EIGENVALUES & EIGENVECTORS

Let A be any square matrix (real or complex). A scalar λ is called an eigenvalue of A if there exists a **nonzero vector** x such that

$$Ax = \lambda x$$

The vector x is an **eigenvector** associated with the **eigenvalue** λ.

Diagonalization of a Matrix:

A square matrix A is said to be **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is a **diagonal matrix** (i.e., A is similar to a diagonal matrix).

Let A be an $n \times n$ matrix. Then A is diagonalizable iff A has n linearly independent eigenvectors.

Applications of Diagonalization

- **Power of Matrices** $\Longrightarrow A^n = PD^nP^{-1}$

- **Solution of System of Linear Differential Equations**

Consider the system of linear differential equations $\dot{X}(t) = A\,X(t)$

Let $\begin{bmatrix} | \\ v_i \\ | \end{bmatrix}$ is the eigenvector corresponding to λ_i

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = C_1 \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} e^{\lambda_1 t} + C_2 \begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} e^{\lambda_2 t} + \cdots + C_n \begin{bmatrix} | \\ v_n \\ | \end{bmatrix} e^{\lambda_n t}$$

Questions from previous year mid-semester examination

Check the diagonalizability of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If it is diagonalizable, then find a diagonal matrix D which is similar to A.

Check the diagonalisability of the following two matrices:

$$\text{(i)}\quad A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & 0 \\ -2 & 1 & 2 \end{bmatrix} \qquad \text{(ii)}\quad B = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}$. Are these two matrices similar? Why?

(a) Let $A = \begin{bmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{bmatrix}$. Then

(i) find the eigenvalues of A.

(ii) Find all the possible values of $a, b \in \mathbb{R}$ so that the vector $(a, b, 2)$ is an eigenvector of A corresponding to the largest eigenvalue of A.

Let A be a 3×3 matrix of real entries and $c \in \mathbb{R}$. Suppose λ_0 is an eigenvalue of A of algebraic multiplicity 2. Then check whether $\lambda_0 + c$ is an eigenvalue of $A + cI$ of algebraic multiplicity 2 or not. Give justification.

LINEAR TRANSFORMATIONS

- **Linear Transformations**
- **Rank and Nullity Theorem**
- **Kernel & Image of Linear Mapping**
- **Matrices as Linear Map**

Linear Mapping (Linear Transformation)

Let X and Y be any two vector space. A mapping $F: X \rightarrow Y$ is called a linear mapping or linear transformation if it satisfies the following two conditions:

- For any two vectors $u, v \in X$, $F(u+v) = F(u) + F(v)$
- For any scalar k and vector $u \in X$, $F(ku) = k\,F(u)$

❖ Remarks

- ➢ The two conditions above can be combined into one: $F(k_1 u + k_2 v) = k_1 F(u) + k_2 F(v)$, where k_1, k_2 are scalars and $u, v \in V$
- ➢ Note that for $k = 0$, $F(0) = 0$. Thus every linear mapping takes the zero vector into the zero vector.

Example 1: Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ with $F(x, y, z) = (x, y, 0)$

Let $u = (a, b, c)$, $\quad v = (a', b', c')$

Then $F(u + v) = F(a + a', b + b', c + c')$

$$= (a + a', b + b', 0) = (a, b, 0) + (a', b', 0)$$

$$= F(u) + F(v)$$

Also, for any scalar k,

$$F(ku) = F(ka, kb, kc) = (ka, kb, 0) = k(a, b, 0) = kF(u)$$

$\Longrightarrow$ F is a linear transformation.

Example 2: Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ with $F(x, y) = (x + 1, y + 2)$.

Here, $F(0,0) = (1,2) \neq (0,0) \quad \Rightarrow F$ is not a linear map.

Example 3: Matrices as Linear Mapping

Any real $m \times n$ matrix A gives a transformation of $\mathbb{R}^n$ into $\mathbb{R}^m$

$$y = Ax, \qquad x \in \mathbb{R}^n, y \in \mathbb{R}^m$$

$$A(u + v) = Au + Av$$

$$A(\lambda u) = \lambda\, Au$$

This is a linear transformation.

Example 4: Let $F:\mathbb{R}^2 \to \mathbb{R}^3$ given by $F(s,t) = \begin{bmatrix} 2s+3t \\ -s+5t \\ 4s-3t \end{bmatrix}$

Is F a linear map?

$$F(s,t) = s\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + t\begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix}$$

$\Longrightarrow$ F is a linear map.

Kernel & Image of Linear Mapping

Let $F: X \to Y$ be a linear mapping

Ker $F = \{x \in X : F(x) = 0\}$

Im $F = \{ y \in Y : \text{ there exists } x \in X \text{ for which } F(x) = y\}$

Example: $F(x, y, z) = (x, y, 0)$

$\mathrm{Im}(F) = \{(a, b, c): c = 0\}$ xy plane

$\mathrm{Ker}(F) = \{(a, b, c) : a = 0, b = 0\}$ z axis

Theorem: Let $F: X \to Y$ be a linear mapping. Then the kernel of F is a subspace of X and image of F is a subspace of Y.

Theorem: Suppose $x_1, x_2, \dots, x_m$ span a vector space X and suppose $F: X \to Y$ is linear. Then $F(x_1), F(x_2), \dots, F(x_m)$ span $\mathrm{Im}(F)$.

Idea: Let $y \in \mathrm{Im}(F)$. Then $\exists\, x \in X$ such that $F(x) = y$

$$x = \sum_{i=1}^{m} \alpha_m x_m \quad \Longrightarrow y = F(x) = \sum_{i=1}^{m} \alpha_m F(x_m)$$

$\Longrightarrow$ The vectors $F(x_1), F(x_2), \dots, F(x_m)$ span $\mathrm{Im}(F)$.

Kernel & Image of Matrix Mapping:

Thus the $\mathrm{Im}(A)$ is precisely the column space of A.

The $\mathrm{Ker}(A)$ is precisely the null space of A.

Consider $A: \mathbb{R}^4 \to \mathbb{R}^3$ with $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$

Take usual basis $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ of $\mathbb{R}^4$

Then Ae_1, Ae_2, Ae_3, Ae_4 span the image of A.

$$\Rightarrow Ae_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}, \quad Ae_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}, \quad Ae_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}, \quad Ae_4 = \begin{bmatrix} a_4 \\ b_4 \\ c_4 \end{bmatrix}.$$

Rank and Nullity of a Linear Mapping

$\boxed{\mathrm{Im}(A)\text{: column space of } A.}$

$\boxed{\mathrm{Ker}(A)\text{: null space of } A.}$

Let $F: X \to Y$ be a linear mapping, then

$$\mathrm{rank}(F) = \dim\big(\mathrm{Im}(F)\big)$$

$$\mathrm{nullity}(F) = \dim(\ker(F))$$

Theorem: Let X be a vector space of finite dimension and let $F: X \to Y$ be a linear map. Then

$$\mathrm{rank}(F) + \mathrm{nullity}(F) = \dim(X)$$

Example 1: Let $F: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear mapping defined as

$$F(x,y,z,t) = (x - y + z + t, \quad 2x - 2y + 3z + 4t, \quad 3x - 3y + 4z + 5t)$$

Find a basis and dimension of, (a) $\mathrm{Im}(F)$, (b) $\mathrm{Ker}(F)$.

(a) **Image of $\boldsymbol{F}$**

$F(e_1), F(e_2), F(e_3), F(e_4)$ span $\mathrm{Im}(F)$

$F(e_1) = (1,2,3)$ $\quad$ $F(e_2)=(-1,-2,-3)$,

$F(e_3) = (1,3,4)$ $\quad$ $F(e_4) = (1,4,5)$

Note that these may not be basis!

Linear Map: $F(x,y,z,t) = (x - y + z + t, \quad 2x - 2y + 3z + 4t, \quad 3x - 3y + 4z + 5t)$

(b) **Kernel of F** $\quad F(x,y,z,t) = 0$ $\quad \therefore \dim(\ker(F)) = \text{nullity}(F) = 2.$

$$x - y + z + t = 0$$
$$2x - 2y + 3z + 4t = 0$$
$$3x - 3y + 4z + 5t = 0$$

Null space of the coefficient matrix is the kernel of F

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $t = \mu_1$ & $y = \mu_2$. $\quad \Longrightarrow z = -2\mu_1 \quad x = \mu_1 + \mu_2$

$$\Longrightarrow \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \mu_1 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \mu_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \text{ \& } \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ form a basis of $\ker(F)$