

# **Module 9**

## **DISTRIBUTION FUNCTION AND ITS PROPERTIES**

- $\mathcal{E}$  : a given random experiment;
- $(\Omega, \mathcal{P}(\Omega), P)$ : probability space associated with  $\mathcal{E}$ ;
- $X : \Omega \rightarrow \mathbb{R}$ : a given random variable;
- $(\mathbb{R}, \mathcal{P}(\mathbb{R}), P_X)$ ; probability space induced by  $X$ ;
- 

$$P_X(A) = P(X^{-1}(A))$$

$$= P(\{\omega \in \Omega : X(\omega) \in A\}), \quad A \in \mathcal{P}(\mathbb{R})$$

$\rightarrow$  induced probability function.

**Definition 1:** The distribution function (or cumulative distribution function) of a random variable  $X$  is a function  $F_X: \mathbb{R} \rightarrow \mathbb{R}$ , defined by

$$\begin{aligned} F_X(x) &= P_X((-\infty, x]) \\ &= P(\{\omega \in \Omega : X(\omega) \leq x\}), \quad x \in \mathbb{R}. \end{aligned}$$

**Remark 1:**

- (a)  $0 \leq F_X(x) \leq 1, \forall x \in \mathbb{R};$
- (b) Clearly the distribution function (d.f.) is determined by induced probability function  $P_X(\cdot)$ . Using advanced probabilistic arguments it can be shown that the induced probability function  $P_X(\cdot)$  is also uniquely determined by d.f.  $F_X(\cdot)$ . Thus studying induced probability space  $(\mathbb{R}, \mathcal{P}(\mathbb{R}), P_X)$  (or induced probability function  $P_X(\cdot)$ ) is equivalent to studying the d.f.  $F_X(\cdot)$ .

**Example 1:** Suppose that, for some real constant  $c$ ,  $P(\{X = c\}) = 1$ . Then

$$F_X(x) = P(\{X \leq x\}) = \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x \geq c \end{cases}.$$

**Example 2:** Suppose

$$P(\{X = 1\}) = P(\{X = -1\}) = \frac{1}{2}.$$

Then

$$F_X(x) = P(\{X \leq x\}) = \begin{cases} 0, & \text{if } x < -1 \\ \frac{1}{2}, & \text{if } -1 \leq x < 1 \\ 1, & \text{if } x \geq 1 \end{cases}.$$

**Result 1:** Let  $F_X(\cdot)$  be the d.f. of a r.v.  $X$ . Then

(a)  $F_X(\cdot)$  is non-decreasing;

(b)  $F_X(\cdot)$  is right continuous, i.e.,  $\lim_{n \rightarrow \infty} F_X(x + \frac{1}{n})$  exists and

$$\lim_{n \rightarrow \infty} F_X(x + \frac{1}{n}) \doteq F_X(x+) = F_X(x), \quad \forall x \in \mathbb{R};$$

(c)  $F_X(-\infty) \doteq \lim_{n \rightarrow \infty} F_X(-n) = 0$  and  $F_X(+\infty) \doteq \lim_{n \rightarrow \infty} F_X(n) = 1$ .

**Proof:**

**(a)** Let  $-\infty < x < y < \infty$ . Then  $(-\infty, x] \subseteq (-\infty, y]$ , and

$$\begin{aligned} F_X(x) &= P(\{X \leq x\}) \\ &= P_X((-\infty, x]) \\ &\leq P_X((-\infty, y]) \\ &\quad \text{(probability function is monotone)} \\ &= F_X(y). \end{aligned}$$



(b) Let  $x \in \mathbb{R}$  be fixed. Define, for  $n = 1, 2, \dots$ ,

$$A_n = \{X \leq x + \frac{1}{n}\} = X^{-1}((-\infty, x + \frac{1}{n}]).$$

Then  $A_n \downarrow$  and

$$\begin{aligned} \bigcap_{n=1}^{\infty} A_n &= X^{-1}(\bigcap_{n=1}^{\infty} (-\infty, x + \frac{1}{n}]) \\ &= X^{-1}((-\infty, x]). \end{aligned}$$

Thus,

$$\begin{aligned} F_X(x) &= P(X^{-1}((-\infty, x])) \\ &= P(\bigcap_{n=1}^{\infty} A_n) \\ &= \lim_{n \rightarrow \infty} P(A_n) \quad (P \text{ is continuous}) \\ &= \lim_{n \rightarrow \infty} P(\{X \leq x + \frac{1}{n}\}) \\ &= \lim_{n \rightarrow \infty} F_X(x + \frac{1}{n}) = F_X(x+). \end{aligned}$$

(c) Define, for  $n = 1, 2, \dots$ ,

$$B_n = \{\omega \in \Omega : X(\omega) \leq -n\} = X^{-1}((-\infty, -n])$$

and

$$C_n = \{\omega \in \Omega : X(\omega) \leq n\} = X^{-1}((-\infty, n]).$$

Then  $B_n \downarrow$ ,  $C_n \uparrow$ ,

$$\bigcap_{n=1}^{\infty} B_n = X^{-1}\left(\bigcap_{n=1}^{\infty} (-\infty, -n]\right) = X^{-1}(\emptyset) = \emptyset,$$

and

$$\bigcup_{n=1}^{\infty} C_n = X^{-1}\left(\bigcup_{n=1}^{\infty} (-\infty, n]\right) = X^{-1}(\mathbb{R}) = \Omega.$$

Thus

$$0 = P(\phi)$$

$$= P\left(\bigcap_{n=1}^{\infty} B_n\right)$$

$$= \lim_{n \rightarrow \infty} P(B_n) \quad (P \text{ is continuous})$$

$$= \lim_{n \rightarrow \infty} P(\{X \leq -n\})$$

**Remark 2:**

- (a) Using advanced mathematical arguments, it can be shown that any function  $G(\cdot)$  satisfying properties (a)-(c) of the above theorem (non-decreasing, right continuous,  $G(-\infty) = 0$ ,  $G(+\infty) = 1$ ) is a d.f of some r.v.

- (b) From the knowledge of calculus we know that any monotone function is either continuous on  $\mathbb{R}$  or it has at most countable number of discontinuities. Thus any d.f  $F_X(\cdot)$  is either continuous on  $\mathbb{R}$  or has only countable number of discontinuities.

(c) We have, for every  $x \in \mathbb{R}$ ,

$$F_X \left( x - \frac{1}{n} \right) \leq F_X(x) = F_X(x+), \quad n \in \mathbb{N}$$

$$\lim_{n \rightarrow \infty} F_X \left( x - \frac{1}{n} \right) \leq F_X(x) = F_X(x+)$$

$$\text{i.e., } F_X(x-) \leq F_X(x) = F_X(x+).$$

(d) For  $x \in \mathbb{R}$

$$\begin{aligned} P(\{X < x\}) &= P_X((-\infty, x)) \\ &= P_X\left(\bigcup_{n=1}^{\infty} \left(-\infty, x - \frac{1}{n}\right]\right) \\ &= \lim_{n \rightarrow \infty} P_X\left(\left(-\infty, x - \frac{1}{n}\right]\right) \\ &\quad \left(\left(-\infty, x - \frac{1}{n}\right] \uparrow\right) \\ &= \lim_{n \rightarrow \infty} F_X\left(x - \frac{1}{n}\right) \\ &= F_X(x-). \end{aligned}$$

Thus

$$P(\{X < x\}) = F_X(x-), \quad \forall x \in \mathbb{R}.$$

(e) For  $-\infty < a < b < \infty$

$$\begin{aligned} P(\{X \leq b\}) &= P(\{X \leq a\}) \\ &\quad + P(\{a < X \leq b\}) \end{aligned}$$

$$\begin{aligned} P(\{a < X \leq b\}) &= P(\{X \leq b\}) \\ &\quad - P(\{X \leq a\}) \end{aligned}$$

$$= F_X(b) - F_X(a).$$

Thus

$$P(\{a < X \leq b\}) = F_X(b) - F_X(a), \quad \forall \quad -\infty < a < b < \infty.$$



Similarly, for  $-\infty < a < b < \infty$ ,

$$\begin{aligned}P(\{a < X < b\}) &= P(\{X < b\}) - P(\{X \leq a\}) \\&= F_X(b-) - F_X(a),\end{aligned}$$

$$\begin{aligned}P(\{a \leq X \leq b\}) &= P(\{X \leq b\}) - P(\{X < a\}) \\&= F_X(b) - F_X(a-),\end{aligned}$$

$$\begin{aligned}P(\{a \leq X < b\}) &= P(\{X < b\}) - P(\{X < a\}) \\&= F_X(b-) - F_X(a-),\end{aligned}$$

$$\begin{aligned}P(\{X > b\}) &= 1 - P(\{X \leq b\}) \\&= 1 - F_X(b),\end{aligned}$$

$$\begin{aligned}P(\{X \geq b\}) &= 1 - P(\{X < b\}) \\&= 1 - F_X(b-).\end{aligned}$$

(f) For  $x \in \mathbb{R}$

$$\begin{aligned} P(\{X = x\}) &= P(\{X \leq x\}) - P(\{X < x\}) \\ &= F_X(x) - F_X(x-). \end{aligned}$$

(g) Let  $D_X$  be the set of discontinuity points of  $F_X$ . Since  $F_X$  is always right continuous,

$$\begin{aligned} x \in D_X &\Leftrightarrow F_X(x) - F_X(x-) > 0 \\ &\Leftrightarrow P(\{X = x\}) > 0. \end{aligned}$$

Thus a d.f. has only jump discontinuities and

$$D_X = \{x \in \mathbb{R} : P(\{X = x\}) > 0\}.$$

**Example 3:** Let  $X$  be a r.v. with d.f

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{3}, & \text{if } 0 \leq x < 1 \\ \frac{1}{2}, & \text{if } 1 \leq x < 2 \\ \frac{2}{3}, & \text{if } 2 \leq x < 3 \\ 1, & \text{if } x \geq 3 \end{cases}.$$

Clearly  $F_X(\cdot)$  is  $\uparrow$ , right continuous,  $F_X(-\infty) = 0$ , and  $F_X(+\infty) = 1$ . Also  $D_X = \{1, 2, 3\}$ .

$$P(\{X = 1\}) = F_X(1) - F_X(1-) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

$$P(\{X = 2\}) = F_X(2) - F_X(2-) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6},$$

$$P(\{X = 3\}) = F_X(3) - F_X(3-) = 1 - \frac{2}{3} = \frac{1}{3},$$

$$P(\{X < 3\}) = F_X(3-) = \frac{2}{3},$$

$$P(\{X \geq 1\}) = 1 - F_X(1-) = 1 - \frac{1}{3} = \frac{2}{3}.$$

For any  $c \notin \{1, 2, 3\}$

$$P(\{X = c\}) = 0,$$

$$P(\{2 < X \leq 3\}) = F_X(3) - F_X(2) = 1 - \frac{2}{3} = \frac{1}{3},$$

$$P(\{1 < X < 2\}) = F_X(2-) - F_X(1) = \frac{1}{2} - \frac{1}{2} = 0$$

$$P(\{1 \leq X < 3\}) = F_X(3-) - F_X(1-) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$P(\{2 \leq X \leq 4\}) = F_X(4) - F_X(2-) = 1 - \frac{1}{2} = \frac{1}{2}.$$

## Quiz

Let  $X$  be a r.v. with d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 1 \\ 1 - \frac{1}{x}, & \text{if } x \geq 1 \end{cases}.$$

Let  $S = \{1, 2, 3, \dots\}$ . Find the value of  $P(\{X \in S\})$ .

## Take Home Problem

1. Let  $G(\cdot)$  be d.f. of some r.v. Show that

(a)

$$G(x-) \leq G(x) = G(x+), \quad \forall x \in \mathbb{R};$$

(b)

$$G(x+) \leq G(y-), \quad \forall -\infty < x < y < \infty.$$

2. Let  $F_X(\cdot)$  be continuous on  $\mathbb{R}$  and let  $S$  be any countable set. Show that  $P(\{X \in S\}) = 0$ .

## Abstract of Next Module

1. We have seen that

$$\begin{aligned} D_X &= \text{set of discontinuity points of } F_X(\cdot) \\ &= \{x \in \mathbb{R} : P(\{X = x\}) > 0\} \\ &\quad (\text{a countable set}) \end{aligned}$$

There may be situations where

$$P(\{X \in D_X\}) = \sum_{x \in D_X} P(\{X = x\}) = 1.$$

We call such r.v.s as discrete r.v.

2. In next module, we will introduce discrete r.v.s and study their properties.



**Thank you for your patience**

