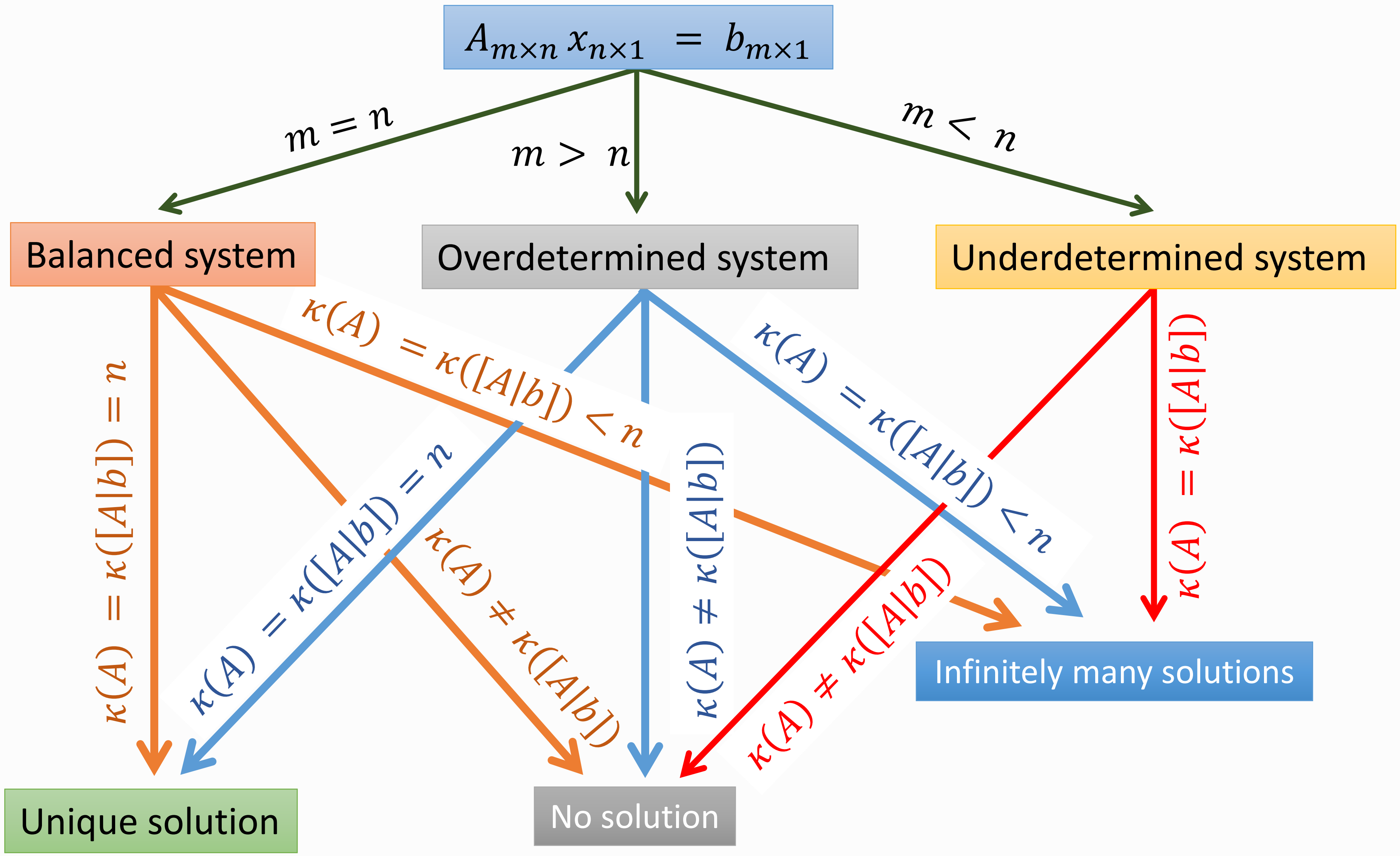
Module 02: NUMERICAL ANALYSIS

Lecture: Iterative Methods for Solving System of Linear Equations

- **Iterative Methods for Solving System of Linear Equations**
  - **Jacobi Method**
  - **Gauss-Seidel Method**

Consistency of a Linear System

$\kappa(A)$: Rank of A



System of Linear Equations $Ax = b$ $\quad A \in \mathbb{R}^{n \times n} \quad x \in \mathbb{R}^{n \times 1} \quad b \in \mathbb{R}^{n \times 1}$

Solution Methods

Direct Methods

Cramer Rule
Gauss Elimination Method
Gauss-Jordan Method
Decomposition Methods

- Deliver Exact Solution
  (in the absence of rounding errors)
- Very Expensive
  (especially for large systems)

Iterative Methods

Jacobi Method
Gauss-Seidel Method
Conjugate Gradient Method
Conjugate Residual Method

- Deliver Approximate Solution
- Less Expensive

Diagonally Dominant Matrix

A matrix $A \in \mathbb{R}^{n\times n}$ is called **diagonally dominant by rows** if

$$|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}|, \qquad i = 1, 2, \dots, n$$

while it is called **diagonally dominant by columns** if

$$|a_{jj}| \geq \sum_{i=1, i\neq j}^{n} |a_{ij}|, \qquad j = 1, 2, \dots, n$$

If the above inequalities hold in a strict sense, A is called **strictly diagonally dominant** (by rows or by columns respectively).

Matrix Norms

A number associated with a matrix that is often requires in analysis of Matrix based algorithm.

Matrix norms give some notion of “size” of a matrix or “distance” between the two matrices.

Some Example: Let $A \in \mathbb{R}^{n \times n}$

Frobenius Norm: $$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2}$$

Row Sum Norm: $$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

Column Sum Norm: $$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

ITERATIVE METHOD

A method for solving the linear system $Ax = b$ is called iterative if it is a numerical method computing a sequence of approximate solutions $x^{(k)}$ that converges to the exact solution x as the number of iterations k goes to ∞.

IDEA FOR DERIVING AN ITERATIVE METHOD

Consider a system of linear equations $Ax = b$

Idea of iterative schemes is based on the splitting $A = P - N$

where P is a non-singular matrix.

$$\text{Given } Ax = b \quad \Rightarrow (P - N)x = b \quad \Rightarrow Px = Nx + b$$

Given $Ax = b \quad \Rightarrow (P - N)x = b \quad \Rightarrow Px = Nx + b$

Consider the iterations with a suitable guess $x^{(0)}$

$$Px^{(k+1)} = Nx^{(k)} + b$$

$$\Longrightarrow x^{(k+1)} = Gx^{(k)} + Hb$$

where $G = P^{-1}N$ is called **iteration matrix** and $H = P^{-1}$.

Definition (Convergence of an Iterative Method):

An iterative method is said to **converge** if for any choice of initial vector $x^{(0)} \in \mathbb{R}^n$, the sequence of approximate solutions $x^{(k)}$ converges to the exact solution x.

Definition (Error):

We call the vector $r_k = b - Ax^{(k)}$ **residual** (respectively **error** $e_k = x^{(k)} - x$) at the kth iteration.

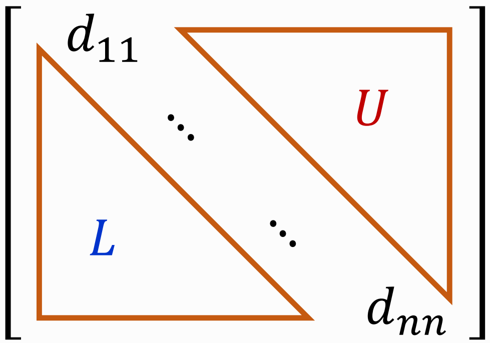
REMARK:

In general, we have no knowledge of e_k because the exact solution x is unknown. However, it is easy to compute the residual r_k, so convergence dedicated on the residual in practice.

Jacobi Iteration Method

Consider a system of linear equations $A_{n\times n}x_{n\times 1} = b_{n\times 1}$

Take splitting of A as $A = L + D + U$

$A =$ 

L: Lower triangular part of A

D : Diagonal entries of A

U : Upper triangular part of A

$A = L + D + U$ $\qquad Ax = b \implies (L + D + U)x = b \quad \implies Dx = -(L + U)x + b$

Assume that D^{-1} exists, then $\Rightarrow x = -D^{-1}(L + U)x + D^{-1}b$

Introducing iterations, the iterative method known as Jacobi iteration method, becomes

$$\boxed{x^{(k+1)} = -D^{-1}(L + U)\, x^{(k)} + D^{-1}b}$$

In component form

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij} x_j^{(k)}\right)} \quad i = 1, 2, \dots, n$$

Jacobi Method

$$x^{(k+1)} = -D^{-1}(L+U)\,x^{(k)} + D^{-1}b$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij}x_j^{(k)}\right)$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k)} - \sum_{j>i} a_{ij}x_j^{(k)}\right)$$

Gauss Seidel Method

To improve Jacobi method, the idea is to use newly computed components $x_j^{(k+1)}$ $(j < i)$

to compute $x_i^{(k+1)}$, that is

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)}\right)$$

Gauss Seidel Method

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij} x_j^{(k+1)} - \sum_{j>i} a_{ij} x_j^{(k)}\right)}$$

The algorithm in matrix form

$$x^{(k+1)} = D^{-1}\left(b - Lx^{(k+1)} - Ux^{(k)}\right)$$

$$\Rightarrow (D+L)x^{(k+1)} = \left(b - Ux^{(k)}\right)$$

$$\boxed{x^{(k+1)} = -(D+L)^{-1}Ux^{(k)} + (D+L)^{-1}b}$$

Example: Consider the following system of equations

$$5x + y + 2z = 13$$

$$x + 3y + z = 12$$

$$-x + 2y + 4z = 8$$

Compute the solution $x^{(1)}$ after one Gauss-Siedel and Jacobi method. Take $x^{(0)} = [1, 1, 1]^T$.

System of Linear Equations

$$5x + y + 2z = 13$$

$$x + 3y + z = 12$$

$$-x + 2y + 4z = 8$$

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j,j\neq i} a_{ij}x_j^{(k)}\right)}$$

Initial guess: $x^{(0)} = [1, 1, 1]^T$

Jacobi method:

$$x^{(1)} = \frac{1}{5}\left(13 - y^{(0)} - 2z^{(0)}\right) = \frac{1}{5}(13 - 1 - 2) = 2$$

$$y^{(1)} = \frac{1}{3}\left(12 - x^{(0)} - z^{(0)}\right) = \frac{1}{3}(12 - 1 - 1) = \frac{10}{3}$$

$$z^{(1)} = \frac{1}{4}\left(8 + x^{(0)} - 2y^{(0)}\right) = \frac{1}{4}(8 + 1 - 2) = \frac{7}{4}$$

System of Linear Equations

$$5x + y + 2z = 13$$

$$x + 3y + z = 12$$

$$-x + 2y + 4z = 8$$

Initial guess: $x^{(0)} = [1, 1, 1]^T$

Gauss-Siedel Method:

$$x^{(1)} = \frac{1}{5}\left(13 - y^{(0)} - 2z^{(0)}\right) = \frac{1}{5}(13 - 1 - 2) = 2$$

$$y^{(1)} = \frac{1}{3}\left(12 - x^{(1)} - z^{(0)}\right) = \frac{1}{3}(12 - 2 - 1) = 3$$

$$z^{(1)} = \frac{1}{4}\left(8 + x^{(1)} - 2y^{(1)}\right) = \frac{1}{4}(8 + 2 - 6) = 1$$

Example: Taking starting values as $[1, 1, 1]^T$, perform 1 iteration using matrix form of Jacobi and Gauss-Seidel Methods for solving the system of equations

$$5x + y + 2z = 13$$

$$x + 3y + z = 12$$

$$-x + 2y + 4z = 8$$

Jacobi Method: $G_J = -D^{-1}(L + U)$

$$= -\begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1/5 & 2/5 \\ 1/3 & 0 & 1/3 \\ -1/4 & 1/2 & 0 \end{bmatrix}$$

Jacobi Method: $$G_J = -\begin{bmatrix} 0 & 1/5 & 2/5 \\ 1/3 & 0 & 1/3 \\ -1/4 & 1/2 & 0 \end{bmatrix}$$

Iterative Scheme: $\boldsymbol{x}^{(k+1)} = G_J \boldsymbol{x}^{(k)} + D^{-1} b$

Starting guess: $[1, 1, 1]^T$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^{(1)} = -\begin{bmatrix} 0 & 1/5 & 2/5 \\ 1/3 & 0 & 1/3 \\ -1/4 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 13 \\ 12 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} -3/5 \\ -2/3 \\ -1/4 \end{bmatrix} + \begin{bmatrix} 13/5 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 10/3 \\ 7/4 \end{bmatrix}$$

$$5x + y + 2z = 13$$

$$x + 3y + z = 12$$

$$-x + 2y + 4z = 8$$

Gauss Seidel Method: $\boldsymbol{x}^{(k+1)} = -(D+L)^{-1}U\boldsymbol{x}^{(k)} + (D+L)^{-1}b$

$$D + L = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 2 & 4 \end{bmatrix}$$

$$\Longrightarrow (D+L)^{-1} = \frac{1}{60}\begin{bmatrix} 12 & -4 & 5 \\ 0 & 20 & -10 \\ 0 & 0 & 15 \end{bmatrix}^T$$

$$\Longrightarrow (D+L)^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ -1/15 & 1/3 & 0 \\ 1/12 & -1/6 & 1/4 \end{bmatrix}$$

$$5x + y + 2z = 13$$
$$x + 3y + z = 12$$
$$-x + 2y + 4z = 8$$

Gauss Seidel Method: $\boldsymbol{x}^{(k+1)} = -(D+L)^{-1}U\boldsymbol{x}^{(k)} + (D+L)^{-1}b$

$$D + L = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 2 & 4 \end{bmatrix} \Longrightarrow (D+L)^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ -1/15 & 1/3 & 0 \\ 1/12 & -1/6 & 1/4 \end{bmatrix}$$

$$\boldsymbol{x}^{(k+1)} = -\begin{bmatrix} 1/5 & 0 & 0 \\ -1/15 & 1/3 & 0 \\ 1/12 & -1/6 & 1/4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{x}^{(k)} + \begin{bmatrix} 1/5 & 0 & 0 \\ -1/15 & 1/3 & 0 \\ 1/12 & -1/6 & 1/4 \end{bmatrix}\begin{bmatrix} 13 \\ 12 \\ 8 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 1/5 & 2/5 \\ 0 & -1/15 & 1/5 \\ 0 & 1/12 & 0 \end{bmatrix}\boldsymbol{x}^{(k)} + \begin{bmatrix} 13/5 \\ 47/15 \\ 13/12 \end{bmatrix}$$

Gauss Seidel Method: $\boldsymbol{x}^{(k+1)} = -(D+L)^{-1}U\boldsymbol{x}^{(k)} + (D+L)^{-1}b$

$$\boldsymbol{x}^{(\boldsymbol{k}+\mathbf{1})} = -\begin{bmatrix} 0 & 1/5 & 2/5 \\ 0 & -1/15 & 1/5 \\ 0 & 1/12 & 0 \end{bmatrix} \boldsymbol{x}^{(k)} + \begin{bmatrix} 13/5 \\ 47/15 \\ 13/12 \end{bmatrix}$$

Starting guess: $[1, 1, 1]^T$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^{(1)} = -\begin{bmatrix} 0 & 1/5 & 2/5 \\ 0 & -1/15 & 1/5 \\ 0 & 1/12 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 13/5 \\ 47/15 \\ 13/12 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Problem: Solve the system of linear equations:

$$4x_1 + 2x_2 + x_3 = 4$$

$$x_1 + 3x_2 + x_3 = 4$$

$$3x_1 + 2x_2 + 6x_3 = 7$$

Using Jacobi and Gauss-Seidel method perform three iterations using the initial guess $x^{(0)} = [0.1, 0.8, 0.5]^T$.

Jacobi Method:

$$x_1^{(k+1)} = \frac{1}{4}\left[4 - 2x_2^{(k)} - x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{3}\left[4 - x_1^{(k)} - x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{6}\left[7 - 3x_1^{(k)} - 2x_2^{(k)}\right]$$

Jacobi Method: Starting Guess: $x^{(0)} = [0.1, 0.8, 0.5]^T$

$$x_1^{(k+1)} = \frac{1}{4}\left[4 - 2x_2^{(k)} - x_3^{(k)}\right] \quad \Longrightarrow x_1^{(1)} = \frac{1}{4}[4 - 2 \times 0.8 - 0.5] = 0.475$$

$$x_2^{(k+1)} = \frac{1}{3}\left[4 - x_1^{(k)} - x_3^{(k)}\right] \quad \Longrightarrow x_2^{(1)} = \frac{1}{3}[4 - 0.1 - 0.5] = 1.1333$$

$$x_3^{(k+1)} = \frac{1}{6}\left[7 - 3x_1^{(k)} - 2x_2^{(k)}\right] \quad \Longrightarrow x_3^{(1)} = \frac{1}{6}[7 - 3 \times 0.1 - 2 \times 0.8] = 0.85$$

Similarly, we can get:

$$x_1^{(2)} = 0.2209 \qquad x_2^{(2)} = 0.8917 \qquad x_3^{(2)} = 0.5514$$

$$x_1^{(3)} = 0.4163 \qquad x_2^{(3)} = 1.0759 \qquad x_3^{(3)} = 0.7590$$

Gauss-Seidel Method:

Starting Guess: $x^{(0)} = [0.1, 0.8, 0.5]^T$

$$x_1^{(k+1)} = \frac{1}{4}\left[4 - 2x_2^{(k)} - x_3^{(k)}\right] \qquad \Longrightarrow x_1^{(1)} = \frac{1}{4}[4 - 2 \times 0.8 - 0.5] = 0.475$$

$$x_2^{(k+1)} = \frac{1}{3}\left[4 - x_1^{(k+1)} - x_3^{(k)}\right] \qquad \Longrightarrow x_2^{(1)} = \frac{1}{3}[4 - 0.475 - 0.5] = 1.0083$$

$$x_3^{(k+1)} = \frac{1}{6}\left[7 - 3x_1^{(k+1)} - 2x_2^{(k+1)}\right] \qquad \Longrightarrow x_3^{(1)} = \frac{1}{6}[7 - 3 \times 0.475 - 2 \times 1.0083] = 0.5931$$

Similarly, we can get:

$$x_1^{(2)} = 0.3476 \qquad x_2^{(2)} = 1.0198 \qquad x_3^{(2)} = 0.6529$$

$$x_1^{(3)} = 0.3269 \qquad x_2^{(3)} = 1.0069 \qquad x_3^{(3)} = 0.6677$$

Convergence of Iterative Methods

- Jacobi Method
- Gauss-Seidel Method

Iterative Methods: $x^{(k+1)} = Gx^{(k)} + Hb$

Necessary and Sufficient Conditions:

The iterative methods converge for any initial guess if and only if all the eigenvalues of the iteration matrix G have absolute value less than 1.

OR

The iterative methods converge if and only if the **spectral radius** (largest absolute eigenvalue) of G is less than 1, i.e., $\rho(G) < 1$.

Lemma: Let A be a square matrix. Then $\lim_{m\to\infty} A^m = \mathbf{0}$ iff $\rho(A) < 1$

Sketch of the proof: Suppose A is diagonalizable. Then there exist a matrix P such that

$$A = PDP^{-1}$$

Where D is a diagonal matrix having the eigenvalues of A on the diagonal. Therefore

$$A^m = PD^mP^{-1}$$

with $$D = \begin{bmatrix} \lambda_1^m & 0 & \cdots & & 0 \\ 0 & \lambda_2^m & & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n^m \end{bmatrix}$$

$\Longrightarrow \lim_{m\to\infty} A^m = 0$ iff all the eignevalues satisfy $|\lambda_i| < 1$ $(\rho(A) < 1)$

The iterative methods $x^{(k+1)} = Gx^{(k)} + Hb$ converge if and only if the spectral radius satisfies $\rho(G) < 1$.

Sketch of the proof: The error is given by $e_k = x^{(k)} - x$

$$\Longrightarrow e_k = \left(Gx^{(k-1)} + Hb\right) - (Gx + Hb)$$

$$\Longrightarrow e_k = G\left(x^{(k-1)} - x\right) \quad \Longrightarrow e_k = Ge_{k-1} \Longrightarrow e_k = G^k e_0$$

$\Longrightarrow \lim_{k\to\infty} e_k = 0$ for any e_0 if and only if $\rho(G) < 1$

Remark : If the spectral radius of G is small, then the convergence is rapid and if the radius of G is close to unity then convergence is very slow.