Proposition, Predicates & Quantifiers

Theoretical Questions

1. Define - Proposition, Truth values, Proposition variables, Propositional constants.
2. Define - Simple/ Atomic/ Primary proposition & Compound/ Molecular/ Composite proposition.
3. Propositional Connectives - Negation, Conjunction, Disjunction, Implication/ Conditional, Equivalence/ Bi-conditional.
4. Define - Logical equivalence.
5. Algebra of Propositions.
   (a) **Idempotent law:**
   - $p \vee p \equiv p$
   - $p \wedge p \equiv p$

   (b) **Associative law:**
   - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
   - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

   (c) **Commutative law:**
   - $p \vee q \equiv q \vee p$
   - $p \wedge q \equiv q \wedge p$

   (d) **Distributive law:**
   - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
   - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

   (e) **Identity law:**
   - $p \vee F \equiv p$
   - $p \wedge T \equiv p$
   - $p \vee T \equiv T$
   - $p \wedge F \equiv F$

   (f) **Complement law:**
   - $p \vee \neg p \equiv T$
   - $p \wedge \neg p \equiv F$
   - $\neg T \equiv F$
   - $\neg F \equiv T$

   (g) **Involution law:**
   - $\neg(\neg p) \equiv p$

   (h) **DeMorgan's Law:**
   - $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$
   - $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$
6. Define - Converse, Contrapositive ans Inverse of an implication $p \to q$.
7. Negation of different compound statements.
   (a) Negation of conjunction.
   (b) Negation of disjunction.

(c) Negation of negation.
(d) Negation of implication.
(e) Negation of bi-conditional.

8. Define - Tautology, Contradiction and Contingency.

9. What do you mean by *Functional complete set of connectives*?

10. Define - Disjunctive normal form (DNF). State the procedure to obtain a DNF from a given logical expression.

11. Define - Conjunctive normal form (CNF). State the procedure to obtain a CNF from a given logical expression.

12. What do you mean by Principal DNF and Principal CNF? State the procedures to obtain them.

13. Define - Predicate, n-ary predicate/ n-place function, Quantifier, Universal quantifier, Existential quantifier, Uniqueness quantifier.

14. Precedence of quantifiers and binding of variables.

15. Translating English sentences into logical expressions.

16. What do you mean by an argument? Define - hypothesis/ premises and conclusion. When an argument is said to be a valid argument?

17. Rules of inference.
    (a) Addition rule
    (b) Simplification rule
    (c) Conjunction rule
    (d) Resolution rule
    (e) Modus ponens
    (f) Modus tollens
    (g) Hypothetical syllogism
    (h) Disjunctive syllogism

18. What do you mean by Fallacies?

19. Rules of Inference for Quantified statements.
    (a) Universal instantiation.
    (b) Universal generalization.
    (c) Existential instantiation.
    (d) Existential generalization.

Problems

- *To be done from **Rosen***

Set, Relation & Mapping

Set Theory

Theoretical Questions

1. Definition of set
2. Representation of set
   - Roster method / Tabular method
   - Set builder's method
3. Finite and Infinite set
4. Null set/ Empty set/ Void set
5. Singleton set
6. Subset
7. Properties of subset
   - Every set is a subset of it's own.
   - Null is a subset of every set.
   - If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
8. Power set **[CU 2010]**
9. Theorem: If a set contains n elements, then it's power set contains 2^n elements. **[CU 2003 2006]**
10. Equal sets
11. Proper subset
12. Universal set / Universe of Discourse (UoD)
13. Operations on sets
    - Complement of a set
    - Union
    - Intersection
    - Difference / Relative Complement
    - Symmetric difference
14. Laws of set algebra
    - **Idempotent law:**
      - $A \cup A = A$
      - $A \cap A = A$
    - **Associative law:**
      - $(A \cup B) \cup C = A \cup (B \cup C)$
      - $(A \cap B) \cap C = A \cap (B \cap C)$

- **Commutative law:**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Distributive law:**
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **Identity law:**
  - $A \cup \mathbf{U} = \mathbf{U}$
  - $A \cup \phi = A$
  - $A \cap \mathbf{U} = A$
  - $A \cap \phi = \phi$
- **Involution law:** $(A^c)^c = A$
- **Complement law:**
  - $A \cup A^c = \mathbf{U}$
  - $A \cap A^c = \phi$
  - $\mathbf{U}^c = \phi$
  - $\phi^c = \mathbf{U}$
- **De-Morgan's law:** **[CU 2008]**
  - $(A \cup B)^c = (A^c \cap B^c)$
  - $(A \cup B)^c = (A^c \cap B^c)$
- Cardinality of a set
- Inclusion and Exclusion Principle
- Class of sets/ Family of sets
- Index and Indexed sets
- Multiset
  - Multiplicity of an element of in a multiset
  - Cardinality of a multiset
- Operation on multiset
  - Union
  - Intersection
  - Difference
  - Sum
- Equivalent / Equipotent sets
- Countable and Uncountable sets
- Countably infinite set and Uncountably infinite set. **[CU 2006]**
- Ordered pair
- Cartesian product
- Properties of Cartesian product
  - $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
  - $(A - B) \times C = (A \times C) - (B \times C)$
  - $(A \cup B) \times C = (A \times C) \cup (B \times C)$
  - $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Problems

1.

Relation

Theoretical Questions

1. Definition - Relation/ Binary relation, Domain, Range.
2. Types of relations
   (a) Universal relation.
   (b) Void relation.
   (c) Identity relation.
   (d) Inverse of a relation.
3. Operations on relations
   (a) Union of two relations
   (b) Intersection of two relations
   (c) Difference of two relations
   (d) Complement of a relation
4. Composition of relations
5. **Properties of relations**
   (a) Reflexive relation & Irreflexive relation
   (b) Symmetric relation
   (c) Antisymmetric relation
   (d) Transitive relation
   (e) Equivalence relation
   (f) Partial order and PoSet

Problems

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a, b) \in R$ iff
   (a) $a = b$
   (b) $a + b = 4$
   (c) $a > b$
   (d) $a|b$
   (e) $gcd(a, b) = 1$
2. If R is a relation on the set $\{1, 2, 3, 4, 5\}$, list the ordered pairs in R when
   (a) $a\,R\,b$ if 3 divides $a - b$
   (b) $a\,R\,b$ if $a + b = 6$
   (c) $a\,R\,b$ if $a - b$ is even
   (d) $a\,R\,b$ if $lcm(a, b)$ is odd
   (e) $a\,R\,b$ if $a^2 = b$
3. If R is a relation on the set $\{1, 2, 3, 4, 5\}$ defined by $(a, b) \in R$ if $a + b \leq 6$
   (a) list the elements of R, R^{-1} ans $\bar{R}$

(b) the domain and range of R and R^{-1} and $\bar{R}$

4. If $R_1 = \{(1,2),\ (2,3),\ (3,4)\}$ *and* $R_2 = \{(1,1),\ (1,2),\ (2,1),\ (2,2),\ (2,3),\ (3,1),\ (3,2),\ (3,3),\ (3,4)\}$ be te relation from $A = \{1,2,3\}$ to $B = \{1,2,3,4\}$, find

   (a) $R_1 \cup R_2$

   (b) $R_1 \cap R_2$

   (c) $R_1 - R_2$

   (d) $R_2 - R_1$

   (e) $R_1 \Delta R_2$

   (f) $R_1 + R_2$

5. If $R = \{(x, x^2)\}$ and $S = \{(x, 2x)\}$ where x is a nonnegative integer, find

   (a) $R \cup S$

   (b) $R \cap S$

   (c) $R - S$

   (d) $S - R$

6. If R_1 and R_2 are relations on the set of positive integers defined by $R_1 = \{(a,b) : b|a\}$ and $R_2 = \{$(a,b) : a is a multiple of b$\}$, find

   (a) $R_1 \cup R_2$

   (b) $R_1 \cap R_2$

   (c) $R_1 - R_1$

   (d) $R_2 - R_1$

7. If the relations R_1, R_2, R_3, R_4 andR_5 are defined on the set of real numbers as $R_1 = \{(a,b) : a \geq b\}$; $R_2 = \{(a,b) : a < b\}$; $R_3 = \{(a,b) : a \leq b\}$; $R_4 = \{(a,b) : a = b\}$; $R_5 = \{(a,b) : a \neq b\}$, Find

   (a) $R_2 \cup R_5$

   (b) $R_3 \cap R_5$

   (c) $R_2 - R_5$

8. If the relations R and S are given by $R = \{(1,1),\ (2,2),\ (3,4)\}$ and $S = \{(1,3),\ (2,5),\ (3,1),\ (4,2)\}$, Find $R \cdot S$, $S \cdot R$, $R \cdot R$, $S \cdot S$, $R \cdot (S \cdot R)$, $(R \cdot S) \cdot R$ and $R \cdot R \cdot R$

9. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, anti-symmetric and/ or transitive, where $(a,b) \in R$ iff

   (a) $a + b = 0$

   (b) $a - b$ is rational

   (c) $ab \geq 0$

   (d) $a = 2b$

   (e) $a = 1$ or $b = 1$

10. If R is a relation on Z defined by

    (a) $a\,R\,b$ iff $2a + 3b = 5n$ for some integer n

    (b) $a\,R\,b$ iff $3a + b$ is a multiple of 4

    Prove that R is an equivalence relation.

11. If R is a relation defined in Z such that $a\,R\,b$ iff $a^2 - b^2$ is divisible by 3 show that R is an equivalence relation

12. If R is the relation on N defined as $a\,R\,b$ iff $\frac{b}{a}$ is an integer, show that R is a partial ordering on A.

13. Show that the following relations are equivalence relations

    (a) R_1 is the relation on the set of integers such that $a\,R_1\,b$ iff $a = b$ or $a = -b$

    (b) R_2 is the relation on the set of integers such that $a\,R_2\,b$ iff $a \equiv b \ (mod\ m)$ where m is a positive integers

    (c) R_3 is the relation on the set of real numbers such that $a\,R_3\,b$ iff $(a - b)$ is an integer

Mapping

Theoretical Questions

1. Define - Function/ Transformation/ Mapping/ Correspondence, Argument/ Pre-image, Image, Domain, Co-domain, Range.

2. Representation of a function

3. Types of functions

    (a) One-to-one/ Injective.

    (b) Onto/ Surjective

    (c) Into

    (d) One-to-one onto/ Bijective

4. Classification of functions

    (a) Algebraic function.

    (b) Logarithmic function.

    (c) Exponential function.

    (d) Inverse function.

    (e) Trigonometric function.

    (f) Transcendental function.

    (g) Identity function.

    (h) Floor and Ceiling function.

    (i) Integer value and absolute value function.

    (j) Remainder function.

5. **Composition of functions**

    - Definition - Composition/ Relative product/ Left image

6. Properties of Composition

    (a) Composition is associative i.e. if $f : A \to B$, $g : B \to C$ and $h : C \to D$ are functions then, $h \circ (g \circ f) = (h \circ f) \circ g$

    (b) When $f : A \to B$ and $g : B \to C$ are functions, then $g \circ f : A \to C$ is an injection, surjection or bijection according as f and g are injective, surjective and bijective respectively.

7. **Inverse of a function** - Definition.
8. Properties
   (a) The inverse of a function f, if exists, is unique.
   (b) The necessary and sufficient condition for the function $f : A \to B$ to be invertible i.e. f^{-1} exists is that f is one-to-one and onto.
   (c) If $f : A \to B$ and $g : B \to C$ are invertible functions, then $g \circ f : A \to C$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. **[CU 2008]**
9. Binary and n-ary operations
10. Closure property
11. Properties of Binary operations
    (a) A binary operation * on a set S is said to be *commutative*, if for any a, $b \in S$,
    a * b = b * a
    (b) A binary operation * on a set S is said to be *associative*, if for any a, $b \in S$,
    (a * b) * c = a * (b * c)
    (c) A binary operation * on a set S is said to be *distributive* over the operation ∘ on S, if for any a, b and $C \in S$,
    a * (b ∘ c) = (a * b) ∘ (a * c)
    (d) If * is a binary operation on a set S and there exists an element $e \in S$ such that e * a = a * e = a, for every $a \in S$, then e is called an *identity element* with respect to *.
    (e) If the identity for a binary operation * on a set S exists, it is unique.
    (f) If * is a binary operation on a set S and $a \in S$ is an element such that a * a = a, then a is called *idempotent* with respect to *.
    (g) If * is a binary operation on a set S having the identity element e and if corresponding to an element $a \in S$, there exists an element $b \in S$ such that a * b = b * a = e, then a is said to be *invertible* and b is said to be *inverse* of a and is denoted by a^{-1}.

Problems

1. Let $A = \{1, 2, 3\}$. Find all one-to-one onto functions $f : A \to A$. **[CU 2014]**
2. Consider a function $h : Z \times Z \to Z$ so that $h(a, b) = (2a + 1)s^b - 1$, where $Z = \{0, 1, 2, 3, ...\}$. Prove that h is an one-to-one function. **[CU 2006]**

Growth of Functions & Assymptotic Notation

Theoritical Questions

1. Define an algorithm. State the generic properties of an algorithm.
2. How the efficiency of an algorithm is measured? State the notions for Time complexity and Space complexity of an algorithm.
3. Define Big **O** notation. Give its geometrical interpretation. **[CU 2013]**
4. Define Big θ notation. Give its geometrical interpretation.
5. Define Big Ω notation. Give its geometrical interpretation.
6. How can you relate Big **O**, Big θ and Big Ω notations? Justify your answer. **[CU 2007, 2008, 2009]**
7. Explain what is meant by order of an algorithm? Discuss the criterion on which the measures of complexities depend. **[CU 2005]**
8. What do you mean by NP-hard and NP-complete problems? **[CU 2005, 2010]**
9. What is satisfiability problem? **[CU 2007]**
10. State Cook's theorem. **[CU 2005, 2006]**

Problems

1. Give Big **O** estimates for factorial function and the logarithm of factorial function where factorial function $f(x)$ is defined as

   $f(x) \, = \, n.(n-1).(n-2).....3.2.1 \;\; \forall \;\; n > 0 \, , \, n \, \in \, Z \, and \, f(0) = 1$ **[CU 2014]**

2. If $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$ where $a_0, a_1, a_2, ..., a_n$ are real numbers then show that $f(x) = \mathbf{O}(x^n)$ **[CU 2012]**
3. Give a Big **O** estimate for $f(x) = (x+1)log(x^2+1) + 3x^2$ **[CU 2010]**
4. The no of operations $f(n)$ required by an algorithm is given by

   $$f(n) = f(n-1) + (n-1) + (n-2) \text{ where } f(1) = 1.$$

   Show that $f(n) = \mathbf{O}(n^2)$ **[CU 2007, 2008, 2009]**

5. Consider the functions $f(x)\,, g(x)\, andh(x)$ such that $f(x) = \mathbf{O}(g(x)) \; and \; g(x) = \mathbf{O}(h(x))$. Show that $f(x) = \mathbf{O}(h(x))$. **[CU 2006]**

Method of Induction

On summation formula

1. $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$
2. $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n(n+1)}{2}^2$
4. $\sum_{i=1}^{n} i^4 = \frac{n\cdot(n+1)\cdot(2n+1)\cdot(3n^2+3n-1)}{30}$
5. $1^2 + 3^2 + 5^2 + ... + (2n+1)^2 = \frac{(n+1)\cdot(2n+1)\cdot(2n+3)}{3}$
6. $1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{n\cdot(2n-1)\cdot(2n+1)}{3}$
7. $2 + 4 + 6 + ... + 2n = (n-1)(n+2)$
8. $4 + 10 + 16 + ... + (6n-2) = n(3n+1)$
9. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n.(n+1)} = \frac{n}{n+1}$
10. $\frac{1}{1.5} + \frac{1}{5.9} + \frac{1}{9.13} + ... + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$
11. $1 + 2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 1$
12. $\sum_{i=0}^{n} ar^i = a + ar + ar^2 + ... + ar^n$
13. $1.1! + 2.2! + 3.3! + ... + n.n! = (n+1)! - 1 \ \forall n \geq 1$
14. $3 + 3\cdot 5 + 3\cdot 5^2 + ... + 3\cdot 5^n = \frac{3\cdot(5^{n+1}-1)}{4} \ \forall n \geq 0$
15. $2 - 2\cdot 7 + 2\cdot 7^2 - ... + 2\cdot(-7)^n = \frac{(1-(-7)^{n+1})}{4} \ \forall n \geq 0$
16. $1^2 - 2^2 + 3^2 + ... + (-)^{n-1} n^2 = (-1)^{n-1} \frac{n\cdot(n+1)}{2}$
17. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + ... + n\cdot(n+1) = \frac{n\cdot(n+1)\cdot(n+2)}{3}$
18. $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + ... + n\cdot(n+1)\cdot(n+2) = \frac{n\cdot(n+1)\cdot(n+2)\cdot(n+3)}{4}$

On divisibility

1. $5^n - 1$ is divisible by 4 $\forall n \geq 1$
2. $5^n - 4n - 1$ is divisible by 16 $\forall n \geq 1$
3. $(x-1)^3 + x^3 + (x+1)^3$ is divisible by 9
4. $5^{2n} - 2^{5n}$ is divisible by 7
5. $10^{n+1} + 10^n + 1$ is divisible by 3
6. $2^n + 3^n + -5^n$ is divisible by 6
7. $11^n - 7^n$ is divisible by 7
8. $n^2 + n$ is divisible by 2 $\forall n \geq 1$

9. $n^3 + 2n$ is divisible by 3 $\forall n > 0$
10. $n^5 - n$ is divisible by 5 $\forall n \geq 0$
11. $n^3 - n$ is divisible by 6 $\forall n \geq 0$
12. $4^{n+1} + 5^{2n-1}$ is divisible by 133 $\forall n > 0$

On inequalities

1. $2^n > n^2 \ \forall n \geq 5$
2. $n! > n^3 \ \forall n \geq 6$
3. $(1 + \frac{1}{2})^n \geq 1 + \frac{n}{2} \ \forall \ n \in N$
4. $\sum_{i=1}^{n}(x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i \ \forall n \geq 1$
5. $3^n < n! \ \forall \ n > 6$

On set theory

1. If S is a finite set with n elements where n is a non-negative integer, then S has 2^n subsets.
2. Prove that, if $A_1, A_2, A_3, ..., A_n$ and B are sets then
   (a) $(A_1 \bigcap A_2 \bigcap ... \bigcap A_n) \bigcup B = (A_1 \bigcup B) \bigcap (A_2 \bigcup B) \bigcap ... \bigcap (A_n \bigcup B)$
   (b) $(A_1 \bigcup A_2 \bigcup ... \bigcup A_n) \bigcap B = (A_1 \bigcap B) \bigcup (A_2 \bigcap B) \bigcup ... \bigcup (A_n \bigcap B)$
3. For any n sets $A_1, A_2, .., A_n$ prove that
   (a) $(\bigcup_{i=1}^{n} A_i)^c = \bigcap_{i=1}^{n} A_i^c$
   (b) $(\bigcap_{i=1}^{n} A_i)^c = \bigcup_{i=1}^{n} A_i^c$
4. $|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| - \sum_{i=1}^{n} \sum_{j=1}^{n} |A_i \bigcap A_j| + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} |A_i \bigcap A_j \bigcap A_j| - ... + (-1)^{n-1} |\bigcap_{i=1}^{n} A_i|$

Inclusion & Exclusion Principle

Theoritical Questions

1. State and prove Principle of Inclusion and Exclusion.
2. State and prove Principle of Inclusion and Exclusion for any three sets A_1, A_2 and A_3.
3. State and prove Principle of Inclusion and Exclusion for any four sets A_1, A_2, A_3 and A_4. **[CU 2012]**
4. State the genralized Principle of Inclusion and Exclusion. **[CU 2005, 2011, 2014]**

Problems

1. In a class containing 50 students 15 play tennis, 20 play cricket and 20 play cricket and 20 play hockey, 3 play tennis and cricket, 6 play cricket and hockey and 5 play tennis and hockey, 7 play no games at all. How many play cricket, tennis and hockey?
2. There are 100 people in a room. In this group, 60 are men, 30 are young, 10 are young men. How many are old women? **[CU 2011]**
3. Among a group of 100 people, 60 are men, 30 are young, 10 are young men. Also it is known that 40 are Indian, 20 are Indian men and 15 are young Indian and 5 are young Indian men. Assuming each person is either an Indian or European, how many are old European women? **[CU 2009]**
4. A survey was conducted among 950 people. Of these 545 are democrats, 545 wear glasses and 500 like ice-cream, 340 of them are democrats who like ice-cream and 330 of them wear glasses and like ice-cream.
   (a) How many of them are not democrats, do not wear glasses and do not like ice-cream?
   (b) how many of them are democrats, who do not wear glasses and do not like ice-cream?
5. In a group of 120 students studying computer science, 84 can program in Pascal, 66 can program in LISP. 45 can program in both Pascal and LISP. How many of the students can not program in either of these languages?
6. Among the integers 1 to 300, Find how many ar not divisible by 3, nor by 5. Find also, how many are divisible by 3 but not by 7.
7. Compute the number of positive integers not exceeding 1000 that are divisible by 7 or 11. **[CU 2007]**
8. Find the no of positive integers not exceeding 100 that are not divisible by 5 or 7. **[CU 2012]**
9. Find how many integers between 1 to 60 are
   (a) Divisible by 2, 3 or 5.
   (b) Not divisible by 2, nor by 3 nor by 5.
   (c) Divisible by 2 but not by 3 nor by 5.
   (d) not divisible by 2 nor by 3.
   (e) Divisible by 2 but not by 5.
10. Consider a set of integers from 1 to 250. Find how many of these numbers are divisible by 3 or 5 or 7. Also indicate how mnany are divisible by 3 or 7 but not by 5 and divisible by 3 or 5.

11. Among the integers from 1 to 1000

    (a) How many of them are not divisible by 3 nor by 5 or 7?

    (b) How many are not divisible by 5 and 7 but divisible by 3?

12. How many integers between 1 and 2000 are divisible by 2, 3, 5 or 7?

13. In a survey, it is reported that of 1000 programmers 650 habitually flow chart their programs. 788 are Cobol programmers, 675 are men, 278 of the women are skilled cobol Programmers. 440 programmers both habitually Flow chart and are skilled in Cobol, 210 women habitually Flow chart and 166 women are both skilled in Cobol and Habitually Flow chart. Would you accpet this data being accurately reported?

14. 30 cars were assembled in a factory. The options available were a radio, an air-conditioner and white wall tires. It is known 15 of the cars have radios, 8 of them have their airconditioners and 8 of them have airconditioners and 6 of them have all 3 options. At least how many cars do not have any option at all?

15. In a survey, 200 people were asked whether they read India Today or Buisness Times. It was found that 1200 read India Today, 900 read Business Times and 400 read both. Find how many read at least one magazine and how many read either?

16. Among 75 children who went to an amusement park, where they could ride on marry-go-round, roller coaster and Ferris wheel. It is known that 20 of them had taken all 3 rides and 55 had taken at least 2 of the 3 rides. Each ride costs Rs. 0.50 and total receipt of park is Rs 70. Determine the no of children who did not try any of the rides?

17. If no of students who got grade A in first exam is equal to that of in second exam. If total no of students who got grade A in exactly one exam is 40 and 4 students did not get grade A in either exam, determine the no of students who got grade A in first exam only, who got grade A in second exam only and who got grade A in both the exams.

18. Find the number of mathematics students at a college tking at least one of the languages French, German and Russian given the following data: 65 study French, 45 study German, 42 study Russian, 20 study French and German, 25 study French and Russian, 15 study German and Russian and 8 study all the three languages.

19. Find the number of Primes within $[1, 100]$ using principle of inclusion and exclusion. **[CU 2006, 2013]**

20. **Sieve of Erathothenes:** How many solutions does $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2 and x_3 are non-negative integers with $x_1 \leq 3$, $x_2 \leq 4$, $x_3 \leq 6$

Pigeonhole Principle

Theoritical Questions

1. State and prove Pigeonhole principle.
2. State and prove generalized Pigeonhole principle.

Problems

1. Let $S = 1, 2, 3, ..., 20$. Let A be any subset of S such that cardinality of A is eleven. Prove that there is a pair of elements in A, whose sum is 21. **[CU 2007 2013]**
2. Show that in any set of six classes there must be two that meet on the same day, assume that no classes are held on weekends. **[CU 2006]**
3. How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?
4. What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?
5. Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.
6. A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists a succession of consecutive days during which the chess master will have played exactly 21 games.
7. Given two disks, one smaller than the other. Each disk is divided into 200 congruent sectors. In the larger disk 100 sectors are chosen arbitrarily and painted red; the other 100 sectors are painted blue. In the smaller disk each sector is painted either red or blue with no stipulation on the number of red and blue sectors. The smaller disk is placed on the larger disk so that the centers and sectors coincide. Show that it is possible to align the two disks so that the number of sectors of the smaller disk whose color matches the corresponding sector of the larger disk is at least 100.

Recurrence Relation

Theoretical questions

1. Define a recurrence relation. What do you mean by it's solution? **[CU 2011]**
2. State the general form of a linear recurrence relation with constants.
3. Define a Homogeneous linear recurrence relation (HDE) and Non-homogeneous linear recurrence relation. **[CU 2010]**

Problems

On HDE

1. Determine whether the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, ...$ where $a_n = 3n$ for every non-negative integer n. What happens when $a_n = 2^n$. **[CU 2010]**
2. $a_n = a_{n-1} + 6a_{n-1}$; $n \geq 2$ with $a_0 = 1, a_1 = 1$
3. $a_n = 3a_{n-1} - 3a_{n-1} + a_{n-3}$; $a_0 = 0, a_3 = 3, a_5 = 10$
4. $a_n = 7a_{n-1} - 10a_{n-2}$; $a_0 = 4, a_1 = 17$
5. $a_n - 8a_{n-1} + 16a_{n-2} = 0$; $a_2 = 16, a_3 = 80$
6. $a_n = a_{n-1} + 2a_{n-2}$; $a_0 = 2, a_1 = 7$
7. $a_{n+2} + 4a_{n+1} + 5a_n$; $n \geq 0$ with $a_0 = 2, a_1 = 8$
8. $a_n = 6a_{n-1} - 8a_{n-2}$; $n \geq 2$ with $a_0 = 4, a_1 = 10$
9. $a_n = 2an - 1 - a_{n-2}$; $n \geq 2$, with $a_0 = 3, a_1 = 1$
10. $a_n = 6a_{n-1} - 9a_{n-2}$; $n \geq 2$ with $a_0 = 1, a_1 = 6$ **[CU 2014]**
11. $a_n = -4a_{n-1} - 4a_{n-2}$; $n \geq 2$ with $a_0 = 6, a_1 = 8$
12. $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$; $n \geq 3$ with $a_0 = 3, a_1 = 6, a_2 = 0$
13. $a_n = 7a_{n-2} + 6a_{n-3}$; $a_0 = 9, a_1 = 10, a_2 = 32$
14. $a_n = 5a_{n-2} - 4a_{n-4}$; $a_0 = 3, a_1 = 2, a_2 = 6, a_3 = 8$
15. $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$; $a_0 = 7, a_1 = -4, a_2 = 8$
16. $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$; $a_0 = -5, a_1 = 4, a_2 = 88$
17. $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$; $a_0 = 5, a_1 = -9, a_2 = 15$
18. **Fibonacci Sequence** $a_n = a_{n-1} + a_{n-2}$; $a_0 = 0, a_1 = 1$ **[CU 2013]**
19. Show that Fibonacci numbers satisfy the recurrence relation $f_n = 5f_{n-4} + 3f_{n-5}$ for $n = 5, 6, 7, ...$ together with initial conditions, $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3$. Use this relation to show that $5f_n$ is divisible by 5, for $n = 1, 2, 3, ...$ **[CU 2012]**
20. The Lucas numbers satisfy the recurrence relation
$L_n = L_{n-1} + L_{n-2}$; $L_0 = 2, L_1 = 1$

(a) Show that, $L_n = f_{n-1} + f_{n+1}$ for $n = 2, 3, ...$ where f_n is the nth Fibonacci number.

(b) Find an explicit formula for Lucas numbers.

On IDE

1. $a_n - 5a_{n-1} + 6a_{n-2} = 1$
2. $a_n - 6a_{n-1} + 8a_{n-2} = 3$
3. $a_{n+2} - 2a_{n+1} + a_n = 3n + 5$
4. $a_{n+2} - 5a_{n+1} + 6a_n = n^2$
5. $a_n + 4a_{n-1} + 4a_{n-2} = n^2 - 3n + 5$
6. $2a_n - 7a_{n-1} + 3a_{n-2} = 2^n$
7. $a_n - 4a_{n-1} + 4a_{n-2} = (n+1)2^n$
8. $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = F(n)$ where
   (a) $F(n) = n^2$
   (b) $F(n) = 2^n$
   (c) $F(n) = n2^n$
   (d) $F(n) = (-2)^n$
   (e) $F(n) = n^2 \cdot 2^n$
   (f) $F(n) = n^3 \cdot (-2)^n$
   (g) $F(n) = 3$
9. $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ where $F(n)\ =$
   (a) n^3
   (b) $(-2)^n$
   (c) $n \cdot 2^n$
   (d) $n^2 \cdot 4^n$
   (e) $(n^2 - 2) \cdot (-2)^n$
   (f) $n^4 \cdot 2^n$
   (g) 2
10. $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$ [*Hint:* $a_n^{(p)} = An2^n + A_0 + A_1 n$]
11. $a_n = 2a_{n-1} + 3 \cdot 2^n$
12. $a_n = 4a_{n-1} - 4a_{n-2} + (n+1) \cdot 2^n$
13. Let a_n be the sum of the first n perfect squares, i.e. $a_n = \sum_{k=1}^{n} k^2$. Show that the sequence a_n satisfies the linear non-homogeneous recurrence relation $a_n = a_{n-1} + n^2$ and the initial condition $a_1 = 1$.

On IDE with initial conditions

1. $a_n - a_{n-1} = n\ ;\ a_0 = 1$ **[CU 2003]**

2. $a_n - 2a_{n-1} = 6\ ; a_1 = 2$

3. $a_n - a_{n+1} = 3^n\ ; a_0 = 1$

4. $a_n - a_{n-1} - 2a_{n-2} = 1\ ;\ a_1 = 1, a_2 = 3$

5. $a_{n+2} - 4a_{n+1} + 4a_n = 2^n\ ;\ a_0 = -2, a_1 = 0$

6. $2a_{n+1} - a_n = (\frac{1}{2})^n\ ;\ a_1 = 2$

7. $a_{n+2} - 6a_{n+1} + 9a_n = 3 \cdot 2^n + 7 \cdot 3^n$ with $a_0 = 1$ and $a_1 = 4$ **[CU 2011]**

8. Determine the values of the constants A and B such that $a_n = An + B$ is a solution of the recurrence relation $a_n = 2a_{n-1} + n + 5\ ;\ a_0 = 4$

9. Solve the simultaneous recurrence relations

   (a) $a_n = 3a_{n-1} + 2b_{n-1}$

   (b) $b_n = a_{n-1} + 2b_{n-1}\ ;\ a_0 = 1, b_0 = 2$

10. Show that if $a_n = a_{n-1} + a_{n-2}\ ;\ a_0 = s$ and $a_1 = t$ where s and t are constants then $a_n = sf_{n-1} + tf_n$ for all positive initegers n .

11. Express solution of the linear non-homogeneous recurrence relation $a_n = a_{n_1} + a_{n-2} + 1\ ;\ n \geq 2$ where $a_0 = 0$ and $a_1 = 1$ in terms of Fibonacci numbers.

    [*Hint:* Let, $b_n = a_n + 1$ and apply previous question the sequence for b_n]

12. $a_n = 2a_{n-1}$ if n is even with $a_0 = 0$
    $= 2a_{n_1} + 1$ if n is odd

Miscelleneous

1. Suppose a person deposit Rs. 10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in account after 30 years?

2. Find a recurrence relation for the no of bit strings that contain the string 01.

3. Find a recurrence relation and give initial condition for the number of bit strings of length n that does not have two consequtive 0s. How many such bit strings are there of length 5?

4. A computer system considers a string od decimal digits a valid codeword if it contains an even no of 0 digits. For example, 1230407869 is valid. Let a_n be the no of valid n digit codeword. Find a recurrence relation for a_n.

Theory of Probability

Theoretical Questions

1. Define the terms -

   Random experiment, Event, Elementary event, Composite event, Sure event, Impossible event, Sample space, Mutually exhaustive event, Mutually exclusive event, Equally likely event

2. State Classical Definition of probability. Also state it's limitations. **[CU 2005]**

3. Find the theoretical limit for the probability of an event.

4. For any two mutually exclusive events A and B, Show that $P(A \cup B) = P(A) + P(B)$. Also extend the proof for three such events.

5. For any n mutually exclusive events $A_1, A_2, A_3, ..., A_n$, Show that $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$. **[CU 2014]**

6. For any two events A and B, Show that$P(A \cup B) = P(A) + P(B) - P(A \cap B)$. **[CU 2010]**

7. For any three events A, B and C show that
   $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

8. For any n events $A_1, A_2, A_3, ..., A_n$ show that,
   $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n} \sum_{j=1} P(A_i \cap A_j) + \sum_{i=1}^{n} \sum_{j=1} \sum_{k-1}^{n} P(A_i \cap A_j \cap A_k) + ... + P(\cap_{i=1}^{n} A_i)$

9. Define Conditional probability of an event A on the condition that the event B has already occurred. **[CU 2006]**

10. State the Compound theorem of probability.

11. State and prove Bayes' theorem on conditional probability. **[CU 2011]**

12. (Boole's inequality)

    (a) If A_1 and A_2 are any two events then $P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$.

    (b) If A_1,A_2,A_3, ... and A_n are any n events then $P(\cup_{i=1}^{n}) \leq \sum_{i=1}^{P}(A_i)$

13. (Bonferroni's inequality)

    (a) If A_1 and A_2 are any two events then $P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$

    (b) If A_1,A_2,A_3, ... and A_n are any n events then $P(\cap_{i=1}^{n}) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$

14. Define independence of two events. Differentiate between - Pairwise independence and Mutual independence.

15. If A and B are any two mutually independent events, then show that

    (a) A and B^c are also independent.

    (b) A^c and B are also independent.

    (c) A^c and B^c are also independent.

16. For any two events A and B show that,
    $P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

17. If A_1,A_2,A_3, ... and A_n are any n independent events such that $P(A_i) = 1 - q_i \forall i = 1, 2, 3, ..., n$, then show that $P(\cup_{i=1}^{n}) = 1 - \Pi_{i=1}^{n} q_i$.

18. For any two events A and B show that, $P(B|A) = 1 - P(\bar{B}|A)$.

19. If $P(A) > P(B)$, then show that $P(A|B) > P(B|A)$.

20. If $A \subseteq B$ then show that $P(A) \leq P(B)$.

21. If $P(A|C) \geq P(B|C)$ and $P(A|\bar{C}) \geq P(B|\bar{C})$, then show that $P(A) \geq P(B)$.

22. If $P(A|B) = 1$, then show that $P(A \cap B \cap C) = P(B \cap C)$.

23. If $P(A \cap B \cap C) = 0$ then show that $P(X|C) = P(A|C) + P(B|C)$ where $X = A + B$.

24. If A, B and C are any three events, then prove that, $P(A+B|C) = P(A|C) + P(B|C) - P(AB|C)$ where $P(C) \neq 0$

25. For any three events A, B and C prove that, $P(A\bar{B}|C) + P(AB|C) = P(A|C)$

26. For any three events A, B and C such that $B \subset C$ and $P(A) > 0$, prove that $P(B|A) \leq P(C|A)$

27. Define - Random variable. What are the different types of a random variable?

28. What do you mean by discrete probability distribution? What is probability mass function?

29. What do you mean by continuous probability distribution? What is probability density function?

30. Define expectation and variance of a random variable.

31. Show that $E(k) = k$ where k is any arbitrarily chosen constant

32. Consider two random variables x and y such that $y = a.x + b$ where a and b are chosen constants. Show that, $E(y) = aE(x) + b$

33. If x_1 and x_2 are two random variables then show that
    (a) $E(x + y) = E(x) + E(y)$
    (b) $E(x + y) = E(x) - E(y)$
    (c) $E(xy) = E(x).E(y)$

34. Consider X to be a discrete random variate following $Bin(n, p)$. Find the expectation (mean) and variance of X.

35. Consider X to be a discrete random variate following $Poi(\lambda)$. Find the expectation (mean) and variance of X.

36. If X follows $Bin(m, p)$ and Y follows $Bin(n, p)$ where X and Y are independent, show that $X + Y$ follows $Bin(m + n, p)$

37. If X follows $Poi(\lambda_1)$ and Y follows $Poi(\lambda_2)$ where X and Y are independent, show that $X + Y$ follows $Poi(\lambda_1 + \lambda_2)$

38. Show that Poisson distribution can be considered as a limiting case of Binomial distribution.

39. Consider X to be a continuous random variate following $N(\mu, \sigma^2)$. Find the expectation (mean) and variance of X

Problems

1. Two dice are thrown simultaneously. Find the probability that the sum of their outputs will be seven. **[CU 2013]**

2. If three dice are rolled, find the probability that exactly one face shows a number less than or equal to 4. **[CU 2012]**

3. An unbiased coin is tossed 3 times. Find the probability of getting more than one HEAD. **[CU 2009]**

4. Calculate the probability "sum is less than 9" in the experiment of rolling two unbiased dice. **[CU 2008]**

5. A bit of string of length 4 is generated so that each of the 16 bit string of length 4 is equally likely. What is the probability that it contains at least two consecutive zeros given that its first bit is zero. **[CU 2006]**

6. Among 10 girls in a class, 3 have blue eyes. Two girls are chosen at random. Find the probability that

   (a) neither has blue eyes

   (b) exactly one has blue eyes

   (c) at least one has blue eyes

   (d) both have blue eyes

   [SU 2010]

7. Consider two fair dice are rolled simultaneously. Compute the expectation of the random variable SUM which represents the sum of the face values appearing on the dice. **[SU 2010]**