Laplace Transform Division by *t*

Laplace transformation: $L(f(t)) = \int_0^{\infty} f(t)e^{-st}dt$

$$L(f(t)) = F(s) .$$

Division by t: If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(u)\, du.$

1. Problem: Find the Laplace transform of $\dfrac{\sin 2t}{t}$.

Solution: We know, $F(s) = L\{\sin 2t\} = \dfrac{2}{s^2+4}$

$$L\left(\frac{\sin 2t}{t}\right) = \int_s^{\infty} F(u)\, du \qquad \Big| \because \ L\left(\frac{f(t)}{t}\right) = \int_s^{\infty} F(u)\, du$$

$$= \int_s^{\infty} \frac{2}{u^2+2^2}\, du = 2.\frac{1}{2}\left[\tan^{-1}\frac{u}{2}\right]_s^{\infty} = \tan^{-1}\infty - \tan^{-1}\frac{s}{2}$$

$$= \pi/2 - \tan^{-1}\frac{s}{2} \qquad = \cot^{-1}\frac{s}{2} .$$

$$\therefore \quad L\left\{\frac{\sin 2t}{t}\right\} = \cot^{-1}\frac{s}{2} \qquad \text{[Answer]}$$

2. Problem: Find the Laplace Transform of $e^{-4t}\,\dfrac{\sin 3t}{t}$.

Solution: $L(\sin 3t) = \dfrac{3}{s^2+9} = F(s)$

$$L\left(\frac{\sin 3t}{t}\right) = \int_s^{\infty} F(u)\, du \qquad \Big| \because \ L\left(\frac{f(t)}{t}\right) = \int_s^{\infty} F(u)\, du$$

$$= \int_s^{\infty} \frac{3}{u^2+3^2}\, du = 3.\frac{1}{3}\left[\tan^{-1}\frac{u}{3}\right]_s^{\infty} = \tan^{-1}\infty - \tan^{-1}\frac{s}{3}$$

$$= \pi/2 - \tan^{-1}\frac{s}{3} = \cot^{-1}\frac{s}{3} .$$

$$\therefore \quad L\left\{e^{-4t}\,\frac{\sin 3t}{t}\right\} = \cot^{-1}\frac{s+4}{3} \quad \text{[Answer]}$$

3. Problem: Prove that $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$.

Solution: Let $f(t) = \cos at - \cos bt$, then

$$F(s) = L\{f(t)\} = L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}.$$

By theorem,

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)du$$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}\right)du$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2u}{u^2+a^2} - \frac{2u}{u^2+b^2}\right)du = \frac{1}{2}\left[\log\left(u^2+a^2\right) - \log\left(u^2+b^2\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[-\log\left(s^2+a^2\right) + \log\left(s^2+b^2\right)\right] = \frac{1}{2}\left[\log\left(s^2+b^2\right) - \log\left(s^2+a^2\right)\right]$$

$$= \frac{1}{2}\log\left[\frac{\left(s^2+b^2\right)}{\left(s^2+a^2\right)}\right].$$

4. Problem: Find the Laplace Transformation of $\int_0^t \frac{\sin t}{t}dt$.

Solution:

We know, $L(\sin t) = \frac{1}{s^2+1} = F(s)$

$$\therefore L\left(\frac{\sin t}{t}\right) = \int_s^\infty F(u)\,du \qquad \because L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \frac{1}{u^2+1}du$$

$$= \left[\tan^{-1} u\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$= \tan^{-1} \frac{1}{s}$$

$$= F(s)$$

$$L\left\{\int_0^t \frac{\sin t}{t}\, dt\right\} = \frac{1}{s} F(s) = \frac{1}{s} \tan^{-1} \frac{1}{s} \qquad \left[\because L\left\{\int_0^t f(u)\, du = \frac{F(s)}{s}\right\}\right]$$

5. **Show that** $\mathcal{L}\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \ln\left(\frac{s+b}{s+a}\right).$

Some problems related to Laplace Transform

Problem: Find the Laplace transformation of $e^{-2t} \cos 3t$

Solution: We know,

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

And $L\{\cos 3t\} = \dfrac{s}{s^2+9}$

$$L\{e^{-2t} \cos 3t\} = \frac{s+2}{(s+2)^2+9}$$

This is the required solution.

Problem: Find the Laplace transformation of $e^{3t}(2\cos 5t - 3\sin 5t)$

Solution: By applying linearity property, we have

$$L\{2\cos 5t - 3\sin 5t\}$$

$$= 2L\{\cos 5t\} - 3L\{\sin 5t\}$$

$$= 2.\frac{s}{s^2+5^2} - 3.\frac{5}{s^2+5^2}$$

$$= \frac{2s-15}{s^2+5^2}$$

$$F(s). \quad \text{(say)}$$

Then by applying first shifting property, we have

$$L\{e^{3t}\ f(t)\} = F(s-3)$$

$$L\{e^{3t}(2\cos 5t - 3\sin 5t)\} = \frac{2(s-3)-15}{(s-3)^2+5^2}$$

This is the required solution.

Problem: Find the Laplace transformation of $t\sinh t$.

Solution: We know, $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$

$$= -\frac{\left(s^2-a^2\right).0 - a.2s}{\left(s^2-a^2\right)^2}$$

$$= \frac{2as}{\left(s^2-a^2\right)^2}$$

Problem: Find the Laplace transformation of $t^2\cos at$

Solution: We know, $L\{\cos at\} = \dfrac{s}{s^2+a^2}$

$$\therefore\ L\{t^2\cos at\} = (-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

Supplementary Problems

LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Find the Laplace transforms of each of the following functions. In each case specify the values of s for which the Laplace transform exists.

(a) $2e^{4t}$	*Ans.* (a) $2/(s-4)$,	$s > 4$
(b) $3e^{-2t}$	(b) $3/(s+2)$,	$s > -2$
(c) $5t-3$	(c) $(5-3s)/s^2$,	$s > 0$
(d) $2t^2 - e^{-t}$	(d) $(4+4s-s^3)/s^3(s+1)$,	$s > 0$
(e) $3\cos 5t$	(e) $3s/(s^2+25)$,	$s > 0$
(f) $10 \sin 6t$	(f) $60/(s^2+36)$,	$s > 0$
(g) $6 \sin 2t - 5\cos 2t$	(g) $(12-5s)/(s^2+4)$,	$s > 0$
(h) $(t^2+1)^2$	(h) $(s^4+4s^2+24)/s^3$,	$s > 0$
(i) $(\sin t - \cos t)^2$	(i) $(s^2-2s+4)/s(s^2+4)$,	$s > 0$
(j) $3\cosh 5t - 4\sinh 5t$	(j) $(3s-20)/(s^2-25)$,	$s > 5$

Evaluate (a) $\mathcal{L}\{(5e^{2t}-3)^2\}$, (b) $\mathcal{L}\{4\cos^2 2t\}$.

Ans. (a) $\dfrac{25}{s-4} - \dfrac{30}{s-2} + \dfrac{9}{s}$, $s > 4$ (b) $\dfrac{2}{s} + \dfrac{2s}{s^2+16}$, $s > 0$

LINEARITY, TRANSLATION AND CHANGE OF SCALE PROPERTIES

Find $\mathcal{L}\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$.

Ans. $\dfrac{72}{s^5} - \dfrac{12}{s^4} + \dfrac{4}{s+3} - \dfrac{10}{s^2+25} + \dfrac{3s}{s^2+4}$