Lecture 7+8

Real and Complex Analysis

MTL122/ MTL503/ MTL506

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1. Open Sets, Closed Sets, and Bounded Sets

Definition 1.1. *Let (X, d) be a metric space, $x \in X$ and $r > 0$. The set*

$$B(x, r) := \{y \in X | d(x, y) < r\}$$

is called the ***open ball*** *with centre x and radius r.*
The set $\bar{B}(x, r) := \{y \in X | d(x, y) \leq r\}$ is called the ***closed ball*** *with centre x and radius r.*

Lemma 1.2. *let x and y be distinct points in a metric space (X, d). Then there is an $\epsilon > 0$ such that $B(x, \epsilon) \cap B(y, \epsilon) = \phi$.*

Proof Since $x \neq y$, it follows that $d(x, y) > 0$. Choose ϵ such that $0 < \epsilon < \frac{d(x,y)}{2}$. Then $B(x, \epsilon) \cap B(y, \epsilon) = \phi$. Indeed, if $z \in B(x, \epsilon) \cap B(y, \epsilon)$, then

$$d(x, z) < \epsilon \text{ and } d(y, z) < \epsilon.$$

Therefore

$$0 < d(x, y) \leq d(x, z) + d(y, z) < \epsilon + \epsilon < \frac{d(x, y)}{2} + \frac{d(x, y)}{2} = d(x, y).$$

That is, $d(x, y) < d(x, y)$, which is absurd. □

Definition 1.3. *Let (X, d) be a metric space. A subset G of X is said to be* ***open*** *if for each $x \in G$, there is an $\epsilon > 0$ such that $B(x, \epsilon) \subset G$.*

Definition 1.4. *Let (X, d) be a metric space. A subset A of X is called a neighbourhood of $x \in X$ if there is an open set $V \subset X$ such that $x \in V \subset A$.*

It is clear that a subset G of a metric space (X, d) is open if G is a neighbourhood of each of its points.

Example 1.5.

(1) *An open ball in a metric space (X, d) is an open set. Indeed, let $B(x, r)$ be an open ball with centre x and radius r and let $y \in B(x, r)$. Then $d(x, y) < r$. Let $\epsilon = r - d(x, y)$. We now show that $B(y, \epsilon) \subset B(x, r)$. Let $z \in B(y, \epsilon)$. Then $(d(y, z) < \epsilon)$. Hence, by the triangle inequality,*

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \epsilon = d(x, y) + r - d(x, y) = r.$$

That is, $z \in B(x, r)$, and so $B(y, \epsilon) \subset B(x, r)$.

(2) *Let (X, d) be a discrete metric space. Then every subset X is open. To see this, let G be a subset of X and $x \in G$. Then, with $0 < \epsilon < 1$, $B(x, \epsilon) = \{x\} \subset G$.*

Theorem 1.6. *Let (X, d) be a metric space.*

(1) X and ϕ are open.
(2) A union of an arbitrary collection of open sets in X is open.
(3) An intersection of a finite collection of open sets in X is open.

We leave this as an exercise.

Proposition 1.7. *Let (X, d) be a metric space. Then a set A in X is open if and only if it is a union of open balls in X.*

Proof. Assume that A is a union of open balls in X, i.e.,$A = \cup_{x\in A} B(x, r_x)$. Since each open ball is an open set and a union of an arbitrary collection of open sets is open, it follows that A is an open set.

Conversely, assume that A is open in X. Then, for each $x \in A$, there is an $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset A$. Obviously $A = \cup_{x\in A} B(x, \epsilon_x)$. □

Definition 1.8. *A subset F of a metric space (X, d) is said to be closed if its complements $X \setminus F$ is open.*

Example 1.9. (1) *A closed ball in a metric space (X, d) is a closed set. Indeed, let $\bar{B}(x, r)$ be a closed ball with centre x and radius r and let $y \in X \setminus \overline{B}(x, r)$. Then $d(x, y) > r$. Let $\epsilon = d(x, y) - r$. We now show that $B(y, \epsilon)) \subset X \setminus \bar{B}(x, r)$. Let $z \in B(y, \epsilon)$. Then $d(y, z) < \epsilon$. Hence, by the triangle inequality,*

$$d(y, z) < \epsilon = d(x, y) - r \iff r < d(x, y) - d(y, z) \leq d(x, z).$$

Hence $z \notin \overline{B}(x, r)$ and so $z \in X \setminus \overline{B}(x, r)$.

(2) *Let (X, d) be a discrete metric space. Then every subset of X is closed. To see this, let A be a subset of X. Since every subset of X is open, $X \setminus A$ is open. Hence $A = X \setminus (X \setminus A)$ is closed.*

Theorem 1.10. *Let (X, d) be a metric space.*

(1) X and ϕ are closed.
(2) An intersection of an arbitrary collection of closed sets in X is closed.
(3) A union of a finite collection of closed sets in X is closed.

Exercise.

Proposition 1.11. *Every singleton set in a metric space (X, d) is closed.*

Proof. Let $x \in X$. We show that the set $\{x\}$ is closed. It suffices to show that the complement $X \setminus \{x\}$ is open. Let $y \in X \setminus \{x\}$. Then $x \neq y$. By Lemma 1.2, there is an $\epsilon > 0$ such that $B(x, \epsilon) \cap B(y, \epsilon) = \phi$. Hence $B(y, \epsilon) \subseteq X \setminus \{x\}$, and so $X \setminus \{x\}$ is open. □

Definition 1.12. *Let S be a subset of a metric space (X, d), and $x \in X$. Then*

a) *$x \in S$ is called an* ***interior point*** *of S if there is an $\epsilon > 0$ such that $B(x, \epsilon) \subset S$. The set of all interior point of a set S is denoted by S° or $int(S)$.*

*b) $x \in X$ is called a **boundary point** of S if for every $\epsilon > 0$ the open $B(x, \epsilon)$ contains points of S as well as points of $X \setminus S$. The set of boundary points of S is denoted by ∂S or $bd(S)$.*

*c) $x \in S$ is called an **isolated point** of S if there exists an $\epsilon > 0$ such that $B(x, \epsilon) \cup S = \{x\}$.*

*d) A point $x \in X$ is called an **accumulation point**(or **limit point**) of S if for every $\epsilon > 0$, the ϵ-ball, $B(x, \epsilon)$, contains a point of S distinct from x. The set of all accumulation points of S is called the **derived set** of S and is denoted by S'. That is, $S' = \{x \in X \ : \ B(x, \epsilon) \setminus \{x\} \cup S \neq \phi \text{ for all } \epsilon > 0\}$.*

e) The closure of the set S, denoted by $\overline{S}$, is the set $\overline{S} = S \cup S'$.

Theorem 1.13. *(Properties of Interior). Let A and B be subsets of a metric space (X, d). Then*

a) $A^\circ \subseteq A$;

b) $A^{\circ\circ} = A^\circ$;

c) If $A \subseteq B$, *then* $A^\circ \subseteq B^\circ$;

d) $(A \cap B)^\circ = A^\circ \cap B^\circ$;

e) $\bigcup_{i \in I} A_i^\circ \subseteq \left(\bigcup_{i \in I} A_i \right)^\circ$;

f) $\left(\bigcap_{i \in I} A_i \right)^\circ \subseteq \bigcap_{i \in I} A_i^\circ$.

Theorem 1.14. *(Properties of Closure). Let A and B be subsets of a metric space (X, d). Then*

a) $A \subseteq \overline{A}$;

b) $\overline{\overline{A}} = \overline{A}$;

c) If $A \subseteq B$, *then* $\overline{A} \subseteq \overline{B}$;

d) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;

e) $\overline{\bigcup_{i \in I} A_i} \subseteq \bigcup_{i \in I} \overline{A_i}$;

f) $\bigcap_{i \in I} \overline{A_i} \subseteq \overline{\bigcap_{i \in I} A_i}$

Theorem 1.15. *A subset C of a metric space (X, d) is closed if and only if it contains all its accumulation points.*

Proof. Assume that C is closed and let $x \in C'$. We want to show that $x \in C$. If $x \notin C$, then $x \in X \setminus C$. Since C is closed, $X \setminus C$ is open. Therefore there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset X \setminus C$. This then implies that $B(x, \epsilon) \cap C = \phi$, which contradicts the fact that $x \in C'$. Thus $C' \subset C$.

To prove the converse, we assume that $C' \subset C$. We want to show that C is closed, or equivalently, that $X \setminus C$ is open. Let $x \in X \setminus C$. Then$x \notin C'$. So there is an $\epsilon > 0$ such that

$$(B(x, \epsilon) \setminus \{x\}) \cap C = \phi.$$

Since $x \notin C$, we have that $B(x, \xi) \cap C = \phi$. Thus $B(x, \xi) \subset X \setminus C$, whence $X \setminus C$ is open. □

Corollary 1.16. *Let C be a subset of a metric (X, d). Then C is closed if and only if $\overline{C} = C$.*

Proof. Assume that C is closed. Then, by Theorem 1.15, $C' \subset C$. Therefore $\overline{C} = C \cup C' \subset C \cup C = C$. But $C \subset C \cup C' = \overline{C}$. Conversely, if $C = \bar{C}$ then C contains all its accumulation points and, consequently, C is closed. □

Definition 1.17. *A subset A of a metric space (X, d) is said to be* ***bounded*** *if $A \subseteq B(x, r)$ for some $x \in X$ and some $r > 0$.*

Proposition 1.18. *A subset A of a metric space (X, d) is bounded if and only if there is a real number $M \geq 0$ such that $d(x, y) \leq M$ for all $x, y \in A$.*

Definition 1.19. *The* ***diameter*** *of a subset A of a metric space (X, d) is defined as*

$$diam(A) := \sup\{d(x, y) \ : \ x, y \in A\}.$$

Note that a subset A of a metric space (X, d) is bounded if and only if $\text{diam}(A) < \infty$.

Proposition 1.20. *Any subset of a discrete metric space (X, d) is bounded.*

Proof. Let A be a subset of X. Clearly, by definition of the discrete metric, $d(x, y) \leq 1$ for all $x, y \in A$. Hence, A is bounded. □

Proposition 1.21. *A finite union of bounded subsets of a metric space (X, d) is bounded.*

Proof. Let $U_1, U_2, ..., U_n$ be open subsets of X. Then, for each $i = 1, 2, ..., n$, there is a $r_i > 0$ such that $d(x, y) \leq r_i$ for all $x, y \in U_i$. Let $r = \max\{r_1, r_2, ..., r_n\}$ and $U = \bigcup_{i=1}^{n} U_i$. For each $i = 1, 2, ..., n$, choose $x_i \in U_i$. Let $s = \max\{d(x_i, x_j) \text{ for all } i, j = 1, 2, ..., n\}$. Let $x, y \in U$. Then $x \in U_i$ and $y \in U_j$ for some $i, j = 1, 2, ..., n$. Therefore

$$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) \leq r + s + r = 2r + s.$$

That is, for all $x, y \in U$, $d(x, y) \leq M$, where $M = 2r + s$ and so U is bounded. □