MTL122 - Real and complex analysis

Assignment-2

Department of Mathematics
Indian Institute of Technology Delhi

Question 1

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Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be sets.

(i) Prove that $\text{Int}(A \cap B) = \text{Int}A \cap \text{Int}B$.

(ii) Prove that $\text{Int}A \cup \text{Int}B \subseteq \text{Int}(A \cup B)$.

(iii) Give an example of two sets A and B such that $\text{Int}A \cup \text{Int}B \neq \text{Int}(A \cup B)$.

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Prove that

(i) If $A \subseteq \mathbb{R}$ is bounded above then $\sup A \in$ Bd(A).

(ii) If $a < b < c$ and the two sets A and B has the property that $A \cap (a,c) = B \cap (a,c)$. Show that $b \in$ Bd(A) if and only if $b \in$ Bd(B).

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$$A + B = \{a + b : a \in A, b \in B\}.$$

Let A be closed set, B be a compact set. Show that $A + B$ is closed.

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$$\bar{d}(x,y) = \begin{cases} d(x,y) & \text{when} \quad d(x,y) < 1 \\ 1 & \text{when} \quad d(x,y) \geq 1. \end{cases}$$

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Question 6

Suppose that $\phi : [0, \infty) \to [0, \infty)$ satisfies $\phi(0) = 0, \phi(r) > 0$ for all $r > 0$ and for all $a, b \in [0, \infty)$:

1) $\phi(a + b) \leq \phi(a) + \phi(b)$
2) If $a \leq b$ then $\phi(a) \leq \phi(b)$.

Let (X, d) be a metric space and let $D : X \times X \to \mathbb{R}$ be defined by $D(x, y) := \phi(d(x, y))$. Prove that D is a metric on X.

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Question 6 Contd...

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$$\begin{aligned} D(x,z) &= \phi(d(x,z)) \\ &\leq \phi(d(x,z)+d(z,y))(why?) \end{aligned}$$

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$$
\begin{aligned}
D(x,z) &= \phi(d(x,z)) \\
&\leq \phi(d(x,z)+d(z,y))\textcolor{red}{(why?)} \\
&\leq \phi(d(x,z))+\phi(d(z,y))\textcolor{red}{(why?)}
\end{aligned}
$$

Question 6 Contd...

- Triangle inequality

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- Hence D is a metric.

Question 7

Let $(X_1, d_1), (X_2, d_2), \ldots$ be a sequence of metric spaces. Let $X = \prod_{n \in \mathbb{N}} X_n$ i.e, X is the set of all sequences $x = (x_1, x_2, \ldots)$ with $x_n \in X_n$ for all $n \in \mathbb{N}$. Prove that the function $d : X \times X \to \mathbb{R}$ defined by

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Solution:

- **Well-defined:** Since $\dfrac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} < 1$. This implies

- $$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}$$

- $$\begin{aligned} d(x,y) &= \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)} \\ &< \sum_{n=1}^{\infty} 2^{-n} \end{aligned}$$

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- $d(x,y) = 0$

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- $d(x,y) = 0 \iff \dfrac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)} = 0 \iff d_n(x_n,y_n) = 0$ for all $n \in \mathbb{N}$

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- $d(x,y)=0 \iff \dfrac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}=0 \iff d_n(x_n,y_n)=0$ for all $n\in\mathbb{N} \iff x_n=y_n$ for all $n\in\mathbb{N}$

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- Clearly d is symmetric as well.(why?)

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Question 7 Contd...

- **Triangle inequality**

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$$d_n(x_n, z_n) \quad \leq \quad d_n(x_n, y_n) + d_n(y_n, z_n) \textcolor{red}{(why?)}$$

Question 7 Contd...

- **Triangle inequality**

$$
\begin{aligned}
d_n(x_n, z_n) &\leq d_n(x_n, y_n) + d_n(y_n, z_n) \text{(why?)} \\
1 + d_n(x_n, z_n) &\leq 1 + d_n(x_n, y_n) + d_n(y_n, z_n) \text{(why?)}
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d_n(x_n, z_n) &\leq d_n(x_n, y_n) + d_n(y_n, z_n)\,\textit{(why?)} \\
1 + d_n(x_n, z_n) &\leq 1 + d_n(x_n, y_n) + d_n(y_n, z_n)\,\textit{(why?)} \\
1 - \frac{1}{1 + d_n(x_n, z_n)} &\leq 1 - \frac{1}{1 + d_n(x_n, y_n) + d_n(y_n, z_n)}
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1 - \frac{1}{1 + d_n(x_n, z_n)} &\leq 1 - \frac{1}{1 + d_n(x_n, y_n) + d_n(y_n, z_n)} \\
\frac{d_n(x_n, z_n)}{1 + d_n(x_n, z_n)} &\leq \frac{d_n(x_n, y_n) + d_n(y_n, z_n)}{1 + d_n(x_n, y_n) + d_n(y_n, z_n)}
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- **Triangle inequality**

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\frac{d_n(x_n, z_n)}{1 + d_n(x_n, z_n)} &\leq \frac{d_n(x_n, y_n) + d_n(y_n, z_n)}{1 + d_n(x_n, y_n) + d_n(y_n, z_n)} \\
&\leq \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n) + d_n(y_n, z_n)} \\
&+ \frac{d_n(y_n, z_n)}{1 + d_n(x_n, y_n) + d_n(y_n, z_n)}
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\frac{d_n(x_n, z_n)}{1 + d_n(x_n, z_n)} &\leq \frac{d_n(x_n, y_n) + d_n(y_n, z_n)}{1 + d_n(x_n, y_n) + d_n(y_n, z_n)} \\
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&+ \frac{d_n(y_n, z_n)}{1 + d_n(x_n, y_n) + d_n(y_n, z_n)} \\
&\leq \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} + \frac{d_n(y_n, z_n)}{1 + d_n(y_n, z_n)}
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- **Triangle inequality**

$$
\begin{aligned}
d_n(x_n, z_n) &\leq d_n(x_n, y_n) + d_n(y_n, z_n)\textcolor{red}{(why?)} \\
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1 - \frac{1}{1 + d_n(x_n, z_n)} &\leq 1 - \frac{1}{1 + d_n(x_n, y_n) + d_n(y_n, z_n)} \\
\frac{d_n(x_n, z_n)}{1 + d_n(x_n, z_n)} &\leq \frac{d_n(x_n, y_n) + d_n(y_n, z_n)}{1 + d_n(x_n, y_n) + d_n(y_n, z_n)} \\
&\leq \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n) + d_n(y_n, z_n)} \\
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\end{aligned}
$$

- Now by multiplying by 2^{-n} then taking sum over $\mathbb{N}$, we will get our required triangle inequality. Hence d is a metric.

Question 8

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Prove that the function $d(m,n) = |\frac{1}{m} - \frac{1}{n}|$ for any $m, n \in \mathbb{N}$ defines a metric on the set of natural numbers. Does this metric extend to $\mathbb{R}^+$.

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Solution:

- Clearly d is non-negative.

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Solution:

- Clearly d is non-negative.
- $d(n,m) = 0$

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- $d(m,n) = |\frac{1}{m} - \frac{1}{k} + \frac{1}{k} - \frac{1}{n}| \leq |\frac{1}{m} - \frac{1}{k}| + |\frac{1}{k} - \frac{1}{n}| \leq d(m,k) + d(k,n).$

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Ans:

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Does this metric extend to $\mathbb{R}^+$.

Ans:Yes.

Question 9

Let A be a subset of a metric space X with closure $\bar{A}$ and boundary of A by ∂A. Show that

(i) Show that $\partial A = \bar{A} \setminus A^\circ$ and ∂A is closed.

(ii) Prove that $X \setminus \bar{A} = (X \setminus A)^\circ$.

(iii) Prove that A is closed if and only if $\partial A \subset A$, and A is open if and only if $\partial A \subset A^c$.

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- Let $x \in \partial A$, i.e. for $\epsilon > 0$, we have $N_\epsilon(x) \cap A \neq \phi$ and $N_\epsilon(x) \cap A^c \neq \phi$.

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- Since $N_\epsilon(x) \cap A^c \neq \phi$,i.e. $N_{\frac{1}{n}}(x)$ always contains element of A^c.

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Question 9 Contd...

- Thus $N_{\frac{1}{n}}(x) \nsubseteq A$.

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- Thus $N_{\frac{1}{n}}(x) \not\subseteq A$. Therefore $x \notin A^{\circ}$.

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- Since every step was following if and only if, Thus we get $\partial A = \bar{A} \setminus A^\circ$.

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- Thus $N_{\frac{1}{n}}(x) \nsubseteq A$. Therefore $x \notin A^{\circ}$. Hence $x \in \bar{A} \setminus A^{\circ}$.
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Now we will prove that ∂A is closed.

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- Since $\partial A = \bar{A} \setminus A^\circ = \bar{A} \cap (A^\circ)^c$,

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Question 9 Contd...

- Thus $N_{\frac{1}{n}}(x) \not\subseteq A$. Therefore $x \notin A^\circ$. Hence $x \in \bar{A} \setminus A^\circ$.
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Question 9 Contd...

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Question 9 Contd...

(iii) To show that A is open if and only if $\partial A \subset A^c$.

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Assume that A is open

- To the contrary suppose that $x \in \partial A$ but $x \notin A^c$.

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Assume that A is open

- To the contrary suppose that $x \in \partial A$ but $x \notin A^c$.i.e. $x \in A$.
- Since A is open,

(iii) To show that A is open if and only if $\partial A \subset A^c$.
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- To the contrary suppose that $x \in \partial A$ but $x \notin A^c$.i.e. $x \in A$.
- Since A is open,there exists $\epsilon > 0$ such that $N_\epsilon(x) \subseteq A \implies N_\epsilon(x) \cap A^c = \phi$.

Question 9 Contd...

(iii) To show that A is open if and only if $\partial A \subset A^c$.
Assume that A is open

- To the contrary suppose that $x \in \partial A$ but $x \notin A^c$.i.e. $x \in A$.
- Since A is open,there exists $\epsilon > 0$ such that $N_\epsilon(x) \subseteq A \implies N_\epsilon(x) \cap A^c = \phi$.
- Thus x cannot be boundary point of A.

Question 9 Contd...

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Assume that A is open

- To the contrary suppose that $x \in \partial A$ but $x \notin A^c$.i.e. $x \in A$.
- Since A is open,there exists $\epsilon > 0$ such that $N_\epsilon(x) \subseteq A \implies N_\epsilon(x) \cap A^c = \phi$.
- Thus x cannot be boundary point of A. Contradiction.

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- To the contrary suppose that $x \in \partial A$ but $x \notin A^c$.i.e. $x \in A$.
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Question 9 Contd...

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- To the contrary suppose that $x \in \partial A$ but $x \notin A^c$.i.e. $x \in A$.
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Assume that $\partial A \subset A^c$

- Let $x \in A$

Question 9 Contd...

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- Thus x cannot be boundary point of A. Contradiction.
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Assume that $\partial A \subset A^c$

- Let $x \in A \implies x$ is not a boundary point.

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Assume that A is open

- To the contrary suppose that $x \in \partial A$ but $x \notin A^c$.i.e. $x \in A$.
- Since A is open,there exists $\epsilon > 0$ such that $N_\epsilon(x) \subseteq A \implies N_\epsilon(x) \cap A^c = \phi$.
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Assume that $\partial A \subset A^c$

- Let $x \in A \implies x$ is not a boundary point.
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Question 9 Contd...

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Assume that A is open

- To the contrary suppose that $x \in \partial A$ but $x \notin A^c$.i.e. $x \in A$.
- Since A is open,there exists $\epsilon > 0$ such that $N_\epsilon(x) \subseteq A \implies N_\epsilon(x) \cap A^c = \phi$.
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Assume that $\partial A \subset A^c$

- Let $x \in A \implies x$ is not a boundary point.
- There exists $\epsilon > 0$ such that $N_\epsilon(x) \cap A = \phi$ or $N_\epsilon(x) \cap A^c = \phi$.
- But $x \in N_\epsilon \cap A \neq \phi$. Thus $N_\epsilon(x) \cap A^c = \phi \implies N_\epsilon(x) \subset A$.

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- To the contrary suppose that $x \in \partial A$ but $x \notin A^c$.i.e. $x \in A$.
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- To the contrary suppose that $x \in \partial A$ but $x \notin A^c$.i.e. $x \in A$.
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- But $x \in N_\epsilon \cap A \neq \phi$. Thus $N_\epsilon(x) \cap A^c = \phi \implies N_\epsilon(x) \subset A$.
- Hence A is open.

Now we will prove that A is closed if and only if $\partial A \subset A$.

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$$A \text{ is closed} \iff A^c \text{ is open}$$

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$$\begin{aligned} A \ \text{ is closed} &\iff A^c \ \text{ is open} \\ &\iff \partial A^c \subset (A^c)^c \quad \color{red}{(why?)} \end{aligned}$$

Question 9 Contd...

Now we will prove that A is closed if and only if $\partial A \subset A$.

$$\begin{aligned} A \ \text{ is closed} \quad &\Longleftrightarrow \quad A^c \ \text{ is open} \\ &\Longleftrightarrow \quad \partial A^c \subset (A^c)^c \quad \textcolor{red}{(why?)} \\ &\Longleftrightarrow \quad \partial A \subset A \quad \textcolor{red}{(why?)} \end{aligned}$$

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(iv)

Question 9 Contd...

Now we will prove that A is closed if and only if $\partial A \subset A$.

$$\begin{aligned} A \text{ is closed} &\iff A^c \text{ is open} \\ &\iff \partial A^c \subset (A^c)^c \quad (why?) \\ &\iff \partial A \subset A \quad (why?) \end{aligned}$$

(iv) Ans:

Now we will prove that A is closed if and only if $\partial A \subset A$.

$$\begin{aligned} A \text{ is closed} &\iff A^c \text{ is open} \\ &\iff \partial A^c \subset (A^c)^c \quad (why?) \\ &\iff \partial A \subset A \quad (why?) \end{aligned}$$

(iv) Ans: No Conterexample

Question 9 Contd...

Now we will prove that A is closed if and only if $\partial A \subset A$.

$$\begin{aligned} A \ \text{ is closed} \quad &\Longleftrightarrow \quad A^c \ \text{ is open} \\ &\Longleftrightarrow \quad \partial A^c \subset (A^c)^c \quad (why?) \\ &\Longleftrightarrow \quad \partial A \subset A \quad (why?) \end{aligned}$$

(iv) Ans: No Conterexample

- Take $A = (-\infty, 0) \cup (0, \infty)$.

Question 9 Contd...

Now we will prove that A is closed if and only if $\partial A \subset A$.

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(iv) Ans: No Conterexample

- Take $A = (-\infty, 0) \cup (0, \infty)$.
- Clealry $\bar{A} = \mathbb{R}$ and $(\bar{A})^\circ = \mathbb{R}$ thus $(\bar{A})^\circ \neq A$.

Now we will prove that A is closed if and only if $\partial A \subset A$.

$$\begin{aligned} A \ \text{ is closed} &\iff A^c \ \text{ is open} \\ &\iff \partial A^c \subset (A^c)^c \quad \color{red}{(why?)} \\ &\iff \partial A \subset A \quad \color{red}{(why?)} \end{aligned}$$

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Question 10 Contd...

Question 10

Let $\mathbb{Q}$, the set of rational numbers, as a metric space with the Euclidean distance $d(p, q) = |p - q|$. Consider the set

$$E = \{p \in \mathbb{Q} : 2 < p^2 < 3\}.$$

Show that E is closed and bounded in $\mathbb{Q}$.

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Solution: We have
$E = \{p \in \mathbb{Q} : 2 < p^2 < 3\}$

Question 10 Contd...

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$E = \{p \in \mathbb{Q} : 2 < p^2 < 3\} = \mathbb{Q} \cap (-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})$.

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- Clearly E is bounded.

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- Clearly there doesn't exist any rational number q such that $q^2 = 2$ or $q^2 = 3$.

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- Let $q \in \mathbb{Q}$ and $q^2 < 2$.

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- Clearly E is bounded.
- Clearly there doesn't exist any rational number q such that $q^2 = 2$ or $q^2 = 3$.
- Let $q \in \mathbb{Q}$ and $q^2 < 2$.
- Take $r_1 = \frac{\sqrt{2}-|q|}{2} > 0$. then $N_{r_1}(q) \cap E = \phi$.

Question 10 Contd...

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Question 10 Contd...

- Let $q \in \mathbb{Q}$ and $q^2 > 3$.
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Question 10 Contd...

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- Take $r_1 = \frac{|q| - \sqrt{3}}{2} > 0$. then $N_{r_2}(q) \cap E = \phi$.
- Thus $q \in \mathbb{Q}$ with $q^2 > 3$ can't be the limit point of E.
- Therefore only possible limit point of E are $q \in \mathbb{Q}$ with $2 < q^2 < 3$.

Question 10 Contd...

- Let $q \in \mathbb{Q}$ and $q^2 > 3$.
- Take $r_1 = \frac{|q|-\sqrt{3}}{2} > 0$. then $N_{r_2}(q) \cap E = \phi$.
- Thus $q \in \mathbb{Q}$ with $q^2 > 3$ can't be the limit point of E.
- Therefore only possible limit point of E are $q \in \mathbb{Q}$ with $2 < q^2 < 3$.
- Hence E is closed.