Assignment 2.

(1.) $\sum_{i=1}^{n} i^2 = \frac{n}{6}(2n+1)(n+1)$ by mathematical induction

check for $n=1$.

$$\sum_{i=1}^{1} i^2 = 1 = \frac{1}{6}(2+1)(2)$$

$$1 = \frac{6}{6} = 1 \checkmark$$

Assuming result true for $n=K$.

$$\sum_{i=1}^{K} i^2 = \frac{K}{6}(2K+1)(K+1)$$

The sum for for $n=K+1$

$$\sum_{i=1}^{K+1} i^2 = \sum_{i=1}^{K} i^2 + (K+1)^2$$

$$= \frac{K(2K+1)(K+1)}{6} + (K+1)^2$$

$$= \frac{K(2K+1)(K+1) + 6(K+1)^2}{6}$$

$$= (K+1)\left(K(2K+1) + 6(K+1)\right) = (K+1)(2K^2 + 7K + 6) =$$

$$= \frac{(K+1)(K+2)(2K+3)}{6} = \boxed{\frac{(K+1)(K+2)\left(2(K+1)+1\right)}{6}}$$

K+1 is exactly same as for n=1. // hence proved.

2(a.)

Maclaurin Polynomial degree $n=2K$

$$f(x) = \cos(2x).$$

$$f(x) = \cos(2x) = f(0) = \cos(0) = 1.$$

$$f'(x) = -2\sin(2x) = f'(0) = 0.$$

$$f''(x) = -4\cos(2x) = f''(0) = -4.$$

$$f^3(x) = 8\sin(2x) = f^3(0) = 0.$$

$$f^4(x) = 16\cos(2x) = f^4(0) = 16.$$

$$f^5(x) = -32\sin(2x) = f^5(0) = 0.$$

$$f^6(x) = -64\cos(2x) = f^6(0) = -64.$$

$$\vdots \quad f^{(n)}(0) = (-1)^K \, 2^n \qquad n = 2K. \quad K = 0,1,2,3.$$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \sim \frac{f^n(0)x^n}{n!}$$

$$P_K(x) = 1 + 0 - \frac{4x^2}{2!} + 0 \sim \frac{(-1)^K (2^{2K} x^{(2K)})}{2K!}$$

$$P_K(x) = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} + \sim \frac{(-1)^K (2^{2K} x^{2K})}{2K!}$$

(2.)(b.) $f(x) = \cos(2)$ $\quad x=2,\ a=0$

$$R_n(x) = \frac{f^{(n+1)}(z)\,(x-a)^{n+1}}{(n+1)!} \qquad x=2,\ a=0.$$

Z between $(0,2)$

$f^1(z) = \cos Z$

$f''(z) = -\sin Z$

$f^3(z) = -\cos Z$

$f^4(z) = \sin Z$

~~for~~ $\forall \leq 1$.

$$|R_n(z)| \leq 1 \left| \frac{Z^{n+1}}{(n+1)!} \right|$$

Calculate $\left| \frac{Z^{n+1}}{(n+1)!} \right|$ until < 0.001

$\frac{2^1}{1!} = 2$ ~~ $\frac{2^3}{3!} = 1.33$

$\frac{2^5}{5!} = 0.266$ ~~ $\frac{2^8}{8!} = .006$

$\frac{2^9}{9!} = \underline{\underline{0.0014}}$.

$\underline{\underline{(3)}}$. $\underline{\text{Taylor series}}$ $f(x) = \ln(x)$ centred at $a = 3$.

$$f(x) = \ln(x) = f(3) = \ln 3.$$

$$f'(x) = \frac{1}{x} = f'(3) = \frac{1}{3}$$

$$f''(x) = -\frac{1}{x^2} = f''(3) = -\frac{1}{3^2}.$$

$$f'''(x) = \frac{2}{x^3} = f'''(3) = \frac{2}{3^3}$$

$$f^4(x) = -\frac{2.3}{x^4} = f^4(3) = -\frac{6}{3^4}.$$

$$f^5(x) = \frac{2.3.4}{x^5} = f^5(3) = \frac{24}{3^5}$$

$$T(x) = \sum_{K=0}^{\infty} \frac{f^{(K)}(3)}{K!}(x-3)^K$$

$$\frac{\ln 3}{0!}(x-3)^0 + \frac{1/3}{1!}(x-3) - \frac{1/(3)^2}{2!}(x-3)^2 + \frac{2/3^3}{3!}(x-3)^3$$

$$- \frac{6/3^4}{4!}(x-3)^4 + \frac{24/3^5}{5!}(x-3)^5 - \cdots$$

$$\ln 3 + \frac{(x-3)}{(3)^1} - \frac{(x-3)^2}{(3)^2\,2} + \frac{(x-3)^3}{(3)^3 \times (3)} - \frac{(x-3)^4}{(3)^4\,(4)}.$$

$$+ \frac{(x-3)^5}{(3^5)(5)}.$$

$$= \ln(3) + \sum_{K=1}^{\infty} \frac{(-1)^{K+1} * \left(\frac{1}{3}\right)^{K} * (x-1)^{K}}{K}$$

Radius of convergence (we can ignore $\ln(3)$.

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(-1)^{n+2} \frac{1}{(3)^{n+1}} (x-1)^{n+1}}{n+1}}{\dfrac{(-1)^{n+1} \frac{1}{(3)^n} (x-1)^n}{n}}$$

$$= \frac{(-1)^{\cancel{n+2}} (x-1)^{\cancel{n+1}}}{(3)^{n+1} (n+1)} * \frac{(3)^n \times (n)}{(-1)^{\cancel{n+1}} \cancel{(x-1)^n}}$$

$$= \frac{(-1)^1 (x-1)\, n}{(3)\,(n+1)} = \text{Divide by } n.$$

$$= \frac{-(x-1)}{(3)\left(1+\frac{1}{n}\right)} \underset{n\to\infty}{=} -\frac{(x-1)}{(3)}$$

Radius $= -\dfrac{(x-1)}{3}$

(4) $\sum_{n=0}^{\infty} a_n$

(a). $\boxed{a_0 = 2}$ $a_{n+1} = \cancel{} \frac{4n+1}{Kn+3} \cdot a_n$ ①

put $n=0 \Rightarrow a_1 = \frac{4\times 0+1}{K\times 0+3} \cdot a_0$

$a_1 = \frac{1}{3}(2) = \boxed{a_1 = \frac{2}{3}}$

In ① put $n=1$.

$a_2 = \frac{4+1}{K+3} \times \frac{2}{3} = \boxed{\frac{10}{(K+3)3} = a_2}$

In ① put $n=2$.

$a_3 = \frac{8+1}{2K+3} \times \frac{10}{(K+3)\times \cancel{3}} = \boxed{\frac{30}{(2K+3)(K+3)} = a_3}$

put $n=3$.

$a_4 = \frac{12+1}{3K+3} \frac{30}{2K+3} = \boxed{\frac{(13)\times 30}{(3K+3)(2K+3)} = a_4}$

$a_5 = \frac{16+1}{4K+3} \frac{(13)\times(30)}{[(3K)+3](2K+3)}$

$n=5$

$a_6 =$

Σa_n

(b)

$n=5$

$$\boxed{a_6 = \frac{(21)(17)(13)\times(30)}{(5K+3)(4K+3)(3K+3)(2K+3)}}$$

$\cancel{\Sigma a_n \neq \frac{2(4n+1)}{(n-1)K}}$

(b) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$a_{n+1} = \frac{4n+1}{Kn+3} a_n$$

$$= \left| \frac{\frac{4n+1}{Kn+3} \cancel{a_n}}{\cancel{a_n}} \right| = \lim_{n\to\infty} \left| \frac{4n+1}{Kn+3} \right|$$

Divide by n.

$$\lim_{n\to\infty} \frac{\left|4+\frac{1}{n}\right|}{\left|K+\frac{3}{n}\right|} \underset{n\to\infty}{\overset{as}{=}} \frac{4}{|K|}$$

converges for $K \geq 5$

diverges for $K \leq 3$

(5.) $\lim_{x \to 0} \frac{1-\cos(x)}{1+x-e^{x}}$

Maclaurin series

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\lim_{x \to 0} \frac{\cancel{1} - \left[\cancel{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right]}{\cancel{1+x} - \left[\cancel{1+x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right]}$$

$$= \frac{\cancel{-}\left[-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right]}{\cancel{-}\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\right]}$$

$$= \frac{-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}}{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}} = \text{Divide by } x^2$$

$$= \frac{-\frac{1}{2} + \underbrace{\frac{x^2}{4!} - \frac{x^4}{6!}}_{\to 0}}{\frac{1}{2} + \underbrace{x^3/3! + x^4/4!}_{\to 0}} = \frac{-1/2}{1/2} = \underline{\underline{-1}}$$