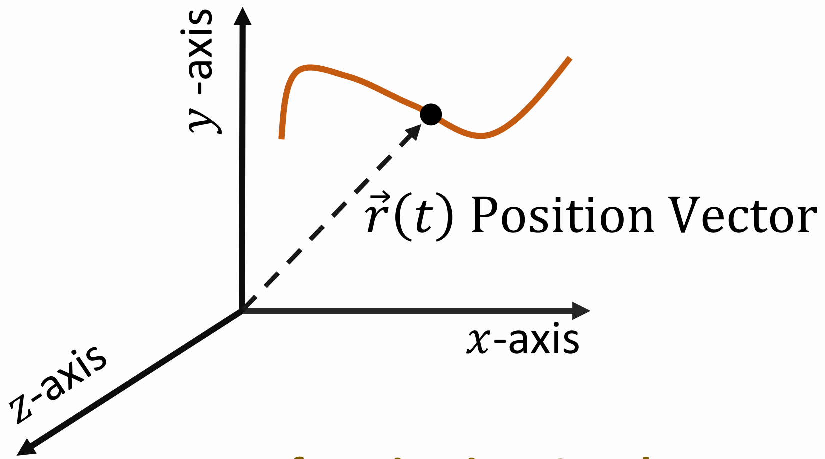
Lecture - 1

- Vector Functions
- Limit, Continuity and Differentiability
- Gradient of a Scalar Function

Vector Functions of One Variable - functions that map a real number to a vector

A vector function, say $\vec{r}(t)$, is written in the form

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \qquad a \leq t \leq b.$$



Vector function in a 3D plane

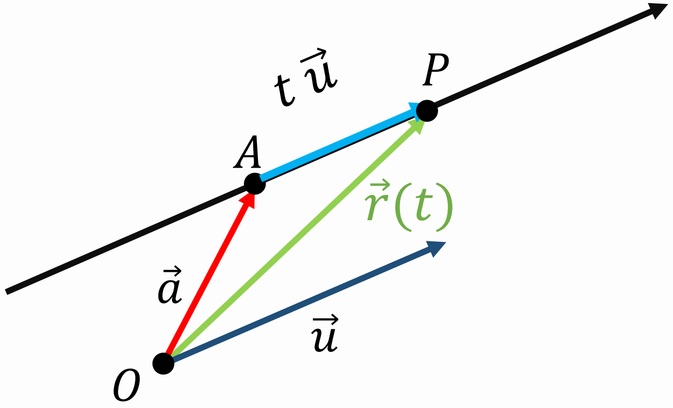
Here x, y and z are real-valued functions of the parameter t

and $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are unit vectors along x, y and z-axes respectively.

In 2D plane, $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}, \qquad a \leq t \leq b.$

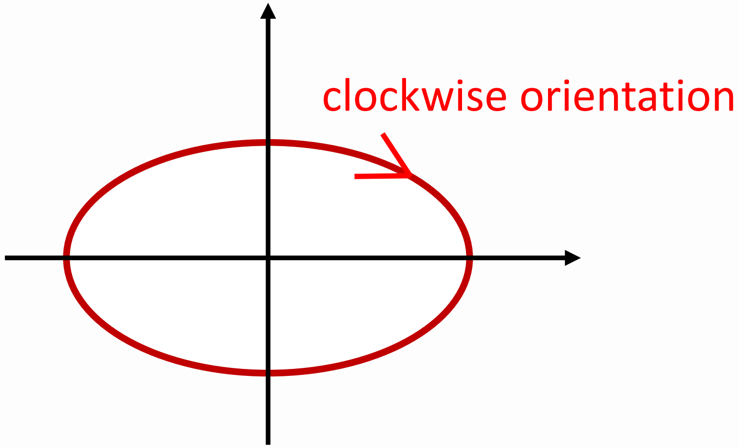
Vector Functions of one Variable

Example 1: Equation of a straight line passing through A with position vector $\vec{a}$ parallel to the vector $\vec{u}$



$$\vec{r}(t) = \vec{a} + t\,\vec{u}, \qquad t \in \mathbb{R}$$

Example 2: Consider $\vec{r}(t) = 3\cos t\,\hat{\imath} - 2\sin t\,\hat{\jmath},\ 0 \le t \le 2\pi$



Example 3: $\vec{r}(t) = 2\cos t\,\hat{\imath} + 2\sin t\,\hat{\jmath} + t\,\hat{k},\ 0 \le t \le 2\pi$ helix

Limit and Continuity of Vector Functions

- **Limit :** $\lim_{t \to a} \vec{r}(t) = \left[\lim_{t \to a} x(t)\right] \hat{\imath} + \left[\lim_{t \to a} y(t)\right] \hat{\jmath} + \left[\lim_{t \to a} z(t)\right] \hat{k}$

  provided $x(t)$, $y(t)$, and $z(t)$ have limits as $t \to a$.

- **Continuity :** A vector-valued function $\vec{r}(t)$ is continuous at $t = a$ if and only if each of its component functions is continuous at $t = a$

Example: Discuss continuity of $\vec{r}(t) = t\,\hat{\imath} + \hat{\jmath} + (2 - t^2)\hat{k}$

Since each component of $\vec{r}(t)$ is continuous for all $t \in \mathbb{R}$

The given vector function of one variable is continuous for all $t \in \mathbb{R}$

Example: Discuss continuity of $\vec{r}(t) = \dfrac{1}{t-2}\,\hat{\imath} + t\,\hat{\jmath} + \ln(t)\,\hat{k}$

The given vector is continuous for all $t > 0$ except $t = 2$

Differentiability of Vector Functions

- **Differentiability :** $\vec{r}(t)$ is said to be differentiable if

$$\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}\,(t)}{\Delta t} \text{ exists.}$$

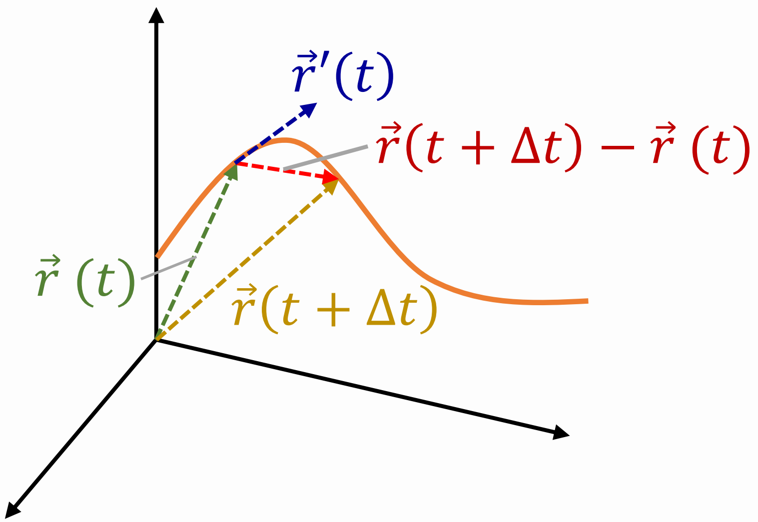
Similar to limit evaluation, differentiation of vector-valued functions can be done on a component-wise as

$$\frac{d\vec{r}(t)}{dt} = \frac{dx(t)}{dt}\,\hat{\imath} + \frac{dy(t)}{dt}\,\hat{\jmath} + \frac{dz(t)}{dt}\,\hat{k}$$

Geometrical Interpretation

$\vec{r}\,'(t)$ is a vector tangent to the curve given by $\vec{r}(t)$ and pointing in the direction of increasing values of t.

Unit tangent vector: $\vec{u} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$



Arc Length of a Curve

Let a curve be given by the vector function $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k},\ a \le t \le b$

Recalls from integral calculus – Parametric equation of the curve $x = x(t), y = y(t), z = z(t)$:

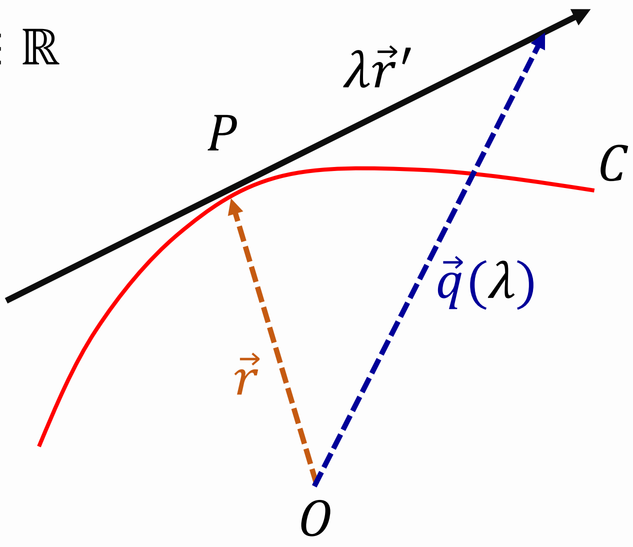
$$\text{Length} = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}\ \text{dt}$$

Note that $|\vec{r}'(t)| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}$ (length of the tangent vector)

Length in terms of position vector $\vec{r}(t) = \int_a^b |\vec{r}'(t)|\ \text{dt}$

Equation of a Tangent to a Curve C at Point P

$$\vec{q}(\lambda) = \vec{r} + \lambda \vec{r}', \qquad \lambda \in \mathbb{R}$$



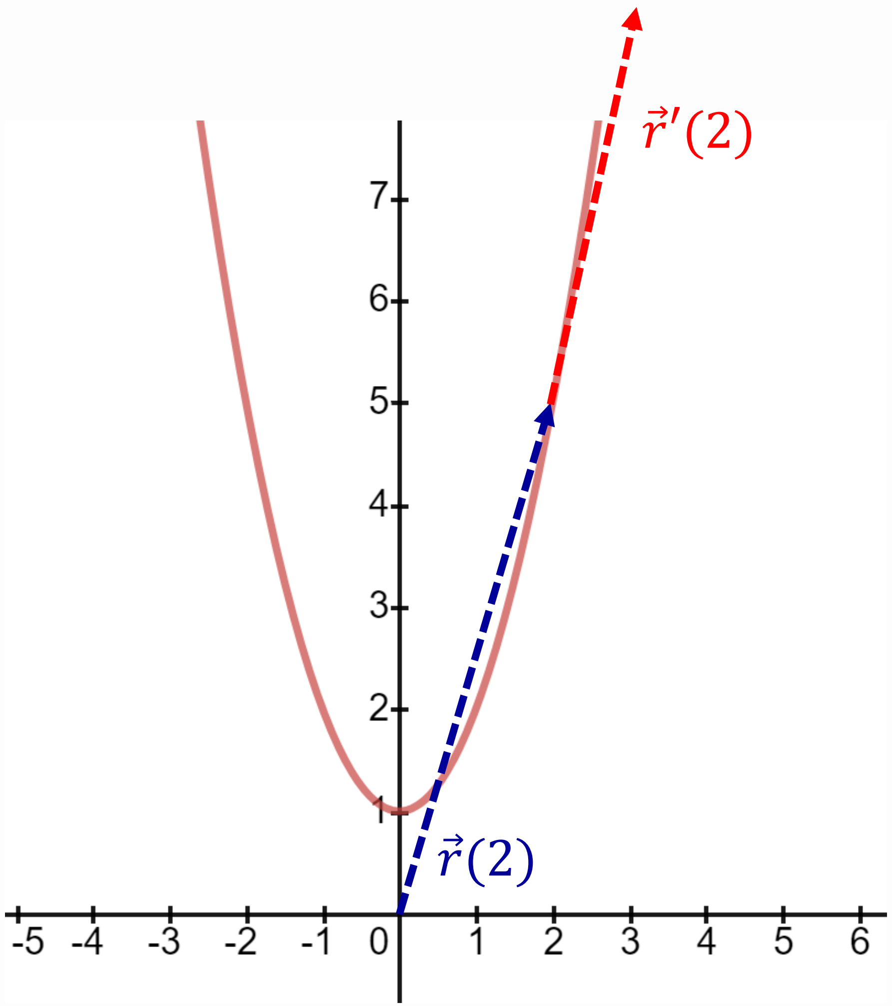
Example: Consider $\vec{r} = t\,\hat{\imath} + (t^2 + 1)\hat{\jmath}$

Tangent vector $\vec{r}' = \hat{\imath} + 2t\,\hat{\jmath}$

Equation of the tangent at $t = 2$:

$$\vec{q}(\lambda) = (2\hat{\imath} + 5\,\hat{\jmath}) + \lambda(\hat{\imath} + 4\,\hat{\jmath})$$

$$= (2 + \lambda)\hat{\imath} + (5 + 4\lambda)\hat{\jmath}$$



Gradient of a Scalar Function (Function of Several Variables)

Let $f(x, y, z)$ be a function of $x, y,$ and z such that f_x, f_y and f_z exist.

The gradient of f, denoted by $\text{grad}\ f$, is the vector

$$\text{grad}\ f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

Vector Function

Nabla or Del operator

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

$$\Longrightarrow \text{grad}\ f = \nabla f$$

Tangent Plane and Normal Line to a Surface

Let a surface S be given by $z = g(x, y)$. Define the function $f(x, y, z) = g(x, y) - z$.

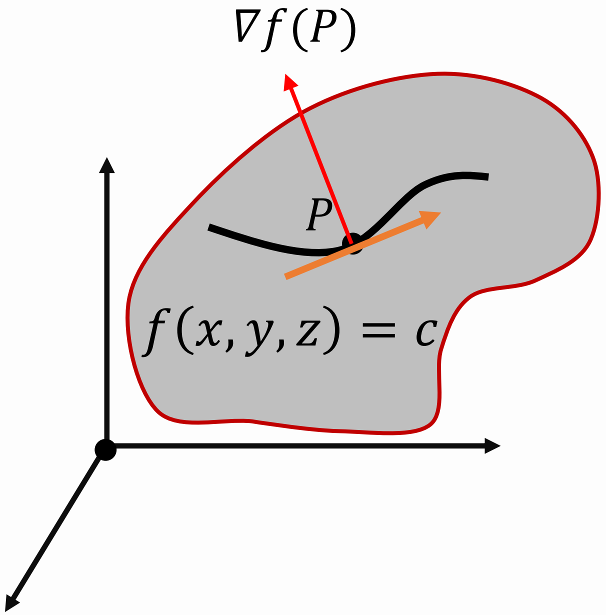
Then the given surface $z = g(x, y)$ can be treated as the level surface of $f(x, y, z)$ given by $f(x, y, z) = 0$.

Note that level surfaces of a function $f(x, y, z)$ are given by $f(x, y, z) = c$

Example: Let $f(x, y, z) = x^2 + y^2 + z^2$

The Level surfaces are concentric spheres centred at the origin.

Let $P(x_0, y_0, z_0)$ be a point on S and let C be a curve on S through P that is defined by the vector-valued function $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$



Since, the curve lies on the surface, we have $f\big(x(t), y(t), z(t)\big) = 0, \ \forall t$

$$\Longrightarrow \ \frac{d}{dt} f\big(x(t), y(t), z(t)\big) = 0 \Longrightarrow f_x(x, y, z)\, x' + f_y(x, y, z)\, y' + f_z(x, y, z)\, z' = 0$$

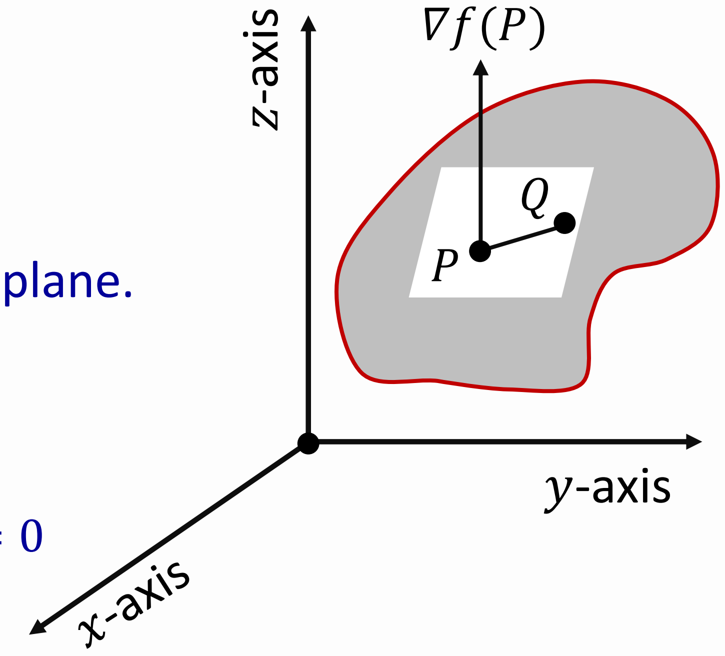
At (x_0, y_0, z_0) we have $\nabla f(x_0, y_0, z_0) \cdot \vec{r}'(t_0) = 0$

$\Rightarrow$ The gradient at P is orthogonal to the tangent vector of every curve on S through P.

Unit normal vector to a surface $f(x, y, z) = c$: $\dfrac{\nabla f}{|\nabla f|}$

The plane through $P(x_0, y_0, z_0)$ that is normal to $\nabla f(x_0, y_0, z_0)$ is called the **tangent plane** to S at P

Let $Q(x, y, z)$ be an arbitrary point in the tangent plane.

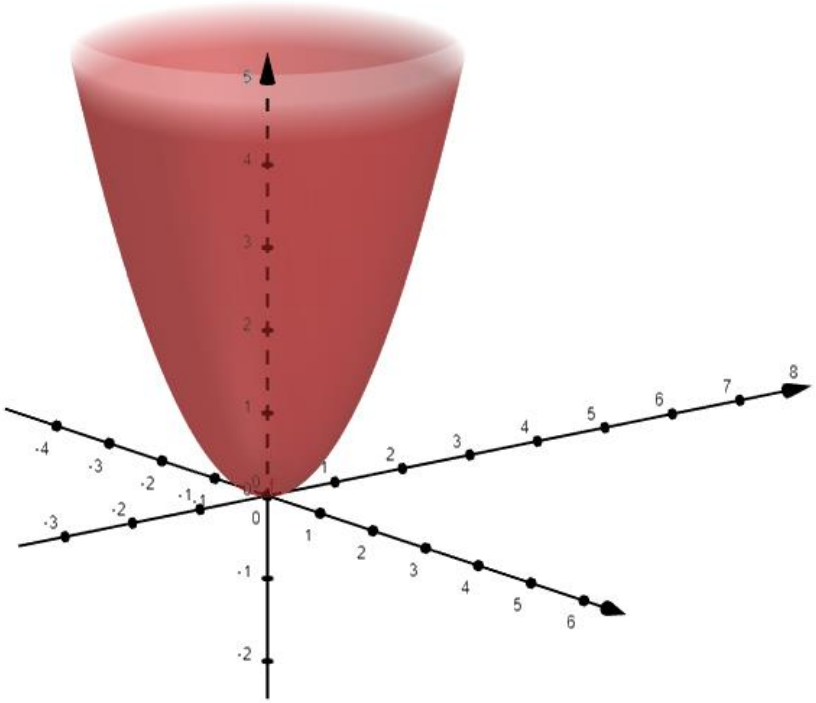


Then the vector $(x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} + (z - z_0)\hat{k}$ lies in the tangent plane.

$$\Rightarrow \left((x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} + (z - z_0)\hat{k}\right) \cdot \left(f_x(P_0)\,\hat{\imath} + f_y(P_0)\,\hat{\jmath} + f_z(P_0)\,\hat{k}\right) = 0$$

$$\boxed{(x - x_0)f_x(x_0, y_0, z_0) + (y - y_0)f_y(x_0, y_0, z_0) + (z - z_0)f_z(x_0, y_0, z_0) = 0}$$

Example : Find the unit normal to the surface $x^2 + y^2 - z = 0$ at the point $(1,1,2)$.



Define $f = x^2 + y^2 - z \quad \Rightarrow \nabla f = 2x\,\hat{\imath} + 2y\,\hat{\jmath} - \hat{k}$

$$\nabla f(1,1,2) = 2\,\hat{\imath} + 2\,\hat{\jmath} - \hat{k}$$

Unit normal vector $\hat{n} = \dfrac{1}{\sqrt{4+4+1}}\left(2\,\hat{\imath} + 2\,\hat{\jmath} - \hat{k}\right)$

$$= \frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} - \frac{1}{3}\hat{k}$$

The other unit normal vector is $-\hat{n} = -\dfrac{2}{3}\hat{\imath} - \dfrac{2}{3}\hat{\jmath} + \dfrac{1}{3}\hat{k}$

SUMMARY

- Vector valued functions $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \qquad a \le t \le b.$
- $\vec{r}'(t)$ is a vector tangent to the curve given by $\vec{r}(t)$
- $\text{grad } f = \dfrac{\partial f}{\partial x}\hat{\imath} + \dfrac{\partial f}{\partial y}\hat{\jmath} + \dfrac{\partial f}{\partial z}\hat{k}$
- $\text{grad } f$ is the normal vector to a surface $f(x, y, z) = c$

Lecture - 2

- Vector and Scalar Fields
- Directional Derivatives

Vector Field **Function that maps a point in space/plane to a vector**

A vector field over a solid region (or a plane) **R** is a function that assigns a vector $\vec{F}(x, y, z)$ $\left(\text{or } \vec{F}(x, y)\right)$ to each point in **R**: $\vec{F}(x, y, z) = f(x, y, z)\hat{\imath} + g(x, y, z)\hat{\jmath} + h(x, y, z)\hat{k}$

Example: Velocity of the air inside a room is defined by a vector field.

Example: Gradient of a function is an example of a vector field:

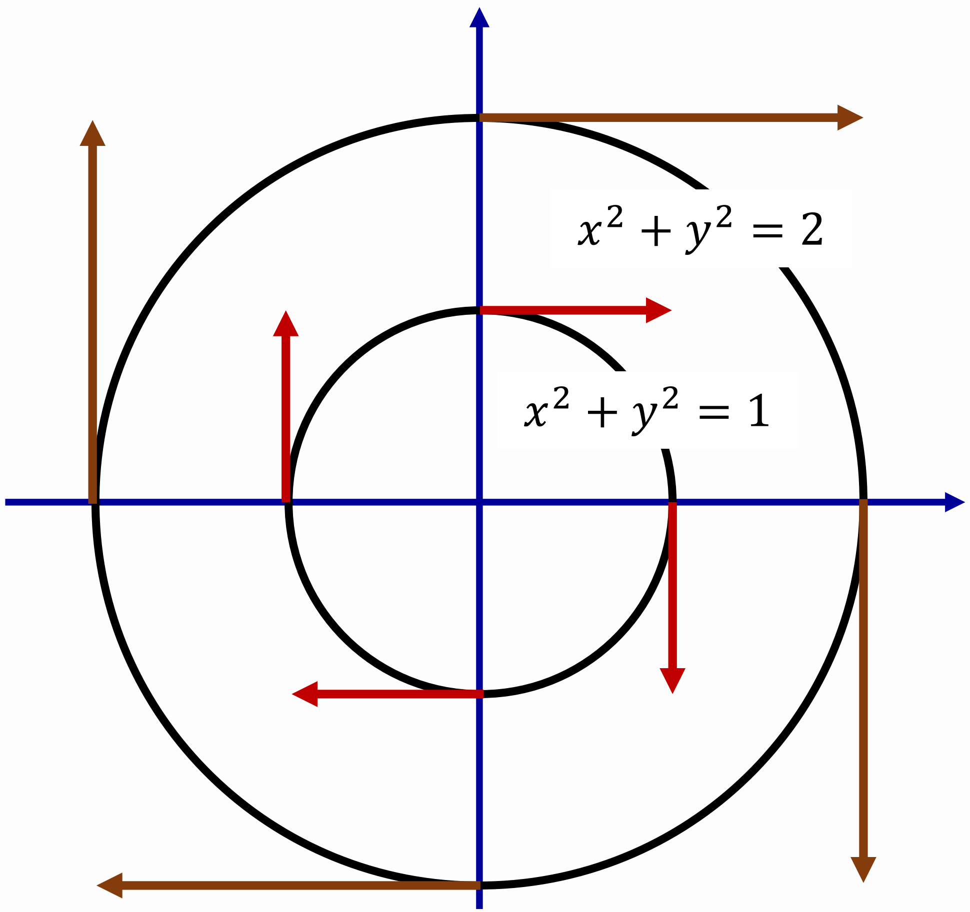
Suppose $f(x, y) = 3x^2y + 2\,xy^3$

$\text{grad } f = \nabla f = (6xy + 2y^3)\,\hat{\imath} + (3x^2 + 6\,xy^2)\,\hat{\jmath}$ **Vector Field (in the plane)**

Example: $\vec{F}(x,y) = y\hat{\imath} - x\hat{\jmath}$

Magnitude of $\vec{F}(x,y)$: $x^2 + y^2$ $\Rightarrow$ vectors of equal magnitude lie on circles $x^2 + y^2 = c$

(level curves)



$\vec{F}(1,0) = -\hat{\jmath}$

$\vec{F}(0,1) = \hat{\imath}$

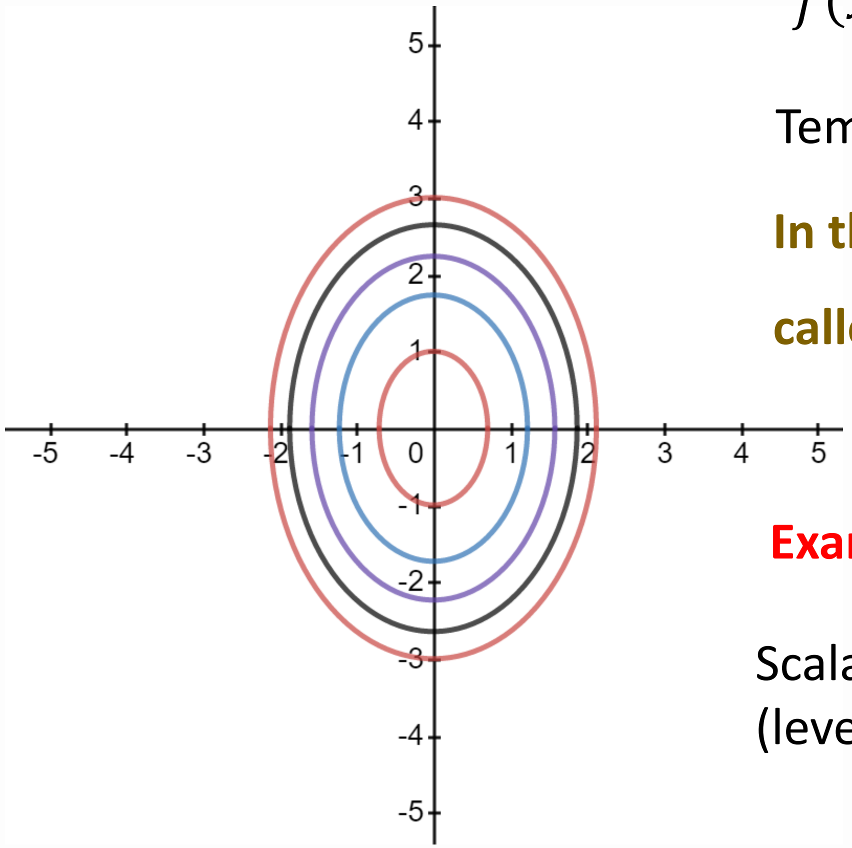
$\vec{F}(-1,0) = \hat{\jmath}$

$\vec{F}(0,-1) = -\hat{\imath}$

Scalar Field

Function that maps a point in space/plane to a scalar

A vector field over a solid region (or a plane) **R** is a function that assigns a scalar to each point in **R**:



$$f(x, y, z) = 3x^2 + 2y^2 + z^2$$

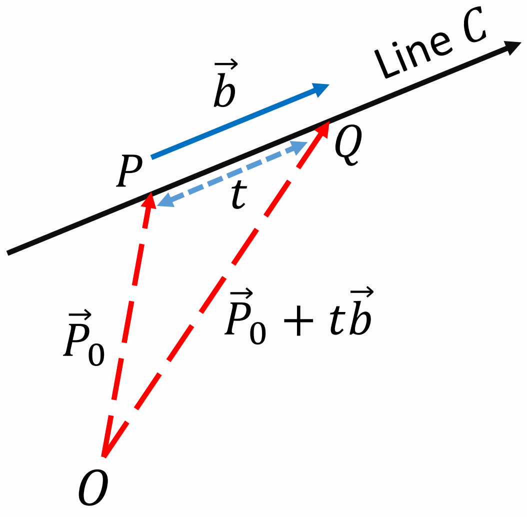
Temperature inside a room is defined by a scalar field.

In the context of vectors, a real valued function of several variables is called a scalar field.

Example: Consider $F(x, y) = 2x^2 + y^2$

Scalar filed may be visualize using level curves of $F(x, y)$
(level surface in case of $F(x, y, z)$)

Directional Derivative of a Scalar Field $f(x, y, z)$ at $P(x_0, y_0, z_0)$ along a Vector $\vec{b}$



Let $|\vec{b}| = 1$. Let C be the line passing through P and parallel to $\vec{b}$

Position vector of the line C is : $\vec{r}(t) = \vec{P}_0 + t\vec{b} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Rate of change of f in the direction $\vec{b}$ is given as

$$\lim_{t\to 0}\frac{f(Q) - f(P)}{t} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right) = \nabla f \cdot \frac{d\vec{r}}{dt} = \nabla f(x_0, y_0, z_0)\cdot\vec{b}$$

At any point P, the directional derivative of f represents the rate of change in f along $\vec{b}$ at the point P, it is denoted by $D_b f = \nabla f\big|_P \cdot \vec{b}$

Example 1: Find the directional derivative of $f(x, y) = 4 - x^2 - \frac{1}{4}y^2$ at $(1, 2)$ in the direction $\vec{u} = \hat{\imath} + \sqrt{3}\,\hat{\jmath}$

$$\nabla f = -2x\,\hat{\imath} - \frac{1}{2}y\,\hat{\jmath} \;\Rightarrow\; \nabla f(1,2) = -2\,\hat{\imath} - \hat{\jmath}$$

Gradient of f at $(1, 2)$

$$\vec{b} = \frac{1}{2}\,\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$$

Unit vector in the direction of $\vec{u}$

$$D_b f = (-2\,\hat{\imath} - \hat{\jmath}) \cdot \left(\frac{1}{2}\,\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}\right) = -1 - \frac{\sqrt{3}}{2}$$

Directional Derivative

Example 2: Find the directional derivative of the scalar field $f = 2x + y + z^2$ in the direction of the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ and evaluate this at the origin.

$$\nabla f = 2\hat{\imath} + \hat{\jmath} + 2z\,\hat{k}$$

$$D_{(1,1,1)}f = \nabla f \cdot \frac{\hat{\imath} + \hat{\jmath} + \hat{k}}{\sqrt{3}} = \left(2\hat{\imath} + \hat{\jmath} + 2z\,\hat{k}\right) \cdot \left(\frac{\hat{\imath}}{\sqrt{3}} + \frac{\hat{\jmath}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}z$$

Value at the origin: $\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \sqrt{3}$

Maximum Rate of Change of a Scalar Field

Rate of change of f in the direction of a unit vector $\vec{b}$: $D_b f = \nabla f \cdot \vec{b} = |\nabla f||\vec{b}| \cos\theta = |\nabla f| \cos\theta$

⇒ Rate of change is maximum when θ is 0, i.e., in the direction of ∇f

⇒ Rate of change is minimum when θ is π, i.e., in the opposite direction of ∇f

⇒ Gradient vector ∇f points in the direction in which f increases most rapidly and

$-\nabla f$ points in the direction in which f decreases most rapidly.

Example: Let $f(x, y, z) = x^2 + y^2 - 2z$. Find the direction of maximum increase of f at $(2, 1, -1)$.

Gradient of f: $2x\,\hat{\imath} + 2y\,\hat{\jmath} - 2\,\hat{k}$

Direction of maximum increase at $(2, -1, 1)$: $4\,\hat{\imath} - 2\,\hat{\jmath} - 2\,\hat{k}$

Note: The above concept of maximum increase/decrease is very useful for optimization problems. Gradient ascent/descent approach is very popular for finding local maximum/minimum.

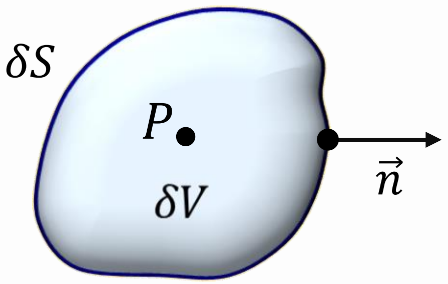
SUMMARY

- Vector Field – Function that maps a point to a vector
- Scalar Field - Function that maps a point to a scalar
- Directional Derivative $D_b f = \nabla f|_P \cdot \vec{b}$

Lecture - 3

- **Divergence of a Vector Field**
- **Curl of a Vector Field**

Divergence of a Vector Field



The divergence of a vector field $\vec{v}$ at a point P is defined as

$$\operatorname{div} \vec{v} = \lim_{\delta V \to 0} \frac{1}{\delta V} \iint_{\delta S} \vec{v} \cdot \vec{n} \; d\sigma$$

Flux of the vector field $\vec{v}$ out of a small closed surface

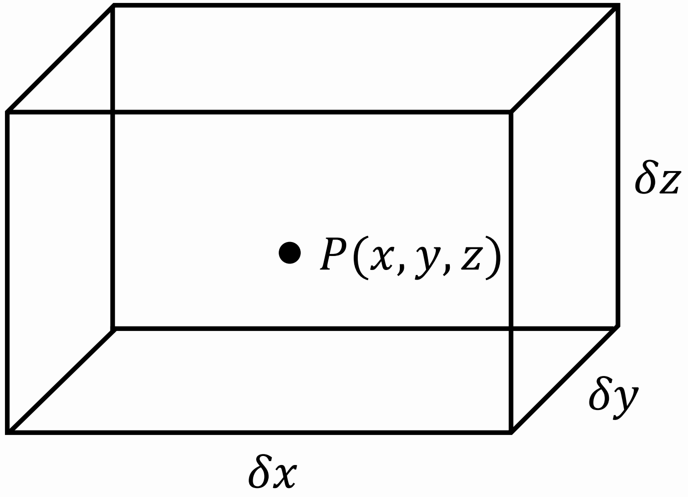
where δV is a small volume enclosing P with surface δS and $\vec{n}$ is the outward pointing normal to δS.

Computation of Divergence

The divergence of a vector field $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ is the scalar field given by

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Physical Interpretation of Divergence of a Vector Field



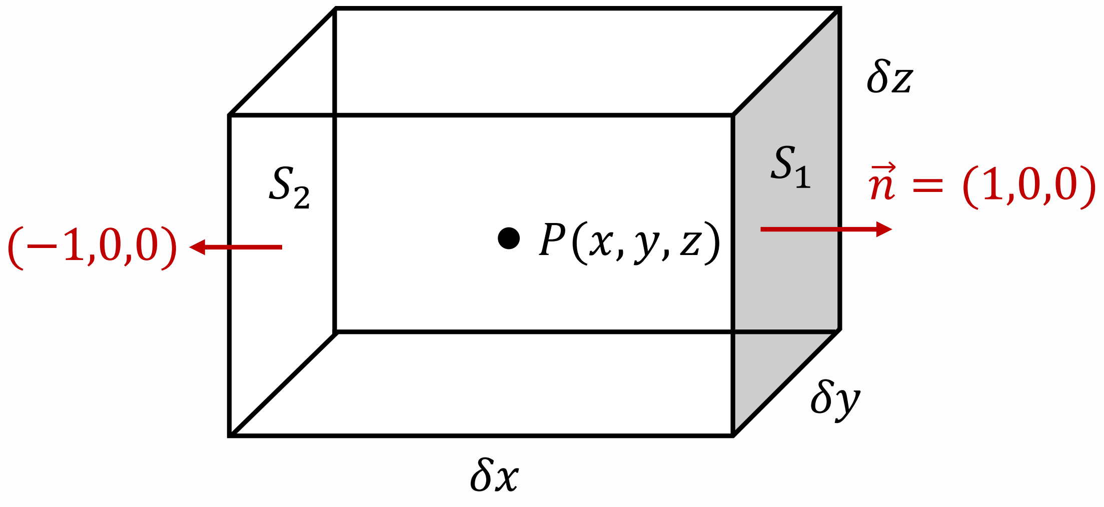
Suppose $\vec{v}(x, y, z)$ is the velocity of a fluid at a point $P(x, y, z)$.

Measure the rate per unit volume at which fluid flows out of this box across its faces:

$$\operatorname{div} \vec{v} = \lim_{\delta V \to 0} \frac{1}{\delta V} \iint_S \vec{v} \cdot \vec{n}\, d\sigma = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}} \frac{1}{\delta x\, \delta y\, \delta z} \left(\sum_{i=1}^{6} \iint_{S_i} \vec{v} \cdot \vec{n}\ d\sigma \right)$$

Flux outward across S_1:

$$\iint_{S_1} \vec{v} \cdot \vec{n}\, d\sigma \approx v_1\left(x + \frac{\delta x}{2}, y, z\right) \delta y\, \delta z$$



Flux outward across S_2:

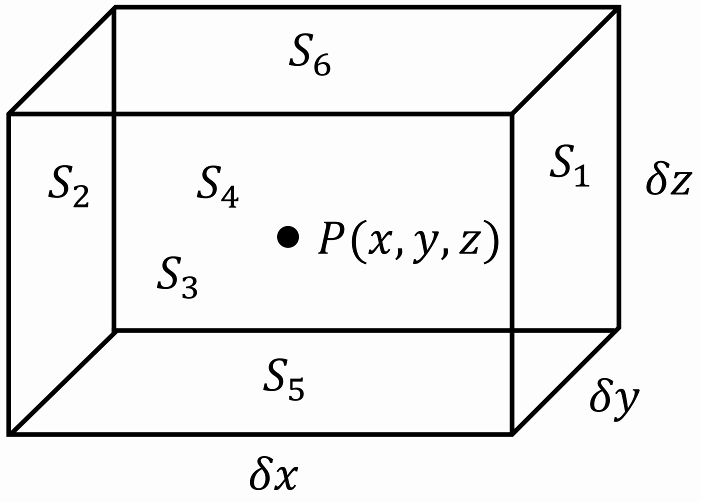
$$\iint_{S_2} \vec{v} \cdot \vec{n}\, d\sigma \approx -v_1\left(x - \frac{\delta x}{2}, y, z\right) \delta y\, \delta z$$

Flux outward across S_1 & S_2:

$$\iint_{S_1+S_2} \vec{v} \cdot \vec{n}\, d\sigma \approx \left(v_1\left(x + \frac{\delta x}{2}, y, z\right) - v_1\left(x - \frac{\delta x}{2}, y, z\right)\right) \delta y \delta z \approx \frac{\partial v_1}{\partial x} \delta x\, \delta y\, \delta z$$

Flux outward across S_1 & S_2:

$$\iint_{S_1+S_2} \vec{v}\cdot\vec{n}\,d\sigma \approx \frac{\partial v_1}{\partial x}\delta x\,\delta y\,\delta z = \frac{\partial v_1}{\partial x}\delta V$$



Similarly from other faces:

$$\iint_{S_3+S_4} \vec{v}\cdot\vec{n}\,d\sigma \approx \frac{\partial v_2}{\partial y}\delta V \qquad \iint_{S_5+S_6} \vec{v}\cdot\vec{n}\,d\sigma \approx \frac{\partial v_3}{\partial z}\delta V$$

Flux per unit volume out of the box $\approx \dfrac{\partial v_1}{\partial x}+\dfrac{\partial v_2}{\partial y}+\dfrac{\partial v_3}{\partial z}$

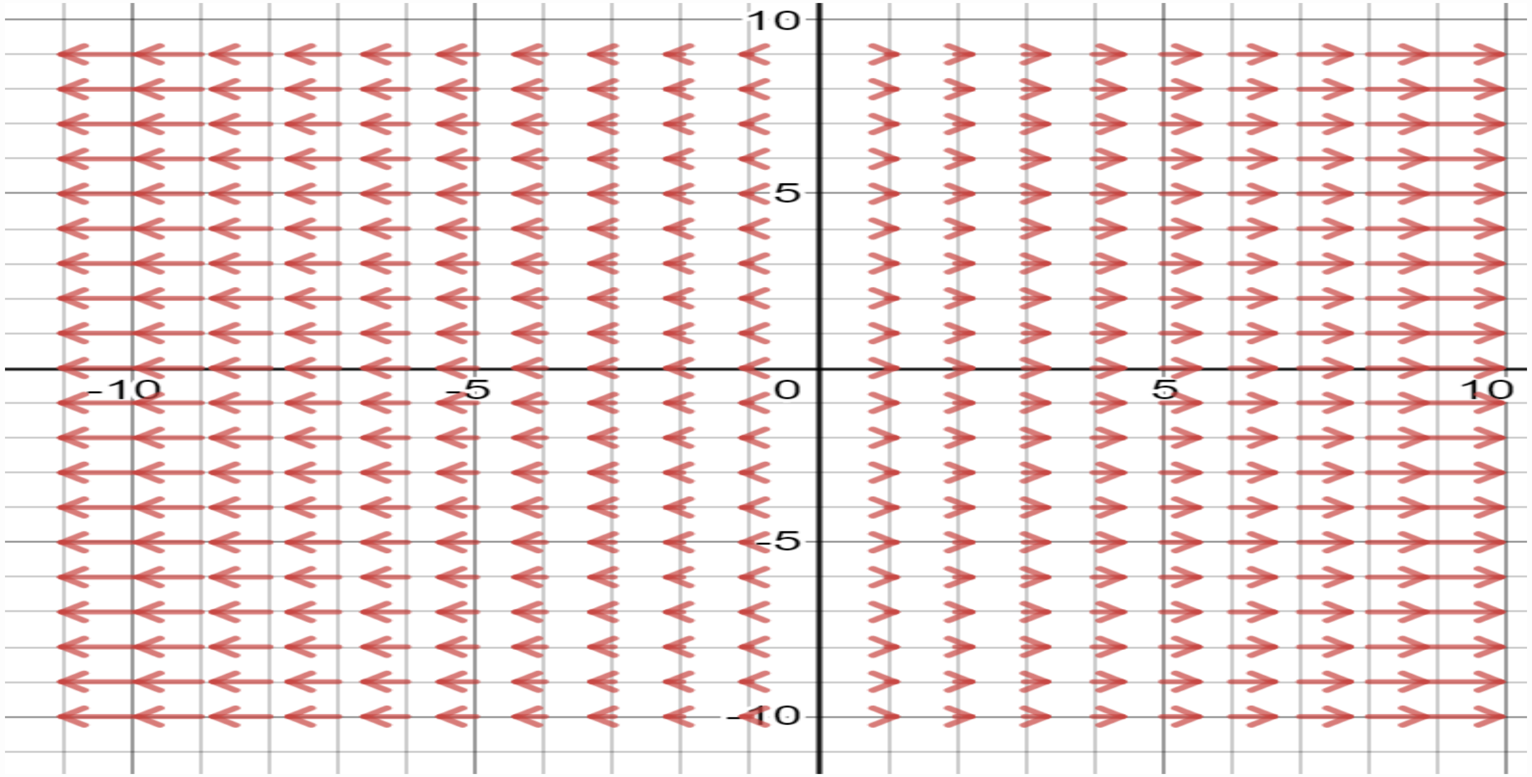
Flux per unit volume at $P(x,y,z) = \dfrac{\partial v_1}{\partial x}+\dfrac{\partial v_2}{\partial y}+\dfrac{\partial v_3}{\partial z} = \mathrm{div}\,\vec{v}$

Divergence can be interpreted as the rate of expression or compression of the vector field.

Example : Consider $\vec{v} = (x, 0,0)$

div $\vec{v} = 1$ (positive)

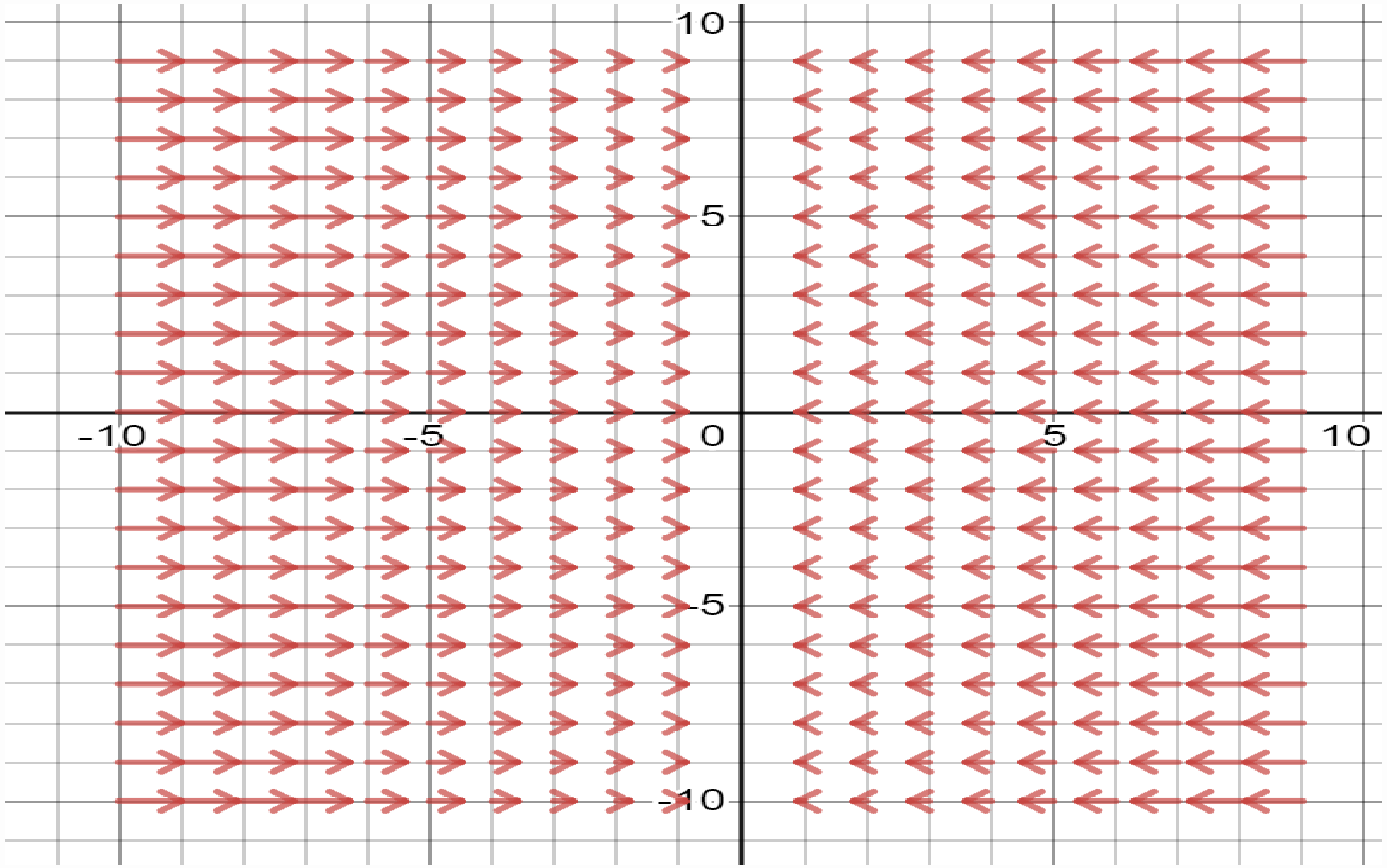
Tendency of fluid is EXPANSION.

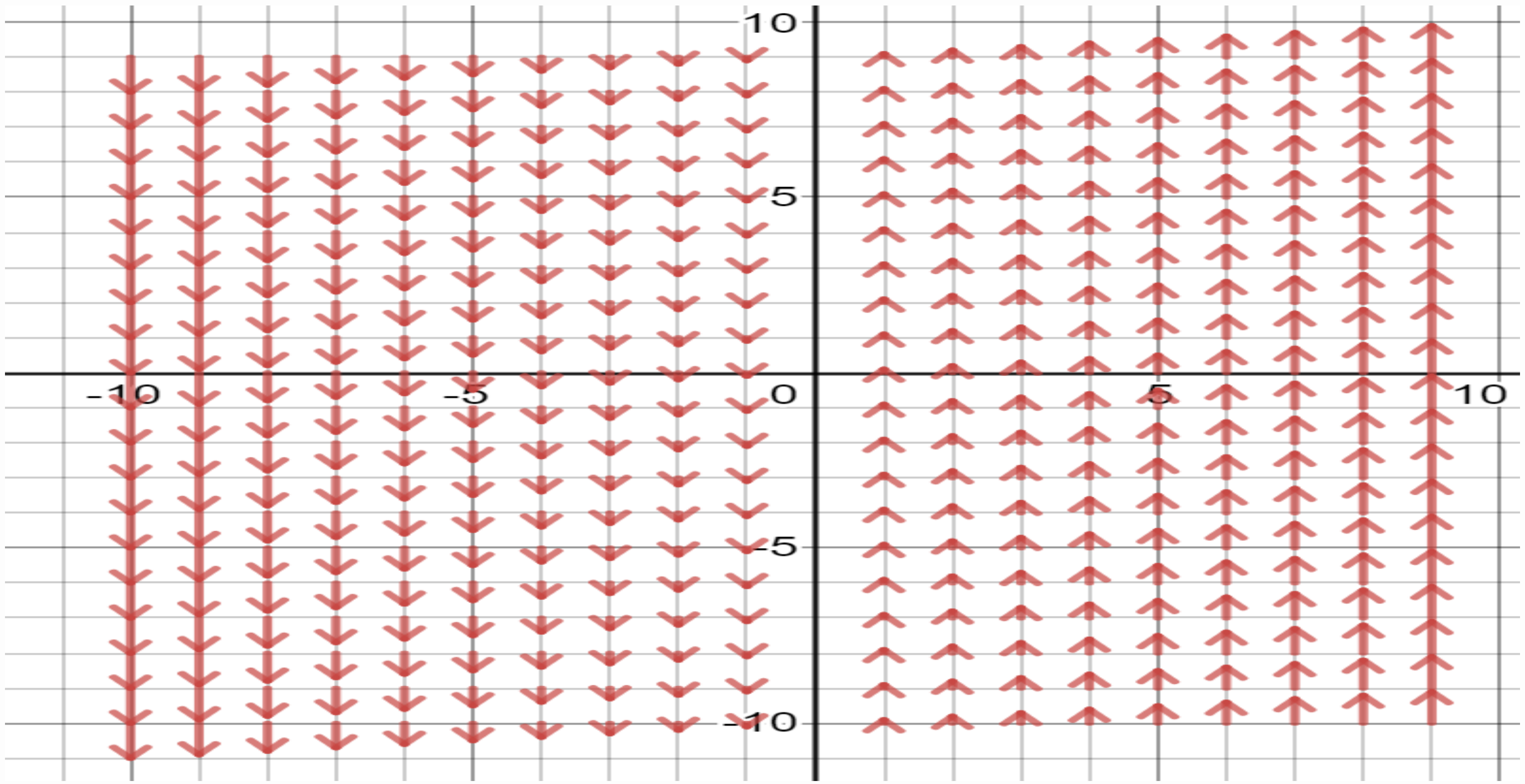


Example : Consider $\vec{v} = (-x, 0, 0)$

div $\vec{v} = -1$ (negative)

Tendency of fluid is COMPRESSION.





Example : Consider $\vec{v} = (0, x, 0)$

div $\vec{v} = 0$

Neither expanding nor contracting.

A vector field $\vec{v}$ for which $\nabla \cdot \vec{v} = 0$ everywhere is said to be **solenoidal**.

The relation div $\vec{v} = 0$ is also known as the **condition of incompressibility**.

Curl of a Vector Field

Curl of a vector $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ field is given by

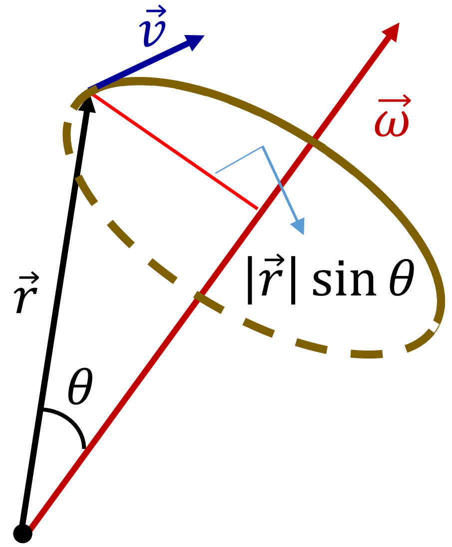
$$\text{curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

Example: Let $\vec{v} = y\,\hat{\imath} + 2xz\,\hat{\jmath} + ze^x\,\hat{k}$

$$\text{curl } \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & 2xz & ze^x \end{vmatrix} = -2x\,\hat{\imath} - ze^x\,\hat{\jmath} + (2z - 1)\,\hat{k}$$

Physical Interpretation of Curl of a Vector Field



Suppose an object rotates with uniform angular velocity $\vec{\omega}$

$$\text{tangential speed} = \text{angular speed} \times \text{radius}$$

$$|\vec{v}| = |\vec{\omega}|\,|\vec{r}|\sin\theta = |\vec{\omega} \times \vec{r}|$$

Note that the direction of $\vec{v}$ is perpendicular to both $\vec{r}$ and $\vec{\omega}$

Since $\vec{v}$ and $\vec{r} \times \vec{\omega}$ both have same direction and same magnitude, we conclude

$$\vec{v} = \vec{\omega} \times \vec{r}$$

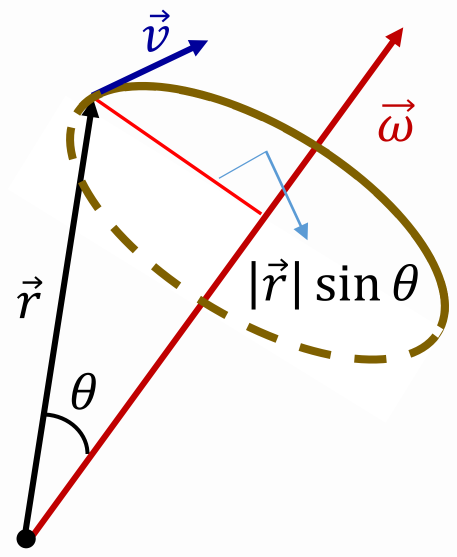
Let $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ and $\vec{\omega} = a\,\hat{\imath} + b\,\hat{\jmath} + c\,\hat{k}$

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\,\hat{\imath} + (cx - az)\,\hat{\jmath} + (ay - bx)\,\hat{k}$$

$$\vec{v} = (bz - cy)\,\hat{\imath} + (cx - az)\,\hat{\jmath} + (ay - bx)\,\hat{k}$$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (bz - cy) & (cx - az) & (ay - bx) \end{vmatrix}$$

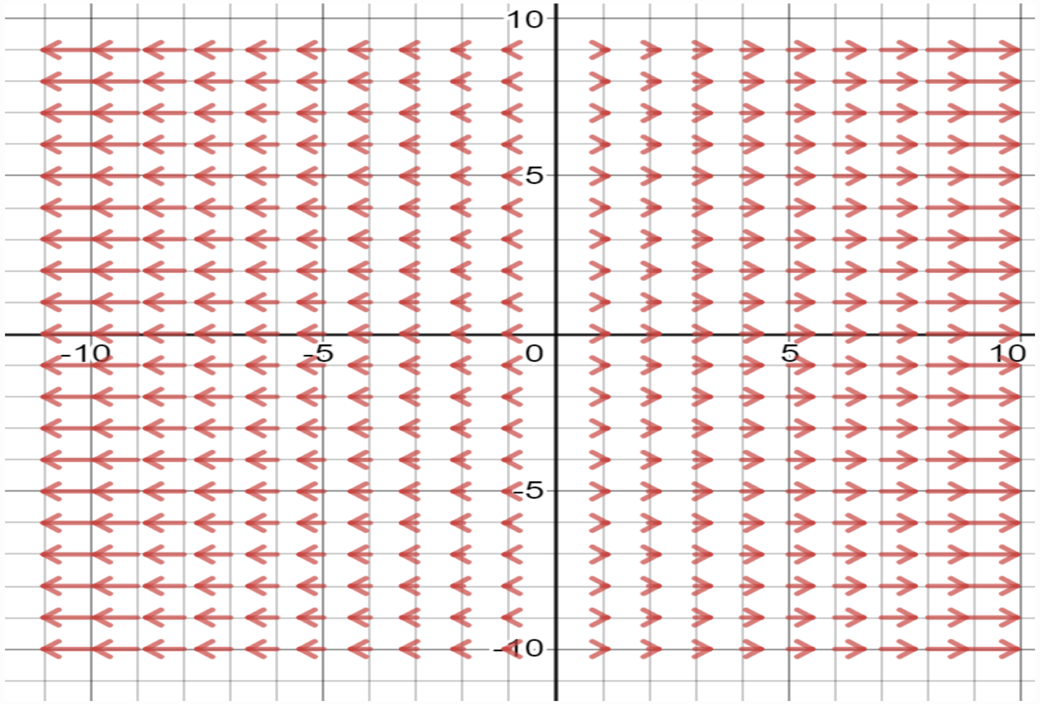
$$= 2a\,\hat{\imath} + 2b\,\hat{\jmath} + 2c\,\hat{k} \;= 2\,\vec{\omega}$$

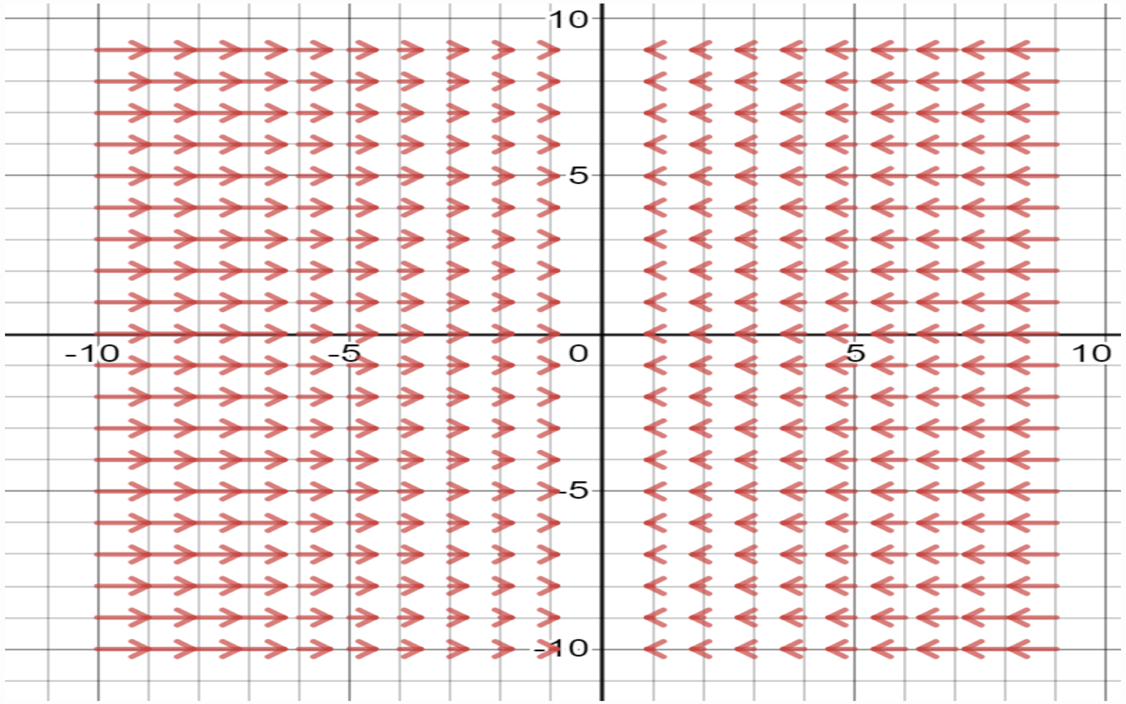


curl $\vec{v}$ signifies the tendency of **ROTATION**.

The vector curl $\vec{v}$ is directed along the axis of rotation with magnitude twice the angular speed.

A vector filed $\vec{v}$ for which $\nabla \times \vec{v}$ is zero everywhere is said to be IRROTATIONAL.



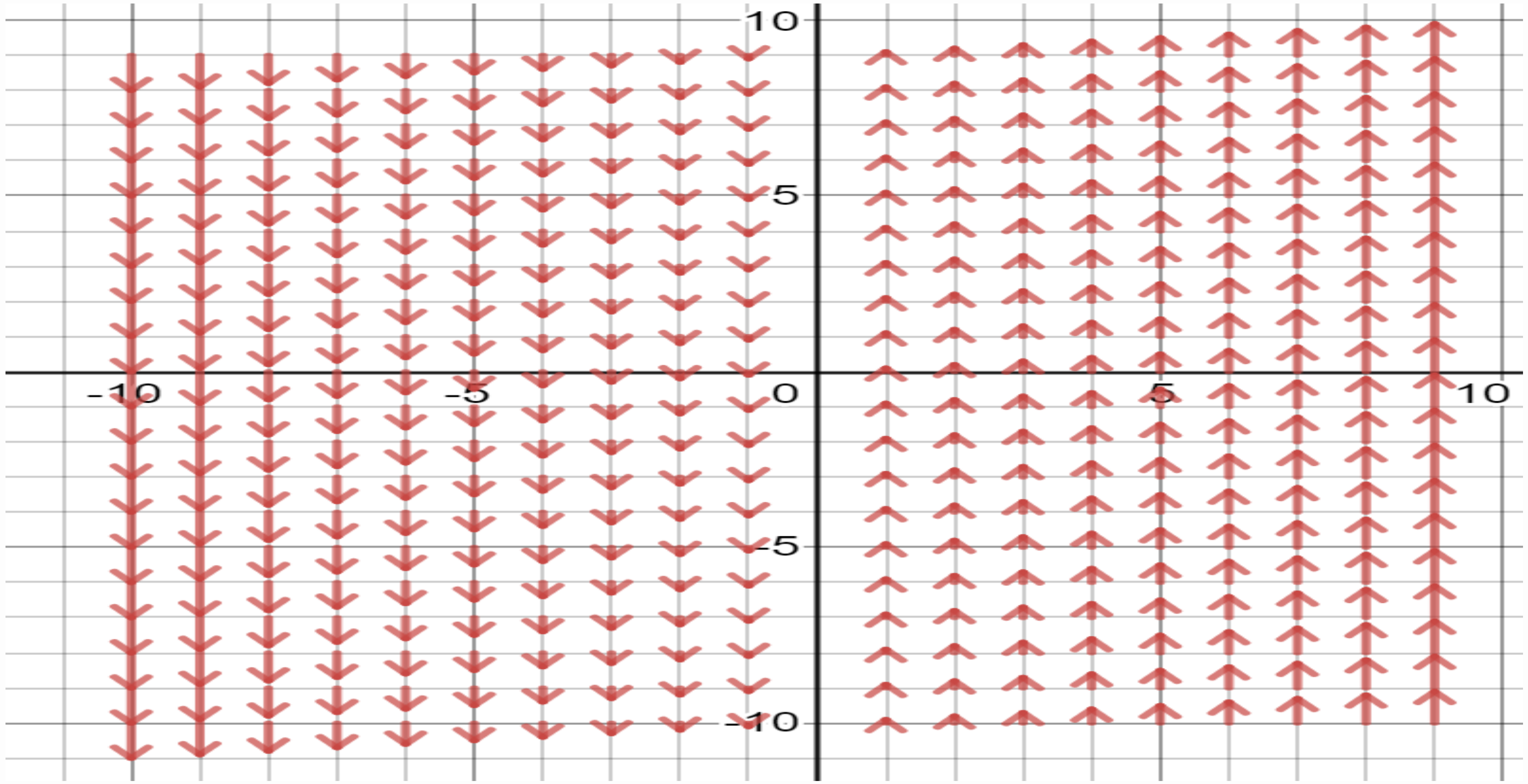


Example: $\vec{v} = (\pm x, 0,0)$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & 0 & 0 \end{vmatrix} = \hat{\imath}.0 - \hat{\jmath}.0 + \hat{k}.0 = \vec{0}$$

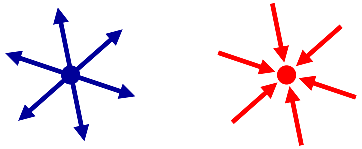
No sense of rotation. **IRROTATIONAL**

Example: $\vec{v} = (0, x, 0)$ $\nabla \times \vec{v} = \hat{k}$ Rotation is about an axis in the $z-$direction.



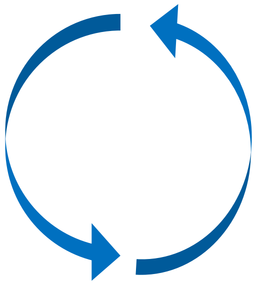
SUMMARY

- Divergence of $\vec{v}$: $\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} + \dfrac{\partial v_3}{\partial z}$



- Expansion or Compression

- curl of $\vec{v}$: $\operatorname{curl} \vec{v} = \nabla \times \vec{v}$



- Sence of Rotation

Lecture - 4

- Smooth and Piecewise Smooth Curves
- Simple Closed Curves
- Line Integrals

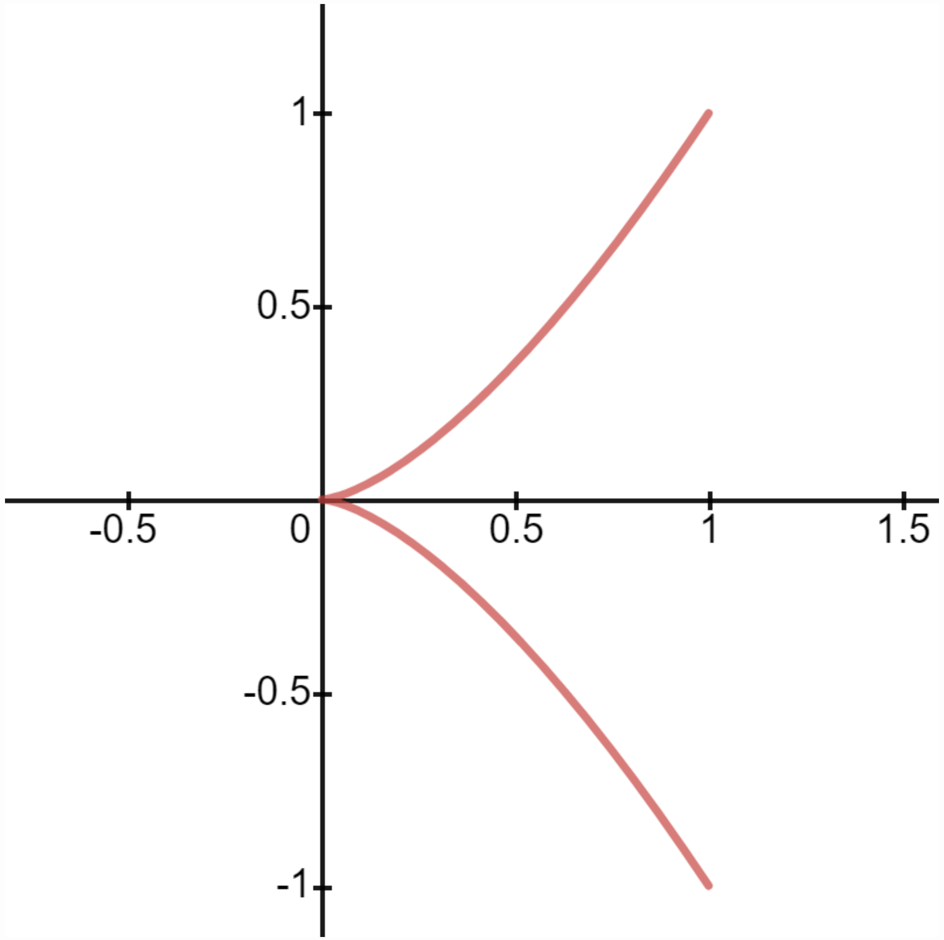
Smooth Curves : Let $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k},\ t \in [a, b]$ denote a curve in space.

If $\vec{r}(t)$ posses a continuous first order derivative (nowhere zero) for the given values of t then the curve is known as smooth.

In other words, the space curve $\vec{r}(t)$ is smooth when $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$ are continuous on $[a, b]$ and not simultaneously zero on (a, b)

Note that the condition **nowhere zero** ensures that the curve has no sharp corners or cusps.

Graph of $\vec{r}(t) = t^2\,\hat{\imath} + t^3\,\hat{\jmath}$



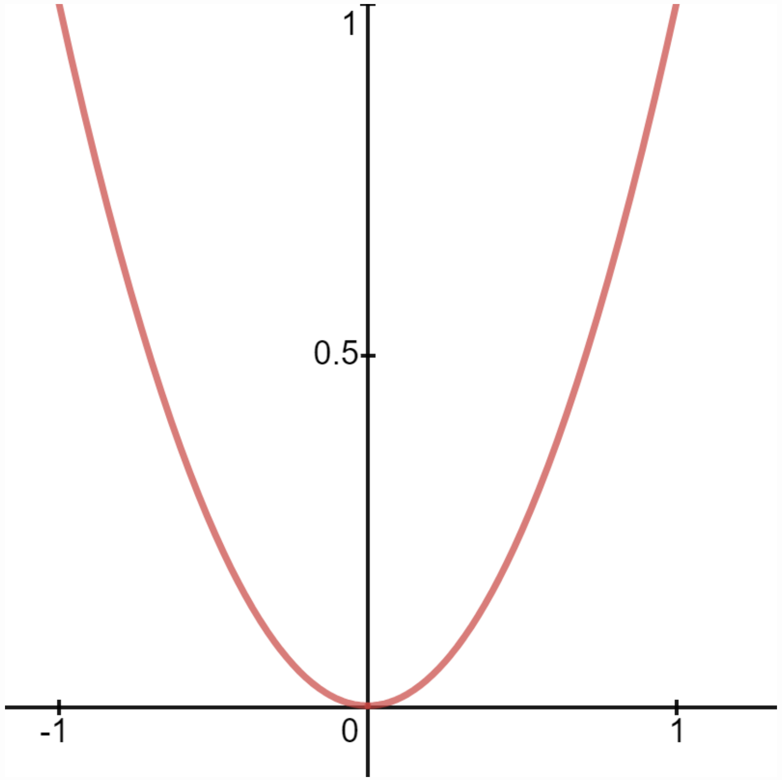
Consider $\vec{r}(t) = t^2\,\hat{\imath} + t^3\,\hat{\jmath},\ t \in [-1, 1]$

Compute $\dfrac{d\vec{r}(t)}{dt} = 2t\,\hat{\imath} + 3t^2\,\hat{\jmath}$

$$\implies \frac{d\vec{r}(t)}{dt} = 0 \ \text{ for } \ t = 0$$

(Indicate non-smoothness)

Graph of $\vec{r}(t) = t^3\,\hat{\imath} + t^6\,\hat{\jmath}$



Note that $\dfrac{d\vec{r}(t)}{dt} = 0$ does not necessarily implies non-smoothness.

However, $\dfrac{d\vec{r}(t)}{dt} \neq 0$ always implies smoothness.

Consider $\vec{r}(t) = t^3\,\hat{\imath} + t^6\,\hat{\jmath},\ t \in [-1, 1] \Longrightarrow \dfrac{d\vec{r}(t)}{dt} = 0 \text{ for } t = 0$

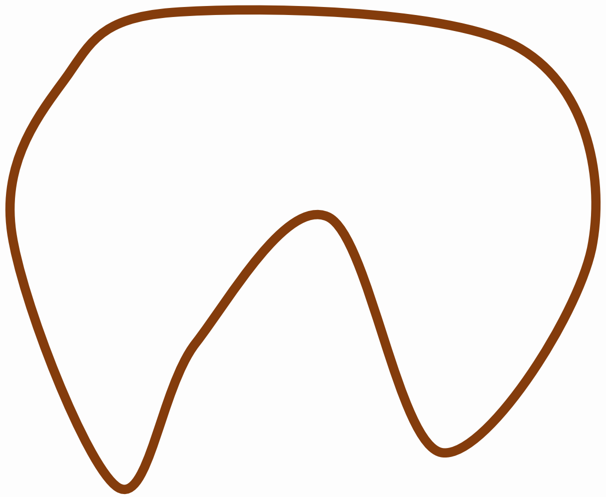
But the curve is smooth

Alternate parameterization: $\vec{r}(t) = t\,\hat{\imath} + t^2\,\hat{\jmath},\ t \in [-1, 1]$

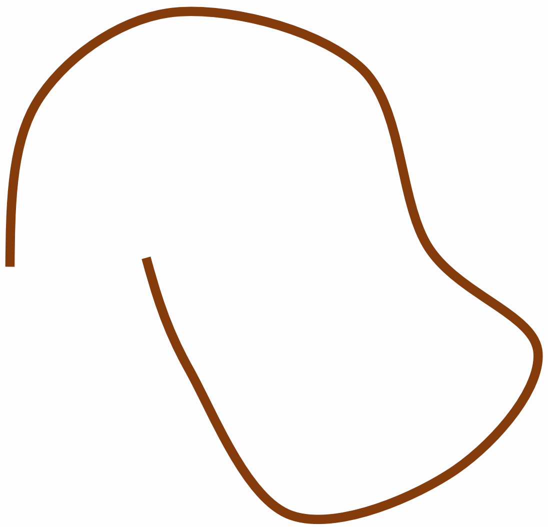
$$\Longrightarrow \frac{d\vec{r}(t)}{dt} \neq 0, \qquad \forall t$$

Piecewise Smooth Curve: If it is made up of a finite number of smooth curves.

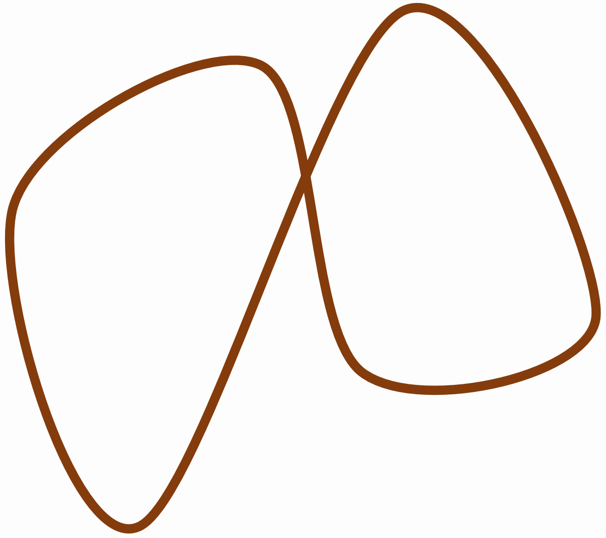
Simple Closed Curve : A curve which does not intersect itself anywhere and initial and end points are same is known as simple closed curve.



Simple closed curve



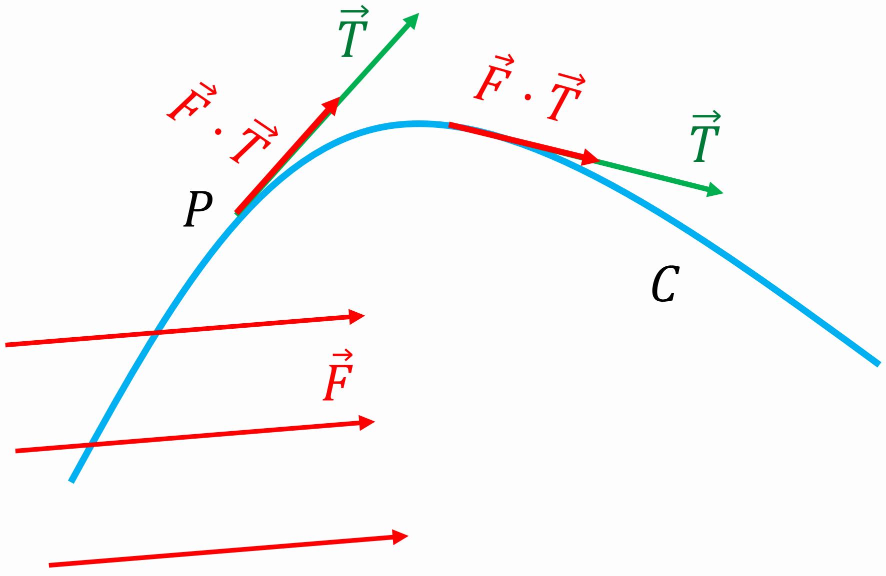
Simple but not closed curve



Closed but not simple

Line Integrals Let a force $\vec{F}$ act upon a particle which is displaced along a given curve C in space.

Let $\vec{T}$ be the unit tangent vector at the point $P(x_i, y_i, z_i)$.



On a small subarc of length Δs_i the work done is

$$\Delta w_i \approx \vec{F}(x_i, y_i, z_i) \cdot \vec{T}(x_i, y_i, z_i) \ \Delta s_i$$

Total work done: $$W = \lim_{N\to\infty} \sum_{i=1}^{N} \Delta w_i$$

$$= \int_C \vec{F} \cdot \vec{T} \ ds$$

Line Integrals Let the curve C be given by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Note that $\vec{T} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ and $ds = |\vec{r}'(t)|\, dt$

$$\vec{F} \cdot \vec{T}\ ds = \vec{F} \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|}\, |\vec{r}'(t)|\, dt = \vec{F} \cdot \vec{r}'(t)\ dt = \vec{F} \cdot d\vec{r}$$

$$W = \int_C \vec{F} \cdot \vec{T}\ ds = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}\big(x(t), y(t), z(t)\big) \cdot \vec{r}'(t)\ dt$$

Evaluation of Line Integrals $\int_C \vec{F} \cdot d\vec{r}$

In Vector form: Note that $\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, a \le t \le b$ and $d\vec{r} = \dfrac{d\vec{r}}{dt}dt$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \frac{d\vec{r}}{dt}dt}$$

In Component form: Suppose $\vec{F} = F_1(x,y,z)\,\hat{\imath} + F_2(x,y,z)\,\hat{\jmath} + F_3(x,y,z)\,\hat{k}$ and $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$

$$\Longrightarrow d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_C F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz}$$

Problem 1: Find the work done by $\vec{F} = (y - x^2)\,\hat{\imath} + (z - y^2)\,\hat{\jmath} + (x - z^2)\,\hat{k}$ over the curve

$\vec{r}(t) = t\hat{\imath} + t^2\hat{\jmath} + t^3\hat{k},\ \ 0 \le t \le 1$ from $(0,0,0)$ to $(1,1,1)$.

Solution: $$\frac{d\vec{r}}{dt} = \hat{\imath} + 2t\hat{\jmath} + 3t^2\hat{k}$$

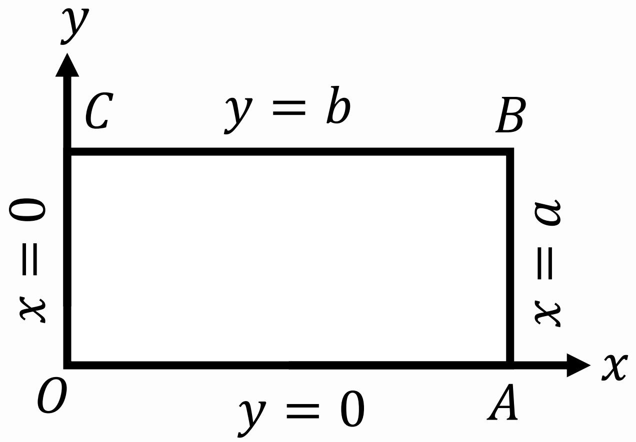
$$\vec{F}\big(\vec{r}(t)\big) = (t^2 - t^2)\,\hat{\imath} + (t^3 - t^4)\,\hat{\jmath} + (t - t^6)\,\hat{k} = (t^3 - t^4)\,\hat{\jmath} + (t - t^6)\,\hat{k}$$

$$\vec{F}\cdot\frac{d\vec{r}}{dt} = 2t(t^3 - t^4) + 3t^2(t - t^6) = 2t^4 - 2t^5 + 3t^3 - 3t^8$$

$$\int \vec{F}\cdot d\vec{r} = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\,dt = \frac{29}{60}$$

Problem 2: Evaluate $\int_C \vec{F} \cdot d\vec{r}$, $\quad \vec{F} = (x^2 + y^2)\,\hat{\imath} - 2xy\,\hat{\jmath}$

C : rectangle in xy plane bounded by $y = 0,\ x = a;\ \ y = b,\ x = 0.$



Solution: $\int_C \vec{F} \cdot d\vec{r} = \int_C (x^2 + y^2)\,dx - 2xy\,dy$

Along OA: $y = 0,\ dy = 0\ \&\ x$ varies from 0 to a.

$$\int \vec{F} \cdot d\vec{r} = \int_0^a x^2\,dx = \frac{a^3}{3}$$

Along AB: $x = a,\ dx = 0\ \&\ y$ varies from 0 to b: $\int \vec{F} \cdot d\vec{r} = \int_0^b -2ay\,dy = -ab^2$

Along BC: $\int \vec{F} \cdot d\vec{r} = \int_a^0 (x^2 + b^2)dx = -\left[\frac{a^3}{3} + ab^2\right]$

Along CO: $\int \vec{F} \cdot d\vec{r} = 0$

$$\int_C \vec{F} \cdot d\vec{r} = -2ab^2$$

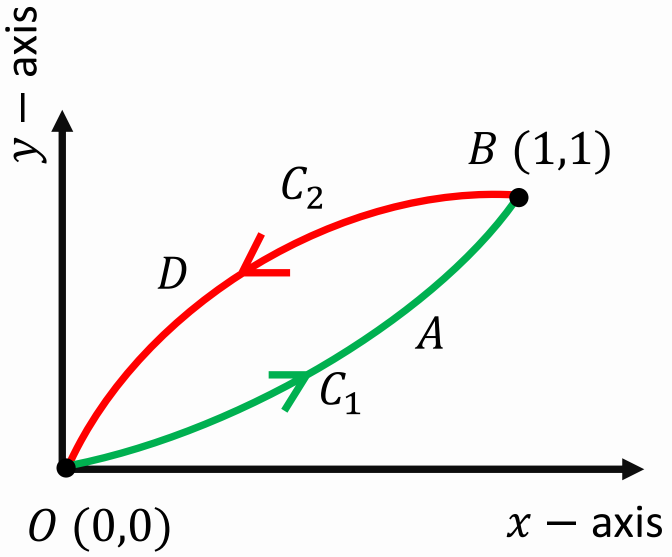
Line Integral as Circulation Let C be an oriented closed curve.

We call the line integral $\oint_C \vec{F} \cdot d\vec{r}$ the circulation of $\vec{F}$ around C.

Problem 3: Find the circulation of $\vec{F}$ around C where

$\vec{F} = (2x + y^2)\hat{\imath} + (3y - 4x)\hat{\jmath}$ and C is the curve

$y = x^2$ from $(0,0)$ to $(1,1)$ and the curve $y^2 = x$ from $(1,1)$ to $(0,0)$.



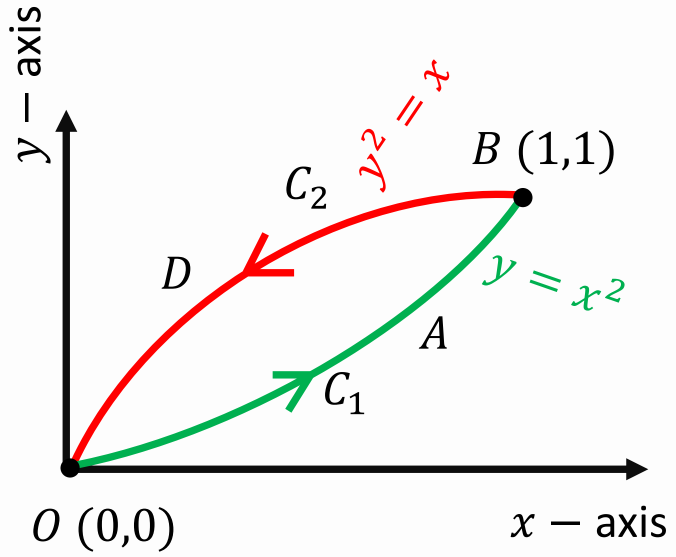
Solution: $\vec{F} \cdot d\vec{r} = (2x + y^2)dx + (3y - 4x)dy$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\vec{F}\cdot d\vec{r} = (2x+y^2)dx + (3y-4x)dy$$

Along OAB: $x^2 = y \Rightarrow 2x\,dx = dy$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 (2x+x^4)\,dx + \int_0^1 (3x^2-4x)\,2x\,dx = \frac{1}{30}$$



Along BDO: $x = y^2 \Rightarrow dx = 2ydy$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = -\int_0^1 (2y^2+y^2)2y\,dy - \int_0^1 (3y-4y^2)\,dy = -\frac{6}{4} - \frac{3}{2} + \frac{4}{3} = -\frac{5}{3}$$

$$\int_C \vec{F}\cdot d\vec{r} = \frac{1}{30} - \frac{5}{3} = -\frac{49}{30}$$

Problem 4: Evaluate $\oint_C \vec{F} \cdot d\vec{r}\,,\qquad \vec{F} = y\,\hat{\imath} - 2x\,\hat{\jmath},\qquad C: x^2 + y^2 = 9$

Solution: Parametric equation of the circle: $x = 3\cos t\,,\ \ y = 3\sin t\,,\qquad 0 \le t \le 2\pi$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_0^{2\pi} (-9\sin^2 t - 18\cos^2 t)\,dt = -9\oint_0^{2\pi} (\sin^2 t + 2\cos^2 t)\,dt$$

$$= -9\oint_0^{2\pi} (1 + \cos^2 t)\,dt \ = -9\oint_0^{2\pi} \left(1 + \frac{1}{2}(1 + \cos 2t)\right) dt$$

$$= -9\left(\frac{3}{2}\,2\pi + 0\right) = -27\,\pi$$

SUMMARY

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be a continuous vector field on a smooth curve C given by

$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

The line integral of $\vec{F}$ on C is given by

$$\int_C \vec{F} \cdot d\vec{r} = \int_C F_1(x, y, z)\, dx + F_2(x, y, z)\, dy + F_3(x, y, z)\, dz$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \frac{d\vec{r}}{dt}\, dt$$

Lecture - 5

- Conservative Field
- Independence of Path

Conservative Vector Field

A vector field $\vec{V}$ is said to be conservative if the vector function can be written as the gradient of a scalar function f, i.e., $\vec{V} = \nabla f$.

The function f is called a potential function or a potential of $\vec{V}$.

Example : Show that the vector field $\vec{F} = (2x + y, x, 2z)$ is conservative.

$\vec{F}$ is conservative if it can be written as $\vec{F} = \nabla f$

$$\Rightarrow \frac{\partial f}{\partial x} = 2x + y, \qquad \frac{\partial f}{\partial y} = x, \qquad \frac{\partial f}{\partial z} = 2z$$

$$\frac{\partial f}{\partial x} = 2x + y, \qquad \frac{\partial f}{\partial y} = x, \qquad \frac{\partial f}{\partial z} = 2z$$

$$\Rightarrow \frac{\partial f}{\partial x} = 2x + y \;\; \Rightarrow f = x^2 + xy + h(y, z) \;\; \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial h}{\partial y}$$

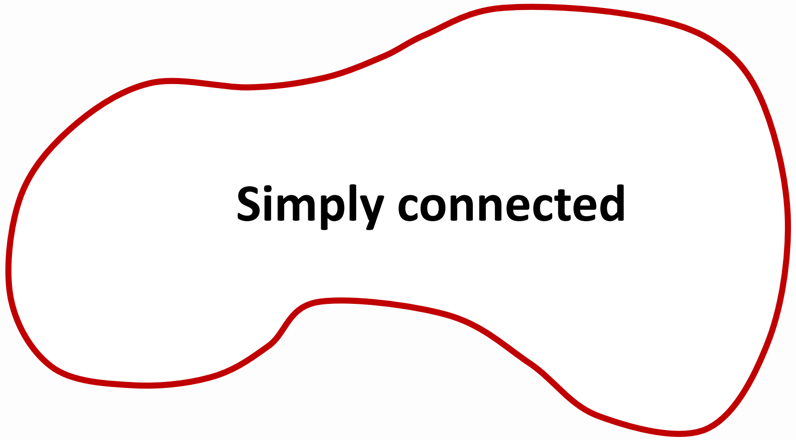
$\Rightarrow x = x + \frac{\partial h}{\partial y} \;\; \Rightarrow \frac{\partial h}{\partial y} = 0 \;\; \Rightarrow h$ is independent of y, i.e., $h = h(z)$

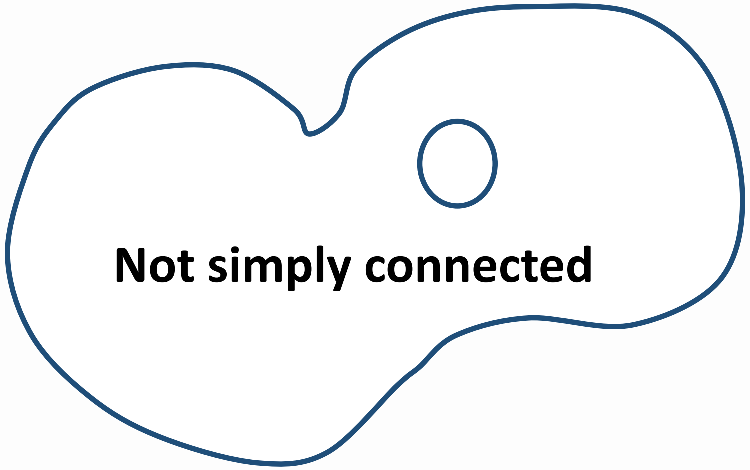
Using the last equation $2z = 0 + \frac{dh}{dz} \;\; \Rightarrow h = z^2 + c$

$$\Rightarrow f = x^2 + xy + z^2 + c$$

Simply Connected domain

A domain D (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is simply connected if it consists of a single connected piece and if every simple, closed curve C in D can be continuously shrunk to a point while remaining in D throughout the deformation.





Test for Conservative Field

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be a vector field whose components have continuous first order partial derivatives in a simply connected domain D.

$\vec{F}$ is conservative if and only if $\nabla \times \vec{F} = 0$ at all points of D

Equivalently, $\vec{F}$ is conservative if and only if

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \quad \& \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \quad \& \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Proof : (conservative $\Longrightarrow \nabla \times \vec{F} = 0$)

$$\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$$

$$= \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$ (assuming that $\vec{F}$ is conservative)

$$\Rightarrow \frac{\partial F_3}{\partial y} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$$ (partial derivatives are continuous)

$$= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial F_2}{\partial z}$$

$$\Rightarrow \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$

Similarly other relations can be proved.

Problem Show that $\vec{F} = (e^x \cos y + yz)\,\hat{\imath} + (xz - e^x \sin y)\,\hat{\jmath} + (xy + z)\,\hat{k}$ is conservative.

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

Solution $F_1 = (e^x \cos y + yz)$ $\quad F_2 = (xz - e^x \sin y)$ $\quad F_3 = (xy + z)$

$$\frac{\partial F_3}{\partial y} = x = \frac{\partial F_2}{\partial z}$$

$$\frac{\partial F_2}{\partial x} = z - e^x \sin y = \frac{\partial F_1}{\partial y}$$

$$\frac{\partial F_1}{\partial z} = y = \frac{\partial F_3}{\partial x}$$

$\Rightarrow \vec{F}$ is conservative that is $\vec{F} = \nabla f$

Path Independence

Let $\vec{F}$ be a vector field defined on a simply connected domain D and suppose that for any two points A and B in D the integral

$$\int_A^B \vec{F} \cdot d\vec{r}$$

is same over all paths from A to B in the domain D.

Then the integral $\int_A^B \vec{F} \cdot d\vec{r}$ is called path independent in D.

Independence of Path and Conservative Vector Fields

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be a vector field whose components are continuous throughout a simply connected region D in space. Then there exists a differentiable function f such that

$$\vec{F} = \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \qquad (\vec{F} \text{ is conservative in } D)$$

if and only if the integral $\int_A^B \vec{F}.d\vec{r}$ is independent of the path in D.

Proof $\vec{F} = \nabla f \Rightarrow$ Path Independence

Let the curve C be given by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

$$\int \vec{F}.d\vec{r} = \int_a^b \vec{F}\big(r(t)\big) \cdot \frac{d\vec{r}}{dt}dt$$

$$= \int_a^b \frac{df}{dt} \cdot dt = f(b) - f(a)$$

$$\Longrightarrow \boxed{\int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A)}$$

SUMMARY

A vector field $\vec{V}$ is said to be conservative $\vec{V} = \nabla f$.

Equivalent Conditions:

1. The field $\vec{F}$ is conservative.

2. $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D

3. $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve in D

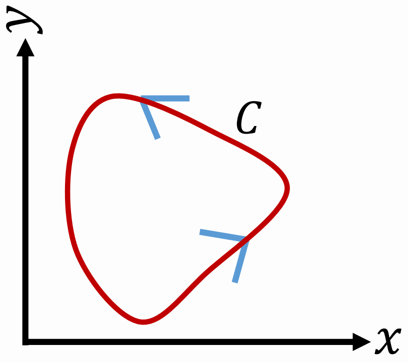
C be a piecewise smooth curve in a simply connected domain D.

Lecture - 6

- **Green's Theorem**

  - Transformation between double integrals and line integral

Green's theorem:



Let R be a region in $\mathbb{R}^2$ whose boundary is a simple closed curve C which is piecewise smooth (oriented counter clockwise – when traversed on C the region R always lies left).

Let $\vec{F} = F_1(x,y)\,\hat{\imath} + F_2(x,y)\hat{\jmath}$ be smooth vector field (F_1 & F_2 are C^1 functions) on both R and C, then

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

Note that the above can also be written as

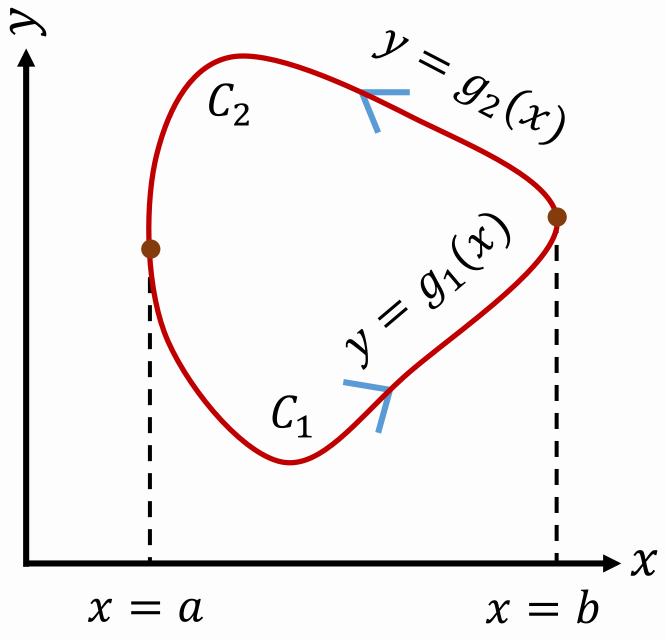
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\operatorname{curl}\vec{F}\right)\cdot\hat{k}\;\,dA$$

Proof: Let C be a simple smooth closed curve in xy plane with the property that lines parallel to axes cut in no more than two points.

$$C_1 : y = g_1(x), \qquad a \leq x \leq b$$

$$C = C_1 \cup C_2$$

$$C_2 : y = g_2(x), \qquad b \geq x \geq a$$

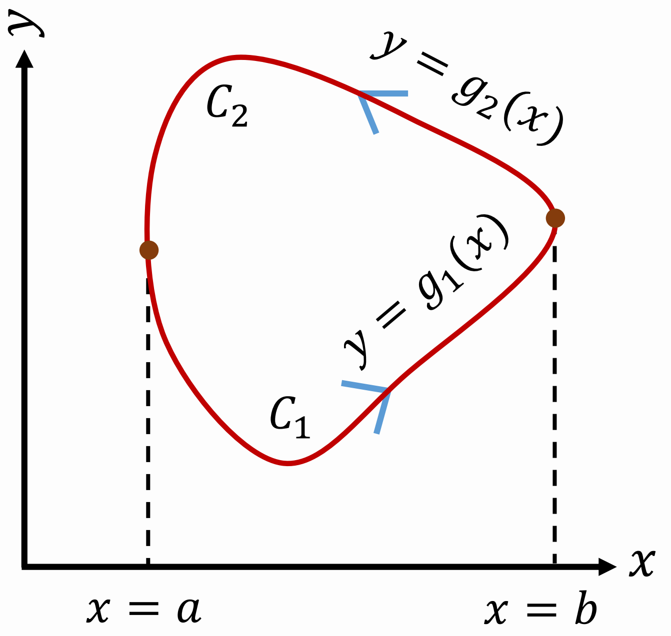


Integrate $\dfrac{\partial F_1}{\partial y}$ with respect to y from $y = g_1(x)$ to $y = g_2(x)$

$$\int_{g_1(x)}^{g_2(x)} \frac{\partial F_1}{\partial y} dy = F_1(x, y)\Big|_{g_1(x)}^{g_2(x)}$$

$$= F_1\,(x, g_2(x)) - F_1(x, g_1(x))$$

$$\int_{g_1(x)}^{g_2(x)} \frac{\partial F_1}{\partial y}\, dy = F_1\,(x, g_2(x)) - F_1(x, g_1(x))$$



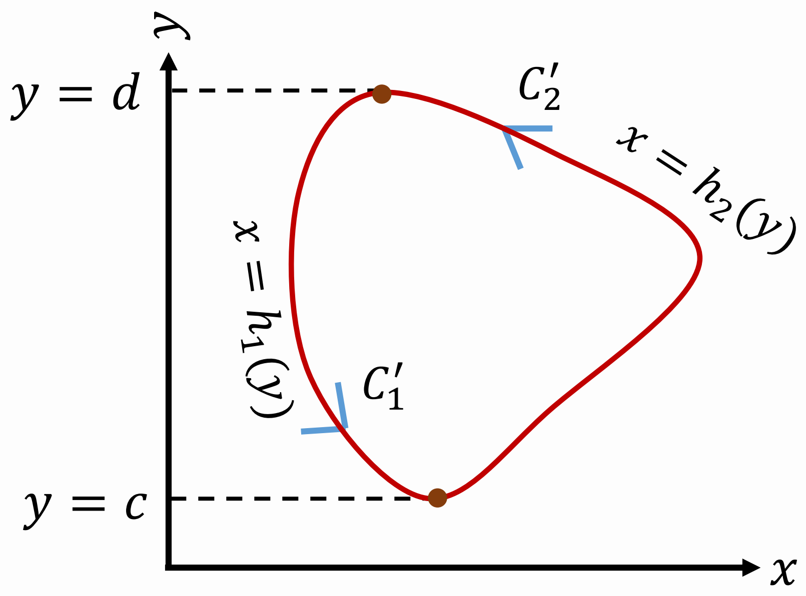
Now integrate with respect to x from a to b :

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial F_1}{\partial y}\, dy\, dx = \int_a^b F_1(x, g_2(x))dx - \int_a^b F_1(x, g_1(x))dx$$

$$= -\int_b^a F_1(x, g_2(x))dx - \int_a^b F_1(x, g_1(x))dx$$

$$= -\int_{C_2} F_1 dx - \int_{C_1} F_1 dx = -\oint_C F_1 dx$$

$$\Rightarrow \oint_C F_1 dx = \iint_R \left(-\frac{\partial F_1}{\partial y}\right) dA$$



$C_1{}': x = h_1(y) \quad d \leq y \leq c \qquad C_2{}': x = h_2(y) \quad \mathrm{c} \leq y \leq d$

Now integrating $\dfrac{\partial F_2}{\partial x}$ first with respect to x and then w.r.t. y :

$$\int_{h_1(y)}^{h_2(y)} \frac{\partial F_2}{\partial x}\, dx = F_2\,(h_2(y), y) - F_2(h_1(y), y)$$

Now integrate with respect to y from c to d:

$$\int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial F_2}{\partial x}\, dx\, dy = \int_c^d F_2(h_2(x), y) dy - \int_c^d F_2(h_1(x), y) dy = \oint_C F_2 dy$$

$$\Rightarrow \oint_C F_2\, dy = \iint_R \frac{\partial F_2}{\partial x}\, dA$$

We have

$$\oint_C F_2 dy = \iint_R \frac{\partial F_2}{\partial x} \, dA$$

$$\oint_C F_1 dx = \iint_R \left(-\frac{\partial F_1}{\partial y}\right) dA$$

$$\boxed{\oint_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA}$$

Problem -1 Verify Green's theorem for the vector field $\vec{F}(x,y) = (x-y)\hat{\imath} + x\,\hat{\jmath}$

The region R is bounded by the circle $C: \vec{r}(t) = \cos t\ \hat{\imath} + \sin t\ \hat{\jmath} \qquad 0 \le t \le 2\pi$

Solution: $F_1 = x - y \implies \dfrac{\partial F_1}{\partial y} = -1 \qquad F_2 = x \implies \dfrac{\partial F_2}{\partial x} = 1$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = 2\iint_R dxdy = 2\pi$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left((\cos t - \sin t)\hat{\imath} + \cos t\,\hat{\jmath}\right)\cdot(-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath})\,dt$$

$$= \int_0^{2\pi} (-\cos t\ \sin t + \sin^2 t + \cos^2 t)\,dt = 2\pi - \frac{1}{2}\int_0^{2\pi} \sin 2t\,dt = 2\pi$$

Problem -2 Evaluate the integral $\oint_C xy\,dy - y^2\,dx$ using Green's theorem.

Here C is the square cut from the first quadrant by the lines $x = 1$ & $y = 1$.

Solution: $$\oint_C \underbrace{xy}_{F_2}\,dy - \underbrace{y^2}_{F_1}\,dx = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

$$= \int_0^1 \int_0^1 (y + 2y)\,dx\,dy$$

$$= \frac{3}{2}$$

Problem-3 Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C x\,dy - y\,dx$.

Solution: Green's theorem: $\frac{1}{2}\oint_C \underbrace{x}_{F_2}\,dy - \underbrace{y}_{F_1}\,dx = \frac{1}{2}\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$

$$= \frac{1}{2}\iint_R [1 - (-1)]dxdy$$

$$= \iint_R dx\,dy$$

$= \text{Area of } R$

Problem - 4 Using Green's theorem, find the area of the ellipse $x = a\cos\theta$, $y = b\sin\theta$

Solution: Using Green's theorem

$$\text{Area of ellipse} = \frac{1}{2}\oint_C xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(b\cos\theta)d\theta - (b\sin\theta)(-a\sin\theta)d\theta$$

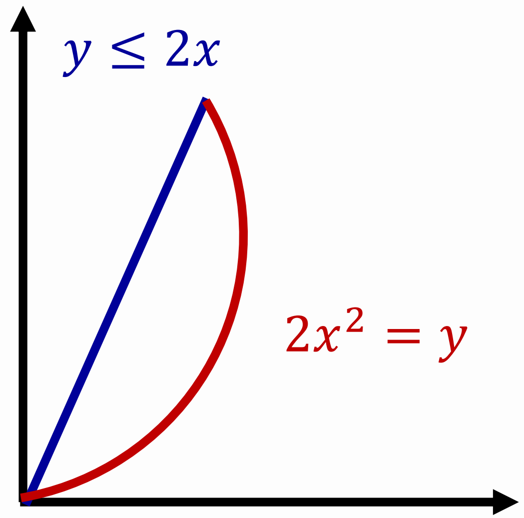
$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)d\theta$$

$$= \pi ab$$

Problem - 5 Evaluate $\oint_C (x^2+y^2)\,dx + 2xy\,dy$, C is the boundary of the region

$$R = \{(x,y): 0 \le x \le 1, 2x^2 \le y \le 2x\}$$

Solution: Using Green's theorem



$$\oint_C (x^2+y^2)\,dx + 2xy\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

$$= \frac{1}{2}\iint_R (2y - 2y)\,dxdy$$

$$= 0$$

Note: $(x^2+y^2)\,\hat{\imath} + 2xy\,\hat{\jmath} = \nabla\left(\frac{1}{3}x^3 + xy^2 + c\right)$ conservative vector field

NOTE: Consider $\vec{F}(x,y) = -\frac{y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$ $\quad R = \{(x,y)\colon\ 0 < x^2+y^2 \le 1\}$

$C\colon\ x = \cos\theta\,, y = \sin\theta$ $\quad \vec{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ $\quad \Rightarrow \frac{d\vec{r}}{d\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$

$$\oint_C \vec{F}.d\,\vec{r} = \int_{\theta=0}^{2\pi} (-\sin\theta)(-\sin\theta) + \cos\theta\cos\theta\ \ d\theta = 2\pi$$

Whereas: $$\iint_R \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right) dxdy$$

$$= \iint_R \left(\frac{(x^2+y^2) - x\,2x}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y\,2y}{(x^2+y^2)^2}\right) dxdy = 0$$

Does it contradict Green's theorem?

SUMMARY

- GREEN's THEOREM

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\mathrm{curl}\,\vec{F}\right)\cdot \hat{k}\,dxdy$$

Lecture - 7

- Smooth Surfaces
- Evaluation of Surface Area
- Surface Integral of a Scalar Function

Smooth Surface

Recall that a curve is called smooth if it has a continuous tangent.

Similarly, a surface is smooth if it has a continuous normal vector.

A surface is called piecewise smooth if it consists of finite number of smooth surfaces.

Example: Surface of a Sphere - a smooth surface

Surface of a cube - a piecewise smooth surface

Does not have a normal vector along any of its edges

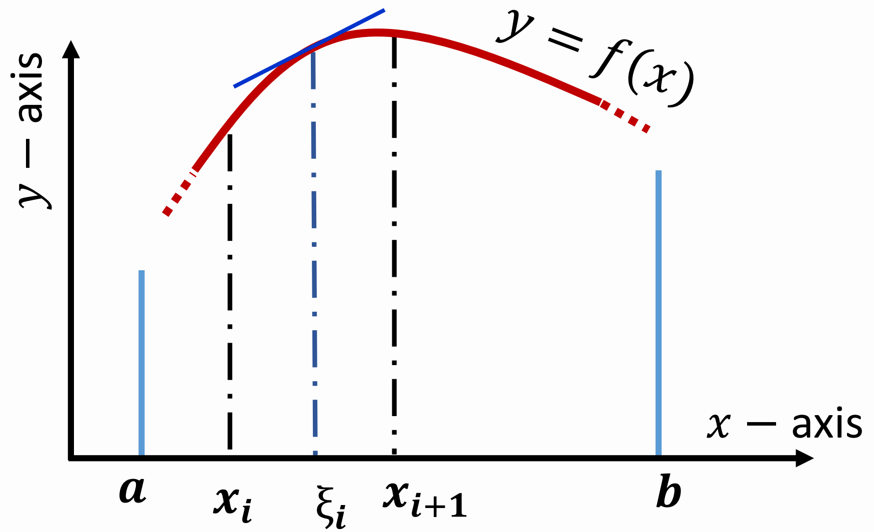
Evaluation of Arc Length (Recall from Integral Calculus)

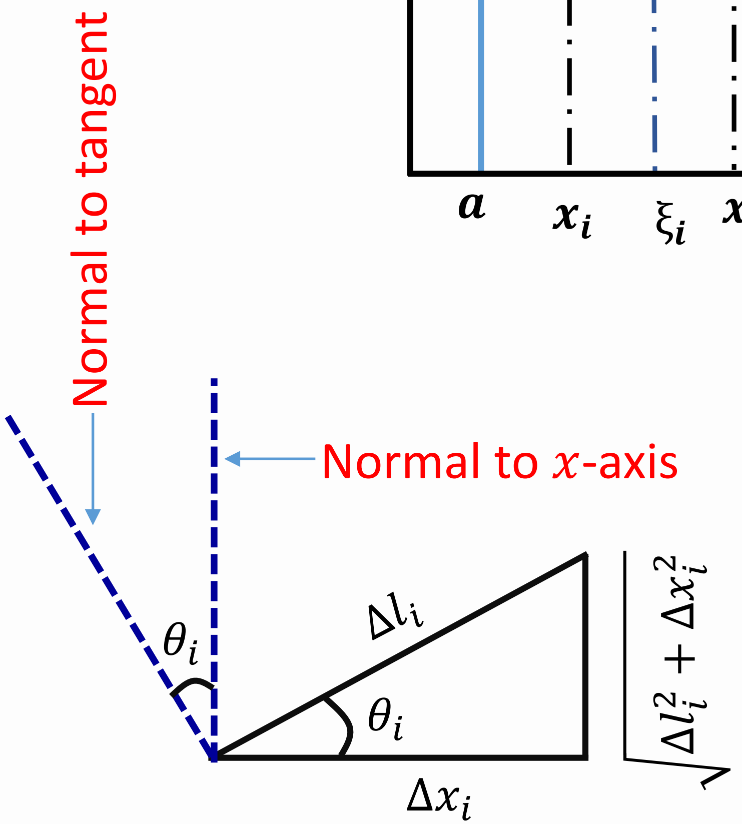
Let θ be the angle of the tangent at ξ_i with the positive x axis

$$\frac{\Delta x_i}{\Delta l_i} = |\cos\theta_i| \Rightarrow \Delta l_i = \frac{1}{|\cos\theta_i|}\Delta x_i$$

Alternatively $\quad f'(\xi_i) = \dfrac{\sqrt{\Delta l_i^2 + \Delta x_i^2}}{\Delta x_i}$

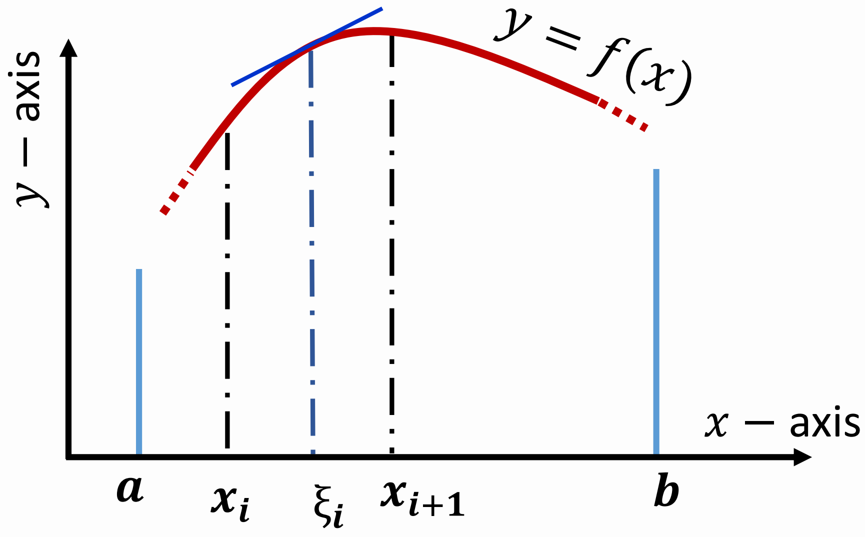
$$\Longrightarrow \Delta l_i = \sqrt{1 + \left(f'(\xi_i)\right)^2}\,\Delta x_i$$





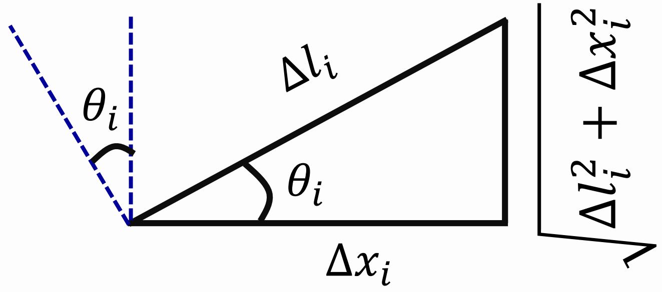
Evaluation of Arc Length (Recall from Integral Calculus)

$$\Delta l_i = \frac{1}{|\cos \theta_i|} \Delta x_i \qquad \Delta l_i = \sqrt{1 + \left(f'(\xi_i)\right)^2}\, \Delta x_i$$



Arc length $L = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta l_i = \int_C dl = \int_a^b \frac{1}{|\cos \theta|} dx$

$$= \int_a^b \sqrt{1 + \left((f'(x)\right)^2}\; dx$$



Arc length differential $\quad dl = \frac{1}{|\cos \theta|} dx = \sqrt{1 + f'^2}\; dx$

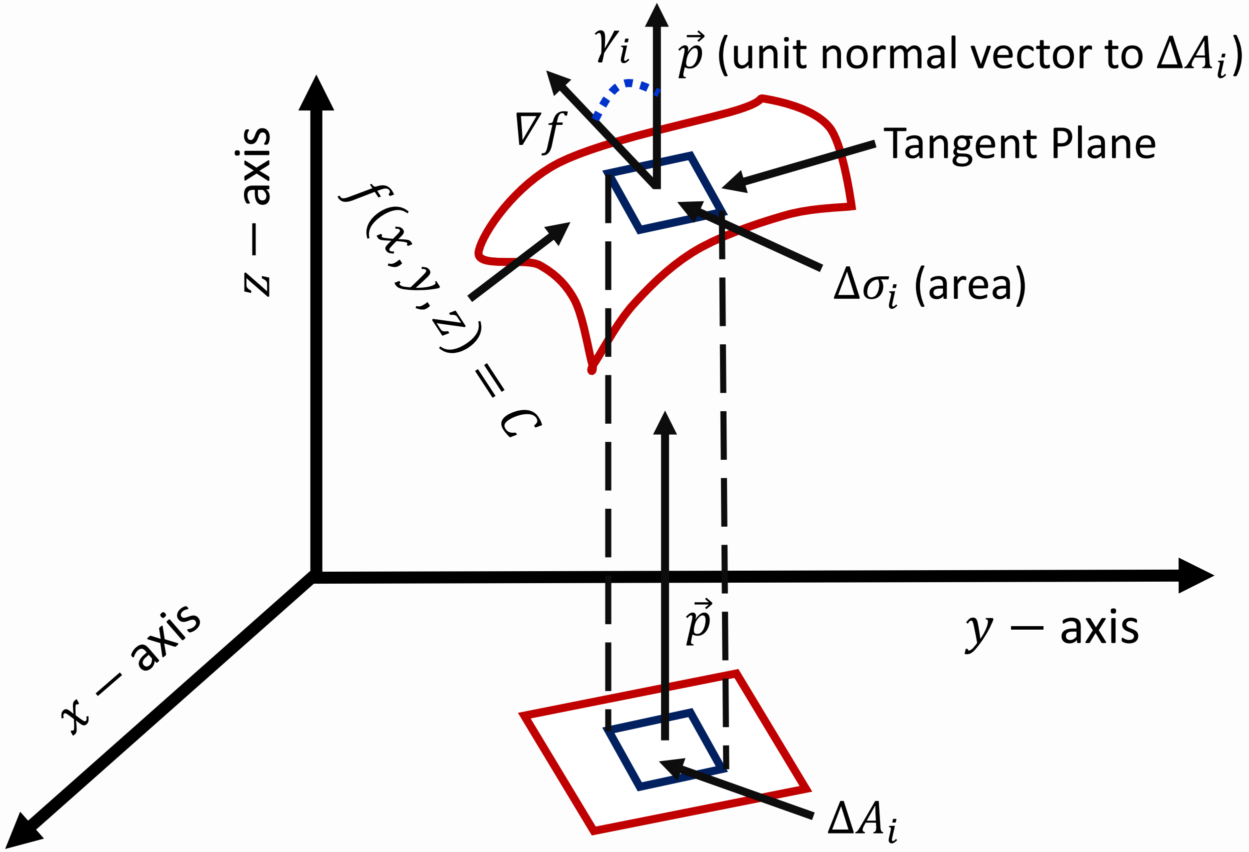
Evaluation of Surface Area

$$\frac{\Delta A_i}{\Delta \sigma_i} = |\cos \gamma_i| \Rightarrow \Delta \sigma_i = \frac{1}{|\cos \gamma_i|} \Delta A_i$$

Surface Area: $S = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta \sigma_i$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{|\cos \gamma_i|} \Delta A_i$$

$$= \iint_R \frac{1}{|\cos \gamma|} dA$$

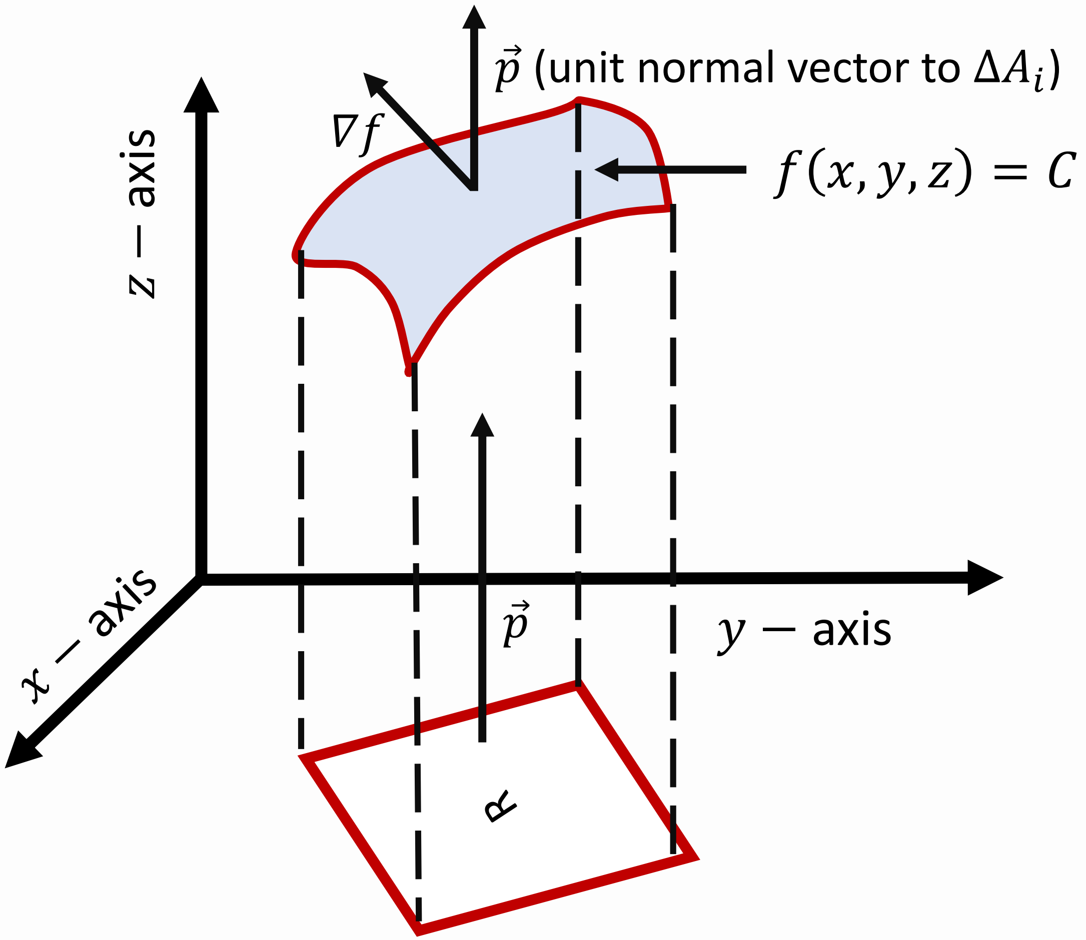


R is the projection of the surface on the xy, yz or zx plane.

$$S = \iint_R \frac{1}{|\cos\gamma|}\, dA$$

Note that : $|\nabla f . \vec{p}| = |\nabla f||\vec{p}||\cos\gamma|$

$$\Rightarrow \frac{1}{|\cos\gamma|} = \frac{|\nabla f|}{|\nabla f . \vec{p}|}$$



The **area of the surface** $f(x, y, z) = C$ **over a closed and bounded plane** $\boldsymbol{R}$:

$$\boxed{S = \iint_S d\sigma = \iint_R \frac{|\nabla f|}{|\nabla f . \vec{p}|}\, dA}$$

R is the projection of S on on the xy, yz or zx plane

$\vec{p}$ is the unit normal to R and $\nabla f . \vec{p} \neq 0$

REMARK: **Recall from Integral Calculus:**

Let $z = g(x, y)$ be the equation of a surface.

Then the surface area (Integral Calculus): $S = \iint\limits_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dx\,dy$

where R is the projection of the surface in the xy plane

In the vector form the same can be calculated using $S = \iint\limits_R \frac{|\nabla f|}{|\nabla f.\vec{p}|} dA$

Let $f = z - g(x, y) = 0 \Rightarrow \nabla f = (-g_x, -g_y, 1)$

$\Rightarrow |\nabla f| = \sqrt{1 + z_x^2 + z_y^2}$ $|\nabla f.\vec{p}| = 1$ (considering $\vec{p}$ as the unit normal to xy plane)

Surface Integral: $\iint\limits_S g(x,y,z)\,d\sigma$

Integrating a function over surface using the idea just developed for calculating surface area.

Suppose, for example, we have electrical charge distribution over the surface $f(x,y,z) = C$

Let the function $g(x,y,z)$ gives the change per unit area (charge density) at each point on S

Total charge on S $= \iint\limits_S g(x,y,z)\,d\sigma = \iint\limits_R g(x,y,z)\dfrac{|\nabla f|}{|\nabla f.\vec{p}|}dA$ **Surface integral of g over S**

NOTE:

- if g gives the mass density of a thin shell of material modeled by S, the integral gives the mass of the shell.
- if $g = 1$ then the integral will simply gives the total area of the surface

Problem - 1 Find the area of the cap cut from the hemisphere $x^2 + y^2 + z^2 = 2$, $z \geq 0$ by the cylinder $x^2 + y^2 = 1$

Solution: Projection of the surface $f(x, y, z) = c$, i.e., $x^2 + y^2 + z^2 = 2$ onto the xy plane : $x^2 + y^2 \leq 1$

Note that $f(x, y, z) = x^2 + y^2 + z^2$

$$\implies \nabla f = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

$$\implies |\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{2}$$

The vector $\vec{p} = \hat{k}$ is normal to the xy plane $\implies |\nabla f.\vec{p}| = |2z| = 2z \quad (\because z \geq 0)$

$x^2 + y^2 + z^2 = 2\,, z \geq 0$

$|\nabla f| = 2\sqrt{2}$

$|\nabla f . \vec{p}| = 2z$

Surface Area: $S = \iint\limits_R \frac{|\nabla f|}{|\nabla f . \vec{p}|} dA = \iint\limits_R \frac{2\sqrt{2}}{2z}\, dA = \sqrt{2} \iint\limits_R \frac{dA}{z}$

$$= \sqrt{2} \iint\limits_R \frac{dA}{\sqrt{2 - (x^2 + y^2)}}$$

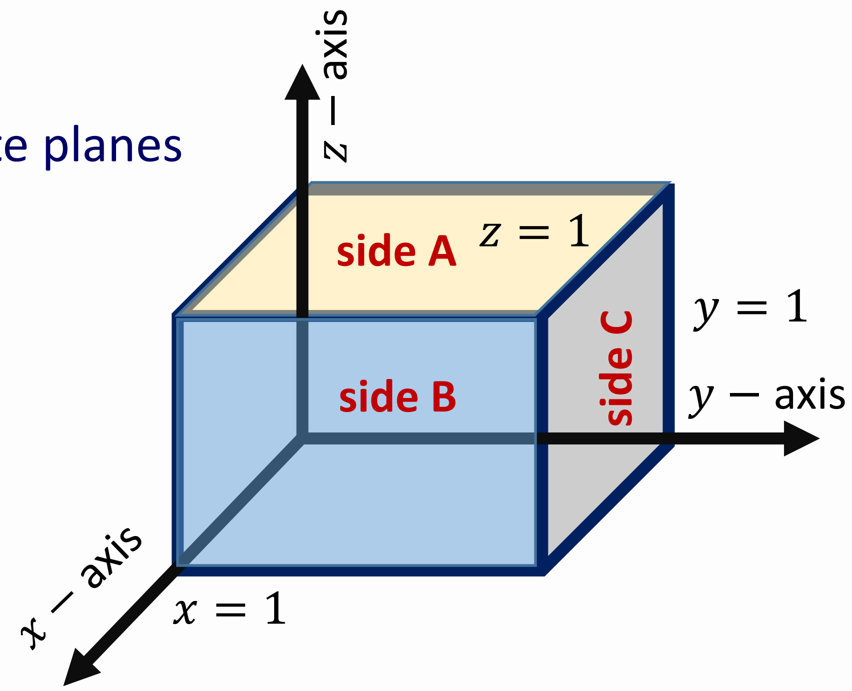
$$= \sqrt{2} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{r\, dr d\theta}{\sqrt{2 - r^2}}$$

$$= \sqrt{2} \int_0^{2\pi} \left[-\sqrt{(2 - r^2)}\right]_{r=0}^{1} d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \left(\sqrt{2} - 1\right) d\theta = 2\pi(2 - \sqrt{2})$$

Problem-2 Integrate $g(x,y,z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1, y = 1$ and $z = 1$

Solution: Note that $xyz = 0$ on the sides that lie in the coordinate planes



The integral over the surface of the cube reduces to

$$\iint\limits_{\substack{\text{cube}\\\text{surface}}} xyz\, d\sigma = \iint\limits_{\text{side A}} xyz\, d\sigma + \iint\limits_{\text{side B}} xyz\, d\sigma + \iint\limits_{\text{side C}} xyz\, d\sigma$$

side A is the surface $f(x,y,z) = z - 1$ over the region

$\mathbb{R}_{xy}: 0 \leq x \leq 1, 0 \leq y \leq 1$ in the xy plane

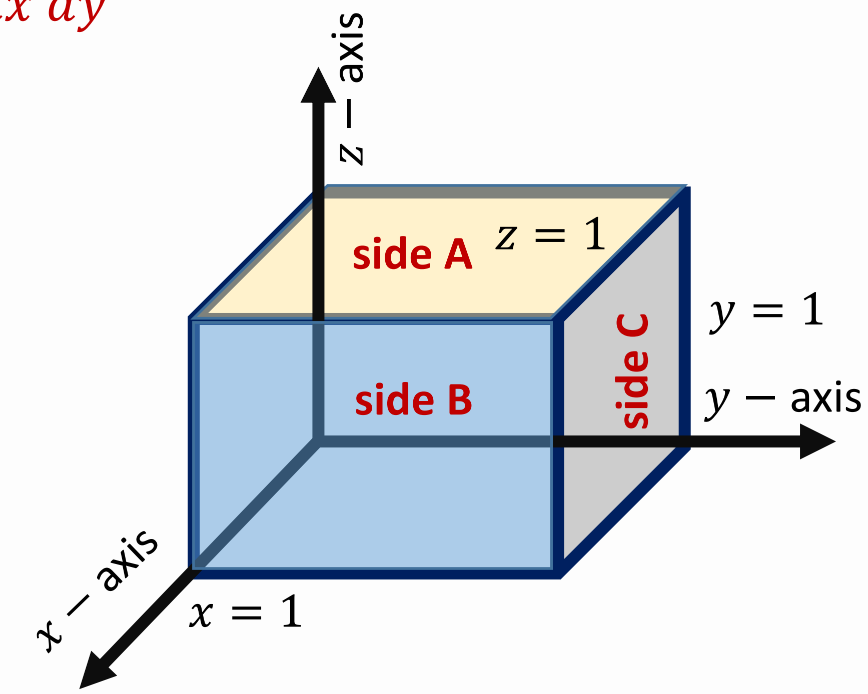
For this surface (side A) and region $\mathbb{R}_{xy}$:

$$\vec{p} = \hat{k}, \nabla f = \hat{k} \Longrightarrow |\nabla f| = 1 \;\;\&\;\; |\nabla f.\vec{p}| = 1$$

$$\Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f.\vec{p}|} dA = dxdy$$

$$\iint_{\text{side A}} xyz\, d\sigma = \int_0^1 \int_0^1 xy(1)\; dxdy = \frac{1}{4}$$

$$d\sigma = dx\, dy$$



Similarly, we obtain

$$\iint_{\text{side B}} xyz\, d\sigma = \frac{1}{4} \qquad \iint_{\text{side C}} xyz\, d\sigma = \frac{1}{4}$$

$$\iint_{\substack{\text{cube} \\ \text{surface}}} xyz\, d\sigma = 3 \times \frac{1}{4} = \frac{3}{4}$$

SUMMARY

- Surface $z = g(x, y)$

$$S = \iint\limits_{R} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dx\, dy$$

- Surface $f = z - g(x, y) = 0$

$$S = \iint\limits_{R} \frac{|\nabla f|}{|\nabla f . \vec{p}|}\ dA$$

$$\boxed{\iint\limits_{S} g(x, y, z)\ d\sigma = \iint\limits_{R} g(x, y, z)\ \frac{|\nabla f|}{|\nabla f . \vec{p}|} dA}$$

Lecture - 8

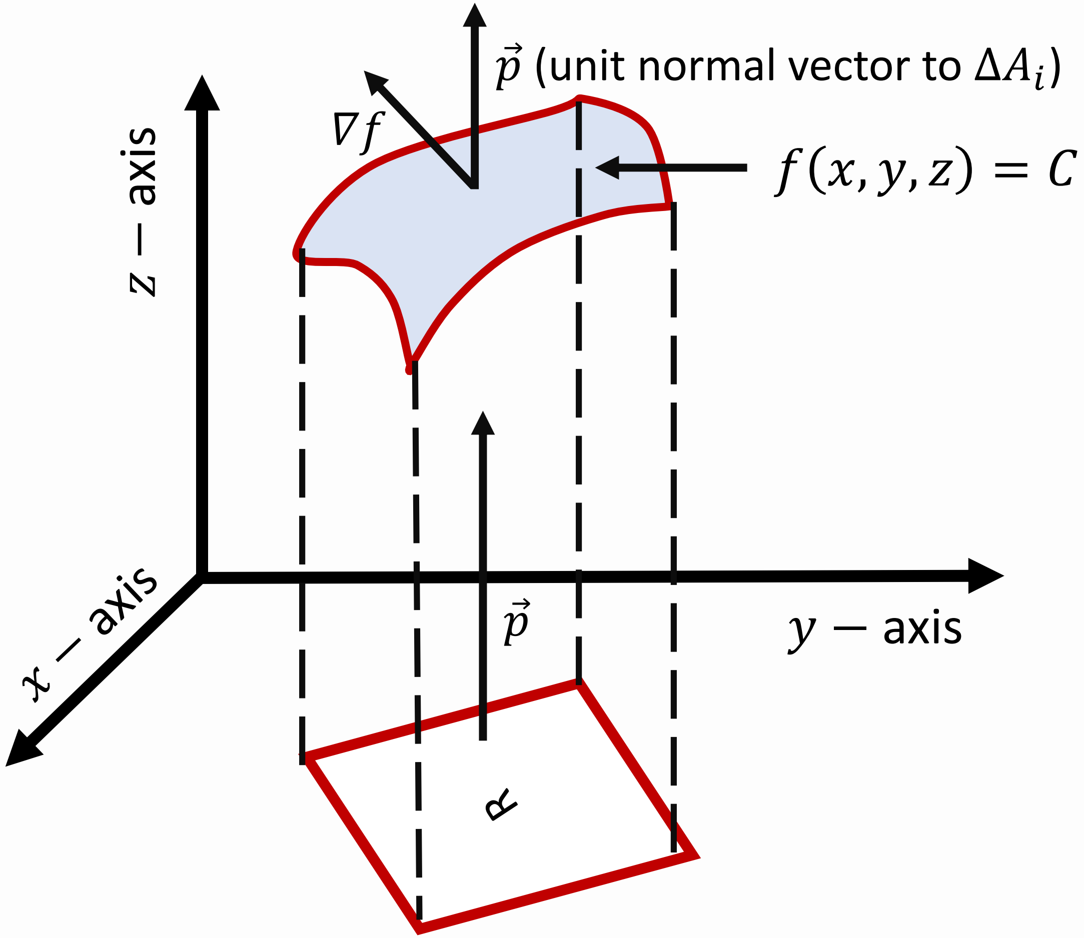
- Orientable Surfaces
- Flux Integrals

Surface integral of g over S

$$\iint_S g(x,y,z)\,d\sigma = \iint_R g(x,y,z)\,\frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\,dA$$

R is the projection of S on on the xy, yz or zx plane

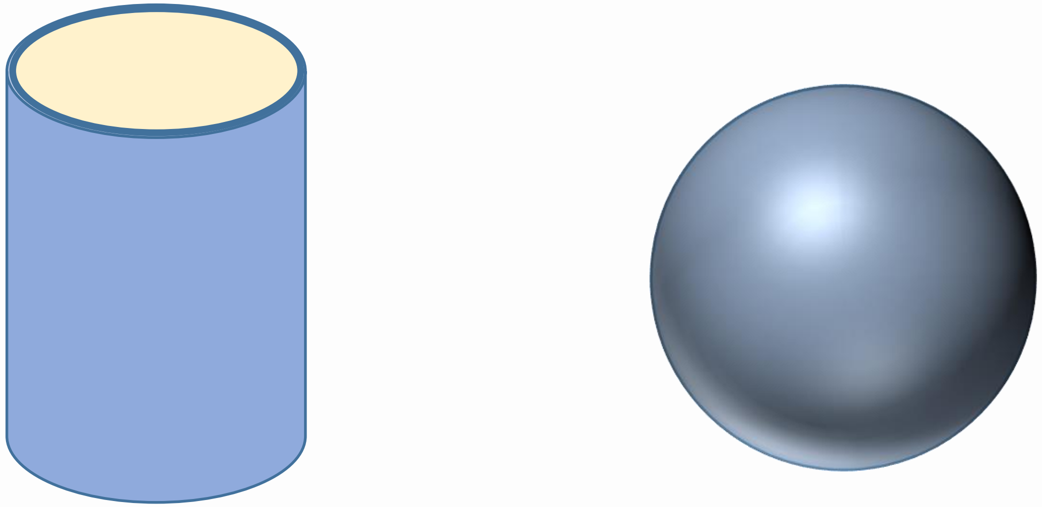
$\vec{p}$ is the unit normal to R and $\nabla f \cdot \vec{p} \neq 0$



Orientable Surface

S is an orientable surface if it has two sides which may be painted in two different colors.

Non-Orientable Surface



Orientable Surfaces

Flux of a vector field $\vec{F}$ through a surface S

The flux of a vector field $\vec{F}$ across an orientable surface S in the direction of $\vec{n}$ (unit normal to S) is given by the integral

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n} \, d\sigma$$

Geometrically, a flux integral is the surface integral over S of the normal component of $\vec{F}$.

If $\vec{F}$ is the continuous velocity field of a fluid and $\rho(x, y, z)$ is the density of the fluid at (x, y, z) then the flux integral

$$\iint_S \rho \ \vec{F} \cdot \vec{n} \ d\sigma$$

represents the mass of the fluid flowing across S per unit of time.

Evaluation of Flux Integral $\iint\limits_S \vec{F} \cdot \vec{n}\, d\sigma$

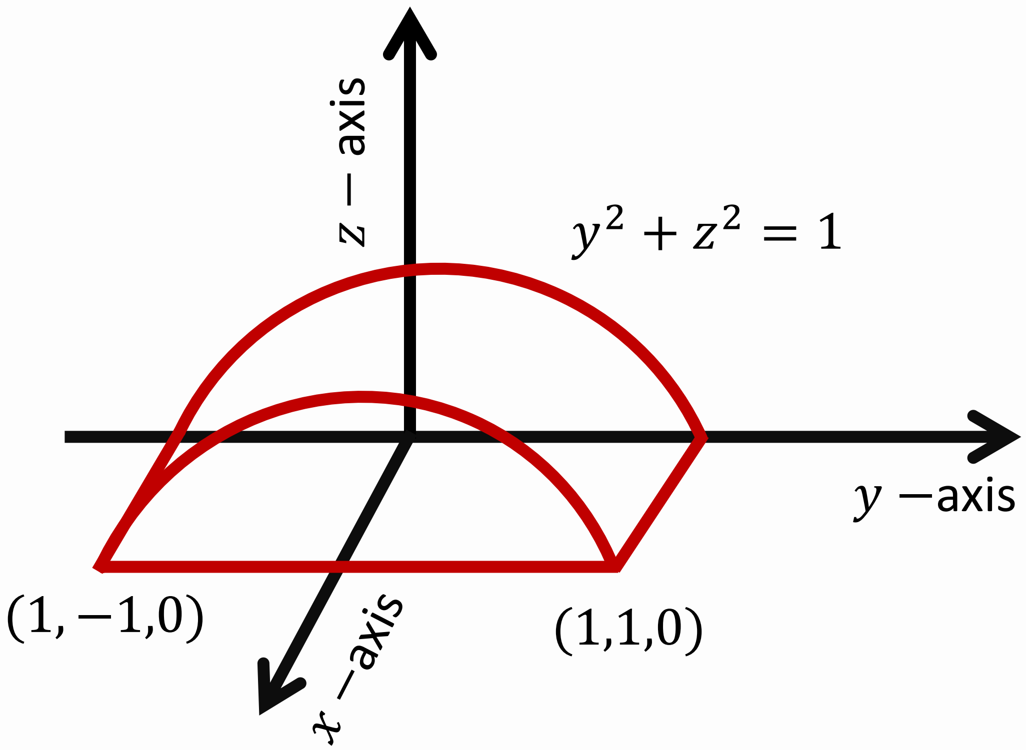
Suppose S is a part of a level surface $f(x, y, z) = C$, then $\vec{n}$ may be taken either of the two unit vectors

$$\vec{n} = \pm \frac{\nabla f}{|\nabla f|}$$

$$\text{Flux} = \pm \iint\limits_R \vec{F} \cdot \left(\frac{\nabla f}{|\nabla f|} \right) \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\, dA$$

$$= \pm \iint\limits_R \vec{F} \cdot \frac{\nabla f}{|\nabla f \cdot \vec{p}|}\, dA$$

Problem-1 Find the flux of $\vec{F} = yz\hat{j} + z^2\hat{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1, z > 0$ by the planes $x = 0$ and $x = 1$.



Solution Surface $f(x, y, z) = C$

$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{2y\hat{j} + 2z\hat{k}}{\sqrt{4(y^2 + z^2)}} = y\hat{j} + z\hat{k} \qquad \vec{p} = \vec{k}$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} dA = \frac{|\nabla f|}{|\nabla f \cdot \vec{k}|} dA = \frac{2}{|2z|} dA = \frac{1}{z} dA$$

Also $\vec{F} \cdot \vec{n} = y^2 z + z^3 = z$

Flux through S: $\iint\limits_{S} \vec{F} \cdot \vec{n}\, d\sigma = \iint\limits_{R_{xy}} z \times \frac{1}{z}\, dA = \iint\limits_{R_{xy}} dA = 2$

Problem-2 Evaluate the integral

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma \quad \text{where} \qquad \vec{F} = 6z\,\hat{\imath} + 6\,\hat{\jmath} + 3y\,\hat{k}$$

and S is the portion of the plane $2x + 3y + 4z = 12$ which is in the first octant.

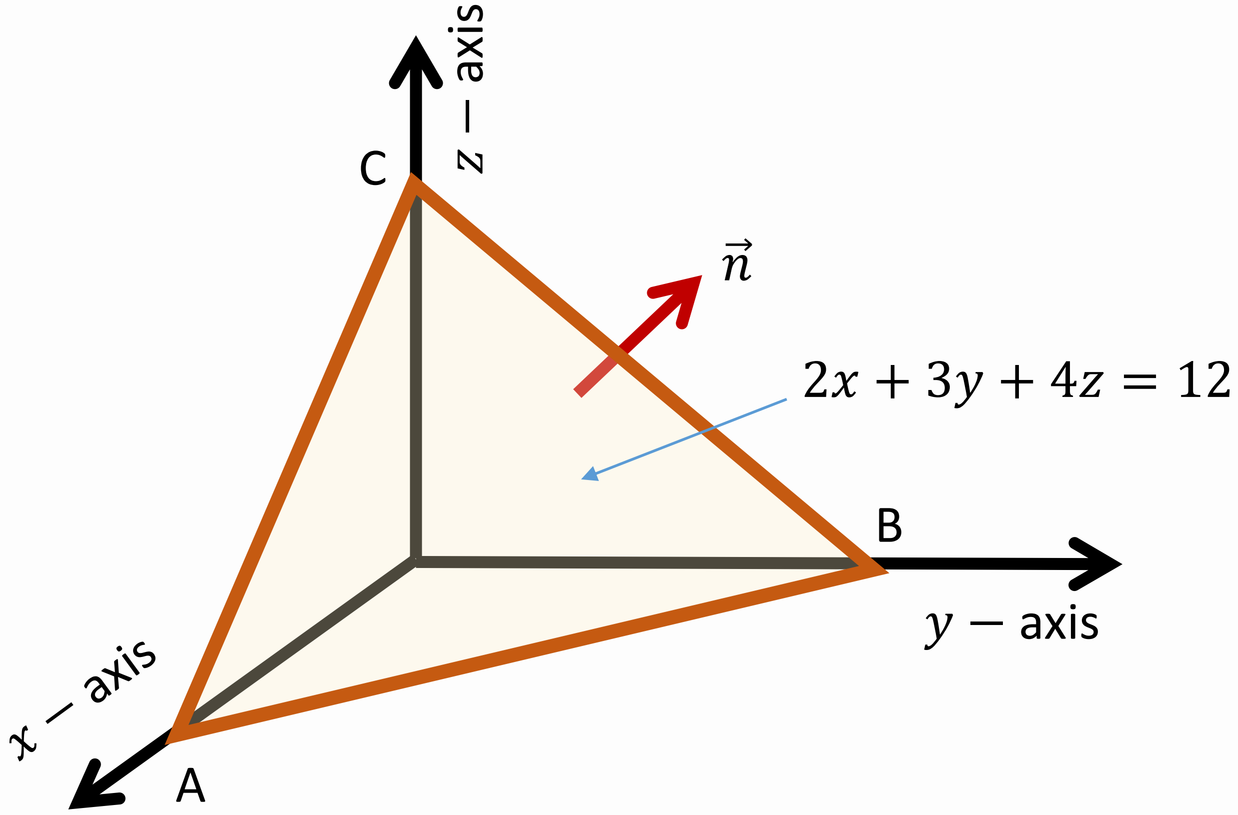
Solution Let $f(x,y,z) = 2x + 3y + 4z \quad \Rightarrow \nabla f = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$

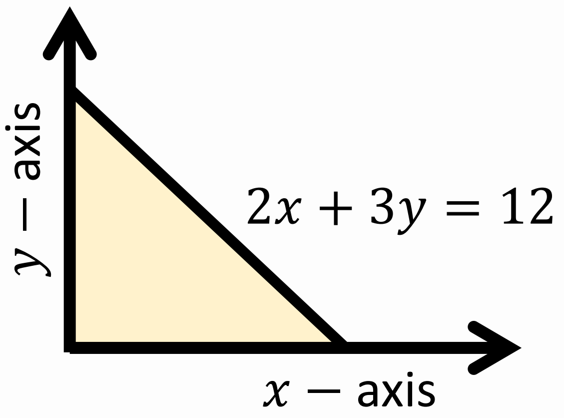
$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{29}}(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) \qquad\qquad \vec{F}\cdot\vec{n} = \frac{1}{\sqrt{29}}(12z + 18 + 12y)$$

$$d\sigma = \frac{|\nabla f|}{\left|\nabla f\cdot\vec{k}\,\right|}dA = \frac{\sqrt{29}}{4}\,dA \qquad\qquad (\vec{p} = \hat{k})$$

We are projecting of S on the xy plane.

The projection R is bounded by x-axis, y-axis and $2x + 3y = 12$





Note that $\vec{F}\cdot\vec{n} = \dfrac{1}{\sqrt{29}}(12z + 18 + 12y)$

Also given surface $2x + 3y + 4z = 12$

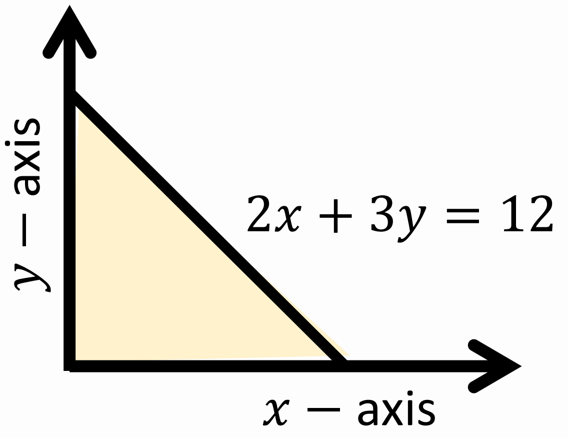
$$d\sigma = \frac{\sqrt{29}}{4}\, dA$$

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_R \frac{1}{\sqrt{29}}(3(12 - 2x - 3y) + 18 + 12y)\left(\frac{\sqrt{29}}{4}\right) dA$$

$$= \frac{1}{4}\iint_R (54 - 6x + 3y)\, dA$$

$$= \frac{1}{4}\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3} (54 - 6x + 3y)\, dy\, dx$$

$$= 138$$



Problem-3 Evaluate the surface integral $\iint_S \vec{F} \cdot \vec{n}\, d\sigma$ where $\vec{F} = z^2\,\hat{\imath} + xy\,\hat{\jmath} - y^2\,\hat{k}$

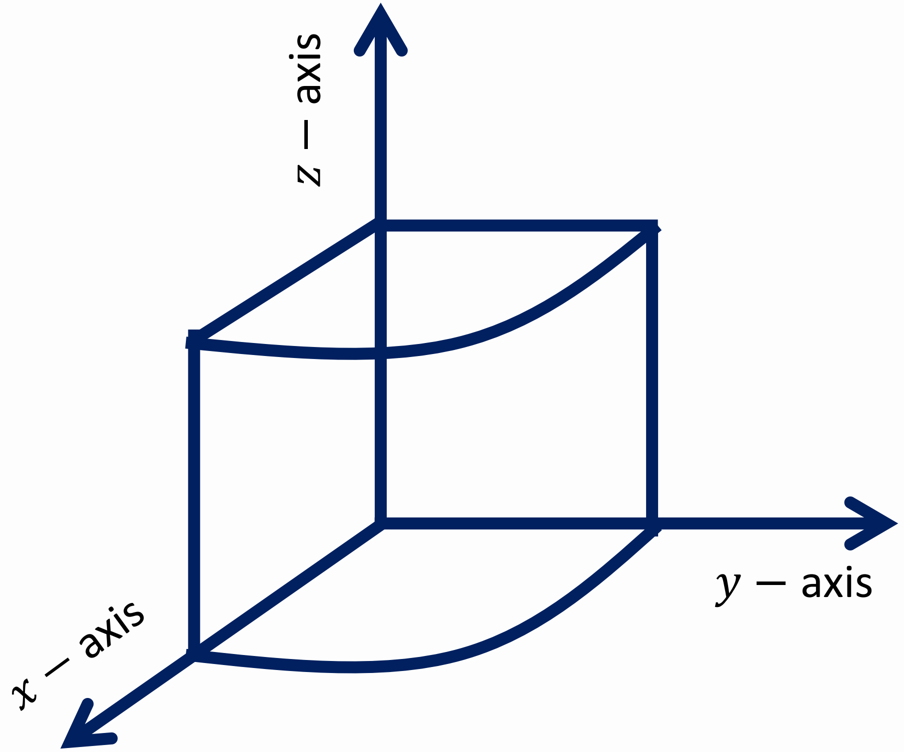
and S is the portion of the surface of the cylinder $x^2 + y^2 = 36$, $0 \le z \le 4$ included in the first octant.

Solution Let $f(x, y, z) = x^2 + y^2 - 36$

$$\Rightarrow \nabla f = 2x\,\hat{\imath} + 2y\,\hat{\jmath} \;\Rightarrow |\nabla f| = \sqrt{4 \times 36} = 12$$

$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{12}(2x\,\hat{\imath} + 2y\,\hat{\jmath})$$

$$= \frac{1}{6}(x\,\hat{\imath} + y\,\hat{\jmath})$$



$$d\sigma = \frac{|\nabla f|}{|\nabla f . \vec{p}|} dA$$

$\vec{p} = i$ (if projection is on yz plane)

$$\nabla f = 2x\,\hat{\imath} + 2y\,\hat{\jmath}$$

$$|\nabla f| = 12$$

$$\vec{F} = z^2\,\hat{\imath} + xy\,\hat{\jmath} - y^2\,\hat{k}$$

$$\vec{n} = \frac{1}{6}(x\,\hat{\imath} + y\,\hat{\jmath})$$

$$d\sigma = \frac{12}{|2x|} dA = \frac{6}{x} dA$$

Therefore $\displaystyle\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_{R_{yz}} \frac{1}{6}(xz^2 + xy^2)\frac{6}{x}\, dA$

$$= \int_{z=0}^{4}\int_{y=0}^{6} (y^2 + z^2)\, dydz = \int_0^4 \left[\frac{y^3}{3} + z^2 y\right]_0^6 dz$$

$$= \int_0^4 (72 + 6z^2) dz = 72 \times 4 + \frac{6}{3} \times 64 = 416$$

SUMMARY

Surface Integrals

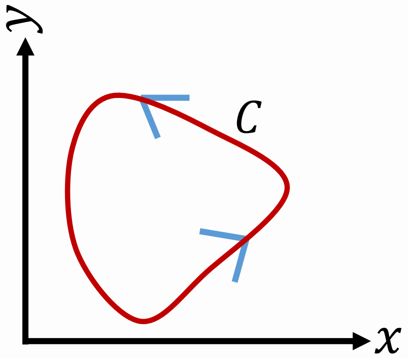
$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_R \vec{F} \cdot \left(\frac{\nabla f}{|\nabla f|}\right) \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\, dA$$

$$= \iint_R \vec{F} \cdot \frac{\nabla f}{|\nabla f \cdot \vec{p}|}\, dA$$

Lecture - 9

- Stokes' Theorem (Generalization of Green's Theorem)

Green's Theorem (Recall):



Let R be a region in $\mathbb{R}^2$ whose boundary is a simple closed curve C

Let $\vec{F} = F_1(x, y)\,\hat{\imath} + F_2(x, y)\hat{\jmath}$ be smooth vector field (F_1 & F_2 are C^1 functions) on both R and C, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

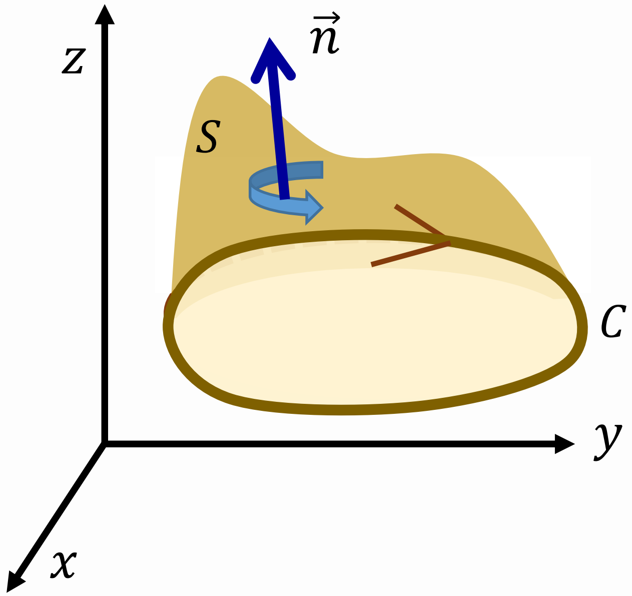
Note that the above can also be written as

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\mathrm{curl}\,\vec{F}) \cdot \hat{k}\ \ dA$$

Stokes' Theorem

Green's theorem in the plane $\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\nabla \times \vec{F}) \cdot \vec{k} \; dxdy$

Let C be a closed curve in 3-D space which forms the boundary of a surface S whose unit normal vector is $\vec{n}$



Then for a continuously differentiable vector field $\vec{F}$, we have

$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \; ds$ where the direction of the line integral around C and the normal $\vec{n}$ are oriented in a right-handed sense

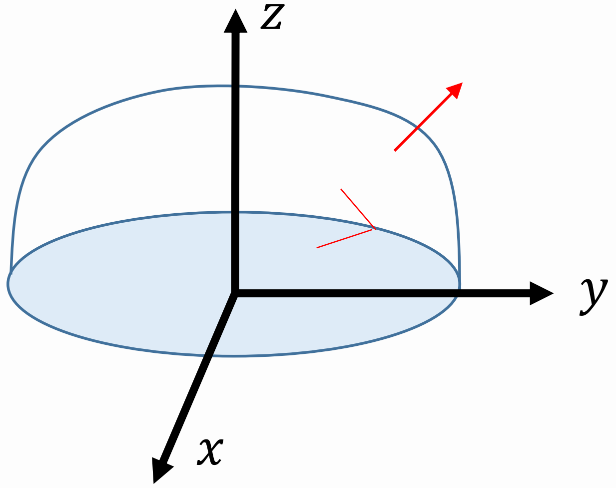
If $\nabla \times \vec{F} = 0$ ($\vec{F}$ is irrotational, or $\vec{F}$ is conservative) then, Stokes' theorem tells us that

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

Problem-1 Verify Stokes' theorem for the hemisphere $S: x^2 + y^2 + z^2 = 9$, $z \geq 0$, its boundary $C: x^2 + y^2 = 9$, $z = 0$ and the field $\vec{F} = y\hat{\imath} - x\hat{\jmath}$

Stokes' theorem: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \; ds$

Solution: Parametric equation of the curve



$$\vec{r}(\theta) = 3\cos\theta\,\hat{\imath} + 3\sin\theta\,\hat{\jmath}, \qquad 0 \leq \theta \leq 2\pi$$

$$\Rightarrow \frac{d\vec{r}}{d\theta} = -3\sin\theta\,\hat{\imath} + 3\cos\theta\,\hat{\jmath}$$

$$\vec{F} = 3\sin\theta\,\hat{\imath} - 3\cos\theta\;\hat{\jmath}$$

$$\vec{F} \cdot \frac{d\vec{r}}{d\theta} = -9\sin^2\theta - 9\cos^2\theta \;= -9$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{\theta=0}^{2\pi} \vec{F} \cdot \frac{d\vec{r}}{d\theta} d\theta$$

$$= \int_0^{2\pi} -9\,d\theta \;= -18\pi$$

$$S: x^2 + y^2 + z^2 = 9$$

$$f = x^2 + y^2 + z^2$$

$$\vec{F} = y\hat{\imath} - x\hat{\jmath}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \hat{\imath}\,(0) + \hat{\jmath}\,(0) + \hat{k}(-1-1) = -2\hat{k}$$

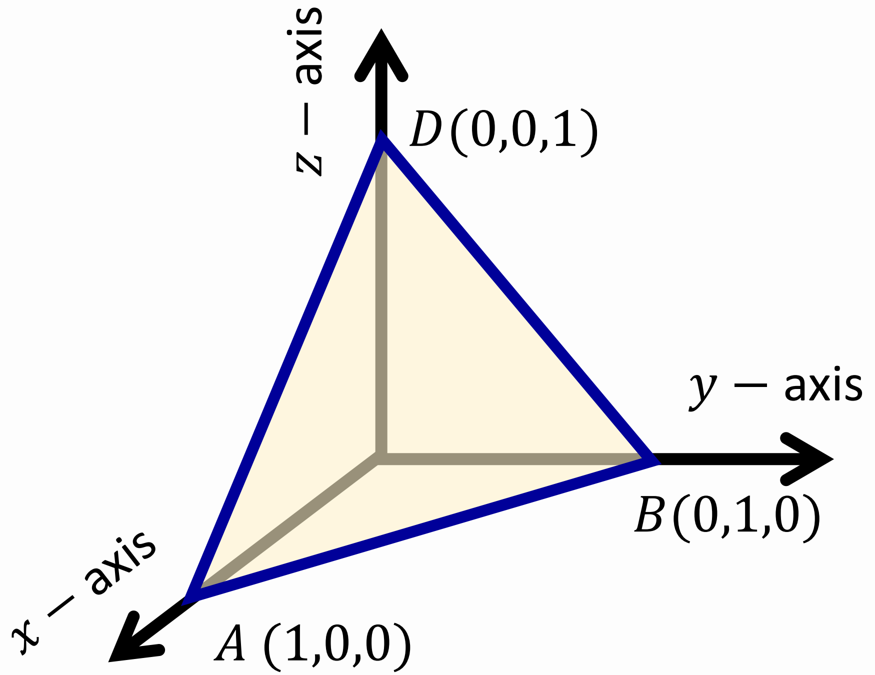
$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}}{\sqrt{4 \times 9}} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{3}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds = \iint_{x^2+y^2\leq 9} -\frac{2z}{3} \frac{|\nabla(x^2+y^2+z^2)|}{|\nabla(x^2+y^2+z^2)\cdot \hat{k}|} dxdy$$

$$= \iint_{x^2+y^2\leq 9} -\frac{2z}{3}\frac{6}{2z} dxdy = -2 \iint_{x^2+y^2\leq 9} dxdy = -18\pi$$

Problem-2 Verify Stokes' theorem for the function $\vec{F} = x\hat{\imath} + z^2\hat{\jmath} + y^2\hat{k}$ over the plane surface $x + y + z = 1$ lying in the first quadrant.

Solution Stokes' theorem: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\ ds$



S: triangle ABD C: lines AB, BD and DA

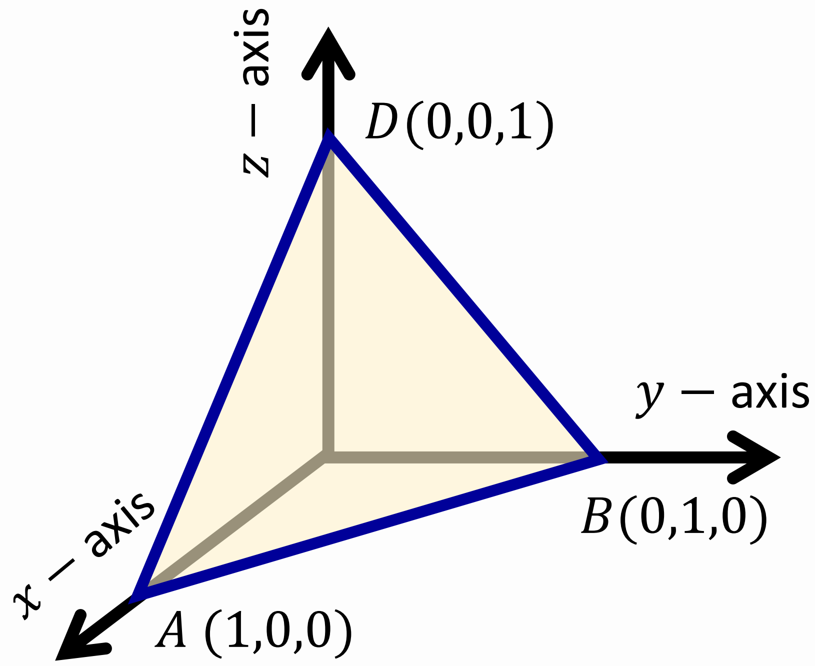
$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \left(x\hat{\imath} + z^2\hat{\jmath} + y^2\hat{k}\right) \cdot \left(\hat{\imath}\,dx + \hat{\jmath}\ dy + \hat{k}\,dz\right)$$

$$= \int_{AB} x\,dx + z^2 dy + y^2 dz + \int_{BD} x dx + z^2 dy + y^2 dz + \int_{DA} x dx + z^2 dy + y^2 dz$$

Equating to the line AB: $\dfrac{x-1}{0-1}=\dfrac{y-0}{1-0}=\dfrac{z-0}{0-0}=t$

$x=1-t \qquad y=t \qquad z=0$

$$\int_{AB} xdx+z^2dy+y^2dz=\int_{t=0}^{1}(1-t)(-dt)=\left[\frac{(1-t)^2}{2}\right]_0^1=-\frac{1}{2}$$



Equating to the line *BD*: $\dfrac{x-0}{0-0}=\dfrac{y-1}{0-1}=\dfrac{z-0}{1-0}=t \qquad x=0 \qquad y=1-t \quad z=t$

$$\int_{BD} xdx+z^2dy+y^2dz=\int_{t=0}^{1}t^2(-dt)+(1-t)^2\,dt=\int_{t=0}^{1}(1-2t)\,dt=0$$

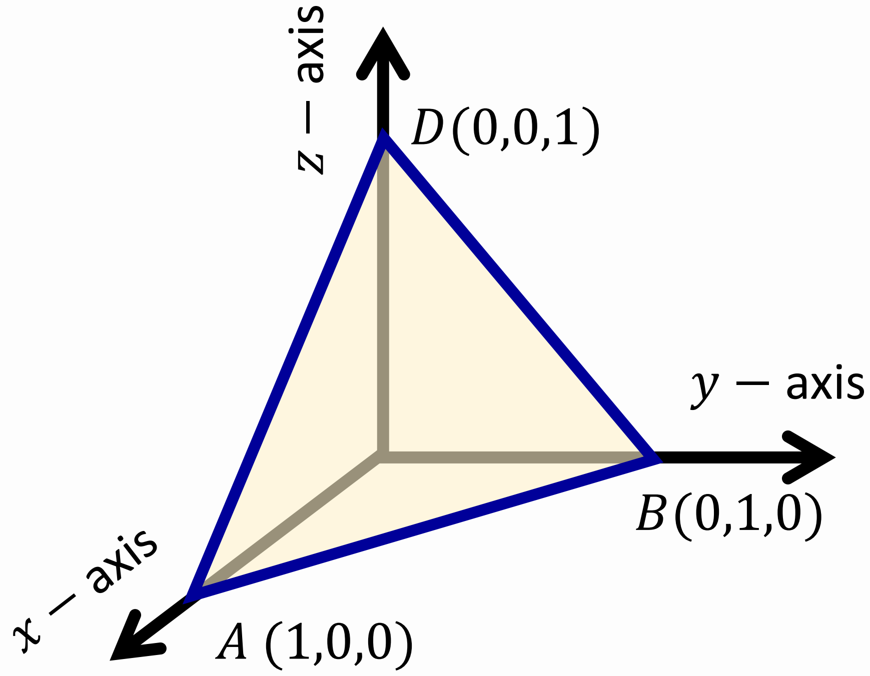
Equating to the line DA: $\dfrac{x-0}{1-0} = \dfrac{y-0}{0-0} = \dfrac{z-1}{0-1} = t$

$$x = t \qquad y = 0 \qquad z = 1-t$$

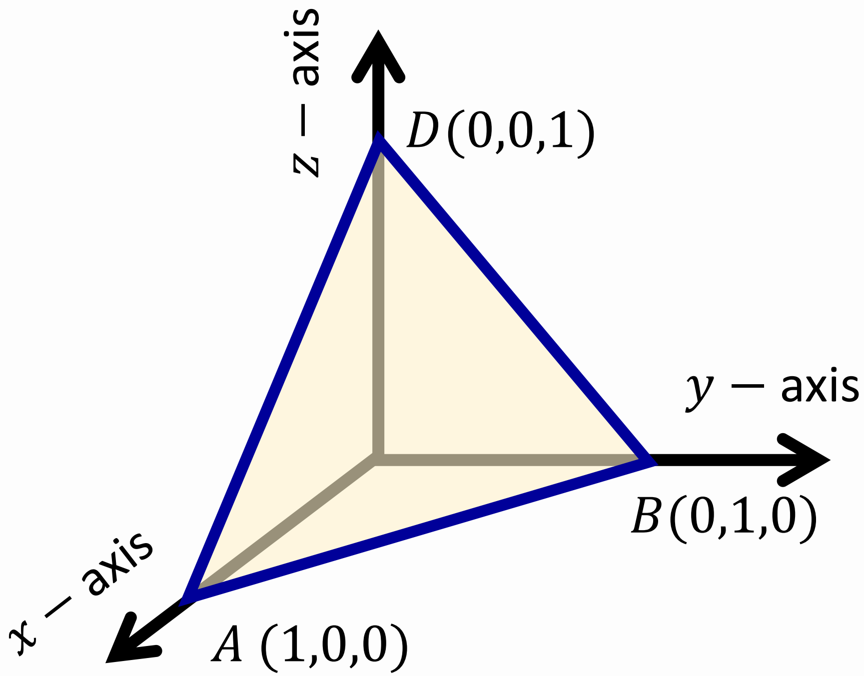
$$\int_{DA} x\,dx + z^2 dy + y^2 dz = \int_{t=0}^{1} t\ dt = \frac{1}{2}$$

We have $\displaystyle\oint_{AB} \vec{F}\cdot d\vec{r} = -\frac{1}{2} \qquad \oint_{BD} \vec{F}\cdot d\vec{r} = 0 \qquad \oint_{DA} \vec{F}\cdot d\vec{r} = \frac{1}{2}$

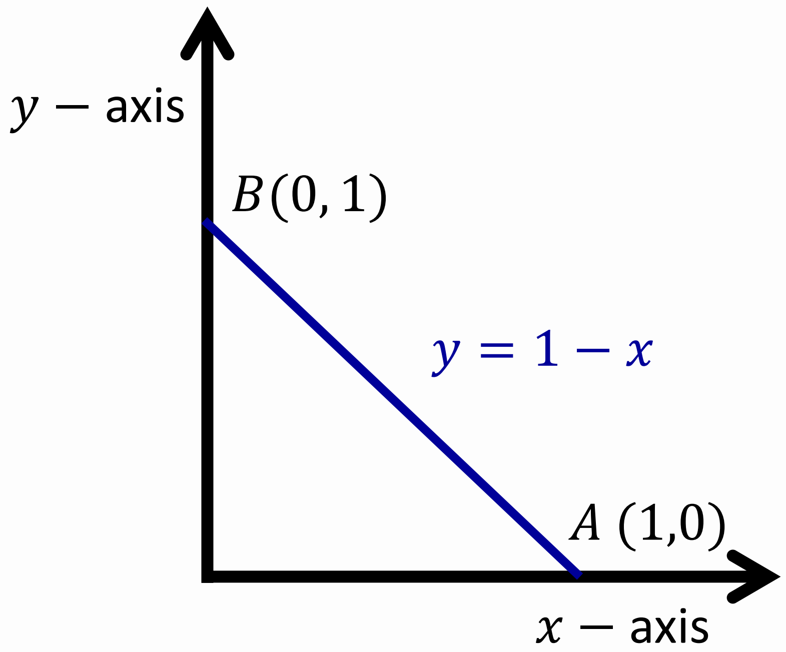
$$\Rightarrow \oint_{C} \vec{F}\cdot d\vec{r} = 0$$



Projecting S on the x-y plane, let R be its projection.



R is bounded by the x-axis, y-axis and straight line AB.



Given surface $f = x + y + z = 1 \Rightarrow \ \nabla f = \hat{\imath} + \hat{\jmath} + \hat{k}$

$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{\hat{\imath} + \hat{\jmath} + \hat{k}}{\sqrt{3}} \qquad \frac{|\nabla f|}{|\nabla f \cdot \hat{k}|} = \frac{\sqrt{3}}{|1|} = \sqrt{3}$$

$$\text{curl}\,\vec{F}\cdot\vec{n} = (2(y-z)\,\hat{\imath})\cdot\left(\frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}(y-z) = \frac{2}{\sqrt{3}}(2y+x-1)$$

$$\vec{F} = x\hat{\imath} + z^2\hat{\jmath} + y^2\hat{k}$$

$$S:\ x+y+z=1$$

$$\vec{n} = \frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}$$

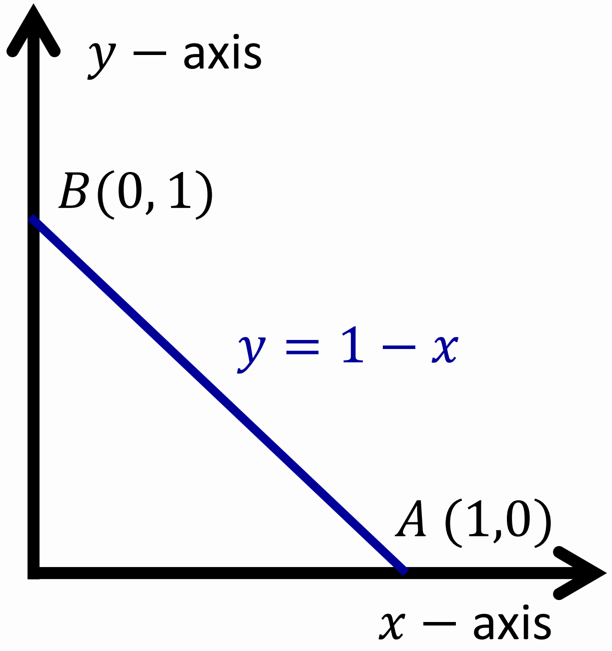
$$\frac{|\nabla f|}{|\nabla f\cdot\hat{k}|} = \sqrt{3}$$

$$\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,ds = \iint_{R_{xy}} \frac{2}{\sqrt{3}}(2y+x-1)\sqrt{3}\ dxdy$$

$$= 2\int_0^1\int_0^{1-x}(2y+x-1)\,dy\,dx$$

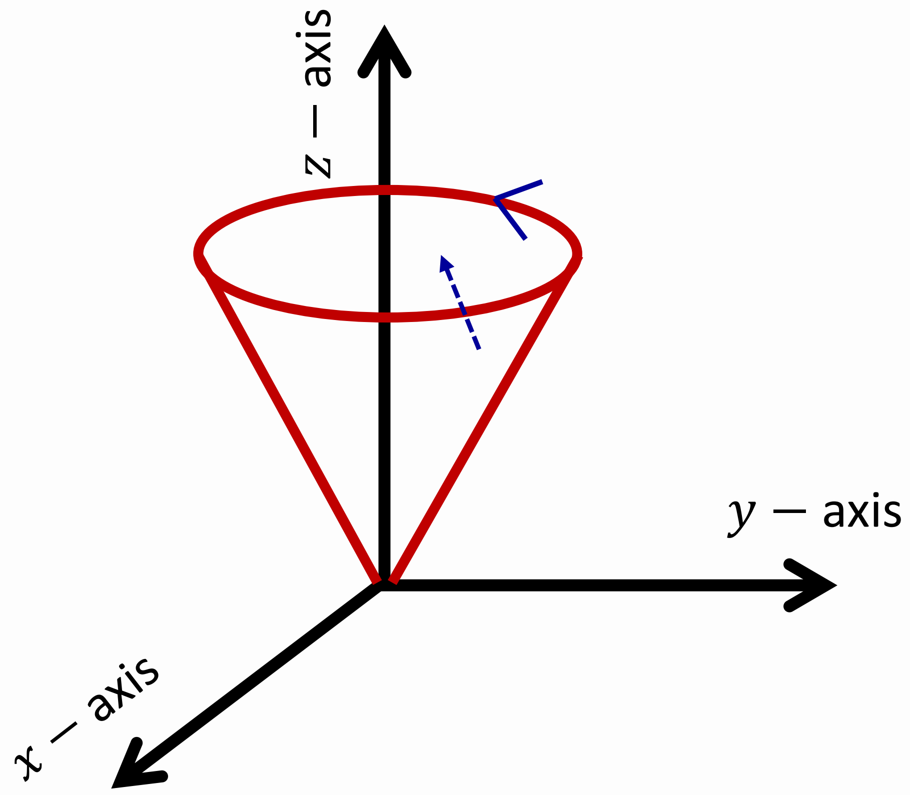
$$= 2\int_0^1 (1-x)^2 + (x-1)(1-x)\ dx$$

$$= 0$$



Problem: Let $\vec{F} = -y\hat{\imath} + x\,\hat{\jmath} - xyz\,\hat{k}$ and let S be the part of cone $z = \sqrt{x^2+y^2}$ for $x^2+y^2 \le 9$.

Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ or $\iint_S (\nabla \times \vec{F})\cdot \vec{n}\, d\sigma$ whichever appears easier. Here $\vec{n}$ is the inner normal vector.



$C: x^2 + y^2 = 9 \;\;\&\;\; z = 3$

$x = 3\cos t\,, \qquad y = 3\sin t\,, \qquad z = 3$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C -y\,dx + x\,dy - xyz\,dz$$

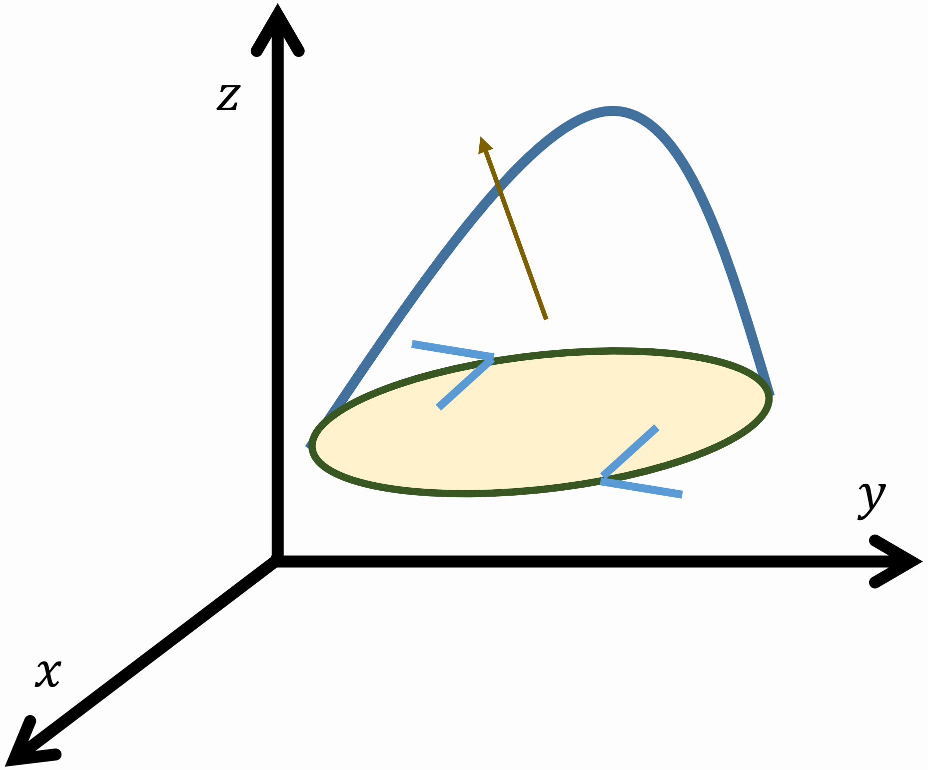
$$= \oint_0^{2\pi} (3\sin t)\,(3\sin t)\,dt \;+ 3\cos t\,(3\cos \mathrm{t})\,\mathrm{dt}$$

$$= 9\oint_0^{2\pi} \mathrm{dt} \;= 18\pi$$

SUMMARY

Stokes' Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \; ds$$



Lecture - 10

- **Divergence Theorem (**volume integrals ↔ surface integrals**)**

Recall Green′s Theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \hat{k} \; dA$$

Its generalization in space $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \; ds$ Stokes’ Theorem

The Divergence Theorem:

Green's Theorem $\oint_C F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$

Define a Vector Field: $\vec{F} = F_2(x,y)\hat{\imath} - F_1(x,y)\hat{\jmath} \Longrightarrow \nabla \cdot \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

Differential element along tangent to C: $d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$

Unit tangent vector to C: $\hat{T} = \frac{dx}{ds}\hat{\imath} + \frac{dy}{ds}\hat{\jmath}$

Unit normal vector to C: $\hat{n} = \frac{dy}{ds}\hat{\imath} - \frac{dx}{ds}\hat{\jmath}$

$$F_1\,dx + F_2\,dy = \left(F_1\frac{dx}{ds} + F_2\frac{dy}{ds}\right)ds = \vec{F}\cdot\hat{n}\,ds$$

Green's Theorem: $\oint_C \vec{F}\cdot\hat{n}\;ds = \iint_D \nabla\cdot\vec{F}\,dA$

The Divergence Theorem (Generalization of Green's Theorem)

Green's Theorem: $\oint_C \vec{F} \cdot \hat{n}\ dt = \iint_D \nabla \cdot \vec{F}\, dA$

Replace the closed curve C $\rightarrow$ a closed surface S in 3D

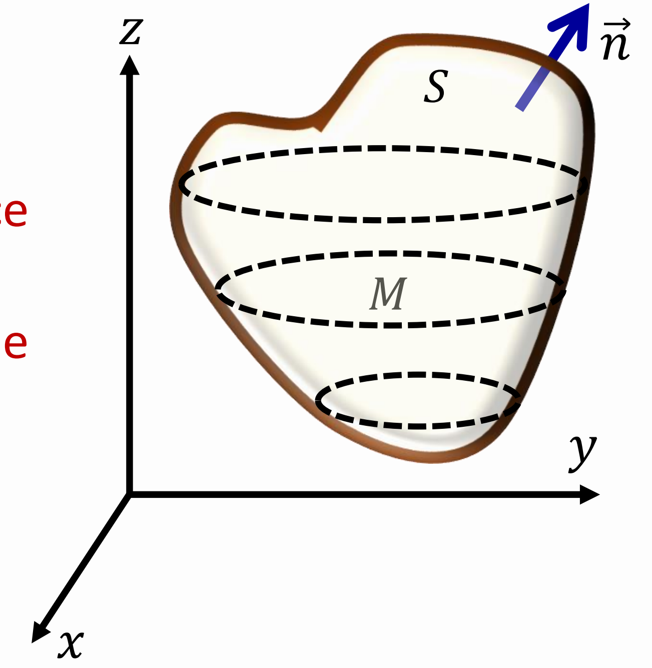
Replace the bounding domain D $\rightarrow$ the bounding volume M

The vector field $\vec{F}(x, y)$ $\rightarrow$ The vector field $\vec{F}(x, y, z)$

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_M \nabla \cdot \vec{F}\, dV$$

The Divergence Theorem

The flux of a vector field $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ across a closed oriented surface S in the direction of the surface's outward unit normal field $\hat{n}$ equals the integral of $\nabla \cdot \vec{F}$ over the region M enclosed by the surface



$$\iint\limits_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint\limits_M \nabla \cdot \vec{F}\, dV$$

Intuitively, it states that sum of all sources minus the sum of all sinks gives the net flow of a region.

Problem-1 Verify Divergence theorem for the field $\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ over the sphere $x^2 + y^2 + z^2 = 9$

Solution: $\vec{n} = \dfrac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{3} \Rightarrow \vec{F} \cdot \vec{n} = \dfrac{1}{3}(x^2 + y^2 + z^2) = 3$

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_S 3\, d\sigma = 3 \iint_S d\sigma = 3\,(4\pi\, 3^2) = 108\,\pi$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$\Longrightarrow \iiint_D \vec{\nabla} \cdot \vec{F}\, dV = \iiint 3\, dV = 3 \times \frac{4}{3}\pi\, 3^3 = 108\,\pi$$

Problem-2 Find the flux of $\vec{F} = xy\,\hat{\imath} + yz\hat{\jmath} + xz\,\hat{k}$ outward through the surface of the cube from the first octant by the planes $x = 2$, $y = 2$ and $z = 2$.

Solution: $\nabla \cdot \vec{F} = y + z + x$

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \vec{\nabla} \cdot \vec{F}\, dV$$

Divergence Theorem

$$= \int_0^2 \int_0^2 \int_0^2 (x + y + z)\, dxdydz$$

$$= 24$$

Problem-3 If V is the volume enclosed by a closed surface S and $\vec{F} = 3x\hat{\imath} + 2y\hat{\jmath} + z\hat{k}$ show that

$$\iint_S \vec{F} \cdot \vec{n}\, ds = 6V$$

Solution: $$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(z) = 6$$

By Gauss Divergence theorem: $$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV$$

$$= 6 \iiint_D dV = 6V$$

Problem-4 Evaluate $\iint_S \left((x^3 - yz)\hat{\imath} - 2x^2y\hat{\jmath} + 2\hat{k}\right) \cdot \hat{n}\, d\sigma$ where S denotes the surface of the cube

bounded by the planes $x = 0, x = 3, y = 0, y = 3, \mathrm{z} = 0, \mathrm{z} = 3$

Solution: $\nabla \cdot \vec{F} = 3x^2 - 2x^2 - 0 = x^2$

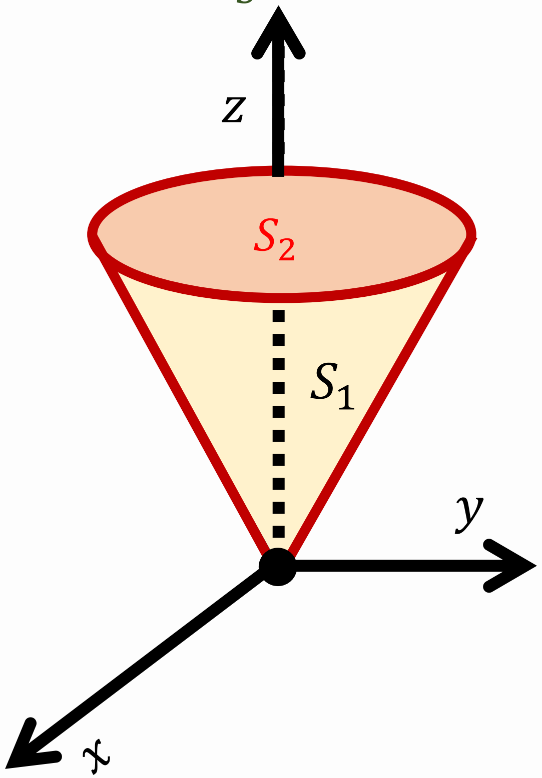
By Gauss Divergence theorem:

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\ dV = \iiint_D x^2\, dxdydz$$

$$= \int_0^3 \int_0^3 \int_0^3 x^2\, dxdydz = 81$$

Problem-5 Let S be given by the cone $z = \sqrt{x^2 + y^2}$ for $x^2 + y^2 \le 1$ together with the disk $x^2 + y^2 \le 1, z = 1$. For $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$, verify the divergence theorem.

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_M \nabla \cdot \vec{F}\, dV$$



Solution Let S_1: $z = \sqrt{x^2 + y^2}, \quad x^2 + y^2 \le 1$

Let S_2: $x^2 + y^2 \le 1, \qquad z = 1$

Surface Integral: $$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_{S_1} \vec{F} \cdot \hat{n}\, d\sigma + \iint_{S_2} \vec{F} \cdot \hat{n}\, d\sigma$$

For S_1: $$\hat{n} = \frac{2x\,\hat{\imath} + 2y\,\hat{\jmath} - 2z\,\hat{k}}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x\,\hat{\imath} + y\,\hat{\jmath} - z\,\hat{k}}{\sqrt{2}\; z} \qquad \vec{F} \cdot \hat{n} = 0$$

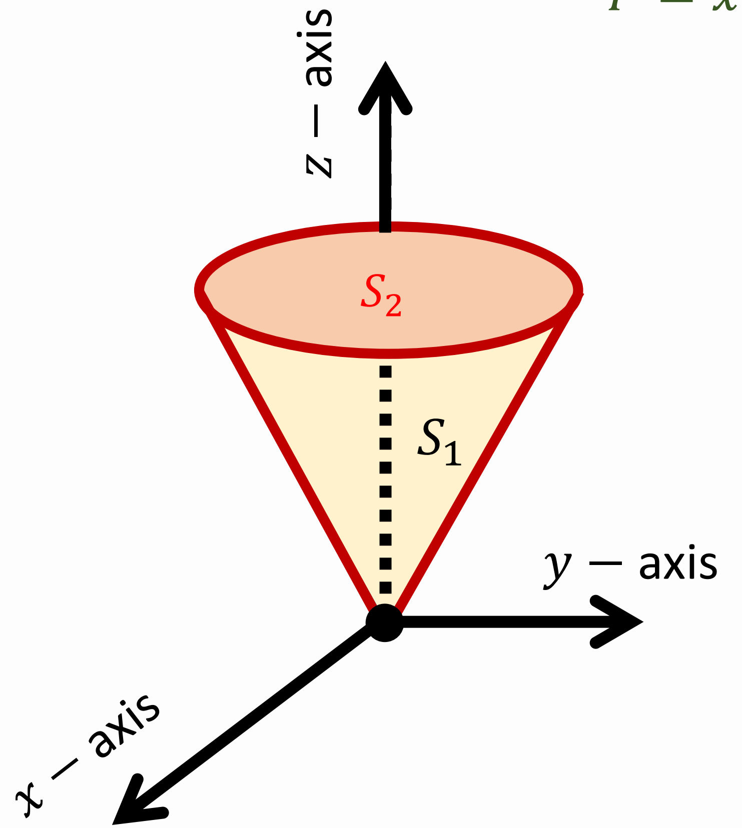
For S_2: $\hat{n} = k$ $\qquad \vec{F} \cdot \hat{n} = z$

S_2: $x^2 + y^2 \le 1, \qquad z = 1$

$\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_{S_1} \vec{F} \cdot \hat{n}\, d\sigma + \iint_{S_2} \vec{F} \cdot \hat{n}\, d\sigma$$

$$= \iint_{S_2} d\sigma = \pi$$



Volume Integral $\iiint_M \nabla \cdot \vec{F}\, dV = 3 \iiint_M dV = 3 \times \pi(1)^2 \frac{1}{3} = \pi$

Volume of a cone of height h and radius $r = \pi r^2 \frac{h}{3}$

SUMMARY

The Divergence Theorem:

$$\iint\limits_{S} \vec{F} \cdot \hat{n} \, d\sigma = \iiint\limits_{M} \nabla \cdot \vec{F} \, dV$$