- Forward and Backward Difference Table
- Newton’s Forward Interpolation Formula
- Newton’s Backward Interpolation Formula

RECALL: Previous Lecture

➢ Polynomial Interpolation

Polynomial interpolation is the method of determining a polynomial that fits a set of given points

➢ Existence and Uniqueness

For $(n+1)$ data points there is one and only one polynomial of order $\leq n$ that passes through all the points.

➢ Error in Interpolating Polynomials

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

Different Methods of Determining Interpolating Polynomials

Although there is a unique nth order polynomial that fits $(n+1)$ data points, there are a variety of mathematical formats in which this polynomial can be expressed.

- Newton's forward and backward interpolating polynomial
- Newton's Divided Difference Formula
- Lagrange Interpolation Formula

Newton's Forward and Backward Interpolation Formula

Let the tabular points $x_0, x_1, \dots, x_n$ be equally spaced, i.e., $x_i = x_0 + ih,\ \ i = 0, 1, \dots, n$

Finite Difference Operator

- The Shift operator: $Ef(x_i) = f(x_i + h)$
- The Forward difference operator: $\Delta f(x_i) = f(x_i + h) - f(x_i)$
- The Backward difference operator: $\nabla f(x_i) = f(x_i) - f(x_i - h)$
- The Central difference operator: $\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$

Newton's Forward and Backward Interpolation Formula

Let $f_i = f(x_i)$ $\qquad f_{i+1} = f(x_i + h)$ $\qquad f_{i+\frac{1}{2}} = f\left(x_i + \frac{h}{2}\right)$

It can easily be verified that

$$\Delta f_i = \nabla f_{i+1} = \delta f_{i+\frac{1}{2}}$$

Also, note that

$$\Delta \equiv E - 1 \qquad (\Delta f_i = E f_i - f_i)$$

$$\nabla \equiv 1 - E^{-1} \qquad (\nabla f_i = f_i - E^{-1} f_i)$$

$$\delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}} \qquad \left(\delta f_i = E^{\frac{1}{2}} f_i - E^{-\frac{1}{2}} f_i\right)$$

Higher Order Differences

$$\Delta^2 f(x_i) = \Delta\big(\Delta f(x_i)\big)$$

$$= \Delta(f_{i+1} - f_i)$$

$$= f_{i+2} - f_{i+1} + (f_{i+1} - f_i)$$

$$= f_{i+2} - 2f_{i+1} + f_i$$

Similarly, $\nabla^2 f_i = \nabla(\nabla f_i) = \nabla(f_i - f_{i-1})$

$$= f_i - f_{i-1} - (f_{i-1} - f_{i-2})$$

$$= f_i - 2f_{i-1} + f_{i-2}$$

Forward Difference Table

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	f_0			
		Δf_0		
			$\Delta^2 f_0$	
x_1	f_1			$\Delta^3 f_0$
		Δf_1		
			$\Delta^2 f_1$	
x_2	f_2			
		Δf_2		
x_3	f_3			

$$\Delta f_i = f_{i+1} - f_i$$

$$\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$$

$$\Delta^3 f_i = \Delta^2 f_{i+1} - \Delta^2 f_i$$

Backward Difference Table

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
x_0	f_0			
		∇f_1		
			$\nabla^2 f_2$	
x_1	f_1			$\nabla^3 f_3$
		∇f_2		
			$\nabla^2 f_3$	
x_2	f_2			
		∇f_3		
x_3	f_3			

$$\nabla f_i = f_i - f_{i-1}$$

$$\nabla^2 f_i = \nabla f_i - \nabla f_{i-1}$$

$$\nabla^3 f_i = \nabla^2 f_i - \nabla^2 f_{i-1}$$

Difference Table – Numerical Example

x	$f(x)$	Δ/∇	Δ^2/∇^2
0	1		
		1	
1	2		-2
		-1	
2	1		

$\Delta f_0 = 1$ $\quad$ $\Delta^2 f_0 = -2$

$\nabla f_2 = -1$ $\quad$ $\nabla^2 f_2 = -2$

Newton's Forward Difference Formula

1. **Linear Interpolation :** The simplest way to connect two data points with a straight line.

Given that :

x_0	x_1
$f(x_0)$	$f(x_1)$

Consider a general equation of straight line

$$P_1(x) = b_0 + b_1(x - x_0)$$

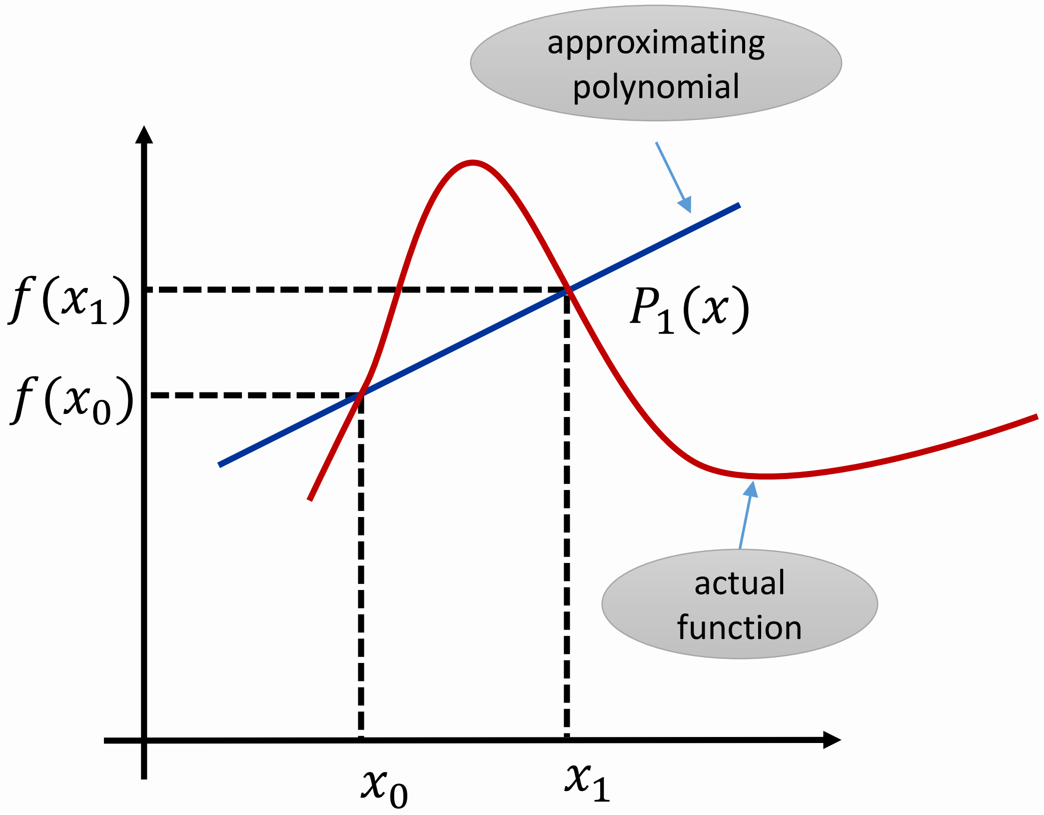
At the point $x = x_0$:

$$P_1(x_0) = f(x_0) = b_0$$

At the point $x = x_1$:

$$P_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$

$$\Rightarrow b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



$$P_1(x) = b_0 + b_1(x - x_0)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Let us consider the equidistant data points, then

$$b_1 = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{\Delta f_0}{h}$$

Interpolating Polynomial

$$\boxed{P_1(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h}}$$

2. Quadratic Interpolation :

Suppose 3 data points are given:

x_0	x_1	x_2
$f(x_0)$	$f(x_1)$	$f(x_2)$

Consider a second order polynomial

$$P_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At the point $x = x_0$: $\quad b_0 = f(x_0)$

At the point $x = x_1$: $\quad b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

In the case of equidistant data points : $\quad b_1 = \dfrac{\Delta f_0}{h}$

$$P_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \qquad b_0 = f(x_0) \qquad b_1 = \frac{\Delta f_0}{h}$$

At the point $x = x_2$: $\quad f(x_2) = f(x_0) + \dfrac{\Delta f_0}{h}\,(x_2 - x_0) + b_2\,(x_2 - x_0)\,(x_2 - x_1)$

$$\Rightarrow f(x_2) = f(x_0) + 2f(x_1) - 2f(x_0) + 2!\,h^2 b_2$$

$$\Rightarrow \frac{f(x_2) - 2f(x_1) + f(x_0)}{2!\,h^2} = b_2 \qquad \Rightarrow b_2 = \frac{\Delta^2 f_0}{2!\,h^2}$$

Interpolating polynomial :

$$\boxed{P_2(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\,h^2}}$$

$$P_1(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h}$$

$$P_2(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\,h^2}$$

Generalized Formula :

We can now write the Newton's forward difference formula based on $(n+1)$ nodal points $x_0, x_1, \dots, x_n$ as:

$$P_n(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\,h^2} + \cdots + (x - x_0)(x - x_1)(x - x_{n-1})\frac{\Delta^n f_0}{n!\,h^n}$$

If we put $\dfrac{x - x_0}{h} = u$ then it takes the following form:

$$P_n(x_0 + hu) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \cdots + \frac{u(u-1)\cdots(u-n+1)}{n!}\Delta^n f_0$$

Alternative Derivation :

$$f(x) = f\left(x_0 + \frac{(x - x_0)}{h}h\right) = f(x_0 + uh)$$

$$= E^u f(x_0)$$

$$= (1 + \Delta)^u f(x_0)$$

$$= f_0 + u\,\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \cdots + \frac{u(u-1)\cdots(u-n+1)}{n!}\Delta^n f_0 + \cdots$$

Neglecting the difference $\Delta^n f_0$ and higher order differences , we get the above generalized formula.

Newton's Backward Difference Formula :

$$\boxed{P_n(x) = f_n + (x - x_n)\frac{\nabla f_n}{h} + \frac{(x - x_n)(x - x_{n-1})}{2!\,h^2}\nabla^2 f_n + \cdots + \frac{(x - x_n)(x - x_{n-1})\ldots(x - x_1)}{n!\,h^n}\nabla^n f_n}$$

$$f(x) = f\left(x_n + \frac{x - x_n}{h}h\right) = f(x_n + hu) = E^u f(x_n) = (1 - \nabla)^{-u} f(x_n) \qquad \nabla \equiv 1 - E^{-1}$$

$$= f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \cdots + \frac{u(u+1)\ldots(u+n-1)}{n!}\nabla^n f(x_n) + \cdots$$

Neglecting the difference $\nabla^{n+1} f(x_n)$ and higher order differences, we get:

$$\boxed{P_n(x_n + hu) = f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + \cdots + \frac{u(u+1)\ldots(u+n-1)}{n!}\nabla^n f_n}$$

➢ Newton's Forward Interpolation Formula

$$P_n(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\,h^2} + \cdots + (x - x_0)(x - x_1)(x - x_{n-1})\frac{\Delta^n f_0}{n!\,h^n}$$

➢ Newton's Backward Interpolation Formula

$$P_n(x) = f_n + (x - x_n)\frac{\nabla f_n}{h} + (x - x_n)(x - x_{n-1})\ \frac{\nabla^2 f_n}{2!\,h^2} + \cdots$$

$$+(x - x_n)(x - x_{n-1}) \ldots (x - x_1)\ \frac{\nabla^n f_n}{n!\,h^n}$$

Example : Using Newton forward and backward interpolation formula, find the quadratic polynomial which takes the following values

x	0	1	2
$f(x)$	1	2	1

Solution : The difference table

x	$f(x)$	$\nabla f/\Delta f$	$\nabla^2 f/\Delta^2 f$
0	1		
		1	
1	2		-2
		-1	
2	1		

Newton's Forward Formula:

x	$f(x)$	$\nabla f/\Delta f$	$\nabla^2 f/\Delta^2 f$
0	1		
		1	
1	2		-2
		-1	
2	1		

$$P_2(x) = f_0 + \frac{(x - x_0)}{h}\Delta f_0 + \frac{(x - x_0)(x - x_1)}{2!\,h^2}\Delta^2 f_0$$

$$= 1 + \frac{x - 0}{1}.1 + \frac{(x - 0)(x - 1)}{2! \cdot 1} \cdot (-2)$$

$$= 1 + x - x^2 + x$$

$$= 1 + 2x - x^2$$

Newton's Backward Formula:

x	$f(x)$	$\nabla f/\Delta f$	$\nabla^2 f/\Delta^2 f$
0	1		
		1	
1	2		-2
		-1	
2	1		

$$P_2(x) = f_2 + \frac{(x - x_2)}{h}\nabla f_2 + \frac{(x - x_2)(x - x_1)}{2!\,h^2}\nabla^2 f_2$$

$$= 1 + \frac{(x - 2)}{1}\cdot(-1) + \frac{(x - 2)(x - 1)}{2!\cdot 1}\cdot(-2)$$

$$= 1 - x + 2 - (x^2 - 3x + 2)$$

$$= 1 + 2x - x^2$$

Note : If we wish to add another data point, say $\left(x_3,\ f(x_3)\right) = (3,\ 10)$ we need to add only another row to the difference table in order to apply forward difference formula.

x	$f(x)$	$\nabla f/\Delta f$	$\nabla^2 f/\Delta^2 f$	$\Delta^3 f/\nabla^3 f$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

$$P_2(x) = 1 + 2x - x^2 + \frac{(x-0)(x-1)(x-2)}{3!}(12)$$

$$= 1 + 2x - x^2 + 2x(x^2 - 3x + 2)$$

$$= 2x^3 - 7x^2 + 6x + 1$$

Example : Construct Newton's forward interpolation polynomial for the following table :

x	4	6	8	10
y	1	3	8	16

Hence evaluate the interpolating polynomial for $x = 5$.

Solution : Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

$$P(x) = 1 + \frac{(x-4)}{2} \cdot 2 + \frac{(x-4)(x-6)}{2!\,2^2} \cdot 3 + 0$$

$$= 1 + x - 4 + \frac{3}{8}(x^2 - 10x + 24)$$

$$= 6 - \frac{11}{4}x + \frac{3}{8}x^2$$

$$P(5) = 6 - \frac{11}{4} \times 5 + \frac{3}{8} \times 25 = 1.625$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

Note : In this example, given 4 data points we get only a second degree polynomial.

Example: Apply Newton's backward difference formula to the data below:

x	1	2	3	4	5
y	1	-1	1	-1	1

Difference Table:

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f_3$	$\nabla^4 f_3$
1	1				
		-2			
2	-1		4		
		2		-8	
3	1		-4		16
		-2		8	
4	-1		4		
		2			
5	1				

Backward Difference Formula

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f_3$	$\nabla^4 f_3$
1	1				
		-2			
2	-1		4		
		2		-8	
3	1		-4		16
		-2		8	
4	-1		4		
		2			
5	1				

$$P_4(x) = f_4 + (x - x_4)\,\frac{\nabla f_4}{h} + \frac{(x - x_4)(x - x_3)}{2!\,h^2}\nabla^2 f_4$$

$$+\frac{(x - x_4)(x - x_3)(x - x_2)}{3!\,h^3}\nabla^3 f_4$$

$$+\frac{(x - x_4)(x - x_3)(x - x_2)(x - x_1)}{4!\,h^4}\nabla^4 f_4$$

$$P_4(x) = 1 + (x - 5)\,2 + \frac{(x-5)(x-4)}{2}4 + \frac{(x-5)(x-4)(x-3)}{6}\,8$$

$$+\frac{(x-5)(x-4)(x-3)(x-2)}{24}\,16$$

$$= \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31$$

Example : Estimate the values of $f(22)$ and $f(42)$ from the following table :

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

Difference table :

x	$f(x)$	Δ/∇	Δ^2/∇^2	Δ^3/∇^3	Δ^4/∇^4	Δ^5/∇^5
20	354					
		-22				
25	332		-19			
		-41		29		
30	291		10		-37	
		-31		-8		45
35	260		2		8	
		-29		0		
40	231		2			
		-27				
45	204					

$f(22)$ (using forward interpolating formula): $x = 22, \quad x_0 = 20, \quad h = 5, \quad u = \dfrac{x - x_0}{h} = \dfrac{2}{5}$

$$P(22) = f_0 + uf_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 f_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f_0$$

$$+ \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 f_0 = 352.223$$

$f(42)$ (using backward interpolation formula): $x = 42, \quad x_5 = 45 \quad u = \dfrac{42 - 45}{5} = -\dfrac{3}{5}$

$$P(22) = f_5 + 4\nabla f_5 + \frac{u(u+1)}{2!}\nabla^2 f_5 + \cdots + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 f_5$$

$$= 218.6630$$

Example : Approximate the function e^x on the interval $[0,1]$ by using polynomial interpolation with

$x_0 = 0,\ x_1 = 0.5,\ x_2 = 1$. Let $P_2(x)$ denote the interpolation polynomial.

$a)$ Find the interpolating polynomial using any method.

$b)$ Find an upper bound for the error magnitude $\max\limits_{0 \le x \le 1} |e^x - P_2(x)|$.

$c)$ Compare the actual error at different points of your choice with the error bound.

$a)$ Difference table :

$$P_2(x) = f_0 + \frac{(x-x_0)}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 f_0$$

x	$f(x)$	Δf	$\Delta^2 f$
0	1		
		0.6487	
0.5	1.6487		0.4208
		1.0696	
1	2.7183		

$$P_2(x) = 1 + (x-0)\frac{0.6487}{0.5} + (x-0)\,(x-0.5)\frac{0.4208}{2\times 0.5^2}$$

$$= 1 + 1.2974\,x + \left(x^2 - \frac{1}{2}x\right) \times 0.8416$$

$$= 0.8416x^2 + 0.8766x + 1$$

$b)$ $$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

$$\max_{x\in[0,1]}|e^x - P_2(x)| \le \frac{1}{6}\max_{t\in[0,1]} e^t \max_{x\in[0,1]}\left|(x-0)\left(x-\frac{1}{2}\right)(x-1)\right|$$

$$= \frac{1}{6} \times e \times \left(\left|(x-0)\left(x-\frac{1}{2}\right)(x-1)\right|\right)_{x=\frac{3\pm\sqrt{3}}{6}} \quad = 0.0218$$

Let $g = x\left(x-\frac{1}{2}\right)(x-1) \quad \Rightarrow g' = \left(x-\frac{1}{2}\right)(x-1) + x(x-1) + x\left(x-\frac{1}{2}\right) = 0$

$$\Rightarrow 3x^2 - 3x + \frac{1}{2} = 0 \quad \Rightarrow x = \frac{3 \pm \sqrt{9 - 4 \times 3 \times \frac{1}{2}}}{2 \times 3} = \frac{3 \pm \sqrt{3}}{6}$$

$b)$ Error bound

$$\max_{x\in[0,1]} |e^x - P_2(x)| \leq \ 0.0218$$

$c)$ Comparison with actual error

x	0.1	0.3	0.6	0.9
$\lvert e^x - P_2(x)\rvert$	0.0091	0.0111	0.0068	0.0110

- Newton’s Divided - Difference Interpolating Polynomial
- **Lagrange's Interpolation Formula**

Recall: Previous Lecture

➢ Newton's Forward Interpolation Formula

$$P_n(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\,h^2} + \cdots + (x - x_0)(x - x_1)(x - x_{n-1})\frac{\Delta^n f_0}{n!\,h^n}$$

➢ Newton's Backward Interpolation Formula

$$P_n(x) = f_n + (x - x_n)\frac{\nabla f_n}{h} + (x - x_n)(x - x_{n-1})\,\frac{\nabla^2 f_n}{2!\,h^2} + \cdots$$

$$+(x - x_n)(x - x_{n-1}) \ldots (x - x_1)\,\frac{\nabla^n f_n}{n!\,h^n}$$

Note that both formulas were derived taking equidistant nodal points!

Newton's Divided - Difference Interpolating Polynomial

Linear Polynomial : Given that :

x_0	x_1
$f(x_0)$	$f(x_1)$

$$P_1(x) = b_0 + b_1(x - x_0)$$

At the point $x = x_0$: $f(x_0) = b_0$

At the point $x = x_1$: $f(x_1) = b_0 + b_1(x_1 - x_0)$

$$\Rightarrow b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_1, x_0] \quad \text{divided difference}$$

$$\boxed{P_1(x) = f(x_0) + f[x_1, x_0](x - x_0)}$$

Quadratic polynomial: Given data points: $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$

In this case, we can fit a polynomial of degree 2: $P_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$

At the point $x = x_0$: $b_0 = f(x_0)$

At the point $x = x_1$: $b_1 = f[x_1, x_0]$

At the point $x = x_2$: $b_2 = \dfrac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} = \dfrac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$

$= f[x_2, x_1, x_0]$

$$\boxed{P_2(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)}$$

Generalized formula for Newton's Divided-Difference Interpolating Polynomial

$$P_n(x) = f(x_0) + (x - x_0)f[x_1, x_0] + (x - x_0)(x - x_1)f[x_2, x_1, x_0] + \cdots$$

$$+(x - x_0)(x - x_1) \ldots (x - x_{n-1})f[x_n, x_{n-1}, \ldots, x_0]$$

This is called Newton's divided difference interpolating polynomial.

Lagrange Interpolating Polynomial

Linear Polynomial : Given data points : $\left(x_0, f(x_0)\right), \ \left(x_1, f(x_1)\right)$

$$\boxed{P_1(x) = \sum_{i=0}^{1} L_i(x) f(x_i)}$$

$$P_1(x) = f(x_0) + f[x_1, x_0](x - x_0)$$

$$P_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$= \underbrace{\frac{(x - x_1)}{(x_0 - x_1)}}_{L_0(x)} f(x_0) + \underbrace{\frac{(x - x_0)}{(x_1 - x_0)}}_{L_1(x)} f(x_1)$$

L_0 & L_1 are called Lagrange polynomials of degree 1.

Linear Polynomial :

$$\boxed{P_1(x) = \sum_{i=0}^{1} L_i(x) f(x_i)}$$

$$L_0(x) = \frac{(x - x_1)}{(x_0 - x_1)} \qquad L_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{1} \frac{x - x_j}{x_i - x_j}; \quad i = 0, 1$$

Generalized Lagrange Interpolating Polynomial

Given data points : $(x_0, f(x_0)),\ (x_1, f(x_1)),\ \dots,\ (x_n, f(x_n))$.

$$\boxed{P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)}$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} = \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)}$$

Lagrange's polynomial of degree n

Example : Using Lagrange and the Newton-divided difference formulas, construct a polynomial of degree 2 or less with the following data:

x	1	2	4
$f(x)$	1	3	3

Solution : Lagrange interpolating polynomial: $P_2(x) = \sum_{i=0}^{2} L_i(x)f(x_i)$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2)(x-4)}{(1-2)(1-4)} = \frac{1}{3}(x-2)(x-4)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-4)}{(2-1)(2-4)} = -\frac{1}{2}(x-1)(x-4)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-2)}{(4-1)(4-2)} = \frac{1}{6}(x-1)(x-2)$$

$$L_0(x) = \frac{1}{3}(x-2)(x-4) \qquad L_1(x) = -\frac{1}{2}(x-1)(x-4) \qquad L_2(x) = \frac{1}{6}(x-1)(x-2)$$

Lagrange interpolating polynomial : $P_2(x) = \sum_{i=0}^{2} L_i(x) f(x_i)$

$$P_2(x) = \frac{1}{3}(x-2)(x-4) \times 1 - \frac{1}{2}(x-1)(x-4) \times 3 + \frac{1}{6}(x-1)(x-2) \times 3$$

$$= \frac{1}{3}(x^2 - 6x + 8) - \frac{3}{2}(x^2 - 5x + 4) + \frac{1}{2}(x^2 - 3x + 2)$$

$$= -\frac{2}{3}x^2 + 4x - \frac{7}{3}$$

Newton's Divided-Difference Formula:

Divided-Difference table

x	$f(x)$	$f[x_i, x_{i-1}]$	$f[x_i, x_{i-1}, x_{i-2}]$
1	1		
		2	
2	3		$-\frac{2}{3}$
		0	
4	3		

$$P_2(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

$$= 1 + 2(x - 1) - \frac{2}{3}(x - 1)(x - 2)$$

$$= -\frac{2}{3}x^2 + 4x - \frac{7}{3}$$

Example: Use the Lagrange and the Newton-divided difference formulas to derive interpolating polynomial from the following table:

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Divided difference table:

x	$f(x)$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot,\cdot,\cdot]$
0	1					
		13				
1	14		-6			
		1		1		
2	15		-2		0	
		-5		1		0
4	5		2		0	
		1		1		
5	6		6			
		13				
6	19					

$$P_5(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

$$= 1 + 13(x) - 6(x)(x - 1) + 1(x)(x - 1)(x - 2)$$

$$= x^3 - 9x^2 + 21x + 1$$

Divided difference table:

x	$f(x)$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot,\cdot,\cdot]$
0	1					
		13				
1	14		-6			
		1		1	0	
2	15		-2			0
		-5		1	0	
4	5		2			
		1		1		
5	6		6			
		13				
6	19					

Lagrange's Interpolation Formula

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

$$P_5(x) = \sum_{i=0}^{5} L_i(x) f(x_i)$$

$$= \frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(-1)(-2)(-4)(-5)(-6)} \times 1 + \frac{(x)(x-2)(x-4)(x-5)(x-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} \times 14$$

$$+ \frac{(x)(x-1)(x-4)(x-5)(x-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \times 15 + \frac{(x)(x-1)(x-2)(x-5)(x-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} \times 5$$

$$+ \frac{(x)(x-1)(x-2)(x-4)(x-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} \times 6 + \frac{(x)(x-1)(x-2)(x-4)(x-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} \times 19$$

$$= x^3 - 9x^2 + 21x + 1$$

Newton's Divided-Difference Interpolating Polynomial

$$P_n(x) = f(x_0) + (x - x_0)f[x_1, x_0] + (x - x_0)(x - x_1)f[x_2, x_1, x_0] + \cdots$$

$$+(x - x_0)(x - x_1)\ldots(x - x_{n-1})f[x_n, x_{n-1}, \ldots, x_0]$$

Lagrange Interpolating Polynomial

$$\boxed{P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)} \qquad L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Numerical Integration

- Trapezoidal Rule
- Simpson's Rule

Numerical Integration

Applications: To find complicated integrals like: $\int_0^1 e^{-x^2}\,dx$ $\int_0^{\pi} x^{\pi}\sin(\sqrt{x})\,dx$

Newton's Cotes Integration formulas :

These formulas are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate.

$$I = \int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx$$ where $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$

The Trapezoidal Rule: (Single Application)

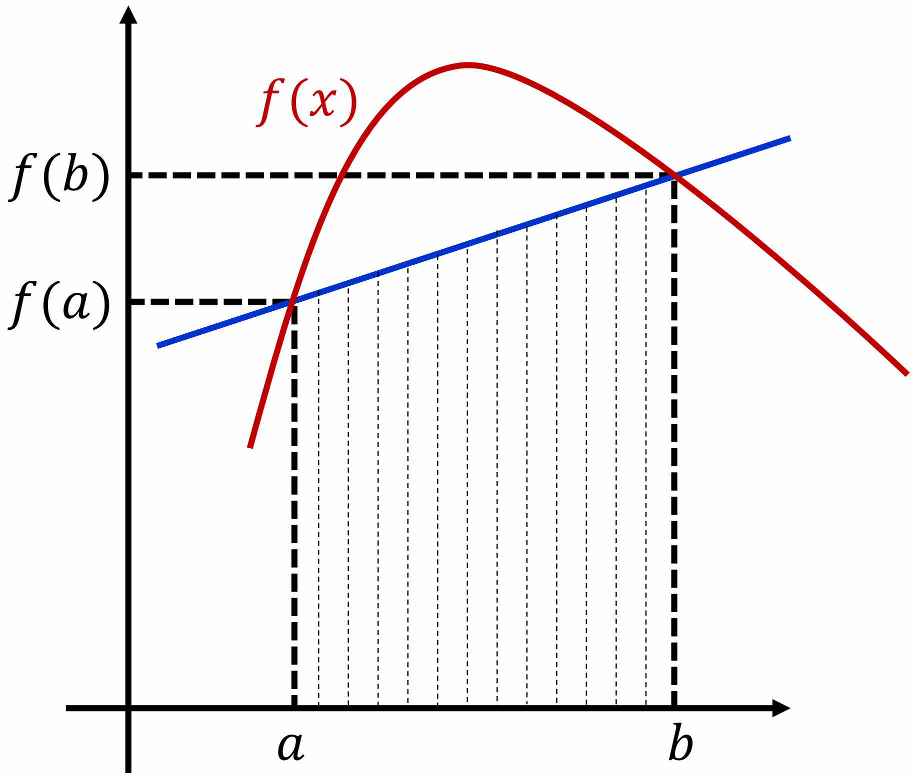
$$I = \int_a^b f(x)\,dx \approx \int_a^b P_1(x)\,dx$$

$$= \int_a^b \left\{ f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right\} dx$$

$$= f(a)(b - a) + \frac{f(b) - f(a)}{b - a}\,\frac{1}{2}\,(b - a)^2$$

$$= f(a)(b - a) + \frac{1}{2}(b - a)\big(f(b) - f(a)\big)$$

$$\Longrightarrow \int_a^b f(x)dx \approx (b - a)\frac{[f(b) + f(a)]}{2}$$



Example : Using trapezoidal rule integral numerically the function

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Compare result with exact value of integral 1.640433.

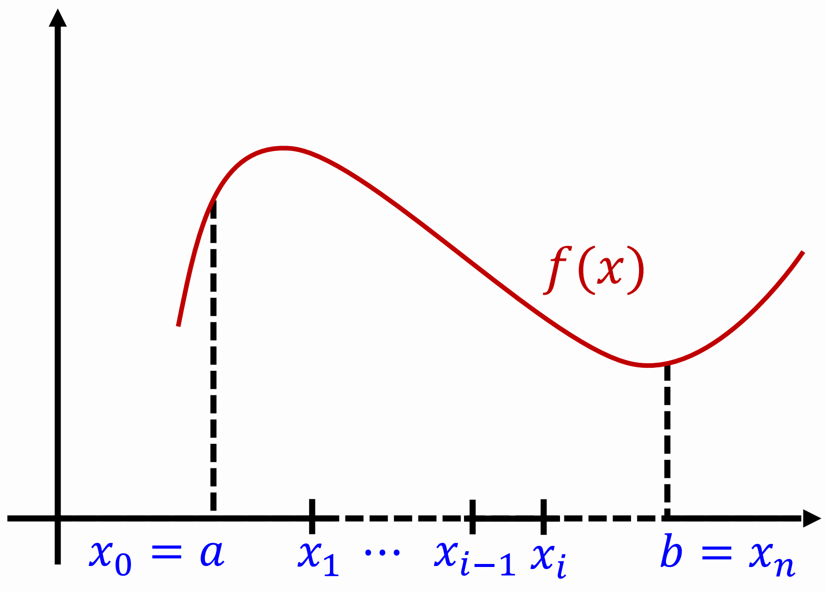
Solution: The function values $f(0) = 0.2$ $f(0.8) = 0.232$

$$\int_0^{0.8} f(x)dx \approx \frac{0.2 + 0.232}{2}(0.8 - 0)$$

$$= 0.1728$$

The Multiple Application of Trapezoidal Rule

To improve accuracy of the trapezoidal rule we divide the integration interval from a to b into a number of segments and apply the method to each segment.



Consider there are $n + 1$ equally spaced base points $x_0, x_1, \ldots, x_n$.

Denote $h = \dfrac{(b-a)}{n}$

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

The Multiple Application of Trapezoidal Rule

$$I = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$\approx h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + \cdots + h\frac{f(x_{n-1}) + f(x_n)}{2}$$

$$= \frac{h}{2}\left[f(x_0) + 2\big(f(x_1) + f(x_2) + \cdots + f(x_{n-1})\big) + f(x_n)\right]$$

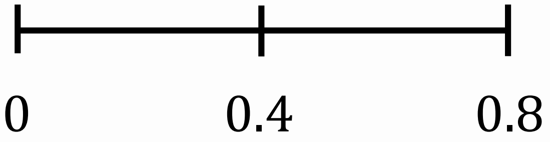
$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

Example : Use the two-segment trapezoidal rule to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$.

Solution:



Exact Value: 1.640433

$$h = \frac{0.8 - 0}{2} = 0.4$$

$$f(0) = 0.2, \quad f(0.4) = 2.456, \quad f(0.8) = 0.232.$$

$$I = \int_0^{0.8} f(x)dx \approx \frac{h}{2}\{0.2 + 2(2.456) + 0.232\} = 1.0688$$

Weighted Mean Value Theorem

Assume f and g are continuous in $[a, b]$. If g never changes sign in $[a, b]$, then

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x) dx \quad \text{where } c \in (a, b) \text{ \& } g \text{ is integrable.}$$

Discrete Mean Value Theorem

Let $f \in C^0[a,\ b]$ and let x_j be $(n+1)$ points in $[a,\ b]$ and C_j be $(n+1)$ constants, all having the same sign. Then there exists $\xi \in [a, b]$ such that

$$\sum_{j=0}^{n} C_j f(x_j) = f(\xi) \sum_{j=0}^{n} C_j$$

In particular, if $C_j = 1 \ \ \forall\, j$, then $\frac{1}{n+1} \sum_{j=0}^{n} f(x_j) = f(\xi)$

Error bounds for the Trapezoidal rule

Single application: We know, $f(x) - P_1(x) = (x - x_0)(x - x_1)\dfrac{f''(t)}{2}$

Integrating above equation from x_0 to $x_1 = x_0 + h$ gives

t depends on x and lies between x_0 & x_1.

$$E = \int_{x_0}^{x_0+h} f(x)dx - \frac{h}{2}[f(x_0) + f(x_1)] = \int_{x_0}^{x_0+h} (x - x_0)(x - x_1)\frac{f''(t)}{2}dx$$

Note that $(x - x_0)(x - x_1)$ does not change the sign in $[x_0, x_0 + h]$

Applying weighted mean value theorem, we get

$$E = \frac{f''(\tilde{t})}{2}\int_{x_0}^{x_0+h} (x - x_0)(x - x_0 - h)\, dx$$

Substitute $x = x_0 = v \quad \Longrightarrow dx = dv.$

Error bounds for the Trapezoidal rule

$$E = \frac{f''(\tilde{t})}{2} \int_{x_0}^{x_0+h} (x - x_0)(x - x_0 - h)dx \qquad \text{where } \tilde{t} \in (x_0,\ x_1)$$

Substitute $x - x_0 = v \quad \Longrightarrow dx = dv.$

$$= \frac{f''(\tilde{t})}{2} \int_0^h v(v - h)\, dx$$

$$= \frac{f''(\tilde{t})}{2} \left[\frac{1}{3}h^3 - \frac{h}{2}h^2\right]$$

$$= -\frac{h^3}{12} f''(\tilde{t})$$

2. Error in multiple application :

$$E = \sum_{i=0}^{n-1} \left\{ -\frac{h^3}{12} f''(\tilde{t}_i) \right\} = -\frac{h^3}{12} \sum_{i=0}^{n-1} f''(\tilde{t}_i)$$

Using discrete mean value theorem

$$= -\frac{h^3}{12}\, n\, f''(\hat{t})$$

where $\hat{t}$ lies between a and b

$$\boxed{E = -\frac{(b-a)}{12} h^2 f''(\hat{t})}$$

Error bounds: Let $M_2 = \max_{[x_0,\, x_n]} |f''(x)|$. Then, $\boxed{|E| \le \frac{(b-a)h^2}{12} M_2}$

Example: Evaluate the following integral using trapezoidal rule with $n = 2, 4$

$$\int_0^1 \frac{dx}{3 + 2x}$$

Calculate numerical values with the exact solution. Find the bound on the error.

Also find the number of sub-intervals required if the error is to be less than 5×10^{-4}

Solution: Case 1: Number of sub-intervals $= 2$

$$\Longrightarrow h = \frac{b - a}{n} = \frac{1 - 0}{2} = 0.5$$

Hence $I_1 = \frac{0.5}{2}\big(f(0) + 2f(0.5) + f(1)\big) = \frac{0.5}{2}\left(\frac{1}{3} + 2 \times \frac{1}{4} + \frac{1}{5}\right) = 0.25833$

Case 2: Number of sub-intervals $= 4$

$$\Longrightarrow h = \frac{1-0}{4} = \frac{1}{4}$$

Hence,

$$I_2 = \frac{1}{4}\frac{1}{2}\left[f(0) + 2\left(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right) + f(1)\right]$$

$$= \frac{1}{8}\left[\frac{1}{3} + 2\left(\frac{2}{7} + \frac{1}{4} + \frac{2}{9}\right) + \frac{1}{5}\right]$$

$$= 0.25615$$

Exact solution: $\frac{1}{2}\ln\frac{5}{3} = 0.25541$

$$E_1 = |0.25541 - 0.258331| = 0.00292$$

$$E_2 = |0.25541 - 0.25615| = 0.00074$$

$$I_1 = 0.25833$$

$$I_2 = 0.25615$$

Error bounds :

$$|E| \le \frac{(b-a)h^2}{12} M_2$$

$$M_2 = \max_{[x_0,\ x_n]} |f''(x)|$$

$$f(x) = \frac{1}{3+2x} \Rightarrow f'(x) = -\frac{2}{(3+2x)^2} \Rightarrow f''(x) = \frac{8}{(3+2x)^3}$$

$$M_2 = \max_{[0,1]} \frac{8}{(3+2x)^3} = \frac{8}{27}$$

Hence, |Error| $\le \frac{(b-a)h^2}{12} M_2 = \frac{1}{12} h^2 \frac{8}{27} = \frac{2h^2}{81}$

For $h = 0.5$, |Error| ≤ 0.00617

For $h = 0.25$, |Error| ≤ 0.00154

Given, $E = 5 \times 10^{-4}$

$$\Rightarrow \frac{(b-a)h^2}{12} M_2 \leq 5 \times 10^{-4}$$

$$\Rightarrow \frac{(b-a)(b-a)^2}{12n^2} \frac{8}{27} \leq 5 \times 10^{-4}$$

$$\Rightarrow \frac{1 \times 8}{12 \times 27 \times 5 \times 10^{-4}} \leq n^2$$

$$\Rightarrow n \geq 7.03$$

Since, n is an integer, we require $n = 8$.

Simpson's 1/3rd Rule

$$I = \int_a^b f(x)dx \approx \int_a^b P_2(x)dx$$

Let $x_0 = a,\ x_1,\ x_2 = b$

$$I \approx \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)\right] dx$$

$$= \frac{1}{2h^2} f(x_0) \int_{x_0}^{x_2} (x-x_1)(x-x_1+x_1-x_2)dx - \frac{1}{h^2} f(x_1) \int_{x_0}^{x_2} (x-x_0)(x-x_0+x_0-x_2)dx$$

$$+ \frac{1}{2h^2} f(x_2) \int_{x_0}^{x_2} (x-x_0)(x-x_0+x_0-x_1)dx$$

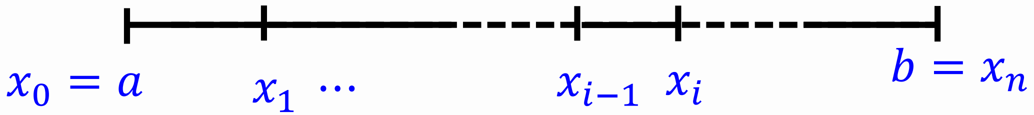
$$I \approx \frac{1}{2h^2} f(x_0) \int_{x_0}^{x_2} (x - x_1)(x - x_1 + x_1 - x_2)dx - \frac{1}{h^2} f(x_1) \int_{x_0}^{x_2} (x - x_0)(x - x_0 + x_0 - x_2)dx$$

$$+ \frac{1}{2h^2} f(x_2) \int_{x_0}^{x_2} (x - x_0)(x - x_0 + x_0 - x_1)dx$$

$$I \approx \frac{f(x_0)}{2h^2}\left[\frac{1}{3}(h^3 + h^3) - h.0\right] - \frac{f(x_1)}{h^2}\left[\frac{1}{3}(2h)^3 - \frac{2h}{3}(2h)^2\right] + \frac{f(x_2)}{2h^2}\left[\frac{1}{3}(2h)^3 + \left(\frac{-h}{2}\right)(2h)^2\right]$$

$$I \approx \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$ Simpson's 1/3rd Rule

Multiple Application of Simpson's Rule



$b - a = nh$

$$I = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$\approx \frac{h}{3}\{f(x_0) + 4f(x_1) + f(x_2)\} + \frac{h}{3}\{f(x_2) + 4f(x_3) + f(x_4)\} + \cdots + \frac{h}{3}\{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\}$$

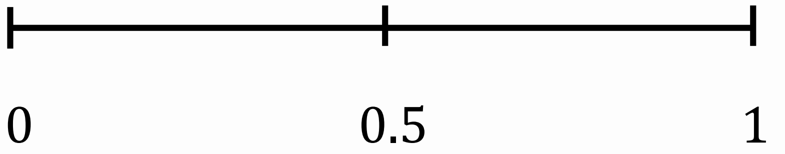
$$= \frac{h}{3}\left\{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)\right\}$$

Error: Single application: $E = -\frac{h^5}{90} f^{(4)}(\xi);\ \ \xi \in (a, b)$

Multiple application: $E = -\frac{b-a}{180} h^4 f^{(4)}(\xi);\ \tilde{\xi} \in (a, b)$

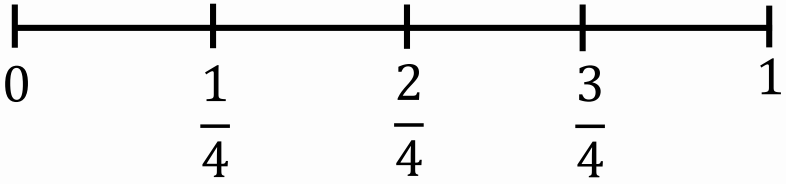
Example: Evaluate $\displaystyle\int_0^1 \frac{dx}{3+2x}$ using Simpson's rule with $n = 2, 4$. Compare with the exact solution.

Solution: For $n = 2$



$$I \approx \frac{h}{3}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

$$= \frac{0.5}{3}\left[\frac{1}{3} + 4 \times \frac{1}{4} + \frac{1}{5}\right] = 0.25556$$



For $n = 4$

Exact solution: $\frac{1}{2}\ln\frac{5}{3} = 0.25541$

$$I \approx \frac{h}{3}\left[f(0) + 4\left\{f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right\} + 2f\left(\frac{1}{2}\right) + f(1)\right] = 0.25542$$

Numerical Integration

Trapezoidal Rule

$$\int_{x_0}^{x_n} f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

$$E = -\frac{(x_n - x_0)}{12} h^2 f''(\xi)$$

$$\xi \in (x_0, x_n)$$

Simpson's 1/3 Rule

$$\int_{x_0}^{x_n} f(x)\,dx \approx \frac{h}{3}\left\{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)\right\}$$

$$E = -\frac{x_n - x_0}{180} h^4 f^{(4)}(\xi); \quad \xi \in (x_0, x_n)$$