

Assignment 2

Real and Complex Analysis

MTL122/ MTL503/ MTL506

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- (1) Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.
- i) Prove that $\text{Int}(A \cap B) = \text{Int}A \cap \text{Int}B$.
 - ii) Prove that $\text{Int}A \cup \text{Int}B \subset \text{Int}(A \cup B)$
 - iii) Give an example of two sets A and B with $\text{Int}(A \cup B) \neq \text{Int}A \cup \text{Int}B$.
- (2) Prove that
- i) If A is bounded above then $\sup A \in \text{Bd}(A)$.
 - ii) If $a < b < c$ and the two sets A and B has the property that $A \cap (a, c) = B \cap (a, c)$. Show that $b \in \text{Bd}(A)$ if and only if $b \in \text{Bd}(B)$.
- (3) Prove or give a counterexample:
- i) The union of infinitely many compact sets is compact.
 - ii) A non-empty subset S of real numbers which has both a largest and a smallest element is compact.
- (4) For $A \subset \mathbb{R}$, $B \subset \mathbb{R}$, let
- $$A + B = \{a + b : a \in A, b \in B\}.$$
- Let A be closed set, B be a compact set. Show that $A + B$ is closed.
- (5) Let (X, d) be a metric space. Define $\bar{d} : X \times X \rightarrow \mathbb{R}$ by $\bar{d}(x, y) = d(x, y)$ when $d(x, y) \leq 1$ and $\bar{d}(x, y) = 1$ when $d(x, y) \geq 1$.
Prove that \bar{d} is a metric on X .
- (6) Suppose that $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfies $\phi(0) = 0$, $\phi(r) > 0$ for all $r > 0$ and for all $a, b \in [0, \infty)$:
- i) $\phi(a + b) \leq \phi(a) + \phi(b)$
 - ii) if $a \leq b$ then $\phi(a) \leq \phi(b)$.
- Let (X, d) be a metric space and let $D : X \times X \rightarrow \mathbb{R}$ be defined by $D(x, y) := \phi(d(x, y))$. Prove that D is a metric on X .
- (7) Let $(X_1, d_1), (X_2, d_2), \dots$ be a sequence of metric spaces. Let $X = \prod_{n \in \mathbb{N}} X_n$, i.e, X is the set of all sequences $x = (x_1, x_2, \dots)$ with $x_n \in X_n$ for all $n \in \mathbb{N}$.

Prove that the function $d : X \times X \rightarrow \mathbb{R}$ defined by

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

is a metric on X .

- (8) Prove that the function $d(m, n) = |m^{-1} - n^{-1}|$ for any $m, n \in \mathbb{N}$ defines a metric on the set of natural numbers. Does this metric extend to \mathbb{R}^+ .
- (9) Let A be a subset of a metric space X with closure \bar{A} and interior of A by A° and boundary of A by δA . Show that
- i) Show that $\delta A = \bar{A} \setminus A^\circ$ and δA is closed.
 - ii) Prove that $X \setminus \bar{A} = (X \setminus A)^\circ$.
 - iii) Prove that A is closed if and only if $\delta A \subset A$, and A is open if and only if $\delta A \subset A^c$.
 - iv) If A is open, does it follow that $(\bar{A})^\circ = A$?
- (10) Let \mathbb{Q} , the set of rational numbers, as a metric space with the Euclidean distance $d(p, q) = |p - q|$. Consider the set
- $$E = \{p \in \mathbb{Q} \mid 2 < p^2 < 3\}.$$
- Show that E is closed and bounded in \mathbb{Q} .