STATS 2103: Probability and Statistics II: Assignment 5

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CHECKLIST

- ☐: Have you shown all of your working, including probability notation where necessary?
- ☐: Have you given all numbers to 3 decimal places unless otherwise stated?
- ☐: Have you included all R output and plots to support your answers where necessary?
- ☐: Have you included all of your R code?
- ☐: Have you made sure that all plots and tables each have a caption?
- ☐: If before the deadline, have you submitted your assignment via the online submission on Canvas?
- ☐: Is your submission a single pdf file - correctly orientated, easy to read? If not, penalties apply.
- ☐: Penalties for more than one document - 10% of final mark for each extra document. Note that you may resubmit and your final version is marked, but the final document should be a single file.
- ☐: Penalties for late submission - within 24 hours 40% of final mark. After 24 hours, assignment is not marked and you get zero.
- ☐: Assignments emailed instead of submitted by the online submission on Canvas will not be marked and will receive zero.
- ☐: Have you checked that the assignment submitted is the correct one, as we cannot accept other submissions after the due date?

$var(Z)$ mean $= 0$ $SD = 1$

Due date: Friday 29th May 2020 (Week 11), 5pm.

$E[Z^2] - E[Z]^2$

Q1: Covariance of normals. Let Z be a standard normal random variable and let $Y_1 = Z$ and $Y_2 = Z^2$.

a. What are $\mathrm{E}[Y_1]$ and $\mathrm{E}[Y_2]$?

$E(Y_1) = 0$

Problem 1. Let Z be a random variable that follows a standard normal distribution $\mathcal{N}(0,1)$. Show that

(a) $\mathbb{E}[Z] = 0$

Answer. The probability density function of Z is :

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

The expected value of Z is :

$$\mathbb{E}[Z] = \int_{-\infty}^{\infty} \underbrace{x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}_{\text{odd function on } \mathbb{R}} \, dx = 0$$

[3 marks]

b. What is $\mathrm{E}[Y_1 Y_2]$? Hint: use the MGF.

[3 marks]

c. What is $\mathrm{cov}(Y_1, Y_2)$?

$cov(X, Y) = E[XY] - E[X]E[Y]$

[2 marks]

[Question total: 8]

mean $= np$ varia $= npq$

Q2: Beta-binomial Suppose that Y has a binomial distribution with parameters n and P, but that P varies from day to day according to a beta distribution with parameters α and β. Show that

a. $\mathrm{E}[Y] = \frac{n\alpha}{\alpha+\beta}$

[4 marks]

b. $\mathrm{var}(Y) = \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Mean	$E[X] = \frac{\alpha}{\alpha+\beta}$

Variance	$\text{var}[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

You may use the expectated value and variance of the beta distribution without deriving them:

https://en.wikipedia.org/wiki/Beta_distribution

[5 marks]

[Question total: 9]

Q3: Vending machine A vending machine can be in two states, (0=working, 1=out of order). If the machine is working on a particular day it will be out of order with probability δ on the next day. If the machine is out of order on a particular day then the probability that it will be working the next day is γ.

a. Write down the one step transition probability matrix for the vending machine.

[4 marks]

$(1-\delta)^3$

b. Assume the machine is working on Monday. What is the probability that the machine will remain working on all of Tuesday, Wednesday and Thursday?

[2 marks]

c. Assume the machine is working on Monday. What is the probability that the machine will be working on Thursday? WNNW + WWWW + WNWW + WWNW

[4 marks]

d. Calculate the equilibrium probabilities for the states of the vending machine.

[5 marks]

[Question total: 15]

[[Assignment total: 32]]

With these assumptions, the one-day transition diagram for the operation of the machine is given by Figure 6.5. The two parameters δ and γ characterize the operation of the system. A reliable, well-maintained machine will be described by a value of δ near 0 and a value of γ near 1.

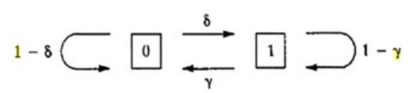

Figure 6.5: Vending Machine

Over a period of many days, the machine will alternate between *up cycles* (successive days in state 0) and *down cycles* (successive days in state 1). Table 6.2 gives some sample data from a computer simulation of this Markov chain, with $\delta = 0.2$ and $\gamma = 0.9$. The simulation data suggests that over a long time period the machine will be up about 81% of the time. This long-term statistic doesn't seem to depend on the initial state of the machine. ■

Table 6.2 Simulation of Vending Machine Operation

	NUMBER OF DAYS SIMULATED				
	10	50	100	500	1000
PROPORTION OF TIME MACHINE UP					
Machine up on day 0	0.90	0.82	0.84	0.842	0.815
Machine down on day 0	0.50	0.86	0.82	0.804	0.812