$$v_z = \frac{\rho g \delta^2 \cos\beta}{2\mu}\left[1-\left(\frac{x}{\delta}\right)^2\right]$$

At the fluid - solid interface, $x = \delta \Rightarrow v_z = 0$

$$\text{Max}(v_z) = \frac{\rho g \delta^2 \cos\beta}{2\mu}$$

$$\Rightarrow v_z = v_{z\,max}\left[1-\left(\frac{x}{\delta}\right)^2\right]$$

Avg. velocity $\langle v_z \rangle = ?$

Here we refer to area, i.e avg. flow rate over flow area.

$$\langle v_z \rangle = \frac{\int_0^W \int_0^\delta v_x \, dx \, dy}{\int_0^W \int_0^\delta dx \, dy}$$

$$= \frac{1}{\delta}\int_0^\delta v_z \, dx$$

$$\therefore \langle v_z \rangle = \frac{\rho g \delta^2 \cos\beta}{3\mu}$$

$Q = W\delta \langle v_z \rangle$ ← Theoretical flow rate

Flow rate

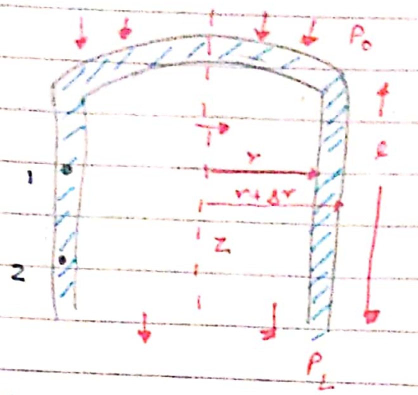
$$Q = \frac{\rho g \delta^3 \cos\beta \, W}{3\mu}$$

2- comp. of the force of the fluid on the surface

$F_z = \int_0^L \int_0^W \tau_{xz} \, dy \, dz$ for newtonian fluid

$$F_z = (\rho g \delta \cos\beta) LW$$

Flow through a circular tube.



It is a 1D - flow

$v_z \neq 0$, $v_r = v_\theta \approx 0$

$v_z = f(r, \theta, z)$

In a pipe if we fix the r, then in the ring like, there is no dependency of v_z on θ as there is ~~no~~ angular symmetry

Consider pipes (1, 2) the ~~rar~~ distance from center line is same, there is no. diff. in velocities at 1, 2 (Law of continuity). Thus, v_z is independent on z.

Thus we consider our shell in r

We have conductive transfer of momentum in r and convective transfer in z- direction

Viscous force is acting on area $2\pi r L$, $2\pi(r+\Delta r)L$ and the total vol. is $2\pi r \Delta r L$

So, now it's clear v_z is only a fn. of r, i.e $v_z = f(r)$

Now WKT,

$$\dot{M} + \Sigma F = 0$$

At the top, the area is $(2\pi r \Delta r)$ and thus the rate of momentum-in is $(2\pi r \Delta r) v_z \rho v_z|_{z=0}$ and out will be $(2\pi r \Delta r) v_z \rho v_z|_{z=L}$

Since v_z isn't a fn. of z, the above two terms gets cancelled out.

Conductive transfer is due to the stress τ_{rz}. The momentum - in is $2\pi r L \tau_{rz}|_r$ and the momentum - out is $2\pi(r+\Delta r) L \tau_{rz}|_{r+\Delta r}$

$$\Sigma F = \underbrace{2\pi r \Delta r P_0 - 2\pi r \Delta r P_L}_{\text{Surface forces}} + \underbrace{2\pi r \Delta r L \rho}_{\text{Body Forces}}$$

$$\Rightarrow \lim_{\Delta r \to 0} \left[\frac{(r\tau_{rz})|_{r+\Delta r} - (r\tau_{rz})|_r}{\Delta r} \right] = \left[\frac{P_0 - P_L}{L} + \rho g \right] r$$

$$\frac{d}{dr}(r\tau_{rz}) = \left(\frac{P_0 - P_L}{L}\right) r$$

$P \equiv p - \rho g z$

$\hookrightarrow P_0 \equiv p_0$

Integrating

$$\Rightarrow \tau_{rz} = \frac{P_0 - P_L}{2L} r + \frac{c_1}{r}$$

For a Newtonian fluid,

$$-\frac{d}{dr}\left(r\mu \frac{dv_z}{dr}\right) = \left(\frac{P_0 - P_L}{L}\right) r$$

τ_{rz} must be finite at $r = 0$

$\Rightarrow c_1 = 0$

$$v_z = -\frac{P_0 - P_L}{4\mu L} \cdot r^2 + c_2$$

$r = R$, $v_z = 0$ $\rightarrow$ No slip cond.

$$v_z = \frac{(P_0 - P_L)}{4\mu L} R^2 \left[1 - \left(\frac{r}{R}\right)^2\right]$$

$v_z = v_{max}$ at $r = 0$

$$\therefore \quad v_z = v_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$ → Parabolic distribution of velocity

$$\langle v_z \rangle = \frac{\int_0^{2\pi}\int_0^R v_z \, r \, dr \, d\theta}{\int_0^{2\pi}\int_0^R r \, dr \, d\theta}$$

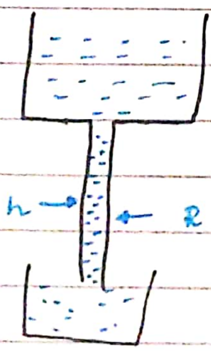
$$= \frac{P_0 - P_L}{8\mu L} R^2$$

Flow rate

$Q = \pi R^2 \langle v_z \rangle$

$$\Rightarrow \quad Q = \frac{\pi (P_0 - P_L) R^4}{8\mu L}$$ → Hagen - Poiseuille eqn.

Capillary Viscometer

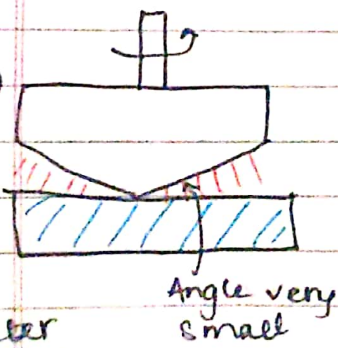


$\mu(T)$ shouldn't be too low then laminar flow isn't possibility

$\mu(T)$ if too high, will take lot of time

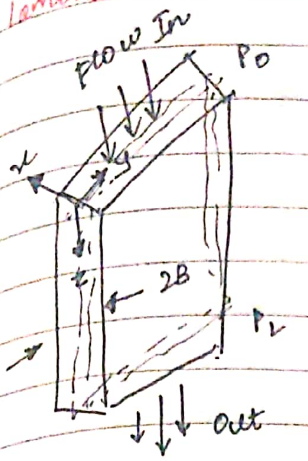
→ Industrial application of Poiseuille eqn.

Cone & Plate Viscometer



→ Very difficult to solve using shell momentum balance

Laminar flow in a narrow slit



$L, W \gg 2B$

$v_z \neq 0$, $v_x = 0$

$v_z \neq f(y)$, $v_z \neq f(z)$

$\langle v_z \rangle = ?$

clearly v_z is a fn: of x
thus a shell is in the x-direction

$L, W, \Delta x$

In $x, x+\Delta x$ we have

conv. , conductive

In - Out $= 0$, $(\tau_{xz}) LW$

In Out

$(\ )|_x$ ① $-$ $(\ )|_{x+\Delta x}$ ②

Surface force $P_0 W\Delta x$ ③ , $P_L W\Delta x$ ④

~~Mass~~ Force due to mass , $WL \cdot \Delta x \rho g$ ⑤

$$\tau_{xz} LW|_x - \tau_{xz} LW|_{x+\Delta x}$$
$$+ P_0 \cdot W\Delta x - P_L W\Delta x - WL\Delta x \rho g = 0$$

$$\Rightarrow (L) \lim_{\Delta x \to 0} \left[\frac{\tau_{xz}|_{x+\Delta x} - \tau_{xz}|_x}{\Delta x} \right]$$
$$= (P_0 - P_L)$$

$$\frac{d}{dx}(\tau_{xz}) = \frac{P_0 - P_L}{L}$$

$$\Rightarrow \tau_{xz} = \left(\frac{P_0 - P_L}{L}\right) x + c_1$$

Using L-S interface cond. , $c_1 = 0$

$$\therefore \tau_{xz} = \left(\frac{P_0 - P_L}{L}\right) x$$ //

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