Lecture 12,13,14 Jan. 2020

Real and Complex Analysis

MTL122/ MTL503/ MTL506

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1. Continuity and Uniform Continuity

Definition 1.1. *Suppose (X, d_X), (Y, d_Y) are metric spaces. A function $f : X \to Y$ is continuous at the point $a \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in X$, $d_X(x, a) < \delta$ implies $d_Y(f(x), f(a)) < \epsilon$.*

*A function is called **continuous** if it is continuous at all $a \in X$.*

In terms of open balls, the definition says that $f(B_\delta(a)) \subset B_\epsilon(f(a))$.

Definition 1.2. *A function $f : A \to Y$ is continuous at $c \in A \subset X$ if for every neighborhood V of $f(c)$ there is a neighborhood U of c such that*

$$x \in U \cap A \implies f(x) \in V.$$

Note that c must belong to the domain of f in order to define the continuity of f at c. If c is an isolated point of A, then the continuity condition holds automatically, since, for sufficiently small $\delta > 0$, the only point $x \in A$ with $d_X(x, c) < \delta$ is $x = c$ and the $0 = d_Y(f(x), f(c)) < \epsilon$. Thus, a function is continuous at every isolated point of its domain, and isolated points are not of much interest.

A function is said to be Lipschitz continuous if there is $C \in \mathbb{R}$ so that

$$d_Y(f(x), f(y)) \le C d_X(x, y) \text{ for all } x, y \in X.$$

We also say that f is Lipschitz with constant C.

Lipschitz continuous map $\implies$ continuous map.

Solution: Let $\epsilon > 0$ and assume $f : X \to y$ be Lipschitz with constant C. Choose $\delta < \frac{\epsilon}{c}$. Now for any $x_1, x_2 \in X$ such that $d_X(x_1, x_2) < \delta$ we have

$$d_Y(f(x_1), f(x_2)) \le C d_X(x_1, x_2) < C\delta < \epsilon.$$

This implies f is continuous.

Converse is not true.

Example 1.3. *The function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is continuous but not Lipschitz.*

Check: f is continuous.

Let $0 < c < \infty$. We will show that there exist choices of $x, y \in [0, \infty)$ so that $c|x - y| < |\sqrt{x} - \sqrt{y}|$. Choose $x = 0$ and $y > 0$ such that $c\sqrt{y} < 1$ which is guaranteed *by Archimedian property. Now we can rewrite this inequality as $cy < \sqrt{y}$. Then we can see that f is not Lipschitz.*

$$c|y-x| = c|y-0| = cy < \sqrt{y} = |\sqrt{y} - \sqrt{0}| = |\sqrt{y} - \sqrt{x}|.$$

Example 1.4. *Let $X = \mathbb{R} = Y$, and let $f : \mathbb{R} \to \mathbb{R}$ be the map*

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \notin \mathbb{Q}. \end{cases} \tag{1.1}$$

Let then d_{Euc} and d_{dis} be the Euclidean and the discrete metrics on $\mathbb{R}$. We have then

- *$f : (X, d_{Euc}) \to (Y, d_{Euc})$ is not continuous;*
- *$f : (X, d_{Euc}) \to (Y, d_{dis})$ is not continuous;*
- *$f : (X, d_{dis}) \to (Y, d_{Euc})$ is continuous;*
- *$f : (X, d_{dis}) \to (Y, d_{dis})$ is continuous.*

This describes the importance of keeping in mind which are the ambient metrics. As an exercise, fill in the details to show each of the points above.

Example 1.5. *The function $x \to d(x, A)$, where $\phi \neq A \subset X$ is continuous.*

Example 1.6. *Since $|x_k - y_k| \leq d_{Euc}(x, y)$ for $x, y \in \mathbb{R}^n$, the projections $x \in \mathbb{R}^n \to x_k$ are Lipschitz continuous.*

One very important property of real-valued continuous functions from a metric space is that they can be added, multiplied, and multiplied by a scalar to get more continuous functions.

Proposition 1.7. *Let (X, d) be a metric space, $f, g : X \to \mathbb{R}$ be continuous, and $c \in \mathbb{R}$. Then $f + g$, cf and $f \cdot g$ are all also continuous functions from X to $\mathbb{R}$.*

Definition 1.8. *A function $f : X \to Y$ is sequentially continuous at $a \in X$ if $x_n \to a$ in X implies that $f(x_n) \to f(a)$ in Y.*

Theorem 1.9. *A function $f : X \to Y$ is continuous at a if and only if it is sequentially continuous at a.*

Proof Suppose that f is continuous at $a \in X$. Let $\epsilon > 0$ be given and suppose that $x_n \to a$. Then there exists $\delta > 0$ such that $d(f(x), f(a)) < \epsilon$ for $d(x, a) < \delta$, and there exists $N \in \mathbb{N}$ such that $d(x_n, a) < \delta$ for $n > N$. It follows that $d(f(x_n), f(a)) < \epsilon$ for $n > N$, so $f(x_n) \to f(a)$ and f is sequentially continuous at a.

Conversely, suppose that f is not continuous at a. Then there exists $\epsilon_0 > 0$ such that for every $n \in \mathbb{N}$ there exists $x_n \in X$ with $d(x_n, a) < \frac{1}{n}$ and $d(f(x_n, f(a)) \geq \epsilon_0$. Then $x_n \to a$ but $f(x_n) \nrightarrow f(a)$, so f is not sequentially continuous at a. $\square$

Recall that the pre-image of U under f is $f^{-1}(U) = \{x : f(x) \in U\}$.
Note that use of f^{-1} does not mean that f has an inverse.

Theorem 1.10. *A map $f : X \to Y$ is continuous if and only if for every open(closed) set $U \subset Y$, the set $f^{-1}(U)$ is open(closed) (in X).*

Proof. Suppose that the inverse image under f of every open set is open. Let $U \subset Y$ open . Then by given condition $f^{-1}(U) \subset X$ is open. Let $y_0 \in Y$ such that $f(x_0) = y_0$. Pick $\epsilon > 0$ and consider $U = B_\epsilon(f(x_0))$. Since $f^{-1}(U)$ is open, that is, $f^{-1}(B_\epsilon(f(x_0)))$ is open, and $x_0 \in f^{-1}(B_\epsilon(f(x_0)))$ so there exists $\delta > 0$ such that $B_\delta(x_0) \subset f^{-1}(B_\epsilon(f(x_0)))$ that is $f(B_\delta(x_0)) \subset B_\epsilon(f(x_0))$. Hence f is continuous.

Conversely, suppose that $f : X \to Y$ is continuous and $U \subset Y$ is open. We have to show that $f^{-1}(U)$ is open. Take $x_0 \in f^{-1}(U)$. Then $f(x_0) \in U$. Since U is open there exists $\epsilon > 0$ such that $B_\epsilon(f(x_0)) \subset U$. For this fixed $\epsilon > 0$ we can use the continuity that is there exists $\delta > 0$ such that $f(B_\delta(x_0)) \subset B_\epsilon(f(x_0))$. That is $B_\delta(x_0) \subset f^{-1}(B_\epsilon(f(x_0))) \subset f^{-1}(U)$, which implies x_0 is an interior point and so $f^{-1}(U)$ is open. □

- Any functions defined on a discrete metric spaces is always continuous.

Theorem 1.11. *The composition of continuous functions is continuous.*

Proof. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous, and $g \circ f : X \to Z$ is their composition. If $W \subset Z$ is open, then $V = g^{-1}(W)$ is open, so $U = f^{-1}(V)$ is open. Now $U = f^{-1}(g^{-1}(W))$ also $(g \circ f)(U) = W$, that is, the pre image of W under $g \circ f$ is U. It follows that $(g \circ f)^{-1}(W) = U$ is open. So $g \circ f$ is continuous. □

1.1. **Uniform Continuity.**

Definition 1.12. *A function $f : X \to Y$ is uniformly continuous if, for every ϵ, there is a $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$, for every $x, y \in X$.*

Spot the difference between uniform continuity and standard continuity? For standard continuity, for any ϵ, we are allowed to find a δ for each x such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$. For uniform continuity you have to pick the same δ for every x. Clearly, any uniformly continuous function is continuous, but, the reverse is not true. The key point of this definition is that δ depends only on ϵ,not on x, y.

- The function $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.

Proof. It is known that $f(x) = 1/x$ is continuous on $(0, 1)$. If we assume $f(x) = 1/x$ is uniformly continuous on $(0, 1)$. Then there exists $\delta > 0$ such that $\forall x, y \in (0, 1)$ with $|x - y| < \delta$, $|\frac{1}{x} - \frac{1}{y}| < 1$. Pick $x \in (0, 1)$ wit $x < \delta$. Then set $y = \frac{x}{2}$. Then $|x - y| = \frac{x}{2} < \frac{\delta}{2} < \delta$, and $|\frac{1}{x} - \frac{1}{y}| = \frac{1}{x} > 1$, a contradiction. □

Exercise 1.13. *If $f : X \to Y$ is a Lipschitz continuous map between two metric spaces then f is uniformly continuous.*

Proof. Let f be a Lipschitz continuous map. By definition, there exists a constant $K > 0$ such that

$$d_Y(f(x), f(y)) \le K d_X(x, y), \text{ for all } x, y \in X.$$

Now for any $\epsilon > 0$, let $\delta(\epsilon) = \frac{\epsilon}{K}$. Then $d_X(x, y) < \delta(\epsilon)$ implies

$$d_Y(f(x), f(y)) \le K d_X(x, y) < K\delta(\epsilon) = \epsilon$$

for all $x, y \in X$. $\square$

Converse is not true.

Example 1.14. *The function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$ but not Lipschitz.*

For any $n \in \mathbb{N}$, $|f(1/n) - f(0)| = \sqrt{1/n} = \sqrt{n}|1/n - 0|$. It follows that f is not Lipschitz.

Given $\epsilon > 0$, let $\delta = \epsilon^2$. Suppose $|x - y| < \delta$, where $x, y \geq 0$. To estimate $|f(x) - f(y),|$ we consider two cases.
In the case $x, y \in [0, \delta)$, we use the fact that f is strictly increasing. Then $|f(x) - f(y)| < f(\delta) - f(0) = \sqrt{\delta} = \epsilon$. Otherwise, when $x \notin [0, \delta)$ or $y \notin [0, \delta)$, we have $\max(x, y) \geq \delta$. Then

$$|\sqrt{x} - \sqrt{y}| = \left| \frac{x - y}{\sqrt{x} + \sqrt{y}} \right| \leq \frac{|x - y|}{2\sqrt{\max(x, y)}} < \frac{\delta}{\sqrt{\delta}} = \sqrt{\delta} = \epsilon.$$

Thus f is uniformly continuous.

2. Compactness in metric spaces

The closed intervals $[a, b]$ of the real line and more generally the close bounded subsets of $\mathbb{R}^n$ are compact. For these type of spaces we have seen interesting and very useful theorems like **Bolzano-Weierstrass theorem, Heine-Borel theorem** etc. Our aim to investigate the generalizations of these to metric spaces.

Let (X, d) be a metric space.

Definition 2.1. *A **covering** of X is a collection of sets whose union is X.*

Definition 2.2. *A collection of open sets $\{U_i : i \in I\}$ in X is an open cover of $Y \subset X$ if $Y \subset \bigcup\limits_{i \in I} U_i$.*

A subcover of $\{U_i : i \in I\}$ is a sub-collection $\{U_j : j \in J\}$ for some $J \subset I$ that still covers Y. It is a finite subcover if J is finite.

Definition 2.3.

*(1) A metric space X is **compact** if every open over of X has a finite subcover.*

*(2) A metric space is **sequentially compact** if every sequence of points in X has a convergent subsequence to a point in X.*

Example 2.4.

(1) $(0, 1]$ *is not sequentially compact and not compact. To show that* $(0, 1]$ *is not compact, it is sufficient to find an open cover of* $(0, 1]$ *that has no finite subcover. Consider* $U_n := (\frac{1}{n}, 2)$, *for* $n \in \mathbb{N}$. *We see,* $\bigcup\limits_{n \in \mathbb{N}} U_n = (0, 2)$ *and* $(0, 1] \subset (0, 2)$. *But if* F *is any finite subset of* $\{U_n : n \in \mathbb{N}\}$. *Let* $N \in \mathbb{N}$ *such*

that $F = \{U_{i_1}, U_{i_2}, ..., U_{i_N}\}$. *Let* $k = \max\{i_1, i_2, ..., i_N\}$. *Then* $U_k \in F$ *and* $U_{i_j} \subseteq U_k$, *for* $i = 1, 2, ..., N$. *and* $(0, 1] \not\subset \bigcup F = U_k = (\frac{1}{k}, 2)$.

(2) $[0, 1]$ *is sequentially compact. In fact,* $[0, 1]$ *is also compact.*

(3) $\mathbb{R}$ *is neither compact nor sequentially compact. It is not sequentially compact or compact follows from the fact that* $\mathbb{R}$ *from Heine-Borel.*

Exercise 2.5.
A nonempty subset K *of a discrete metric space* (X, d) *is compact if and only if* K *is finite.*

Proof. Assume that K is compact. Since eack singleton set in a discrete metric space is open, the collection $\mathcal{C} = \{\{x\} : x \in K\}$ is an open cover for K. Since K is compact, there are elements $x_1, x_2, ..., x_n$ in K such that $K \subseteq \bigcup_{i=1}^{n} \{x_i\}$. Hence K is finite.
Conversely, assume that K is finite. Then K is clearly compact as any finite set is compact. □

Theorem 2.6. *A closed subset* F *of a compact metric space* (X, d) *is compact.*

Easy.

Theorem 2.7. *Every compact subset* K *of a metric* (X, d) *is closed and bounded.*

Proof. If $K = \phi$ then it is bounded. If not, the let $x_0 \in K$. Then let $K \subseteq \bigcup_{n\in\mathbb{N}} B_n(x_0)$. By compactness there is a finite subset $F \in \mathbb{N}$ such that $K \subseteq \bigcup_{n\in F} B_n(x_0)$. Let N be the largest integer in F. Such an integer exists because F is finite and non-empty. Then $K \subset B_N(x_0)$ and it follows that K is bounded.
Let K be a compact subset of (X, d). We shall prove $X \setminus K$ is open. Fix a point $p \in X \setminus K$. For each $x \in K$ consider $\delta_x = \frac{1}{2} d(p, x)$. Then $\{B_{\delta_x}(x)\}_{x\in K}$ forms an open cover for K. Since K is compact, there exist $x_1, x_2, ..., x_k$ such that $K \subseteq \bigcup_{i=1}^{k} B_{\delta_{x_i}(x_i)}$. Then $V = \bigcap_{i=1}^{k} B_{\delta_{x_i}}(p)$ is an open set and contains p . Also since $B_{\delta_{x_i}}(x_i) \cap B_{\delta_{x_i}}(p) = \phi$ for all i and so $V \subset X \setminus K$. Hence K is closed as $X \setminus K$ is open. □

We know by Heine-Borel Theorem that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded. That is, the converse of Theorem (2.7) holds if $X = \mathbb{R}$. But in general the converse doesn't hold.

Example 2.8. K *be an infinite subset of a discrete metric space* (X, d). *Then* K *is closed and bounded but not compact. (Exercise)*

Definition 2.9. *A subset* A *of a metric space* X *is totally bounded if, for every* $\epsilon > 0$, *there exist* $x_1, x_2, ..., x_k \in A$, *with* k *finite, so that* $\{B_\epsilon(x_i) : 1 \leq i \leq k\}$ *is an open* *cover of X, that is,*

$$A \subseteq \bigcup_{i=1}^{k} B_\epsilon(x_i)$$

Example 2.10. *The set $(0,1)$ is totally bounded. For any $\epsilon > 0$ by Archimedean property, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Then consider the points $\epsilon, 2\epsilon, ...(N-1)\epsilon$, and the balls $B_\epsilon(n\epsilon)$, $n = 0, 1, 2, ..., N$. Then $(0,1) \subset \bigcup_{n=1}^{N} B_\epsilon(n\epsilon)$.*

- A totally bounded set is bounded.

Let B be a totally bounded set. We can therefore cover B with finite number of ϵ-balls, in particular with 1-balls.

$$B \subset \bigcup_{i=1}^{N} B_1(a_i).$$

Since there are only a finite number of these balls, we can find the maximum distance between their centres. $D := \max\{d(a_i, a_j)\}$. Now given any two points $x, y \in B$, they must be covered by two of these balls $B_1(a_I)$ and $B_1(a_J)$, say. Therefore, using the triangle inequality twice,

$$d(x,y) \leq d(x, a_I) + d(a_I, a_J) + d(a_J, y) \leq 1 + D + 1.$$

That is $D + 2$ is an upper bound for the distance between points in B.

But the converse is not true, i.e. bounded $\not\Rightarrow$ totally bounded.

Example 2.11. *Take $U = \{e_n | n \in \mathbb{N}\} \subset l^\infty(\mathbb{R})$, where $e_n = (e_{nj})_{j\in\mathbb{N}}$ with $e_{nj} = 1$, $j = n$ and 0 otherwise. Then since $\forall e_i, e_j \in U$ $d(e_i, e_j) = 1$ so obviously this set is bounded. But for $\epsilon = 1$ we cannot find finite number of open balls with radius ϵ that cover U- as each ball will contain only one element of U. Hence U is not totally bounded.*

Theorem 2.12. *X is totally bounded if and only if every sequence in X has a Cauchy subsequence.*

Proof. First assume that X is totally bounded, and let $\{x_n\}$ be a sequence in X. X is a union of finitely many sets of diameter less than 1. We pick one of these sets that contains infinitely many elements of (x_n) and call it S_1. Choose n_1 such that $x_{n_1} \in S_1$. Again since S_1 is totally bounded, it is a union of finitely many sets of diameter less than $1/2$. From here also we pick up a set that contains infinitely many elements of (x_n) and call it S_2. Choose $n_2 > n_1$ such that $x_{n_2} \in S_2$. Continuing this way we get a decreasing sequence $\{S_k\}$ of sets, that is $S_k \supset S_{k+1}$, $\forall k$, of diameter less than $1/k$ and a strictly increasing sequence $\{n_k\}$ in $\mathbb{N}$ such that $x_{n_k} \in S_k$ for all k. Since $d(x_{n_j}, x_{n_l}) < 1/k$ for all $j, l \geq k$, the subsequence is Cauchy.

Conversely, suppose X is not totally bounded. We will construct a sequence in X with no Cauchy subsequence. We can choose $\epsilon > 0$ such that X is not finite union of open balls of radius ϵ. Choose $x_1 \in X$. Since $X \neq B_\epsilon(x_1)$, we can choose $x_2 \notin B_\epsilon(x_1)$.

Similarly, we can choose $x_3 \in \bigcup_{i=1}^{2} B_\epsilon(x_i)$. Continue in this way, getting a sequence $\{x_n\}$ in X such that x_n in X such that

$$x_n \notin \bigcup_{i<n} B_\epsilon(x_i), \text{ for all } n.$$

It follows that if $n \neq k$ then $d(x_n, x_k) \geq \epsilon$. Therefore, $\{x_n\}$ has no Cauchy subsequence. □

Theorem 2.13. *For a metric space (X, ρ), the following are equivalent:*

(1) X is compact.

(2) X is sequentially compact.

(3) X is complete and totally bounded.

Proof. (1) $\implies$ (2).

Let (x_n) be a sequence in a compact metric space X. If no subsequence of (x_n) converges in X, then for each $y \in X$ there is some $r_y > 0$ and a positive integer n_y such that $x_n \notin B_{r_y}(y)$ for all $n \geq n_x$. Since $X = \bigcup_{y\in X} B_{r_y}(y)$, and X is compact so there are finitely many $y_1, y_2, ...y_k \in X$ such that $X = \bigcup_{i=1}^{k} B_{r_{y_i}}(y_i)$. Take $n_0 = \max\{n_{y_1}, n_{y_2}, ..., n_{y_k}\}$. Then $x_{n_0+1} \notin B_{r_{y_i}}(y_i)$, $\forall i = 1, 2, ...k$. So $x_{n_0+1} \notin X$. A contradiction.

(2) $\implies$ (3).

Suppose that every sequence in X has a convergent subsequence. Since a Cauchy sequence having a convergent subsequence is it self convergent so X is complete. Next assume that X is not totally bounded. Then there is some $\epsilon > 0$ such that X cannot be covered by finitely many open balls of radius ϵ. Let $x_1 \in X$ and consider $B_\epsilon(x_1)$ for some $\epsilon > 0$. Find $x_2 \in X$ such that $x_2 \notin B_\epsilon(x_1)$. Inductively find $x_{n+1} \in X$ such that $x_{n+1} \notin \bigcup_{i=1}^{n} B_\epsilon(x_i)$. Then by construction $d(x_n, x_m) \geq \epsilon$ for all $n, m = 1, 2, ...$, so (x_n) cannot have any convergent subsequence. A contradiction. Hence X is totally bounded.

(3) $\implies$ (1).

Suppose that X is complete and totally bounded. Assume that X is not compact. Consider an open cover of $X = \bigcup_{\alpha\in\Omega}\{U_\alpha\}$ without any finite subcover. Since X is totally bounded, cover X by finitely many open balls of radius 1. Then for at least one of these open balls of radius 1, say $B_1(x_1)$ there exists no finite subcover. Now $B_1(x_1) \subset X$ is totally bounded, cover $B_1(x_1)$ by finitely many open balls of radius 1/2. Then for at least one of these open balls of radius 1/2 there exists no finite subcover. So there exists $x_2 \in B_1(x_1)$ such that $B_{1/2}(x_2)$ has no finite subcover. Continuing this we get a sequence (x_n) in X with $x_{n+1} \in B_{1/2^n}(x_n)$ such that $B_{1/2^n}(x_n)$ has no finite subcover. Now

$$d(x_n, x_m) \leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \leq \sum_{j=n}^{m-1} \frac{1}{2^j} \leq \frac{1}{2^{n-1}}$$

for all $m > n$. So (x_n) is a Cauchy sequence in the complete metric space X. Let $x_n \to x \in X$. Then $x \in U_\alpha$ for some $\alpha \in \Omega$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U_\alpha$. For this $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon/2$ for all $n \geq N$. We choose $m > N$ so large that $1/2^m < \epsilon/2$. Now let $y \in B_{1/2^m}(x_m)$, then

$$d(x, y) \leq d(x, x_m) + d(x_m, y) < \frac{\epsilon}{2} + \frac{1}{2^m} < \epsilon.$$

This implies $y \in B_\epsilon(x)$, that is $B_{1/2^m}(x_m) \subset B_\epsilon(x) \subset U_\alpha$. Thus U_α constitutes a open subcover of $B_{1/2^m}(x_m)$. A contradiction to our construction of $B_{1/2^m}(x_m)$.

□

Theorem 2.14. *Let (X, d_1) and (Y, d_2) be metric spaces and $f : X \to Y$ be a continuous function. Then for each compact subset $C \subset X$, $f(C) \subset Y$ is compact.*

Proof. Let $\{U_i : i \in I\}$ be an open cover of $f(C)$, and for each $i \in I$, define $V_i = f^{-1}(U_i)$. Notice that since f is continuous, each V_i is open and $\{V_i :\in I\}$ is an open cover of C. Since C is compact so it has a finite subcover $\{V_{i_1}, ...V_{i_n}\}$. Then $\{U_{i_1}, U_{i_2}, ..., U_{i_n}\}$ is a finite subcover of $f(C)$. This proves $f(C)$ is compact. □

Theorem 2.15. *Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous function of metric spaces, and suppose in addition that X is compact. Then f is uniformly continuous.*

Proof. Let (X, d) be a compact metric space and (Y, ρ) be a metric space. Suppose $f : X \to Y$ is continuous. We want to show that it is uniformly continuous.
Let $\epsilon > 0$. We want to find $\delta > 0$ such that $d(x, y) < \delta \implies \rho(f(x), f(y)) < \epsilon$ for all $x, y \in X$. Since f is continuous at each $x \in X$ then there is some $\delta_x > 0$ so that

$$f(B(x, \delta_x) \subseteq B(f(x), \epsilon/2).$$

Now $\{B(x, \delta_{x/2})\}_{x\in X}$ is an open cover of X so there is a finite subcover $\{B(x_i, \delta_{x_i}/2)\}_{i=1}^n$. Take $\delta = \min_i(\frac{\delta_{x_i}}{2})$. Suppose for $x, y \in X$, $d(x, y) < \delta$. Since $x \in X$ so $x \in B(x_i, \delta_{x_i}/2)$ for some i. We claim then $y \in B(x_i, \delta_{x_i})$.(Check). Then for $d(x, y) < \delta$ we have

$$\rho(f(x), f(y)) \leq \rho(f(x), f(x_i)) + \rho(f(x_i), f(y)) < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Theorem 2.16. *Let (X, d_X) be a compact metric space. Then (X, d_X) is complete.*

Remark 2.17. *(Exercise)*

(1) The product of two compact metric spaces is compact.
(2) A finite union of compact sets is compact.