Laplace Transformation of multiplication by power of 't':

If $L\{f(t)\} = f(s)$ then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

$= (-1)^n f^n(s)$

1. find the Laplace Transformation of —

(i) $L\{t\sin at\}$ (ii) $L\{t^2 \cos at\}$.

Solutions:

(i) We know that,

$L\{f(t)\} = f(s)$

Here, $L\{\sin at\} = f(s) = \frac{a}{s^2+a^2}$

Now, $L\{t\sin at\} = (-1)^1 \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$

$= -\frac{(s^2+a^2)\cdot 0 - a\cdot(2s+0)}{(s^2+a^2)^2}$

$= \frac{-0+2as}{(s^2+a^2)^2}$

$= \frac{2as}{(s^2+a^2)^2}$.

(ii) $L\{t^2 \cos at\}$

We know, $L\{\cos at\} = f(s) = \frac{s}{s^2+a^2}$

Now,

$$L\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left\{\frac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{ds}\left(\frac{a^2-s^2}{(s^2+a^2)^2}\right)$$

$$= \frac{(s^2+a^2)^2.(-2s) - (a^2-s^2).2(s^2+a^2).2s}{(s^2+a^2)^4}$$

$$= \frac{-(s^4+2s^2a^2+a^4)(2s) - (4s^3+4sa^2)(a^2-s^2)}{(s^2+a^2)^4}$$

$$= \frac{-2s^5 - 4s^3a^2 - 2sa^4 - 4s^3a^2 + 4s^5 - 4sa^4 + 4s^3a^2}{(s^2+a^2)^4}$$

$$= \frac{2s^5 - 6sa^4 - 4s^3a^2}{(s^2+a^2)^4}$$

$$= \frac{2s(s^4 + 3a^4 - 2s^2a^2)}{(s^2+a^2)^4}$$

$$= \frac{2s(s^4 + a^4 + 2s^2a^2 - 4a^4 - 4s^2a^2)}{(s^2+a^2)^4}$$

Exercise:

(1) Find the Laplace Transformation of

• $L\{t^2 \sin t\}$.

Ans: $\frac{6s^2-2}{(s^2+1)^3}$

(2) Evaluate:

(i) $L\{t \cosh 3t\}$ Ans: $\frac{s^2+9}{(s^2-9)^2}$

(ii) $L\{t^3 \cos t\}$ Ans: $\frac{6s^4-36s^2+6}{(s^2+1)^4}$.

Laplace Transformation of Division by 't'

(1) Show that. $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \ln\left(\frac{s+b}{s+a}\right)$

(2) Show that, $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$

(3) Find $L\left\{\frac{\sinh t}{t}\right\}$.

Ans: $\frac{1}{2}\ln\left(\frac{s+1}{s-1}\right)$

(4) Evaluate $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt$

Ans: $\ln 3/2$.