Recall: Previous Lectures

- Linear Independence of Vectors

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0 \quad \Longrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$$

- Linear Span

$$\text{SPAN}\,(v_1, v_2, \ldots, v_n) = \left\{ \sum_{i=1}^{n} \lambda_i v_i \mid \lambda_i \in \mathbb{R} \right\}$$

- Spanning Set

The set $\{v_1, v_2, \ldots, v_n\}$ is said to form a **spanning set** of a vector space V if for any $v \in V, \exists$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathbb{R}$ such that $v = \sum_{i=1}^{n} \alpha_i v_i$.

Dimension: Maximum number of linearly independent vectors in a vector space V

Basis: Set of these linearly independent vectors

LECTURE – 7

Rank of a Matrix

Def. The rank of a matrix is the number of nonzero rows (number of pivots) in its reduced row echelon form

$$A = \begin{bmatrix} 1 & 2 & -2 & -1 & 1 \\ 2 & 4 & -4 & 0 & 3 \\ -1 & -2 & 3 & 3 & 4 \\ 3 & 6 & -7 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & -2 & -1 & 1 \\ 0 & 0 & \boxed{1} & 2 & 5 \\ 0 & 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RANK $(A) = 3$

- The rank of a $m \times n$ matrix cannot be greater than n or m, i.e., $\text{Rank}(A) \leq \min(m, n)$.

$$A = \begin{bmatrix} 1 & 2 & -2 & -1 & 1 \\ 2 & 4 & -4 & 0 & 3 \\ -1 & -2 & 3 & 3 & 4 \\ 3 & 6 & -7 & 1 & 1 \end{bmatrix}$$

$$Ax = 0 \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -9.5 \\ 0 \\ -4 \\ -0.5 \\ 1 \end{bmatrix}$$

Case-1: $\alpha_1 = 1 \ \& \ \alpha_2 = 0$ $\quad Ax = 0 \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ -2 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -4 \\ 3 \\ -7 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies -2C_1 + C_2 = 0$$

Case-2: $\alpha_1 = 0 \;\&\; \alpha_2 = 1$

$$Ax = 0 \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -9.5 \\ 0 \\ -4 \\ -0.5 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & -2 & -1 & 1 \\ 2 & 4 & -4 & 0 & 3 \\ -1 & -2 & 3 & 3 & 4 \\ 3 & 6 & -7 & 1 & 1 \end{bmatrix}$$

$$Ax = 0 \Longrightarrow \quad x_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ -2 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -4 \\ 3 \\ -7 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-9.5\, C_1 - 4\, C_3 - 0.5\, C_4 + C_5 = \mathbf{0}$$

$$\Longrightarrow \; C_5 = 9.5\, C_1 + 4\, C_3 + 0.5\, C_4$$

Def. The **rank** of a matrix is the number of linearly independent rows

(or number of linearly independent columns).

Column Space, $\mathrm{C}(A)$: SPAN of columns vectors of A

Row Space, $\mathrm{R}(A)$: SPAN of row vectors of A

Def. The **rank** of a matrix is the dimension of the row space

(or column space) of A

Rank – Nullity Theorem

$$Ax = 0$$

$$A = \begin{bmatrix} 1 & 2 & -2 & -1 & 1 \\ 2 & 4 & -4 & 0 & 3 \\ -1 & -2 & 3 & 3 & 4 \\ 3 & 6 & -7 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & -2 & -1 & 1 \\ 0 & 0 & \boxed{1} & 2 & 5 \\ 0 & 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -9.5 \\ 0 \\ -4 \\ -0.5 \\ 1 \end{bmatrix}$$

RANK $(A) = 3$ NULLITY $(A) = 2$

Nullity of A = Dim (Null Space) = number of free variables $= (n - r)$

Rank of $A = r$

$$\boxed{\text{Rank } (A) + \text{Nullity } (A) = n}$$

Problem 1: Find the rank of $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

$$A \sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \mathrm{Rank}(A) = 2.$$

Problem 2: Find the rank of A given by

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 1 & 4 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 7 & 7 \end{bmatrix}$$

$$\Rightarrow \mathrm{Rank}(A) = 3.$$

Rank in terms of Determinants:

- **Submatrix:** Suppose A is any matrix of order $m \times n$, then a matrix obtained by leaving some rows and columns from A is called a submatrix of A
- **Rank:** An $m \times n$ matrix A has rank $r \geq 1$ iff A has $r \times r$ submatrix with nonzero determinant, whereas the determinant of every square submatrix with $(r+1)$ or more rows is zero
- In particular, if A is a square $n \times n$ matrix, it has rank n iff $\det(A) \neq 0$
- Rank of a **zero** matrix is $\mathbf{0}$

Example 1: $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$

$|A| = 0,$ as $R_1 = R_3$.

All 2×2 submatrices have 0 determinant.

Rank is 1.

Example 2: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

Here, $|A| = 0 \ \Rightarrow \mathrm{Rank}(A) < 3.$

Also, $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \neq 0 \ \Rightarrow \mathrm{Rank}(A) = 2.$

Example 3: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$|A| = 1 \neq 0 \Rightarrow \mathrm{Rank}(A) = 3$

Conclusion:

RANK (A) $=$ number of pivots

$=$ number of linearly independent rows

$=$ number of linearly independent columns

$=$ dim(C(A))

$=$ dim(R(A))

Rank in terms of determinants

LECTURE – 8 & 9

EIGENVALUES & EIGENVECTORS

Eigenvalues and Eigenvectors

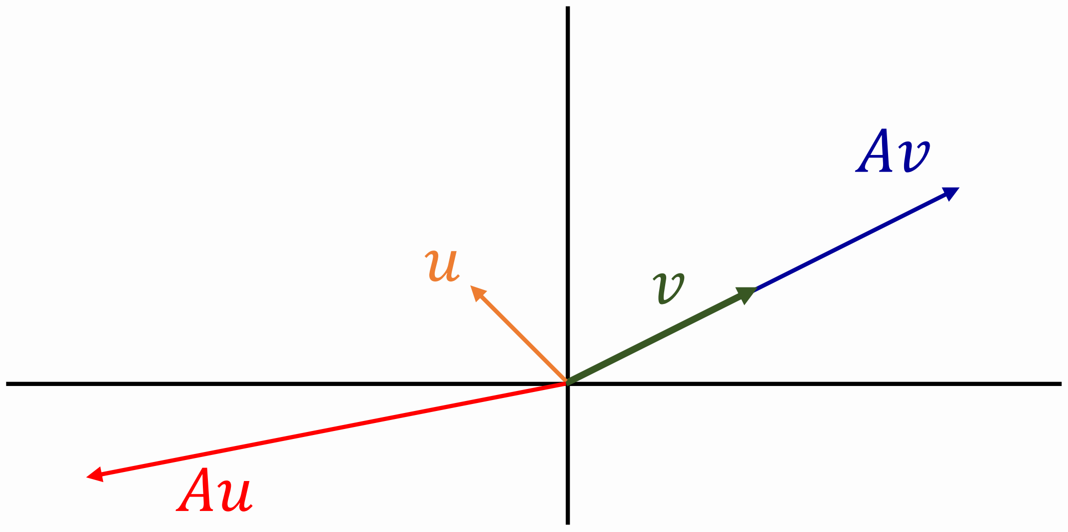
Consider $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



Definition: Let A be any square matrix (real or complex). A scalar λ is called an eigenvalue of A if there exists a **nonzero vector** x such that

$$Ax = \lambda x$$

The vector x is an **eigenvector** associated with the **eigenvalue** λ.

- ❖ **Geometrically**, an eigenvector of a matrix A is a nonzero vector x in $\mathbb{R}^n$ such that the vectors x and Ax are parallel.

- ❖ **Algebraically**, an eigenvector x is a non-trivial solution of the equation $Ax = \lambda x$, i.e., an eigenvector x is a nonzero vector in the null space of $(A - \lambda I)$.

How to Find Eigenvalues and Eigenvectors:

The null space $\mathrm{Null}(A - \lambda I)$ is called the **eigenspace** of A corresponding to eigenvalue λ

- Consider $(A - \lambda I)x = 0$ $\rightarrow$ Two unknowns λ and x.

- Note that $(A - \lambda I)x = 0$ has a non-trivial solution x iff λ satisfies the equation $\det(A - \lambda I) = 0 \Rightarrow c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$

  The above equation is called the characteristic equation of A.

- Roots of the characteristic equation are **eigenvalues**.

- Eigenvectors of A can be determined by solving the homogeneous system of equations $(A - \lambda I)x = 0$ for each eigenvalue λ.

Problem - 1 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$

Characteristic equation: $|A - \lambda I| = 0 \quad \Longrightarrow (\lambda - 3)(\lambda + 2) = 0$

Eigenvalues: $\lambda_1 = 3, \quad \lambda_2 = -2$

Eigenvector corresponding to $\lambda_1 = 3$: $(A - 3I)x = 0 \quad x = [1, 1]^T$

Eigenvector corresponding to $\lambda_2 = -2$: $(A + 2I)x = 0 \quad x = [1, -4]^T$

Problem - 2 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Characteristic equation: $|A - \lambda I| = 0$

Eigenvalues: $\lambda_1 = 1, \qquad \lambda_2 = 1$

Eigenvector corresponding to $\lambda_{1,2} = 1$: $(A - I)x = 0 \qquad x = [1, 0]^T$

Cayley-Hamilton Theorem:

Every square matrix satisfies its own characteristic equation

Characteristic Equation

$$\det(A - \lambda I) = 0 \quad \Longrightarrow c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

$$c_0A^n + c_1A^{n-1} + \cdots + c_nI = 0$$

Problem 3: Let $A = \begin{bmatrix} 11 & -6i \\ 4i & 1 \end{bmatrix}$. Verify Cayley Hamilton theorem for A.

Characteristic polynomial of A:

$$\begin{vmatrix} 11-\lambda & -6i \\ 4i & 1-\lambda \end{vmatrix} = 0 \implies \lambda^2 - 12\lambda - 13 = 0$$

By Cayley-Hamilton theorem

$$A^2 - 12A - 13I = \begin{bmatrix} 145 & -72i \\ 48\,i & 25 \end{bmatrix} - \begin{bmatrix} 132 & -72i \\ 48\,i & 12 \end{bmatrix} + \begin{bmatrix} -13 & 0 \\ 0 & -13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 4: Use Cayley-Hamilton theorem to find A^{-1} when $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$.

Characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 5-\lambda \end{vmatrix} = 0 \quad \Longrightarrow \lambda^2 - 7\lambda - 2 = 0$$

By Cayley-Hamilton theorem

$$A^2 - 7A - 2I = 0$$

$$\Longrightarrow A(A - 7I) = 2I$$

$$\Longrightarrow A^{-1} = \frac{1}{2}(A - 7I) = \frac{1}{2}\begin{bmatrix} -5 & 4 \\ 3 & -2 \end{bmatrix}$$

Problem 5: Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$.

Characteristic equation: $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda(\lambda - 3) = 0$$

Eigenvalues: $\lambda_1 = 0, \lambda_2 = 3.$

- **Eigenvector ($\boldsymbol{\lambda_1 = 0}$):** $(A - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$$

- **Eigenvector ($\lambda_2 = 3$):** $(A - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix}$ & $\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$

Note that eigenvectors are linearly independent

Problem 6: Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Characteristic equation:

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 1)(\lambda - 5)^2 = 0$$

Eigenvalues: $\lambda_1 = 1, \qquad \lambda_2 = \lambda_3 = 5$

- **Eigenvector** $(\lambda_1 = 1)$:

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

- **Eigenvector** $(\boldsymbol{\lambda_2 = \lambda_3 = 5})$:

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 = \alpha_1 \quad \& \quad x_3 = \alpha_2 \qquad x_1 = -\alpha_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Problem 7: Find a basis for the eigenspace of $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Characteristic equation: $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 & 0 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = 0 \qquad \Rightarrow (2-\lambda)^4 = 0$$

Eigenvalues: $\lambda = 2, 2, 2, 2$

Eigenvectors $(\lambda = 2)$:

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus a basis of eigenspace: $\{(1,0,0,0)^T\}$.

Problem - 8 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$

$$\det(A - \lambda I) = 0 \quad \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda^2 - 4\lambda + 5 = 0$$

$$\Rightarrow \lambda = 2 \pm i$$

A real matrix may have complex eigenvalues

Eigenvector corresponding to $\lambda_1 = 2 + i$:

$$(A - \lambda I)x = 0 \quad \Rightarrow \begin{bmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvector corresponding to $\lambda_1 = 2 + i$:

$$A = \begin{bmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix} = \begin{bmatrix} -1-i & 1 \\ -2 & 1-i \end{bmatrix}$$

$$R_2 \leftarrow R_2 - R_1 \times (1-i)$$

$$\sim \begin{bmatrix} -1-i & 1 \\ 0 & 0 \end{bmatrix}$$

$$(1+i)x_1 = x_2 \implies x_2 = (1+i) \;\&\; x_1 = 1$$

A eigenvector corresponding to λ_1: $\begin{bmatrix} 1 \\ (1+i) \end{bmatrix}$

Eigenvector corresponding to $\lambda_1 = 2 - i$:

$$A = \begin{bmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix} = \begin{bmatrix} -1+i & 1 \\ -2 & 1+i \end{bmatrix} \sim \begin{bmatrix} -1+i & 1 \\ 0 & 0 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_1 \times (1+i)$

$$(1-i)x_1 = x_2 \implies x_2 = (1-i) \;\; \& \;\; x_1 = 1$$

A eigenvector corresponding to λ_1: $\begin{bmatrix} 1 \\ (1-i) \end{bmatrix}$

If A is a real matrix and has a complex eigenvalue λ, then the conjugate $(\bar{\lambda})$ is also an eigenvalue. Thus, we have $Ax = \lambda x$ and $A\bar{x} = \bar{\lambda}\,\bar{x}$

Conclusion:

Eigenvalues & Eigenvectors $Ax = \lambda x$

Cayley-Hamilton Theorem $c_0 A^n + c_1 A^{n-1} + \cdots + c_n I = 0$

- Eigenvectors corresponding to distinct eigenvalues are linearly independent
- A real matrix may have complex eigenvalues
- Both the eigenvalues and eigenvectors occur as complex conjugate pairs