Analysis and Realization of an Exponentially-Decaying Impulse Response Model for Frequency-Selective Fading Channels

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***Abstract*—We analyze a simple frequency-selective fading model that is formulated as a sequence of independent discrete-time impulse responses, each with independent Gaussian variates scaled by an exponential power-delay profile. This model is useful for simulating propagation path responses in a scattering environment. Various expressions and relationships are derived for the frequency correlation function, correlation bandwidth, rms delay spread, and decay time. Realization techniques are also discussed, and it is concluded that the most straightforward and efficient approach is to generate a block of samples by multiplying i.i.d. complex Gaussian variates by the square root of the exponential power-delay profile and then taking the FFT.**

***Index Terms*—Correlation bandwidth, decay time, exponential power-delay profile, fading model, frequency correlation function, frequency-selective, rms delay spread.**

I. Introduction

In the simulation of many types of wideband communication systems, it is essential to have a realistic model of the propagation channel that includes variation over frequency. In a rich scattering environment, a channel formed by the combination of several paths can be modeled as a random process that can produce deep fades. For narrowband communication, these fades can be considered as "flat" or frequency-nonselective, whereby the response is approximately constant over frequency. However, for wideband systems, there is considerable variation of the frequency response over the bandwidth, giving rise to frequency-selective fading [1, Ch. 7].

Propagation in a scattering environment is often characterized by an impulse response with exponentially-decaying power-delay profile, where the individual components of each instantiation are independent complex Gaussian variates (independent scattering) [2]. This model is usually valid over some bandwidth that is much wider than the operating band of the system of interest. In this letter, we develop and analyze this model, and we discuss various approaches to realization.

Throughout this letter, the time evolution of the channel is implicitly represented as a sequence of independent impulse response snapshots, which exhibit frequency-selective block fading. As such, this model is useful for burst or packet mobile communications, as well as realizing an ensemble of responses to evaluate static systems. More elaborate models that include continuous-time variations for arbitrary power-delay profiles can be found in [2].

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II. Model

Here we formulate the analytical model of a channel $H(k)$ that induces selective fading over frequencies $k = 0, 1, \ldots, K-1$. For convenience, we assume a discrete-time baseband impulse response at the outset, where the sampling rate is assumed high enough to capture all of the significant temporal detail. For computational efficiency, we then consider decimation (filtering and downsampling) from this high sampling rate to the bandwidth of interest. (An exemplary set of physical parameter values will be presented later in Section IV.) If the decay rate is slow enough compared to the operating bandwidth, then the downsampled baseband impulse response still has independent complex Gaussian components with an exponentially-decaying power-delay profile (see the Appendix). Thus, we write each normalized impulse response as

$$h(n) = \sqrt{\frac{1-\alpha}{1-\alpha^K}}\alpha^{n/2}v(n), \quad n = 0, 1, \ldots, K-1 \tag{1}$$

where $0 < \alpha < 1$ and $v(n)$ is a set of K independent identically distributed (i.i.d.) Gaussian variates with zero mean $[E\{v(n)\} = 0]$ and unit variance $[E\{|v(n)|^2\} = 1]$. The power-delay profile of this channel impulse response is given by

$$E\{|h(n)|^2\} = \frac{1-\alpha}{1-\alpha^K}\alpha^n, \quad n = 0, 1, \ldots, K-1 \tag{2}$$

which is seen to decay exponentially at a rate of $10\log_{10}\alpha$ dB per sample. Also, we see that this response is normalized such that the power-delay profile (2) sums to unity. Note that we have implicitly assumed here that the impulse response is quasi-stationary in the sense that the particular realization is constant, but the channel evolves over time (or space) as an independent sequence of such realizations.

The channel frequency response is given by the discrete Fourier transform (DFT) of (1) as follows:

$$H(k) = \sum_{n=0}^{K-1} h(n)e^{-j2\pi nk/K}. \tag{3}$$

The frequency correlation function (FCF) expresses the correlation between the channel frequency response at one frequency with the response at another frequency spaced some distance away. For our discrete-time formulation, the FCF is defined as

$$\Gamma(k, l) \equiv E\{H(k)H^*(k-l)\} \tag{4}$$

where l is the frequency spacing index. Substituting (1) and (3) into (4) and using the i.i.d. property of $v(n)$, we calculate the FCF

$$\Gamma(l) = \frac{1-\alpha}{1-\alpha e^{-j2\pi l/K}}. \tag{5}$$

Here, we have dropped the k index because for the independent scattering model of (1), the FCF is wide-sense stationary (WSS), depending only on the frequency spacing l (sometimes called the "spaced-frequency" correlation function). As a consequence of the chosen normalization, we have $\Gamma(0) = 1$. We note in passing that, as is well known [2], (5) is the DFT of the power-delay profile (2). Also note that since our frequency response model is a WSS Gaussian process, it is therefore ergodic. (Experimental procedures for determining the validity of these assumptions for real data are developed in [3].)

It is useful to parameterize the FCF by the normalized correlation bandwidth B over which the magnitude of the correlation drops to 1/2 and relate this to the decay rate α. Setting $|\Gamma(BK)| = 1/2$ in (5) and solving for B shows that the normalized correlation bandwidth is given by

$$B = \frac{1}{2\pi}\cos^{-1}\left(\frac{8\alpha - 3 - 3\alpha^2}{2\alpha}\right) \qquad \frac{4-\sqrt{7}}{3} \le \alpha < 1 \tag{6a}$$

$$\xrightarrow[\alpha\to 1]{} \frac{\sqrt{3}}{2\pi}\cdot\frac{1-\alpha}{\sqrt{\alpha}}. \tag{6b}$$

Conversely, for any value of B, the corresponding value of α can be calculated as

$$\alpha = \frac{4-\cos(2\pi B) - \sqrt{[4-\cos(2\pi B)]^2 - 9}}{3} \qquad 0 < B \le \tfrac{1}{4} \tag{7a}$$

$$\xrightarrow[B\to 0]{} 1 - \frac{2\pi B}{\sqrt{3}}. \tag{7b}$$

(The reason for the lower bound on α in (6a) and consequent upper bound on B in (7a) is that, otherwise, the FCF is greater than 1/2 over the entire range $0 \le l \le K$, and hence, the 50% correlation bandwidth does not exist.)

Fig. 1 shows an example of the FCF (5) for $K = 32$ points and $B = 0.2$ ($\alpha = 0.5136$). The real part (solid) is symmetric and the imaginary part (dashed) is antisymmetric, while the magnitude (dotted) falls off to 1/2 for $l = 6.4$ in agreement with the value $BK = 6.4$ used in this example (in practice, l is restricted to integer values). Note that as a consequence of the discrete-time formulation, the FCF reaches its minimum value at the Nyquist frequency index $K/2$, and it is not modeled beyond this point.

From (5), we have for large K,

$$|\Gamma(l)|^2 \xrightarrow[K\to\infty]{} \frac{1}{1 + \left|\frac{\alpha}{1-\alpha}\cdot\frac{2\pi}{K}l\right|^2} \tag{8}$$

which is of Lorenzian shape, also known as the Cauchy distribution or "Witch of Agnesi" curve, being the Fourier transform of a double-sided exponential. In this case, $|\Gamma(l)|$, as well as $|\Gamma(l)|^2$, is quadratic for small l, and this is the characteristic behavior that is so often observed in experimental data [3], [4].

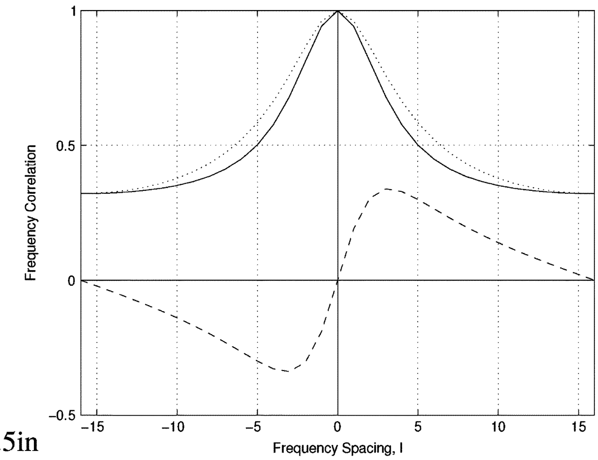

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Fig. 1. Frequency correlation function for exponential power-delay profile, showing the real (solid) and imaginary (dashed) parts as well as the magnitude (dotted) for $K = 32$ points and normalized correlation bandwidth $B = 0.2$ ($\alpha = 0.5136$).

It is useful to relate the above-defined normalized correlation bandwidth to other channel characterizations. The exponential power-delay profile (2) has a mean delay of

$$\bar{\tau} = \frac{\sum_{n=0}^{K-1} n\alpha^n}{\sum_{n=0}^{K-1}\alpha^n} \approx \frac{\alpha}{1-\alpha}, \quad K\alpha^K \ll 1 \tag{9}$$

and variance

$$\sigma_\tau^2 = \frac{\sum_{n=0}^{K-1}(n-\bar{\tau})^2\alpha^n}{\sum_{n=0}^{K-1}\alpha^n} \approx \frac{\alpha}{(1-\alpha)^2}, \quad K^2\alpha^K \ll 1. \tag{10}$$

Therefore, the asymptotic rms delay spread is

$$\sigma_\tau \approx \frac{\sqrt{\alpha}}{1-\alpha} \xrightarrow[\alpha\to 1]{} \frac{\sqrt{3}}{2\pi B}, \quad K^2\alpha^K \ll 1 \tag{11}$$

where the last relationship follows from (6b) for α close to 1 (B small). Therefore, we have $B\sigma_\tau \approx \sqrt{3}/(2\pi) = 0.2757$, and this is exactly the result that is obtained for a continuous-time exponential. In fact, simple calculations show that for large K

$$0.2757 = \frac{\sqrt{3}}{2\pi} < B\sigma_\tau \le \sqrt{\frac{3}{32}} = 0.3062 \tag{12}$$

over the entire feasible range $0 < B \le 0.25$, i.e., $B\sigma_\tau$ only varies about 11%. So a reasonable rule of thumb is therefore $\sigma_\tau \approx 0.3/B$.

Another relationship can be easily derived between the normalized correlation bandwidth B and the decay time of the power-delay profile. Here, we consider the 60-dB decay time, denoted by n_{60} (in samples), as a useful reference measure. Setting $\alpha^{n_{60}} = 10^{-6}$ and using (6b), we have for α close to 1 (small B)

$$n_{60} = \frac{6\ln(10)}{\ln(1/\alpha)} \xrightarrow[\alpha\to 1]{} \frac{3\sqrt{3}\ln(10)}{\pi B} = \frac{3.8084}{B} \tag{13}$$

and this limiting expression again is exactly the result that is obtained for a continuous-time exponential. It can be easily shown that

$$3.8084 = \frac{3\sqrt{3}\ln(10)}{\pi} < Bn_{60} \leq \frac{1.5\ln(10)}{\ln(3) - \ln(4-\sqrt{7})} = 4.3425 \tag{14}$$

over the entire feasible range $0 < B \leq 1/4$, i.e., Bn_{60} only varies about 14%. Therefore, we can take $n_{60} \approx 4/B$.

III. Computational Considerations

The desired discrete-time frequency correlation is expressed in (5). Several methods can be conceived for realizing impulse responses (1) or, alternatively, frequency responses (3) that result in such correlation. A block of K frequency-domain samples could be generated directly by multiplying a vector of i.i.d. complex Gaussian random variables by the square root of the Toeplitz correlation matrix formed by (5), which would entail about $O(K)$ multiplies/sample (neglecting the matrix rooting). Alternatively, one could consider an Lth-order autoregressive (AR) model [5], requiring L multiplies/sample. However, AR filtering in the frequency domain must approximate an exponential power-delay profile in the time domain, which is a nonrational function and therefore requires a very large order for sufficient accuracy. In fact, the characteristic poles calculated in [5] are not characteristic at all, and they just reflect the truncation effects of a bad approximation. (Similar pole patterns can be easily demonstrated using the exponential power-delay profile model developed here.) For the same reason, moving average (MA), and more generally, ARMA, models (or equivalently "spectral" factorization in the time domain), fail to provide accurate representation. Therefore, we are led to a time-domain formulation, whereby a block of K samples is generated by multiplying K i.i.d. complex Gaussian variates by the square root of the desired power-delay profile (2), and then using the fast Fourier transform (FFT) to produce the frequency-domain realization. This is very efficient, requiring only the multiplication of two K-point arrays and a K-point FFT, resulting in $O[1+\log_2(K)]$ multiplies/sample. This is the method adopted for the simulation results in this letter.

IV. Simulation

As an application example, suppose that the actual wideband RF impulse response is assumed to be sampled at 10.24 GHz, corresponding to about 0.1 ns (3 cm) resolution, with a total length of 2048 samples (200 ns or 60 m). We consider an operating bandwidth of 160 MHz, which implies decimation by a factor of 64, producing a downsampled impulse response of length $K = 32$. Now let us further assume that the desired nominal correlation bandwidth is 32 MHz, which, from the results of Section II, implies an rms delay spread of approximately $0.3/(32 \text{ MHz}) = 9.4$ ns and a 60-dB decay time of approximately $4/(32 \text{ MHz}) = 125$ ns. The corresponding normalized correlation bandwidth to be used in the baseband simulation is then $B = 32/160 = 0.2$, which, in terms of the downsampled impulse response, gives a normalized rms delay spread of approximately $0.3/B = 1.5$ (decimated) samples and a normalized 60-dB decay time of approximately $4/B = 20$ (decimated) samples.

Fig. 2(a) shows one realization of an impulse response magnitude generated by the above time-domain method for $K = 32$ samples with normalized correlation bandwidth $B = 0.2$ $(\alpha = 0.5136)$, and it is seen to exhibit the desired exponentially-decaying mean, which by design has independent Rayleigh fading at each time point. The observed 60-dB decay time is close to the aforementioned approximate value of $4/B = 20$ samples for $B = 0.2$. The frequency response of this realization appears in Fig. 2(b), showing significant variations of magnitude and phase over normalized frequency intervals equal to the correlation bandwidth $B = 0.2$, as would be expected.

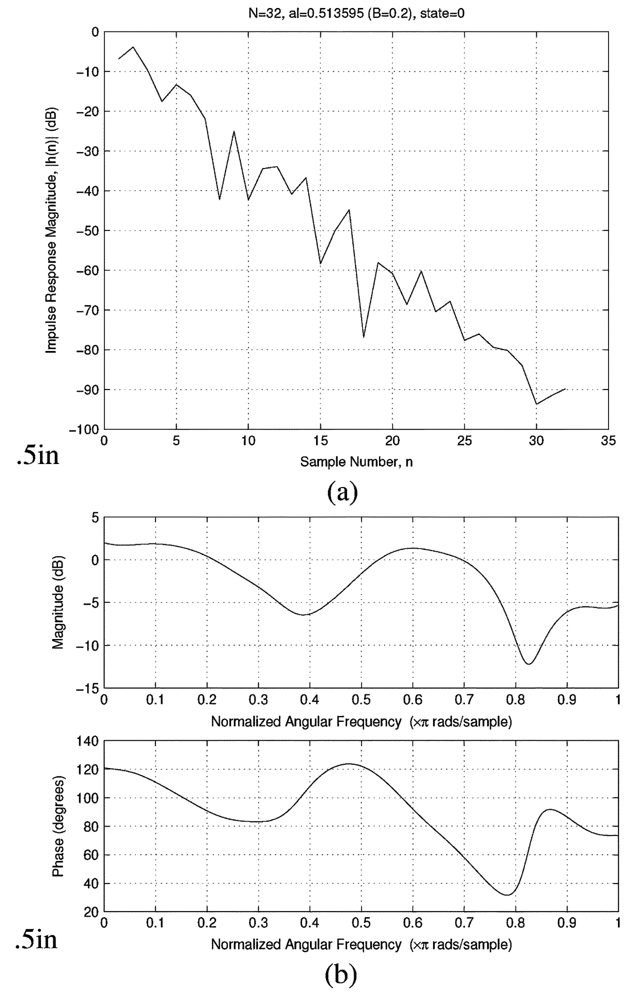

Fig. 2. Typical realization of simulated (a) channel impulse response and (b) frequency response for exponential mean power-delay profile with $K = 32$ points and normalized correlation bandwidth $B = 0.2$ $(\alpha = 0.5136)$.

V. Conclusion

In this letter, a frequency-selective fading model that assumes independent time-domain fading and Lorenzian frequency correlation induced by a nominally exponential power-delay profile was analyzed. An expression was derived for the frequency correlation function and correlation bandwidth B in terms of the exponential decay rate of the power-delay profile. An expression was also derived for the rms delay spread, and for a long impulse response was shown to be well approximated as $\sigma_\tau \approx 0.3/B$. Another relationship was derived between the decay time and correlation bandwidth, where it was shown that the 60-dB decay time in samples is approximately determined as $n_{60} \approx 4/B$.

Investigation of several possible realization techniques showed that the most straightforward and efficient approach is to generate a block of samples by multiplying i.i.d. complex Gaussian variates by the square root of the exponential power-delay profile and then taking the FFT.

APPENDIX
DECIMATION OF RANDOM SEQUENCE WITH EXPONENTIAL POWER PROFILE

Here we substantiate the fact that a decimated (filtered and downsampled) version of an independent sequence of random variates with exponential power-delay profile retains the same characteristics, provided that the decay rate is slow enough compared to the bandwidth. First define the (unnormalized) random sequence

$$h(n) = \alpha^{n/2} v(n), \quad n \geq 0 \tag{15}$$

where $0 < \alpha < 1$ determines the exponential decay rate and $v(n)$ is a sequence of i.i.d. complex Gaussian variates with $E\{|v(n)|^2\} = 1$ and $E\{v(n)v^*(m)\} = 0, m \neq n$. Therefore, $h(n)$ has an exponential power-delay profile

$$E\{|h(n)|^2\} = \alpha^n. \tag{16}$$

The sequence $h(n)$ is then lowpass filtered with coefficients g_l to produce the sequence

$$\hat{h}(n) = \sum_{l=0}^{n} g_l \alpha^{\frac{n-l}{2}} v(n-l) \tag{17}$$

where we have assumed that g_l, as well as $h(n)$, is causal. The autocorrelation of $\hat{h}(n)$ is then derived as

$$\begin{aligned} R_{\hat{h}}(m,n) &\equiv E\{\hat{h}(n)\hat{h}^*(n-m)\} \\ &= E \sum_{k=0}^{n} \sum_{l=0}^{n-m} g_k g_l \alpha^{\frac{n-k}{2}} \alpha^{\frac{n-m-l}{2}} \\ &\quad \times v(n-k) v^*(n-m-l) \\ &= \sum_{l=0}^{n-m} g_{l+m} g_l \alpha^{n-m-l}, \quad 0 \leq m \leq n \\ &\xrightarrow[n \to \infty]{} \alpha^n \rho(m) \end{aligned} \tag{18}$$

with

$$\rho(m) \equiv \sum_{l=0}^{\infty} g_{l+m} g_l \alpha^{-l-m}, \quad 0 \leq m \leq n \tag{19}$$

where we have used the i.i.d. property of $v(n)$, and the limiting case of the last step is justified for any filter that decays fast enough compared to α^{-l}, and certainly for any finite impulse response (FIR) filter. Thus, beyond a short transient period, the filtered sequence also has an exponential power-delay profile, $E\{|\hat{h}(n)|^2\} = \alpha^n \rho(0)$. As the decay rate approaches zero, we have

$$\rho(m) \xrightarrow[\alpha \to 1]{} R_g(m) \equiv \sum_{l=0}^{\infty} g_{l+m} g_l \tag{20}$$

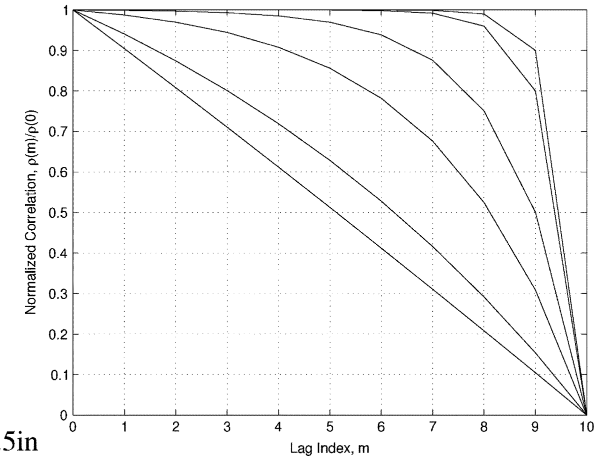

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Fig. 3. Normalized correlation of decimated random sequence for $L = 10$ and (right to left) $\alpha = 0.1, 0.2, 0.5, 0.7, 0.9, 0.99$.

where $R_g(m)$ is the autocorrelation of the filter impulse response.

As a simple example, suppose that g_l is a moving-average filter

$$g_l = \begin{cases} 1/L, & 0 \leq l \leq L-1 \\ 0, & l \geq L \end{cases} \tag{21}$$

which has a $\sin(\pi L f)/\sin(\pi f)$ frequency response with normalized bandwidth $1/L$. For this lowpass filter, we then have

$$\begin{aligned} \rho(m) &= \frac{1}{L} \sum_{l=0}^{L-m-1} \alpha^{-L-m} \\ &= \frac{1}{L} \cdot \frac{\alpha^{-m} - \alpha^{-L}}{1 - \alpha^{-1}}, 0 \leq m \leq L && \text{(22a)} \\ &\xrightarrow[\alpha \to 1]{} 1 - m/L. && \text{(22b)} \end{aligned}$$

Fig. 3 plots the normalized correlation $\rho(m)/\rho(0)$ from (22a) for several values of α. Note that as $\alpha \to 1$, $\rho(l)$ converges to (22b), the triangular autocorrelation of (21). Moreover, in all cases, $\rho(m) = 0$ for $m \geq L$. Therefore, when downsampled by L, we have $\rho(kL) = 0, k \neq 0$, and so we see that the i.i.d. property is retained.

For more sophisticated filters with better selectivity, the i.i.d. property may only be approximately achieved, unless the filters are constrained to meet the Nyquist criterion [1, Sec. 6.2.1].

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