Lecture - 23

MTL - 122

Differentiability

$f : \mathbb{D} \to \mathbb{C}$, $z_0 \in \operatorname{int}(D)$

- f is differentiable at z_0

if $\displaystyle\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists *

$\displaystyle\lim_{(x,y) \to (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0)}{\sqrt{h^2 + k^2}}$ * $\mathbb{R}^2$

$h = z - z_0 = \Delta z_0$ $\in \mathbb{C}$

$$\boxed{f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}}$$ ✓

Chain Rule

$f \to z_0$

$g \to f(z_0)$

$$(g \circ f)'(z_0) = g'(f(z_0))\, f'(z_0)$$

Ex. Is there any diff between the differentiability in $\mathbb{R}^2$ & $\mathbb{C}$.

$f : \mathbb{C} \to \mathbb{C}$, $f(z) = |z|^2$

$\mathbb{R}^2$

$f(x, y) = x^2 + y^2$

Differentiable everywhere.

$\mathbb{C}$

$f(z) = |z|^2$

$|z|^2 = z\bar{z}$

$z_0 \in \mathbb{C}$, $h \in \mathbb{C}$

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

$$= \frac{|z_0 + h|^2 - |z_0|^2}{h}.$$

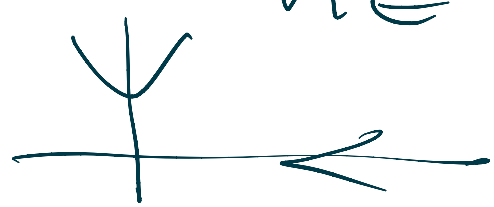
$$= \frac{z_0\bar{h} + \bar{z}_0 h + h\bar{h}}{h}$$

$$= z_0\left(\frac{\bar{h}}{h}\right) + \bar{z}_0 + \bar{h}$$

$h \to 0$

$$\frac{\bar{h}}{h} \to 1, \quad \to -1$$

$h \in \mathbb{C}$



For this limit to exist $z_0 = 0$

- $f(z) = |z|^2$ is differentiable at $z_0 = 0$.

Properties

$c_0 + ic_1$

- $\frac{dc}{dz} = 0$

- $\frac{d}{dz}\left(f(z)+g(z)\right) = f'(z)+g'(z)$

- $\frac{d}{dz}\left(f(z)F(z)\right) = f(z)F'(z) + F(z)f'(z)$

- $F(z) \neq 0 \quad \frac{d}{dz}\left(\frac{f(z)}{F(z)}\right)$

$$= \frac{F(z)f'(z) - f'(z)F(z)}{\left(F(z)\right)^2}.$$

A necessary condition

- $\underline{f}$ is differentiable at z_0.

$$f(x+iy) = u(x,y) + i v(x,y)$$

$$z_0 = x_0 + i y_0.$$

- $$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

$$\& \quad = f'(z_0).$$

Horizontal direction.

$$z = x + i y_0 \longrightarrow z_0$$

$$x \longrightarrow x_0. \qquad h = z - z_0$$

$$f'(z_0) = \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}$$

$$= u_x(x_0, y_0) + i v_x(x_0, y_0)$$

Vertical direction

$$z = x_0 + iy \longrightarrow z_0$$

$$y \longrightarrow y_0$$

$$z - z_0 = i(y - y_0)$$

$$f'(z_0) = \lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} - $$

$$+ \cancel{i} \lim_{y \to y_0} \frac{v(x_0, y) - v(x_0, y_0)}{\cancel{i}(y - y_0)}.$$

$$f'(z_0) = -i u_y(x_0, y_0) + v_y.$$

$$\left.\boxed{\begin{aligned} u_x &= v_y \\ v_x &= -u_y. \end{aligned}}\right\} \text{Cauchy Riemann (CR).}$$

- $f(z) = f(x+iy) = u(x,y) + iv(x,y)$

is differentiable at $z_0 = x_0 + iy_0$

Then the p.d of u & v exists at $z_0 = (x_0, y_0)$

and

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right| \text{ at } z_0.$$

Ex.

$$f(z) = \frac{\bar{z}^2}{z}, \quad z \neq 0$$
$$= 0, \quad z = 0.$$

$$u(x,y) = \frac{x^3 - 3xy^2}{x^2+y^2}$$

$$v(x,y) = \frac{y^3 - 3x^2y}{x^2+y^2}$$

at $z = 0$

$$\begin{cases} u_x = 1 & u_y = 0 \\ = v_y & = -v_x \end{cases}$$

$\Rightarrow$ Satisfying C-R eqns. ✓

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{\frac{x^3 - 3xy^2}{x^2+y^2} + i\frac{y^3 - 3x^2y}{x^2+y^2}}{x+iy} \rightarrow \text{does not exist.}$$

Along $y = x$

$$\frac{f(z) - f(0)}{z - 0}$$

$$= \frac{-x - ix}{x + ix} \rightarrow -1.$$

x axis.

$$\frac{f(z)-f(0)}{z-0}$$

$$= \frac{x-0}{x} \longrightarrow 1$$

$\Rightarrow$ f is not differentiable at $z=0$

Theo. $f = u+iv$, $B(z_0, r)$ open subset D

- u_x, u_y, v_x, v_y exist on $B(z_0, r)$
- u_x, u_y, v_x, v_y are continuous at z_0
- u, v satisfy CR equs.

f is diff at z_0.

& $f'(z_0) = u_x(z_0) + i v_x(z_0)$

Ex: Show that the fn $f(z) = \frac{1}{z}$, is differentiable everywhere except $z = 0$.

$$f(x+iy) = \frac{1}{x+iy}$$

$$= \frac{x - iy}{x^2+y^2}$$

$$u(x,y) = \frac{x}{x^2+y^2}$$

$$v(x,y) = -\frac{y}{x^2+y^2}$$

$x = y = 0$ $\Rightarrow$ f is not dif' at $z_0 = 0$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{-x^2+y^2}{(x^2+y^2)^2} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} = \dfrac{2xy}{(x^2+y^2)^2} = -\dfrac{\partial u}{\partial y} \end{cases}$$

u_x, v_y, v_x v_y

$(x,y) = (0,0)$

Polar form

$$w = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$$

$$\boxed{r u_r = v_\theta \;,\; v_\theta = -r v_r.}$$

- $$f'(z_0) = \lim_{r \to r_0} \frac{f(re^{i\theta_0}) - f(r_0 e^{i\theta_0})}{re^{i\theta_0} - r_0 e^{i\theta_0}}$$

$$= e^{-i\theta_0} \left[[\quad] + i [\,] \right]$$

Chain rule.

- Analytic fn s.

f(z) is said to be analytic in an open set of complex plane if f(z) has a derivative at each point of that set.

Remark.

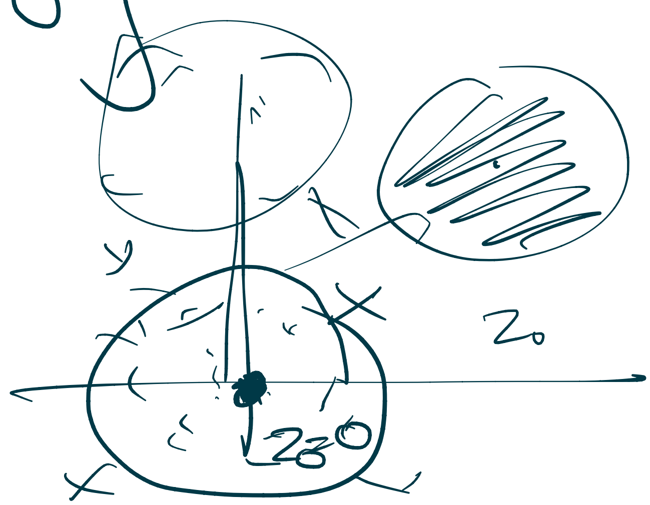
- f is analytic at a point z_0 if it is analytic in a ngb of z_0.

Ex. • $f(z) = \frac{1}{z}$ $(z \neq 0)$

$\rightarrow \mathbb{C} \setminus \{0\}$

- $f(z) = |z|^2$

not analytic



The point where

- f ceases to be analytic $\rightarrow$ singular points

- $\boxed{z^n}$, $n \in \mathbb{N}_+$ $\rightarrow$ entire fn's.

- $\boxed{\mathbb{C}}$ —

$f(z) = \frac{1}{z}$, $z = 0$ singular.