Convergence of Iterative Methods

- Jacobi Method
- Gauss-Seidel Method

Iterative Methods: $x^{(k+1)} = Gx^{(k)} + Hb$

Necessary and Sufficient Conditions:

The iterative methods converge for any initial guess if and only if all the eigenvalues of the iteration matrix G have absolute value less than 1.

OR

The iterative methods converge if and only if the **spectral radius** (largest absolute eigenvalue) of G is less than 1, i.e., $\rho(G) < 1$.

Lemma: Let A be a square matrix. Then $\lim_{m\to\infty} A^m = \mathbf{0}$ iff $\rho(A) < 1$

Sketch of the proof: Suppose A is diagonalizable. Then there exist a matrix P such that

$$A = PDP^{-1}$$

Where D is a diagonal matrix having the eigenvalues of A on the diagonal. Therefore

$$A^m = PD^mP^{-1}$$

with $$D = \begin{bmatrix} \lambda_1^m & 0 & \cdots & & 0 \\ 0 & \lambda_2^m & & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n^m \end{bmatrix}$$

$\Longrightarrow \lim_{m\to\infty} A^m = 0$ iff all the eignevalues satisfy $|\lambda_i| < 1$ $(\rho(A) < 1)$

The iterative methods $x^{(k+1)} = Gx^{(k)} + Hb$ converge if and only if the spectral radius satisfies $\rho(G) < 1$.

Sketch of the proof: The error is given by $e_k = x^{(k)} - x$

$$\Rightarrow e_k = \left(Gx^{(k-1)} + Hb\right) - (Gx + Hb)$$

$$\Rightarrow e_k = G\left(x^{(k-1)} - x\right) \Rightarrow e_k = Ge_{k-1} \Rightarrow e_k = G^k e_0$$

$$\Rightarrow \lim_{k\to\infty} e_k = 0 \text{ for any } e_0 \text{ if and only if } \rho(G) < 1$$

Remark : If the spectral radius of G is small, then the convergence is rapid and if the radius of G is close to unity then convergence is very slow.

Vector Norm: Let $x, y \in \mathbb{R}^n$. The norm of a vector is number that measures "size" or "length" of a vector. It satisfies

(i) $\|x\| > 0$ for $x \neq 0$ and $\|x\| = 0$ for $x = 0$

(ii) $\|\lambda x\| = |\lambda|\|x\|, \quad \forall\, \lambda \in \mathbb{R}$

(iii) $\|x + y\| \leq \|x\| + \|y\|$

Matrix Norm: Let $A, B \in \mathbb{R}^{n\times n}$. Similar to vector norm, matrix norm also satisfies the following properties

(i) $\|A\| > 0$ for $A \neq 0$ and $\|A\| = 0$ for $A = 0$

(ii) $\|\lambda A\| = |\lambda|\|A\|, \quad \forall\, \lambda \in \mathbb{R}$

(iii) $\|A + B\| \leq \|A\| + \|B\|$

Note: For any vector norm, we can also define a corresponding matrix norm (called induced matrix norm) as

$$\|A\| = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} \qquad \Longrightarrow \|Ax\| \leq \|A\|\|x\|$$

Example of Matrix Norms

Row Sum Norm: $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$

Frobenius Norm: $\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2}$

Column Sum Norm: $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$

Convergence Theorems (Iterative Methods for Solving $Ax = b$) Sufficient Conditions :

1. If any **norm of iteration matrix G** is less than 1, i.e. $\|G\| < 1$, then the iterative methods converge for any initial guess.

2. If A is **strictly diagonally dominant** by rows (or by columns) then the Jacobi and Gauss-Siedel methods converge for any initial guess.

1. If any **norm of iteration matrix $\boldsymbol{G}$** is less than 1, i.e. $\|G\| < 1$, then the iterative methods converge for any initial guess.

Note that $\quad Gx = \lambda x \quad \Longrightarrow \|\lambda x\| = \|Gx\|$

$$\Longrightarrow |\lambda|\|x\| \leq \|G\|\|x\|$$

$$\Longrightarrow |\lambda| \leq \|G\|$$

$$\Longrightarrow \rho(G) \leq \|G\|$$

It clearly shows that if $\|G\| < 1$ then spectral radius $\rho(A) < 1$

2. If A is **strictly diagonally dominant** by rows (or by columns) then the Jacobi and Gauss-Seidel methods converge for any initial guess.

Sketch of the proof (Jacobi): $x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b = -D^{-1}(A-D)\, x^{(k)} + D^{-1}b$

$$= (I - D^{-1}A)\, x^{(k)} + D^{-1}b$$

The scheme will converge if $\|(I - D^{-1}A)\|_\infty < 1 \Leftrightarrow$ (Row sum norm)

$$\Leftrightarrow \max_i \frac{1}{|a_{ii}|} \sum_{j=1, j\neq i}^{n} |a_{ij}| < 1 \Leftrightarrow \frac{1}{|a_{ii}|} \sum_{j=1, j\neq i}^{n} |a_{ij}| < 1, \forall i \Leftrightarrow \sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}|, \forall i$$

If A is diagonally dominant by rows then $\|(I - D^{-1}A)\|_\infty < 1$ and hence Jacobi method will converge. Similarly one can prove if A is diagonally dominant by columns. Convergence of Gauss-Seidel also follows similar steps.

Example: Consider the following system of equations

$$5x + y + 2z = 13$$

$$x + 3y + z = 12$$

$$-x + 2y + 4z = 8$$

Discuss the convergence of the Jacobi and Gauss-Seidel methods?

The coefficient matrix is strictly diagonally dominant by rows and hence both the methods will converge for any initial guess.

Example: Analyse the convergence of Gauss-Seidel method for solving the system.

$$2x + y + z = 4$$
$$x + 2y + z = 4$$
$$x + y + 2z = 4$$

Solution: The coefficient matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$G_{GS} = -(L + D)^{-1}U = -\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} 4 & 0 & 0 \\ -2 & 4 & 0 \\ -1 & -2 & 4 \end{bmatrix}$$

$$G_{GS} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & \frac{1}{4} & \frac{-1}{4} \\ 0 & \frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

$$\| G_{GS} \|_1 = \max\{0, 7/8, 9/8\} > 1$$

$$\| G_{GS} \|_\infty = \max\{1, 1/2, 1/2\} = 1$$

$$\| G_{GS} \|_F = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{16} + \frac{1}{16} + \frac{1}{4} + \frac{9}{64}} = \sqrt{\frac{32 + 8 + 10}{64}} = \sqrt{\frac{50}{64}} < 1$$

⇒ The Gauss-Siedel method will converge.

A comparative Study of Jacobi and Gauss-Seidel Method

Consider the following system of linear equations

$$5x + y + 2z = 13$$

$$x + 3y + z = 12$$

$$-x + 2y + 4z = 8$$

Exact Solution: [1.6364 3.1818 0.8182]

Spectral Radius of the Iteration Matrices:

Gauss-Seidel Method ρ_{GS}: **0.1667**

Jacobi Method ρ_J: **0.4860**

Iteration	Jacobi Method
1	2.0000 3.3333 1.7500
2	1.2333 2.7500 0.8333
3	1.7167 3.3111 0.9333
4	1.5644 3.1167 0.7736
5	1.6672 3.2206 0.8328
6	1.6228 3.1667 0.8065
⋮	
13	1.6365 3.1819 0.8182
14	1.6363 3.1818 0.8182
15	**1.6364 3.1818 0.8182**
16	1.6364 3.1818 0.8182

Exact Solution: [1.6364 3.1818 0.8182]

ρ_{GS} = **0.1667**

Iterative Method: $x^{(k+1)} = Gx^{(k)} + \text{Hb}$

ρ_J = **0.4860**

Initial Guess = [1 1 1]

Iteration	Gauss-Seidel Method
1	2.0000 3.0000 1.0000
2	1.6000 3.1333 0.8333
3	1.6400 3.1756 0.8222
4	1.6360 3.1806 0.8187
5	1.6364 3.1816 0.8183
6	**1.6364 3.1818 0.8182**
7	1.6364 3.1818 0.8182

Iteration	Jacobi Method
1	2.6000 4.0000 2.0000
2	1.0000 2.4667 0.6500
3	1.8467 3.4500 1.0167
4	1.5033 3.0456 0.7367
5	1.6962 3.2533 0.8531
6	1.6081 3.1502 0.7974
⋮	
15	1.6364 3.1819 0.8182
16	1.6363 3.1818 0.8182
17	**1.6364 3.1818 0.8182**
18	1.6364 3.1818 0.8182

Exact Solution: [1.6364 3.1818 0.8182]

ρ_{GS} = **0.1667**

Iterative Method: $x^{(k+1)} = Gx^{(k)} + \text{Hb}$

ρ_J = **0.4860**

Initial Guess = [0 0 0]

Iteration	Gauss-Seidel Method
1	2.6000 3.1333 1.0833
2	1.5400 3.1256 0.8222
3	1.6460 3.1773 0.8229
4	1.6354 3.1806 0.8186
5	1.6365 3.1817 0.8183
6	**1.6364 3.1818 0.8182**
7	1.6364 3.1818 0.8182

REMARK

Gauss Seidel as by construction seems to be faster than Jacobi method. However this is not true in general.

There are examples where Jacobi converges faster than Gauss Seidel.

If we change the order of the equations the iterative methods may not be convergent.

For example, consider the following system of linear equations

$$x + 3y + z = 12$$

$$-x + 2y + 4z = 8$$

$$5x + y + 2z = 13$$

Exact Solution: [1.6364 3.1818 0.8182]

Initial Guess = [1 1 1]

Iteration	Jacobi Method	Gauss-Seidel Method
1	8.0000 2.5000 3.5000	8.0000 6.0000 -16.5000
2	1.0000 1.0000 -14.7500	10.5000 42.2500 -40.8750
3	23.7500 34.0000 3.5000	-73.8750 48.8125 166.7813
4	-93.5000 8.8750 -69.8750	-301.2188 -480.1719 999.6328
⋮		
10	$10^4 \times$ [-2.5925 -0.6066 -2.9968]	$10^6 \times$ [0.5066 -1.4169 -0.5580]

$\rho_{GS} = \mathbf{3.8730}$

$\rho_J = \mathbf{2.7233}$

Remark: Without analyzing iteration matrix, **it** is difficult to conclude that which of the methods converges faster.

Consider the following examples of coefficient matrices:

Example 1:

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 6 & 4 & 2 \\ -1 & 5 & 2 \end{bmatrix}$$

$$\rho_J = 1.3121$$

$$\rho_{GS} = 0.4000$$

Observation: Jacobi method fails to converge while Gauss-Seidel method converges.

Example 2:

$$A = \begin{bmatrix} -3 & 3 & -6 \\ -4 & 7 & -8 \\ 5 & 7 & -9 \end{bmatrix}$$

$$\rho_J = 0.8133$$

$$\rho_{GS} = 1.1111$$

Observation: Gauss-Seidel method fails to converge while Jacobi method converges.

Example 3:

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 2 & -9 & 0 \\ 0 & -8 & -6 \end{bmatrix} \qquad \rho_J = 0.4438 \qquad \rho_{GS} = 0.0185$$

Observation: Jacobi method is more slowly convergent than Gauss-Seidel.

Example 4:

$$A = \begin{bmatrix} 7 & 6 & 9 \\ 4 & 5 & -4 \\ -7 & -3 & 8 \end{bmatrix} \qquad \rho_J = 0.6411 \qquad \rho_{GS} = 0.7746$$

Observation: Gauss-Seidel method is more slowly convergent than Jacobi.

CONCLUSIONS

Convergence of Iterative Methods

Necessary and Sufficient Conditions:

The iterative methods converge if and only if the spectral radius (largest absolute eigenvalue) of G is less than 1, i.e. $\rho(G) < 1$.

Sufficient Conditions

1. If any **norm of iteration matrix $\boldsymbol{G}$** is less than 1, i.e. $||G|| < 1$, then the iterative methods converge for any initial guess.

2. If A is **strictly diagonally dominant** by rows (or by columns) then the Jacobi and Gauss-Siedel methods converge for any initial guess.

Determination of Roots of Algebraic and Transcendental Equations

- Bisection Method
- Fixed Point Iteration Method
- Newton-Raphson Method
- Secant Method

Bisection Method

It is based on the following theorem for zeroes of continuous functions:

Theorem: Given a continuous function $f : [a, b] \to \mathbb{R}$ such that $f(a)f(b) < 0$, then $\exists\ \alpha \in (a, b)$ such that $f(\alpha) = 0$.

Outline of the Algorithm

Choosing $I_0 = [a, b]$, so that $f(a)f(b) < 0$.

The bisection method generates a sequence of subinterval $I_k = \left[a^{(k)}, b^{(k)}\right], k \geq 0$

such that $I_k \subset I_{k-1}, k \geq 1$ and the property $f\left(a^{(k)}\right)f\left(b^{(k)}\right) < 0$.

Pseudocode

Set $a^{(0)} = a\,, b^{(0)} = b$ and $x^{(0)} = \dfrac{a+b}{2}$

For $k \geq 0$

if $f\left(a^{(k)}\right) f\left(x^{(k)}\right) < 0$

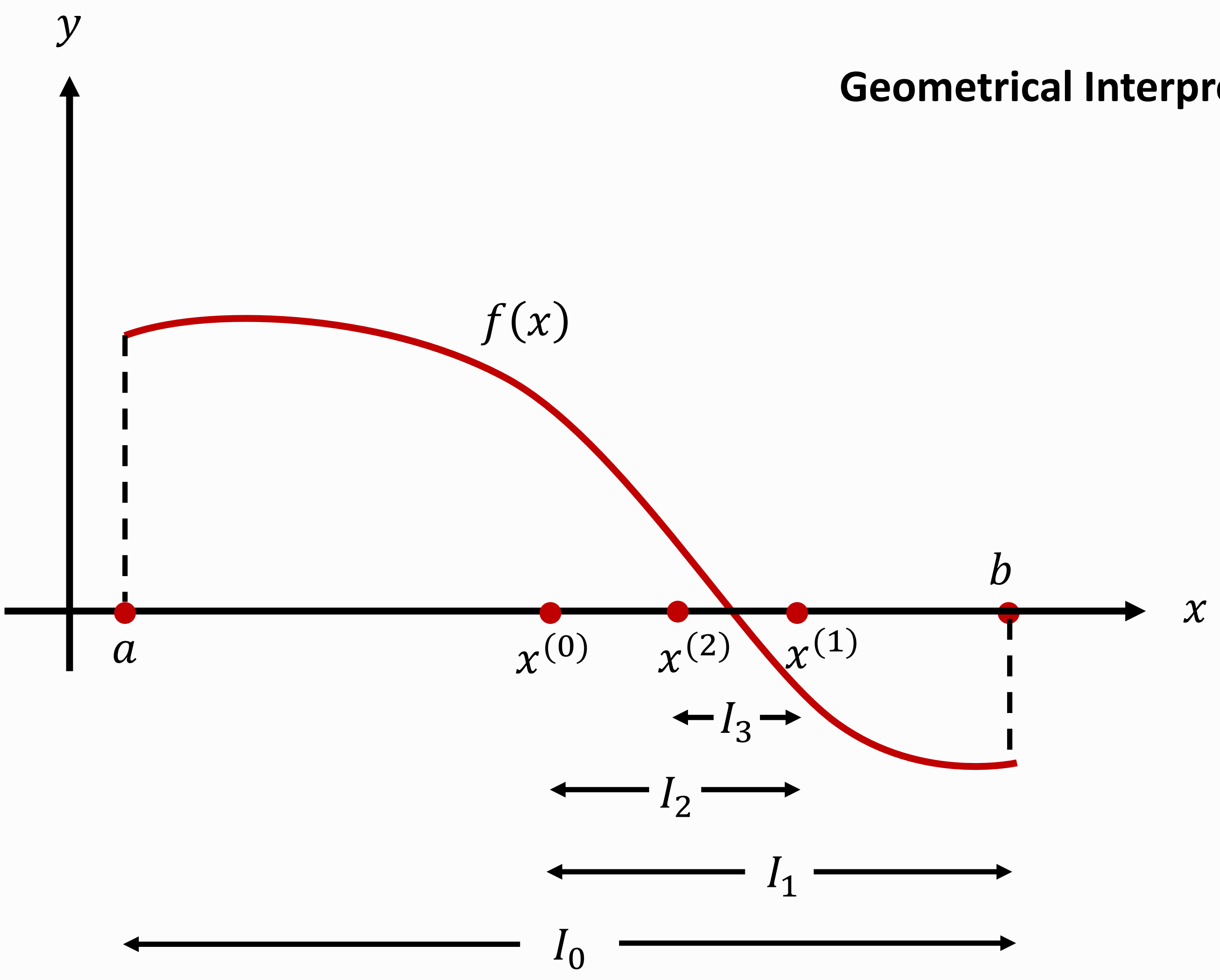
set $a^{(k+1)} = a^{(k)}$ $\quad b^{(k+1)} = x^{(k)}$

if $f\left(x^{(k)}\right) f\left(b^{(k)}\right) < 0$

set $a^{(k+1)} = x^{(k)}$ $\quad b^{(k+1)} = b^{(k)}$

Set $x^{(k+1)} = \dfrac{a^{(k+1)} + b^{(k+1)}}{2}$

Geometrical Interpretation of Bisection Method



Convergence of Bisection Method

Let $|I_k| = \left|b^{(k)} - a^{(k)}\right|$ $\quad f(\alpha) = 0$

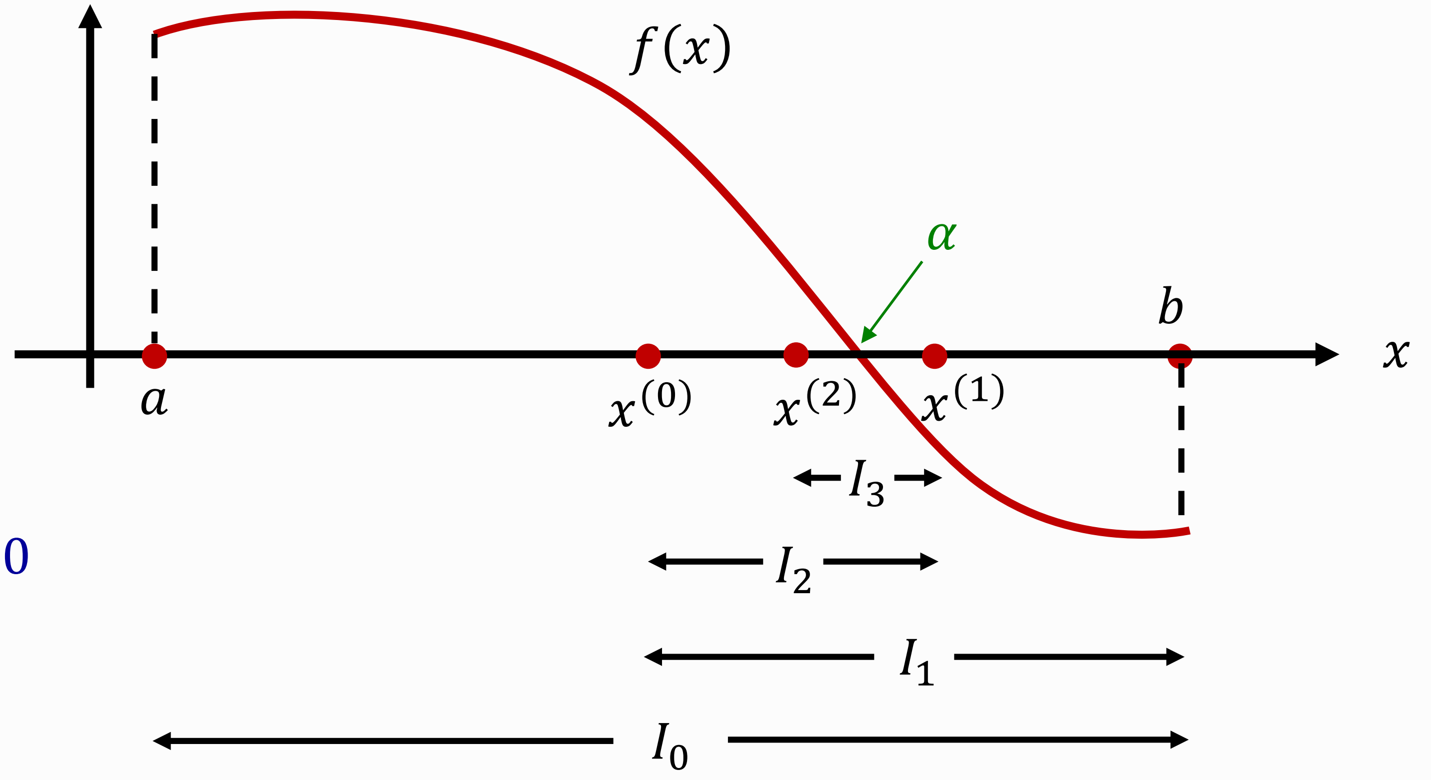
Note that $|I_k| = \dfrac{|I_{k-1}|}{2}$

$$\Rightarrow |I_k| = \frac{|I_0|}{2^k};\ k \geq 0 \ \Rightarrow |I_k| = \frac{b-a}{2^k};\ k \geq 0$$

Denoting error $\quad e^{(k)} = x^{(k)} - \alpha$

$$\Rightarrow \left|e^{(k)}\right| < \frac{|I_k|}{2} = \frac{(b-a)}{2^{(k+1)}};\ k \geq 0 \ \Rightarrow \lim_{k\to\infty}\left|e^{(k)}\right| = 0$$

The bisection method is globally convergent!



Example : Perform five iterations of the bisection method to obtain he smallest positive root of the equation

$$f(x) := x^3 - 5x + 1 = 0$$

Actual Roots:

$2.12841, \quad -2.33005, \quad 0.20163$

Solution : $f(0) = 1 \;\&\; f(1) = -3 \quad \Rightarrow f(0)f(1) < 0$

Initialization $\quad a^{(0)} = 0 \qquad b^{(0)} = 1 \qquad x^{(0)} = \dfrac{1+0}{2} = 0.5$

Observe $\quad f\left(a^{(0)}\right) f\left(x^{(0)}\right) < 0$

$a^{(0)} = 0 \quad b^{(0)} = 1 \quad x^{(0)} = \frac{1+0}{2} = 0.5 \quad f\left(a^{(0)}\right) f\left(x^{(0)}\right) < 0$

$f(x) = x^3 - 5x + 1$

Iteration	$a^{(k)}$	$x^{(k)}$	$b^{(k)}$	Observation
1	0 $(f > 0)$	0.25 $(f < 0)$	0.5 $(f < 0)$	$f\left(a^{(k)}\right) f\left(x^{(k)}\right) < 0$
2	0 $(f > 0)$	0.125 $(f > 0)$	0.25 $(f < 0)$	$f\left(x^{(k)}\right) f\left(b^{(k)}\right) < 0$
3	0.125 $(f > 0)$	0.1875 $(f > 0)$	0.25 $(f < 0)$	$f\left(x^{(k)}\right) f\left(b^{(k)}\right) < 0$
4	0.1875 $(f > 0)$	0.21875 $(f < 0)$	0.25 $(f < 0)$	$f\left(a^{(k)}\right) f\left(x^{(k)}\right) < 0$

Root lies in (0.1875,0.21875)

Approximate root after

5 iterations:

$x^{(5)} = 0.203125$

Fixed Point Iteration Method:

Idea of general iteration method:

Rewrite $f(x) = 0$ to the form $x = g(x)$ and set up the iterations

$$x^{(k+1)} = g\left(x^{(k)}\right), \qquad k = 0, 1, 2, \dots$$

Convergence of the method will depend on the function $g(x)$.

Remark: The point x^* is called a fixed point of the function g is $x^* = g(x^*)$.

$$f(x) = 0 \Leftrightarrow x = g(x)$$

Note that the choice of g is not unique. For instance, we me take:

$$g(x) = x - f(x)$$

$$g(x) = x + 2f(x)$$

$$g(x) = x - \frac{f(x)}{f'(x)} \qquad \text{assuming } f'(x) \neq 0$$

Sufficient condition for convergence

If $g(x)$ is continuous in some interval $[a, b]$ that contains the root and $|g'(x)| \leq \rho < 1$ in this interval, then for any choice of $x^{(0)}$ from $[a, b]$ the sequence $x^{(k)}$ will converge to the root of the equation $f(x) = 0$.

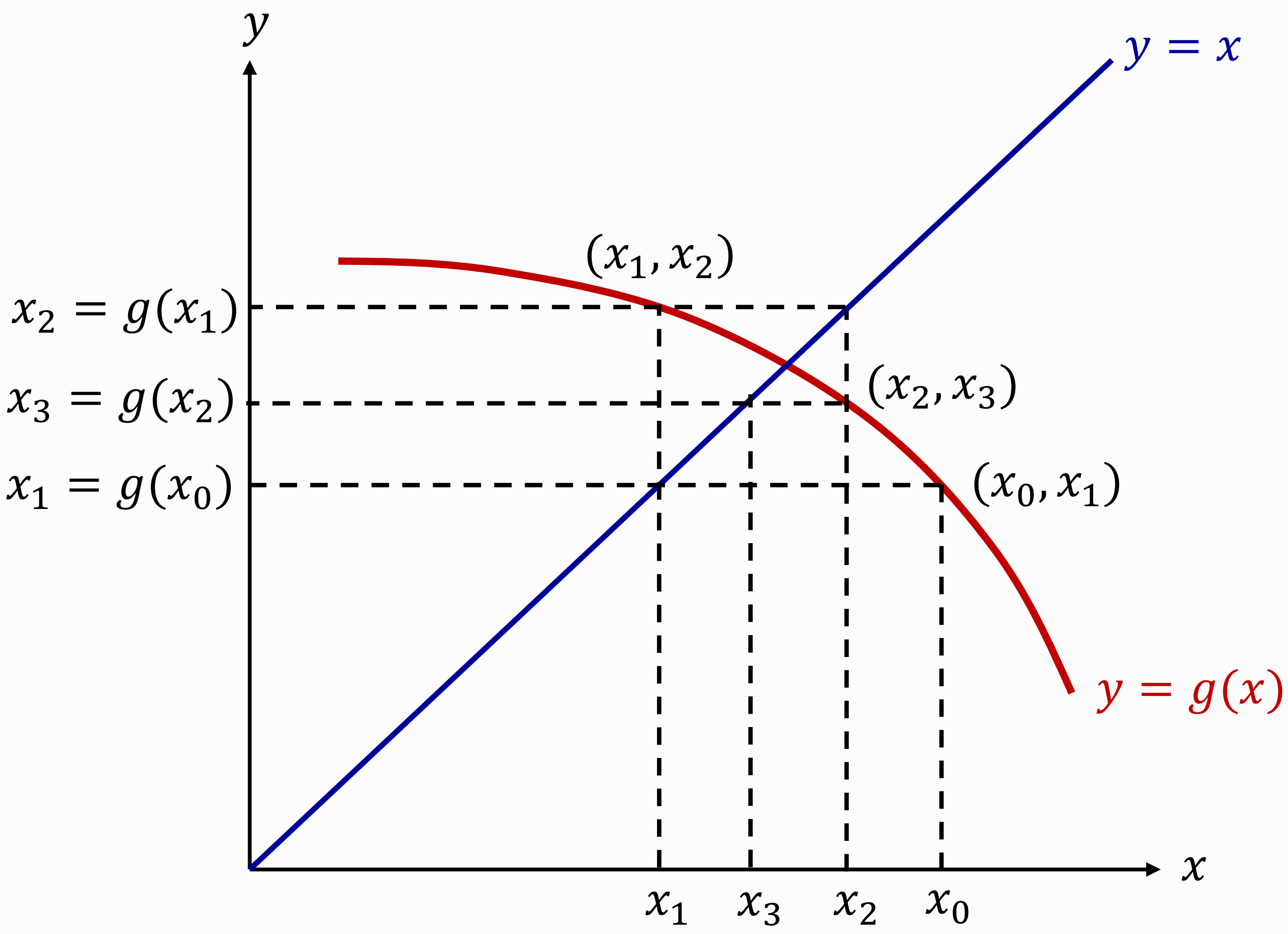
Proof: Consider $\left|x^{(k+1)} - x^*\right| = \left|g\left(x^{(k)}\right) - g(x^*)\right| = \left|g'(\xi)\left(x^{(k)} - x^*\right)\right|, \ \xi \in \left(x^{(k)}, x^*\right)$ using MVT

Since $|g'(x)| \leq \rho$, we get

$$\left|x^{(k+1)} - x^*\right| \leq \rho\left|x^{(k)} - x^*\right| \leq \rho^2\left|x^{(k-1)} - x^*\right| \leq \cdots \leq \rho^{k+1}\left|x^{(0)} - x^*\right|$$

Since $\rho < 1$, we have $\rho^k \to 0$ as $k \to \infty$.

Geometrical Interpretation $x^{(k+1)} = g\left(x^{(k)}\right)$



Example : Consider $x^3 - 5x + 1 = 0$

Case 1: Rewrite the equation $x = g(x) = \dfrac{(1 + x^3)}{5}$

Iteration method becomes:

$$x^{(k+1)} = \frac{\left(1 + \left(x^{(k)}\right)^3\right)}{5} \qquad\qquad g'(x) = \frac{3x^2}{5}$$

Root lies in the interval $(0, 1)$ so we can choose $x^{(0)} = 0.5$ as initial guess

$x^{(0)} = 0.5$ $\quad x^{(1)} = 0.2250$ $\quad x^{(2)} = 0.2023$

$x^{(3)} = 0.2017$ $\quad x^{(4)} = 0.2016$ $\quad x^{(5)} = 0.2016$

Case 2: Now take initial guess $\boldsymbol{x^{(0)}=2.5}$ in the above example.

$$x^{(k+1)}=\frac{\left(1+\left(x^{(k)}\right)^3\right)}{5}$$

$x^{(1)}=3.325$ $x^{(2)}=7.552$

$x^{(3)}=86.3419$ $x^{(4)}=1.2873\times10^5$ $x^{(5)}=4.2669\times10^{14}$

The iterations are diverging toward plus infinity.

Remark : Note that $g'(x)=\dfrac{3x^2}{5}$ in above both the cases.

- In case 1, in the interval containing the root and initial guess, $|g'|<1$ and hence convergence is guaranteed.
- In case 2, in the interval containing the root and initial guess, $|g'|>1$ and hence convergence is NOT guaranteed.

Case 3: Rewrite the equation as $x = g(x) = \dfrac{-1}{x^2 - 5}$

Now taking the initial guess $x^{(0)} = 2.5$, we get

$x^{(0)} = 2.5$ $\quad$ $x^{(1)} = -0.80$ $\quad$ $x^{(2)} = 0.2294$

$x^{(3)} = 0.2021$ $\quad$ $x^{(4)} = 0.2016$ $\quad$ $x^{(5)} = 0.2016$

Remark :

Note that, $|g'| = \dfrac{2|x|}{(x^2 - 5)^2}$

In the interval containing the root and initial guess $|g'| > 1$ but the sequence converges as this is the sufficient condition for convergence not necessary.

CONCLUSIONS

- **Bisection Method**

  The Bisection Method is an iterative approach that narrows down an interval that contains a root of the function $f(x)$. Convergence is always guaranteed.

- **Fixed Point Iteration Method** $\boldsymbol{f(x) = 0 \Leftrightarrow x = g(x)}$

  $$x^{(k+1)} = g\left(x^{(k)}\right), \qquad k = 0, 1, 2, \ldots$$

  Convergence is guaranteed if $|g'(x)| \leq \rho < 1$