Assignment 5

Real and Complex Analysis

MTL122/ MTL503/ MTL506

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(1) Let (X, d) be a metric space and Y, Z be subsets of X such that $Y \subset Z \subset \overline{Y}$. If Y is connected then Z is connected.

(2) Any product of path connected spaces is path connected.

(3) Prove that if A and B are connected subsets of $\mathbb{R}$ then $A \cap B$ is a connected subset of $\mathbb{R}$. Find two connected subsets A and B of $\mathbb{R}^2$ such that $A \cap B$ is not connected.

(4) Let $f : X \to Y$ be a function from a connected metric space (X, d) to a metric space (Y, d_{disc}) with the discrete metric. Show that f is continuous if and only if it is constant.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For any $a \in \mathbb{R}$, let $f_a : \mathbb{R} \to \mathbb{R}$ be the shifted function $f_a(x) := f(x - a)$.

a) Show that f is continuous if and only if whenever $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to 0, f_{a_n} converge pointwise to f.

b) Show that f is uniformly continuous if and only if whenever $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to 0, f_{a_n} converge uniformly to f.

(6) Show that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

(7) Give an example of a sequence of discontinuous functions f_k converging uniformly to a limit function f that is continuous.

(8) Suppose $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly to a function g on $\mathbb{R}$ and suppose that $h : \mathbb{R} \to \mathbb{R}$ is a bounded function on $\mathbb{R}$. Prove that $\sum\limits_{k=1}^{\infty} hg_k$ converges uniformly to hg.