V, V)$. Recall that null(T) is a subspace of V and range(T) is a subspace of W. Is it true that $V = \text{null}(T) + \text{range}(T)$. Is it a direct sum? Give a proof or a counterexample.

2. We proved in class that $V \cong W$ when $\dim V = \dim W$. Give an example of two vector spaces V, W such that $\dim V = \dim W$ and $T \in \mathcal{L}(V, W)$ such that T is not an isomorphism.

3. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\text{rank}(T^2) = \text{rank}(T)$. Prove that the range and null space of T are disjoint, i.e., have only the zero vector in common.

4. Find two linear operators T and U on $\mathbb{R}^2$ such that $TU = 0$ but $UT \neq 0$.

5. Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that range $S \subset$ null T. Prove that $(ST)^2 = 0$.

6. Let V be a vector space and T a linear transformation from V into V. Prove that the following two statements about T are equivalent.

   (a) The intersection of the range of T and the null space of T is the zero subspace of V.

   (b) If $T(T(a)) = 0$, then $T(a) = 0$.

7. Let T be the linear operator on $\mathbb{R}^3$ defined by

   $$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 - x_2 + x_3).$$

   Is T invertible? If so, find T^{-1}.

8. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

9. Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

10. Suppose $p \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.

11. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except the entries in row j, column j, equal 1 for $1 \leq j \leq \dim \text{range}(T)$.

12. Suppose T is a function from V to W. The **graph** of T is the subset of $V \times W$ defined by

    $$\text{graph of } T = \{(v, Tv) \in V \times W \mid v \in V\}.$$

    Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

13. Give an example of a vector space V and subspaces U_1, U_2 of V such that $U_1 \times U_2$ is isomorphic to $U_1 + U_2$ but $U_1 + U_2$ is not a direct sum.

Problem Sheet 4, Math 104

Topics to revise

- Quotient Spaces
- Dual spaces
- Annilator of subspace, U°

Exercises

1. Suppose $\varphi \in \mathcal{L}(V, \mathbb{F})$ and $\varphi \neq 0$. Prove that $\dim V/(\operatorname{null} \varphi) = 1$.
2. Suppose U is a subspace of V and $v_1 + U, \ldots, v_m + U$ is a basis of V/U and $u_1, \ldots, u_n$ is a basis of U. Prove that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V.
3. Explain why every linear functional is either surjective or the zero map.
4. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. Suppose φ_1, φ_2 denotes the dual basis of the standard basis of $\mathbb{R}^2$ and ψ_1, ψ_2, ψ_3 denotes the dual basis of the standard basis of $\mathbb{R}^3$.
   (a) Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.
   (b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as a linear combination of ψ_1, ψ_2, ψ_3.
5. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.
6. Suppose $T \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathcal{P}_5(\mathbb{R}))$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, where φ is the linear functional on $\mathcal{P}_5(\mathbb{R})$ defined by $\varphi(p) = p(8)$. Prove that $\operatorname{range} T = \{p \in \mathcal{P}_5(\mathbb{R}) \mid p(8) = 0\}$.
7. Show that $(\mathcal{P}(\mathbb{R}))'$ and $\mathbb{R}^\infty$ are isomorphic.
8. The ***double dual space*** of V, denoted V'', is defined to be the dual space of V'. In other words, $V'' = (V')'$. Define $\Lambda : V \to V''$ by
   $$(\Lambda v)(\varphi) = \varphi(v)$$
   for $v \in V$ and $\varphi \in V'$.
   (a) Show that Λ is a linear map from V to V''.
   (b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.
   (c) Show that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.
9. Suppose U and W are subsets of V with $U \subseteq W$. Prove that $W^0 \subseteq U^0$.
10. Suppose V is finite-dimensional and U is a subspace of V. Show that $U = V$ if and only if $U^0 = \{0\}$.
11. Suppose U, W are subspaces of V. Show that $(U + W)^0 = U^0 \cap W^0$.
12. Suppose V is finite-dimensional and U and W are subspaces of V. Prove that $(U \cap W)^0 = U^0 + W^0$.

Problem Sheet 5, Math 104

Topics to revise

- LU Factorization
- Some applications giving rise to Linear Systems Problems
- Eigenvalue and eigenvector of an operator

Exercises

1. Let $A = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 18 & 26 \\ 3 & 16 & 30 \end{pmatrix}$.

   (a) Determine the LU factors of A.

   (b) Use the LU factors to solve $Ax_1 = b_1$ as well as $Ax_2 = b_2$, where $b_1 = \begin{pmatrix} 6 \\ 0 \\ -6 \end{pmatrix}$ and $b_2 = \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix}$.

   (c) Use the LU factors to determine A^{-1}.

2. Let A and b be the matrices
$$A = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 4 & 8 & 12 & -8 \\ 2 & 3 & 2 & 1 \\ -3 & -1 & 1 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 60 \\ 1 \\ 5 \end{pmatrix}.$$

   (a) Explain why A does not have an LU factorization.

3. Give an example of an invertible matrix which does not have an LU decomposition.

4. Suppose A is invertible and has a LU decomposition, show that the decomposition is unique. That is , if A is invertible and $A = LU = L_1U_1$, then show that $L = L_1$ and $U = U_1$.

5. Determine the value(s) of ξ for which $\begin{pmatrix} \xi & 2 & 0 \\ 1 & \xi & 1 \\ 0 & 1 & \xi \end{pmatrix}$ fails to have an LU factorization.

6. Let $A \in \mathbb{R}^{n\times n}$ and (λ, x) be an eigenpair for A. Explain the geometrical meaning of $Ax = \lambda x$.

7. Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

   (a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

   (b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$.

8. Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

9. (a) Suppose $T \in \mathcal{L}(V)$ is invertible.

      i. Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\dfrac{1}{\lambda}$ is an eigenvalue of T^{-1}.

      ii. Prove that T and T^{-1} have the same eigenvectors.

10. Suppose n is a positive integer and $T \in \mathrm{Hom}(\mathbb{F}^n)$ is defined by
$$T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n);$$
in other words, T is the operator whose matrix (with respect to the standard basis) consists of all 1's. Find all eigenvalues and eigenvectors of T.

11. Suppose $T \in \mathcal{L}(V)$ and there exists a positive integer n such that $T^n = 0$.

    (a) Prove that $I - T$ is invertible and that
$$(I - T)^{-1} = I + T + \cdots + T^{n-1}.$$

    (b) Explain how you would guess the formula above.

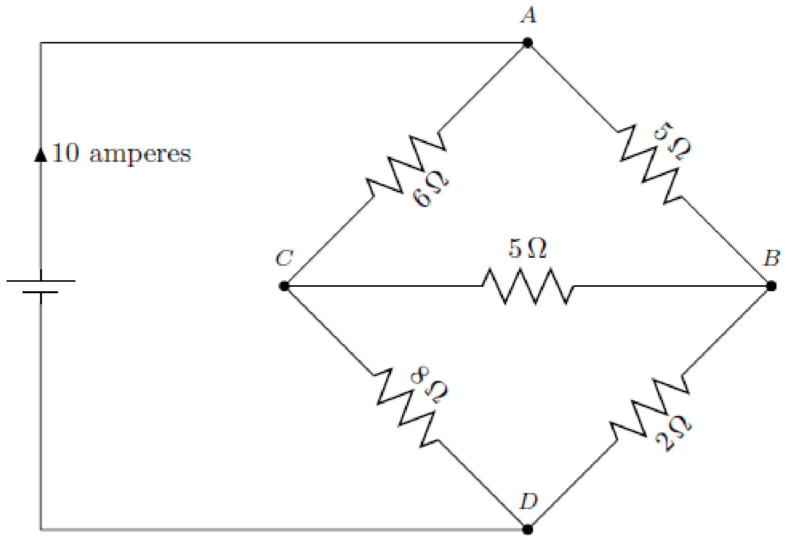
12. (a) Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operator is invertible.

    (b) Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

13. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Find all the eigenvalues of P and Prove that $V = KerP \oplus RangeP$.

Applications to Linear systems

1. Consider the electrical circuit shown below



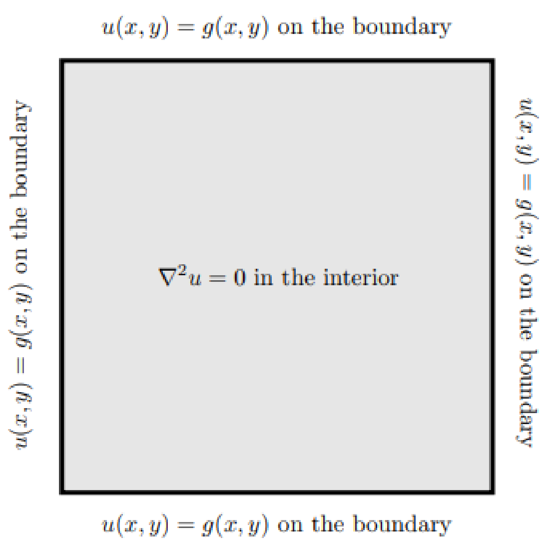
10 amperes

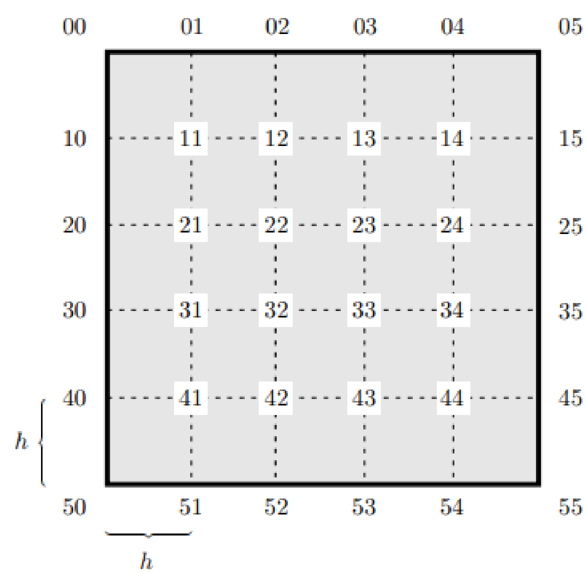
As discussed in the class, find the current I_{AB} flowing from node A to B, and also the currents I_{BC}, I_{AC}, I_{BD}, and I_{CD} by reducing the above data in the form of a system of linear equations and then solving it using LU Factorization.

2. For a given function f the equation $\nabla^2 u = f$ is called Poisson's equation. Consider Poisson's equation on a square in two dimensions with Dirichlet boundary conditions. That is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \text{ with } u(x,y) = g(x,y) \text{ on boundary }.$$

Discretize the problem by overlaying the square with a regular mesh containing n^2 interior points at equally spaced intervals of length h (see the figure below). Let $f_{ij} = f(x_i, y_j)$, and define f to be the vector $f = (f_{11}, f_{12}, \ldots, f_{1n} | f_{21}, f_{22} \ldots, f_{2n} | \ldots | f_{n1}, f_{n2}, \ldots, f_{nn})^T$. Show that the discretization of Poisson's equation produces a system of linear equations of the form $Lu = g - h^2 f$, where L is the discrete Laplacian, g is the column vector corresponds to the boundary values for each mesh points and u is the unknown column vector, i.e., the required values of internal grid points.





For simplicity, take here $n = 4$. Approximate $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ at the interior grid points (x_i, y_j) by using the second-order centered difference formula

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i, y_i)} = \frac{u(x_i + h, y_j) - 2u(x_i, y_j) + u(x_i - h, y_j)}{h^2} + O(h^2)$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{(x_i, y_i)} = \frac{u(x_i, y_j + h) - 2u(x_i, y_j) + u(x_i, y_j - h)}{h^2} + O(h^2).$$

Adopt the notation $u_{ij} = u(x_i, y_j)$ and $f_{ij} = f(x_i, y_j)$, and add the expressions in above equations using $\nabla^2 u \mid_{(x_i,y_j)} = f(x_i, y_j)$ for interior points (x_i, y_j) to produce

$$h^2 f_{ij} + 4u_{ij} = (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}) + O(h^4) \quad \text{for} \quad i, j = 1, 2, \ldots, n.$$

Problem Sheet 6, Math 104

Topics to revise

- Transpose of a linear transformation
- Upper triangulization and Diagonalization of linear operators of finite dimensional vector spaces over $\mathbb{C}$.
- Simultaneous triangulization and simultaneous diagonalization
- Inner product spaces, Orthonormal bases, Gram-Schimdt Orthogonalization process.

Exercises

1. Let T be the linear operator on $M_{n\times n}(\mathbb{R})$ defined by $T(A) = A^t$.
   (a) Find all eigenvalues of T.
   (b) Describe the eigenvectors corresponding to each eigenvalue of T.
2. Show that similar matrices need not have the same eigenvectors by giving an example of two matrices that are similar but have different eigenspaces.
3. Let V be the vector space of $n \times n$ matrices with entries in $\mathbb{C}$. For a matrix $A \in V$ define a linear operator $T_A : V \to V$ such that $T_A(B) = AB - BA$. If A is diagonalizable, is it true that T_A is diagonalizable?
4. Let V be the vector space of $n \times n$ matrices with entries in $\mathbb{C}$. For a matrix $A \in V$ define a linear operator $T_A : V \to V$ such that $T_A(B) = AB - BA$. Suppose A, A' are both diagonal matrices, show that T_A and $T_{A'}$ are simultaneously diagonalizable.
5. Compute $\lim_{n\to\infty} A^n$ for $A = \begin{pmatrix} 7/5 & 1/5 \\ -1 & 1/2 \end{pmatrix}$.
6. Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \operatorname{null} T \oplus \operatorname{range} T$. Is the converse true?
7. Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if
$$V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I),$$
for every $\lambda \in \mathbb{C}$.
8. Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that $E(\lambda, T) = E\left(\frac{1}{\lambda}, T^{-1}\right)$ for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.
9. Prove that $A = c_{n\times 1} d^T_{1\times n}$ is diagonalizable if and only if $d^T c \neq 0$.
10. Explain why the following "proof" of the Cayley–Hamilton theorem is not valid. $p(\lambda) = det(A - \lambda I) \implies p(A) = det(A - AI) = det(0) = 0$.
11. Suppose $R, T \in \mathcal{L}(\mathbb{F}^3)$ each have $2, 6, 7$ as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(\mathbb{F}^3)$ such that $R = S^{-1}TS$.
12. Suppose $T \in \mathcal{L}(\mathbb{C}^3)$ is such that 6 and 7 are eigenvalues of T. Furthermore, suppose T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$. Prove that there exists $(x, y, z) \in \mathbb{C}^3$ such that $T(x, y, z) = (17 + 8x, \sqrt{5} + 8y, 2\pi + 8z)$.
13. Suppose V is a finite dimensional inner product space (with inner product $\langle \bullet, \bullet \rangle$) and $T \in \mathcal{L}(V)$. Define $\langle u, v\rangle_1 = \langle Tu, Tv\rangle$. If T is injectie, show that $\langle \bullet, \bullet \rangle_1$ is an inner product on V.
14. Suppose $T \in \mathcal{L}(V)$ such that $||T(v)|| \leq ||v||$ for all $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.
15. Find an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$ where the inner product is defined as $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx$.

Problem Sheet 7, Math 104

Topics to revise

- Linear Functionals, Riesz representation theorem, Orthogonal complement, Projection opreator.
- Adjoint of an operator, self adjoint, unitary and normal operators

Exercises

1. Suppose U is a subspace of $\mathbb{R}^4$ defined by $U = \text{span}\left\{\begin{bmatrix}1\\2\\3\\-4\end{bmatrix}, \begin{bmatrix}-5\\4\\3\\2\end{bmatrix}\right\}$. Find an orthonormal basis of U and an orthonormal basis of $U^\perp$.
2. Suppose U is a subspace of V with basis $u_1, \ldots, u_m$ and we extend it to a basis $u_1, \ldots, u_m, w_1, \ldots, w_n$. The Gram-Schimdt process produced a orthonormal basis $e_1, \ldots, e_m, f_1, \ldots, f_m$. Show that $e_1, \ldots, e_m$ is an orthonormal basis for U and $f_1, \ldots, f_m$ is an orthonormal basis for $U^\perp$.
3. Suppose V is a finite dimensional and $P \in \mathcal{L}(V)$ such that $P^2 = P$ and every vector in the null space $\text{null}\, P$ is orthogonal to every vector in the range P. Prove that there exsits a subspace U of V such that $P = P_U$ (recall that P_U is the projection operator on U.
4. Supose V is finite dimensional and $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U and $U^\perp$ are both invariant under T if and only if $P_U T = T P_U$.
5. Suppose $T \in \mathcal{L}(\mathbb{R}^n)$ given by $T\left(\begin{bmatrix}z_1\\z_2\\\vdots\\z_n\end{bmatrix}\right) = \begin{bmatrix}0\\z_1\\\vdots\\z_{n-1}\end{bmatrix}$. Find the formula and matrix representation of the adjoint T^*.
6. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.
7. Prove that $\dim \text{null}\, T^* = \dim \text{null}\, T + \dim W - \dim V$ and $\dim \text{range}\, T^* = \dim \text{range}\, T$ for every $T \in \mathcal{L}(V, W)$.
8. SUppose $S, T \in \mathcal{L}(V)$ are self adjoint operators. Prove that ST is self adjoint if and only if $ST = TS$.
9. Suppose $P \in \mathcal{L}(V)$ such that $P^2 = P$. Prove that there is a subspace U of V such that $P = P_U$ if and only if P is self adjoint.
10. Fix $n \in \mathbb{N}$. In the inner product space of continuous real valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$$

let $V = \text{span}(1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx)$.

(a) Define $D \in \mathcal{L}(V)$ by $Df = f'$. Show that $D^* = -D$. Conclude that D is normal but not self adjoint.

(b) Define $T \in \mathcal{L}(V)$ by $Tf = f''$. Show that T is self adjoint.

Problem Sheet 8, Math 104

Topics to revise

- Minimizing distance, Least square problems.
- positive operators, square root of an operator, isometry.
- Spectral theorems

Exercises

1. In $\mathbb{R}^4$, let
$$U = \operatorname{span}\big((1,1,0,0),(1,1,1,2)\big).$$
Find $u \in U$ such that $\big\|u-(1,2,3,4)\big\|$ is as small as possible.

2. Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = p'(0) = 0$, and
$$\int_0^1 |2+3x-p(x)|^2 dx$$
as small as possible.

3. Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in null P is orthogonal to every vector in range P. Prove that there exists a subspace U of V such that $P = P_U$.

4. In physics, Hooke's law states that (within certain limits) there is a linear relationship between the length x of a spring and the force y applied to (or exerted by) the spring. That is, $y = cx + d$, where c is called the spring constant. Use the following data to estimate the spring constant (the length is given in inches and the force is given in pounds).

Length x	Force y
3.5	1.0
4.0	2.2
4.5	2.8
5.0	4.3

5. (True/False)
   - The set of all least squares solutions is precisely the set of solutions to the system of normal equations $A^TAx = A^Tb$.
   - There is a unique least squares solution if and only if $rank(A) = n$, in which case it is given by $x = (A^TA)^{-1}A^Tb$
   - If $Ax = b$ is consistent, then the solution set for $Ax = b$ is the same as the set of least squares solutions.

6. Show that a vector x is a least-squares solution of $Ax = b$ if and only if x is a solution of $A^TAx = A^Tb$

7. Let $T \in \mathcal{L}(V)$ be normal. Prove that null T^k = null T and range T^k = range T for all integers k.

8. In problem 10 in problem sheet 7, show that the functions in V form an orthogonal set in V under the inner product defined. Construct an orthonormal set using the set in V.

9. Suppose T is a self adjoint operator on a finite dimensional inner product space and $2, 3$ are teh only eigenvalues of T. PRove that $T^2 - 5T + 6I = 0$.

10. Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Prove that T is normal if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal. Do the same for self adjoint operators for vector spaces over real field $\mathbb{R}$.

11. Let V be a complex inner product space. Prove that every normal operator on V has a square root.

Problem Sheet 9, Math 104

Topics to revise

- Singular Value Decomposition and its applications
- Minimal Polynomial
- Jordan Forms

Exercises

1. Give an example of $T \in \mathcal{L}(V)$ such that 0 is the only eigenvalue of T and the singular values 0.
2. Find the singular values of the differentiation operator $D \in \mathcal{P}(\mathbb{R}^2)$ defined by $Dp = p'$, where the inner product on $\mathcal{P}(\mathbb{R}^2)$ is given by
3. Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if 0 is not a singular value of T.
4. Suppose $T \in \mathcal{L}(V)$. Prove that $\dim \operatorname{range} T$ equals the number of nonzero singular values of T.
5. Define $N \in \mathcal{L}(V)$ by $N(x_1, x_2, x_3, x_4, x_5) = (2x_2, 3x_3 - x_4, 4x_5, 0)$. Find a square root of $I + N$.
6. Define $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(w, z) = (-z, w)$. Find the generalized eigenspaces corresponding to the distinct eigenvalues of T.
7. Suppose $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$. Find the minimal and characteristic polynoomials of the operator on $\mathbb{C}^n$ whose matrix (with respect to standard basis) is
$$A = \begin{bmatrix} 0 & & & & -a_0 \\ & & & & \vdots \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ & & & \ddots & -a_{n-2} \\ & & & 1 & -a_{n-2} \end{bmatrix}.$$
8. Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V that is a Jordan basis for T. Describe the matrix of ^{2} with respect to this basis.
9. Suppose V is an inner product space and $T \in \mathcal{L}(V)$ is normal. Prove that the minimal polynomial of T has no repeated zeros.
10. Suppose V is a complex vector space. Suppose $T \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that the characteristic polynomial of P is $z^m(z-1)^n$, where $m = \dim \operatorname{null} P$ and $n = \dim \operatorname{range} P$.