LECTURE – 2 & 3

Gauss Elimination Method

System of Linear Equations: Solution Methods

- Method of Determinants: Cramer's rule
- Matrix Inversion Method: $Ax = b \Rightarrow x = A^{-1}b$
- Gauss Elimination Method

direct method
(exact solution)

- Iterative Method – Jacobi & Gauss-Seidel method

approximate
solution

System of Linear Equations: Gauss Elimination Method

Elementary Row Operations

- Interchange of i-th and j-th rows $\left(R_i \leftrightarrow R_j\right)$
- Multiplication of the i-th row by a nonzero number λ $(R_i \leftarrow \lambda R_i)$
- Addition of λ times the j-th row to the i-th row $\left(R_i \leftarrow R_i + \lambda R_j\right)$

Gauss Elimination: Example - 1

$$x_1 + x_2 + x_3 = 4$$
$$2x_1 + 3x_2 + x_3 = 7$$
$$x_1 + 2x_2 + 3x_3 = 9$$

Augmented Matrix

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 7 \\ 1 & 2 & 3 & 9 \end{array}\right]$$

$$R_2 \leftarrow R_2 - 2R_1$$
$$R_3 \leftarrow R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 2 & 5 \end{array}\right]$$

$$R_3 \leftarrow R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

Gauss Elimination: Example - 1

$$[A|b] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

Echelon form

Back substitution:

$$x_3 = 2$$

$$x_2 = -1 + x_3 = 1$$

$$x_1 = 4 - x_2 - x_3 = 1$$

Number of Pivots = Number of Unknowns $\Longrightarrow$ Unique Solution

OR every column has a pivot

Gauss Elimination: Example - 2

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 7 \\ 1 & 2 & 0 & 9 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 5 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

Equations are inconsistent and hence the solution does not exist.

right-most column has a pivot

$\Longrightarrow$ No Solution

Gauss Elimination: Example - 3

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 7 \\ 1 & 2 & 0 & 3 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 - R_1$

number of pivots (r)

$<$ numer of unknowns (n)

number of free variable = (n − r)

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Equations are Consistent

x_3 Free variable

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 Free variable

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 3x_2 + x_3 &= 7 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

Choose $x_3 = \alpha$

$x_2 = -1 + \alpha$

$x_1 = 5 - 2\alpha$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

particular solution

solution of $Ax = 0$

(Null Space)

$$x = x_p + x_h$$

Solution of System of Linear Equations: $A_{m\times n}x_{n\times 1} = b_{m\times 1}$

Echelon Form:

$$[\tilde{A}|\tilde{b}] = \left(\begin{array}{ccccccccc|c}
\boxtimes & * & * & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & \boxtimes & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & \boxtimes & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
0 & \cdots & 0 & 0 & 0 & \boxtimes & * & \cdots & * & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes \\
\vdots & & & & & & & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes
\end{array}\right)$$

pivot rows (r)

zero rows $(m-r)$

$\boxtimes$ – pivot element $\neq 0$ $\quad$ $\otimes$ & $*$ – other elements (may be zero)

Def. Rank (A) = *r* (number of pivots)

$$[A|b]\sim\left(\begin{array}{ccccccccc|c}
\boxtimes & * & * & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & \boxtimes & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & \boxtimes & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
0 & \cdots & 0 & 0 & 0 & \boxtimes & * & \cdots & * & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes \\
\vdots & & & & & & & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes
\end{array}\right)
\begin{array}{l}
\left.\vphantom{\begin{matrix}a\\a\\a\\a\\a\\a\end{matrix}}\right\}(r) \\
\left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}(m-r)
\end{array}$$

- If $\otimes \neq 0$ the equations become inconsistent and hence the system has no solution

OR in terms of rank: $\text{Rank}\,(A) \neq \text{Rank}([A|b])$

$$[A|b] \sim \left(\begin{array}{ccccccccc|c}
\boxtimes & * & * & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & \boxtimes & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & \boxtimes & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
0 & \cdots & 0 & 0 & 0 & \boxtimes & * & \cdots & * & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes \\
\vdots & & & & & & & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes
\end{array}\right)
\begin{array}{l}
\left.\rule{0pt}{6em}\right\} (r) \\
\left.\rule{0pt}{3em}\right\} (m-r)
\end{array}$$

Def. Rank (A) = *r* (number of pivots)

If $\otimes = 0$

The system will be consistent

($Ax = 0$ is always consistent)

- If $\otimes = 0$ and number of pivot elements (r) = number of unknowns (n)

  OR each column has a pivot

  Then the system has a unique solution

  OR in terms of rank: $\text{Rank}\,(A) = \text{Rank}([A|b]) = n$

Def. Rank (A) = *r* (number of pivots)

$$[A|b] \sim \left(\begin{array}{ccccccccc|c}
\boxtimes & * & * & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & \boxtimes & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & \boxtimes & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
0 & \cdots & 0 & 0 & 0 & \boxtimes & * & \cdots & * & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes \\
\vdots & & & & & & & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes
\end{array}\right)
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} (r) \\
\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} (m-r)
\end{array}$$

- If $\otimes = 0$ and number of pivot elements $(r) <$ number of unknowns (n)

  Then the system has infinitely many solutions

  OR in terms of rank: $\text{Rank}\ (A) = \text{Rank}([A|b]) < n$

Problem -1 Solve the system of equations $Ax = b$ with

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 2 & 4 & -4 & 0 & 3 & 2 \\ -1 & -2 & 3 & 3 & 4 & 3 \\ 3 & 6 & -7 & 1 & 1 & \beta \end{array}\right]; \qquad \beta \in \mathbb{R}$$

$R_2 \to R_2 - 2R_1$ $\qquad$ $R_3 \to R_3 + R_1$ $\qquad$ $R_4 \to R_4 - 3R_1$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & -1 & 4 & -2 & \beta - 3 \end{array}\right]$$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & -1 & 4 & -2 & \beta - 3 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 4 & -2 & \beta - 3 \end{array}\right]$$

$R_4 \to R_4 + R_2$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 6 & 3 & \beta + 1 \end{array}\right]$$

$R_4 \to R_4 - 3R_3$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta + 1 \end{array}\right]$$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta + 1 \end{array}\right]$$

Case – I: $\beta \neq -1 \quad \Longrightarrow$ No Solution

Case – II: $\beta = -1$

$$\begin{array}{ccccccc} x_1 & & x_3 & x_4 & & & \\ \hline \end{array}$$

dependent variables

$$[A|b] = \begin{array}{c} \begin{array}{cccccc} x_1 & & x_3 & x_4 & & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccc} & x_2 & & & x_5 & \end{array} \end{array}$$

free variables

Case – II: $\beta = -1$

Take $x_2 = \alpha_1, \quad x_5 = \alpha_2, \quad$ then

dep. variables $x_1 \quad x_3 \quad x_4$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free variables $x_2 \quad x_5$

$$x_4 = -\tfrac{1}{2}\alpha_2 \qquad x_3 = 4 - 4\alpha_2$$

$$x_1 = 9 - 2\alpha_1 - 9.5\alpha_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -9.5 \\ 0 \\ -4 \\ -0.5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} + \underbrace{\alpha_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -9.5 \\ 0 \\ -4 \\ -0.5 \\ 1 \end{bmatrix}}_{x_h}$$

$$x = x_p + x_h$$

$$Ax_p = b \qquad Ax_h = 0$$

Solution of nonhomogeneous linear system are of the form $x = x_p + x_h$, where x_p is any fixed solution of $Ax = b$ and x_h runs through all the solutions corresponding to homogeneous system $Ax = 0$.

Remarks:

- Free variable(s) is (are) responsible for infinitely many solutions

- An invertible matrix has no free variable $(Ax = b \Longrightarrow x = A^{-1}b)$ unique solution

- Vectors that generate solutions of $Ax = 0$ are $[-2, 1, 0, 0, 0]^T$ & $[-9.5, 0, -4, -0.5, 1]^T$

- These generators are called BASIS of solution space of $Ax = 0$ (NULL Space)

- NULL Space is a vector space (next lecture)

SUMMARY:

System of Linear Equations

- Gauss Elimination
- Echelon form
- Solution (Consistency & Inconsistency)
- Free variables – Infinitely many solutions
- $x = x_p + x_h$