11 THEORY OF PROBABILITY

11.1 INTRODUCTION

The word *probability* literally denotes 'chance', and the *theory of probability* deals with laws governing the chances of occurrence of phenomena which are unpredictable in nature.

Although historically the probability theory originated from the games of chance played by tossing coins, throwing dice, drawing cards etc., its importance has enormously increased in recent years. Today, the notions of probability find important applications in almost all disciplines—physics, chemistry, biology, psychology, education, economics, business, industry, engineering etc. The concept of probability plays a vital role in statistics.

11.2 RANDOM EXPERIMENT, OUTCOME, EVENT

(1) **Random Experiment:** When a coin is tossed (as is done before the start of a cricket match), either Head or Tail appears. But the result of any toss cannot be predicted in advance, and is said to 'depend on chance'. Similarly, when a die is tossed and the number coming on the uppermost face is observed, one of the numbers 1, 2, 3, 4, 5, 6 definitely appears, but nobody can assert with certainty which result will materialise in any particular toss.

The word *experiment* is used to describe an act which can be repeated under some given conditions. *Random experiments* are those experiments whose results depend on chance (The word 'random' may be taken as equivalent to 'depending on chance').

Example 11.1 *Each of the following may be called a random experiment:*

(a) Tossing a coin (or several coins).
(b) Throwing a die (or several dice).
(c) Drawing cards from a pack.
(d) Drawing balls from a box containing given numbers of white and black balls.
(e) Studying the distribution of boys and girls in families having three children.

(2) **Outcome:** The result of a random experiment will be called an outcome. The possible outcomes of a random experiment may often be described in several ways.

Example 11.2

(a) In the random experiment of tossing a coin, there are 2 possible outcomes— '*Head*' and '*Tail*'; or in symbols

$$H, \quad T$$

(b) In the random experiment of throwing a six-faced die an observing the number of points that appear, the possible outcomes at

$$1, 2, 3, 4, 5, 6$$

In the same experiment, the possible outcomes could also be stated as

'*Odd number of points*'; '*Even number of points*'

(c) In the random experiment of two tosses of a coin (or, a single toss of two coins simultaneously), we may list the possible outcome as

$$HH, \quad HT, \quad TH, \quad TT$$

where the two letters indicate the results of the first and the second tosses (or coins) respectively. If our interest lies in the number of heads turned up, the possible outcomes may also be stated as

2 *heads*, 1 *head*, 0 *head*

(d) There are 36 possible outcomes in the random experiment of a single throw of 2 dice (or, two throws of one die), as follows:

(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

each pair of numbers indicating the results of the first and the second dice (or throws) respectively. If we are interested in the sum of the points on the two dice, the possible outcomes may again be listed as

$$2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10, \quad 11, \quad 12,$$

because the sum of points on the two dice may vary from 2 to 12.

(e) In the random experiment of studying the distribution of boys (B) and girls (G) in families having 3 children, there are 8 possible outcomes

$$BBB, \quad BBG, \quad BGB, \quad BGG, \quad GBB, \quad GBG, \quad GGB, \quad GGG$$

(the symbol BGG, for instance, denoting the outcome 'eldest child boy, 2nd and 3rd girls'); or, in terms of the number of boys, 4 possible outcomes.

3 *boys*, 2 *boys*, 1 *boy*, 0 *boy*

or, in terms of the number of girls, 4 possible outcomes

0 *girl*, 1 *girl*, 2 *girls*, 3 *girls*

(3) Event: In the theory of probability, the term event is used to denote any phenomenon which occurs in a random experiment. In effect, one or more outcomes are said to constitute an "event". Events may be 'elementary' or 'composite'. An event is said to be *elementary*, if it cannot be decomposed into simpler events. A *composite* event is an aggregate of several elementary events.

[**Note:** Readers will note that the words "outcome" and "event" may appear to have been used almost in the same sense. In the axiomatic theory of probability. (Section 11.13), 'outcome' is an undefined concept connected with a random experiment, just in the same way as 'point' is undefined to geometry. However, a slight distinction is made between the two words. The 'outcomes' are generally thought of as the ultimate results of a random experiment which cannot be split up further. Hence these are sometimes referred to as 'elementary outcomes.' Any "event" (i.e., phenomenon) connected with a random experiment is then defined as an aggregate (or 'set' in the mathematical sense) of certain specified outcomes. In particular, a set containing only a single outcome is called an *elementary event.*

For example, when a die is thrown, there are six possible outcomes 1, 2, 3, 4, 5, 6. The aggregate of three outcomes 1, 3, 5 may be said to form the event 'odd number', the aggregate of two outcomes 5, 6 may be called the event 'at least five', and the aggregate (if we use the word in a general sense) of just one outcome 6 may be called the event 'six']

Example 11.3

(a) When a coin is tossed, we may speak of the events '*Head*' and '*Tail*', each of which is an elementary event.

(b) When 2 coins are tossed, the event 'both heads, is an elementary event (HH). But 'one head and one tail' is a composite event consisting of the elementary events HT and TH.

(c) When a die is tossed, the event 'odd number of points' is composite, because it may be split up into elementary events 1, 3, 5.

(d) When two dice are thrown, the event 'total 8 points' is composite, consisting of the elementary events (2, 6), (3, 5), (4, 4), (5, 3) and (6, 2). But the event 'total 12' is elementary, consisting of the only outcome (6, 6), which cannot be decomposed.

Tree Diagram provides a systematic method of enumerating the elementary events of a random experiment. This diagram shows the outcomes at each stage of the experiment in an ordered form through the different 'branches' of the 'tree'.

Example 11.4 *For the possible distribution of boys and girls in families having 3 children, the tree diagram is shown below:*

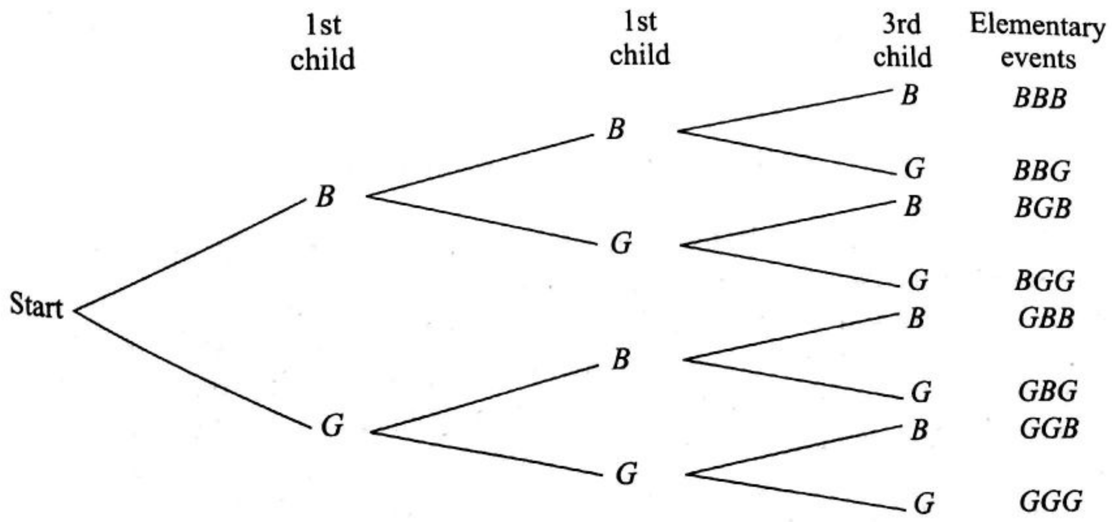


Fig. 11.1 Tree Diagram

The 1st child may have two possible outcomes, viz. Boy or Girl. Thus, two branches emerge from "Start", giving out B and G. Corresponding to B here, there are two possible outcomes B or G for the 2nd child, represented by the two branches emerging from the branch B of the 1st child. Similarly, from the branch G of the 1st child, there are two branches for the 2nd child, showing B and G. From each B and G for the 2nd child, 2 branches again spread out for the 3rd child B or G. When all different branches of the tree are followed from Start, the ordered results yield all possible elementary events.

Example 11.5 *There are 3 identical boxes:*

Box I contains 2 white and 5 red balls

Box II contains 4 white and 1 red balls

Box III contains 8 white balls only.

One box is selected at random; 2 balls are then successively taken out of it, and their colour noted. The elementary events arising out of the experiment, regard being had as to the number of the selected box and the colours of the balls drawn, can be represented by the following tree diagram:

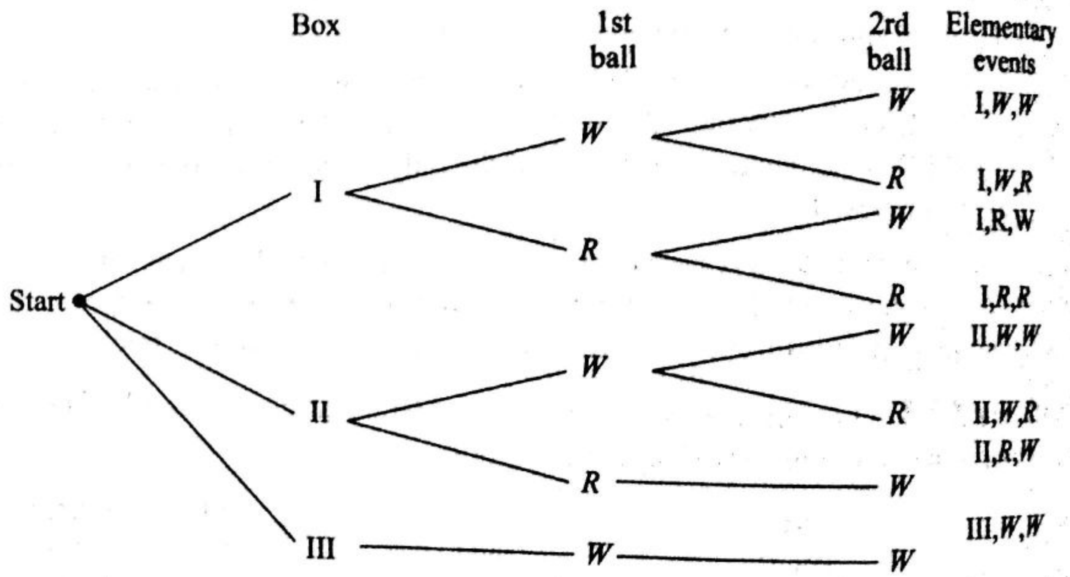


Fig. 11.2 Tree Diagram

Any of the 3 boxes may be selected, leading to 3 branches, I, II, III from start. When box I is selected, the Ist ball may be either White (W) or Red (R) giving out 2 branches; similarly for box II. But when box III is selected, the 1st ball can be only W, because there are no red balls in box III. As regards the colour of the 2nd ball, we see that when the 1st ball from box I is W, the 2nd ball may have any of the two possible colours W and R, giving out 2 branches; when the 1st ball is R, the 2nd may also be either W or R. Again, from box II, when the 1st ball is W, the 2nd ball may be either W or R, leading to 2 branches; but when the 1st ball is R, the 2nd can only be W (because in box II there was only red ball). From box III, there is only one branch W for the 1st ball, and the 2nd ball must also be W, leading to only one branch. The last column shows the complete set of elementary events.

11.3 IMPORTANT TERMINOLOGY

(1) **Mutually Exclusive:** Events are said to be 'mutually exclusive', when two or more of them can not occur simultaneously. This means that mutually exclusive events can occur only one at a time, and the occurrence of any event signifies impossibility of the remaining events in any particular performance of the random experiment.

[**Note:** Mutually exclusive events can always be connected by the words "either ... or". Events A, B, C, D are mutually exclusive, only if *either A or B or C or D* can occur.]

Example 11.6

(a) In tossing a coin, the elementary events 'Head' and 'Tail' are mutually exclusive. Because in any toss either Head occurs, or Tail occurs; and the occurrence of Head as well as Tail in any toss is impossible. The occurrence of any of them signifies non-occurrence of the other.

(b) When a die is thrown, let the symbols $E_1, E_2, E_3, E_4, E_5, E_6$ denote respectively the appearance of points 1, 2, 3, 4, 5, 6, on the face coming uppermost. These events are mutually exclusive, because the appearance of a particular face implies non-occurrence of any of the remaining faces. Only one of the faces can appear in any particular throw of the die. Again, let

A denote the event 'Odd number of points'

B denote the event 'Even number of points'

These events are mutually exclusive because the result of any throw shows either an odd number or an even number, but not both odd and even numbers simultaneously. (Note that $E_1, E_2, ..., E_6$ are elementary events, while A and B are composite events).

(c) When 2 coins are tossed, let

A_1 denote the event HH

A_2 denote the event HT

A_3 denote the event TH

A_4 denote the event TT

These 4 events are mutually exclusive because the result of any toss yields either A_1 or A_2 or A_3 or A_4. The result cannot be, for instance, HH as well as TH.

Again, in this experiment, let

B_1 denote the event 'Two heads'

B_2 denote the event 'One head'

B_3 denote the event 'No head'

Events B_1, B_2 and B_3 are mutually exclusive; because when 2 coins are tossed they show either both heads, or only one head (and one tail), or no head (i.e., both tails). Note that here B_1 and B_2 are elementary events, while B_2 is a composite event, composed of the elementary events HT and TH.

In the same experiment, let

C_1 denote the event 'Same result on the 2 coins'

C_2 denote the event 'Different results on the 2 coins'

The two events are mutually exclusive (Note that both C_1 and C_2 are composite events).

(2) **Exhaustive:** Several events are said to form an *exhaustive set*, if at least one of them must necessarily occur. The complete group of all possible elementary events of a random experiment gives an exhaustive set of events.

Example 11.7

(a) In Example 11.6(a), the two events 'Head' and 'Tail' form an exhaustive set, because one of these two must necessarily occur.

(b) In Example 11.6(b), the six events, E_1, E_2, E_3 ... E_6 form an exhaustive set. Similarly, the events A and B are also exhaustive.

(c) In Example 11.6(c), each of the groups of events (A_1, A_2, A_3, A_4), (B_1, B_2, B_3) and (C_1, C_2) forms an exhaustive set.

(d) Let us consider only 4 of the events E_2, E_3, E_4, E_6, as defined in Example 11.6(b). These evens are mutually exclusive, because if one of them occurs, another cannot. However, the four events are not exhaustive, because the complete group of elementary events of the experiment includes some other events, viz. E_1 and E_5.

(e) When a die is thrown, let

A denote the event 'Odd number of points'

B denote the event 'At least 2 points'

These two events together form an exhaustive set; because the possible outcomes of this experiment can be described by one or the other of the events A and B. However, they are *not* mutually exclusive, because the occurrence of event A does not necessarily imply non-occurrence of B.

(3) **Trial:** Any particular performance of the random experiment is usually called a *trial*.

(4) **Cases Favourable to an Event:** Among all the possible outcomes of a random experiment, those cases which entail occurrence of an event A are called '*cases favourable to A*'.

Example 11.8

(a) When a die is thrown, these are six possible outcomes, viz 1, 2, 3, 4, 5, 6. Among these, 3 cases (viz. 1, 3, 5) are favourable to the event 'odd number of points', and 3 cases (viz. 2, 4, 6) are favourable to the event 'even number of points'.

(b) When 2 coins are tossed, out of the 4 possible outcomes *HH, HT, TH, TT*, there are 3 cases favourable to the event 'at least one head', (viz. *HH, HT, TH*), and 2 cases favourable to the event 'one head' (viz. *HT, TH*).

(c) When a card is drawn from a full pack, there are 52 possible outcomes; because any of the 52 cards may be drawn. Out of these only 4 cases are favourable to the event 'ace', viz. the cases when the card drawn is either an ace of spade, or an ace of heart, or of diamond, or of club. There are 13 cases favourable to the event 'spade' because the drawing of any of the 13 spade cards will entail drawing a spade. Similarly, there are 26 cases favourable to the event 'red card'.

(5) **Equally Likely:** The outcomes of a random experiment are said to be '*equally likely*', if after taking into consideration all relevant evidence, none of them can be expected in preference to another.

Example 11.9

(a) If a coin is unbiased (i.e., is perfectly homogenous and is uniform in both faces), there is no reason to expect that, for instance, heads will appear more often than tails, or vice versa. The two outcomes. Head and Tail are therefore equally likely.

(b) If a die is unbiased (i.e., is a perfect geometrical cube in shape and made of homogeneous material) there is no reason to suspect that in any throw of the die some particular face will come up more frequently than another face. The six outcomes showing points 1, 2, 3, 4,, 5, 6 are therefore considered 'equally likely.'

(c) If two unbiased coins are thrown, the elementary events *HH*, *HT*, *TH* and *TT* are equally likely. But the outcomes '2 heads' '1 head' and 'no head' are not equally likely.

(d) Out of a full pack of 52 cards, two cards may be drawn in $^{52}C_2 = \dfrac{52 \times 51}{1 \times 2}$ = 1326 ways, and the drawing of a pair of cards gives rise to an outcome. Since the drawing of 2 cards may reveal any of these 1326 possible combinations, the outcomes are equally likely.

11.4 TECHNIQUES OF COUNTING

Some mathematical methods are shown below, which are often helpful for determining without direct enumeration the number of outcomes of a random experiment or the number of cases favourable to an event. These are referred to as "*Combinatorial Methods*"

1. *Fundamental Principle of Counting*: If several processes can be performed in the following manner: the first process in p ways, the second in q ways, the third in r ways, and so on, then the total number of ways in which the whole process can be performed in the order indicated is given by the product.

$$p \,.\, q \,.\, r \ldots \qquad (11.4.1)$$

2. *Permutation*: The total number of ways of arranging (called *permutation*) n distinct objects taken r at a time is given by

$$^nP_r = n(n-1)\,(n-2) \ldots (n-r+1) \qquad (11.4.2)$$

3. *Arrangement in a Line or Circle*: The total number of ways in which n distinct objects can be arranged among themselves is

   (i) in a line $\quad n\,! = 1\,.\,2\,.\,3 \ldots n \qquad (11.4.3)$

   (ii) in a circle $\quad (n-1)\,! \qquad (11.4.4)$

4. *Permutation with Repetition*: The number of ways of arranging n objects, among which p are alike, q are alike, r are alike, etc. is

$$\frac{n\,!}{p\,!\,q\,!\,r\,! \ldots} \qquad (11.4.5)$$

5. *Combination*: The total number of possible groups (called *combination*) that can be formed by taking r objects out of n distinct objects is given by

$$^nC_r = \frac{n(n-1)(n-2)\ \dots\ (n-r+1)}{r\,!}$$

$$= \frac{n!}{r\,!\,(n-r)!} \qquad (11.4.6)$$

6. *Combination (any number at a time)*: The total number of ways of forming groups by taking any number from n distinct objects is

$$^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n = 2^n - 1 \qquad (11.4.7)$$

7. *Choosing Balls from an Urn*: The total number of ways of choosing a white balls and b black balls from an urn containing A white and B black balls is

$$^AC_a \,.\, {}^BC_b \qquad (11.4.8)$$

This may be extended to more than two categories of balls.

8. *Ordered Partitions (Distinct objects)*: The total number of ways of distributing n distinct objects into r compartments marked 1, 2, ..., r is

$$r^n \qquad (11.4.9)$$

The number of ways in which the n objects can be distributed so that the compartments contain respectively $n_1, n_2, \dots, n_r$ object is

$$\frac{n!}{n_1\,!\,n_2\,!\dots n_r\,!} \qquad (11.4.10)$$

9. *Ordered Partitions (Identical objects)*: The total number of ways of distributing n identical objects into r compartments marked 1, 2, ..., r is

$$^{n+r-1}C_{r-1} \qquad (11.4.11)$$

If none of the compartments should remain empty, the total number of ways of distributing the balls is

$$^{n-1}C_{r-1} \qquad (11.4.12)$$

10. *Sum of Points on the Dice*: When n dice are thrown, the number of ways of getting a total of r points is given by the

$$\text{Coefficient of } x^r \text{ in } (x + x^2 + x^3 + x^4 + x^5 + x^6)^n \qquad (11.4.13)$$

11. *Dearrangements and Matches*: If n objects numbered 1, 2, 3, ..., n are distributed at random in n places also numbered 1, 2, 3, ..., n, a "*match*" is said to occur if an object occupies the place corresponding to its number. The number of permutations in which no match occurs is

$$t_n = n\,!\left\{1 - \frac{1}{1\,!} + \frac{1}{2\,!} - \frac{1}{3\,!} + \dots + (-1)^n \frac{1}{n\,!}\right\} \qquad (11.4.14)$$

This is also known as "*derangement*".

The number of permutations of n objects in which exactly r matches occur is

$$^nC_r\ t_{n-r} = \frac{n\,!}{r\,!}\left\{1 - \frac{1}{1\,!} + \frac{1}{2\,!} - \frac{1}{3\,!} + \dots + \frac{(-1)^{n-r}}{(n-r)\,!}\right\} \qquad (11.4.15)$$

11.5 CLASSICAL (OR 'A PRIORI') DEFINITION OF PROBABILITY

If a random experiment has n possible outcomes, which are *mutually exclusive, exhaustive* and *equally likely*, and m of these are favourable to an event A, then the 'probability' of the event is defined as the ratio m/n. In symbols

$$P(A) = \frac{m}{n} \tag{11.5.1}$$

$$\textit{Probability of an event} = \frac{\textit{Number of outcomes favourable to the event}}{\textit{Total number of mutually exclusive, exhaustive and equally likely outcomes of the random experiment}}$$

It may be noted that 'probability', as defined above, is only a ratio of two numbers, in which the numerator (m) is the number of favourable cases and the denominator (n) is the total number of possible outcomes satisfying certain conditions. Therefore, for the calculation of probability the undernoted steps should be followed:

(Step 1): Enumerate all the possible outcomes of the experiment, such that they satisfy the 3 criteria of being 'mutually exclusive', 'exhaustive', and 'equally likely'. Count the number (n) of such outcomes.

(Step 2): Check how many of these cases are favourable to the event for which the probability is desired. Let this number be m.

(Step 3): Divide m by n, and the result gives the probability of the event.

Probability, as defined by (11.5.1), always lies between 0 and 1.

$$0 \leq p \leq 1 \tag{11.5.2}$$

The minimum value of p, viz. 0, is attained when none of the outcomes is favourable to the event, i.e., $m = 0$. The event is then said to be '*impossible*'. The maximum value of p, viz. 1, is attained when all the possible outcomes are favourable to the event, i.e, $m = n$. The event is then said to be '*certain*'.

Defects of Classical Definition

(i) It is based on the feasibility of subdividing the possible outcomes of the experiment into 'mutually exclusive', 'exhaustive' and 'equally likely' cases. Unless this can be done, the formula is inapplicable.

(ii) The phrase 'equally likely' appearing in the classical definition, is synonymous with 'equally probable', which means that we are trying to define probability in terms of equal probabilities. How do you know whether the probabilities are equal, before you can measure them ? The definition is thus circular in nature.

(iii) The definition has only limited applications in coin-tossing, die-throwing and similar games of chance. Using this definition, we cannot, for example, find the probability that an Indian aged 25 will die before reaching the age 50 (such probabilities are required to be calculated for fixing the premium rates in life insurance). Thus, it may not be practically possible to enumerate all the outcomes of a random experiment.

(iv) The definition fails, when the number of possible outcomes is infinitely large.

Example 11.10 *What is the probability of obtaining 'Head' in a single toss of an unbiased coin ?*

Solution When a coin is tossed there are two possible outcomes viz. Head and Tail. These two outcomes are mutually exclusive and exhaustive. Moreover, since the coin is unbiased, the outcomes are also equally likely. Out of these two mutually exclusive, exhaustive and equally likely outcomes, only one case is favourable to the event 'Head'. Thus, using the classical definition we have

$$\text{Probability of obtaining 'Head'} = \frac{m}{n} = \frac{1}{2}$$

Example 11.11 *Two unbiased coins are tossed. What is the probability of obtaining (a) both heads, (b) one head and one tail, (c) both tails, (d) at least one head?*

Solution The experiment has 4 possible outcomes, viz. *HH, HT, TH, TT* (the two letters in each case denoting the results on the 1st and 2nd coins respectively). These are mutually exclusive, exhaustive and equally likely (Examples 11.6c, 11.7c, 11.9c). Thus $n = 4$. The cases favourable to the events are as follows:

Event	*Favourable cases*	*Number of favourable cases*
(a) *Both heads*	*HH*	1
(b) *One head and one tail*	*HT, TH*	2
(c) *Both tails*	*TT*	1
(d) *At least one head*	*HH, HT, TH*	3

Applying the classical definition (11.5.1)

(a) *P (both heads)* = 1/4

(b) *P (one head and one tail)* = 2/4 = 1/2

(c) *P (both tails)* = 1/4

(d) *P (both least one head)* = 3/4

Example 11.12 *A die is tossed and the number of points appearing on the uppermost face is observed. What is the probability of obtaining (a) an even number, (b) an odd number, (c) less than 3, (d) a "six" ?*

Solution In this experiment there are 6 possible outcomes, viz. 1, 2, 3, 4, 5, 6. These are mutually exclusive and exhaustive. If the die is assumed to be unbiased, then the 6 outcomes are also equally likely, Thus $n = 6$. Out of them.

3 cases (viz., 2, 4, 6) are favourable to 'even number'
3 cases (viz., 2, 3, 5) are favourable to 'odd number'
2 cases (viz., 1, 2) are favourable to 'less than 3'
1 case (viz., 6) is favourable to 'six'

Thus,

P(even number) = 3/6 = 1/2
P(odd number) = 3/6 = 1/2
P(less than 3) = 2/6 = 1/3
P(six) = 1/6

Example 11.13 *When two unbiased coins are tossed. what is the probability of obtaining (a) 3 heads, (b) not more than 3 heads?*

Solution There are 4 possible outcomes, viz. HH, HT, TH, TT. These are mutually exclusive, exhaustive and equally likely. Hence $n = 4$. Since none of these cases is favourable to the event '3 heads', $m = 0$. Therefore

$$P(\text{3 heads}) = .0/4 = 0$$

(Note that in a toss of two coins, 3 heads are impossible).

Again, we see that all the four outcomes are favourable to the event 'not more than 3 heads' ($m = 4$). So

$$P(\text{not more than 3 heads}) = 4/4 = 1$$

(Note that in a toss of 2 coins it is certain that we get not more than 3 heads; in fact, at most 2 heads may be obtained.)

Example 11.14 *Two coins are tossed. Find the probability of getting both heads or both tails.*

Solution Assuming that the coins are unbiased, there are 4 outcomes, viz. HH, HT, TH, TT, which are mutually exclusive, exhaustive and equally likely. Out of them, 2 cases, viz. HH and TT, are favourable to the event "both heads or both tails". Hence by the classical definition of probability.

$$p = 2/4 = 1/2$$

Example 11.15 *Two dice with faces marked 1, 2, 3, 4, 5, 6 are thrown simultaneously and the points on the dice are multiplied together. Find the probability that the product is 12.*

Solution If two dice are thrown, there are 36 possible outcomes (see Example 11.2(d), page 2), shown as pairs of numbers corresponding to the points on the 1st and the 2nd dice. These outcomes are mutually exclusive and exhaustive, and also equally likely on the assumption that the dice are unbiased. Of them only the following 4 cases are favourable to the event "product 12"—(2, 6), (3, 4), (4, 3), (6, 2). So, using the classical definition,

$$p = \frac{4}{36} = \frac{1}{9}.$$

Example 11.16 *A bag contains 6 white and 4 black balls. One ball is drawn. What is the probability that it is white ?*

Solution Let us put a serial number on each ball as follows. White balls: 1, 2, 3, 4, 5, 6 and Black balls: 7, 8, 9, 10. There are 10 outcomes as regards the number on the selected ball, because any of the 10 balls could be drawn. Since the balls are assumed to be identical except in colour, any of the balls is as likely to appear as any other ball. Out of these 10 possible outcomes, which are mutually exclusive, exhaustive and equally likely, only 6 cases (viz. when the number of the selected ball is 1, or 2, ... or 6) are favourable to the event "white ball". Hence,

$$p = \frac{6}{10} = \frac{3}{5}.$$

Note that here

$$\text{Probability} = \frac{\text{Number of white balls}}{\text{Total number of balls}}$$

Example 11.17 *If 2 balls are drawn one after another from a bag containing 3 white and 5 black balls, what is the probability that*

(i) the fist ball is white and the 2nd is black;

(ii) one ball is white and the other is black ?

[D.M. '78]

Solution Let us put serial numbers on the balls as follows: Whites – 1, 2, 3; Black – 4, 5, 6, 7, 8. In order to find the total number of possible outcomes, we see that the 1st ball may be selected in 8 ways, because any of the 8 balls may be drawn. Corresponding to each way of drawing the 1st ball the 2nd ball may be drawn in 7 ways (because there remain only 7 balls in the bag after the 1st ball has been drawn). Hence the 2 balls may be drawn in $8 \times 7 = 56$ ways. Since the balls are identical in all respects except in colour, these 56 cases are mutually exclusive, exhaustive and equally likely.

(i) As regards the number of favourable cases, the 1st ball will be white only if any of the balls numbered 1, 2, 3 is drawn i.e., in 3 ways. When the 1st ball has been obtained in any of these 3 ways, the 2nd ball will be black only if any of the balls numbered 4, 5, 6, 7, 8 is drawn, i.e., in 5 ways. Hence the number of cases favourable to the event is $3 \times 5 = 15$. So,

$$p = \frac{15}{56}.$$

(ii) We have seen that the number of ways of drawing a white ball and a black ball in the order (white, black) is 15. Similarly, it can be shown that the number of ways of drawing a white ball and a black ball in the order (black, white) is $5 \times 3 = 15$. Hence the number of cases favourable to the event "one ball is white and the other black", irrespective of the order in which they are drawn, is $(15 + 15) = 30$. Therefore

$$p = \frac{30}{56} = \frac{15}{28}$$

Example 11.18 *Two cards are drawn from a full pack of 52 cards. Find the probability that (i) both are red cards, (ii) one is a heart and the other is a diamond.*

Solution *(First method) (Ignoring the order of drawing).* Two cards may be drawn out of the pack of 52 cards in ${}^{52}C_2 = 1326$ ways. These outcomes are mutually exclusive, exhaustive and equally likely.

(i) The number of cases favourable to the event "both red cards" in ${}^{26}C_2 = 325$, because the pack contains only 26 red cards.

$$p = \frac{325}{1326} = \frac{25}{102}$$

(ii) In order to find the number of cases favourable to the event "1 heart and 1 diamond", we see that 1 heart can be drawn in 13 ways and similarly 1 diamond can be drawn in 13 ways (because each pack contains 13 cards of each suit: Spades, Hearts, Diamonds, Clubs). Since any of the heart card can be combined with any of diamond card to give a group of 1 heart and 1 diamond, the number of cases favourable to the event is $13 \times 13 = 169$.

$$p = \frac{169}{1326} = \frac{13}{102}$$

(Second method) (Considering the order of drawing) The 1st card may be drawn in 52 ways, and corresponding to each way of drawing the 1st card the 2nd card may be drawn in 51 ways. Hence the total number of cases, considering the order, is $52 \times 51 = 2652$.

(i) Since the 1st red card can be drawn in 26 ways, and thereafter the 2nd red card in 25 ways, the number of cases favourable to "two red cards" is $26 \times 25 = 650$.

$$p = \frac{650}{2652} = \frac{25}{102}$$

(ii) Here the number of cases favourable to the order (heart diamond) is $13 \times 13 = 169$; similarly the number of cases favourable to the order (diamond, heart) is also 169. Hence, the number of cases favourable to "one heart and one diamond" is $(169 + 169) = 338$

$$p = \frac{328}{2652} = \frac{13}{102} \text{ (as before)}$$

Example 11.19 *What is the probability that all 3 children in a family have different birthdays? (Assume, 1 year = 365 days).*

Solution The 1st child may be born on any of 365 days of the year; the 2nd also on any of the 365 days, and similarly the 3rd child. Hence, the total number of possible ways in which the 3 children have birthdays is $365 \times 365 \times 365$. These cases are mutually exclusive, exhaustive and equally likely. As regards the number of favourable cases out of these, we note that the 1st child may have any of the 365 days of the year as its birthday. In order that the 2nd child has a birthday different from that of the 1st, it should have been born on any of the 364 remaining days of the year; similarly the 3rd should be born on any of the remaining 363 days. So, the number of cases favourable to the event "different birthdays" is $365 \times 364 \times 363$.

$$p = \frac{365 \times 364 \times 363}{365 \times 365 \times 365} = 0.992$$

Example 11.20 *Five persons A, B, C, D, E occupy seats in a row at random. What is the probability that A and B sit next to each other?*

Solution Five persons can arrange themselves in a row, without restriction, in $5\,! = 1 \times 2 \times 3 \times 4 \times 5 = 120$ ways. Considering A and B together, they can arrange themselves in $4\,! \times 2 = (1 \times 2 \times 3 \times 4) \times 2 = 48$ ways; because A may be to the left or to the right of B.

$$p = \frac{48}{120} = \frac{2}{5}$$

[**Note:** The phrase 'at random' denotes 'with equal probability'.]

Example 11.21 *A batch contains 10 articles of which 4 are defective. If 3 articles are chosen at random, what is the probability that none of them is defective?*

Solution Total number of ways of selecting 3 articles (without restriction) out of 10 is ${}^{10}C_3 = 120$. If none of the selected articles is defective, they must form a group out of the 6 non-defective articles. Hence the number of favourable cases is ${}^{6}C_3 = 20$.

$$p = \frac{20}{120} = \frac{1}{6}$$

Example 11.22 *10 distinguishable balls are distributed at random into 4 boxes. What is the probability that a specified box contains exactly 2 balls?*

Solution The first ball may go to any of the four boxes and hence may be distributed in 4 ways. The 2nd ball may also be distributed in 4 ways, the 3rd ball in 4 ways, ..., the 10th ball in 4 ways. Hence the total number of ways in which the 10 balls may be distributed into the 4 boxes is $4 \times 4 \times 4$ ... (10 times) $= 4^{10}$ (see 11.4.9). In order to find the number of favourable cases, we see that the specified box may receive any of the ${}^{10}C_2$ groups of 2 balls out of 10. When two such balls have gone into that box the remaining 8 balls may be distributed in any manner into the remaining 3 boxes in 3^8 ways. Hence the total number of favourable cases is ${}^{10}C_2 \times 3^8 = 45 \times 3^8$. Therefore,

$$p = 45 \times \frac{3^8}{4^{10}}$$

Example 11.23 *If 10 persons are arranged at random (i) in a line (ii) in a ring, find the probability that 2 particular persons will be next to each other.*

Solution

(i) Ten persons can be arranged in a line in 10 ! ways (see formula 11.4.3), which are mutually exclusive, exhaustive and equally likely. To find the number of favourable cases, i.e., when the two particular persons remain together, we consider the two as one (supporting they are fastened together), so that the number of permutations is 9 !. But the two persons can arrange themselves in 2 ! ways without affecting the position of any of the remaining persons. Hence the total number of favourable cases is 9 ! 2 !. Using the classical definition,

$$p_1 = \frac{9\,!\,2\,!}{10\,!} = \frac{1}{5}$$

(ii) Ten persons can arrange themselves in a *ring* in 9 ! ways (see formula 11.4.4). Arguing as before, the number of favourable cases is 8 ! 2 !. Hence

$$p_2 = \frac{8\,!\,2\,!}{9\,!} = \frac{2}{9}$$

Example 11.24 *X and Y stand in a line at random with 10 other people. What is the probability that there are 3 people between X and Y?*

(C.U., B.A., (Econ) '66; B.Sc. (Math.) 70; B.U., B.A., (Econ) '67

Solution *(First method)* There are 12 people including X and Y and they can arrange themselves in $n = 12$! ways. These n arrangements are mutually exclusive, exhaustive and equally likely. To find the number of arrangements (m) favourable to the event (viz. 3 people standing between X and Y), let us first consider the case when X occupies place 1 and Y occupies place 5 (there are 3 people in between), so that the remaining 10 places can be filled up in 10 ! ways.

Place	1	2	3	4	5	6	7	8	9	10	11	12
	X				Y							

If X and Y interchange their positions, we have another set of 10! arrangements. Thus, with X and Y in places (1, 5) the total number of arrangements favourable to the event is (2 × 10 !).

In fact, with X and Y occupying any two specified places, the number of arrangement is (2 × 10)!. Among the total number of n arrangements, only those cases will be favourable to the event when X and Y occupy the following 8 pairs of places:

(1, 5), (2, 6), (3, 7), (4, 8), (5, 9), (6, 10), (7, 11), (8, 12)

Hence, the total number of arrangements favourable to the event is $m = 8(2 \times 10\,!)$. By the classical definition of probability (11.5.1),

$$p = \frac{m}{n} = \frac{8(2 \times 10\,!)}{12\,!} = \frac{8 \times 2}{12 \times 11} = \frac{4}{33}$$

(*Second method*) Of the 12 places, X and Y could occupy any 2 places in $^{12}C_2 = 66$ ways, which are mutually exclusive, exhaustive and equally likely. There will be 3 persons between X and Y, if the latter occupy places (1, 5), (2, 6), (3, 7), (4, 8), (5, 9), (6, 10), (7, 11), (8, 12). Hence the number of favourable cases is 8. So, the required probability = 8/66 = 4/33.

Example 11.25 *Twelve persons, amongst whom are X and Y, are seated at random at a round table, What is the probability that there are 3 persons between X and Y?*

Solution X can occupy any of the 12 seats, and correspondingly Y any of the remaining 11 seats. Hence X and Y can occupy seats in $12 \times 11 = 132$ ways. There will be 3 persons in between if X and Y occupy seats (1, 5), (2, 6), (3, 7), (4, 8), (5, 9), (6, 10), (7, 11), (8, 12), (9, 1), (10, 2), (11, 3), (12, 4). Since X and Y may interchange their seats, the total number of favourable cases is $12 \times 2 = 24$.

$$\text{Required probability} = \frac{24}{132} = \frac{2}{11}$$

(Note that the permutation of n persons in a *ring* or *circle* is $(n-1)\,!$ but around a '*round table*' is $n\,!$; because in the former the positions are relative to their neighbours, but in the latter relative to their seat numbers.

Example 11.26 *A lady declares that by tasting a cup of tea with milk, she can discriminate whether milk or tea infusion was first added to the cup. In order to test these assertion, 10 cups of tea are prepared—5 in one way and 5 in the other, and presented to the lady for judgement in a random order. Assuming that the lady has no discrimination power, calculate the probability that she would judge correctly all the cups, it being known to her that 5 are of each kind.*

What is the probability, if the tea cups were presented to the lady in 5 pairs—each pair consisting of cups of each kind—in a random order?

Solution Since the 5 cups of each kind, prepared with milk (M) or tea (T) infusion first added, are identical, the total number of permutations of the 10 cups is (see formula 11.4.5).

$$\frac{10\,!}{5\,!\,5\,!} = 252.$$

Thus, there are 252 different possible ways of presenting the cups to the lady, and these are mutually exclusive, exhaustive and equally likely. Only one of these agrees with the lady's assertion, suppose $M\,T\,T\,T\,M\,M\,T\,M\,T\,M$. So the required probability is 1/252.

In the second case, when the cups are presented in 5 pairs, the total number of permutations is

$$2 \times 2 \times 2 \times 2 \times 2 = 32$$

because each pair can be permuted in 2 ways—either (M, T) or (T, M). So there are now 32 possible ways of presenting the cups to the lady, As before, only one of these agrees with the lady's statement, say ($T\,M$), ($M\,T$), ($M\,T$), ($T\,M$), and ($M\,T$). The required probability is therefore 1/32.

Ans. 1/252, 1/32.

Example 11.27 *A box contains twenty tickets of identical appearance, the tickets being numbered 1, 2, 3, ..., 20. If 3 tickets are chosen at random, find the probability that the numbers on the drawn tickets are in arithmetic progression.*

[C.U., B.Sc. '71]

Solution Three tickets can be drawn out of 20 in ${}^{20}C_3$ ways. The total number of possible outcomes, which are mutually exclusive, exhaustive and equally likely, is therefore

$$n = {}^{20}C_3 \; \frac{20 \times 19 \times 18}{1 \times 2 \times 3} = 1140$$

The 3 numbers on the drawn tickets will be in A.P., if they have a common difference of either 1, or 2, or 3, ... or at most 9.

With a common difference 1, there are 18 sets, viz.

(1, 2, 3), (2, 3, 4), (3, 4, 5), ... (18, 19, 20),

With a common difference 2, there are 16 sets, viz.

(1, 3, 5), (2, 4, 6), (3, 5, 7), ... (16, 18, 20),

With a common difference 3, there are 14 sets, viz.

(1, 4, 7), (2, 5, 8), (3, 6, 9), ... (14, 17, 20),

and so on. Proceeding this way, finally

With a common difference 9, there are only 2 sets, viz.

(1, 10, 19), (2, 11, 20).

Therefore, the total number of sets of 3 numbers in A.P., whatever be the common difference, i.e., number of favourable cases, is

$$m = 18 + 16 + 14 + \ldots + 2 = 90$$

Therefore, the required probability is

$$p = \frac{m}{n} = \frac{90}{1140} = \frac{3}{38}$$

Example 11.28 *Four cards are drawn at random from a full pack. What is the probability that they belong to (i) 4 different suits, (ii) different suits and denominations ?*

Solution The 4 cards may be assumed to have been drawn one by one without replacement. The first card can be drawn in 52 ways, the second card in 51 ways, the third in 50 and the fourth in 49 ways. So, the total number of ways in which the cards can be drawn (attention being paid to the order) is $52 \times 51 \times 50 \times 49$. (formula 11.4.1)

(i) In order to find the number of cases favourable to the event 'one card drawn from each suit', we see that the first drawn card may belong to any suit, and hence may be chosen in 52 ways. There now remain 51 cards of which 12 belong to the same suit as the first card and 39 belong to other suits. Since the 4 cards are to belong to different suits, the second card should come from the 39. Thus the second card may be drawn in 39 ways; similarly the third card in 26 ways and the fourth card in 13 ways. The number of favourable cases is therefore $52 \times 39 \times 26 \times 13$. The required probability is

$$\frac{52 \times 39 \times 26 \times 13}{52 \times 51 \times 50 \times 49} = \frac{2197}{20825}$$

(see Example 11.50)

(ii) In order to calculate the number of favourable cases, the first card can, as before, be drawn in 52 ways. The second card must not belong to the same suit and denomination as the first and can be drawn in 36 ways (because in the remaining 51 cards, 39 belong to other suits of which 3 are of the same denominating as the first and should be excluded). Similarly, the third card may be drawn in 22 ways and the fourth in 10 ways. The number of favourable cases is then $52 \times 36 \times 22 \times 10$. The required probability is

$$\frac{52 \times 36 \times 22 \times 10}{52 \times 51 \times 50 \times 49} = \frac{264}{4165}$$

Example 11.29 *A group of 2n boys and 2n girls is divided at random into two equal batches. Find the probability that each batch will be equally divided into boys and girls.*

Solution The group of $4n$ boys and girls will be divided into two equal batches, if $2n$ out of them are selected to form one batch (the remaining $2n$ will form another batch). This selection can be done in ${}^{6n}C_{2n}$ ways. In order that each batch consists of equal numbers of boys and girls, the first batch of $2n$ selected persons should contain n boys and n girls. So, the number of favourable cases is ${}^{2n}C_n \,.\, {}^{2n}C_n = ({}^{2n}C_n)^2$. Hence, the required probability is

$$({}^{2n}C_n)^2/({}^{4n}C_{2n}).$$

Example 11.30 *From a pack of 52 cards, an even number of cards is drawn. Show that the probability that these consist half of red and half black is*

$$\frac{\left\{\dfrac{52\,!}{(26\,!)^2} - 1\right\}}{(2^{51} - 1)}$$

Solution An even number of cards drawn may either be 2. or 4, or 6, ... or 52. From the full pack of 52 cards, 2 cards can be drawn in ${}^{52}C_2$ ways, 4 cards in ${}^{52}C_4$ ways, and so on. So, the total number of ways of drawing as even number of cards is

$${}^{52}C_2 + {}^{52}C_4 + {}^{52}C_6 + \dots + {}^{52}C_{52} = (2^{51} - 1)$$

Assuming that all the different ways of drawing an even number of cards are equally probable irrespective of their number, these cases are mutually exclusive, exhaustive and equally likely.

When 2 cards are drawn, the number of ways of getting 1 red and 1 black cards is (see formula 11.4.8) ${}^{26}C_1 \times {}^{26}C_1 = ({}^{26}C_1)^2$. When 4 cards are drawn, the number of ways of getting 2 red and 2 black cards is similarly ${}^{26}C_2 \times {}^{26}C_2 = ({}^{26}C_2)^2$; and so on. Hence the number of favourable cases is

$$({}^{26}C_1)^2 + ({}^{26}C_2)^2 + ({}^{26}C_3)^2 + \dots + ({}^{26}C_{26})^2$$

$$= {}^{52}C_{26} - 1 = \frac{52\,!}{26\,!\,26\,!} - 1$$

Using the classical definition of probability, the result follows.

[**Note:** Binomial coefficients satisfy the following relations:

(a) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$

(b) ${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots = 2^{n-1}$

(c) $({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \dots + ({}^nC_n)^2 = {}^{2n}C_n$]

Example 11.31 *10 dissimilar balls are distributed at random into 4 boxes marked A, B, C, D. Find the probability that these boxes contain respectively 2, 4, 4, 0 balls.*

Solution The total number of possible ways of distribution is 4^{10}. The number of ways of distributing the balls so that Boxes A, B, C, D contain respectively 2, 4, 4, 0 balls is (see formulae 11.4.9 and 11.4.10)

$$\frac{10!}{2!4!4!0!} = 3150$$

Hence, the required probability is $3150/4^{10}$.

Example 11.32 *10 identical balls are distributed at random into 4 boxes marked A, B, C, D. Find the probability that these boxes contain respectively 2, 4, 4, 0 balls.*

Solution The total number of possible ways of distribution is (see formula 11.4.11)

$$^{10+4-1}C_{4-1} = {}^{13}C_3 = 286$$

Since the given distribution of balls (namely 2, 4, 4, 0 in that order) is only a particular case of these 286 possible cases, which are mutually exclusive, exhaustive and equally likely, the number of favourable cases is 1. Hence the required probability is

$$p = \frac{1}{286}$$

Example 11.32A *15 identical objects are distributed at random into 4 boxes numbered 1, 2, 3, 4. Find the probability that (i) each box contains at least 2 objects, (ii) no box is empty.*

Solution 15 identical objects may be distributed in 4 boxes in (see formula 11.4.11)

$$^{15+4-1}C_{4-1} = {}^{18}C_3 = 816$$

ways. These $n = 816$ cases are mutually exclusive, exhaustive and equally likely.

(i) If each box is to contain at least 2 objects, we first place 2 objects in each box. The remaining $15 - 8 = 7$ objects may then be distributed in any manner among the 4 boxes in

$$^{7+4-1}C_{4-1} = {}^{10}C_3 = 120$$

ways. Thus, the number of cases favourable to the event "at least two objects in each box" is $m = 120$. Hence, by the classical definition, the required probability is

$$p_1 = \frac{120}{816} = \frac{5}{34}$$

(ii) No box is empty, if each box contains at least 1 object. As before, the number of ways of distributing the 15 objects so that each box contains at least 1 object is obtained by first placing 1 object in each box, and then considering the number of ways of distributing the remaining $15 - 4 = 11$ objects in any manner. The number of favourable cases is therefore, by formula (11.4.11), given by

$$^{11+4-1}C_{4-1} = {}^{14}C_3 = 364.$$

i.e., $m = 364$. Note that we may also apply formula (11.4.12) and obtain

$$^{15-1}C_{4-1} = {}^{14}C_3 = 364$$

Hence, the required probability is

$$p_2 = \frac{384}{816} = \frac{91}{204}$$

Ans. (i) 5/34, (ii) 91/204.

Example 11.33 *What is the probability of obtaining a sum of 10 in a single throw with 5 dice?*

Solution When n dice are thrown, there are 6^n possible outcomes, among which the number of outcomes giving a total of r spots, is given be (see 11.4.13) the coefficient of x^r in the expansion of

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^n = \frac{x^n(1-x^6)^n}{(1-x)^n} = x^n(1-x^6)^n(1-x)^{-n}$$

i.e. the coefficient of x^{r-n} in the expansion of $(1-x^6)^n(1-x)^{-n}$.

In the present case, $r = 10$, $n = 5$. Therefore, the number of cases favourable to a total of 10 spots on the 5 dice is the

Coefficient of x^{10-5} in the expansion of $(1-x^6)^5(1-x)^{-5}$

$$= \text{Coefficient of } x^5 \text{ in } (1 - 5x^6 + \ldots)(1 + 5x + \ldots + {}^{4+t}C_4 x^t + \ldots)$$

$$= {}^9C_4 = \frac{9\times 8\times 7\times 6}{1\times 2\times 3\times 4} = 126$$

Therefore, the required probability $= \dfrac{126}{6^5} = \dfrac{126}{7776} = \dfrac{7}{432}$

Example 11.34 *Five different objects numbered 1, 2, 3, 4, 5 are placed at random into 5 places also marked 1, 2, 3, 4, 5. What is the probability that (i) no object occupies the place corresponding to its number, (ii) exactly 2 objects are in their correct places ?*

Solution 5 objects can be distributed in 5 places in 5 ! = 120 ways (see formula 11.4.3). These are mutually exclusive, exhaustive and equally likely.

(i) The number of arrangements in which no match occurs is (formula 11.4.14)

$$t_5 = 5!\left\{1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right\}$$

$$= 120\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$$

$$= 120 \times \frac{11}{30} = 44$$

Hence, the probability of no match is 44/120 = 11/30.

(ii) The two objects occupying the correct places may be any one of ${}^5C_2 = 10$ groups. Corresponding to any of these, the number of arrangements in which the remaining 3 objects are not in their proper places is

$$t_3 = 3!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right\}$$

$$= 6\left(\frac{1}{2} - \frac{1}{6}\right) = 2$$

Hence, the number of favourable cases is $10 \times 2 = 20$. The required probability of exactly 2 matches is therefore 20/120 = 1/6.

$P(A)$	denotes '*probability* of occurrence of *event A*'.
$\bar{A}$	denotes '*non-occurrence of event A*' (or, 'event *complementary to A*', i.e. *opposite* A)
$P(\bar{A})$	denotes '*probability* that event *A fails to occur*'.
$P(A + B)$ or $P(A \cup B)$	denotes 'probability of occurrence of *at least one* of the events *A* and *B*', (i.e., either *A*, or *B*, or both). In particular, if *A* and *B* are mutually exclusive, both *A* and *B* cannot occur simultaneously, and then $P(A + B)$ signifies the probability of occurrence of *either A or B*'.

$P(AB)$ or $P(A \cap B)$	denotes 'probability of occurrence of *A as well as B*', or 'probability of the *compound event AB*'. This is also called *Compound Probability*
$P(A/B)$	denotes 'probability of *occurrence of A, assuming that B has actually occurred*'.

$P(A/B)$ is called *Conditional Probability*. In contrast, $P(A)$ may be called *Unconditional Probability*, which means the probability of occurrence of A, irrespective of whether B has occurred or not.

11.6 THEOREMS OF PROBABILITY

(I) Theorem of Total Probability

If two events A and B are mutually exclusive, then the probability of occurrence of *either* A or B is given by the sum of their probabilities, i.e.,

Probability of (A or B) = Probability of A + Probability of B.

In symbols,

$$P(A + B) = P(A) + P(B) \tag{11.6.1}$$

This is also known as *Addition Theorem.*

Proof

Let us suppose that a random experiment has n possible outcomes, which are mutually exclusive, exhaustive and equally likely. If m_1 of these cases are favourable to the event. A, and m_2 cases are favourable to the event B, then the probabilities of these events are, by the classical definition,

$$P(A) = \frac{m_1}{n}, \quad P(B) = \frac{m_2}{n}$$

However, since the events A and B are mutually exclusive (i.e., both of them cannot occur simultaneously), the m_1 cases favourale to A are completely distinct from the m_2 cases favourable to B. The number of cases favourbale to 'either A or B' is therefore $(m_1 + m_2)$

$$\therefore \quad P(A + B) = \frac{m_1 + m_2}{n}$$

But,

$$\frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B)$$

Hence,

$$P(A + B) = P(A) + P(B)$$

(Proved)

Deductions from Theorem of Total Probability:

1. Theorem of Total Probability can be extended to any number of mutually exclusive events. If events $A_1, A_2, ..., A_k$ are mutually exclusive, then the probability of occurrence of any of them is given by the sum of the probabilities of the events.

$$P(A_1 + A_2 + ... + A_k) = P(A_1) + P(A_2) + ... + P(A_k) \tag{11.6.2}$$

In particular, for three mutualy exclusive events A, B, C, the probability of occurrence of either A or B or C is

$$P(A + B + C) = P(A) + P(B) + P(C) \tag{11.6.3}$$

2. The probability of the event complementary to A is given by

$$P(\bar{A}) = 1 - P(A) \tag{11.6.4}$$

or $$P(A) = 1 - P(\bar{A}) \quad (11.6.5)$$

Because, the event A and the complementary event $\bar{A}$ are mutually exclusive. So, using (11.6.1),

$$P(A + \bar{A}) = P(A) + P(\bar{A})$$

Also, the events A and $\bar{A}$ are exhaustive, so that one of them must necessarily occur; i.e., the event that any of them occurs is a certainty.

$\therefore$ $$P(A + \bar{A}) = 1$$

Eqating the right hand sides of the above two relations,

$$P(A) + P(\bar{A}) = 1$$

Hence the results.

Formula (11.6.4) gives us a method of finding the probability of the complementary event. Again, in many problems it is found comparatively easier to calculate the probability of complementary event by direct methods, and then the probability of the event is given by (11.6.5); see Examples 11.41, 11. 48 (ii), 11.53 (iii), 11.54.

3. The probability of occurrence of at least one of the 2 events A and B (which may not be mutually exclusive), is given by

$$P(A + B) = P(A) + P(B) - P(AB) \quad (11.6.6)$$

Because, the event $(A + B)$ means the occurrence of one of the following mutually exclusive events: $AB, A\bar{B}$, $\bar{A}B$. Therefore, by the extended Theorem of Total Probability (11.6.2)

$$P(A + B) = P(AB + A\bar{B} + \bar{A}B)$$
$$= P(AB) + (A\bar{B}) + P(\bar{A}B)$$

Again, coccurrence of either AB or $A\bar{B}$ signifies occurrence of A. Hence by (11.6.1),

$$P(A) = P(AB) + P(A\bar{B}); \text{ i.e.,}$$
$$P(A\bar{B}) = P(A) - P(AB)$$

Similarly, $$P(\bar{A}B) = P(B) - P(AB)$$

Using these relations,

$$P(A + B) = P(AB) + \{P(A) - P(AB)\} + \{P(B) - P(AB)\}$$
$$= P(A) + P(B) - P(AB)$$

The probability of occurrence of at least one of the 3 events A, B, C (which may not be mutually exclusive), is

$$P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) \quad (11.6.7)$$

4. Boole's inequality

$$P(A + B) \leq P(A) + P(B) \quad (11.6.8)$$

This follows from (11.6.6). Since the probability of an event can never be negative, the minimum value of $P(AB)$ is zero. So, the R.H.S. of (11.6.6) may have the maximum value $P(A) + P(B)$. Hence the result. The equality sign holds, when $P(AB) = 0$ i.e., A and B are mutually exclusive events.

5. Boneferroni's inequality:

$$P(A\,B) \geq P(A) + P(B) - 1 \quad (11.6.9)$$

This also follows from (11.6.6). Since the probability of an event can never exceed 1, the maximum value of $P(A + B)$ is 1; i.e.,

$$1 \geq P(A) + P(B) - P(AB)$$

Transposing, the result folows.

(II) Theorem of Compound Probability

The probability of occurrence of the event *A as well as B* is given by the product of (unconditional) probability of *A* and conditional probability of *B*, assuming that *A* has actualy occurred. i.e.,

Probability of (*A and B*) = Probability of *A* × Conditional probability of *B*, assuming *A*.

In symbols,
$$P(AB) = P(A) \cdot P(B/A) \tag{11.6.10}$$

This is also known as *Multiplication Theorem.*

Proof

Suppose that a random experiment has n mutually exclusive, exhaustive and equally likely outcomes, among which m cases are favourbale to an event A. So, the unconditinal probability of A is

$$P(A) = \frac{m}{n}$$

Out of these m cases, let m_1 cases be favourable to anotehr event B also, i.e., the number of cases favourable to A as well as B is m_1. Hence, by the 'classical definition of probability.

$$P(AB) = \frac{m_1}{n}$$

It may be noticed that once event A is known to have actually occurred, the occurrence of B as well is limited to only m_1 cases out of m (in which A occurs). So, the conditional probability of B, assuming that A has already occurred, is

$$P(B/A) = \frac{m_1}{m}$$

We find that
$$\frac{m_1}{n} = \frac{m}{n} \cdot \frac{m_1}{m}$$

i.e.,
$$P(AB) = P(A) \,.\, P(B/A)$$

Theorem of Compund Probability can be extended to several events. For example, the probability of occurrence of the event *A as well as B as well as C* is given by

$$P(ABC) = P(A) \,.\, P(B/A) \,.\, P(C/AB) \tag{11.6.10a}$$

where $P(C/AB)$ denotes the conditional probability of event C, assuming that both A and B have actually occurred. Similarly for four events A, B, C, D,

$$P(ABCD) = P(A).P(B/A).P(C/AB).P(D/ABC) \tag{11.6.10b}$$

Independent Events

Several events are considered to be "*independent*" in the probability sense, or *statistically independent*, if the probability of occurrence of any of them remains unaffected by the supplementary knowledge regarding occurrence or non-occurrence of any number of the remaining events.

If 2 dice are tossed, the event (A) 'six' on Die I and (B) 'six' on Die II, are independent; because the probability of obtaining a six suppose, on Die II, which is $\frac{1}{6}$ does not change, if it be known that Die I has resulted in a six or not. This means that the unconditional probability of B is the same as conditional probabilities, given A or $\bar{A}$.

$$P(B) = P(B/A) = P(B/\bar{A}) \tag{11.6.11}$$

Similarly, if 2 balls are drawn *with replacement* from a box containing 4 white and 6 black balls, the events (A) 'white ball is 1st draw' and (B) 'white ball in 2nd draw' are independent. However, if the drawing is *without replacement,* A and B will be *dependent events;* because $P(B/A) = \frac{1}{3}$ $P(B/\bar{A}) = \frac{4}{9}$ and they are not equal.

Deductions from Theroem of Compound Probability

(1) If events A and B are *independent,* the probability of occurrence of A as well as B is given by the product of their probabilities.

$$P(AB) = P(A).P(B) \tag{11.6.12}$$

Because, in this case $P(B/A)$ is unaffected by the supplementary knowledge regarding the occurrence or non-occurrence of A, so that $P(B/A) = P(B)$. Hence, from (11.6.10) the result follows.

Relation (11.6.12) is also taken as a suficient condition for independence of two events. If A and B are independent, (11.6.12) holds. Conversely, for any two events A and B, if (11.6.12) holds, the events are said to be 'independent'.

Two events A and B are said to be "independent", if the probability of occurrence of A as well as B is given by the product of the probability of A and the probability of B. In symbols,

$$P(AB) = P(A).P(B)$$

For three independent events, A, B, C

$$P(ABC) = P(A).P(B).P(C) \tag{11.6.13}$$

The result may be extended to any number of events. If events $A_1, A_2, ..., A_k$ are independent, then the probability that all of them occur simultaneusly, is

$$P(A_1A_2 \ldots A_k) = P(A_1) \,.\, P(A_2) \ldots P(A_k) \tag{11.6.14}$$

(2) The occurrence of an event B may be associated with the occurrence or with the non-occurrence of another event A. This means that event B may be assumed to be composed of two mutually exclusive compound events AB and $\bar{A}B$. (note that in either case B occurs). Hence, by the theorem of total probability

$$P(B) = P(AB) + P(\bar{A}B)$$

Again by the theorem of compound probability,

$$P(AB) = P(A).P(B/A)$$

and $$P(\bar{A}B) = P(\bar{A}).P(B/\bar{A}). \text{ Hence}$$

$$P(B) = P(A).P(B/A) + P(\bar{A}).P(B/\bar{A}) \tag{11.6.15}$$

(3) The conditional probability of an event B, on the assumption that another event A has actually occurred, is given by

$$P(B/A) = \frac{P(AB)}{P(A)} \tag{11.6.16}$$

This follows from the theorem of compound probability. Similarly,

$$P(A/B) = \frac{P(AB)}{P(B)} \tag{11.6.17}$$

Thus the Theorem of Compound Probability enables us to find a formula for the calculation of conditional probability.

Important Formulae and Results

1. *Addition Theorem of Probability:* If events A and B are 'mutually exclusive', then
$$P(A + B) = P(A) + P(B)$$
2. In general, for any two events A and B,
$$P(A + B) = P(A) + P(B) - P(AB)$$
3. For three mutually exclusive events A, B, C
$$P(A + B + C) = P(A) + P(B) + P(C)$$
This can be extended to any number of mutually exclusive events.
4. For any three events A, B, C
$$P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$
5. Probability of the complementary event:
$$P(\bar{A}) = 1 - P(A)$$
6. Boole's inequality:
$$P(A + B) \leq P(A) + P(B)$$
The sign of equality holds, when A and B are mutually exclusive.
7. *Multiplication Theorem of Probability:* For any two events A and B,
$$P(AB) = P(A) \,.\, P(B/A) = P(B).P(A/B)$$
8. If events A and B are 'independent', then
$$P(AB) = P(A) \,.\, P(B)$$
Conversely, if this relation holds, then the events A and B are said to be "independent".
9. For any three events A, B, C
$$P(ABC) = P(A).P(B/A).P(C/AB)$$
10. For three *independent* events $A, B, C,$
$$P(ABC) = P(A).P(B).P(C)$$
This can be extended to any number of independent events.
11. If evetns A and B are independent, then
$$P(A) = P(A/B) = P(A/\bar{B})$$
$$P(B) = P(B/A) = P(B/\bar{A})$$
This is the conditional and unconditional probability are equal.
12. $$P(A + B) = 1 - P(\bar{A}\,\bar{B}) = 1 - P(\bar{A}).P(\bar{B}),$$
if A and B are independent.

13. $$P(B) = P(AB) + P(\bar{A}B)$$
$$= P(A).P(B/A) + P(\bar{A}).P(B/\bar{A})$$

14. *Conditional Probability:*

$$P(B/A) = \frac{P(AB)}{P(A)}; \text{ provided } P(A) \neq 0.$$

$$P(A/B) = \frac{P(AB)}{P(B)}; \text{ provided } P(B) \neq 0.$$

Example 11.35 *Given* $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(AB) = \frac{1}{4}$,

(a) find the values of the following probabilities:

$P(\bar{A})$, $P(A + B)$, $P(A/B)$, $P(\bar{A}B)$, $P(\bar{A}\ \bar{B})$, $P(\bar{A} + B)$

(b) State whether the events A and B are (i) mutually exclusive, (ii) exhaustive, (iii) equally likely, (iv) independent.

Solution

(a)
$$P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$$
$$P(A + B) = P(A) + P(B) - P(AB)$$
$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$$
$$P(A/B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$$
$$P(\bar{A}B) = P(B) - P(AB) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$
$$P(\bar{A}\ \bar{B}) = 1 - P(A + B) = 1 - \frac{7}{12} = \frac{5}{12}$$
$$P(\bar{A} + B) = P(\bar{A}) + P(B) - P(\bar{A}B)$$
$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{12} = \frac{3}{4}$$

(b) (i) No; because $P(AB) \neq 0$. Here, $P(AB) = \frac{1}{4}$

(ii) No; because $P(A + B) \neq 1$. Here, $P(A + B) = \frac{7}{12}$

(iii) No; because $P(A)$ and $P(B)$ are not equal. Here $P(A) = \frac{1}{2}$, and $P(B) = \frac{1}{3}$

(iv) No; because $P(AB) \neq P(A).P(B)$. Here $P(AB) = \frac{1}{4}$, but $P(A) . P(B) = \frac{1}{6}$

Example 11.36 *Given that* $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{3}$ *and* $P(A + B) = \frac{3}{4}$, *find* $P(A/B)$ *and* $P(B/A)$. *Are A and B independet?* [W.B.H.S., '78]

Solution Using the relation

$$P(A+B) = P(A) + P(B) - P(AB),$$

$$\frac{3}{4} = \frac{3}{8} + \frac{5}{8} - P(AB)$$

Solving, we get $P(AB) = \frac{1}{4}$. Hence,

$$P(A/B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{4}}{\frac{5}{8}} = \frac{2}{5};$$

$$P(B/A) = \frac{P(AB)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{8}} = \frac{2}{3}$$

We have, $P(AB) = \frac{1}{4}$ and $P(A)\,P(B) = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$.

Since $P(AB) \neq P(A).P(B)$, so events A and B are not independent.

Also, note that $P(A) \neq P(A/B)$ and $P(B) \neq P(B/A)$.

Example 11.37 *For any two events A and B, prove that*

$$P(AB) \le P(A) \le P(A+B) \le P(A) + P(B)$$

Solution From the Multiplication Theorem, we have

$$P(AB) = P(A)P(B/A)$$

Since the factor $P(B/A)$ on the R.H.S. is a 'probability', it is neither negative, nor greater than 1. Hence, $P(AB)$ can never exceed $P(A)$;

i.e., $$P(AB) \le P(A) \quad \text{...(i)}$$

Again,
$$\begin{aligned} P(A+B) &= P(A) + P(B) - P(AB) \\ &= P(A) + P(AB + \bar{A}B) - P(AB) \\ &= P(A) + P(AB) + P(\bar{A}B) - P(AB) \\ &= P(A) + P(\bar{A}B) \end{aligned}$$

(because AB and $\bar{A}B$ are two mutually exclusive and exhaustive forms of the event B). But $P(\bar{A}B)$ cannot be negative; hence

$$P(A) \le P(A+B) \quad \text{...(ii)}$$

Using the relation $P(A+B) = P(A) + P(B) - P(AB)$, we see that since $P(AB)$ is non-negative, $P(A+B)$ cannot exceed $P(A) + P(B)$;

i.e. $$P(A+B) \le P(A) + P(B) \quad \text{...(iii)}$$

Combining the inequalities (i), (ii) & (iii), the results follow.

Example 11.38 *If events A and B are independent, prove that $\bar{A}$ and $\bar{B}$ are also independent.*

Solution From the definition of independence (11.6.12),

$$P(AB) = P(A).P(B).$$

Therefore,
$$\begin{aligned} P(A+B) &= P(A) + P(B) - P(AB) \\ &= P(A) + P(B) - P(A).P(B) \end{aligned}$$

Now, since the event $\bar{A}\bar{B}$ is complementary to the event $A + B$, (i.e. non-occurrence of both A and B, is complementary to occurrence of at least one of A and B), hence

$$\begin{aligned} P(\bar{A}\bar{B}) &= 1 - P(A+B) \\ &= 1 - \{P(A) + P(B) - P(A).P(B)\} \\ &= \{1 - P(A)\}\{1 - P(B)\} \\ &= P(\bar{A}).P(\bar{B}) \end{aligned}$$

We have shown that

$$P(\bar{A}\bar{B}) = P(\bar{A}).P(\bar{B})$$

So, events $\bar{A}$ and $\bar{B}$ are independent.

Example 11.39 *The odds in favour of an event A are 3:4. The odds against another independent event B are 7:4. What is the probability that at least one of the events will happen?*

Solution

"Odds *in favour of* A are $a : b$" signifies $\quad P(A) = \dfrac{a}{a+b}$

"Odds *against* B are $a : b$" signifies $\quad P(B) = \dfrac{b}{a+b}$

Here, the probabilities of occurrence of A and B are $P(A) = \dfrac{3}{7}$ and $P(B) = \dfrac{4}{11}$. Also, since A and B are independent, we have $P(AB) = P(A).\ P(B) = \dfrac{3}{7} \times \dfrac{4}{11} = \dfrac{12}{77}$. The probability of occurrence of at least one of the events A and B is given by

$$\begin{aligned} P(A+B) &= P(A) + P(B) - P(AB) \\ &= \frac{3}{7} + \frac{4}{11} - \frac{12}{77} = \frac{7}{11} \end{aligned}$$

Example 11.40 *A card is drawn from each of two well-shuffled packs of cards. Find the probability that at least one of them is an ace.* [C.U., B.Sc. (Math) '73]

Solution Let us denote by

A = event that the card from Pack I is an ace,

B = event that the card from Pack II is an ace,

It is required to find $P(A + B)$.

(First method) $\quad P(A+B) = P(A) + P(B) - P(AB)$

Since there are 4 aces in a pack of 52 cards, $P(A) = 4/52 = 1/13$. Similarly, $P(B) = 1/13$. The events A and B are independent, because the drawing of an ace or otherwise from one pack does not in any way affect the probability of drawing an acc from another pack. So,

$$P(AB) = P(A).P(B) = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$$

Substituting the values,

$$P(A+B) = \frac{1}{13} + \frac{1}{13} - \frac{1}{169} = \frac{25}{169}$$

(Second method) The event complementary to 'drawing at least one ace' is that 'none of the drawn cards is an ace', i.e. the card drawn from Pack I is a non-ace as well as the card from Pack II is

a non-ace (in symbols $\bar{A}\bar{B}$). So,

$$P(A + B) = 1 - P(\bar{A}\bar{B})$$

Since A and B are independent, so also are $\bar{A}$ and $\bar{B}$. Hence,

$$P(\bar{A}\bar{B}) = P(\bar{A}).P(\bar{B})$$
$$= \{1 - P(A)\}.\{1 - P(B)\}$$
$$= \left(1 - \frac{1}{13}\right)\left(1 - \frac{1}{13}\right) = \frac{144}{169}$$

$$\therefore \quad P(A + B) = 1 - \frac{144}{169} = \frac{25}{169}.$$

Example 11.41 *An article manufactured by a company consists of two parts I and II. In the process of manufacture of part I, 9 out of 100 are likely to be defective. Similarly, 5 out of 100 are likely to be defective in the manufacture of part II. Calculate the probability that the assembled article will not be defective.* [C.A., Nov. 76]

Solution Let

A = event that part I is not defective,
B = event that part II is not defective.

We have to find the compound probability $P(AB)$ that part I is not defective, as well as part II is not defective. But

$$P(A) = \text{Probability that part I is not defective}$$
$$= 1 - \text{Probability that part I is defective}$$
$$= 1 - \frac{9}{100} = 0.91$$

Similarly, $$P(B) = 1 - \frac{5}{100} = 0.95.$$

Assuming that event A and B are independent, the required probability is

$$P(AB) = P(A).P(B) = 0.91 \times 0.95 = 0.8645$$

Example 11.42 *One urn contains 2 white and 2 black balls; a second urn contains 2 white and 4 black balls. (i) If one ball is chosen from each urn, what is the probability that they will be of the same colour? (ii) If an urn is selected at random and one ball is drawn from it, what is the probability that it will be a white ball?*

[C.U., M.Com. '75]

Solution

(i) The event (E) 'both the drawn balls are of the same colour' has two mutually exclusive forms, either E_1 (both white) or E_2 (both black). So, by the theorem of total probability.

$$P(E) = P(E_1) + P(E_2)$$

But E_1 is a compound event formed by two *independent* events of drawing a white ball from each urn. Hence $P(E_1) = \frac{2}{4} \times \frac{2}{6} = \frac{1}{6}$. Similarly, E_2 is also a compound event, and $P(E_2) = \frac{2}{4} \times \frac{4}{6} = \frac{1}{3}$. Hence, the required probability is $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$.

(ii) A white ball can be selected in two mutually exclusive ways:

(*A*) when urn I is selected and a white ball is drawn from it;

(*B*) when urn II is selected and a white ball is drawn from it.

To find the probability of event *A*, we note that the selection of urn I itself depends on chance and has probability $\frac{1}{2}$. Once urn I has been selected, the selection of a white ball from it again depends on chance and has probability $\frac{2}{4}$. Therefore, by the theorem of compound probability, $P(4) = \frac{1}{2} \times \frac{2}{4} = \frac{1}{4}$. In the same way, $P(B) = \frac{1}{2} \times \frac{2}{6} = \frac{1}{6}$. Now, using the theorem of total probability, the required probability of drawing a white ball (irrespective of the choice of urn) is $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$.

Example 11.43 *A salesman has a 80% chance of making a sale to each customer. The behaviour of successive customers is assumed to be independent. If two customers X and Y enter the shop, what is the probability that the salesman will make a sale?*

Solution Let *A* and *B* denote the events 'sale to customer *X*' and 'sale to customer *Y*' respectively. We have to find the probability that a sale is made to at least one of the two customers *X* and *Y*, i.e. $P(A + B)$. Since the complementary event is that no sale is made, i.e. no sale to *X* as well as no sale to *Y*, hence

$$P(A + B) = 1 - P(\bar{A}\bar{B}) = 1 - P(\bar{A}).P(\bar{B})$$

because the events *A* and *B* are independent. But $P(\bar{A}) = 1 - P(A) = 1 - \frac{80}{100} = \frac{1}{5}$. Similarly, $P(\bar{B}) = \frac{1}{5}$. Hence, the required probability is $1 - \frac{1}{5} \times \frac{1}{5} = \frac{24}{25}$.

Example 11.44 *There is a 50–50 chance that a contractor's firm A will bid for the construction of a multi-storeyed building. Another firm B submits a bid and the probability is 3/4 that it will get the job, provided firm A does not bid. If firm A submits a bid, the probability that firm B will get the job is only 1/3. What is the probability that firm B will get the job?*

Solution Let

A = event that firm *A* submits the bid;

B = event that firm *B* gets the job.

We are given $P(A) = 1/2$, $P(B/\bar{A}) = 3/4$ and $P(B/A) = 1/3$. Using (11.6.15),

$$\begin{aligned} P(B) &= P(AB) + P(\bar{A}B) \\ &= P(A).P(B/A) + P(\bar{A}).P(B/\bar{A}) \\ &= \left(\frac{1}{2} \times \frac{1}{3}\right) + \left(\frac{1}{2} \times \frac{3}{4}\right) = \frac{13}{24} \end{aligned}$$

Example 11.45 *Two players A and B toss a die alternately. He who first throws a "six" wins the game. If A begins, what is the probability that he wins? What is the probability of B winning the game?*

Solution A wins the game if any of the following events happens—

(E_1) A gets a six in the first toss;

(E_2) A fails, B fails, and then A gets a six.

(E_3) A fails, B fails, then again A fails, B fails, and then A gets a six.

This may continue indefinitely.

Since the probability of getting six is $\frac{1}{6}$ in each trial, and the probability of failing to get a six is $\frac{5}{6}$ and also the elementary events in successive trials are independent,

$$P(E_1) = \frac{1}{6}$$

$$P(E_2) = \frac{5}{6}.\frac{5}{6}.\frac{1}{6} = \frac{1}{6}.\left(\frac{5}{6}\right)^2$$

$$P(E_3) = \frac{5}{6}.\frac{5}{6}.\frac{5}{6}.\frac{5}{6}.\frac{1}{6} = \frac{1}{6}\left(\frac{5}{6}\right)^4$$

and so on. Using the theorem of total probability,

Probability that A wins

$$= P(E_1) + P(E_2) + P(E_3) + \dots \dots$$

$$= \frac{1}{6} + \frac{1}{6}\left(\frac{5}{6}\right)^2 + \frac{1}{6}\left(\frac{5}{6}\right)^4 + \dots \dots$$

$$= \frac{\frac{1}{6}}{1-\left(\frac{5}{6}\right)^2} = \frac{6}{11}$$

[**Note :** Sum of infinite G.P. series ($|r| < 1$) $a + ar + ar^2 + \dots = \frac{a}{1-r}$]

B wins the game, if either

(F_1) A fails to get a six in the first trial, but in the next trial B gets a six.

(F_2) A fails, B fails, and then A fails, but in the next trial B gets a six.

(F_3) A fails B fails, A fails, B fails, A fails, and then B gets a six.

and so on. Proceeding as before,

Probability that B wins

$$= P(F_1) + P(F_2) + P(F_3) + \dots \dots$$

$$= \frac{5}{6}.\frac{1}{6} + \frac{5}{6}.\frac{5}{6}.\frac{5}{6}.\frac{1}{6} + \frac{5}{6}.\frac{5}{6}.\frac{5}{6}.\frac{5}{6}.\frac{5}{6}.\frac{1}{6} + \dots \quad \dots$$

$$= \frac{5}{36} + \frac{5}{36}\left(\frac{5}{6}\right)^2 + \frac{5}{36}\left(\frac{5}{6}\right)^4 + \dots \quad \dots$$

$$= \frac{\frac{5}{36}}{1-\left(\frac{5}{6}\right)^2} = \frac{5}{11}$$

Ans. $\frac{6}{11}, \frac{5}{11}$

Example 11.46 *If $P(X) = i) = P_4$ and $P(Y = j) = q_i$, $(i, j = 1, 2, ..., n)$ where X and Y are two mutually independent random variables, prove that $P(X + Y = n) = \sum_{i=1}^{n-1} p_i q_{n-i}$* [C.U., B.A. (Econ) '72, '75]

Solution The event $X + Y = n$ can occur in the following mutually exclusive ways:

$$(X = 1, Y = n - 1), (X = 2, Y = n - 2), \ldots (X = n - 1, Y = 1)$$

Therefore, by the Theorem of Total Probability,

$$P(X + Y = n) = P(X = 1, Y = n - 1) + P(X = 2, Y = n - 2) + \ldots \ldots + P(X = n - 1, Y = 1)$$

$$= \sum_{i=1}^{n-1} P(X = i, Y = n - i)$$

However, since X and Y are independent, by (11.6.12, page 23),

$$P(X = i, Y = n - i = P(X = i) \times P(Y = n - i) = P_i q_{n-}$$

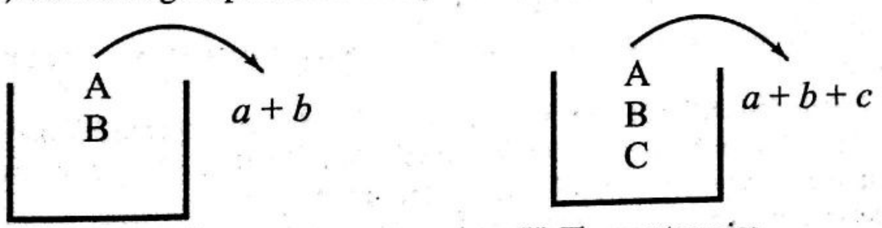
Hence the result.

11.7 DRAWING WITHOUT REPLACEMENT

A box contains A white and B black balls. If $(a + b)$ balls are drawn at random, the probability that among them exactly a are white and b are black is [see Fig. 11.3(i)]

$$\frac{{}^{A}C_a \cdot {}^{B}C_b}{{}^{A+B}C_{a+b}} \tag{11.7.1}$$

There are ${}^{A+B}C_{a+b}$ possible ways of forming groups of $(a + b)$ balls out of a total of $(A + B)$ balls, and these groups are mutually exclusive, exhaustive and equally likely.



(i) Two categories (ii) Three categories

Fig. 11.3 Drawing with Replacement (Urn Models)

However, a group of a white balls can be obtained in ${}^{A}C_a$ ways because there are A white balls in the box. Similarly, a group of b black balls can be obtained in ${}^{B}C_b$ ways. Since any of the ${}^{A}C_a$ groups of a white balls can be combined with any of the ${}^{B}C_b$ groups of b black balls, the number of groups of $(a + b)$ balls favourable to the event is ${}^{A}C_a \cdot {}^{B}C_b$. Hence, by the classical definition, the required probability is given by (11.7.1).

[**Note:** (i) The phrase, 'at random signifies that all possible drawings are 'equally likely'.
(ii) If instead of drawing the $(a + b)$ balls all at a time, the balls are drawn one by one, $(a + b)$ times in succession (a ball drawn once *not having been returned* to the box), then the same probability (11.7.1) is obtained. Hence, such drawing is also call '*drawing without replacement*', as distinct from '*drawing with replacement*' (Section 11.8).]

If the box contains balls of 3 different categories A of them white, B black and C red, and $(a + b + c)$ balls are drawn, then the probability that among them exactly a are white, b black and c red, is [see Fig. 11.3(ii)]

$$\frac{{}^{A}C_{a}.{}^{B}C_{b}.{}^{C}C_{c}}{{}^{A+B+C}C_{a+b+c}} \tag{11.7.2}$$

Example 11.47 *A bag contains 8 white and 6 black balls. If 5 balls are drawn at random, what is the probability that 3 are white and 2 black?*

Solution 5 balls can be drawn out of 14 in ${}^{14}C_5$ ways, and these cases are mutually exclusive, exhaustive and equally likely. However, a group of 3 white balls can be drawn out of 8 in ${}^{8}C_3$ ways, and 2 black balls out of 6 in ${}^{6}C_2$ ways. So the number of favourable cases is ${}^{8}C_3.{}^{6}C_2$. By the classical definition

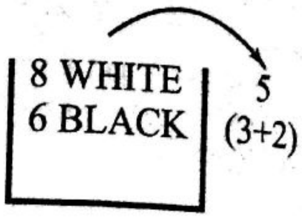


Fig. 1.4 Drawing Balls

$$P = \frac{{}^{8}C_3.{}^{6}C_2}{{}^{14}C_5}$$

But $${}^{8}C_3 = \frac{8\times7\times6}{1\times2\times3} = 56;\ {}^{6}C_2 = \frac{6\times5}{1\times2} = 15;\ \text{(see p. 56)}$$

$${}^{14}C_5 = \frac{14\times13\times12\times11\times10}{1\times2\times3\times4\times5} = 14\times13\times11$$

$$\therefore \quad P = \frac{56\times15}{14\times13\times11} = \frac{60}{143}$$

(*Second method*) We may assume that the balls have been drawn one by one without replacement.

Of the 5 balls drawn, the first one may be any of the 14 in the bag, the 2nd may be one of the remaining 13, and so on. So, the total number of ways in which 5 balls may be drawn, considering the order in which they appear, is $14 \times 13 \times 12 \times 11 \times 10$.

Now, let us consider the number of ways in which 3 white and 2 black balls may be obtained in a particular order; for example, the first 3 are white and the last 2 are black. This number is $(8 \times 7 \times 6) \times (6 \times 5)$. Hence the probability of drawing only white balls in the first 3 draws and only black balls in the last 2 draws is

$$\frac{(8\times7\times6)\times(6\times5)}{14\times13\times12\times11\times10} = \frac{6}{143}$$

But, obviously this is the probability of having 3 white and 2 black balls in any other order. In our problem, the order is immaterial, and hence the required probability of having 3 white and 2 black balls is, by the Theorem of Total Probability,

$$\frac{6}{143} + \frac{6}{143} + \frac{6}{143} + \dots (k \text{ times}) = \frac{6}{143} \times k$$

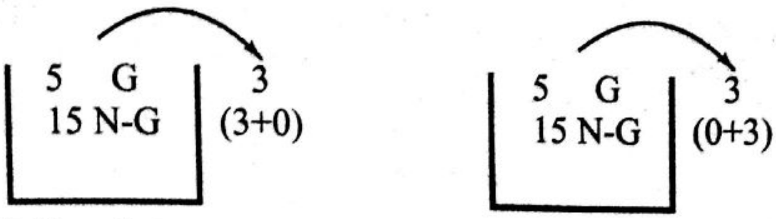
where k is the number of arrangements (i.e., permutations) in which 3 white and 2 black balls may appear. Since this is given by

$k = \dfrac{5!}{3!\,2!} = 10$, the required probability is $\dfrac{6}{143} \times 10 = \dfrac{60}{143}$

Example 11.48 *Five men in a company of 20 are graduates. If 3 men are picked out of the 20 at random, what is the probability that they are all graduates? What is the probability of at least one graduate?* [C.U., B.A. (Econ) '73]

Solution (i) Applying (11.7.1), let

A = Number of graduates in the company = 5
a = Number of graduates in the sample = 3
B = Number of non-graduates in the company = 20 – 5 = 15
b = Number of non-graduates in the sample = 0



(i) All graduates (ii) None graduate

Fig. 11.5 Drawing from Two Categories

The probability that all are graduates is

$$p_1 = \frac{{}^5C_3 . {}^{15}C_0}{{}^{20}C_3}$$

But,

$$ {}^5C_3 = \frac{5 \times 4 \times 3}{1 \times 2 \times 3} = 10$$

$$ {}^{15}C_0 = 1; {}^{20}C_3 = \frac{20 \times 19 \times 18}{1 \times 2 \times 3} = 1440,$$

$\therefore$

$$p_1 = \frac{10 \times 1}{1140} = \frac{1}{114}$$

(ii) In order to find the probability of at least one graduate, it will be easier to find the probability of the complementary event, viz. that 'none is a graduate' (i.e. all 3 are non-graduates), so that $A = 5$, $a = 0$, $B = 15$, $b = 3$. Applying (11.7.1), the complementary probability is

$$\frac{{}^5C_0 . {}^{15}C_3}{{}^{20}C_3} = \frac{1 \times 455}{1140} = \frac{91}{228}$$

Hence, by (11.6.5), the required probability is

$$p_2 = 1 - \frac{91}{228} = \frac{137}{228}$$

(*Alternative method*)

(iii) The event 'at least one graduate' can be split up into three mutually exclusive events:

1. exactly 1 graduate and 2 non-graduates.
2. exactly 2 graduates and 1 non-graduate.
3. exactly 3 graduates and 0 non-graduate.

The probabilities of these cases are, rspectively

$$\frac{{}^5C_1 \times {}^{15}C_2}{{}^{20}C_3} = \frac{525}{1140}; \quad \frac{{}^5C_2 \times {}^{15}C_1}{{}^{20}C_3} = \frac{150}{1140};$$

$$\frac{{}^5C_3 \times {}^{15}C_0}{{}^{20}C_3} = \frac{10}{1140}$$

By the Theorem of Total Probability (11.6.2), the required probability p_2 is given by the sum of these probabilities. Hence,

$$p_2 = \frac{525}{1140} + \frac{150}{1140} + \frac{10}{1140} = \frac{137}{228}$$

Ans. $\frac{1}{114}, \frac{137}{228}$

Example 11.49 *A bag contains 8 red and 5 white balls. Two successive draws of 3 balls are made without replacement. Find the probability that the first drawing will give 3 white balls and the second 3 red balls.* [CA., May '78]

Solution Let A denote the event 'first drawing gives 3 white balls', and B denote the event 'second drawing gives 3 red balls'. We have to find the probability of the event "A and B", i.e. "A as well as B", which in symbols is $P(AB)$.

$$P(AB) = P(A) \,.\, P\left(\frac{B}{A}\right)$$

Using (11.7.1)

$$P(A) = \frac{{}^{8}C_0 \,.\, {}^{5}C_3}{{}^{18}C_3} = \frac{1 \times 10}{286} = \frac{5}{143}$$

In order to find the conditional probability $P\left(\frac{B}{A}\right)$, we assume that event A has actually happened, i.e. white balls have been taken out, so that there remain 8 red and 2 white balls in the bag after the first drawing. The probability of getting 3 red balls now is

$$P\left(\frac{B}{A}\right) = \frac{{}^{8}C_8 \,.\, {}^{2}C_0}{{}^{10}C_8} = \frac{56 \times 1}{120} = \frac{7}{15}$$

Hence, the required probability is $P(AB) = \frac{5}{143} \times \frac{7}{15} = \frac{7}{429}$

Example 11.50 *Four cards are drawn at random from a full pack. What is the probability that they belong to different suits?*

Solution In the pack of 52 cards, there are 13 cards of each suit—13 spades, 13 hearts, 13 diamonds and 13 clubs. If the 4 drawn cards are to belong to four different suits, we should have 1 card from each suit. Hence the required probability is (extension of formula 11.7.2).

13 SPADE
13 HEART
13 DIAMOND
13 CLUB

4
(1+1+1+1)

Fig. 11.6 Drawing from Four Categories

$$\frac{{}^{13}C_1 \,.\, {}^{13}C_1 \,.\, {}^{13}C_1 \,.\, {}^{13}C_1}{{}^{52}C_4} = \frac{2197}{20825}$$ (see Example 11.28)

11.8 REPEATED TRIALS—DRAWING WITH REPLACEMENT

In a certain experiment, the probability of occurrence of an event is p and consequently the probability of its non-occurrence is $1 - p = q$, suppose. In n repeated trials of the experiment, if p remains a constant in each trial, then the probability that the event occurs i time is

$$^nC_r\, p^r\, q^{n-r} \tag{11.8.1}$$

where $p + q = 1$.

The performance of an experiment is usually called a "trial"; the occurrence of the event is called a "success" and its non-occurrence a "failure". Thus, the probability of r successes (and mutually $n - r$ failures) in n *independent* trials is given by (11.8.1), where p is the probability of success *in each trial*.

Example 11.51 *Find the probability that there will be exactly successes in n independent trials (n ≥ r), where p is the probability of success in a single trial.*

Solution Let us, at the beginning, consider the event in which the first r trials result in success in each trial, and the remaining $n - r$ trials result in failures only. Now, the probability of success is p in each trial, so that the probability of failure is $1 - p = q$, say. Since the n trials are *independent*, the probability of r consecutive success, followed by $n - r$ consecutive failures is, by the theorem of compound probability, given by

$$\underbrace{p \times p \times \ldots\, p}_{r \text{ times}} \times \underbrace{q \times q \times \ldots \times q}_{(n-r) \text{ times}} = p^r q^{n-r}$$

The probability of obtaining r successes (and hence $n - r$ failures) in any other specified order is similarly $p^r q^{n-r}$. However, we are not interested in any particular arrangement of the r successes and the $n - r$ failures, but overall number of r successes in n trials, whatever be the arrangement in which they appear. But, r successes and $n - r$ failures in n trails may arise in $\dfrac{n!}{r!(n-r)!} = {}^nC_r$ mutually exclusive ways, in each of which the probability is the same, viz. $p^r q^{n-r}$. Hence, by the theorem of total probability, the probability that there will be r successes, irrespective of the order in which they appear, is

$$p^r q^{n-r} + p^r q^{n-r} + \ldots\ ({}^nC_r \text{ times}) = {}^nC_r\, p^r q^{n-r}.$$

[**Note:** This problem may be identified with drawing n balls from a box in which the proportions of white and black balls are p and q. If balls are drawn one by one, n times in succession, the ball drawn each time *being returned* to the box before the next drawing, then the probability of obtaining r white balls is given by (11.8.1). Such experiments are, therefore, called "*drawing with replacement*". [Also see Note (ii) Section 11.7, page 31].

Example 11.52 *A coin is tossed 10 times. Find the probability of getting (i) exactly 6 heads, and (ii) 9 heads and 1 tail.*

Solution Let us describe the appearance of 'head' in one toss of the coin as "success". Then, assuming that the coin is unbiased,

$$p = \text{probability of success in each trial} = \frac{1}{2}$$

$$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$$

Also, since the probability of occurrence of a head or a tail in any tossing is not affected by the results of any other tossing, the trials are *independent*. Therefore, we apply (11.8.1), with $n = 10, p = \frac{1}{2}, q = \frac{1}{2}$. (i) Here, $r = 6$. Therefore, the probability of 6 heads is

$$^{10}C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^{10-6} = \frac{210}{2^{10}} = \frac{105}{512}$$

(ii) We have to find the probability of 9 successes (and obviously the remaining 1 is a failure). Putting $r = 9$, the required probability is

$$^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^{10-9} = \frac{10}{2^{10}} = \frac{5}{512}$$

Example 11.53 *The probability that an entering college student will be a graduate is 0.4. Determine the probability that out of 5 entering students, (i) none, (ii) one, (iii) at least one, will be a graduate.*

Solution Let the event 'an entering college student will be a graduate' be called a "success". Then p = probability of success in each trial = 0.4. Assuming that the success or failure of one student does not affect the result of any other student, the trials may be considered as *independent*.

Hence, putting $n = 5, p = 0.4, q = 1 - p = 0.6$ in (11.8.1)

(i) Probability that none is a graduate

$= $ probability of 0 success $= {}^3C_0\,(0.4)^0\,(0.6)^{5-0}$

$= (1)\,(1)\,(0.6)^5 = 0.07776$

(ii) Probability that one is a graduate

$= $ probability of one success $= {}^5C_1\,(0.4)^1\,(0.6)^{5-1}$

$= 5\,(0.4)\,(0.6)^4 = 0.2592$

(iii) The complementary event is 'none is a graduate'. Hence, applying (11.6.5). Probability that at least one is graduate

$= 1-$ (probability that none is a graduate)

$= 1 - .07776$, from (i) above.

$= .92224.$

Example 11.54 *A machine produces on the average 2 per cent defectives. If 4 articles are chosen randomly, find the probability that there are at least 2 defective articles.*

Solution Denoting the occurrence of a defective article as "success", we find that

$$p = \text{probability of a defective article} = 20\% = \frac{2}{100} = 02$$
$$q = 1 - 0.2 = .98$$

Since the occurrence of a defective article does not affect the probability of another article being defective or not, hence the trials are independent. Therefore, applying (11.8.1), we have to find the probability of at least 2 successes in 4 trials. The complementary event consists of two mutually exclusive cases, viz. (i) exactly 0 success in 4 trials, and (ii) exactly 1 success in 4 trials. The probabilities of these cases are respectively

$$^4C_0\,(.02)^0\,(.98)^{40} = (1)\,(1)\,(.98)^4 = .922 \text{ (approx.)}$$
$$^4C_1\,(.02)^1\,(.98)^{-1} = (4)\,(.02)\,(.98)^3 = .075 \text{ (approx.)}$$

By the Theorem of Total Probability, the probability of the complementary event is, therefore, .922 + .075 = .997. The required probability is 1 – .997 = .003. *Ans.* 0.003

11.9 BAYES' THEOREM

An event A can occur only if one of the mutually exclusive and exhaustive set of events $B_1, B_2, \ldots B_n$ occurs. Suppose that the unconditional probabilities

$$P(B_1), P(B_2), \ldots P(B_n)$$

and the conditional probabilities

$$P, \left(\frac{A}{B_1}\right), P\left(\frac{A}{B_2}\right), \ldots P\left(\frac{A}{B_n}\right)$$

are known. Then the conditional probability $P\left(\frac{B_i}{A}\right)$ of a specified event B_i, when A is stated to have actually occurred, is given by

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i).P\left(\frac{A}{B_i}\right)}{\sum_{i=1}^{n} P(B_i).P\left(\frac{A}{B_i}\right)} \qquad (11.9.1)$$

This is known as *Bayes' Theorem.*

Proof

The event A can happens in n mutually exclusive ways $B_1 A, B_2 A, \ldots B_n A$, i.e. either when B_1 has occurred, or $B_2, \ldots$ or B_n. So by the theorem of total probability (11.6.2)

$$\begin{aligned} P(A) &= P(B_1A + B_2A + \ldots + B_nA) \\ &= P(B_1A) + P(B_2A) + \ldots + P(B_nA), \qquad \because \text{mutually exclusive} \\ &= P(B_1) \,.\, P\left(\frac{A}{B_1}\right) + P(B_2) \,.\left(\frac{A}{B_2}\right) + \ldots + P(B_n) \,.\, P\left(\frac{A}{B_n}\right), \\ &\qquad \text{using the Multiplication Theorem (11.6.10)} \\ &= \sum_{i=1}^{n} P(B_i) \,.\, P\left(\frac{A}{B_i}\right) \end{aligned}$$

Again, $P(AB_i) = P(A) \,.\, P\left(\frac{B_i}{A}\right)$

$$P(B_i A) = P(B_i) \,.\, P\left(\frac{A}{B_i}\right)$$

Since the events AB_i and $B_i A$ are equivalent, their probabilities are equal. Hence

$P(A) \,.\, P\left(\frac{B_i}{A}\right) = P(B_i) \,.\, P\left(\frac{A}{B_i}\right)$, so that

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i) \, . \, P\left(\frac{A}{B_i}\right)}{P(A)}$$

Substituting for $P(A)$ from above, the theorem is proved.

Formula (11.9.1) is also known as "Bayes" formula for probabilities of hypothesis", because $B_1, B_2, \ldots B_n$ may be considered as hypothesis which account for the occurrence of A. The probabilities $P(B_1), P(B_2), \ldots P(B_n)$ are called '*a priori*' probabilities of the hypothesis (i.e. probabilities prior to any knowledge as to the occurrence or non-occurrence of A); while $P\left(\frac{B_1}{A}\right), P\left(\frac{B_2}{A}\right), \ldots P\left(\frac{B_n}{A}\right)$ are known as a '*a posteriori*' probabilities of the same hypothesis (i.e. probabilities after the occurrence of A is definitely known).

Example 11.55 *Two boxes contain respectively 4 white and 2 black, and 1 white and 3 black balls. One ball is transferred from the first box into the second, and then one ball is drawn from the latter. It turns out to be black. What is the probability that the transferred ball was white?*

Solution There are two hypotheses:

B_1 = the transferred ball was white;
B_2 = the transferred bass was black.

The event A which is stated to have actually happened after the occurrence of B_1 or B_2, is

A = the ball drawn from the 2nd box is black.

We have to find the probability $P\left(\frac{B_i}{A}\right)$.

Now, $P(B_1) = \frac{4}{6} = \frac{2}{3}, P(B_2) = \frac{2}{6} = \frac{1}{3}$.

Also, $P\left(\frac{A}{B_1}\right)$ = Probability that the ball drawn from the 2nd box is *black*, assuming that the transferred ball was *white*.

$$= \frac{3}{5}$$

since after transfer the 2nd box contains 5 balls (2 white and 3 black).

Similarly, $P\left(\frac{A}{B_2}\right)$ = Probability that the ball drawn from the 2nd box is *black*, assuming that the transferred ball was *black*

$$= \frac{4}{5}$$

Using Baye's formula (11.9.1)

$$P\left(\frac{B_1}{A}\right) = \frac{P(B_1) \, . \, P\left(\frac{A}{B_1}\right)}{P(B_1) \, . \, P\left(\frac{A}{B_1}\right) + P(B_2) \, . \, P\left(\frac{A}{B_n}\right)} = \frac{\frac{2}{5} \times \frac{3}{5}}{\frac{2}{5} \times \frac{3}{5} \times \frac{1}{3} \times \frac{4}{5}} = \frac{3}{5} \qquad \textit{Ans. } \frac{3}{5}$$

Example 11.56 *Three identical boxes, I, II, III contain respectively 4 white and 3 red balls, 3 white and 7 red balls, and 2 white and 3 red balls. A box is chosen at random and a ball is drawn out of it. If the ball is found to be white, what is the probability that Box II was selected ?*

Solution Here A is the *observed* event that the 'drawn ball is white'. There are three hypotheses B_1, B_2, B_3, namely 'Box I was chosen', 'Box II was chosen', 'Box III was chosen' respectively. We have to find the probability of the hypothesis B_2, given that A has happened, i.e. $P\left(\frac{B_2}{A}\right)$.

Since the 3 boxes are identical in appearance.

$$P(B_1) = \frac{1}{3}, P(B_2) = \frac{1}{3}, P(B_n) = \frac{1}{3}$$

Also, $P\left(\frac{A}{B_1}\right)$ = Probability of getting a white ball, assuming that Box I was selected

$$= \frac{4}{7}$$

Similarly, $P\left(\frac{A}{B_2}\right) = \frac{3}{10}, P\left(\frac{A}{B_3}\right) = \frac{2}{5}$

Hypothesis

(B_i)	$P(B_i)$	$P\left(\frac{A}{B_i}\right)$	$P(B_i).P\left(\frac{A}{B_i}\right)$
B_1	$\frac{1}{3}$	$\frac{4}{7}$	$\frac{4}{21}$
B_2	$\frac{1}{3}$	$\frac{3}{10}$	$\frac{1}{10}$
B_3	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{2}{15}$
Total	1	—	$\frac{89}{210}$

Note that $\Sigma P(B_i) = 1$, because B_i's are mutually exclusive and exhaustive.
Using Baye's formula (11.9.1).

$$P\left(\frac{B_2}{A}\right) = \frac{P(B_2).P\left(\frac{A}{B_2}\right)}{\Sigma P(B_i).P\left(\frac{A}{B_i}\right)} = \frac{\frac{1}{10}}{\frac{89}{210}} = \frac{21}{89}$$

[**Note:** (i) In problems involving Bayes' Theorem, some event A is known to have definitely occurred, through certain hypotheses $B_1, B_2, \ldots B_n$, and we are required to find the probability of one of these hypotheses say $P\left(\frac{B_i}{A}\right)$, after the occurrence of A is known.

(ii) In the numerator of Bayes' formula (11.9.1), start with the unconditional probability $P(B_i)$; the other factor is $P\left(\frac{A}{B_i}\right)$—not $P\left(\frac{B_i}{A}\right)$, which we are going to find, but A and B_i interchanging places. The denominator is the sum of all such products for all the hypotheses. Observe that the numerator and the denominator differ only in respect of a Σ sign.]

Example 11.57 *In a bolt factory, the machines M_1, M_2, M_3 manufacture respectively 25, 35 and 40 per cent of the total product. Of their output 5, 4 and 2 per cent respectively are defective bolts. One bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine M_3?*

Solution The observed event A is that the selected bolt is defective. The hypotheses B_1, B_2, B_3 are that the selected bolt comes from machines M_1, M_2, M_3 respectively. We have to find $P\left(\frac{B_3}{A}\right)$.

$$P(B_1) = 25\% = 0.25,\ P(B_2) = 35\% = 0.35,\ P(B_3) = 40\% = 0.40$$

$P\left(\frac{A}{B_1}\right)$ = Probability that a bolt is defective, if it comes from machine M_1 = 5% = .05

Similarly, $P\left(\frac{A}{B_2}\right) = 4\% = .04$, $P\left(\frac{A}{B_3}\right) = 2\% = .02$

B_i	$P(B_i)$	$P\left(\frac{A}{B_1}\right)$	$P(B_i) . P\left(\frac{A}{B_1}\right)$
B_1	0.25	.05	.0125
B_2	0.35	.04	.0140
B_3	0.40	.02	.0080
Total	1.00	—	.0345

Using Bayes' formula,

$$P\left(\frac{B_3}{A}\right) = \frac{P(B_3) . P\left(\frac{A}{B_3}\right)}{\Sigma P(B_i) . P\left(\frac{A}{B_i}\right)} = \frac{.0080}{.0345} = \frac{16}{69}$$

11.10 MATHEMATICAL EXPECTATION

Suppose that a random experiment has n mutually exclusive and exhaustive outcomes corresponding to which a variable x takes the values

$$x_1, \quad x_2, \quad \dots \quad x_n$$

with probabilities

$$p_1, \quad p_2, \quad \dots \quad p_n.$$

respectively. Then the *mathematical expectation* (or *Expected Value*) of the variable x, denoted by $E(x)$, is defined as

$$E(x) = p_1x_1 + p_2x_2 + \dots + p_nx_n$$
$$= \Sigma p_i\, x_i \qquad (11.10.1)$$

where $\Sigma p_i = 1$. That is, the mathematical expectation is the sum of products of the different possible values of the variable and the corresponding probabilities.

If $E(x) = m$, the mathematical expectation of $(x - m)^2$ is called the Variance of x, and usually written as Var (x) or σ^2.

$$\text{Var}(x) = E(x - m)^2$$
$$= p_1(x_1 - m)^2 + p_2(x_2 - m)^2 + \dots + p_n(x_n - m)^2$$
$$= \Sigma p_i(x_i - m)^2 \qquad (11.10.2)$$

It can be shown that

$$\sigma^2 = \Sigma p_i x_i^2 - m^2 \qquad (11.10.3)$$

where $m = E(x)$. The square-root of variance is called *Standard Deviation.* It is always considered positive and usually denoted by the symbol σ.

Theorems on Mathematical Expectation

Theorem I

The mathematical expectation of the sum of two variables x and y is the sum of their mathematical expectations.

$$E(x + y) = E(x) + E(y) \qquad (11.10.4)$$

Theorem II

The mathematical expectation of the product of two *independent* variables x and y is the product of their mathematical expectations.

$$E(x\,.\,y) = E(x)\,.\,E(y) \qquad (11.10.5)$$

provided x and y are independent.

The theorems may be extended to more than two variables.

Example 11.58 *(i) An unbiased die is thrown. What is the mathematical expectation of the number of points?*

(ii) If 4 dice are thrown, find the expected sum of points.

Solution

(i) The variable x is the 'number of points', which takes the values 1, 2, 3, 4, 5, 6 with probabilities $\frac{1}{6}$ for each.

$$\therefore \quad E(x) = \left(\frac{1}{6}\times 1\right) + \left(\frac{1}{6}\times 2\right) + \left(\frac{1}{6}\times 3\right) + \left(\frac{1}{6}\times 4\right) + \left(\frac{1}{6}\times 5\right) + \left(\frac{1}{6}\times 6\right)$$

$$= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5$$

(ii) Let x, y, z, w represent the points obtained on the 4 dice. The sum of the points on the dice is

$$s = x + y + z + w$$

The mathematical expectation of the sum is the sum of mathematical expectations.

$$E(s) = E(x) + E(y) + E(z) + E(w)$$

Since x, y, z, w represent individual points on the 4 dice, we have from (i) above,

$$E(x) = 3.5, E(y) = 3.5, E(z) = 3.5, E(w) = 3.5$$

Hence, the mathematical expectation of the sum of points on the 4 dice is

$$E(s) = 3.5 + 3.5 + 3.5 + 3.5 = 14.$$

Ans. 3.5, 14.

Example 11.59 *A box contains 4 white and 6 black balls. If 3 balls are drawn at random, find the mathematical expectation of the number of white balls.*

Solution Here the variable x is the number of white balls obtained among the 3 balls drawn. The number of white balls may be 0, 1, 2, 3.

$$\text{Probability of 0 white ball} = \frac{^4C_0 \cdot {}^6C_3}{^{10}C_3} = \frac{1 \times 20}{120} = \frac{1}{6}$$

$$\text{Probability of 1 white ball} = \frac{^4C_1 \cdot {}^6C_2}{^{10}C_3} = \frac{4 \times 15}{120} = \frac{1}{2}$$

$$\text{Probability of 2 white balls} = \frac{^4C_2 \cdot {}^6C_1}{^{10}C_3} = \frac{6 \times 6}{120} = \frac{3}{10}$$

$$\text{Probability of 3 white balls} = \frac{^4C_3 \cdot {}^6C_0}{^{10}C_3} = \frac{4 \times 1}{120} = \frac{1}{30}$$

The possible values of x and the corresponding probabilities p are shown below:

x	p	px
0	$\frac{1}{6}$	0
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{3}{10}$	$\frac{3}{5}$
3	$\frac{1}{30}$	$\frac{1}{10}$
Total	1	$\frac{6}{5}$

(Note that $\Sigma p_i = 1$)

$$\therefore \quad E(x) = \Sigma p_i x_i = \frac{6}{5} = 1.2$$

Example 11.60 *A man purchases a lottery ticket, in which he may win the first prize of Rs 10,000 with probability .0001 or the second prize of Rs 4,000 with probability .0004. Find this mathematical expectation.*

Solution The probability that the man does not get any prize is $1 - .0001 - .0004 = .9995$. Applying (11.10.1).

$$E(x) = (.0001 \times 10.000 + .0004 \times 4.000 + .9995 \times 0) \text{ Rs}$$
$$= \text{Rs } 260$$

Example 11.61 *Two dice, with faces numbered 1 to 6, are thrown and their points are added. The thrower is given Rs 40 for a score of 12, but he has to pay Rs 2 if the score is less than 12. Find his expectation per throw.*

Solution With 2 dice, the probability of getting a total of 12, i.e. 6 on die I and 6 on die II, is $\frac{1}{36}$. Since the total score on the dice cannot exceed 12, the probability of getting a score less than 12 is $1 - \frac{\cancel{35}\ 1}{36} = \frac{35}{36}$. Considering his loss as a negative gain, we find that the thrower gets,

Rs 40 with probability $\frac{1}{36}$

Rs (– 2) with probability $\frac{35}{36}$

$$\therefore \quad \text{Expectation} = \frac{1}{36} \times 40 + \frac{35}{36} \times (-2) = -\frac{5}{6} \text{ Re}$$

i.e., he expects to lose Re $\frac{5}{6}$ per throw.

Example 11.62 *A man tosses a coin twice and is to receive Rs 2 if a head is shown, and lose Re 1 for a tail. Find this expected gain and the variance.*

Solution Assuming that the coin is unbiased, the possible outcomes, their probabilities and the corresponding values of gain (x) and are shown below:

Outcome	*Probability (p)*	*Gain (x)*	*px*	*px*^{2}
2 heads	$\frac{1}{4}$	$2 \times 2 = 4$	1	4
1 head and 1 tail	$\frac{1}{2}$	$1 \times 2 + 1\,(-1) = 1$	$\frac{1}{2}$	$\frac{1}{2}$
2 tails	$\frac{1}{4}$	$2\,(-1) = -2$	$-\frac{1}{2}$	1
Total	1	—	1	$\frac{11}{2}$

$$\text{Expected Gain } (m) = \Sigma p_i x_i = 1$$

$$\text{Variance } (\sigma^2) = \Sigma p_i x_i^2 - m^2 = \frac{11}{2} - 1^2 = \frac{9}{2} = 4.5$$

Ans. 1 Re, 4.5 Rs^2

Example 11.63 *A man has the choice of running either a hot-snack stall or an ice-cream stall at a seaside resort during the summer season. If it is a fairly cool summer he should make Rs 5000 by running the hot-snack stall, but if the summer is quite hot he can only expect to make Rs 1000. On the other hand, if he operates the ice-cream stall, his profit is estimated at Rs 6500 if the summer is hot, but only Rs 1000 if it is cool. There is a 40 per cent chance of the summer being hot. Should he opt for running the hot-snack stall or the ice-cream stall? Give mathematical arguments.* [I.C.W.A., Dec. '75]

Solution Probability of hot summer $= 40\% = \frac{40}{100} = 0.4$

Probability of cool summer $= 1 - 0.4 = 0.6$

The given data are tabulated below:

Condition of Summer Weather	*Probability (p)*	*Gain (x) from Hot-snack Stall*	*Gain (y) from Ice-cream Stall*
(i) Hot	0.4	Rs 1000	Rs 6500
(ii) Cool	0.6	Rs 5000	Rs 1000
Total	1 .0	—	—

Expected gain from hot-snack stall $= E(x)$

$$= 0.4 \times 1000 + 0.6 \times 5000$$
$$= 400 + 3000 = 3400 \text{ Rs}$$

Expected gain from ice-cream stall $= E(y)$

$$= 0.4 \times 6500 + 0.6 \times 1000$$
$$= 2600 + 600 = 3200 \text{ Rs}$$

Since the expected gain from the hot-snack stall is larger, the man should opt for running the hot-snack stall.

Example 11.64 *If $y = ax + b$, where a and b are constants, prove that $E(y) = a.E(x) + b$.*

Solution Let the variable x take the values $x_1, x_2, \dots x_n$ with probabilities $p_1, p_2, \dots p_n$ respectively, where $\Sigma p_i = 1$. The values assumed by the variable y are respectively

$$y_1 = ax_1 + b, y_2 = ax_2 + b, \dots y_n = ax_n + b$$
$$E(y) = p_1(ax_1 + b) + p_2(ax_2 + b) + \dots + p_n(ax_n + b)$$
$$= (p_1ax_1 + p_2ax_2 + \dots + p_nax_n) + (p_1b + p_2b + \dots + p_nb)$$
$$= a(p_1x_1 + p_2x_2 + \dots + p_nx_n) + b(p_1 + p_2 + \dots + p_n)$$
$$= a\,\Sigma p_i x_i + b\,\Sigma p_i$$
$$= a \,.\, E(x) + b \text{ (since } \Sigma p_i = 1)$$

(Proved)

Example 11.65 *A coin is tossed repeatedly until a head appears. Find the expected number of tosses required to obtain the first head.*

Solution Let p denote the probability of getting a head in a single toss, and $q = 1 - p$ be the probability of getting a tail. Consider the variable x = Number of tosses required to obtain the first head. x can take the values 1, 2, 3, ... ∞. In general, r tosses will be required, if the first $r - 1$ tosses result in tails only and the last toss gives a head. Since the successive trials are independent, the probability of such an occurrence, i.e.

$$P(x = r) = \underbrace{(q \cdot q \dots q)}_{r-1 \text{ times}} \cdot p = p \cdot q^{r-1}$$

Hence, $$E(x) = \sum_{r=1}^{\infty} (p.q^{r-1}).r = p\sum_{r=1}^{\infty}(r.q^{r-1})$$
$$= p(1 + 2q + 3q^2 + \dots \dots)$$
$$= p.(1-q)^{-2}$$

Since $(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots$ for $|x| < 1$.

$$= \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

Ans. $\frac{1}{p}$

11.11 OTHER APPROACHES TO PROBABILITY THEORY

In view of the inherent defects of classical definition (page 10), the probability theory has also been considered from other points of view.

Frequency (or 'Empirical') Theory: In N trials of a random experiment, if an event is found to occur f times, the relative frequency of occurrence of the event is $\frac{f}{N}$. If this relative frequency approaches a limiting value p, as N increases indefinitely, then p is called the "probability" of the event. In mathematical language.

$$p = \lim_{N \to \infty} \left(\frac{f}{N}\right) \tag{11.11.1}$$

The frequency definition of probability has the following has the following defects: (i) It is based on the stability of relative frequency $\frac{f}{N}$, as N becomes large. Hence, for its calculation a large number of experiments are to be performed under identical conditions. (ii) The limiting value is only a mathematical concept, and 'probability' cannot be measured exactly, even with a very large number of cases.

Example 11.66 1000 patients suffering from a disease, were treated with a certain medicine, and 880 of them were found to have been cured. The relative frequency of cures is therefore $\frac{880}{1000} = 0.880$. In another batch of 1500 patients, the number of cures with the medicine is 1300. So the relative frequency from the combined group is $\frac{(880+1300)}{2500} = 0.872$. A third batch of patients shows 450 cures, and therefore the relative frequency of cures in the total of 3000 patients is $\frac{(880+1300+450)}{3000} = 0.877$.

According to the frequency definition of probability, by continuing in this manner, the relative frequencies calculated from a larger group is expected to come closer and closer to a number, which is called the 'probability' of cure with the medicine. We may state that the estimate of this probability is about 0.88.

(b) Axiomatic Theory: This modern theory introduces 'probability' simply as a *number* associated with each event. It is based on certain axioms, which express the rules for operating with such numbers. This means that the probabilities of our events can be perfectly arbitrary, except that they must satisfy the axioms. The advantage of the axiomatic theory is that it embraces all situations irrespective of whether the possible outcomes of a random experiment are "equally likely" or not. The classical theory can be derived frm the axiomatic theory as a special case.

Some elementary concepts of the mathematical 'Set Theory' which is used in the Axiomatic Theory of Probability are given below. A more elaborate discussion will be found in "*Set Theory (with Problems and Exercises)*" by the present author.

11.12 SET THEORY

Set

Any well-defined collection of distinct objects is called a *set*. Each object of the set is called its *element* or *member*.

Illustration 1.

(a) The set of vowels in the English alphabet.
(b) The set of students who appeared at the B.Com. examination of Calcutta University in the year 1979.
(c) The set of numbers which are multiples of 4.
(d) The set of lines parallel to a given straight line.

A set is specified either by listing its members or by stating some property held by all members of the set and no non-member. The members of the set or the statement specifiying the members are written down within curly brackets "{ }". Sets are usually denoted by capital letters, such as A, B, S, R, etc. The elements of a set are denoted by small letters, such as a, b, c, p, etc. If p is an element of a set A, we write

$$p \in A$$

which is read "*p belongs to A*".

Illustration 2.

(a) $A = \{1, 3, 5, 7, 9)$ means that A is a set whose elements are 1, 3, 5, 7, 9. The arrangement of elements in the set is immaterial, e.g. $\{7, 1, 3, 9, 5\}$. We may also write the set in the form

$$A = \{x: x \text{ is a positive odd number, } x \leq 10\}$$

This is read "*A* is the set of elements *x*, such that *x* is a positive odd number and *x* less than or equal to 10." Note that the symbol ":" is used to denote "such that".

(b) $B = \{x: x \text{ is a river in Uttar Pradesh}\}$
(c) $N = \{1, 2, 3, 4, ...\}$ is the set of natural numbers.
(d) $R = \{x: x \text{ is real}, -2 \leq x \leq 1\}$ is the set of all real numbers lying between -2 and 1, both inclusive.

Subset

If each element of a set A is also an element of a set B, then A is called a *subset* of B, and we write

$$A \subseteq B \quad \text{or} \quad B \supseteq A$$

This is read as "*A is contained in B*" or "*B contains A*" respectively. Note that if $A \subseteq B$, then the elements of B may be either exactly the same as those of A, or the elements of B include all the elements of A and some more elements. The sign $\subseteq$ is similar to $\leq$ (i.e. less than or equal to) and has a some what similar interpretation in set theory.

If $A \subseteq B$ and $B \subseteq A$, i.e. A is a subset of B and B is a subset of A, then the sets A and B are said to be *equal* or *identical*, and we write

$$A = B$$

In such a case, sets A and B have exactly the same elements. If sets A and B are not equal, i.e. the elements of A are not the same as the elements of B, then we write $A \neq B$.

If $A \subseteq B$ but $A \neq B$, i.e. A is a subset of B but the sets are not equal, then A is said to be a *proper subset* of B. We write $A \subset B$.

Illustration 3.

(a) The set $A = \{e, i, a\}$ is a 'subset' of the set $B = \{a, e, i, o, u\}$, because all the elements of A are also elements of B. Thus $A \subseteq B$.

(b) The set $C = \{3, 7, 8\}$ is a subset of the set $D = \{8, 3, 7\}$, since each element of C is also an element of D. Thus $C \subseteq D$. Similarly, we see that $D \subseteq C$. Hence, $C = D$, i.e. the sets C and D are 'equal'.

(c) $E = \{2, 4, 6\}$ is a 'proper subset' of $S = \{1, 2, 3, 4, 5, 6\}$ because each element of E belongs to S, but $E \neq S$.

(d) The set $P = \{2, 8, 12\}$ is not a subset or $Q = \{1, 4, 5, 8, 12\}$ because the element 2 of P does not belong to Q.

Universal Set (or Space)

For many purposes we limit our discussion to subsets of some particular set, which is called the *universal set.* Thus any set is a subset of the universal set. Using geometrical terminology, the universal set is also called *Space,* and its elements are called *points.* The symbol S is generally used to denote the universal set.

Null Set

It is useful to consider a set which contains no elements at all. This is called the *Null Set* (or *Empty Set*), and is denoted by the symbol ϕ. It will be seen that the null set ϕ is somewhat equivalent to 0 (zero) in the number system. Note that the null set ϕ is distinct from the set $\{0\}$ which is a set containing only one element, viz. 0.

Illustration 4.

(a) In the study of income of Indian population, the set of incomes of all citizens of India is the universal set, and the incomes of different groups or classes of people are its subsets.

(b) THe set $\{x$: x is a person who can jump to a height of 3 miles$\}$ is the null set, because there is no person who can jump to such a height.

Finite and Infinite sets

A set is said to be *finite,* if it is empty or contains a finite number of elements, (i.e. if the number of different elements of the set can be counted and the counting process can be completed). Otherwise, the set is said to be *infinite.*

Infinite sets are of two types—(i) *countably infinite,* and (ii) *uncountably infinite.* An infinite set is called *countable,* if its elements can be written down in some order. Otherwise an infinite set is called *uncountable.*

Illustration 5.

(a) The set of days of the week {*Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday*} is finite, because the total number of elements of the set is a specified number 7.

(b) The set of even numbers $\{2, 4, 6, 8, \ldots \ldots\}$ is countably infinite, because the total number of elements is not finite and the elements can be arranged in an increasing order.

(c) The set $\{x: x \text{ is real}, 0 \leq x \leq 1\}$ is uncountably infinite, because the possible elemetns of the set is neither finite in number nor can be arranged in a sequence.

(d) Let $R = \{x: x \text{ is a mountain peaks in the world}\}$. Although it is difficult to count all the mountain peaks in the whole world, R is still a finite set.

Venn Diagram

A universal set S is often represented geometrically by the set of points inside a rectangle, and the subsets of S, such as A and B, are represented by sets of points inside closed areas and shaded. Such a diagrammatic representation of sets is known as *Venn Diagram* (Fig. 11.7).

Set Operations

Union

The *union* of sets A and B is the set of all elements which belong *either to A, or to B, or to both* (i.e. the set of elements which belong to *at least one* of the sets A and B). This is written as

$$A \cup B$$

and read "*A union B*". Some authors also write $A + B$ and call this the set-theoretic "*sum* of A and B", or simply "*A plus B*".

Illustration 6.

Let $A = \{1, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$. Then $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$.

Note: (i) By definition $A \cup B = B \cup A$ (*i.e.* the union of A and B is identical with the union of B and A).

(ii) A and B are subsets of $A \cup B$.

(iii) The union of several sets, viz. $A_1 \cup A_2 \cup \ldots \cup A_n$ is sometimes abbreviated as

$$\bigcup_{i=1}^{n} A_i$$

Intersection

The *intersection* of sets A and B is the set of all elements which belong to *both A and B* (i.e. the set of elements which are *common* to the sets A and B). This is written as

$$A \cap B$$

and read "*A intersection B*". Some authors write AB and call this the set-theoretic "*product* of A and B".

Illustration 7.

(a) Let $A = \{1, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$. Then $A \cap B = \{4, 6\}$.

(b) Let $C = \{1, 3, 5\}$ and $D = \{2, 4, 6\}$. Then $C \cap D = \phi$. (i.e. the intersection of C and D is the null set, since they have no common elements).

Note: (i) By definition $A \cap B = B \cap A$ (i.e. the intersection of sets A and B is identical with the intersection of B and A).

(ii) $A \cap B$ is a subset of A and also a subset of B.

(iii) The intersection of several sets, viz. $A_1 \cap A_2 \cap \ldots \cap A_n$ is sometimes abbreviated as

$$\bigcap_{i=1}^{n} A_i$$

Complement

The *complement* of a set A is the set of all elements of the universal set S, which *do not belong* to A. This is written as

$$A' \text{ or } \bar{A} \text{ or } A^{\circ}$$

and read "*A prime*" or "*A bar*" or "*A complement*" respectively.

Illustration 8.

Let the universal set be $S = \{1, 2, 3, 4, 5, 6\}$ and $A = \{2, 4\}$. Then $A' = \{1, 3, 5, 6\}$.

Note: (i) $A \cup A' = S$ and $A \cap A' = \phi$.
(i.e. the union of a set and its complement is the universal set; and the intersection of a set and its complement is the null set).
(ii) $S' = \phi$, and $\phi' = S$.
(i.e. universal set and null set are complementary to each other).
(iii) $(A')' = A$
(i.e. the complement of the complement of a set is the set itself).

Difference

The *difference* of two sets A and B is the set of all elements which *belong to A but do not belong to B*. This is written as

$$A - B$$

and read "*A difference B*", or simply "*A minus B*".

Illustration 9.

Let $A = \{1, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$. Then $A - B = \{1, 3, 5\}$ and $B - A = \{2, 8\}$.

Note: (i) $A - B$ is a subset of A; and $B - A$ is a subset of B.
(ii) $A - B \neq B - A$
(i.e. the difference of A and B is not identical with the difference of B and A).
(iii) $A - B = A \cap B'$
(i.e. the difference of A and B is identical with the intersection of A and B complement).

Disjoint Sets

Two sets A and B are said to be *disjoint*, if they have no common element (i.e. not a single element of B is contained in A, and none of A is contained in B). If A and B are disjoint sets then $A \cap B = \phi$.

Note: (i) Any set A and its complement A' are disjoint, because $A \cap A' = \phi$.
(ii) Any set A and the null set ϕ are disjoint, because $A \cap \phi = \phi$.

Illustration 10.

(a) Let $A = \{2, 6, 7, 9\}$ and $B = \{1, 3, 4, 5, 8\}$. Then A and B are disjoint sets, since no element is common to both A and B.
(b) The set of odd numbers and the set of even numbers are disjoint, since no number can be both odd and even.
(c) Let $A = \{a, b, c, f\}$ and $B = \{d, e, f, g, h\}$. Then A and B are not disjoint, since the element f is common to both.

Venn Diagrams for some important cases are shown below:

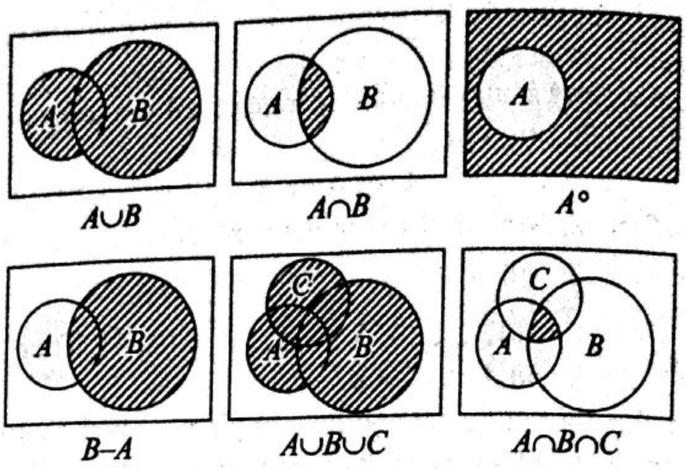


Fig. 11.7 Venn Diagrams

Algebra of Sets

Under the operations of union, intersection and complement, sets satisfy various laws, some of which have already been noted earlier. These are referred to as the algebra of sets:

1. Commutative Laws:
   $A \cup B = B \cup A$ $\quad\quad$ $A \cap B = B \cap A$
2. Associative Laws:
   $(A \cup B) \cup C = A \cup (B \cup C)$ $\quad\quad$ $(A \cap B) \cap C = A \cap (B \cap C)$
3. Distributive Laws:
   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\quad\quad$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. De Morgan's Laws:
   $(A \cup B)' = A' \cap B'$ $\quad\quad$ $(A \cap B)' = A' \cup B'$

Besidees, we have:

5. $A \cup A = A$ $\quad\quad$ $A \cap A = A$
6. $A \cup \phi = A$, $A \cup S = S$ $\quad\quad$ $A \cap S = A$, $A \cap \phi = \phi$
7. $A \cup A' = S$ $\quad\quad$ $A \cap A' = \phi$
8. $(A')' = A$ $\quad\quad$ $S' = \phi$, $\phi = S$

11.13 SET AND PROBABILITY

Sample Space and Sample Point

In the axiomatic development of probability theory, we start with certain undefined objects, called '*outcomes*'. The 'set' of all possible outcomes of a given random experiment is called '*sample space*'. Any particular outcome (i.e. an element of the sampel space) is called a *Sample Point*. It may be noted that the sample space is not a geometrical space at all and has no dimension. It is the 'universal set' of outcomes. Often the sample space of a random experiment can be given in several ways, but there is usually one which provides the maximum information.

A sample space which is finite or countably infinite is called a '*discrete sample space*'. An uncountably infinite sample space is called a '*continuous sample space*'.

Illustration 1.

(a) In the random experiment of tossing a coin, the sample space S is the set

$$S = \{H, T\}$$

It has two sample points, viz. H and T.

(b) In the random experiment of tossing a die, the sample space is usually given as

$$S = \{1, 2, 3, 4, 5, 6\}.$$

We could also show the sampel space as $S_1 = \{\text{even, odd}\}$.

(c) In the experiment of tossing two coins

$$S = \{HH, HT, TH, TT\}$$

Note that here the sample points are ordered pairs of results (called 2-tuples) one from each coin. The sample space could also be given as $S_1 = \{0, 1, 2\}$ the sample points indicating the total number of heads obtained from the coins.

(d) In drawing a ball from an urn containing 2 white and 3 red balls, we could regard the balls as numbered, say white balls 1 and 2; red balls 3, 4 and 5. The sample space is then

$$S = \{1, 2, 3, 4, 5\}$$

(e) Suppose 2 balls are drawn from the above urn one at a time without replacement and the order in which they are drawn is taken into consideration. We shall write (4, 1) to denote the outcome when the first ball drawn bears the number 4 and the second ball bears the number 1. The sample space has now 20 points as follows:

$$S = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 3), 2, 4), (2, 5), \\ (3, 1), (3, 2), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 5), \\ (5, 1), (5, 2), (5, 3), (5, 4)\}$$

It will possibly be easier to understand if we number these 20 elements of S serially, and write

$$S = \{e_1, e_2, e_3, \dots \dots e_{20}\}$$

Note that here the sample points e_i are 2-tuples.

(f) When a coin is tossed 3 times (or 3 coins are tossed simultaneously), the sample space S contains 8 sample points.

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

The sample points are now 3-tuples.

(g) Toss a coin repeatedly until a head appears. Then

$$S = \{H, TH, TTH, TTTH, TTTTH, \dots \dots\}$$

If the number of times the coin need be tossed is counted, the sample space may be given as

$$S_1 = \{1, 2, 3, 4, 5, \dots \dots\}$$

(h) Suppose we observe the lifetime (in hours) of electric lamps of a particular brand, which are known to have a maximum life of 2000 hours. The sample space is

$$S = \{x\text{: } x \text{ is the lifetime in hours, } 0 \le x \le 2000\}$$

The sample spaces mentioned at (a) to (f) above are finite and at (g) are countably infinite. These are 'discrete' sampel spaces. The sample space at (h) is uncountably infinite and is therefore a 'continuous' sample space.

Event

An *event* is a 'set' of outcomes of a random experiment. In other words, event is a subset of the sample space S. Events may be 'elementary' or 'composite'. An *elementary event* is the set which contains a single sample points. A *composite event* contains more than one sample point.

In particular, the sample space S, which is a subset of itself, is also an event, called the *sure event* or *certain event*. The *impossible event* is the null set ϕ, i.e. the subset of S which contains no sample point at all.

Subevent

If $A \subseteq B$, where A and B are events of the sample space S, then A is called a *subevent* of B.

Illustration 2.

(a) In tossing a coin, the sample space is $S = \{H, T\}$. Let A = event that head appears; B = event that tail appears. Then,

$$A = \{H\}, B = \{T\}$$

Events A and B are elementary. Note the distinction between the sample point H, which is an element of S, and the event $\{H\}$, which is a subset of S.

(b) If the experiment of tossing a die is described by the sample space $S = \{1, 2, 3, 4, 5, 6\}$, then

A = event that an odd number appears $= \{1, 3, 5\}$
B = event that more than 4 appears $= \{5, 6\}$
C = event that "six" appears $= \{6\}$

Events A and B are composite, but event C is elementary.

(c) In drawing 2 balls from an urn containing 2 white and 3 red balls (*Illustration 1*-e),

E = event that the second ball is white
$= \{(1, 2), (2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)\}$

(d) In tossing two dice,

A = event that the sum of the points on the dice is 8
$= \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$

(e) In the random experiment of tossing 3 coins (*Illustration 1*-f)

A_1 = event that the 2nd coin shows Head
= set of sample points which show H in the second place
$= \{HHH, HHT, THH, THT\}$
A_2 = event that exactly 2 heads appear $= \{HHT, HTH, THH\}$
A_3 = event that all heads or all tails appear $= \{HHH, TTT\}$

(f) In the random experiment of tossing a coin until the first head (*Illustration 1*-g),

A = event that at most 3 tosses are required $= \{H, TH, TTH\}$

(g) In the random experiment of observing the lifetime of electric bulbs (*Illustration 1*-h)

B = Event that a lamp burns not less than 500 hours
$= \{x: 500 \le x \le 2000\}$

Remarks

If the sample space S is finite or countably infinite every subset of S is an event.

However, if the sample space is uncountable, then for technical reasons, certain subsets of S cannot be 'events'.

Events Derived by Set Operations

We can combine events to form new events using the various set operations. Thus if A and B are events, then

$A \cup B$ denotes the event "*at least one* on A and B", i.e. "either A or B or both". This is also written as $A + B$.

$A \cap B$ denotes the event "A *as well as* B", i.e. "both A and B". This is also written as AB, and called *the* "*compound event*".

A' denotes the event "not-A", i.e. "opposite A". This is also written as $\bar{A}$, and called the "*complementary event*".

$A - B$ denotes the event "A but not B", i.e. "both A and not-B". This may also be written as $A \cap B'$.

Mutually Exclusive Events

Two events A and B are said to be *mutually exclusive* or *disjoint*, if $A \cap B = \phi$ (*i.e.* A and B cannot occur simultaneously).

Note: (i) Events A and A' are mutually exclusive, because $A \cap A' = \phi$.

(ii) Any event A and the impossible evert ϕ are mutually exclusive, because $A \cap \phi = \phi$.

Illustration 3.

(a) In the experiment of tossing a die, we have $S = \{1, 2, 3, 4, 5, 6\}$. Let

$A = \{1, 3, 5\}$ = event that an odd number appears.

$B = \{2, 4, 6\}$ = event that an even number appears.

$C = \{5, 6\}$ = event that more than 4 appears.

Then

$A \cup C = \{1, 3, 5, 6\}$ = event that an odd number *or* more than 4 appears.

$B \cap C = \{6\}$ = event that a number which is even *as well as* more than 4 appears.

$A' = S - A = \{2, 4, 6\}$ = event that an odd number *does not* appear.

$A' \cup C = \{2, 4, 5, 6\}$ = event that an odd number does not appear *or* more than 4 appears.

$A \cap B = \phi$ = event that an odd number *as well as* an even number appears (this is impossible).

(b) Let the sample sapce $S = \{HH, HT, TH, TT\}$ describe the random experiment of tossing 2 coins, and

A = event that at least one head appears = $\{HH, HT, TH\}$.

B = event that the 2nd coin shows tail = $\{HT, TT\}$.

Then

$A \cup B$ = event that at least one head appears *or* the 2nd coin shows tail = $\{HH, HT, TH, TT\} = S$.

$A \cap B$ = event that at least one head appears *and* the 2nd coin shows tail = $\{HT\}$.

B = event that the 2nd coin *des not* show tail = $S - B$ = $\{HH, TH\}$.

$A - B$ = event that at least one head appears *but* the 2nd coin does *not* show tail = $\{HH, TH\}$.

(c) In tossing 3 coins, $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ suppose. Let

A_1 = event that the 2nd coin shows head $= \{e_1, e_2, e_5, e_6\}$

A_2 = event that exactly 2 heads appear $= \{e_2, e_3, e_5\}$

A_4 = event that 2 or more heads appear $= \{e_1, e_2, e_3, e_5\}$

$\therefore A_1 \cup A_2$ = event that *either* the 2nd coin shows head *or* exactly 2 heads appear $= \{e_1, e_2, e_3, e_5, e_6\}$.

A_2' = event that exactly 2 heads *do not* appear $= \{e_1, e_4, e_6, e_7, e_8\}$

Since, $A_2 \subseteq A_4$, we have $A_2 \cup A_4 = A_4$ and $A_2 \cap A_4 = A_2$. Hence

$A_1 \cap A_2 \cap A_4 = A_1 \cap (A_2 \cap A_4) = A_1 \cap A_2 = \{e_2, e_5\}$

= event that the 2nd coin shows head *as well as* 2 heads appear.

11.14 AXIOMS OF PROBABILITY

Let S be a sample space of a random experiment. If to each event A of the set of all possible events of S, we associate a real number $P(A)$, then $P(A)$ is called the "probability" of event A, if the following axioms hold:

Axiom 1. For every event A

$$P(A) \geq 0 \tag{11.14.1}$$

Axiom 2. For the sure event S

$$P(S) = 1 \tag{11.14.2}$$

Axiom 3. For any finite number or countably infinite number of mutually exclusive events $A_1, A_2, \ldots$ of S

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots \tag{11.14.3}$$

In particular, for two mutually exclusive events A and B

$$P(A \cup B) = P(A) + P(B) \tag{11.14.4}$$

It should be noted that we can speak of the 'probability' only if the event is a subset of a specified sample space S, and to each subset of S a real number, satisfying the axioms, can be assigned. A sample space on which 'probability' has been defined is called a *probability space.*

Deductions from the Axioms

Theorem I

The probability of the impossible event is zero.

$$P(\phi) = 0 \tag{11.14.5}$$

Proof

Any event A and the impossible event ϕ are mutually exclusive.

Also $A \cup \phi = A$. Hence, by Axiom 3,

$P(A) = P(A \cup \phi) = P(A) + P(\phi)$

$\therefore$ $P(\phi) = 0$.

Theorem II

The probability of the complementary event is

$$P(A') = 1 - P(A) \tag{11.14.6}$$

Proof

A and A' are mutually exclusive events, and

$$A \cup A' = S.$$

Hence, $P(S) = P(A \cup A') = P(A) + P(A')$, by Axiom 3.

i.e. $$1 = P(A) + P(A'), \qquad \therefore \quad P(A') = 1 - P(A)$$

Theorem III

The probability of an event lies between 0 and 1.

$$0 \leq P(A) \leq 1 \tag{11.14.7}$$

Proof

By Axiom 1, $0 \leq P(A)$. Also, from (11.14.6), we have $P(A) = 1 - P(A')$. Since $P(A')$ is a probability, it cannot be negative (Axiom 1); therefore $P(A) \leq 1$. Combining both the inequalities, the result follows.

Theorem IV

If $A = A_1 \cup A_2 \cup \ldots \cup A_n$, where $A_1, A_2, \ldots A_n$ are mutually exclusive events, then

$$P(A) = P(A_1) + P(A_2) + \ldots + P(A_n) \tag{11.14.8}$$

In particular, if $A = S$, the sample space, then

$$P(A_1) + P(A_2) + \ldots + P(A_n) = 1 \tag{11.14.9}$$

Proof

Relation (11.14.8) follows from Axiom 3 for the case when the number of mutually exclusive events is finite, say n. In addition, using Axiom 2, the result (11.14.9) follows.

Theorem V

If $A \subseteq B$ (i.e. event A implies event B), then

$$P(A) \leq P(B) \tag{11.14.10}$$

Proof

If $A \subseteq B$, then events A and $A' \cap B$ are mutually exclusive, and their union $A \cup (A' \cap B) = B$. Hence, by Axiom 3,

$$P(B) = P(A) + P(A' \cap B)$$

Since by Axiom 1, $P(A' \cap B)$ cannot be negative, hence

$$P(B) \geq P(A).$$

Theorem VI

For any two events A and B,

$$P(A) = P(A \cap B) + P(A \cap B') \tag{11.14.11}$$

$$P(B) = P(A \cap B) + P(A' \cap B) \tag{11.14.12}$$

Proof

Events $A \cap B$ and $A \cap B'$ are mutually exclusive, and their union is the event A. Hence by Axiom 3, relation (11.14.11) can be proved. Similarly, event B is the union of mutually exclusive events $A \cap B$ and $A' \cap B$, and result (11.14.12) can be proved.

Theorem VII

For any two events A and B (which may or may not be mutually exclusive),

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{11.14.13}$$

Proof

Events $A \cap B'$, $A \cap B$ and $A' \cap B$ are mutually exclusive, and their union is the event $A \cup B$. Hence, by Axiom 3,

$$P(A \cup B) = P(A \cap B') + P(A \cap B) + P(A' \cap B)$$

But from (11.14.11) and (11.14.12), we have

$$P(A \cap B') = P(A) - P(A \cap B) \text{ and } P(A' \cap B) = P(B) - P(A \cap B).$$

Substituting these values,

$$\begin{aligned} P(A \cap B) &= [P(A) - P(A \cap B)] + P(A \cap B) + [P(B) - P(A \cap B)] \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

Theorem VIII

For any three events A, B, C

$$\begin{aligned} P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) \\ - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned} \quad (11.14.14)$$

Proof

In (11.14.13) let us replace the event B by $B \cup C$. Then

$$\begin{aligned} P[A \cup (B \cup C)] &= P(A) + P(B \cup C) - P[A \cap (B \cup C)] \\ &= P(A) + [P(B) + P(C) - P(B \cap C)] - P[(A \cap B) \cup (A \cap C)], \end{aligned}$$

by Distributive law.

Again using (11.14.13) for the union of events $A \cap B$ and $A \cap C$,

$$\begin{aligned} P[(A \cap B \cup (A \cap C)] &= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)] \\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \end{aligned}$$

Hence the result follows.

11.15 FINITE PROBABILITY SPACE AND ASSIGNMENT OF PROBABILITIES

Suppose that a sample space is finite and consists of n elementary outcomes:

$$S = \{e_1, e_2, \dots e_n\}$$

Then a total number of 2^n possible events can be obtained from S. For example, in the random experiment of tossing 2 coins and observing the sequence of Heads and Tails, $S = \{HH, HT, TH, TT\} = \{e_1, e_2, e_3, e_4\}$, suppose. Here, the sample space S contains 4 outcomes and thereofore $2^4 = 16$ events

$$\phi, \{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_1, e_4\},$$
$$\{e_2, e_3\}, \{e_2, e_4\}, \{e_3, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\},$$
$$\{e_1, e_3, e_4\}, \{e_2, e_3, e_4\} \text{ and } \{e_1, e_2, e_3, e_4\} \text{ i.e. } S.$$

In order to define "probability" we need therefore specify 2^n values $P(A)$ for all 2^n events A, so that Axioms 1, 2, and 3 are satisfied.

In the finite probability space, it will however be sufficient if we assign only n real numbers $p_1, p_2, \dots p_n$ to the n elementary events

$$A_1 = \{e_1\},\ A_2 = \{e_2\}, \dots A_n = \{e_n\}$$

respectively, such that

(i) $p_i \geq 0$ $(i = 1, 2, \dots n)$ (11.15.1)

(ii) $p_1 + p_2 + \dots + p_n = 1$ (11.15.2)

This means that the 'numbers' p_i are arbitrary positive proper fractions, or some zeros, whose total is 1. These 'numbers' $p_1, p_2, \dots p_n$ are called *"probabilities"* associated with the elementary events $A_1, A_2, \dots A_n$ respectively.

The probability of an event A can then be computed as the sum of the probabilities of those elementary events whose union constitutes A. For example, if $A = \{e_1, e_3, e_4\}$, we can write

$$A = \{e_1, e_3, e_4\} = \{e_1\} \cup \{e_3\} \cup \{e_4\} = A_1 \cup A_3 \cup A_4$$

because the elementary events $\{e_i\}$ are mutually exclusive. Therefore by Axiom 3,

$$P(A) = P(A_1) + P(A_3) + P(A_4)$$
$$= p_1 + p_3 + p_4$$

11.16 FINITE EQUIPROBABLE SAMPLE SPACE AND CLASSICAL DEFINITION

A finite sample space $S = \{e_1, e_2, \ldots e_n\}$ is said to have "*equally likely*" outcomes, if the probabilities $p_1, p_2, \ldots, p_n$ assigned to all the n elementary events of S are equal; i.e. $p_1 = p_2 = \ldots = p_n$. The sample space S is then said to be "*equiprobable*".

By (11.15.1) and (11.15.2), since the probabilities $p_1, p_2, \ldots, p_n$ cannot be negative and their total must be 1, each p_i must have the same value $\frac{1}{n}$, i.e.

$$p_1 = p_2 = \ldots = p_n = \frac{1}{n}$$

Therefore, if an event A has m sample points, then the probability of A is

$$P(A) = \frac{1}{n} + \frac{1}{n} + \ldots (m \text{ times})$$

or, $$P(A) = \frac{m}{n} \qquad (11.16.1)$$

Thus, if we have a finite equiprobable sample space of n elementary outcomes then the computation of probability of an event reduces to the computation of the number (m) of sample points which belong to the event A. The probability of A is then given by (11.16.1).

$$P(A) = \frac{\text{Number of outcomes favourable to } A}{\text{Total number of possible outcomes}} = \frac{m}{n}$$

This is equivalent to the 'classical' definition of probability (11.5.1).

The axiomatic definition of probability is thus more general, from which the classical definition can be obtained as a special case, i.e. when the sample space is finite and equiprobable.

Summary of Axiomatic Theory of Probability (Discrete Case)

I. (*Finite Case*) Let us suppose that a random experiment has a finite number (n) of possible 'elementary outcomes' $e_1, e_2, \ldots, e_n$. The set $S = \{e_1, e_2, \ldots e_n\}$ is called a 'Sample Space' connected with the random experiment and its elements e_i are called 'Sample Points'. Any set, say $\{e_1, e_2, e_5, e_7\}$, which is a subset of S, is called an "Evant". The sets $\{e_i\}$ consisting of single elements are called 'elementary events', while sets which consist of more than one element are called 'composite events'. In particular, the null set ϕ is called the 'impossible event' and the universal set S is called the 'sure event'.

Suppose that corresponding to the elementary events $\{e_1\}, \{e_2\}, \ldots \{e_n\}$ we are given real numbers $p_1, p_2 \cdots p_n$ respectively such that $p_i \geq 0$ and $\sum_1^n p_i = 1$. The numbers p_i are called "probabilities" assigned to the elementary events $A_i = \{e_i\}$. The probability of any event, say

$$A = \{e_1, e_2, \ldots p_m\}$$

is then given by the sum of the probabilities associated with those outcomes which belong to the event A (strictly speaking, with those elementary events whose union constitutes the given event A).

$$P(A) = p_1 + p_2 + \ldots + p_m$$

In particular, if the possible outcomes $e_1, e_2, \ldots e_n$ are "equally likely", the probabilities $p_1, p_2, \ldots p_n$ are equal and all have the same value $1/n$. The probability of the event A consisting of m sample points is then given by

$$P(A) = \frac{m}{n}$$

Thus the classical definition of probability is obtained from the axiomatic theory as a special case.

II. (*Countably Infinite Case*) The theories here are exactly the same as in Case I, with the following modifications:

The number of possible elementary outcomes $e_1, e_2, \ldots$ is countably infinite, and the sample space $S = \{e_1, e_2, \ldots\}$ is an infinite set. The probabilities $p_1, p_2, \ldots$ assigned to the elementary events are, as before, non-negative and their sum equals 1, i.e. $p_i \geq 0$ and $\sum_1^\infty p_i = 1$.

However, since the number of possible outcomes is infinite, the classical definition cannot be derived from this case.

Example 11.67 *There are 3 children in a family. What is the probability that they include (i) exactly 2 girls, (ii) not move than one girl?*

Solution The sample space is finite and contains 8 sample points

$$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB. GGG\}$$

If we assume that the birth of boys and girls are equally likely, then S is equiprobable, so that the probability assigned to each elementary event is $\frac{1}{8}$.

Probability that there are exactly 2 girls

$$= P(\{BGG, GBG, GGB\}) = \frac{1}{8} \times 3 = \frac{3}{8}$$

Probability that not more than one is a girl

$$= P(\{BBB, BBG, BGB, GBB\}) = \frac{1}{8} \times 4 = \frac{1}{2}$$

Example 11.68 *Two urns marked I and II contain respectively 3 white, 2 black balls and 2 white, 4 black balls. If one ball is drawn from each urn, what is the probability that both are white balls?*

Solution Let us assume that the balls are numbered as follows:

Urn I: White balls 1, 2, 3; Black balls 4, 5.

Urn II: White balls 6,7; Black balls 8, 9, 10, 11.

The sample space S contains 30 sample points, each of which is an ordered pair of the form (2, 9), where the first number refers to the ball from Urn I and the second from Urn II. If the balls are identical in all respects except in colour, we may assume that the sample space is equiprobable, so that the probability assigned to each elementary event is $\frac{1}{30}$. The event A that both the balls are white is

$$A = \{(1, 6), (1, 7), (2, 6), (2, 7), (3, 6), (3, 7)\}$$

which contains 6 sample points. Thus $P(A) = \frac{1}{30} \times 6 = \frac{1}{5}$.

11.17 CONDITIONAL PROBABILITY

Let A and B be two events, such that $P(A) \neq 0$. Then the *Conditional probability* of B, *given that* A has occurred, is defined as

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} \qquad (11.17.1)$$

It follows that

$$P(A \cap B) = P(A) \, . \, P\left(\frac{B}{A}\right) \qquad (11.17.2)$$

This is known as Multiplication Theorem of probability.

Similarly, if $P(B) \neq 0$, the conditional probability of A, given that B has occurred, is

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} \qquad (11.17.3)$$

from which we get

$$P(A \cap B) = P(B) \, . \, P\left(\frac{A}{B}\right) \qquad (11.17.4)$$

Equating the right hand sides of (11.17.2) and (11.17.4),

$$P(A) \, . \, P\left(\frac{B}{A}\right) = P(B) \, . \, P\left(\frac{A}{B}\right) \qquad (11.17.5)$$

provided $P(A) \neq 0$, $P(B) \neq 0$. Thus we find that

$$P\left(\frac{A}{B}\right) = \frac{P(A)}{P(B)} P\left(\frac{B}{A}\right) \qquad (11.17.6)$$

Example 11.69 *There are 3 children in a family. Find the probability that all the children are boys, (i) if no prior information is available about the children, (ii) if it is known that the two eldest are boys, (iii) if it is known that at least two of them are boys.*

Solution The sample space is shown in Example 11.67. let A, C, D denote the events that 'all 3 are boys', 'eldest two are boys', and 'at least 2 of the children are boys' respectively.

Then

$$A = \{BBB\}, C = \{BBB, BBG\}, D = \{BBB, BBG, BGB, GBB\}$$

We are required to find (i) $P(A)$, (ii) $P\left(\frac{A}{C}\right)$, (iii) $P\left(\frac{A}{D}\right)$. By definition, $P(C) \neq 0$ and $P(D) \neq 0$,

$$P\left(\frac{A}{C}\right) = \frac{P(A \cap C)}{P(C)}, \quad P\left(\frac{A}{D}\right) = \frac{P(A \cap D)}{P(D)}$$

Since the sample space is finite (size 8) and equiprobable, and events A, C, D contain 1, 2 and 4 sample points respectively

$$P(A) = \frac{1}{8}, P(C) = \frac{2}{8}, P(D) = \frac{4}{8}.$$

Also, $P(A \cap C) = P(\{BBB\}) = \frac{1}{8}$, $P(A \cap D) = P(\{BBB\}) = \frac{1}{8}$, Thus,

Probability that all the children are boys $= P(A) = \frac{1}{8}$.

Probability that all the children are boys, given that the two eldest are boys

$$= P\left(\frac{A}{C}\right) = \frac{1}{8} + \frac{2}{8} = \frac{1}{2}$$

Probability that all the children are boys, given that at least 2 of them are boys

$$= P\left(\frac{A}{D}\right) = \frac{1}{8} + \frac{4}{8} = \frac{1}{4}$$

Example 11.70 *If A and B are mutually exclusive events and $P(A \cup B) \neq 0$, then prove that*

$$P\left(\frac{A}{A \cup B}\right) = \frac{P(A)}{P(A) + P(B)}$$

[C.U., B.Sc. '81]

Solution By the definition of conditional probability, and since $P(A \cup B) \neq 0$ (given),

$$P\left(\frac{A}{A \cup B}\right) = \frac{P[A \cap (A \cup B)]}{P(A \cup B)} \qquad \text{(i)}$$

Also, since A and B are known to be mutually exclusive,

$$A \cap B = \phi \quad \text{and} \quad P(A \cup B) = P(A) + P(B) \qquad \text{(ii)}$$

By the Algebra of sets (Distributive Law),

$$A \cap (A \cup B) = (A \cap A) \cup (A \cap B)$$
$$= A \cup \phi = A$$

$$\therefore \quad P[A \cap (A \cup B)] = P(A) \qquad \text{(iii)}$$

Substituting from (ii) and (iii) on the R.H.S. of (i), the result follows.

11.18 INDEPENDENT EVENTS

An event B is considered to be *independent* of an event A, if the probability that B occurs is not influenced by the knowledge that A has occurred. In other words, the unconditional probability of B equals the conditional probability of B, given A, i.e.

$$P(B) = P\left(\frac{B}{A}\right) \tag{11.18.1}$$

Using (11.17.5), we see that in case $P(B) = P\left(\frac{B}{A}\right)$, then $P(A) = P\left(\frac{A}{B}\right)$. Thus, if B is independent of A, then A is also independent of B. We therefore say that events "*A and B are independent.*" From the Multiplication Theorem (11.17.2) it follows that $P(A \cap B) = P(A) \,.\, P(B)$. This equation is taken as a formal definition for independence of two events.

Events A and B are said to be "independent" (or stochastically independent), if and only if

$$P(A \cap B) = P(A) \,.\, P(B) \tag{11.18.2}$$

Otherwise, they are said to be "*dependent*" events,

If events A and B are independent, it can be shown (Example 11.72 that their complementary events A' and B' are also independent. Similarly, events A and B' and also events A' and B are independent. This means that if $P(A \cap B) = P(A) \,.\, P(B)$ then the following relations are also satisfied:

$$P(A' \cap B') = P(A') \,.\, P(B')$$
$$P(A \cap B') = P(A) \,.\, P(B')$$
$$P(A' \cap B) = P(A') \,.\, P(B).$$

Thus, we may replace any one or both the events by their complements A', B' on both sides of (11.18.2) and still get a true result.

Note: If A and B are *two* independent events then (11.18.2) holds. Conversely, if this relation holds, the events must be independent (in the sense that any combination of A or A' with B or B' will be independent). The relation (11.18.2) is therefore called a 'necessary and sufficient' condition for independence of two events.

Pairwise Independent Events

Several events $A_1, A_2, ..., A_n$ are said to be "*pairwise independent*", if every pair of these events are independent; i.e.

$$P(A_i \cap A_j) = P(A_i) \,.\, P(A_j) \tag{11.18.3}$$

for all values of $i, j = 1, 2, ..., n, (i \neq j)$.

For example, 3 events A, B, C will be said to be 'pairwise independent' if the following relations hold:

$$P(A \cap B) = P(A) \,.\, P(B)$$
$$P(A \cap C) = P(A) \,.\, P(C)$$
$$P(B \cap C) = P(B) \,.\, P(C)$$

Mutually Independent Events

Several events $A_1, A_2, ... A_n$ are said to be "*mutually independent*" (or simply *independent*), if the probability of the joint occurrence of any number of these events is equal to the product of their probabilities.

For example, 3 events A, B, C will be said to be 'independent' if every one of the following relations holds:

$$P(A \cap B) = P(A) \,.\, P(B)$$
$$P(A \cap C) = P(A) \,.\, P(C)$$
$$P(B \cap C) = P(B) \,.\, P(C) \quad (11.18.4)$$
and
$$P(A \cap B \cap C) = P(A) \,.\, P(B) \,.\, P(C)$$

In general, for the mutual independence of n events, $2^n - n - 1$ such conditions must be satisfied.

It may be noted here that if 3 or more events $A_1, A_2, ..., A_n$ are mutually independent, then the relation

$$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)\, P(A_2), ...\, P(A_n) \quad (11.18.5)$$

obviously holds (Besides some other relations must also hold).

If events $A_1, A_2, ..., A_n$ are independent, then their complements $A_1', A_2', ..., A_n'$ are also independent. In particular

$$P(A_1' \cap A_2' \cap ... \cap A_n') = P(A_1')\, P(A_2'), ...\, P(A_n') \quad (11.18.6)$$

Note:

1. If n events are mutually independent, they are necessarily pairwise independent. The converse is not true. Pairwise independent events may not be mutually independent.
2. If 3 or more events are independent, then the relation (11.18.5) must hold. But the converse is not true in general. In other words, if only (11.18.5) holds, the events $A_1, A_2, ... A_n$ ($n \geq 3$) may not be independent. This relation is therefore a necessary, but not sufficient, condition for independence of *more than two* events.
3. If n events A_i are independent, then any combination of these events, taken any number at a time, is also independent; e.g. events A_2, A_4 and A_5 will be independent.
4. If n events A_i are independent and we replace some or all of them by their complements, then any combination of the new group of events, taken any number at a time, will also be independent; e.g. events A_1' A_3, A_5' and A_7' will be mutually independent. In particular, the n complementary events A_i' will also be independent.

Example 11.71 *A fair coin is tossed 3 times in succession. Show that the events 'first toss gives a head' and 'second toss gives a head' are independent.*

Solution The sample space $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ is equiprobable, and contains 8 sample points.

Let A and B denote respectively the events 'first toss gives a head' and 'second toss gives a head'. Then

$$P(A) = P(\{HHH, HHT, HTH, HTT\}) = \frac{4}{8} = \frac{1}{2}$$

$$P(B) = P(\{HHH, HHT, THH, THT\}) = \frac{4}{8} = \frac{1}{2}$$

$$P(A \cap B) = P(\{HHH, HHT\}) = \frac{2}{8} = \frac{1}{4}$$

We find that $\quad P(A) \,.\, P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(A \cap B)$

So, A and B are independent events.

Example 11.72 *Let A and B be two independent events. Then show that (i) A and B^c, (ii) A^c and B^c are also independent.* [C.U., B.Com. (Hons) '81]

Solution Since A and B are independent events, we have

$$P(A \cap B) = P(A) \,.\, P(B) \quad (1)$$

Here A^c and B^c denote the complements of A and B respectively.

(i)
$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B), && \text{by (11.14.11)} \\ &= P(A) - P(A)\,.\,P(B), && \text{by (1) above} \\ &= P(A)\,.\,\{1 - P(B)\} \\ &= P(A)\,.\,P(B^c), && \text{by (11.14.6)} \end{aligned}$$

Since $P(A \cap B^c) = P(A)\,.\,P(B^c)$, the events A and B^c are independent.

(ii)
$$\begin{aligned} P(A^c \cap B^c) &= P(A \cup B)^c, \text{ by De Morgan's Law} \\ &= 1 - P(A \cup B), \\ &= 1 - [P(A) + P(B) - P(A \cap B)], && \text{by (11.14.13)} \\ &= 1 - P(A) - P(B) + P(A)\,.\,P(B), && \text{by (1)} \\ &= [1 - P(A)]\,[1 - P(B)] \\ &= P(A^c)\,.\,P(B^c) \end{aligned}$$

Thus events A^c and B^c are independent.

Example 11.73 *(a) $A_1, A_2, \ldots A_n$ are independent events such that $P(A_i) = 1 - q_i$, $i = 1, 2, \ldots n$. Prove that*

$$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - q_1 q_2 \cdots q_n.$$

(b) In supplies of 3 components, viz. base, neck and switch, for an electric lamp, the percentages of defectives on a day were 5, 20 and 10 respectively. An assembled lamp is considered defective if at least one of the 3 components is defective. If components are selected randomly, what is the probability that an assembled lamp would be defective? [W.B.H.S. '79]

Solution

(a) Since $A_1, A_2, \ldots A_n$ are independent events, their complements $A_1', A_2', \ldots A_n'$ are also independent, having probabilities $P(A_i') = 1 - P(A_i) = q_i$ (given, $i = 1, 2 \ldots n$. So,

$$\begin{aligned} P(A_1' \cap A_2' \cap \ldots \cap A_n') &= P(A_1')\,.\,P(A_2'), \ldots P(A_n') \\ &= q_1 q_2 \cdots q_n \end{aligned}$$

$$\begin{aligned} \therefore \quad P\left(\bigcap_{i=1}^{n} A_i\right) &= P(A_1 \cup A_2 \cup \ldots \cup A_n) \\ &= 1 - P(A_1 \cup A_2 \cup \ldots \cup A_n)', \text{ since } P(A) = 1 - P(A') \\ &= 1 - P(A_1' \cap A_2' \cap \ldots \cap A_n'), \text{ by De Morgan's law} \\ &= 1 - q_1 q_2 \cdots q_n \end{aligned}$$

(b) Let A, B, C denote the events 'base is defective', 'neck is defective', 'switch is defective' respectively. Then (given)

$$P(A) = 5\% = \frac{1}{20}, \; P(B) = 20\% = \frac{1}{5}, \; P(C) = 10\% = \frac{1}{10}.$$

The probabilities of complementary events are

$$P(A') = \frac{19}{20}, \; P(B') = \frac{4}{5}, \; P(C') = \frac{9}{10}.$$

We assume that the events A, B, C are independent.

$\therefore$ Probability that the lamp is defective

$$= \text{Probability that at least one component is defective}$$
$$= P(A \cup B \cup C) = P(A' \cap B' \cap C')'$$
$$= 1 - P(A' \cap B' \cap C')$$
$$= 1 - P(A') \,.\, P(B') \,.\, P(C'), \text{ since } A, B, C \text{ are assumed independent}$$
$$= 1 - \frac{19}{20} \times \frac{4}{5} \times \frac{9}{10}$$
$$= \frac{79}{250} = 0.316$$

Comparison of Classical Theory and Axiomatic Theory

(i) In the classical theory, all theorems and results are obtianed by logical arguments. In the axiomatic theory, all results are derived from the axioms by using the mathematical properties of sets.

(ii) The classical theory is based upon the concept of "equally likely cases" when the number of possible outcomes is only finite. The axiomatic theory is quite general and embraces all cases whether equally likely or not and irrespective of whether the number of possible outcomes is finite or infinite.

(iii) The classical theory defines "event" simply as a phenomenon which may arise in the performance of the random experiment. The axiomatic theory defines the "event" strictly according to mathematical principles as a 'set', which is in effect a subset of the universal set of all possible outcomes, called the Sample Space.

(iv) In the classical theory, "probability" is defined as a ratio (i.e. fraction) of two positive whole numbers (the numerator may be zero) showing the number of cases favourable to the event and the total number of all possible outcomes which are equally likely. In the axiomatic theory, "probability" is simply a non-negative number associated with the event, i.e. probability is a set-function obeying the three axioms.

(v) The 'addition theorem' of classical theory is not derived in the axiomatic theory, but simply accepted as a rule by Axiom 3 (see 11.14.4). The 'multiplication theorem' is derived as a rule (see 11.17.2) which follows from the definition of conditional probability in axiomatic theory.

(vi) The concepts of "conditional probability" and "independent events" are introduced in the classical theory by logical arguments, whereas in the axiomatic theory these are defined by mathematical statements.

11.19 RANDOM VARIABLE

Let S be a sample space of some given random experiment. It has been observed that the outcomes (i.e. sample points of S) are not always numbers. We may however assign a real number to each sample point according to some definite rule. Such an assignment given us a "*function defined on the sample space S*". This function is called a *random variable* (or *stochastic variable*).

Random Variable X may be defined as a function which assigns a real number $X(e)$ to each sample point e of a given sample space S.

Illustration 1.
In the random experiment of tossing 2 coins, let the sample space be $S = \{HH, HT, TH, TT\}$. If X is the random variable denoting the 'number of heads', then we have assigned a number to each sample point as follows:

$$X(HH) = 2, \quad X(HT) = 1, \quad X(TH) = 1, \quad X(TT) = 0$$

(**Note:** Many other random variables could also be proposed on this sample space, e.g. the square of the number of tails, the difference of the numbers of heads and tails.)

Illustration 2.
Let $S = \{1, 2, 3, 4, 5, 6\}$ be a sample space of the random experiment of throwing a die. The sample points are 1, 2, 3, 4, 5, 6. We introduce the random variable X = number of points obtained when the die is thrown. Then

$$X(1) = 1, \; X(2) = 2, \quad X(3) = 3, \quad X(4) = 4, \quad X(5) = 5, \quad X(6) = 6$$

Suppose, we introduce another random variable Y on the same sample space, Y = (Number of points less 3)^2. Then

$$Y(1) = (1-3)^2 = 4 \qquad Y(4) = (4-3)^2 = 1$$
$$Y(2) = (2-3)^2 = 1 \qquad Y(5) = (5-3)^2 = 4$$
$$Y(3) = (3-3)^2 = 0 \qquad Y(6) = (6-3)^2 = 9$$

These are shown in the table below:

Sample point	1	2	3	4	5	6
Random variable Y	4	1	0	1	4	9

A random variable which assumes a finite number or a countably infinite number of values is called a *Discrete random variable.* If the random variable assumes an uncountably infinite number of values, it is called a *Continuous random variable.*

Illustration 3.
The random variable X in Illustration 1 takes only 3 values 0, 1, 2. In *Illustration 2*, X takes 6 values 1, 2, ... 6, and Y takes 4 values 0, 1, 4, 9. These are all discrete random variables with a finite number of values.

Discrete Probability Distribution

Let X be a discrete random variable which can assume the values $x_1, x_2, x_3, \ldots$ (arranged in an increasing order of magnitude) with probabilities $p_1, p_2, p_3 \ldots$ respectively ($\Sigma p_i = 1$). The specification of the set of values x_i together with their probabilities p_i ($i = 1, 2, 3, \ldots$) defines the *discrete probability distribution* of X.

Let us write $f(x)$ to denote the probability that X takes a specified value x.

$$f(x) = P(X = x) \tag{11.19.1}$$

The function $f(x)$ is called the *probability mass function* (p.m.f.), or simply *probability function,* of the discrete random variable X. It satisfies two conditions

$$\text{(i) } f(x) \geq 0, \quad \text{(ii) } \Sigma f(x) = 1 \tag{11.19.2}$$

the summation in (ii) being taken over all values of $x = x_1, x_2, x_3, \ldots$

$$f(x) = p_i \text{ when } x = x_i (i = 1, 2, \ldots) \tag{11.19.3}$$
$$= 0 \quad \text{otherwise}$$

Example 11.74 *Find the probability distributions of the random variables in Illustrations 1 and 2.*

Solution

(i) Assuming that the coin is unbiased, the sample space is finite and equiprobable.

$$P(X = 0) = P(\{TT\}) = \frac{1}{4}$$

$$P(X = 1) = P(\{HT, TH\}) = \frac{2}{4} = \frac{1}{2}$$

$$P(X = 2) = P(\{HH\}) = \frac{1}{4}$$

The probability distribution of X is

x	0	1	2	Total
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

(ii) In *Illustration* 2, assuming that the die is unbiased, the probability distribution of X is

x	1	2	3	4	5	6	Total
$f(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Again, $P(Y = 0) = P(\{3\}) = \frac{1}{6}$

$$P(Y = 1) = P(\{2, 4\}) = \frac{2}{6} = \frac{1}{3}$$

$$P(Y = 4) = P(\{1, 5\}) = \frac{2}{6} = \frac{1}{3}$$

$$P(Y = 9) = P(\{6\}) = \frac{1}{6}$$

The probabiity distribution of Y is therefore

y	0	1	4	9	Total
$f(y)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	1

Note the distinction between capital letters X, Y which denote the random variables, and small letters x, y which denote any arbitrary values of these random variables.

Example 11.75 *Find the probability distribution of the number of tails when a coin is thrown repeatedly until the first head appears.*

Solution Here $S = \{H, TH, TTH, TTTH, \ldots\}$

Sample point	H	TH	TTH	$TTTH$	...
X	0	1	2	3	...

Assuming that the coin is unbiased,

$$P(X = 0) = P(\{H\}) = \frac{1}{2}$$

$$P(X = 1) = P(\{TH\}) = \left(\frac{1}{2}\right)^2$$

$$P(X = 2) = P(\{TTH\}) = \left(\frac{1}{2}\right)^3$$

$$P(X = 3) = P(\{TTTH\}) = \left(\frac{1}{2}\right)^4 \text{ etc.}$$

The probability distribution of X is

x	0	1	2	3	...	Total
$f(x)$	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$	...	1

Continuous Probability Distribution

If X is a continuous random variable, the number of possible values which it can assume is uncountably infinite, and hence the probability function cannot be defined in the same manner as for a discrete random varaible. In this case we introduce a function $f(x)$ such that it satisfies two conditions.

$$\text{(i) } f(x) \geq 0 \quad \text{(ii) } \int_{-\infty}^{\infty} f(x)\, dx = 1. \tag{11.19.4}$$

The probability that X lies between two specified values c and d, is defined as

$$P(c \leq X \leq d) = \int_c^d f(x)\, dx \tag{11.19.5}$$

It may be shown that this definition of probability satisfies all the Axioms of probability (page 409). The function $f(x)$ is called the *probability density function* (p.d.f.), or simply *density function,* of the continuous random variable X. Any function satisfying both the conditions (11.19.4) may be accepted as a density function.

Example 11.76 *(a) Is the following a probability density function?*

$$f(x) = \begin{cases} 2x, & 0 < x \leq 1 \\ 4 - 2x, & 1 < x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

(b) If the random variable X has the probability density function

$$f(x) = \begin{cases} \frac{1}{4}, & -2 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

obtain $P\{(2x + 3) > 5\}$. (Here P denotes probability)

[C.U., B.Sc. (Econ) '81 (New)]

Solution

(a) If the function $f(x)$ is to be a p.d.f. it must satisfy the two conditions (i) and (ii) at (11.19.4). The function $f(x)$ defined here clearly satisfies (i). It is not negative at any value of x.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^{\infty} f(x)dx$$

$$= 0 + \int_0^1 (2x)dx + \int_1^2 (4 - 2x)dx + 0$$

(the first and the last integrals being zero, since $f(x)$ itself is defined to be zero in those two intervals)

$$= \left[x^2\right]_0^1 + \left[4x - x^2\right]_1^2.$$
$$= 1 + 1 = 2$$

Since condition (ii) is not satisfied, the given function $f(x)$ is not a probability density function.

(b) The event $(2x + 3) > 5 \Rightarrow 2x > 2 \Rightarrow x > 1$. Therefore

$$P\{(2x+3) > 5\} = P\{x > 1\}$$
$$= \int_1^2 f(x)\, dx, \text{ since } f(x) = 0 \text{ for } x > 2.$$
$$= \int_1^2 \left(\frac{1}{4}\right) dx = \frac{1}{4}$$

[*Ans.* No.; $\frac{1}{4}$]

11.20 CUMULATIVE DISTRIBUTION FUNCTION (C.D.F.)

The probability that a random variable X takes a value less than or equal to a specified value x is a function of x, and is usually denoted by $F(x)$. This function is called *cumulative distribution function* (c.d.f.), or simply *distribution function*, of the random variable X.

$$F(x) = P(X \le x) \tag{11.20.1}$$

Since $F(x)$ represents probability, it must lie between 0 and 1.

$$0 \le F(x) \le 1 \tag{11.20.2}$$

For the discrete probability distribution of X, which assumes a finite number of values $x_1, x_2, \dots x_n$ with probabilities $p_1, p_2 \dots p_n$ respectively, the c.d.f. is given as follows:

$$\begin{aligned} F(x) &= 0 && \text{when } x < x_1 \\ &= p_1 && \text{when } x_1 \le x < x_2 \\ &= p_1 + p_2 && \text{when } x_2 \le x < x_3 \\ &= p_1 + p_2 + p_3 && \text{when } x_3 \le x < x_4 \\ &\dots \quad \dots \\ &= p_1 + p_2 + \dots + p_n = 1 && \text{when } x_n \le x \end{aligned} \tag{11.20.3}$$

Thus, the probability that a continuous random variable takes any specified value is zero. As a consequence,

$$P(c \le X \le d) = P(c \le X < d)$$
$$= P(c < X \le d) = P(c < X < d).$$

Example 11.77 *An unbiased coin is thrown three times. If the random variable X denotes the number of heads obtained, find the cumulative distribution function (c.d.f.) of X.*

Solution It can be easily shown that the probabilities of obtaining 0 head, 1 head, 2 heads, 3 heads are respectively $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$. The probability distribution of X is therefore

x	0	1	2	3	Total
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

From this table, we can find the probabilities of obtaining 0 or less head, 1 or less head, 2 or less heads, 3 or less heads, as follows:

$$F(0) = P(X \le 0) = P(X = 0) = \frac{1}{8}$$
$$F(1) = P(X \le 1)$$
$$= P(X = 0) + P(X = 1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$
$$F(2) = P(X \le 2)$$
$$= P(X = 0) + P(X = 1) + P(X = 2)$$
$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$$
$$F(3) = P(X \le 3)$$
$$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$
$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

Thus we have

x	0	1	2	3
$F(x)$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{7}{8}$	1

In general, if x denotes any arbitrary real number, the probability that the number of heads (X) is x or less can be given as

$$F(x) = 0, \qquad \text{when } x < 0$$

because we cannot have less than 0 head.

$$F(x) = \frac{1}{8}, \qquad \text{when } 0 \le x < 1$$

For example $F(0.7) = \frac{1}{8}$, because the probability of having 0.7 or less head is the same as that of having 0 head.

$$F(x) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}, \text{ when } 1 \le x < 2$$

For example, $F(11.34) = \frac{1}{2}$, because the probability of having 11.34 or less heads is the same as that of having 0 or 1 head. Similarly,

$$F(x) = \frac{1}{8}+\frac{3}{8}+\frac{3}{8}=\frac{7}{8}, \qquad \text{when } 2 \le x < 3$$

and $$F(x) = \frac{1}{8}+\frac{3}{8}+\frac{3}{8}+\frac{1}{8} = 1, \qquad \text{when } 3 \le x$$

For example $F(8.75) = 1$, because the probability of having 8.75 heads or less is the same as that of having 0, 1, 2, 3 heads.

The c.d.f. of X is thus given as follows:

$$\begin{aligned} F(x) &= 0, && \text{when } x < 0 \\ &= \frac{1}{8}, && \text{when } 0 \le x < 1 \\ &= \frac{1}{2}, && \text{when } 1 \le x < 2 \\ &= \frac{7}{8}, && \text{when } 2 \le x < 3 \\ &= 1, && \text{when } 3 \le x \end{aligned}$$

Example 11.78 *The p.d.f. of a continuous random variable is*

$$y = k(x-1)(2-x); \qquad (1 \le x \le 2)$$

Determine

(i) the value of the constant k;

(ii) the cumulative distribution function;

(iii) the probability that X is less than $\frac{5}{4}$;

(iv) the probability that X is greater than $\frac{3}{2}$;

(v) the probability that X lies between $\frac{5}{4}$ *and* $\frac{3}{2}$.

Solution $y = k(x-1)(2-x) = k(3x - x^2 - 2)$

(i) Using the second relation of (11.19.4),

$$\begin{aligned} 1 &= \int_1^2 k(3x - x^2 - 2)\, dx \\ &= k \cdot \left[\frac{3x^2}{2} - \frac{x^3}{3} - 2x\right]_1^2 \\ &= k\left[\left(6 - \frac{8}{3} - 4\right) - \left(\frac{3}{2} - \frac{1}{3} - 2\right)\right] = \frac{k}{6} \end{aligned}$$

Thus, $\frac{k}{6} = 1$; i.e. $k = 6$

The complete p.d.f. is therefore $y = 6(3x - x^2 - 2)$

(ii) $F(x) = \int_1^x 6(3t - t^2 - 2)\, dt$, by (11.20.5)

$$= 6\left[\frac{3t^2}{2} - \frac{t^3}{3} - 2t\right]^x$$

$$= 6\left[\left(\frac{3x^2}{2} - \frac{x^3}{3} - 2x\right) - \left(\frac{3}{2} - \frac{1}{3} - 2\right)\right]$$
$$= 5 - 12x + 9x^2 - 2x^3$$

(iii) Probability that X is less than $\frac{5}{4}$

$$= P\left(X \leq \frac{5}{4}\right) = F\left(\frac{5}{4}\right), \text{ by (11.20.1)}$$
$$= 5 - 12 \times \frac{5}{4} + 9 \times \frac{25}{16} - 2\ \frac{125}{64}$$
$$= \frac{5}{32}$$

(iv) Probability that X is less than $\frac{3}{2}$

$$= P\left(X < \frac{3}{2}\right) = P\left(X \leq \frac{3}{2}\right) = F\left(\frac{3}{2}\right)$$
$$= 5 - 12 \times \frac{3}{2} + 9 \times \frac{9}{4} - 2 \times \frac{27}{8}$$
$$= \frac{1}{2}$$

$\therefore$ Probability that X is greater than $\frac{3}{2}$

$$= P\left(X > \frac{3}{2}\right) = 1 - P\left(X \leq \frac{3}{2}\right)$$
$$= 1 - F\left(\frac{3}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

(v) Probability that X lies between $\frac{5}{4}$ and $\frac{3}{2}$

$$= P\left(\frac{5}{4} \leq X \leq \frac{3}{2}\right) = F\left(\frac{3}{2}\right) - F\left(\frac{5}{4}\right)$$
$$= \frac{1}{2} - \frac{5}{32} = \frac{11}{32}$$

11.21 JOINT DISTRIBUTION OF TWO VARIABLES (DISCRETE)

Let S be a sample space of some given random experiment. We may assign two real numbers $X(e)$ and $Y(e)$ to each sample point e of S according to some given rules (Section 11.19). We then have two random variables X and Y defined on the sample space S.

The possible pairs of values (x, y) assumed by the two random variables X and Y together with the probabilities for all the pairs of values, gives the *Joint Distribution* of X and Y, or *Bivariate Distribution.*

Example 11.79 *Four unbiased coins are tossed. If X and Y denote respectively the 'number of heads' and the 'longest run of heads', construct the joint distribution of X and Y.*

Solution The sample space $S = \{e_1, e_2, \ldots e_{16}\}$ consisting of the following 16 sample points is equiprobable. Hence, the probability of each elementary event is $\frac{1}{16}$ (p. 404). The values (x, y) assumed by X and Y are shown against each sample point.

Sample point	x	y
e_1 = *HHHH*	4	4
e_2 = *HHHT*	3	3
e_3 = *HHTH*	3	2
e_4 = *HHTT*	2	2
e_5 = *HTHH*	3	2
e_6 = *HTHT*	2	1
e_7 = *HTTH*	2	1
e_8 = *HTTT*	1	1
e_9 = *THHH*	3	3
e_{10} = *THHT*	2	2
e_{11} = *THTH*	2	1
e_{12} = *THTT*	1	1
e_{13} = *TTHH*	2	2
e_{14} = *TTHT*	1	1
e_{15} = *TTTH*	1	1
e_{16} = *TTTT*	0	0

The distinct pairs of values (x, y) when systematically arranged, show the following joint distribution of X and Y:

(x, y)	*Probability*
0, 0	$\frac{1}{16}$
1, 1	$\frac{4}{16}$
2, 1	$\frac{3}{16}$
2, 2	$\frac{3}{16}$
3, 2	$\frac{2}{16}$
3, 3	$\frac{2}{16}$
4, 4	$\frac{1}{16}$
Total	1

However, it is preferable to show the values x and y in a two-way table, with the probabilities written in the cells as in the table below:

Joint Distribution of X and Y

x \ y	0	1	2	3	4	Total
0	$\frac{1}{16}$					$\frac{1}{16}$
1		$\frac{4}{16}$				$\frac{4}{16}$
2		$\frac{3}{16}$	$\frac{3}{16}$			$\frac{6}{16}$
3			$\frac{2}{16}$	$\frac{2}{16}$		$\frac{4}{16}$
4					$\frac{1}{16}$	$\frac{1}{16}$
Total	$\frac{1}{16}$	$\frac{7}{16}$	$\frac{5}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	1

EXERCISES

1. Define and illustrate the following terms—Mutually exclusive events, Exhaustive set of events, Independent events, and Complementary events.
[C.U., M. Com. '79; B.Com. (Hons) '80]
2. Give the *classical definition* of probability. What are its limitations?
[C.U., B.Com. (Hons) '82; I.C.W.A., June '81; W.B.H.S. '82]
3. State and prove the *Addition Theorem* of probability for two mutually exclusive events. What modification would you make if the events are *not* mutually exclusive?
4. With usual notations, prove that
$$P(A + B) = P(A) + P(B) - P(AB).$$
[W.B.H.S. '78; I.C.W.A., June '81]
5. (a) Establish the relation
$$P(A + B) \leq P(A) + P(B).$$
When does the sign of equality hold? [W.B.H.S. '78]
(b) If events A and B are not mutually exclusive, show that
$$P(AB) \geq P(A) + P(B) - 1$$
6. What is meant by *Compound Event* in probability ? State and prove the Theorem of Compound Probability.
[C.U., B.Com. (Hons) '81; B.Sc. (Econ) '81]
7. State and prove the *Multiplication Theorem* of probability. How is the result modified when the events are *independent*? [C.U., M.Com. '80]
8. Give the different definitions of probability and state their limitations if any.
[C.U., M.Com. '81]
9. State whether each of the following statements is *True* or *False*:
(a) The probability of an event cannot be zero.
(b) The probability of a complementary event must not exceed one.
(c) The probability of the simultaneous occurrence of two events can never exceed the sum of the probabilities of these events.
(d) The probability of occurrence of at least one of two events is less than the probability of occurrence of each of these events.
(e) If two events are mutually exclusive, their complements are also mutually exclusive.
(f) Two events are said to be "independent", if the occurrence of one of them is unaffected by the occurrence of the other event.
(g) The conditional probability of an event, given another event, can never be less than the probability of the joint occurrence of these events.
(h) Two mutually exclusive events must be independent.
(i) The mathematical expectation of the product of two random variables is given by he product of their mathematical expectations.
(j) In a Bernoullian series of trials, the probability of success must remain a constant in each trial.
10. Examine the following relations and state which of them are *incorrect* (A and B are arbitrary events):

(a) $P(A + B) = P(A) + P(B)$
(b) $P(AB) \le P(A) + P(B)$
(c) $P(A + B) \ge P(A)$
(d) $P(A \text{ and } B) \le P(A \text{ or } B)$
(e) $P(B) \le P(A \text{ and } B)$
(f) $P(A \cup B) + P(A \cap B) = P(A) + P(B)$
(g) $P(A \cap B) = P(A \cup B) + P(A \cup \bar{B}) + P(\bar{A} \cup B)$
(h) $P\left(\frac{A}{B}\right) . P\left(\frac{B}{A}\right) = 1$
(i) $P\left(\frac{A}{B}\right) \le \frac{P(A)}{P(B)}$
(j) $P\left(\frac{\bar{A}}{B}\right) = 1 - P\left(\frac{A}{B}\right)$

11. State which of the following statements are *Correct* and which are *Wrong*:
(a) For any event A, $P(A)$ is a non-negative real number.
(b) If A and B are mutually exclusive events, then $P(A) + P(B) = 1$.
(c) If event A imples event B, then $P(A) \le P(B)$.
(d) If $P\left(\frac{A}{B}\right) = P(A)$, then $P\left(\frac{B}{A}\right) = P(A)$.
(e) If $P(A) \ne 0$ and $P(B) \ne 0$, then $\frac{P(A)}{P(B)} = \frac{P\left(\frac{A}{B}\right)}{P\left(\frac{B}{A}\right)}$
(f) If A and B are independent events, then $P(AB) = P(A) . P(B)$.
(g) If events A and B are independent, then A^c and B^c are dependent.
(h) If A and B are independent events, then A and B are also independent.
(i) If $P(ABC) = P(A) . P(B) . P(C)$, then events A, B, C are independent.
(j) The probability mass function $f(x)$ of a discrete random variable X may exceed 1.

12. Let A be the event that a person is a male and B the event that a person is more than 6 feet tall. Give, in words, the events whose probabilities are represented by
(i) $P(A \text{ or } B)$
(ii) $P(A \text{ or } B) - P(A \text{ and } B)$,
(iii) $1 - P(A)$. *[W.B.H.S. '80]*

13. In each of the following cases, pick out the correct alternative:
(a) If an event cannot happen, the probability of the event will be: $+1, -1, 0$, none of these. *[C.A., Nov. '81]*
(b) When two perfect coins are tossed simultaneously, the probability of getting at least one head is: $\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, 1$. *[C.A., May '81]*

(c) A dice is thrown two times, and the sum of numbers on the faces up is noted. The probability of this sum being 11 is: $\frac{1}{6}$, $\frac{1}{36}$, $\frac{1}{18}$, none of these. *[C.A., Nov. '82]*

(d) Out of 120 tickets numbered consecutively from 1 to 120, one is drawn at random. What is the probability of getting a number which is a multiple of 5? $\frac{1}{24}, \frac{1}{8}, \frac{1}{5}, \frac{1}{16}$. *[C.A., May '81]*

(e) If two cards are drawn at random from a good deck, the probability that both the cards are of the same colour is: $\frac{1}{4}, \frac{1}{2}$, 1, none of these.

(f) If A_1, A_2, A_3 are equally likely, mutually exclusive and exhaustive, then $P(A_1)$ equals: 1, 0, $\frac{1}{2}$, $\frac{1}{3}$. *[W.B.H.S. '82]*

(g) If $P(A) = 0.2$, $P(B) = 0.1$, and $P(C) = 0.3$, and A, B, C are mutually independent events, then the probabiilty that all the three events do not happen simultaneously is: .006, 0.504, 0.994, none of these.

(h) The probability of having at least one 'six' from 3 throws of a perfect dice is: $\left(\frac{5}{6}\right)^3$, $1 - \left(\frac{5}{6}\right)^3$, $\left(\frac{1}{6}\right)^3$, none of these. *[W.B.H.S. '79]*

(i) The probabilities of occurrence of events A and B are respectively 0.25 and 0.50. Both the events occur simultaneously with probability 0.14. The probability that neither A nor B occurs is: 0.39, 0.25, 0.11, none of these.

14. For two events A and B, let $P(A) = 0.4$, $P(A + B) = 0.7$ and $P(B) = p$. For what value of p are A and B (i) mutually exclusive, (ii) independent?

15. Three perfect coins are tossed together. What is the probability of getting at least one head? *[C.A., Nov. '81]*

16. If four unbiased coins are tossed, find the probability that there should be two tails. *[C.U., B.Com. (Hons) '82]*

17. A card is drawn at random from a well-shuffled pack of cards. What is the probability that it is a heart or a queen ? *[C.A. May '82]*

18. In a single cast with two dice find the chance of throwing 7 (i.e. for throwing two numbers whose sum is 7). *[B.U., B.Com. '76]*

19. An ordinary die is tossed twice and the difference between the numbers of spots turned up is noted. Find the probability of a difference of 3. *[C.U., B.A. (Econ) '66]*

20. Three balls are drawn at random from a bag containing 6 blue and 4 red balls. What is the chance that two balls are blue and one is red? *[B.U., B.Com. '77]*

21. An urn contains 8 white and 3 red balls. If 2 balls are drawn at random, find the probability that (i) both are white, (ii) both are red, (iii) one is of each colour. *[C.U., B.A. (Econ) '73]*

22. Five persons A, B, C, D, E speak at a meeting. What is the probability that A speaks immediately before B?

23. The nine digits 1, 2, 3, ..., 9 are arranged in random order to form a nine-digit number. Find the probability that 1, 2 and 3 appear as neighbours in the order mentioned. *[C.U., B.A. (Econ) '72]*
24. What is the chance that a leap year, selected at random, will contain 53 Sundays? *[C.U., B.Sc. '77]*
25. Three cards are drawn at random one after another from a full pack of Playing Cards. What is the probability that (i) the first two are spades and the third is a heart, (ii) two are spades and one is a heart?
26. A four-digit number is formed by the digits 1, 2, 3, 4 with no repetition. Find the probability that the number is (i) odd, (ii) divisible by 4.
27. Four dice are thrown. Find the probability that the sum ofthe numbers appearing will be 18.
28. There are 4 persons in a company. Find the probability that (i) all of them have different brithdays, (ii) at least 2 of them have the same birthday, (iii) exactly 2 of them have the same birthday (Assume 1 year = 365 days).
29. (i) If $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{5}$, $P(A + B) = \frac{1}{2}$, find $P(AB)$ $P\left(\frac{A}{B}\right)$ and $P\left(\frac{B}{A}\right)$.

    (ii) If $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{5}$, $P(AB) = \frac{1}{3}$, find $P(A + B)$, $P(\overline{A}\overline{B})$, $P(\overline{A} + \overline{B})$ and $P(A\overline{B})$.

    (iii) If A and B are independent events, and $P(A) = \frac{2}{3}$, $P(B) = \frac{3}{5}$, find $P(A + B)$, $P\left(\frac{\overline{A}}{B}\right)$ and $P\left(\overline{A}B\right)$.
30. Given $P(A\overline{B}) = \frac{1}{3}$, $P(A + B) = \frac{2}{3}$, find $P(B)$. What is $P(A)$, if $P(AB) = \frac{1}{6}$? *[C.U., B.Sc. '73]*
31. The probability that a contractor will get a plumbing contract is $\frac{2}{3}$, and the probability that he will not get an electric contract is $\frac{5}{9}$. If the probability of getting at least one contract is $\frac{4}{5}$, what is the probability that he will get both the contracts? *[C.A., May '79]*
32. From a set of 1000 cards serially numbered 1, 2, 3, ..., 1000, one card is drawn at random. Find the probability that the number found is a the multiple of (i) 12 or 18, (ii) 12 and 18.
33. The probability that Asok can solve a problem in Business Statistics is $\frac{4}{5}$, that Amal can solve it is $\frac{2}{3}$, and that Abdul can solve it is $\frac{3}{7}$. If all of them try independently, find the probability that the problem will be solved. *[C.U., B.Com. (Hons) '82]*

34. The odds against student X solving a Business Statistics problem are 8 to 6, and the odds in favour of student Y solving the same problem are 14 to 16. (i) What is the chance that the problem will be solved if they both try, independently of each other? (ii) What is the probability that neither solves the problem? *[C.A., Nov. '79]*

35. Two sets of candidates are competing for the position on the Board of Directors of a company. The probabilities that the first and second sets will win are 0.6 and 0.4 respectively. If the first set wins, the probability of introducing a new product is 0.8, and the corresponding probability if the second set wins is 0.3. What is the probability that the new product will be introduced? *[C.A., Nov. '78]*

36. Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, and 1 girl and 3 boys. One child is selected at random from each group. Find the chance that the selected group consists of 1 girl and 2 boys. *[C.U., B.Sc. '77]*

37. The probability that a teacher will give a surprise test during any class meeting is $\frac{1}{5}$. If a student is absent on two days, what is the probability that he will miss at least one test? *[C.U., B.Sc. '76]*

38. Three lots contain respectively 10%, 20% and 25% defective articles. One article is drawn at random from each lot. What is the probability that among them there is (i) exactly one defective, (ii) at least one defective?

39. A candidate is selected for interview for 3 posts. For the first post there are 3 candidates, for the second post there are 4, and for the third there are 2. What are the chances of getting at least one post? *[C.A., May '80]*

40. An urn contains 2 white and 2 black balls. Balls are drawn successively at random without replacement. What is the probability that a black ball appears (i) for the first time in the 3rd drawing, (ii) for the 2nd time in the 4th drawing ? *[C.U., B.Sc. '76]*

41. Five different letters are put inside 5 addressed envelopes by an illiterate servant. What is the probability that only 2 letters are placed in the correct envelopes ?

42. There are two identical boxes containing respectively 4 white and 3 red balls, and 3 white and 7 red balls. A box is chosen at random and a ball is drawn from it. Find the probability that the ball is white. *[C.U., B.Sc. (Math) '67]*

43. What is the chance of getting at least one defective item, if 3 items are drawn randomly from a lot containing 10 items, of which just 2 are defective ? *[W.B.H.S., '78]*

44. A lot contains 10 items of which 3 are defective. Three items are chosen from the lot at random one after another without replacement. Find the probability that all the three are defective. *[C.A., May '81]*

45. A packet of 10 electronic components is known to include 3 defectives. If 4 components are randomly chosen and tested, what is the probability of finding among them not more than one defective? *[C.U., B.Com. (Hons) '80; M.Com. '79]*

46. A bag contains 7 red and 5 white balls. 4 balls are drawn at random. What is the probability that (i) all of them are red; (ii) 2 of them are red and 2 white? *[C.U., M.Com. '72]*

47. A subcommittee of 6 members is to be formed out of a group consisting of 7 men and 4 ladies. Calculate the probability that the subcommittee will consist of (i) exactly 2 ladies, and (ii) at least 2 ladies. *[C.U., B.A. (Econ) '75]*
48. A bag contains 8 red balls and 5 white balls. Two successive draws of 3 balls are made without replacement. Find the probability that the first drawing will give 3 white balls and the second 3 red balls. *[C.A., May '78]*
49. A bag contains 8 white and 4 red balls. Two balls are first drawn at random and then without replacement another two balls are drawn at random. What is the probability of obtaining one white and one red ball in each drawing? *[C.U., M.Com. '80]*
50. In a bridge game, North and South have 9 spades between them. Find the probability that either East or West has no spades. (There are only 13 spades in a pack of 52 cards and each player has 13 cards. The players are designated by the positions they occupy, viz. North, South, East, West.) *[C.U., B.Sc. (Math) '68]*
51. A manufacturer supplies cheap clocks in lots of 50. A buyer, before taking a lot, tests a random sample of 5 clocks and if all are good, he accepts the lot. What is the probability that he accepts a lot containing 10 defective clocks? *[C.U., B.Com(Hons) '67]*
52. 5 cards are drawn from a pack of 52 well-shuffled cards. Find the probabiity that 4 are aces and 1 is a king. *[C.U., B.A. (Econ) '69]*
53. A bag contains 50 tickets numbered 1, 2, 3, ..., 50, of which 5 are drawn at random and arranged in ascending order of their numbers $x_1 < x_2 < ... < x_5$. What is the probability that $x_3 = 30$? *[C.U., B.Sc. (Math.) '66]*
54. A bag contains 5 red and 4 black balls. A ball is drawn at random from the bag and put into another bag which contains 3 red and 7 black balls. A ball is drawn randomly from the second bag. What is the probability that it is red? *[C.U., B.A.(Econ) '70]*
55. An experiment succeeds twice as often as it fails. What is the probability that in the next 6 trails there will be at least 4 successes?
56. If 20 dates are named at random, what is the probability that 3 of them will be Sundays?
57. A factory produces articles among which 20% are defective. If 5 articles are selected at random from a day's production, find the probability that there will be (i) exactly 2 defectives, (ii) not less than 2 defectives.
58. The incidence of occupational disease is such that on the average 20% of workers suffer from it. If 10 workers are selected at random, find the probability that (i) exactly 2 workers suffer from the disease, (ii) not more than 2 workers suffer from the disease. *[C.U., B.A. (Econ) '74]*
59. A marksman firing at a target hits the bulls-eye once in 3 shots on an average. If he fires 4 times, what is the probability of hitting the bulls-eye (i) twice, (ii) 3 times, (iii) not at all?
60. A bag contains 5 white and 2 black balls. The experiment of drawing a ball from the bag is performed 3 times. Calculate the probability of drawing 2 white and 1 black balls, if the ball is (i) replaced, (ii) not replaced, after each drawing.

61. A box contains 7 white and 5 black balls. If 3 balls are drawn simultaneously at random, what is the probability that they are not all of the same colour? Calculate the probability of the same event for the case where the balls are drawn in succession with replacement between drawings. *[C.U., M.Com. '81]*

62. Two dice are thrown n times in succession. What is the probability of obtaining doubel-six at least once? *[C.U., B.Sc. '75]*

63. (a) Define *mathematical expectation.* *[C.A., Nov. '82]*
    (b) An urn contains 7 white and 3 red balls. Two balls are drawn together at random from this urn. Find the mathematical expectation of the number of white balls drawn. *[C.U., B.A. (Econ) '78]*

64. A man draws 2 balls from a bag containing 3 white and 5 black balls. If he is to receive Rs 14 for every white ball which he draws and Rs 7 for every black ball, what is his expectation? *[C.U., B.Sc. '77]*

65. A bag contains 5 white and 7 black balls. Find the expectation of a man who is allowed to draw two balls from the bag and who is to receive one rupee for each black ball and two rupees for each white ball drawn. *[C.U., B.Com. (Hons) '82]*

66. If a person gains or loses an amount equal to the number appearing when a balanced die is rolled once, accordig to whether the number is even or odd, how much money can he expect per game in the long run? *[C.U., B.Sc. (Econ) '82]*

67. A and B play for a prize of Rs 99. The prize is to be won by a player who first throws a '3' with one die. A first throws, and if he fails B throws, and if he fails A again throws, and so on. Find their respective expectations. *[C.A., Nov. '81]*

68. If x is a discrete random variable and k an arbitrary constant, show that

$$E(kX) = k \,.\, E(X).$$

69. (a) For two jointly distributed random variables X and Y, prove that

$$E(X + Y) = E(X) + E(Y).$$

    (b) Three coins whose faces are marked 1 and 2, are tossed. What is the expectation of the total value of numbers on their faces? *[C.A., Nov. '82]*

70. (a) If X and Y are *independent* random variables, show that

$$E(XY) = E(X) \,.\, E(Y).$$

    (b) A number is chosen at random from the set 1, 2, 3, ..., 100 and another number is chosen at random from the set 1, 2, 3, ..., 50. What is the expected value of the product ? *[C.U., B.Com. (Hons) '81]*

71. Three boxes of the same appearance have the following proportions of white and black balls—Box I: 1 white and 2 black; Box II: 2 white and 1 black; Box III: 2 white and 2 black. One of the boxes is selected at random and one ball is drawn randomly from it. It turns out to be white. What is the probability that the third box is chosen? *[C.U., B.Sc. '79]*

72. The contents of three urns are as follows: 1 white, 2 black, 3 red balls; 2 white, 1 black, 1 red; 4 white, 5 black, 3 red. One urn is chosen at random and 2 balls are drawn. They happen to be white and red. What is the probability that they came from the second urn? *[C.U., B.Sc. (Math) '67]*

73. Let A, B, C be the subsets of a sample space S defined by
$S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 3, 5, 7\}$
$B = \{1, 2, 3, 4, 5\}$, $C = \{5, 6, 7, 8\}$.
Find the set (i) $A \cup B$, (ii) $A \cap B$, (iii) $A \cup B \cup C$, (iv) $(A \cup B)'$, (v) $(A \cap B)'$, (vi) $C'' \cap B'$, (vii) $A \cap B \cap C$.

74. Let $S = \{a, b, c, d, e\}$ be the Universal set and let $A = \{a, b, d\}$ and $B = \{b, d, e\}$ be two of its subsets. Find $(A \cap B)'$ and $(A \cup B)'$.
[C.U., B. Com. (Hons) '82]

75. Explain the terms—Sample space; Sample point; Event; Elementary and Composite events, as used in the axiomatic theory of probability.

76. Suppose that an unbiased coin is tossed 3 times in succession, and that one is interested to know the number of *Heads* that fall. Using this illustration, explain the terms—Sample space, Event, Random variable. *[I.C.W.A., June '82]*

77. Suppose that a random experiment consists of throwing a coin 3 times in succession. Write down the sample space S. Mention two mutually exclusive *events* of this sample space. *[C.U., B.Com. (Hons) '82]*

78. A biased coin is tossed 3 times. Write down the sample space, and the event "the second toss shows a head". If the probability of getting a head with the coin is $\frac{1}{3}$, find the probability of this event.

79. Explain the axiomatic theory of probability and show how the classical definition can be derived from this as a special case.

80. Let S be a sample space and let A be any event in the field of events F. State three axioms to define a probability function $P(A)$ on the field F. Using these axioms prove that
(i) $P(A) + P(A') = 1$
(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for *any two* events A and B in F.
[C.U., B.Com. (Hons) '82]

81. Assuming the result for two events and using set algebra, show that for 3 events A, B, C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

82. A and B are two events of a sample space on which a probability function has been defined.
(i) If $P(A) = 1/2$, $P(B) = 3/8$ and $P(A \cap B) = 1/4$, find the values of $P(A \cup B)$, $P(B')$, $P(A' \cap B')$, $P(A' \cup B')$, $P(A' \cap B)$, $P(A/B)$, $P(B/A')$.
(ii) If $P(A \cup B) = 3/4$, $P(A) = 1/2$, $P(B') = 5/8$, find the values of $P(A \cap B)$, $P(A' \cup B)$, $P(A'/B)$.
(iii) If $P(A \cap B) = 1/4$, $P(A \cup B) = 3/4$, $P(B) = 1/3$, find $P(A \cup B')$, $P(A \cap B')$ $P(A/B')$
(iv) If $P(A) = 1/4$, $P(B/A) = 1/2$, $P(A/B) = 3/4$, find $P(B)$, $P(A \cup B)$, $P(A^c/B^c)$.

83. Let A_1 and A_2 be two events related to an experiment E. If it is given that $P(A_1) = 1/2$, $P(A_2) = 1/3$, and $P(A_1 \cap A_2) = 1/4$, then find $P([A_1 \cup A_2]^c)$ where c stands for the complement. *[C.U., B.Sc. '76]*

84. A box contains 40 envelops of which 25 are ordinary (not meant for airmail) and 16 are unstamped, while the number of unstamped ordinary envelopes is

10. What is the probability that an envelope chosen from the box is a stamped air-mail envelope? *[C.U., B.Sc. '76]*

85. Let A, B, C be three events related to an experiment. Under what conditions will the events be (i) exhaustive, (ii) mutually exclusive, (iii) independent ?

86. Let A, B, C be three arbitrary events. Find expressions for the following events, using the usual set-theory notations:
   (i) only A occurs,
   (ii) both A and B, but not C occur,
   (iii) all these events occur,
   (iv) at least one event occurs,
   (v) at least two events occur. *[C.U., B.Sc. (Econ) '81]*

87. One card is drawn at random from a well-shuffed pack of 52 playing cards. Let A denote the event that the card drawn is a Heart, and B denote the event that it is a face card (King, Queen or Jack). Assuming natural assignment of probabilities, evaluate (i) $P(A \cup B)$, (ii) $P(A \cap B')$.

88. The probability that a construction job will be finished in time is 17/20; the probability that there will be no strike is 3/4; and the probability that the construction job will be finished in time, assuming that there will be no strike, is 14/15. Find the probability that
   (i) The construction job will be finished in time and there will be no strike;
   (ii) there will be strike or the job will not be finished in time.

89. What is *conditional probability*? *[W.B.H.S. '78; I.C.W.A. June '82; C.U., M.Com. '76 & B.Sc. (Econ) '81]*

90. Given the information that family has 2 children and that at least one of these two children is a boy, find the probability that both are boys. *[C.U., B.Sc. '76]*

91. There are two events A and B. The probability that B happens is x when A has happened; and y when A has failed to happen. If the probability that A happens is p, find the probability that B happens

92. (a) Show that in general, $P(B/A) = 1 - P(\bar{B} / A)$.
   (b) If $P(A) > P(B)$, show that $P(A/B) > P(B/A)$.

93. When are *two* events said to be 'independent'? *[W.B.H.S. '78; C.U., B.Sc. (Econ) '81; I.C.W.A., June '82]*

94. It is 9 to 5 against a person who is 50 years living till he is 70 and 8 to 6 against a person who is 60 living till he is 80. Find the probability that at least one of them will be alive after 20 years. *[C.A., Nov. '81]*

95. Prove that if A and B are mutually exclusive events and $P(A) \neq 0$, $P(B) \neq 0$, then A and B are not stochastically independent. *[C.U., B.Sc. (Econ) '81]*

96. Distinguish between *pairwise independence* and *mutual independence* of events. *[C.U., B.Sc. (Econ) '82]*

97. (a) Define stochastic independence of three events.
   (b) Suppose that all the four possible outcomes e_1, e_2, e_3, e_4 of an experiment are equally likely. Define the events A, B, C as
   $A = \{e_1, e_4\}$, $B = \{e_2, e_4\}$, $C = \{e_3, e_4\}$.

What can you say about the dependence or independence of the events A, B, C? *[C.U., B.Sc. '76]*

98. One shot is fired form each of three guns. Let A, B, C denote the events that the target is hit by the first, second and the third gun respectively. Assuming that A, B, C are mutually independent events and the $P(A) = 0.5$, $P(B) = 0.6$, $P(C) = 0.8$, find the probability that at least on hit is registered.

99. There are three men aged 60, 65 and 70 years. The probability to live 5 years more is 0.8 for a 60 year old, 0.6 for a 65 year old, and 0.3 for a 70 year old person. Find the probability that at least two of the three persons will remain alive 5 years hence. *[C.U., B.Com. (H) '81]*

100. The probabilities of occurrence of 3 independent events are p, q, r. Find the probabilities of (i) occurrence of at least one of the events, (ii) occurrence of exactly one of the events (whichever occurs).

101. If $P(A \cap B \cap C) = 0$, show that
$$P(A \cup B/C) = P(A/C) + P(B/C)$$

102. If A, B, C are independent events, prove that A and $B \cup C$ are independent.

103. If events $A_1, A_2, \dots A_n$ are independent, show that
$$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - P(A_1').P(A'_2). \dots P(A_n')$$

104. A random experiment consists of 3 independent tosses of a fair coin. Let X be the random variable whose value for any outcome is the number of heads obtained. (i) Construct a table showing the probability distribution of X. Also show the cumulative distribution function (c.d.f.) of X.

105. A continuous random variable X follows uniform distribution with probability density function (p.d.f.)
$$f(x) = 1/2;\ (4 \le x \le 6)$$
(a) Find the probabilities (i) $P(4 \le X \le 5)$, (ii) $P(X < 4.2)$ (iii) $P(X < 3.2)$, (iv) $P(3 \le X \le 4.5)$, (v) $P(X > 5.5)$.
(b) Find the cumulative distribution function (c.d.f.) and hence determine that value of k such that $P(X < k) = 0.7$.

106. A random variable X has the probability density function
$$\begin{aligned} f(x) &= cx^2,\ 0 \le x \le 1 \\ &= 0 \quad \text{otherwise} \end{aligned}$$
Find the value of the constant c; and the probabilities $P(0 \le X \le \frac{1}{2})$, $P\left(X > \frac{3}{4}\right)$, $P\left(\frac{1}{4} < X < \frac{3}{4}\right)$. What is the cumulative distribution function?

107. (a) Explain the concept of *probability density function.*
(b) Show that $f(x)$ defined by
$$\begin{aligned} f(x) &= x; \quad 0 \le x \le 1 \\ &= k - x;\ 1 \le x \le 2 \\ &= 0 \quad \text{elsewhere,} \end{aligned}$$
is a probability density function for a suitable value of the constant k. *[C.U., B.Sc. (Math.) '73]*

108. A continuous random variable X has the density function

$$f(x) = \frac{1}{2} - ax; \ (0 \le x \le 4)$$

where a is constant. (i) Determine the value of a and the probability that X lies between 2 and 3. (ii) Also find the cumulative distribution function (c.d.f.) and calcualte the probabilities $P(X \le 1)$, $P(X \ge 2.5)$, $P(|X - 2| < 0.5)$.

109. The length of life X (in hours) of a certain electronic components is assumed to follow a continuous distribution with density function $f(x) = k/x^2$; $(1000 \le x \le 1500)$.

Determine the constant k, and find the probability that a component selected at random will function for at least 1200 hours.

110. (a) Define the *distribution function* or a random variable X and state some of its properties. *[C.U., B.Sc. (Econ) '81]*

(b) Given the probability density function

$$f(x) = \frac{1}{2}e^{-x/2}, \ x > 0$$

evaluate the distribution function. *[C.U., B.Sc. (Econ) '82]*

ANSWERS

9. True—(b), (c), (g), (j); False—(a), (d), (e), (f), (h), (i).
10. (a), (e), (g), (h)
11. Correct—(a), (c), (e), (f), (h); Wrong—(b), (d), (g), (i), (j).
12. (i) The person is either a male or more than 6 ft. tall;
    (ii) The person is either a male not more than 6 ft. tall, or a female but more than 6 ft. tall;
    (iii) The person is not a male.
13. (a) 0; (*b*) 3/4; (c) 1/18; (d) 1/5; (e) none of these; (f) 1/3; (g) 0.994; (h) $1 - (5/6)^3$; (i) 0.39
14. (i) 0.3; (ii) 0.5
15. 7/8
16. 3/8
17. 4/13
18. 1/6
19. 1/6
20. 1/2
21. 28/55, 3/55, 24/55
22. 1/5
23. 1/72
24. 2/7
25. 13/850, 39/850
26. 1/2, 1/4
27. 5/81
28. $(364 \times 363 \times 362)/365^3 = 0.984$; $1 - 0.984 = 0.016$; $(6 \times 364 \times 363)/365^3$
29. (i) 3/20, 3/8, 3/5; (ii) 23/30, 7/30, 2/3, 1/6; (iii) 13/15, 1/3, 1/5
30. 1/3, 1/2;
31. 14/45
32. 0.111, 0.027
33. 101/105
34. 73/105, 32/105
35. 0.6
36. 13/32
37. 9/25
38. 3/8, 23/50
39. 3/4
40. 1/6, 1/2
41. 1/6
42. 61/140
43. 8/15
44. 1/120
45. 2/3
46. 7/99, 14/33
47. 5/11, 53/66
48. 7/429
49. 112/495
50. 11/115
51. ${}^{40}C_5/{}^{50}C_5$
52. $4/{}^{52}C_5$