12 THEORETICAL DISTRIBUTIONS-BINOMIAL, POISSON, NORMAL

12.1 RANDOM VARIABLE AND PROBABILITY DISTRIBUTION

Random Variable has been defined as a 'function' which assumes real values on the outcomes of a random experiment (see Section 11.19). In repeated performances of the random experiment, the random variable will in general assume different values with a definite probability associated with each value or interval of values.

Probability Distribution (or simply *Distribution*) of a random variable is a statement specifying the set of its possible values together with their respective probabilities. When a random experiment is theoretically assumed to serve as a model, the probabilities can be given as a function of the random variable. The probability distribution concerned is then generally known as *theoretical distribution*. We shall discuss here mainly four important theoretical distributions, viz. Binomial, Poisson, Hypergeometric and Normal.

12.2 DISCRETE PROBABILITY DISTRIBUTION

Let a discrete random variable X assume the values $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n$ respectively, where $\Sigma p_i = 1$. The specification of the set of values x_i together with their probabilities p_i ($i = 1, 2, ..., n$) defines the probability distribution of the discrete random variable X, or in short, *discrete probability distribution* of X.

Table 12.1 Discrete Probability Distribution

Values (x)	x_1	x_2	x_3	...	...	x_n	Total
Probabilities (p)	p_1	p_2	p_3	...	...	p_n	1

Often, the probability (p) that the random variable X assumes a specified value (x) can be expressed in terms of a general mathematical expression $f(x)$.

$$f(x) = \text{Probability that } X \text{ assumes the value } x$$
$$= P(X = x) \qquad (12.2.1)$$

The function $f(x)$ is known as "*probability mass function*" (p.m.f), or sometimes '*probability function*' of the discrete random variable X.

If $f(x)$ is known, it is unnecessary to write the probabilities $p_1, p_2, ..., p_n$, because p_i may be obtained from $f(x)$ on putting $x = x_i$, i.e., $p_i = f(x_i)$. The discrete probability distribution is then defined simply by stating the mathematical expression for $f(x)$, along with the set of possible values $x_1, x_2, ..., x_n$.

Table 12.2 Theoretical Distribution (Discrete)

x	x_1	x_2	x_3	...	x_n	Total
$p = f(x)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	...	$f(x_n)$	1

The discrete random variable X may also assume countably infinite number of possible values $x_1, x_2, ..., x_n, ...$ with p.m.f. $f(x)$.

The probability mass function $f(x)$ must satisfy the two conditions

$$\text{(i) } f(x) \geq 0, \qquad \text{(ii) } \Sigma f(x) = 1 \qquad (12.2.2)$$

where the summation is taken over all possible value x.

Illustration 1. The following table shows the discrete, probability distribution of a random variable X:

Value (x)	3	6	7	11	Total
Probability (p)	0.2	0.5	0.2	0.1	1

Illustration 2. If X denotes the 'number of points obtained when an unbiased die is thrown', the probability distribution of X is:

No. of Points (x)	1	2	3	4	5	6	Total
Probability (p)	1/6	1/6	1/6	1/6	1/6	1/6	1

This distribution may also be written as

$$f(x) = 1/6; \quad (x = 1, 2, ..., 6)$$

Illustration 3. If X denotes the 'number of heads obtained in 3 tosses of an unbiased coin', the probability distribution of X is:

No. of Heads (x)	0	1	2	3	Total
Probability (p)	1/8	3/8	3/8	1/8	1

These probabilities may also be given by the p.m.f. (Example 12.8)

$$f(x) = {}^3C_x \left(\frac{1}{2}\right)^3; \quad (x = 0, 1, 2, 3)$$

Illustration 4. An unbiased coin is thrown repeatedly until a head appears. If X denotes the 'number of tails preceding the first head' then its probability distribution is given below:

No. of Tails (x)	0	1	2	3	...	r	....	Total
Probability (p)	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$	...	$\left(\frac{1}{2}\right)^{r+1}$	...	1

This distribution can be written by the p.m.f.

$$f(x) = \left(\frac{1}{2}\right)^{x+1} ; \quad (x = 0, 1, 2, ..., \infty)$$

[**Note:** (i) A discrete random variable (r.v.) can assume either a 'finite' number of possible values (see *Illustrations 1, 2, 3*) or a 'countably infinite' number of values (*Illustration 4*), with a definite probability (i.e. a non-negative real number) associated with each district value of the variable, such that the total probability is 1. The 'probability distribution' of a discrete r.v. shows how the total probability 1 is distributed over the different possible values of the variable. The probability of an interval of values is then given by the sum of the probability associated with those values which lie in that interval.

It will be shown later that a continuous random variable assumed an uncountably infinite' number of possible values. Probabilities are now associated with intervals of values, but not with each individual value of the variable. The probability is obtained by integrating the '*probability density function*' (p.d.f.) over the limits of the interval.

(ii) Generally, the capital letter X is used to denote the '*random variable*' and the small letter x to denote '*any specified value*' of the random variable. If however no confusion arises, the small letter x is often used for both.]

Example 12.1 *X is a discrete random variate having probability mass function:*

x	0	1	2	3	4	5	6	7
$P = (X = x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Determine the constant k; (ii) Find $P(X < 6)$; (iii) What will be $P(X \geq 6)$?

[W.B.H.S. '81]

Solution (i) For a discrete random variable X, the probability mass function $f(x) = P(X = x)$ must satisfy the relations

$$\text{(I) } f(x) \geq 0, \qquad \text{(II) } \Sigma f(x) = 1$$

where the summation is taken over all possible values x. Using (II) here we have

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + (7k^2 + k) = 1$$

or, $$10k^2 + 9k = 1$$

or, $$10k^2 + 9k - 1 = 0$$

or, $$(k + 1)(10k - 1) = 0;$$

$\therefore$ $$k = -1, \frac{1}{10}.$$

By virtue of (I), $k = -1$ is impossible. (Note that the probabilities corresponding to 1, 2, 3, 4 become negative). Hence $k = \frac{1}{10} = 0.1$

(ii)
$$\begin{aligned} P(X < 6) &= P(X = 0) + P(X = 1) + ... + P(X = 5) \\ &= 0 + k + 2k + 2k + 3k + k^2 \\ &= 8k + k^2 = 8(0.1) + (0.1)^2 = 0.81 \end{aligned}$$

(iii) $P(X \geq 6) = 1 - P(X < 6)$, **since the events $X < 6$ and $X \geq 6$ are complementary.**
$$= 1 - 0.81 = 0.19$$

Ans. 0.1, 0.81, 0.19

12.3 EXPECTATIONS—MEAN, VARIANCE, MOMENTS (DISCRETE DISTRIBUTION)

Let a discrete random variable x assume the values $x_1, x_2, \dots, x_n$ with probabilities $p_1, p_2, \dots, p_n$ respectively. Then the *expectation*, or '*expected value*' of x—written $E(x)$—is defined as the sum of products of the different values of x and the corresponding probabilities.

$$E(x) = \Sigma p_i x_i \qquad (12.3.1)$$

The expected value of x^2 is similarly defined as the sum of products of the squares of values and the corresponding probabilities.

$$E(x^2) = \Sigma p_i x_i^2 \qquad (12.3.2)$$

In general, the expected value of any function $g(x)$ is defined as

$$E[g(x)] = \Sigma p_i g(x_i)$$

Hence, the expected value of a constant k is the constant k itself.

$$E(k) = k,$$

where k is a constant; because $E(k) = \Sigma p_i k = k\Sigma p_i = k$.

If the p.m.f. $f(x)$ is given, then the expectations are defined as follows:

$$E(x) = \Sigma x \,.\, f(x) \qquad (12.3.3)$$

$$E(x^2) = \Sigma x^2 \,.\, f(x) \qquad (12.3.4)$$

Mean of a probability distribution is the expected value of x.

$$\text{Mean } (\mu) = E(x) \qquad (12.3.5)$$

Variance is the expected value of $(x - \mu)^2$, where μ is the mean.

$$\text{Variance } (\sigma^2) = E(x - \mu)^2$$

It may be shown that

$$\sigma^2 = E(x^2) - \mu^2 \qquad (12.3.6)$$

The positive square-root of variance gives the *standard deviation* (σ).

Moments of a discrete distribution are defined as follows:

$$\begin{aligned} &r\text{-th moment about } A: && \mu'_r = X(x-A)^r = \Sigma(x-A)^r f(x) \\ &r\text{-th raw moment:} && \mu'_r = E(x^r) = \Sigma x^r \,.\, f(x) \\ &r\text{-th central moment:} && \mu_r = E(x-\mu)^r = \Sigma(x-\mu)^r \,.\, f(x) \end{aligned} \qquad (12.3.7)$$

where $\mu = E(x)$ is the mean of the distribution. Note that

$$\mu'_0 = \mu_0 = 1: \qquad \mu'_1 = E(x) = \mu$$

$$\mu_1 = E(x - \mu) = \Sigma(x - \mu) \,.\, f(x) = \Sigma x \,.\, f(x) - \mu\Sigma f(x)$$

$$= \mu - \mu = 0$$

The central moments can be obtained from non-central moments using the relations (see 7.2.2)

$$\begin{aligned} \mu_2 &= \mu'_2 - \mu_1'^2 \\ \mu_3 &= \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3 \\ \mu_4 &= \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu_1'^2 - 3\mu_1'^4 \end{aligned} \qquad (12.3.8)$$

Example 12.2 *A random variable has the following probability distribution:*

x	*4*	*5*	*6*	*8*
Probability	*0.1*	*0.3*	*0.4*	*0.2*

Find the expectation and the standard deviation of the random variable.

[C.U., B.Com. (Hons) '80]

Solution (Note that the sum of the probabilities is 1; $\Sigma p_i = 1$)

$$\text{Mean } (\mu) = E(x) = \Sigma p_i\, x_i$$
$$= (0.1 \times 4) + (0.3 \times 5) + (0.4 \times 6) + (0.2 \times 8)$$
$$= 0.4 + 1.5 + 2.4 + 1.6 = 5.9$$
$$E(x^2) = \Sigma p_i\, x_i^2 = (0.1 \times 4^2) + (0.3 \times 5^2) + (0.4 \times 6^2) + (0.2 \times 8^2)$$
$$= 1.6 + 7.5 + 14.4 + 12.8 = 36.3$$
$$\text{Variance } (\sigma^2) = E(x^2) - \mu^2, \text{ using (12.3.6)}$$
$$= 36.3 - (5.9)^2 = 36.3 - 34.81 = 1.49$$
$$\text{S.D. } (\sigma) = \sqrt{1.49} = 1.22$$

Ans. 5.9, 1.22

Example 12.3 *For what value of a will the function*

$$f(x) = ax; \qquad x = 1, 2, 3, \ldots, n$$ *be the probability mass function of a discrete random variable x? Find the mean and variance of x.*

[W.B.H.S. '78, '82]

Solution The conditions for any function $f(x)$ to be a p.m.f. are

(i) $f(x) \geq 0$, (ii) $\Sigma f(x) = 1$

From (i), we have $a > 0$, because none of the values of x is negative.

From (ii),
$$1 = \Sigma f(x) = \sum_{x=1}^{n} ax = a \sum_{1}^{n} x = a \,.\, \frac{n(n+1)}{2}$$

$$\therefore \quad a = \frac{2}{n(n+1)}$$

$$\text{Mean} = E(x) = \Sigma x\, f(x) = \sum_{x=1}^{n} x \,.\, ax = a \sum_{1}^{n} x^2$$
$$= a(1^2 + 2^2 + \ldots + n^2)$$
$$= \frac{2}{n(n+1)} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)}{3}$$

Again,
$$E(x^2) = \Sigma x^2 \,.\, f(x) = a \sum_{1}^{n} x^3$$
$$= a(1^3 + 2^3 + \ldots + n^3)$$
$$= \frac{2}{n(n+1)} \left\{\frac{n(n+1)}{2}\right\}^2 = \frac{n(n+1)}{2}$$

$$\therefore \quad \text{Variance} = E(x)^2 - \{E(x)\}^3 = \frac{n(n+1)}{2} - \left(\frac{2n+1}{3}\right)^2$$
$$= \frac{(n-1)(n+2)}{18}$$

Example 12.4 *A random variable X is defined as follows:*

$$Prob\,(X = 1) = p, \quad Prob\,(X = 0) = 1 - p$$

where $0 < p < 1$. Find the mean, variance and the central moments μ_2, μ_3, μ_4 of the distribution.

Solution The probability distribution of X is as follows:

Value (x)	1	0	Total
Probability	p	$1-p$	1

The raw moments are

$$\mu_1' = E(X) = p \times 1 + (1-p) \times 0 = p + 0 = p$$
$$\mu_2' = E(X^2) = p \times 1^2 + (1-p) \times 0^2 = p \times 1 + (1-p) \times 0 = p$$
$$\mu_3' = E(X^3) = p \times 1^3 + (1-p) \times 0^3 = p$$
$$\mu_4' = E(X^4) = p \times 1^4 + (1-p) \times 0^4 = p$$

Using (12.3.8)

$$\mu_2 = p - p^2 = p(1-p)$$
$$\mu_3 = p - 3p \,.\, p + 2p^3 = p - 3p^2 + 2p^3 = p(1-p)(1-2p)$$
$$\mu_4 = p - 4p \,.\, p + 6p \,.\, p^2 - 3p^4 = p(1-p)(1-3p+3p^2)$$

Also, Mean (μ) $= E(x) = \mu_1 = p$

Variance $(\sigma^2) = \mu_2 = p(1-p)$.

Example 12.5 *Obtain the expectation of the number of tails preceding the first head in an indefinite series of tosses of the same coin.*

[C.U., B.Sc.(Econ) '81 (New)]

Solution Let the random variable X denote the number of tails preceding the first head. Suppose that p denotes the probability of getting a head in one toss of the coin and $q\,(= 1-p)$ denotes the probability of a tail. Then the probability that x tails precede the first head is

$P(X = x)$ = Probability of x successive tails, followed by a head,

$= q \,.\, q \,.\, q ,\ldots, q \,.\, p$ (q's nothing x times)

$= q^x \,.\, p$

The p.m.f of X is thus given by

$$f(x) = pq^x; \quad (x = 0, 1, 2, 3, \ldots \infty)$$

[Check: (i) $f(x) \geq 0$; because none of p and q is negative

(ii) $$\sum_0^\infty f(x) = pq^0 + pq^1 + pq^2 + pq^3 + \ldots \ldots$$
$$= p(1 + q + q^2 + q^3 + \ldots \ldots$$
$$= p(1-q)^{-1} = p \,.\, p^{-1} = 1]$$

The expectation of the number of tails preceding the first head is

$$E(x) = \Sigma x \,.\, f(x) = \sum_0^\infty x \,.\, pq^x$$
$$= 0 + pq + 2pq^2 + 3pq^2 + \ldots \ldots$$
$$= pq(1 + 2q + 3q^2 + \ldots \ldots)$$
$$= pq(1-q)^{-2} = pq \,.\, p^{-2} = qp^{-1}$$
$$= q/p.$$

Ans. q/p

12.4 UNIFORM DISTRIBUTION (DISCRETE)

If a discrete random variable x assumes n possible values $x_1, x_2, ..., x_n$ with *equal probabilities*, then the probability that it takes any particular value is a constant $1/n$. The probability distribution defined by the probability mass function (p.m.f).

$$f(x) = 1/n; \quad (x = x_1, x_2, ..., x_n) \tag{12.4.1}$$

is known as *Uniform Distribution*.

Table 12.3 Uniform Distribution (Discrete)

x	x_1	x_2	...	x_n	Total
$f(x)$	$1/n$	$1/n$	...	$1/n$	1

The number of points obtained in a single throw of an unbiased die follows uniform distribution with p.m.f

$$f(x) = 1/6, (x = 1, 2, 3, ..., 6).$$

No. of Points x	1	2	3	4	5	6	Total
Probability $f(x)$	1/6	1/6	1/6	1/6	1/6	1/6	1

[**Note:** (i) The p.m.f of a discrete uniform distribution is a constant (independent of x), and this constant is one divided by the number of possible values of x.

(ii) In a discrete uniform distribution, the possible values of the random variable may be any real numbers, and the successive values may not have a common difference. However, there must be a finite number of possible values.

Example 12.6 *If a discrete random variable x follows uniform distribution and assumes only the values 8, 9, 11, 15, 18, 20, find the probabilities (i) $P(x = 9)$, (ii) $P(x = 12)$, (iii) $P(x < 15)$, (iv) $P(x \le 15)$, (v) $P(x > 15)$, (vi) $P(|x - 14| < 15)$.*

Solution Since x has a discrete uniform distribution with 6 possible values, the probability that it takes any particular value is a constant $\frac{1}{6}$ and the probability that it takes a value other than those given, is zero. Therefore,

(i) $$P(x = 9) = \frac{1}{6}$$

(ii) $$P(x = 12) = 0$$

(iii) Now, $x < 15$ implies that x takes only 3 values, viz. 8, 9, 11.

Hence, $$P(x < 15) = 3 \times \frac{1}{6} = \frac{1}{2}$$

(iv) Again, $x \le 15$ implies that the variable takes 4 values (15 included), viz 8, 9, 11, 15. Hence

$$P(x \le 15) = 4 \times \frac{1}{6} = \frac{2}{3}$$

(v) $x > 15$ implies only 2 possible values of x viz. 18 and 20.

$$P(x > 15) = 2 \times \frac{1}{6} = \frac{1}{3}$$

(vi) $|x - 14| < 5$ implies that the difference of x and 14 is to be less than 5; so x takes only the values 11, 15 and 18. Hence,

$$P(|x - 14| < 5) = 3 \times \frac{1}{6} = \frac{1}{2}$$

Ans. $\frac{1}{6}, 0, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{1}{2}$

Example 12.7 *Find the mean and the standard deviation of the uniform distribution $f(x) = 1/n$; $(x = 1, 2, ..., n)$*

Solution Mean $(\mu) = E(x) = \Sigma x \cdot f(x)$

$$= \sum_{x=1}^{n} x \cdot (1/n) = (1/n) \sum_{x=1}^{n} x$$

$$= (1/n)(1 + 2 + ... + r)$$

$$= \frac{n(n+1}{2n} = \frac{n+1}{2}$$

$$E(x^2) = \Sigma x^2 \cdot f(x) = \sum_{x=1}^{n} x^2 \cdot (1/n)$$

$$= (1/n)(1^2 + 2^2 + ... + n^2)$$

$$= \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}$$

Using (12.3.6),

$$\sigma^2 = E(x^2) - \mu^2 = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n^2 - 1}{12}$$

$\therefore$ S.D. $(\sigma) = \sqrt{(n^2 - 1)/12}$ *Ans.* $(n + 1)/2$, $\sqrt{(n^2 - 1)/12}$

12.5 BINOMIAL DISTRIBUTION

Binomial distribution is a discrete probability distribution and is defined by the p.m.f.

$$f(x) = {}^nC_x\, p^x q^{n-x}; \quad (x = 0, 1, 2, ..., n) \qquad (12.5.1)$$

where p and q are positive fraction $(p + q = 1)$.

Suppose that we have a series of n independent trails in each of which the probability of occurrence of an event is fixed and constantly p. Then the probability that the event occurs exactly r times in n trials (Section 11.8) is ${}^xC_r\, p^r q^{n-r}$, where $q = 1 - p$ and r may assume any of the values 0, 1, 2, ..., n. In general, the occurrence of the event is called "*success*" and its non-occurrence is known as "*failure*". The generalised various of the theory is as follows:

In a series of *n independent trials*, if the probability of 'success' in each trial is a *constant p*, and the probability of 'failure' is q, then the probability of x successes (and obviously $n - x$ failures) is given by the Binomial distribution (12.5.1.)

Table 12.4 Binomial Distribution

x	0	1	2	...	n	Total
$f(x)$	q^n	${}^nC_1 pq^{n-1}$	${}^nC_2 p^2q^{n-2}$	...	p^n	1

Note: (i) In the expression for $f(x)$, besides the factor ${}^nC_x$, the power of p (i.e., probability of success) is the number of successes x , and the power of q (i.e., probability of failure) is the number of failures $n - x$.

(ii) The sum of the powers of p and q is always n, whatever be the number of successes.

(iii) These are $(n + 1)$ possible values of x, viz. 0, 1, 2, ..., n.

(iv) The total probability for all the $(n + 1)$ possible successes is 1.]

The distribution is known as 'binomial', because the probabilities are given by the binomial series.

$$(q + p)^n = q^n + {}^nC_1 pq^{n-1} + {}^nC_2 p^2q^{n-2} + ... + p^n. \tag{12.5.2}$$

A series of independent trials of a random experiment with a constant probability of success in each trial is called a "*Bernoullian series*", and this distribution is known as "Bernoulli's distribution", after the name of its discoverer James Bernoulli.

The two constants n and p appearing in the expression for $f(x)$, are known as '*parameters*' of the binomial distribution. If the values of parameters are known, the distribution is completely known $(q = 1 - p)$.

I should be noted that the binomial distribution holds under the following conditions:

(1) *The result of any trial can be classified only under two categories*; e.g., either head or tail in a throw of one coin; either a "six" or a "non-six" in one throw of a die; either a spade or a non-spade when one card is drawn; either a defective or a non-defective when an article is examined from a lot; i.e., in general, occurrence (called *success*) or non-occurrence (called *failure*) of a certain event.

(2) *The probability of success in each trial remains a constant,* and does not change from one trial to another. For example, the probability of obtaining a head in successive thrown of a coin is always $\frac{1}{2}$; the probability of obtaining a defective article from a batch does not change in successive drawings *with replacement*, and practically remains a constant even in drawings *without replacement,* when the batch is large.

(3) *The trials are independent,* so that the probability of success in any trial remains unaffected by the results of other trials. For example, in successive throws of a coin the occurrence of a head at any trial will in no way affect the probability of a head or a tail in any subsequent trial; or if several coins are thrown together, the occurrence of a head or a tail in any particular coin does not alter the probability of occurrence of a head in any other coin.

Important Properties

(1) Binomial distribution is a discrete probability distribution, where the random variable assumes a finite number of values 0, 1, 2, ..., n. The distribution is specified by two parameters n and p.

(2) Mean = np, variance = npq (12.5.3)

Standard deviation $(\sigma) = \sqrt{npq}$

(3) Binomial distribution may have either one or two modes. When $(n + 1)p$ is not an integer, Mode is the largest integer contained therein. However, when $(n + 1)p$ is itself an integer, there are two modes, viz. $(n + 1)p$ and $(n + 1)p - 1$.

(4) Skewness $(\gamma_1) = \dfrac{q-p}{\sqrt{npq}}$; Kurtosis $(\gamma_2) = \dfrac{1-6pq}{npq}$ (12.5.4)

When $p = q = \dfrac{1}{2}$, the distribution becomes symmetrical for all values of n.

i.e., $f(x) = f(n-x) = {}^nC_x\left(\dfrac{1}{2}\right)^n$.

(5) If x follows binomial distribution with parameters (n_1, p) and y follows binomial distribution with parameters $(n_2, -p)$, and x and y are statistically independent, then $(x + y)$ also follows binomial distribution with parameters $(n_1 + n_2, p)$. The result can be extended to several independent binomial variates with a common p.

(6) Binomial distribution may be obtained as a limiting case of Hypergeometric distribution.

Example 12.8 *Three coins are tossed. Find the probabilities of (i) 0 head, 1 head, 2 heads, 3 heads; (ii) more than one head; (iii) at least 1 head.*

Solution Here, the 'random experiment' consists in tossing 3 coins and observing the number of heads. The 'random variable' x is the number of heads obtained in a toss of 3 coins.

If we denote the occurrence of head as "success", and the coins are assumed to be unbiased, then

$$p = \text{Probability of "success" in a single trial}$$
$$= \text{Probability of head with a single coin} = \frac{1}{2}$$
$$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$$
$$n = \text{Number of independent trials} = 3$$

Since the value of p is constant for each coin and the trials are independent, the variable x follows binomial distribution with parameters $n = 3$, $p = \dfrac{1}{2}$. Therefore, the probability of x successes is

$$f(x) = {}^nC_x p^x q^{n-x} = {}^3C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{3-x} = {}^3C_x\left(\frac{1}{2}\right)^3$$

(i) Putting the values of $x = 0, 1, 2$, respectively,

$$f(0) = \text{Probability of 0 success}$$
$$= {}^3C_0\left(\frac{1}{2}\right)^3 = 1 \times \frac{1}{8} = \frac{1}{8}$$

$f(1)$ = Probability of 1 success

$$= {}^3C_1\left(\frac{1}{2}\right)^3 = 3\times\frac{1}{8} = \frac{3}{8}$$

$f(2)$ = Probability of 2 successes

$$= {}^3C_2\left(\frac{1}{2}\right)^3 = 3\times\frac{1}{8} = \frac{3}{8}$$

$f(3)$ = Probability of 3 successes

$$= {}^3C_3\left(\frac{1}{2}\right)^3 = 1\times\frac{1}{8} = \frac{1}{8}$$

(ii) Prob (more than one success) = $f(2)+f(3)$

$$= \frac{3}{8}+\frac{1}{8} = \frac{1}{2}$$

(iii) Prob (at least one success) = 1 – Prob (0 success)

$$= 1-f(0) = 1-\frac{1}{8} = \frac{7}{8}$$

Example 12.9 *Five coins are tossed 3200 times. Find the expected frequencies of the distribution of heads and tails, and tabulate the result. Calculate the mean number of heads and standard deviation.* [C.A., May '77]

Solution We assume that the coins are unbiased. The probability of getting a head (or tail) is $\frac{1}{2}$ for each coin at each trial. Also the successive trials are independent. So, the conditions of binomial distribution are fulfilled. The probabilities of 0 head, 1 head, ... 5 heads are then given by the successive terms in the expansion of the binomial expression

$$\left(\frac{1}{2}+\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^5 + {}^5C_1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 + {}^5C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$$

$$+ {}^5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 + {}^5C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^5$$

$$= \frac{1}{32}+\frac{5}{32}+\frac{10}{32}+\frac{10}{32}+\frac{5}{32}+\frac{1}{32}$$

The probabilities of the different numbers of heads and tails in a single toss of 5 coins, and the expected frequencies in 3200 tosses are shown below:

Number of Heads	*Number of* Tails	*Probability*	*Expected Frequency = 3200 × Probability*
0	5	1/32	100
1	4	5/32	500
2	3	10/32	1000
3	2	10/32	1000
4	1	5/32	500
5	0	1/32	100
Total		1	3200

The mean number of heads, and the standard deviation are now calculated from the following frequency distribution (see Examples 5.57 and 6.29):

Number of Heads (x)	0	1	2	3	4	5	Total
Frequency (f)	100	500	1000	1000	500	100	3200

Mean ($\bar{x}$) = 2.5, Standard deviation (σ) = $\sqrt{5/2}$ = 1.12

[**Note:** that the mean and s.d. calculated from the frequency distribution agree exactly with the formulae (12.5.3), viz.

$$np = 5 \times \frac{1}{2} = 2.5, \text{ s.d.} = \sqrt{npq} = \sqrt{5 \times \frac{1}{2} \times \frac{1}{2}} = \sqrt{\frac{5}{2}}$$

Example 12.10 *The overall percentage of failures in a certain examination is 40. What is the probability that out of a group of 6 candidates at least 4 passed the examinations?* [I.C.W.A. Dec.'74]

Solution Let us denote the event of a candidate passing the examination as "success". It is required to find the probability of at least 4 successes.

$$\begin{aligned} q &= \text{Probability of failure of a candidate} \\ &= 40\% = 40/100 = 2/5 \\ p &= \text{Probability of success in a single trial} \\ &= 1 - q = 1 - 2/5 = 3/5 \\ n &= \text{Number of candidates in the group} = 6 \end{aligned}$$

Using binomial distribution, the probability of x successes is

$$f(x) = {}^nC_x\, p^x\, q^{n-x} = {}^6C_x \left(\frac{3}{5}\right)^x \left(\frac{3}{5}\right)^{6-x}$$

Since 'at least 4' in a group of 6 implies 'either 4, or 5, or 6', the probability of at least 4 successes is given by the sum of probabilities $f(4) + f(5) + f(6)$. Putting $x = 4, 5, 6$ in the expression for $f(x)$, we have

$$f(4) = {}^6C_4 \left(\frac{3}{5}\right)^4 \left(\frac{3}{5}\right)^2 = 15 \times \frac{3^4 \times 2^2}{5^6} = \frac{4860}{15625}$$

$$f(5) = {}^6C_5 \left(\frac{3}{5}\right)^5 \left(\frac{3}{5}\right)^1 = 6 \times \frac{3^5 \times 2}{5^6} = \frac{2916}{15625}$$

$$f(6) = {}^6C_6 \left(\frac{3}{5}\right)^6 \left(\frac{3}{5}\right)^0 = 1 \times \frac{3^6 \times 1}{5^6} = \frac{729}{15625}$$

Thus, the required probability is

$$f(4) + f(5) + f(6) = \frac{4860 + 2916 + 729}{15625} = \frac{1701}{3125}$$

Example 12.11 *In 10 independent throws of a defective dice, the probability that an even number will appear 5 times is twice the probability that an even number will appear 4 times. Find the probability that an even number will not appear at all in 10 independent throws of the dice.*

Solution Let the occurrence of an even number be called a "success". We write

p = probability of getting an even number (i.e., success) in a single trial,
q = probability of not getting an even number (i.e., failure)
n = number of throws = 10

Using binomial distribution, the probability of x successes in 10 throws is

$$f(x) = {}^{10}C_x\, p^x\, q^{10-x}$$

It is given that

$$f(5) = 2 \times f(4)$$

or, $${}^{10}C_5\, p^5\, q^5 = 2 \times {}^{10}C_4\, p^4\, q^6$$

or, $$\frac{10!}{5!\,5!}\, p^5\, q^5 = 2 \times \frac{10!}{4!\,6!}\, p^4\, q^6$$

Simplifying, we get $3p = 5q$

or, $3p = 5(1-p)$

Solving we get $p = \frac{5}{8}$, so that $q = 1 - p = 1 - \frac{5}{8} = \frac{3}{8}$

$\therefore$ Probability that an even number will not appear at all (i.e. 0 success) is

$$f(0) = q^{10} = \left(\frac{3}{8}\right)^{10}$$

Ans. $\left(\frac{3}{8}\right)^{10}$

Example 12.12 *Find the mean and the standard deviation of binomial distribution with parameters n and p.*

Solution The p.m.f. of binomial distribution is

$$f(x) = {}^nC_x\, p^x\, q^{n-x};\ (x = 0, 1, 2, \ldots, n)$$

Hence, by definition (12.3.3),

$$\text{Mean} = E(x) = \sum_x x\, f(x) = \sum_{x=0}^{n} x \,.\, {}^nC_x\, p^x\, q^{n-x}$$

$$= 0 \times ({}^nC_0\, p^0\, q^n) + 1 \times ({}^nC_1\, p\, q^{n-1}) + 2 \times ({}^nC_2\, p^2\, q^{n-2}) + 3 \times ({}^nC_3\, p^3\, q^{n-3}) + \ldots + n \times ({}^nC_n\, p^n\, q^0)$$

$$= 0 + 1 \times npq^{n-1} + 2 \times \frac{n(n-1)}{1 \times 2}\, p^2\, q^{n-2} + 3 \times \frac{n(n-1)(n-2)}{1 \times 2 \times 3}\, p^2\, q^{n-3} + \ldots + n \times p^n$$

$$= npq^{n-1} + n(n-1)\, p^2\, q^{n-2} + \frac{n(n-1)(n-2)}{1 \times 2}\, p^3\, q^{n-3} + \ldots + n\, p^n$$

$$= np[q^{n-1} + (n-1)\, pq^{n-2} + \frac{n(n-1)(n-2)}{1 \times 2}\, p^2\, q^{n-3} + \ldots + p^{n-1}]$$

$$= np\, [q^{n-1} + {}^{n-1}C_1\, pq^{n-2} + {}^{n-1}C_2\, pq^{n-3} + \ldots + {}^{n-1}C_{n-1}\, p^{n-1}]$$

$= np(q+p)^{n-1}$, replacing n by $n-1$ in (12.5.2)

$= np(1)^{n-1}$, because $p + q = 1$

$= np(1) = np$

Therefore, Mean $(\mu) = np$

Again, by (12.3.6) $\sigma^2 = E(x^2) - \mu^2$

But, $$E(x^2) = Ex^2 \,.\, f(x) = \Sigma\{x(x-1) + x\} \,.\, f(x)$$

$$= \Sigma x(x-1)\, f(x) + \Sigma x \,.\, f(x)$$

$$= \Sigma x(x-1) \,.\, f(x) + \mu.$$

Now, $\Sigma x(x-1) \cdot f(x) = \sum_{x=0}^{n} x(x-1) \cdot {}^nC_x p^x q^{n-x};$

$$= 0(-1)\{{}^nC_0 p^0 q^n\} + 1\times 0\{{}^nC_1 pq^{n-1}\} + 2\times 1\{{}^nC_2 p^2 q^{n-2}\} + 3\times 2\{{}^nC_3 p^3 q^{n-3}\} + \dots + n(n-1)\{{}^nC_n p^n q^0\}$$

$$= 0 + 0 + 2\times 1\left\{\frac{n(n-1)}{1\times 2} p^2 q^{n-2}\right\} + 3\times 2\left\{\frac{n(n-1)(n-2)}{1\times 2\times 3} p^3 q^{n-3}\right\} + \dots + n(n-1)\{1\cdot p^n\}$$

$$= n(n-1)p^2 q^{n-2} + n(n-1)(n-2)p^3 q^{n-3} + \dots + n(n-1)p^n$$

$$= n(n-1)p^2[q^{n-2} + (n-2)pq^{n-3} + \dots + p^{n-2}]$$

$$= n(n-1)p^2[q^{n-2} + {}^{n-2}C_1 pq^{n-3} + \dots + {}^{n-2}C_{r-2} p^{n-2}]$$

$= n(n-1)p^2(q+p)^{n-2}$, replacing n by $n-2$ in (12.5.2)

$= n(n-1)p^2(1)^{n-2}$, since $p+q=1$

$= n(n-1)p^2$

$\therefore \quad E(x)^2 = n(n-1)p^2 + \mu$, and hence

$$\sigma^2 = E(x^2) - \mu^2$$

$$= n(n-1)p^2 + \mu - \mu^2 = n(n-1)p^2 + np - (np)^2$$

$$= n^2p^2 - np^2 + np - n^2p^2 = -np^2 + np$$

$= np(1-p) = npq,$ since $q = 1-p$.

$\therefore$ Standard Deviation $(\sigma) = \sqrt{npq}$

Ans. mean $= np$, s.d.$= \sqrt{npq}$

Example 12.13 *Arithmetic mean and standard deviation of a binomial distribution are respectively 4 and $\sqrt{8/3}$. Find the values of q and p.* [W.B.H.S. '81]

Solution Using the formulae for mean and s.d. of the binomial distribution

$$np = 4, \qquad \sqrt{npq} = \sqrt{8/3}$$

Squaring the second relation, $npq = 8/3$. Now putting $np = 4$ and solving we have $q = 2/3$. Hence, $p = 1 - q = 1/3$.

Putting the value of p in $np = 4$, we get $n = 12$. *Ans.* $n = 12$, $p = 1/3$.

Example 12.14 *Find the mode of binomial distribution with parameters n and p.*

Solution The 'mode' of a discrete distribution is the value of the variate which corresponds to the maximum probability.

In the binomial distribution, the random variable X assumes only the values 0, 1, 2, ..., n and the probabilities are given by

$$f(x) = P(X = x) = {}^nC_x p^x q^{n-x}; \ (x = 0, 1, 2, \dots, n)$$

The mode is that value (or values) among 0, 1, 2, ..., n corresponding to which $f(x)$ is the maximum.

$$\frac{f(x-1)}{f(x)} = \frac{{}^nC_{x-1} p^{x-1} q^{n-x+1}}{{}^nC_x p^x q^{n-x}} \ (x \neq 0)$$

$$= \frac{n!}{(x-1)!(n-x+1)!} p^{x-1} \cdot q^{n-x+1} \quad \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \frac{xq}{(n-x+1)p} = \frac{xq}{(n+1)p - xp} \qquad \text{... (1)}$$

We find that:

(a) $f(x-1) < f(x)$

whenever $xq < (n+1)p - xp$

or, $x(p+q) < (n+1)p$; i.e. $x < (n+1)p$ ... (2)

(b) $f(x-1) = f(x)$

whenever $xq = (n+1)p - xp$; i.e. $x = (n+1)p$ ... (3)

(c) $f(x-1) > f(x)$

whenever $xq > (n+1)p - xp$; i.e. $x > (n+1)p$ ... (4)

We discuss the following cases:

Case I When $(n+1)p$ is not an integer.

Let r be the largest integer in $(n+1)p$. Since in a binomial distribution, x assumes only integral values, the result (3) is impossible.

From (2),

$$f(x-1) < f(x), \quad \text{when } x = 1, 2, 3, ..., r$$

or, $f(0) < f(1)$, $f(1) < f(2)$, ... $f(r-1) < f(r)$

or, $f(0) < f(1) < f(2) < ... f(r-1) < f(r)$ ... (5)

From (4),

$$f(x-1) > f(x), \quad \text{when } x = r+1, r+2, ..., n$$

or, $f(r) < f(r+1)$, $f(r+1) > f(r+2)$, ... $f(n-1) > f(n)$

or, $f(r) > f(r+1) > f(r+2) > ... > f(n)$ ... (6)

Combining the results at (5) and (6),

$$f(0) < f(1) < ... < f(r-1) < f(r) > f(r+1) > ... > f(n).$$

We find that the probability $f(r)$ corresponding to $x = r$ is the maximum, and therefore the mode is r.

Thus, when $(n+1)p$ is not an integer, the mode is the largest integer contained in $(n+1)p$.

Case II When $(n+1)p$ is an integer, say $(n+1)p = r$.

From (2),

$$f(x-1) < f(x), \quad \text{when } x = 1, 2, 3, ..., r-1$$

or, $f(0) < f(1) < ... < f(r-1)$. ... (7)

From (3),

$$f(x-1) = f(x), \quad \text{when } x = r$$

or, $f(r-1) = f(r)$ ... (8)

From (4),

$$f(x-1) > f(x), \quad \text{when } x = r+1, r+2, ..., n$$

or, $f(r) > f(r+1) > ... > f(n)$ ... (9)

Combining the results at (7), (8) and (9),

$$f(0) < f(1) < ... < f(r-1) = f(r) > f(r+1) > ... > f(n).$$

We find that the probabilities $f(r)$ and $f(r-1)$ corresponding to $x = r$ and $x = r-1$ respectively are equal, and have the largest values. There are now two modes $r = r-1$, and the distribution is '*bimodal*' (i.e., have two modes).

Thus, when $(n + 1)p$ is not an integer, say r, the binomial distribution has two modes, viz. r and $r - 1$.

12.6 POISSON DISTRIBUTION

Poisson distribution is a discrete probability distribution and is defined by the probability mass function (p.m.f.)

$$f(x) = \frac{e^{-m} . m^x}{x\,!}; \quad (x = 0, 1, 2, ..., \infty) \qquad (12.6.1)$$

where m is positive. The quantity e is a mathematical constant and is given by the infinite series

$$e = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \frac{1}{3\,!} + ... = 2.718 \text{ approx} \qquad (12.6.2)$$

The constant m appearing in (12.6.1) is known as 'parameter' of the Poisson distribution. It may be noted that the variable assumes an infinite number of values $x = 0, 1, 2, 3, ..., \infty$ with probabilities as shown below:

Table 12.5 Poisson Distribution

x	0	1	2	3	...	Total
$f(x)$	e^{-m}	$e^{-m} . m$	$\frac{e^{-m} . m^2}{2\,!}$	$\frac{e^{-m} . m^3}{3\,!}$	...	1

This distribution is named after its discoverer S.D. Poisson.

Important Properties

(1) Poisson distribution is a discrete probability distribution where the random variable assumes a countably infinite number of values 0, 1, 2, ..., ∞. The distribution is completely specified, when the parameter m (positive) is known.

(2) Mean = m, Variance = m

Standard deviation $(\sigma) = \sqrt{m}$ (12.6.3)

(3) Poisson distribution may have either one or two modes (like the binomial distribution). When m is not an integer, mode is the largest integer contained in m. However, when m is itself an integer, there are two modes, viz m and $m - 1$.

(4) Skewness $(\gamma_1) = \frac{1}{\sqrt{m}}$, Kurtosis $(\gamma_2) = \frac{1}{m}$ (12.6.4)

Thus, Poisson distribution is positively skew and leptokurtic.

(5) If x and y are independent Poisson variables with parameters m_1 and m_2 respectively then $(x + y)$ also follows Poisson distribution with parameter $(m_1 + m_2)$.

(6) Poisson distribution may be used as an approximation to binomial distribution, when p is small, and n is large, but np finite.

Some examples of Poisson variable are:

(i) Number of printing mistakes per page (or typographical errors per page by a skilled typist);

(ii) Number of goals scored in a foot ball match;

(iii) Number of telephone calls received in a telephone box per unit interval of time during a busy period;

(iv) Number of suicides (or deaths from a rare disease) per year in a given region;

(v) Number of bacteria present in a given liquid culture per unit area observed under the microscope;

(vi) Number of radio-active atoms decaying in a given interval of time;

(vii) Number of defects per unit area of sheet material (e.g. paper, cloth, or metal sheet);

(viii) Number of cars passing through a road crossing per unit time interval (e.g. 1 minute) during a busy period.

Example 12.15 *A random variable x follows Poisson distribution with parameter 3, Find the probabilities that the variable assumes the values (i) 0, 1, 2, 3; (ii) less than 3; (iii) at least 2, (Given that $e^{-3} = .0498$).*

Solution For the Poisson distribution with parameter $m = 3$, the probability of x successes is

$$f(x) = \frac{e^{-m}.\, m^x}{x\,!} = \frac{e^{-3}.\, 3^x}{x\,!}$$

(i) Putting $x = 0, 1, 2, 3$ successively, the required probabilities are

$$f(0) = \frac{e^{-3}.\, 3^0}{0\,!} = \frac{e^{-3}.\, 1}{1} = e^{-3} \text{ (since } 0\,! = 1 \text{ and } 3^0 = 1)$$
$$= .0498$$
$$f(1) = \frac{e^{-3}.\, 3^1}{1\,!} = \frac{e^{-3}.\, 3}{1} = 3\,.\; e^{-3}$$
$$= 3 \times .0498 = .1494$$
$$f(2) = \frac{e^{-3}.\, 3^2}{2\,!} = \frac{e^{-3}.\, 9}{2} = 4.5 \times e^{-3}$$
$$= 4.5 \times .0498 = .2241$$
$$f(3) = \frac{e^{-3}.\, 3^3}{3\,!} = \frac{e^{-3}.\, 27}{6} = 4.5 \times e^{-3}$$
$$= 4.5 \times .0498 = .2241$$

(ii) Since 'less than 3' implies 'either 0, or 1, or 2', the probability of less than 3 successes is given by the sum of the probabilities

$$f(0) + f(1) + f(2) = .0498 + .1494 + .2241 = 0.4233$$

(iii) Since 'at least 3' implies 'either 2, or 3, or 4, ..., ∞, the probability of at least 2 successes is given by the sum of the probabilities of the infinite number of terms. But since the sum of the probabilities for all values of x, viz. 0, 1, 2,, ..., ∞ is 1, i.e.,

$$f(0) + f(1) + f(2) + \;...\; = 1$$

hence
$$f(2) + f(3) + f(4) + \;...\; = 1 - f(0) - f(1)$$
$$= 1 - .0498 - .1494 = .8008$$

Example 12.16 *A business firm receives on an average 2.5 telephone calls per day during the time period 10.00 – 10.05 a.m. Find the probability that on a certain day, the firm receives (i) no call; (ii) exactly 4 calls, during the same period. (Assume Poisson distribution; given $e^{-2.5} = .0821$).*

Solution The random variable x is the 'number of telephone calls received during the period 10.00 – 10.05 a.m.'. Since x is assumed to follow Poisson distribution, the parameter m is equal to the mean of the distribution; i.e., $m = 2.5$. Hence, the probability of x calls is

$$f(x) = \frac{e^{-2.5}(2.5)^x}{x\,!}$$

(i) Probability of no call during the period (i.e., $x = 0$) is

$$f(0) = \frac{e^{-2.5}(2.5)^0}{0\,!} = \frac{e^{-2.5}\times 1}{1} = e^{-2.5} = .0821$$

(ii) Probability of exactly 4 calls during the period (i.e., $x = 4$) is

$$f(4) = \frac{e^{-2.5}(2.5)^4}{4\,!} = \frac{.0821\times 39.0625}{24} = .1336$$

Ans. .0821, .1336

Example 12.17 *Let x be distributed in the Poisson form. If $P(x = 1) = P(x = 2)$, what is $P(x = 0 \text{ or } 1)$. Also find $E(x)$.*

Solution For Poisson distribution, the p.m.f. is

$$f(x) = \frac{e^{-m}m^x}{x\,!}$$

The probability that x assumes the value 1 is then given by $f(1)$, i.e., the value of $f(x)$ when $x = 1$.

$$P(x = 1) = f(1) = \frac{e^{-m}m^1}{1\,!} = e^{-m}\cdot m$$

Similarly, the probability that x assumes the value 2 is $f(2)$.

$$P(x = 2) = f(2) = \frac{e^{-m}.\,m^2}{2\,!} = \frac{e^{-m}m^2}{2}$$

But it is given that

$$P(x = 1) = P(x = 2)$$

or,

$$e^{-m}.\,m = \frac{e^{-m}.\,m^2}{2}$$

Dividing both sides by $(e^{-m}\cdot m)$, we have

$$1 = \frac{m}{2};\quad \text{i.e., } m = 2$$

Therefore, the probability that x assumes the value 0 or 1 is

$$\begin{aligned} P(x = 0 \text{ or } 1) &= P(x = 0) + P(x = 1) \\ &= f(0) + f(1) \\ &= e^{-m}(1 + m) = e^{-2}(1 + 2) = 3e^{-2} \end{aligned}$$

$$E(x) = m = 2.$$

Ans. $3e^{-2}$, 2.

Example 12.18 *Find the mean, and variance of Poisson distribution.*

[C.U., B.Sc.(Econ.)'81; M.Com. '75, '77]

Solution Poisson distribution is defined by the probability mass function (p.m.f.)

$$f(x) = e^{-m} \frac{m^x}{x!} \quad (x = 0, 1, 2, \ldots, \infty)$$

$$\begin{aligned}
\text{Mean} &= E(x) = \sum_{x=0}^{\infty} x \cdot f(x) \\
&= 0 \,.\, f(0) + 1 \,.\, f(1) + 2 \,.\, f(2) + 3 \,.\, f(3) + \ldots \text{ to infinity.} \\
&= 0 + 1 \,.\, e^{-m} \frac{m}{1!} + 2 \,.\, e^{-m} \frac{m^2}{2!} + 3 \,.\, e^{-m} \frac{m^3}{3!} + \ldots \\
&= e^{-m} \,.\, m + e^{-m} \frac{m^2}{2!} + e^{-m} \,.\, \frac{m^2}{2!} + \ldots \\
&= m \,.\, e^{-m} \left(1 + \frac{m}{1!} + \frac{m^2}{2!} + \ldots\right) \\
&= m \,.\, e^{-m} \,.\, e^m \\
&= m \\
\text{Variance} &= E(x^2) - \{E(x)\}^2 \\
&= E(x^2) - m^2, \quad \text{since } E(x) = m \\
&= E\{x(x-1) + x\} - m^2 \\
&= E\{x(x-1)\} + m - m^2
\end{aligned}$$

Now,

$$\begin{aligned}
&= E\{x(x-1)\} = \sum_{x=0}^{\infty} x(x-1) \,.\, f(x) \\
&= 0 \,.\, f(0) + 1 \times 0 \,.\, f(1) + 2 \times 1 \,.\, f(2) + 3 \times 2 \,.\, f(3) + \ldots \\
&= 2 \,.\, f(2) + 3 \times 2 \,.\, f(3) + 4 \times 3 \,.\, f(4) + \ldots \\
&= 2 \,.\, e^{-m} \frac{m^2}{2!} + 3 \times 2 \,.\, e^{-m} \frac{m^3}{3!} + 4 \times 3 \,.\, e^{-m} \frac{m^4}{4!} + \ldots \\
&= e^{-m} \,.\, m^2 + e^{-m} \,.\, \frac{m^3}{1!} + e^{-m} \,.\, \frac{m^4}{2!} + \ldots \\
&= m^2 \,.\, e^{-m} \left(1 + \frac{m}{1!} + \frac{m^2}{2!} + \ldots\right) \\
&= m^2 e^{-m} \,.\, e^m = m^2
\end{aligned}$$

$$\therefore \quad \text{Variance} = m^2 + m - m^2 = m$$

Example 12.19 *Find the mode of Poisson distribution.*

[C.U., B.Sc. (Econ.). '81 (New)]

Solution In the Poisson distribution with parameter m, the random variable X assumes the values $0, 1, 2, \ldots \infty$ and the probabilities are given by

$$f(x) = P(X = x) = e^{-m} \frac{m^x}{x!}; \quad (x = 0, 1, 2, \ldots, \infty)$$

The mode is that value (or values) among $0, 1, 2, 3, \ldots, \infty$ corresponding to which the probabilities $f(x)$ is the maximum.

$$\frac{f(x-1)}{f(x)} = e^{-m} \cdot \frac{\dfrac{m^{x-1}}{(x-1)!}}{e^{-m} \cdot \dfrac{m^x}{x!}} \qquad (x \neq 0)$$

$$= \frac{x}{m} \qquad \text{... (1)}$$

We find that:

(a) $f(x-1) < f(x)$, whenever $x < m$ .... (2)

(b) $f(x-1) = f(x)$, whenever $x = m$ .... (3)

(c) $f(x-1) > f(x)$, whenever $x > m$ .... (4)

Case I When m is not an integer,

Let r be the integral part of m. Since in a Poisson distribution x assumes only non-negative integral values, the result (3) is impossible here.

From (2), $f(x-1) < f(x)$, when $x = 1, 2, 3, ..., r$

$\therefore$ $f(0) < f(1) < f(2) < ... < f(r)$ ... (5)

From (4), $f(x-1) > f(x)$, when $x = r+1, r+2, ...$

$\therefore$ $f(r) > f(r+1) > f(r+2) > ... ...$ ... (6)

Combining the results (5) and (6), we have

$$f(0) < f(1) < ... < f(r-1) < f(r) > f(r+1) > ...$$

Thus, the mode of Poisson distribution is r, i.e., the largest integer contained in the parameter m.

Case II When m is an integer, say $m = r$

From (2), $f(x-1) < f(x)$, when $x = 1, 2, 3, ..., (r-1)$

$\therefore$ $f(0) < f(1) < f(2) < ... < f(r-1)$ ... (7)

From (3), $f(x-1) = f(x)$, when $x = r$

$\therefore$ $f(r-1) = f(r)$ ... (8)

From (4), $f(x-1) > f(x)$, when $x = r+1, r+2, ...$

$\therefore$ $f(r) > f(r+1) > f(r+2) > ... ...$ ... (9)

Combining the results (7), (8) and (9)

$$f(0) < f(1) < ... < f(r-1) = f(r) > f(r+1) > ... ...$$

We find that two terms $f(r)$ and $f(r-1)$ are equal and the largest of all. Thus the Poisson distribution is bimodal when m is an integer, the modes being m and $m-1$.

Note: This method is identical with that for binomial distribution:

12.7 POISSON APPROXIMATION TO BINOMIAL DISTRIBUTION

Poisson distribution (12.6.1) may be obtained as a limiting case of Binomial distribution (12.5.1) under the following conditions:

(i) the number of trials n is infinitely large; i.e., $n \to \infty$;

(ii) the probability of success (p) is extremely small; i.e., $p \to 0$;

(iii) the mean $np = m$ is finite.

Under these conditions it can be shown that the probability of x successes in Binomial distribution can be closely approximated by the probability of x successes in Poisson distribution with parameter $m = np$. i.e.,

$$^nC_x\, p^x\, q^{n-x} = \frac{e^{-m}\, m^x}{x\,!} \tag{12.7.1}$$

Proof $$^nC_x\, p^x\, q^{n-x} = \frac{n(n-1)(n-2)\,...\,(n-x+1)}{x\,!}\, p^x(1-p)^{n-x}$$

$$= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{x-1}{n}\right)}{x\,!}\,(np)^x\left(1-\frac{np}{n}\right)^{n-x}$$

$$= \frac{1}{x\,!}\left[1.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{x-1}{n}\right)\right] m^x\left(1-\frac{m}{n}\right)^{n-x}$$

As $n \to \infty$, each of the factors $\left(1-\frac{1}{n}\right), \left(1-\frac{2}{n}\right), ..., \left(1-\frac{x-1}{n}\right)$ tends to 1 and $\left(1-\frac{m}{n}\right)^{n-x}$ tends to the limiting value e^{-m}. Thus, under the conditions stated above.

$$^nC_x\, p^x\, q^{n-x} \simeq \frac{1}{x\,!}\,[1, 1, 1, ...\, 1]\, m^x\, e^{-m}$$

$$= e^{-m}, \frac{m^x}{x\,!}$$

Since the probabilities of Poisson distribution are easier to compute than those of Binomial distribution, for practical purposes the Poisson approximation is used when p is less than 0.1 and np is not very large (say, less than 10).

Poisson distribution thus funds important application in such phenomenon where the probability of occurrence of an event (success) is extremely small, but the number of opportunities is infinitely large, so that the mean is something finite. Hence this distribution is sometimes known as the "distribution of rare events".

Example 12.20 *In turning out certain toys in a manufacturing process in a factory, the average number of defectives is 10%. What is the probability of getting exactly 3 defectives in a sample of 10 toys chosen at random, by using the Poisson approximation to the binomial distribution? (Take e = 2.72).*

[I.C.W.A., Dec. '78]

Solution Let the occurrence of a defective toy be called a "success". The number of defectives (x) follows binomial distribution with parameters $n = 10$ and $p = 10\% = 0.1$. If this distribution is to be approximated by Poisson distribution

$$f(x) = e^{-m}\,.\,\frac{m^x}{x\,!}$$

we have $m = np = 10 \times 0.1 = 1$. Hence the probability of 3 defective in the sample is

$$f(3) = e^{-1}\,.\,\frac{1^3}{3\,!} = e^{-1}\,.\,\frac{1}{6} = \frac{1}{6e}$$

$$= \frac{1}{6 \times 2.72} = 0.61$$

[**Note:** Using binomial distribution, the true probability is ${}^{10}C_3(0.1)^3(0.9)^7] = .057$ (upto 3 decimals. Even for a small $n = 10$, this agrees fairly well with the Poisson approximation 061 (the difference being only .004).]

Example 12.21 *2% of the items made by a machine are defective. Find the probability that 3 or more items are defective in a sample of 100 items (Given $e^{-1} = 0.368$, $e^{-2} = 0.135$, $e^{-3} = .0498$).* [Dip. Management '77]

Solution The number of defectives (x) follows binomial distribution. Since $p = 2\% = 0.2$ is small, and $n = 100$ is fairly large, making $np = 100 \times .02 = 2$ a finite quantity, we use Poisson approximation to binomial distribution with $m = np = 2$.

$$f(x) = e^{-m} \cdot \frac{m^x}{x\,!} = e^{-2}\frac{2^x}{x\,!}$$

Probability of 3 or more defectives

$$= f(3) + f(4) + f(5) + \dots \dots$$

$$= 1 - [f(0) + f(1) + f(2)]$$

$$= 1 - e^{-2}\left[1 + 2 + \frac{2^2}{2\,!}\right]$$

$$= 1 - 0.135 \times 5$$

$$= 0.325$$

12.8 HYPERGEOMETRIC DISTRIBUTION

Suppose that a box contains N balls among which A balls are white and the remaining $N - A$ are black. If n balls are drawn at random *without replacement*, the probability of obtaining x white balls (and obviously $n - x$ black balls) among them is given by

$$f(x) = \frac{{}^{A}C_x \; {}^{N-A}C_{n-x}}{{}^{N}C_n}\,;\ (x = 0, 1, 2, \dots, m) \tag{12.8.1}$$

where m is the smaller of the positive integers n and A. The probability distribution defined by the probability mass function (12.8.1) is known as *Hypergeometric Distribution*.

It may be noted that (12.8.1) satisfies both the conditions for a p.m.f. viz. (i) $f(x) \geq 0$; because the numerator and denominator are positive integers; (ii) $\Sigma f(x) = 1$; because the L.H.S. represents the sum of probabilities for all mutually exclusively and exhaustive events $x = 0, 1, 2, \dots, m$.

[The relation (ii) may also be obtained by using the identity

$$\sum_{x=0}^{m} {}^{A}C_x \, {}^{N-A}C_{n-x} = {}^{N}C_n$$

i.e. $\;{}^{A}C_0 \cdot {}^{A}C_n + {}^{A}C_1 \cdot {}^{N-A}C_{n-1} + {}^{A}C_2 \cdot {}^{N-A}C_{n-2} + \dots + {}^{A}C_m \cdot {}^{N-A}C_{n-m} = {}^{N}C_n$

This may be established by equating the coefficients of y^n on both sides of the identity $(1 + y)^A . (1 + y)^{N-A} = (1 + y)^N$, where N and A are positive integers.]

The p.m.f. of Hypergeometric distribution may also be given in another form. Let us put $A = Np$ in (12.8.1), so that $p = A/N$ is the proportion of white balls in the box. Then $N - A = N(1 - p) = Nq$ (suppose), where $q = 1 - p$ is the proportion of black balls. Thus we can write

$$f(x) = \frac{{}^{Np}C_x \; {}^{Nq}C_{n-x}}{{}^{N}C_n} \; ; (x = 0, 1, 2, ..., m) \tag{12.8.2}$$

where $p + q = 1$ (p and q being positive proper fractions), and m is the smaller of the numbers n and Np.

Hypergeometric distribution has three parameters, viz. N, n and A in (12.8.1), or N, n and p in (12.8.2).

[**Note:** The model for hypergeometric distribution (12.8.2) is somewhat analogous to that of binomial distribution (12.5.1). If n balls are drawn at random from a box containing Np white and Nq black balls ($p + q = 1$), then the probability of obtaining exactly x white balls is given by

$${}^{Np}C_x \; {}^{Nq}C_{n-x} / {}^{N}C_n$$

if the drawing is made *without replacement*, and

$${}^{n}C_x \, p_x \, q^{n-x}$$

if the drawing is made x *with replacement*.]

Important Properties:

1. Hypergeometric distribution is a discrete probability distribution where the random variable assumes a finite number of values 0, 1, 2, ..., m. It has three parameters—(N, n, A) in (12.8.1) and (N, n, p) in (12.8.2).
2. For the hypergeometric distribution (12.8.1)

$$\text{Mean} = \frac{nA}{N}; \quad \text{Variance} = \frac{nA(N-A)(N-n)}{N^2(N^2-1)} \tag{12.8.3}$$

These expressions take convenient forms on putting $A = Np$. Thus, for the hypergeometric distribution (12.8.2),

$$\text{Mean} = np, \quad \text{Variance} = npq\left(\frac{N-n}{N-1}\right) \tag{12.8.4}$$

[These results may be compared with those of binomial distribution at (2.5.2)]

3. The binomial distribution may be obtained as a limiting case of Hypergeometric distribution when $N \to \infty$

Proof The p.m.f. of hypergeometric distribution is

$$\frac{{}^{Np}C_x \; {}^{Nq}C_{n-x}}{{}^{N}C_n} = \frac{\dfrac{Np(Np-1)\,...\,(Np-x+1)}{x\,!} \cdot \dfrac{Nq(Nq-1)\,...\,(Nq-n+x+1)}{(n-x)\,!}}{\dfrac{N(N-1)\,...\,(N-n+1)}{n\,!}}$$

$$= \frac{n\,!}{x\,!(n-x)\,!} \cdot \frac{\overbrace{Np.(Np-1)\ldots(Np-x+1)}^{x \text{ factors}} \cdot \overbrace{Nq.(Nq-1)\ldots(Nq-n+x+1)}^{n-x \text{ factors}}}{\underbrace{N(N-1)\ldots\ldots\ldots(N-n+1)}_{n \text{ factors}}}$$

$$= {}^nC_x \cdot \frac{p \cdot \left(p - \frac{1}{N}\right) \cdots \left(p - \frac{x-1}{N}\right) \cdot q \cdot \left(q - \frac{1}{N}\right) \cdots \left(q - \frac{n-x-1}{N}\right)}{1 \cdot \left(1 - \frac{1}{N}\right) \cdots \left(1 - \frac{n-1}{N}\right)}$$

(dividing each factor in the numerator and denominator by N)

$$\therefore \quad \underset{N \to \infty}{Lt} \frac{{}^{Np}C_x \cdot {}^{Nq}C_{n-x}}{{}^NC_n} = {}^nC_x \cdot \frac{(p \cdot p \ldots p)(q \cdot q \ldots q)}{1 \cdot 1 \ldots \ldots 1}$$

$$= {}^nC_x \, p^x \, q^{n-x}$$

this is the p.m.f. of binomial distribution. (Proved)

Example 12.22 *Two defective tube-lights are mixed with 8 non-defective tube-lights by mistake. A sample of 3 tubes is taken at random from the lot and tested for the number of defectives. Find the probability distrib··tion of the 'number of defective tube-lights in the sample' and tabulate the probabilities. What are the mean and standard deviation of the number of defective obtained in the sample?*

Solution The 'number of defectives (x) in the sample' of 3 tubes may be $x = 0, 1, 2$ (because the whole lot contains only 2 defectives). Since the tubes may be assumed to have been drawn without replacement, x follows Hypergeometric distribution.

$$f(x) = \frac{{}^2C_x \cdot {}^3C_{3-x}}{{}^{10}C_3} = \frac{1}{120} \left({}^2C_x \, {}^3C_{3-x}\right); \ (x = 0, 1, 2)$$

Putting $x = 0, 1, 2$, and simplifying.

$$f(0) = \frac{7}{15}, \quad f(1) = \frac{7}{15}, \quad f(2) = \frac{1}{15}.$$

The probability distribution of the number (x) of defective tube-lights is:

x	0	1	2	Total
Probability	7/15	7/15	1/15	1

Here $N = 10$, $n = 3$, and the proportion of defectives in the lot is $p = 2/10 = 0.2$. Using (12.8.4)

$$\text{Mean} = 3 \times 0.2 = 0.6$$

$$\text{Variance} = 3 \times 0.2 \times 0.8 \left(\frac{10-3}{10-1}\right) = \frac{3.36}{9}$$

$$\therefore \quad \text{Standard deviation} = \sqrt{3.36/9} = 1.83/3 = 0.61$$

12.9 MULTINOMIAL DISTRIBUTION

The binomial distribution is applicable only in cases when the possible outcomes are of 2 types—'success' and 'failure'. In general, if the possible outcomes of a random experiment are of k types, say $E_1, E_2, \ldots, E_k$ with constant probabilities $p_1, p_2, \ldots, p_k$ respectively, then the probability that in n independent trials E_1 occurs x_1 times, E_2 occurs x_2 times, ..., E_k occurs x_k times is given by

$$\frac{n!}{x_1!\,x_2!\dots x_k!}\,(p_1)^{x_1}(p_2)^{x_2}\dots(p_k)^{x_k} \tag{12.9.1}$$

where $x_1 + x_2 + \dots + x_k = n$ and $p_1 + p_2 + \dots + p_k = 1$.

The discrete probability distribution defined by the probability function (12.9.1) is called *Multinomial Distribution*. The binomial distribution may be obtained as a special case of Multinomial distribution, when $k = 2$.

Example 12.23 *A die is rolled 6 times. Find the probability that each of the six faces appears exactly once.*

Solution In a single throw of the die, there are 6 possible outcomes, viz. 1, 2, ..., 6 with probability 1/6 for each (the die is assumed to be unbiased); and 6 independent trials are made because the die is rolled 6 times. Hence

$$p_1 = \frac{1}{6};\ p_2 = \frac{1}{6};\ p_3 = \frac{1}{6};\ p_4 = \frac{1}{6};\ p_5 = \frac{1}{6};\ p_6 = \frac{1}{6}$$

$k = 6$ and $n = 6$. We have to find the probability that

$$x_1 = 1,\quad x_2 = 1,\quad x_3 = 1,\quad x_4 = 1,\quad x_5 = 1,\quad x_6 = 1$$

Substituting the values in (12.9.1), the required probability is

$$\frac{6!}{1!1!1!1!1!1!}\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1$$

$$= 720\left(\frac{1}{6}\right)^6 = \frac{720}{6^6} = \frac{5}{324}$$

Example 12.24 *A box contains 4 white, 3 black and 5 red balls. A ball is drawn from the box at random, its colour noted, and then the ball is replaced. If 6 balls are drawn in this manner (i.e., with replacement), find the probability that (i) 3 are white, 1 black and 2 red, (ii) 2 are white and 4 red.*

Solution Here the 6 trials are independent and the outcome of any trial may be one of the 3 types—white, black, red. Since there are 12 balls in the box (4 white, 3 black, 5 red), therefore, in a single trial.

$$p_1 = \text{probability of white ball} = \frac{4}{12} = \frac{1}{3}$$

$$p_2 = \text{probability of black ball} = \frac{3}{12} = \frac{1}{4}$$

$$p_3 = \text{probability of red ball} = \frac{5}{12}$$

(i) Probability that among $n = 6$ balls drawn, $x_1 = 3$, $x_2 = 1$ and $x_3 = 2$ is

$$\frac{6!}{6!1!2!}\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right)^1\left(\frac{5}{12}\right)^2$$

$$= \frac{720}{6\times 2}\left(\frac{1}{27}\right)\left(\frac{1}{4}\right)\left(\frac{25}{144}\right) = \frac{125}{1296}$$

(ii) Probability that 2 are white and 4 red ($x_1 = 2, x_2 = 0, x_3 = 4$) is

$$\frac{6\,!}{2\,!\,0\,!\,4\,!}\left(\frac{1}{3}\right)^2\left(\frac{1}{4}\right)^0\left(\frac{5}{12}\right)^4$$

$$= 15 \times \frac{1}{9} \times 1 \times \frac{625}{20736} = \frac{3125}{62208}$$

12.10 JOINT DISTRIBUTION OF TWO VARIABLES

Let us suppose that a random variable x assumes 3 possible values x_1, x_2, x_3 corresponding to each of which another random variable y assumes 4 possible values y_1, y_2, y_3, y_4. There are then 12 pairs of values (x_i, y_j), $i = 1, 2, 3; j = 1, 2, 3, 4$. Let p_{ij} denote the probability assigned to the pair (x_i, y_j).

$$p_{ij} = \text{Prob}\ (x = x_i,\ y = y_i) \qquad (12.10.1)$$

All the possible values of the pair (x, y) and the corresponding probabilities can be shown in a *Joint Distribution* (section 11.21) as follows:

Table 12.6 Joint Distribution of x and y

y/x	y_1	y_2	y_3	y_4	Total
x_1	p_{11}	p_{12}	p_{13}	p_{14}	$p_{1\cdot}$
x_2	p_{21}	p_{22}	p_{23}	p_{24}	$p_{2\cdot}$
x_3	p_{31}	p_{32}	p_{33}	p_{34}	$p_{3\cdot}$
Total	$p_{\cdot 1}$	$p_{\cdot 2}$	$p_{\cdot 3}$	$p_{\cdot 4}$	1

Obviously, the sum of the probabilities p_{ij} in all rows and columns is 1.

$$p_{11} + p_{12} + \ldots + p_{33} + p_{34} = 1$$

i.e.,

$$\sum_i \sum_j p_{ij} = 1 \qquad (12.10.2)$$

We use the symbols $p_{1\cdot}, p_{2\cdot}, p_{3\cdot}$ to denote the row-totals and $p_{\cdot 1}, p_{\cdot 2}, p_{\cdot 3}, p_{\cdot 4}$ to denote the column-totals. These are known as *marginal probabilities*.

[Note: that p_1 is the total of all p's with the first subscript 1, and $p_{\cdot 1}$ is the total of all p's whose second subscript is 1, etc.]

$$p_{i\cdot} = p_{i1} + p_{i2} + p_{i3} + p_{i4} = \sum_j p_{ij} \qquad (12.10.3)$$

$$p_{\cdot j} = p_{1j} + p_{2j} + p_{3j} + p_{4j} = \sum_i p_{ij}$$

The sum of the marginal probabilities for x (or for y) is therefore 1.

$$p_{1\cdot} + p_{2\cdot} + p_{3\cdot} = \sum_i p_{i\cdot} = 1$$

$$(12.10.4)$$

$$p_{\cdot 1} + p_{\cdot 2} + p_{\cdot 3} + p_{\cdot 4} = \sum_j p_{\cdot j} = 1$$

Marginal Distribution

Let us consider only the random variable x, which assumes the values x_1, x_2, x_3. The event $x = x_1$, for example, occurs in 4 mutually exclusive ways, in association with the values assumed by y, viz. as (x_1, y_1), (x_1, y_2) (x_1, y_3) (x_1, y_4). So, by the theorem of total probability,

$$
\begin{aligned}
P(x = x_1) &= P(x = x_1, y = y_1) + P(x = x_1, y = y_2) \\
&\qquad + P(x = x_1, y = y_3) + P(x = x_1, y = y_4) \\
&= p_{11} + p_{12} + p_{13} + p_{14} \\
&= p_{1\cdot}
\end{aligned}
$$

Similarly, $P(x = x_2) = p_{2\cdot}$ and $P(x = x_3) = p_{3\cdot}$

Thus the total of probabilities in any row (as shown in the margin) gives the probability of the corresponding value $x = x_i$.

The probability distribution of a random variable, obtained from the joint distribution *irrespective of the values assumed by the other variable,* is called the *Marginal Distribution*. It shows the possible values of the variable and their 'marginal probabilities'.

Table 12.7 Marginal Distribution of x

x	x_1	x_2	x_3	Total
Probability	$p_{1\cdot}$	$p_{2\cdot}$	$p_{3\cdot}$	1

Table 12.8 Marginal Distribution of y

y	y_1	y_2	y_3	y_4	Total
Probability	$p_{\cdot 1}$	$p_{\cdot 2}$	$p_{\cdot 3}$	$p_{\cdot 4}$	1

Conditional Distribution

By the definition of conditional probability (Section 11.17)

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

Suppose that A and B represent the events $x = x_1$ and $y = y_3$ respectively. Then the conditional probability of the event $x = x_1$, given the $y = y_3$ has happened, is

$$P(x = x_1 / y = y_2) = \frac{P(x = x_1,\ y = y_3)}{P(y = y_3)} = \frac{p_{13}}{p_{\cdot 3}}$$

Similarly $P\,(x = x_2/y = y_3) = \dfrac{p_{23}}{p_{\cdot 3}}$, $P\,(x = x_3/y = y_3) = \dfrac{p_{33}}{p_{\cdot 3}}$

This means that once $y = y_3$ is known to have occurred, we fix our attention to the entries in the two-way table against y_3. Then the conditional probabilities of x_1, x_2, x_3 are obtained on dividing the probabilities p_{13}, p_{23}, p_{33} (shown against y_3) by their total ($p_{\cdot 3}$).

The probability distribution of a random variable, *given that the other variable has a specified value*, is called *Conditional Distribution*. It shows the possible values of the variable and their conditional probabilities.

Table 12.9 Conditional Distribution of x
(given $y = y_3$)

x	x_1	x_2	x_3	Total
Probability	$p_{13}/p_{\cdot 3}$	$p_{23}/p_{\cdot 3}$	$p_{33}/p_{\cdot 3}$	1

[**Note:** The probabilities shown here are those given in the two-way table against $y = y_3$, but adjusted to a total on division by their total $p_{\cdot 3}$]

Table 12.10 Conditional Distribution of y
(given $x = x_2$)

y	y_1	y_2	y_3	y_4	Total
Probability	$p_{21}/p_{2\cdot}$	$p_{22}/p_{2\cdot}$	$p_{23}/p_{2\cdot}$	$p_{24}/p_{2\cdot}$	1

Similarly, conditional distributions of x (or of y) can be stated for other given values.

Example 12.25 *The following tables gives the joint distribution of x and y.*

x \ y	2	3	7	Total
1	*.10*	*.25*	*.05*	*.40*
3	*.30*	*.15*	*.15*	*.60*
Total	*.40*	*.40*	*.20*	*1*

(i) How many pairs of values of x and y are possible? Write them down and show the probability of each pair separately.
(ii) Show the marginal distribution of x and y.
(iii) Show the conditional distribution of x, given y = 2.
(iv) Show the conditional distribution of y, given x = 3.
(v) Find the probabilities $P(x < y)$, $P(2x + y \geq 9)$.

Solution

(i) There are 2 possible values of x, and 3 possible values of y, so that the number of possible pairs (x, y) is $2 \times 3 = 6$.

(x, y)	(1, 2)	(1, 3)	(1, 7)	(3, 2)	(3, 3)	(3, 7)	Total
Probability	.10	.25	.05	.30	.15	.15	1

(ii) *Marginal Distribution of x*

x	1	3	Total
Probability	.40	.60	1

Marginal Distribution of y

y	2	3	7	Total
Probability	.40	.40	.20	1

(iii) *Conditional Distribution of x (given y = 2)*

x	1	3	Total
Probability	$\frac{.10}{.40} = .25$	$\frac{.30}{.40} = .75$	1

(iv) *Conditional Distribution of y (given x = 3)*

y	2	3	7	Total
Probability	$\frac{.30}{.60} = .50$	$\frac{.15}{.60} = .25$	$\frac{.15}{.60} = .25$	1

(v) The event $x < y$ consists of the pairs of values (1, 2), (1, 3), (1, 7), (3, 7). Hence $P(x < y)$ is given by the sum of the probabilities of these pairs.

$$P(x < y) = .10 + .25 + .05 + .15 = 0.55$$

The event $2x + y \geq 9$ is satisfied by the pairs (1, 7), (3, 3), (3, 7).

$$\therefore \quad P(2x + y \geq 9) = .05 + .15 + .15 = .035$$

Independent Variables

The random variables x and y are said to be *"independent"*, if

$$p_{ij} = p_{i\cdot} \times p_{\cdot j} \tag{12.10.5}$$

i.e. $$P(x = x_i,\ y = y_i) = P(x = x_i) \times P(y = y_i)$$

for each and every pair of values $(x_i, y_i) : i = 1, 2, ... m ; j = 1, 2, ... n$. This means that if x and y are independent, then each entry p_{ij} in the joint distribution is the product of the corresponding marginal probabilities $p_{i\cdot}$ and $p_{\cdot f}$. If the condition (12.10.5) fails even for one pair of values (x_i, y_i), then the variables x and y will not be 'indepependent'.

Illustration In the following joint distribution, the variables x and y are 'independent'.

x \ y	0	2	4	Total
2	.06	.15	.09	.30
5	.14	.35	.21	.70
Total	.20	.50	.30	1

Note: That each cell probability is the product of the corresponding marginal probabilities of x and of y.

Bivariate Expectation and Covariance

If the joint distribution of x and y is given, the expectation of x (or of y) or of any function of a single variable can be calculated from the corresponding marginal distribution. Thus

$$E(x) = \sum_i p_i . x_i \qquad E(x^2) = \sum_j p_i . x_i^2$$

$$E(y) = \sum_j p_{.j}\, y_j \qquad E(y^2) = \sum_j p_{j} . y_j^2 \tag{12.10.6}$$

The variances of x and y are then obtained, as before

$$\text{Var}(x) = E(x^2) - \{E(x)\}^2$$
$$\text{Var}(y) = E(y^2) - \{E(y)\}^2 \tag{12.10.7}$$

In general, the bivariate expectation of any function of the random variables x and y is defined as the sum of products of all possible values of the function and the corresponding bivariate probabilities. For example

$$E(x + y) = \sum_i \sum_j p_{ij}\,(x_i + y_j)$$

$$E(xy) = \sum_i \sum_j p_{ij}\,(x_i\, y_j)$$

$$E(x - a) = \sum_i \sum_j p_{ij}\,(x_i - a) = \Sigma p_i (x_i - a)$$

$$E\{(x - a)(y - b)\} = \sum_i \sum_j p_{ij}\,(x_i - a)(y_j - b),$$

where a and b are constants.

If m_1, m_2 denote the means of x and y respectively, the expectation of $(x - m_2)$ $(y - m_2)$ is particularly important, and is called the *Covariance of x and y*, written as Covariance of x and y, written as Covariance (x, y).

$$\text{Cov}(x, y) = E\{(x - m_1)(y - m_2)\}$$
$$= E(xy) - E(x).E(y) \tag{12.10.8}$$

In case the variables x and y are independent

$$\text{Cov}(x, y) = 0 \tag{12.10.8a}$$

The following expectations are especially important:

(i) $E(x) = m_1; \qquad E(y) = m_2$

(ii) $E(x - m_1) = 0; \qquad E(y - m_2) = 0$

(iii) $E(x - m_1)^2 = \text{Var}(x); \quad E(y - m_2)^2 = \text{Var}(y)$

(iv) $E\{(x - m_1)(y - m_2)\} = \text{Cov}(x, y)$ (12.10.9)

Theorems of Expectation

Theorem I The expectation of the sum of two jointly distributed random variables x and y is the sum of the expectations. (12.10.10)

$$E(x + y) = E(x) + E(y)$$

Proof Let x and y be jointly distributed discrete random variables, and suppose that x takes numerically different values $x_1, x_2, \ldots x_m$ while y takes numerically different values $y_1, y_2, \ldots y_3$. Let p_{ij} denote the probability that x assumes the value x_i and simultaneously y assumes the values y_j.

$$P(x = x_i, y = y_j) = p_{ij}$$

The marginal probabilities are then given by

$$P(x = x_i) = \sum_j p_{ij} = p_{i\cdot}$$

$$P(y = y_j) = \sum_i p_{ij} = p_{\cdot j}$$

Hence, $$E(x) = \Sigma p_i \,.\, x_i \quad E(y) = \Sigma p_{\cdot j}\, y_j$$

Since x and y can assume m and n different values, there are mn mutually exclusive cases for their sum $x + y$. In general, when x assumes the value x_i and y assumes the value y_j, the sum $x + y$ assumes the values $x_j + y_j$. Therefore,

$$P(x + y = x_i + y_j) = P(x = x_i, y = y_j) = p_{ij}$$

Hence, by definition of expectation

$$\begin{aligned} E(x + y) &= \sum_i \sum_j p_{ij}\,(x_i + y_j) \\ &= \sum_i \sum_j p_{ij}\, x_i + \sum_i \sum_j p_{ij}\, y_j \\ &= \sum_i x_i \sum_j p_{ij} + \sum_j y_j \sum_i p_{ij} \\ &= \sum_i x_i p_i + \sum_j y_j p_{\cdot j} \\ &= E(x) + E(y) \end{aligned}$$

This theorem can be extended to more than two variables.

Theorem II The expectation of the product of two *independent* random variables x and y is the product of their expectations.

$$E(x, y) = E(x) \,.\, E(y) \qquad (12.10.11)$$

provided x and y are independent.

Proof Let x and y be independent discrete random variables and suppose that x takes m numerically different values $x_1, x_2, \ldots x_m$ and y takes n numerically different values $y_1, y_2, \ldots y_n$. Let us write

$$P(x = x_i, y = y_j) = p_{ij}$$

We have, as before,

$$E(x) = \sum_i p_i \,.\, x_i \text{ and } E(y) = \sum_j p_{\cdot j}\, y_j$$

Since x and y can assume m and n different values there are mn mutually exclusive cases for the product xy. In general, when x assumes the value x_i and y assumes the value y_j, the product xy assumes the value $x_i y_j$. Therefore,

$$P(xy = x_i y_j) = P(x = x_i, y = y_j) = p_{ij}$$

If x and y are *independent* random variables, we have

$$P(x = x_i, y = y_j) = P(x = x_i) \,.\, P(y = y_j)$$

i.e. $$p_{ij} = p_{i\cdot}\, p_{\cdot j}$$

Hence, by the definition of expectation

$$E(xy) = \sum_i \sum_j p_{ij}\,(x_i\, y_j)$$

$$= \sum_i \sum_j p_{i\cdot} p_{\cdot j} x_i y_j$$

$$= \sum_i p_{i\cdot} x_i \sum_j p_{\cdot j} y_j$$

$$= E(x) \,.\, E(y)$$

This theorem can be extended to several *mutually independent* random variables.

Correlation Coefficient

This *Correlation Coefficient* between x and y, which is denoted by the symbol ρ_{xy}, is defined as

$$\rho_{xy} = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y} \qquad (12.10.12)$$

where σ_x and σ_y are the standard deviations of x and y respectively. It has the following important properties :

(1) The correlation coefficient is independent of origin and scale of the variables. If $u = ax + b$ and $v = cy + d$, where a, b, c, d are constants, then

$$\rho_{uv} = \pm\, \rho_{xy}$$

according as a and c have the same sign or opposite signs.

(2) The correlation coefficient lies between – 1 and + 1.

$$-1 \leq \sigma_{xy} \leq +1$$

(3) If the variables x and y are independent, then $\rho_{xy} = 0$. But, if $\rho_{xy} = 0$, the variables may not be independent; they are then said to be *uncorrelated*.

Example 12.26 *The following table gives the joint distribution of x and y:*

x \ y	0	1	2
1	0.3	0.2	0.1
2	0.1	0.0	0.3

(i) Are x and y independent?

(ii) Determine the correlation coefficient between x and y. (W.B.H.S. '78)

Solution After completing the row and column-totals, the joint distribution of x and y is

x \ y	0	1	2	Total
1	0.3	0.2	0.1	0.6
2	0.1	0.0	0.3	0.4
Total	0.4	0.2	0.4	1

We find that $P(x = 1, y = 0) = 0.3$, $P(x = 1) = 0.6$ and $P(y = 0) = 0.4$. Since here $P(x = 1, y = 0) \neq P(x = 1) \,.\, P(y = 0)$ the variables are not independent.

From the marginal distribution of x,

$$E(x) = 0.6 \times 1 + 0.4 \times 2 = 1.4$$

$$E(x^2) = 0.6 \times 1^2 + 0.4 \times 2^2 = 2.2$$

$\therefore \quad Var(x) = 2.2 - (1.4)^2 = 2.2 - 1.96 = 0.24; \quad \sigma_x = \sqrt{0.24}$

Similarly from the marginal distribution of y,

$$E(y) = 0.4 \times 0 + 0.2 \times 1 + 0.4 \times 2 = 1.0$$
$$E(y^2) = 0.4 \times 0^2 + 0.2 \times 1^2 + 0.4 \times 2^2 = 1.8$$

$\therefore \quad Var(y) = 1.8 - (1.0)^2 = 0.8; \quad \sigma_y = \sqrt{0.8}$

Again, from the joint distribution, we have

$$\begin{aligned} E(xy) &= (1 \times 0) \times 0.3 + (1 \times 1) \times 0.2 + (1 \times 2) \times 0.1 \\ &\quad + (2 \times 0) \times 0.1 + (2 \times 1) \times 0.0 + (2 \times 2) \times 0.3 \\ &= 0 + 0.2 + 0.2 + 0 + 0 + 1.2 = 1.6 \\ Cov(x, y) &= E(xy) - E(x).E(y) = 1.6 - 1.4 \times 1.0 = 0.2 \end{aligned}$$

Now, using (12.10.9).

$$\rho_{xy} = \frac{0.2}{\sqrt{0.24 \times 0.8}} = \frac{0.2}{0.438} = 0.46$$

Example 12.27 *The marginal distributions of x and y are given in the following table:*

$y \backslash x$	5	7	Total
3			$\frac{1}{3}$
6			$\frac{2}{3}$
Total	$\frac{1}{2}$	$\frac{1}{2}$	1

If cov (x, y) = E(xy) – E(x). E(y) is known to be $-\frac{1}{2}$, *obtain the cell probabilities. Hence calculate* $P(x > y)$ *and* $P(x = \frac{5}{y} = 6)$. [C.U., B.Sc. '74]

Solution We find that $E(x) = \frac{1}{2} \times 5 + \frac{1}{2} \times 7 = 6$

$$E(y) = \frac{1}{3} \times 3 + \frac{2}{3} \times 6 = 5$$

Let us write p for the unknown cell probability $P(x = 5, y = 3)$. Then the other cell probabilities can be obtained from the marginal totals.

$y \backslash x$	5	7	Total
3	p	$\frac{1}{3} - p$	$\frac{1}{3}$
6	$\frac{1}{2} - p$	$\frac{1}{6} + p$	$\frac{2}{3}$
Total	$\frac{1}{2}$	$\frac{1}{2}$	1

Using these cells probabilities,

$$E(xy) = p\,(5 \times 3) + (\frac{1}{3} - p)\,(7 \times 3) + (\frac{1}{2} - p)\,(5 \times 6) + (\frac{1}{6} + p)\,(7 \times 6)$$
$$= 15\,p + (7 - 21p) + (15 - 30p) + (7 + 42p)$$
$$= 6p + 29$$

$\therefore$ $$\text{Cov}\,(x, y) = E(xy) - E(x).E(y)$$
$$= (6p + 29) - 6 \times 5 = 6p - 1$$

By the given condition,

$$6p - 1 = -\frac{1}{2}\ ; \qquad \therefore\ p = \frac{1}{12}$$

The other cell probabilities are then

$$\frac{1}{2} - p, = \frac{1}{4},\ \frac{1}{2} - p = \frac{5}{12},\ \frac{1}{6} + p = \frac{1}{4}.$$

The completed joint distribution is

y \ x	5	7	Total
3	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$
6	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{2}{3}$
Total	$\frac{1}{2}$	$\frac{1}{2}$	1

$$P\,(x > y) = \text{Sum of the probabilities of the pairs } (5, 3), (7, 3), (7, 6)$$
$$= \frac{1}{12} + \frac{1}{4} + \frac{1}{4} = \frac{7}{12}$$

$$P\,(x = 5/y = 6) = \frac{P\,(x = 5,\ y = 6)}{P\,(y = 6)} = \frac{\frac{5}{12}}{\frac{2}{3}} = \frac{5}{8}$$

Example 12.28 *Show that the correlation coefficient of x and y is zero if x and y are independence. Is the converse also true? Give reasons for your answer.* [W.B.H.S., '80]

Solution Let the variables x and y be independent and have means m_1, m_2 and variances σ_x^2, σ_y^2 respectively. Then,

$$\text{Cov}\,(x, y) = E\{(x - m_1)\,(y - m_2)\}$$
$$= E(x - m_1)\,.\,E(y - m_2), \text{ since independent}$$
$$= 0.0 = 0$$

Therefore, the correlation coefficient is

$$\rho_{xy} = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y} = \frac{0}{\sigma_x \sigma_y} = 0$$

This shows that the correlation coefficient is zero, if x and y are independent.

However, if $\rho_{xy} = 0$, we have

$$\text{Cov}(x, y) = 0$$

or, $$E(xy) - E(x) \cdot E(y) = 0$$

or, $$E(xy) = E(x) \cdot E(y)$$

or, $$\sum_i \sum_j p_{ij}\, x_j\, y_j = \sum_i p_i \cdot x_i \sum_i p_{\cdot j}\, y_j$$

This does not ensure that $p_{ij} = p_j \cdot p_{\cdot j}$, which is condition for independence of x and y. That is, the condition

$$P(x = x_j, y = y_j) = P(x = x_j) \cdot P(y = y_j)$$

is not necessarily true for all pairs of values (x_i, y_j).

Therefore, if $\rho_{xy} = 0$, the variables x and y may not be independent. The converse is not true.

Example 12.29 *If $u = ax + b$ and $v = cy + d$, where a, b, c, d are constants, show that $\rho_{uv} = \pm \rho_{xy}$.*

Solution Let us suppose that the random variables x and y have means m_1, m_2 and variances σ_1^2, σ_2^2 respectively;

i.e. $$E(x) = m_1,\ E(y) = m_2$$

$$E(x - m_1)^2 = \sigma_1^2,\ E(y - m_2)^2 = \sigma_2^2$$

Also, $$\text{Cov}(x, y) = E\{(x - m_1)(y - m_2)\}$$

and $$\rho_{xy} = \frac{\text{Cov}(x, y)}{\sigma_1 \sigma_2}$$

If we write $$u = ax + b,\ v = cy + d$$

then
$$\begin{aligned} m_u &= E(u) = E(ax + b) = E(ax) + E(b) \\ &= a.E(x) + b = am_1 + b \end{aligned}$$

Similarly, $$m_v = E(v) = E(cy + d) = cm_2 + d$$

Hence,
$$\begin{aligned} u - m_u &= (ax + b) - (am_1 + b) = a(x - m_1) \\ v - m_v &= (cy + d) - cm_2 + d) = c(y - m_2) \end{aligned}$$

The variances of u and v are

$$\begin{aligned} \sigma_u^2 &= E(u - m_u)^2 = E\{a^2(x - m_1)^2\} \\ &= a^2.E(x - m_1)^2 = a^2 \cdot \sigma_1^2 \end{aligned}$$

Similarly, $$\sigma_v^2 = c^2 \cdot \sigma_2^2$$

Since by definition, the standard deviation is always positive

$$\sigma_u = |a| \cdot \sigma_1, \quad \sigma_v = |c| \cdot \sigma_2$$

Again,
$$\begin{aligned} \text{Cov}(u, v) &= E\{(u - m_u)(v - m_v)\} \\ &= E\{ac(x - m_1)(y - m_2)\} \\ &= ac.E\{(x - m_1)(y - m_2)\} \\ &= ac.\text{Cov}(x, y) \end{aligned}$$

The correlation coefficient between u and v is

$$\begin{aligned} \rho_{uv} &= \frac{\text{Cov}(u, v)}{\sigma_u \sigma_v} = \frac{ac.\text{Cov}(x, y)}{|a|\sigma_1 |c| . \sigma_2} \\ &= \frac{ac}{|a|.|c|} \cdot \frac{\text{Cov}(x, y)}{\sigma_1 \sigma_2} = \frac{ac}{|a|.|c|}\, \rho_{xy} \end{aligned}$$

When a and c have the same sign (i.e. both are positive, or both negative), the product ac is positive and is the same $|a|$. $|c|$; so that $\rho_{uv} = \rho_{xy}$.

When a and c have opposite signs (i.e. one is positive and the other is negative), the product ac is negative. But $|a|$. $|c|$ is always positive and has the same numerical values as ac; Hence $\rho_{uv} = -\rho_{xy}$.

Thus, $\rho_{uv} = \pm \rho_{xy}$

[*Illustration* If the correlation coefficient between x and y is -0.7, then the correlation coefficient between (i) $u = 2x + 3$ and $v = 5y - 8$ is also -0.7; (ii) $u = -2x + 3$ and $v = 5v$ is $+0.7$.]

Example 12.30 *Prove that the correlation coefficient p between two jointly distributed random variables x and y always like between – 1 and + 1.*

Solution Let the random variables x and y have means m_1, m_2 and variances σ_1^2, σ_2^2 respectively. For any real value of k, the square quantity $\{(x - m_1) + k(y - m_2)\}^2$ can never be negative, and hence its expected value

$$E\{(x - m_1) + k(y - m_2)\}^2 \geq 0$$

Expanding the L.H. and taking expectations

$$E(x - m_1)^2 + k^2.E(y - m_2)^2 + 2k.E\{(x - m_1)(y - m_2)\} \geq 0$$

or, $$\sigma_1^2 + k^2.\sigma_2^2 + 2k \,.\, \mathrm{Cov}(x, y) \geq 0$$

or, $$\sigma_1^2 + k^2.\sigma_2^2 + 2k \,.\, \rho\sigma_1\sigma_2 \geq 0$$

or, $$(k^2\sigma_2^2 + 2k\,\sigma_2.\rho\sigma_1 + \rho^2\sigma_1^2) + \sigma_1^2(1 - \rho^2) \geq 0$$

i.e. $$(k\sigma_2 + \rho\sigma_2)^2 + \sigma_1^2(1 - \rho^2) \geq 0$$

Since this is true for any value of k, we have for $k = -\dfrac{\rho\sigma_1}{\sigma_2}$, the first term is zero and hence

$$\sigma_1^2(1 - \rho^2) \geq 0$$

or $$(1 - \rho^2) \geq 0$$

or, $$\rho^2 \leq 1$$

or, $$-1 \leq \rho \leq +1 \qquad \text{(Proved)}$$

Variance of the Sum (Difference)

If x and y are jointly distributed random variables, then

$$\mathrm{Var}(x + y) = \mathrm{Var}(x) + \mathrm{Var}(y) + 2\,\mathrm{Cov}(x, y)$$
$$\mathrm{Var}(x - y) = \mathrm{Var}(x) + \mathrm{Var}(y) - 2\,\mathrm{Cov}(x, y) \qquad (12.10.13)$$

Theorem $\mathrm{Var}(ax + by) = a^2\,\mathrm{Var}(x) + b^2\,\mathrm{Var}(y) + 2ab\,\mathrm{Cov}(x, y)$ (12.10.14)

where a and b are constants.

Proof Suppose that x and y have means m_1, m_2 and variances σ_1^2, σ_2^2 respectively.

If we write

$$z = ax + by$$

then $$E(z) = E(ax) + E(by) = a \,.\, E(x) + b.E(y)$$

or, $$m_z = am_1 + bm_2, \text{ where } m_z \text{ is the mean of } z$$

$\therefore$ $$z - m_z = (ax + by) - (am_1 + bm_2) = a(x - m_1) + b(y - m_2)$$

By definition,

$$\mathrm{Var}(z) = E(z - m_z)^2$$
$$= E[a(x - m_1) + b(y - m_2)]^2$$
$$= E[a^2(x - m_1)^2 + b^2 + (y - m_2)^2 + 2ab(x - m_1)(y - m_2)]$$

$$= a^2 . E(x - m_1)^2 + b^2 (y - m_2)^2 + 2ab.E\{(x - m_1)(y - m_2)\}$$

i.e. $\text{Var}(ax + by) = a^2 . \text{Var}(x) + b^2.\text{Var}(y) + 2ab.\text{Cov}(x, y)$ (Proved)

It may be noted that the results at (12.10.13) may be obtained from (12.10.14) on putting $a = 1, b = 1$ and $a = 1, b = -1$.

When x and y are independent, we have $\text{Cov}(x, y) = 0$, and hence

$$\text{Var}(x + y) = \text{Var}(x) + \text{Var}(y)$$
$$\text{Var}(x - y) = \text{Var}(x) + \text{Var}(y) \quad (12.10.15)$$

Thus the variance of the sum or difference of two independent random variables is the sum of their variances.

It may be noted from (12.10.12) that

$$\text{Cov}(x, y) = \rho_{xy}\, \sigma_x\, \sigma_y$$

and so the results (12.10.13) may also be written in the form

$$\text{Var}(x + y) = \sigma_x^2 + \sigma_y^2 + 2\rho_{xy}\, \sigma_x\, \sigma_y$$
$$\text{Var}(x - y) = \sigma_x^2 + \sigma_y^2 - 2\rho_{xy}\, \sigma_x\, \sigma_y \quad (12.10.16)$$

If x and y are uncorrelated, i.e. $\rho_{xy} = 0$ (they may not be independent) then

$$\text{Var}(x + y) = \sigma_x^2 + \sigma_y^2 = \text{Var}(x - y) \quad (12.10.17)$$

Example 12.31 *If the random variables x and y have the same standard deviation, show that u = x + y and v = x − y are uncorrelated.*

Solution Let the variables x and y have means m_1, m_2 and a common standard deviation σ.

$$E(x) = m_1,\ E(y) = m_2,\ E(x - m_1)^2 = E(y - m_2)^2 = \sigma^2$$

The means of u and v are then

$$m_u = E(x) + E(y) = m_1 + m_2$$
$$m_v = E(x) - E(y) = m_1 - m_2$$

Therefore,

$$u - m_u = (x + y) - (m_1 + m_2) = (x - m_1) + (y - m_2)$$
$$v - m_v = (x - y) - (m_1 - m_2) = (x - m_1) - (y - m_2)$$

By definition, the covariance of u and v is

$$\begin{aligned}\text{Cov}(u, v) &= E\{(u - m_u)(v - m_v)\}\\ &= E[\{(x - m_1) + (y - m_2)\}\ \{(x - m_1) - (y - m_2)\}]\\ &= E[(x - m_1)^2 - (y - m_2)^2]\\ &= E(x - m_1)^2 - E(y - m_2)^2\\ &= \sigma^2 - \sigma^2\\ &= 0\end{aligned}$$

If σ_u and σ_v denote the standard deviations of u and v respectively, the correlation coefficient between u and v is

$$\rho = \frac{Cov(u, v)}{\sigma_u\, \sigma_v} = \frac{0}{\sigma_u\, .\, \sigma_v} = 0$$

i.e. u and v are uncorrelated.

Example 12.32 *Let x and y be independent random variables with standard deviations σ_x and σ_y. Show that the correlation co-efficient between x and x + y is*

$$\frac{\sigma_x}{\sqrt{(\sigma_x^2 + \sigma_y^2)}}$$

Solution Let m_x and m_y denote the means of x and y respectively.

$$E(x) = m_x, \qquad E(y) = m_y$$
$$E(x - m_y)^2 = \sigma_x^2 \qquad E(y - m_y)^2 = \sigma_y^2$$

If we write $u = x + y$, we have to find the correlation coefficient between x and u.

$$\rho_{xu} = \frac{Cov\,(x,u)}{\sigma_x \sigma_u} \qquad \text{...(i)}$$

The mean of u is

$$E\,(u) = E(x + y) = E(x) + E(y)$$

or,

$$m_u = m_x + m_y$$

Hence,

$$u - m_u = (x + y) - (m_x + m_y) = (x - m_x) + (y - m_y)$$

Now,

$$\text{Var}\,(u) = \text{Var}\,(x + y) = \text{Var}\,(x) + \text{Var}\,(y) + 2.\text{Cov}\,(x, y)$$
$$= \sigma_x^2 + \sigma_y^2,$$

since x and y are independent, Cov $(x, y) = 0$

i.e.

$$\sigma_u = \sqrt{(\sigma_x^2 + \sigma_y^2)} \qquad \text{...(ii)}$$

Again,

$$\text{Cov}\,(x, u) = E\{(x - m_x)(u - m_u)\}$$
$$= E\,[(x - m_x)\{(x - m_x) + (y - m_y)\}]$$
$$= E(x - m_x)^2 + E\{(x - m_x)(y - m_y)\}$$
$$= \sigma_x^2 + \text{Cov}\,(x, y)$$
$$= \sigma_x^2, \text{ since Cov}\,(x, y) = 0 \qquad \text{...(iii)}$$

Substituting from (ii) and (iii) in (i), the required correlation coefficient is

$$\rho_{z,\,x+y} = \frac{\sigma_x^2}{\sigma_x \cdot \sqrt{(\sigma_x^2 + \sigma_y^2)}} = \frac{\sigma_x}{\sqrt{(\sigma_x^2 + \sigma_y^2)}}$$

Example 12.33 *If X and Y are independent binomial variables with parameters (m, p) and (n, p) respectively, show that the their sum X + Y has a binomial distribution with parameters (m + n, p).*

Solution The probability that X assumes an arbitrary value x is

$$P(X = x) = {}^mC_x\, p^x\, q^{m-x} \qquad \text{...(i)}$$

for all values of $x = 0, 1, 2, \ldots m$, where $p + q = 1$. Similarly, the probability that Y assumes an arbitrary value y is

$$P(Y = y) = {}^nC_y\, p^y q^{n-y} \qquad \text{...(ii)}$$

for all values of $y = 0, 1, 2, \ldots n$.

Let us write $Z = X + Y$. Since the random variable X can assume the values 0, 1, 2, ... m and Y can assume the values 0, 1, 2, ...n, their sum Z can assume the values 0, 1, 2, ..., $(m + n)$.

The event that $Z = X + Y$ can assume an arbitrary value $z(z = 0, 1, 2, \ldots m + n)$ can materialise in the following mutually exclusive ways, viz. $(X = 0, Y = z)$, $(X = 1, Y = z - 1)$, ..., $(X = z, Y = 0)$. Therefore by the Addition Theorem of Probability.

$$P\,(Z = z) = P\,(X = 0, Y = z) + P\,(X = 1, Y = z - 1) + \ldots + P\,(X - z, Y = 0)$$

$$= \sum_{r \to 0}^{z} P\;(X = r, Y = z - r)$$

$$= \Sigma\, P(X = r)\,.\,P\,(Y = z - r),$$

since X and Y are independent.

Using (i) and (ii), we have

$$P(Z = z) = \sum_{r \to 0}^{z} {}^{m}C_r\, p^r\, q^{m-r}\, {}^{n}C_{z-r}\, p^{z-r}\, q^{n-(z-r)}$$

$$= p^z\, a^{m+n-z} \sum_{r \to 0}^{z} {}^{m}C_r \,.\, {}^{n}C_{z-r}$$

$$= {}^{m+n}C_z\, p^z\, q^{m+n-z} \quad (z = 0, 1, 2, \ldots .\, m + n)$$

This shows that the random variable $Z = X + Y$ has a binomial distribution with parameters $(m + n, p)$.

Example 12.34 *If X and Y are independent Poisson variates, find the conditional distribution of X, given X + Y. What are the mean and variance of this conditional distribution?*

Solution If X and Y are Poisson variates with parameters m and n respectively, then

$$P(X = x) = \frac{e^{-m} m^x}{x!}\,; \qquad P(Y = y) = \frac{e^{-n} n^y}{y!} \qquad \ldots(i)$$

$$x = 0, 1, 2, \ldots \infty\,; \qquad y = 0, 1, 2, \ldots .\, \infty$$

We know that the sum of two independent Poisson variates also follows a Poisson distribution. Therefore, $Z = X + Z$ follows Poisson distribution with parameter $m + n$. This means that the probability of Z to assume a specified value z is given by

$$P(Z = z) = \frac{e^{-(m+n)}\,(m+n)^z}{z!} \qquad \ldots(ii)$$

$$z = 0, 1, 2, \ldots \infty$$

The conditional probability that X takes an arbitrary value x, when $Z = X + Y$ is known to have assume a fixed value z is, by definition,

$$P(X = x/Z = z) = \frac{P(X = x, Z = z)}{P(Z = z)} \qquad \ldots(iii)$$

$$\left[\text{Remember the formula } P(A/B) = \frac{P(A \cap B)}{P(B)}\right]$$

But the event $(X = x$ and $Z = z)$ is the same as $(X = x$ and $X + Y = z)$, which is equivalent to $(X = x$ and $Y = z - x)$. Therefore,

$$P(X = x, Z = z) = P(X = x) \cdot P(Y = z - x)$$

since X and Y are independent

$$= \frac{e^{-m} m^x}{x!} \cdot \frac{e^{-n} n^{z-x}}{(z-x)!} \text{ using (i)}$$

$$= \frac{e^{-(m+n)} \,.\, m^x n^{2-x}}{x!(z-x)!}\,; \qquad \ldots(iv)$$

$$x = 0, 1, 2, \ldots z$$

(Note: Since the values of a Poisson variable are only non-negative whole numbers, x and z are necessarily whose numbers. Also x cannot exceed z because the value attained by X can not be larger than that of $X + Y$).

Substituting from (ii) and (iv) in (iii), and simplifying

$$P(X = x/Z = z) = \frac{z!}{x!(z-x)!} \cdot \frac{m^x n^{z-x}}{(m+n)^z}$$

$$= {}^zC_x \left(\frac{m}{m+n}\right)^x \left(\frac{.n}{m+n}\right)^{z-x}$$

$$(x = 0, 1, 2, \ldots z)$$

This is seen to be a binomial distribution with parameters z and $\frac{m}{(m+n)}$. Thus, the conditional distribution of X, given $X + Y$, is a binomial distribution. The mean and variance of this distribution are obtained by using the formulae for those of binomial distribution.

$$\text{Mean} = z \cdot \frac{m}{m+n}; \text{ Variance} = z \cdot \frac{m}{m+n} \cdot \frac{n}{m+n}$$

$$= \frac{zmn}{(m+n)^2}.$$

12.11 CONTINUOUS PROBABILITY DISTRIBUTION

Let x be a continuous random variable, which can assume any value in the interval (a, b), i.e. $a \le x \le b$. Since the number of possible values of x is uncountable infinite, we cannot assign a probability to each value of the variable, as is done in discrete probability distributions (Section 12.2). Therefore, in a continuous probability distribution we have to assign probabilities to intervals, and not to individual values.

As stated earlier (Section 11.19) for a continuous probability distribution let $f(x)$ be a non-negative function such that

$$P(c \le x \le d) = \int_c^d f(x)\, dx \qquad (12.11.1)$$

i.e. the probability of x lying in any given interval (c, d) can be obtained by integrating the function between the two limits of the interval. The function $f(x)$ is called the *probability density function* (p.d.f.) or simply *density function*, of the continuous random variable x. It satisfies the two conditions

$$\text{(i) } f(x) \ge 0 \qquad \text{(ii) } \int_c^b f(x)\, dx = 1 \qquad (12.11.2)$$

where $a \le x \le b$ is the range of x.

A continuous probability distribution is defined by stating the mathematical expression for the p.d.f. $f(x)$ and specifying the range (a, b). The curve represented by the equation

$$y = f(x)$$

is known as the '*probability curve*'. Geometrically, the integral of the p.d.f. represents the area under the probabilities curve, and hence

$P(c \le x \le d)$ = Area under probability curve between the vertical lines at c and d. (12.11.3)

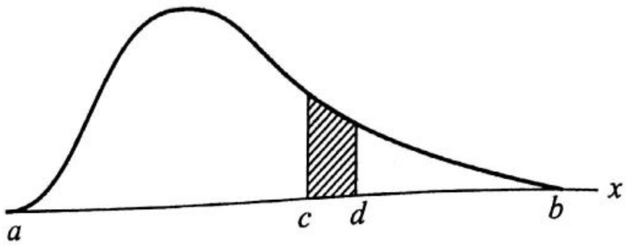


Fig. 12.1 Probability Curve

The cumulative distribution function (c.d.f.) $F(x)$, defined by (11.20.5) represents the probability that the random variable takes a value less than or equal to a specified value x. Geometrically, $F(x)$ shows the area under probability curve to the left of the ordinate at any specified value. For example,

$F(c) = P(x \le c)$
= Area under probability curve to the left of the vertical line at c. (12.11.4)

If the values of $F(x)$ are known for all values of x, then the probability distribution is completely given. Because then the probabilities of intervals can be obtained from the relation

$P(c \le x \le d)$ = Area between the ordinates at c and d
= (Area to the left of the ordinate at d) *minus* (Area to the left of the ordinate at c)
$= F(d) - F(c)$ (12.11.5)

Putting $c = d$, we see that

$P(c \le x \le c)$ = Area between the ordinates at c and c, which is zero.

i.e. $P(x = c) = 0$ (12.11.5a)

Thus, as shown in (11.20.8) earlier, for a continuous probability distribution, the probability that a random variable takes any specific value is zero.

Expectation–Mean, Variance (Continuous Distribution)

For a continuous probability distribution with p.d.f. $f(x)$ in the range (a, b), the expectation mean, variance, moments etc. are defined as follows :

$$\text{Mean } (\mu) = E(x) = \int_a^b x.f(x)\,dx \qquad (12.11.6)$$

$$\text{Variance } (\sigma^2) = E(x-\mu)^2 = \int_a^b (x-\mu)^2 \,.\, f(x)\,dx \qquad (12.11.7)$$

$$= E(x^2) - \mu^2 = \int_a^b x^2 f(x)\,dx - \mu^2 \qquad (12.11.8)$$

In general, the expectation of any function $g(x)$ is defined as

$$E\{g(x)\} = \int_a^b g(x) \,.\, f(x)\, dx \tag{12.11.9}$$

The moments are defined as follows:

r-th moment about A: $\mu_r' = E(x-A)^r = \int_a^b (x - A^r) \,.\, f(x)\, dx$

r-th raw moment: $\mu_r' = E(x^r) = \int_a^b x^r \,.\, f(x)\, dx$

r-th central moment: $\mu_r = E(x-\mu)^r = \int_a^b (x - \mu^r)\, f(x)\, dx$ (12.11.10)

The relations (12.3.8) should be used to find the central moments from non-central moments.

12.12 UNIFORM DISTRIBUTION (CONTINUOUS)

A continuous random variable x is said to follow *Uniform distribution*, if the probabilities associated with intervals of equal length are equal at any part of the range of values. The uniform distribution is defined by the density function

$$f(x) = \frac{1}{b-a}; \quad (a \le x \le b) \tag{12.12.1}$$

Since the probability curve for this distribution looks like a rectangle of height $\frac{1}{(b-a)}$ over the range $(a \le x \le b)$, the distribution is also known as "*Rectangular Distribution*". The total area under the curve is 1 and the probability that x lies between any two specified values c and d (within the range) is

$$P(c \le x \le d) = \frac{d-c}{b-a} \tag{12.12.2}$$

The cumulative distribution function (c.d.f.) is given by the part area of the rectangle to the left of the ordinate at x, i.e.

$$F(x) = \frac{x-a}{b-a}; \quad (a \le x \le b) \tag{12.12.3}$$

Example 12.35 *If a continuous random variable x follows rectangular distribution in the range (2, 7), find the probabilities:*

(i) $P(2, 5 \le x \le 4)$, *(ii)* $P(x \le 3.4)$

(iii) $P(x > 6)$ *(iv)* $P(x = 4.5)$.

Solution Using (12.12.2),

$$P(2.5 \le x \le 4) = \frac{4-2.5}{7-2} = \frac{1.5}{5} = 0.3$$

Using (12.12.3), $P(x \le 3.4) = \dfrac{3.4-2}{7-2} = \dfrac{1.4}{5} = 0.28$

$$P(x > 6) = 1 - P(x \le 6) = 1 - \frac{6-2}{7-2} = 0.2$$

Since x is a continuous random variable, by (12.11.5a),

$$P(x = 4.5) = 0$$

Ans. 0.3, 0.28, 0.2, 0

12.13 NORMAL DISTRIBUTION

Normal distribution, or *Gaussian distribution*, is a continuous probability distribution and is defined by the density function (p.d.f.)

$$f(x) = \frac{1}{\sigma\sqrt{2x}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \; ; (-\infty < x < \infty) \qquad (12.13.1)$$

where μ = mean, σ = standard deviation. [π and e are two mathematical constants having the approximate values $\frac{22}{7}$ and 2.718 (see 12.6.2). respectively]. μ and σ are known as '*parameters*' of this distribution

Although at first glance the expression (12.13.1) may look very complicated, this is the most useful distribution in theoretical statistics because of its many important characteristics. A random variable x is said to '*follow normal distribution*', or to be '*normally distributed*' if the probability density function is of the form (12.13.1). The probability curve of normal distribution is known as *Normal Curve*. The curve is symmetrical and bell-shaped (see Fig. 12.3) and the two tails extend to infinitely on either side.

If a random variable x is normally distributed with mean μ and standard deviation σ, then $z = \dfrac{(x-\mu)}{\sigma}$ is called the "*standard normal variable*" or "*normal deviate*". It has the density function

$$p(z) = \frac{1}{\sqrt{2x}}\, e^{-\frac{z^2}{2}} \; ; (-\infty < z < \infty) \qquad (12.13.2)$$

The continuous probability distribution defined by (12.13.2) is known as *Standard Normal Distribution*. This is in fact a special case of normal distribution with mean 0 and standard deviation 1.

Area under Normal Curve

As in all continuous probability distributions, the total area under the normal curve is 1; and the probability that x lies between c and d, denoted by $P(c \le x \le d)$, is given by the area under the curve between the vertical lines at c and d. This is also equal to the area under '*Standard normal curve*' between the vertical lines at the standardized values (page 230, Section 7.4) of c and d; i.e.

$P(c \le x \le d)$ = Area under '*standard normal curve*' between the vertical lines at c' and d'. (12.13.3) where $c' = \dfrac{(c-\mu)}{\sigma}$ and $d' = \dfrac{(d-\mu)}{\sigma}$.

Extensive tables showing the areas under standard normal curve are available for this purpose.

The cumulative distribution function (c.d.f.) of standard normal distribution, viz.

$\Phi(z)$ = Probability that the standard normal variable takes a value less than or equal to z.

= Area under '*standard normal curve*' to the *left* of the ordinate at z. (12.13.4)

Mathematically,

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}\, dz \qquad (12.13.4a)$$

and hence $\Phi(z)$ is also known as *Probability Integral* of standard normal distribution. The values of $\Phi(z)$ are given in statistical tables only for positive values of z. For negative values the relation

$$\Phi(-z) = 1 - \Phi(z) \qquad (12.13.5)$$

is used to find the area; because, due to the symmetry of standard normal curve about 0.

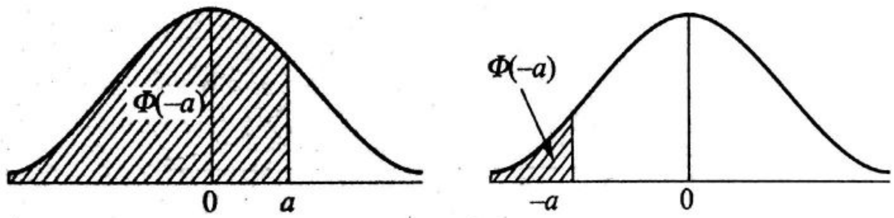


Fig. 12.2 Probability Integral of Standard Normal Distribution

(*Area between* $z = -a$ and $z = 0$) (12.13.6)

= (*Area between* $z = 0$ and $z = a$)

Thus, probabilities of normal distribution can be calculated by using the relation

$$P(c \le x \le d) = \Phi(d') - \Phi(c') \qquad (12.13.7)$$

where c' and d' are the standardized values of c and d.

Properties of Normal Distribution

(i) The normal distribution has two parameters μ and σ.

(ii) Mean = μ; Standard Deviation = σ.

(iii) Mean, median and mode are equal, each being μ.

Mean = Median = Mode = μ

(iv) The quartiles are equidistant from mean. Approximately;

$Q_1 = \mu - 0.67\,\sigma$, $Q_3 = \mu + 0.67\,\sigma$

$\therefore$ Quartile deviation = $0.67\,\sigma$

Mean deviation = $0.80\,\sigma$

(v) All *odd* order central moments are zero.

$\mu_1 = \mu_3 = 0$

The second order and fourth order central moments are

$$\mu_2 = \sigma^2, \quad \mu_4 = 3\sigma^4$$

In general, $\mu_{2r+1} = 0, \quad \mu_{2r} = 1.3.5 \ldots (2r-1)\,\sigma^{2r}$

(vi) $$\beta_1 = 0, \quad \beta_2 = 3;$$

Skewness $(\gamma_1) = 0$; Kurtosis $(\gamma_2) = 0$

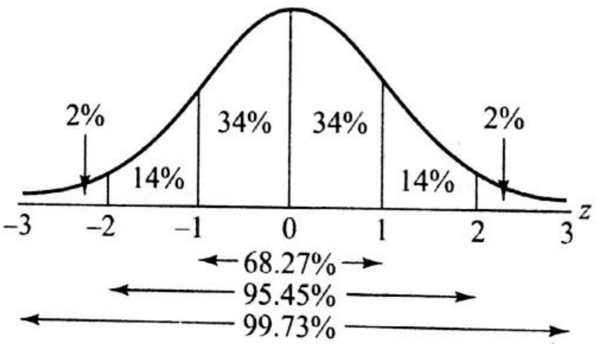
(vii) The normal curve is bell-shaped and symmetrical about the line $x = \mu$. The two tails of the curve extend to infinity on both sides of the mean. The maximum ordinate is at $x = \mu$ and given by

$$y = \frac{1}{\sigma\sqrt{2x}}$$

[This is obtained from (12.13.1) by putting $x = \mu$]. As the distance from the mean increases, the curve comes closer and closer to the horizontal axis but never meets it.

(viii) The points of inflection of the normal curve are at $x = \mu \pm \sigma$. This means that at these points the normal curve changes its curvature from concave to convex and vice versa.

(ix) The percentage distribution of area under the 'standard normal curve' is broadly shown in Fig. 12.3.



Area between $z = \pm 1$ is 68.27%
Area between $z = \pm 2$ is 95.45%
Area between $z = \pm 3$ is 99.73%

Fig. 12.3 Area under Standard Normal Curve

The last result is especially useful in the theory of Statistical Quality Control (S.Q.C.). It implies that if a random variable x is normally distributed with mean μ and s.d. σ, then almost all the values of x will lie between the limits $\mu - 3\sigma$ to $\mu + 3\sigma$, i.e. Mean ± 3 (S.D.).

(x) If x and y are independent normal variates with means μ_1 and μ_2, and variances σ_1^2 and σ_2^2 respectively, then $(x + y)$ is also a normal variate with mean $(\mu_1 + \mu_2)$ and variance $(\sigma_1^2 + \sigma_2^2)$.

Importance of Normal Distribution

The normal distribution has many important algebraic properties for which it is used widely in statistical theories. In most cases of physical, biological and psychological measurements, the data are found to follow normal distribution. *Statistical Quality*

Control (S.Q.C.) methods in the manufacturing industry and the *Theory of Errors* of observations in physical measurements are also based on normal distribution. The normal distribution is used (Section 12.14) to serve as approximations to the binomial and Poisson distributions. In the theory of sampling, it has been found that any statistic based on a large sample approximately follows normal distribution. The result considerably simplifies the work of testing statistics hypotheses and is also useful to find the confidence limits of parameters.

Example 12.36 *The height distribution of a group of 10,000 men is normal with mean height 64.5″ and s.d. 4.5″. Find the number of men whose height is (a) less than 69″ but greater than 55.5″, (b) less than 55.5″, and (c) more than 73.5″.*

Solution The mean (μ) and standard deviation (σ) of the normal distribution are given to be $\mu = 64.5''$ and $\sigma = 4.5''$.

(a) Percentage of men whose height lies between 55.5″ and 69″

= Area under standard normal curve between the vertical lines at the corresponding standardized values, viz.

$$z = \frac{55.5 - 64.5}{4.5} = -2 \quad \text{and} \quad z = \frac{69 - 64.5}{4.5} = 1.$$

From the percentage distribution of area under the standard normal curve (Fig. 12.3), it is found that the area between $z = -2$ and $z = 1$ is (14% + 34% + 34%) = 82%. This means that 82% of the total number of 10,000 men are expected to have heights between 55.5″ and 69″. Hence, the required number of men is 82% of 10,000 i.e. $\frac{82}{100} \times 10{,}000 = 8.200$.

(b) Percentage of men whose height is less than 55.5″

= Area under standard normal curve to the *left* of the Standardized value

$$z = \frac{55.5 - 64.5}{4.5} = -2.$$

= 20% (see Fig. 12.3).

The number of men is therefore, 20% of 10,000 i.e. 200.

(c) Percentage of men whose height is *more* than 73.5″.

= Area under standard normal curve to the *right* of the Standardized value

$$z = \frac{73.5 - 64.5}{4.5} = 2.$$

= 20%

Hence, the number of men whose height is more than 73.5″ is 20% of 10,000. i.e. 200.

Ans. (a) 8,200; (b) 200; (c) 200.

Example 12.37 *The mean weight of 500 male students at a certain college is 151 lbs. and the standard deviation is 15 lbs. Assuming that the weights are normally distributed, find how many students weight (i) between 120 and 155 lbs., (ii) more than 155 lbs. [Given $\Phi(0.27) = 0.6064$ and $\Phi(2.07) = 0.9808$, where $\Phi(t)$ denotes the area under standard normal curve to the left of the ordinate at t.]*

Solution The mean (μ) and the standard deviation (σ) are:

$$\mu = 151 \text{ lbs.,} \quad \sigma = 15 \text{ lbs.}$$

(i) Proportion of students whose weights lie between 120 and 155 lbs. = Area under standard normal curve between the vertical lines at the standardized values, viz.

$$z = \frac{(120-151)}{15} = -2.07 \quad \text{and} \quad z = \frac{(155-151)}{15} = 0.27. \quad \text{Using (12.13.7).}$$

$$\begin{aligned} P(120 \le x \le 155) &= \Phi(0.27) - \Phi(-2.07) \\ &= \Phi(0.27) - \{1 - \Phi(2.07)\}, \quad \text{by} \quad (12.13.5) \\ &= 0.6064 - 1 + 0.9808 \\ &= 0.5872 \quad \text{[see Fig. 12.4 (a)]} \end{aligned}$$

The number of students whose weights lie between 120 and 155 lbs. is therefore $0.5872 \times 500 = 294$ (approx.)

(ii) Proportion of students who weigh more than 155 lbs. is

$$\begin{aligned} P(x > 155) &= 1 - P(x \le 155) \\ &= 1 - \Phi(0.27) = 1 - 0.6064 = 0.3964 \end{aligned}$$

The number of students who are expected to weigh more than 155 lbs is $0.3936 \times 500 = 197$ (approx.)

Ans. 294, 197

Example 12.38 *A sample of 100 dry battery cells tested to find the length of life produced the following results : $\bar{x} = 12$ hours, $\sigma = 3$ hours. Assuming that the data are normally distributed, what percentage of battery cells are expected to have life (i) more than 15 hours, (ii) less than 6 hours, and (iii) between 10 and 14 hours?*

[I.C.W.A. June, '79]

Given	*z*	*2.5*	*2*	*1*	*0.67*
	Area	*.4938*	*.4772*	*.3413*	*.2487*

Solution [**Note:** The given table shows the area under standard normal curve between the ordinates at $z = 0$ and the specified values of z. This is clear from the fact that the area increases with larger values of z, and the area upto $z = 2.5$ is almost 0.5. Remember that the area under standard normal curve between $z = -3$ and $z = +3$ is almost 1, and hence the area from $z = 0$ upto $z = 3$ may be taken to be 0.5].

The percentage of battery cells with life between specified limits will be given by the proportionate are under the standard normal curve between their standardized values. Here, mean = 12, s.d. = 3.

$$\text{Standardized value of 15 hours} = \frac{(15-12)}{3} = +1$$

$$\text{Standardized value of 6 hours} = \frac{(6-12)}{3} = -2$$

$$\text{Standardized value of 10 hours} = \frac{(10-12)}{3} = -0.67$$

$$\text{Standardized value of 14 hours} = \frac{(14-12)}{3} = +0.67$$

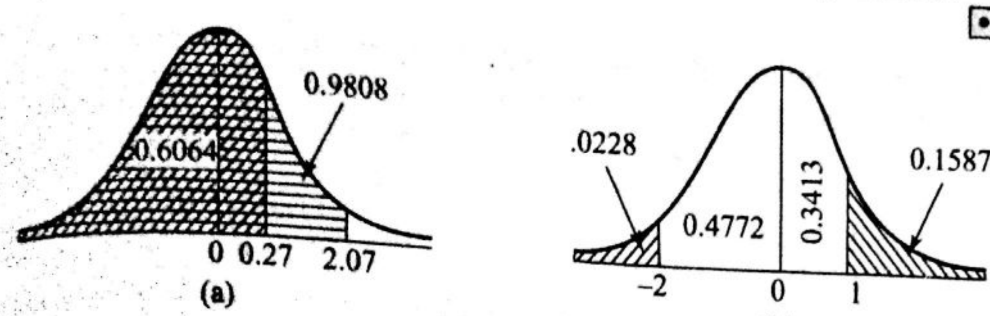


Fig. 12.4 Area under Standard Normal Curve

Proportion of battery cells with life *more* than 15 hours

= Area under standard normal curve to the *right* of $z = +1$
= (Total area to the right of $z = 0$) – (Area between $z = 0$ and $z = +1$)
= $0.5 - 0.3413$ [see Fig. 12.4(b)]
= $0.1587 = 15.87\%$

Proportion with life *less* than 6 hours

= Area under the curve to the *left* side of $z = -2$
= Area to the *right* of $z = +2$, (since the curve is symmetrical about $z = 0$)
= $0.5 - 0.4772$
= $0.0228 = 2.28\%$

Proportion with life between 10 and 14 hours

= Area under the curve between $z = -0.67$ and $z = +0.67$
= $2 \times$ (Area between $z = 0$ and $z = +0.67$), since the curve is symmetrical
= 2×0.2487 [see Fig. 12.5 (a)]
= $0.4974 = 49.74\%$

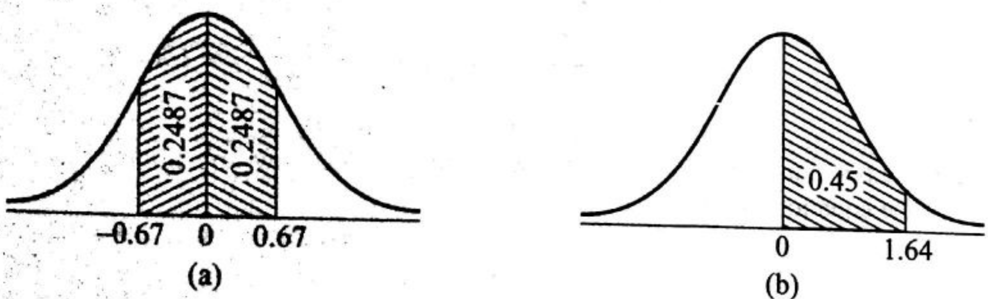


Fig. 12.5 Area under Standard Normal Curve

Hence the required percentage of battery cells with life (i) more than 15 hours is 15.87, (ii) less than 6 hours is 2.28, and (iii) between 10 and 14 hours is 49.74.

Ans. 15.87, 2.28. 49.74

Example 12.39 *The mean of a normal distribution is 50 and 5% of the values are greater than 60. Find the standard deviation of the distribution (Given that the area under standard normal curve between z = 0 and z = 1.64 is 0.45).*

(I.C.W.A., Dec. '79)

Solution The probability that x takes a value greater than 60 is 5%, i.e. .05. This must be the area under standard normal curve to the right of the ordinate at the standardized value

$$z = \frac{60 - 50}{\sigma} = \frac{10}{\sigma}$$

Since the area to the right of $z = 0$ is 0.5 and the area between $z = 0$ and $z = 1.64$ is given to be 0.45, hence the area to the right of $z = 1.64$ is $0.5 - 0.45 = .05$ [see Fig. 12.5].

Thus, $$\frac{10}{\sigma} = 1.64; \quad \text{or, } \sigma = \frac{10}{1.64} = 6.1.$$

Example 12.40 *Assuming that the height distribution of a group of men is normal, find the mean and standard deviation, if 84% of the men have heights less than 65.2 inches and 68% have height lying between 65.2 and 62.8 inches.*

(I.C.W.A. June '78)

Solution Let μ denote the mean and σ the standard deviation: and z_1 and z_2 represent the standardized values of 62.8″ and 65.2″,

i.e. $$z_1 = \frac{62.8 - \mu}{\sigma} \quad \text{and} \quad z_2 = \frac{65.2 - \mu}{\sigma} \quad ...(i)$$

(Note that since 65.2 is greater than 62.8, z_2 will be greater than z_1).

Since 84% of the men have height *below* 65.2″, this must be the area under the standard normal curve to the *left* of the standardized value z_2. But from the percentage distribution of area under the curve (Fig. 12.3) we find that the area to the left of $z = 1$ is 84%. Hence, z_2 must be identical with 1, i.e. $z_2 = 1$.

Again, it is given that 68% of the men have height between 65.2″ and 62.8″; i.e. the percentage of area under the standard normal curve between $z_2 = 1$ and a smaller value z_1 is 68%. From Fig. 12.3 it is found that the area between $z = 1$ and $z = -1$ is 68%. Hence, z_1 must be identical with -1, i.e. $z_1 = -1$.

Thus, we find that $z_1 = -1$ and $z_2 = 1$.

or, $$\frac{62.8 - \mu}{\sigma} = -1 \text{ and } \frac{65.2 - \mu}{\sigma} = 1, \text{ by (i)}$$

or, $$62.8 - \mu = -\sigma \quad \text{and} \quad 65.2 - \mu = \sigma$$

Adding the two equations, $128.0 - 2\mu = 0$, so that $\mu = 64$. Now, substituting this value of μ in one of the equations, $\sigma = 1.2$.

Ans. mean = 64.0; s.d. = 1.2 (inches)

Example 12.41 *For a certain normal distribution, the first moment about 10 is 40, and the fourth moment about 50 is 48. Find the arithmetic mean and the standard deviation of the distribution.*

[Given: For a normal distribution $\mu_{2n} = 1.3.5 \ldots. (2n - 1)\, \sigma^{2n}$]

(I.C.W.A., Dec. '78)

Solution

$$\text{Mean} = A + \text{first moment about } A, \text{ (see 7.1.10)}$$
$$= 10 + 40 = 50$$

Since the mean is 50, the fourth moment about 50 is the fourth central moment μ_4. Again, putting $n = 1, 2$ in the given formula,

$$\mu_2 = \sigma^2, \quad \mu_4 = (1 \times 3)\, \sigma^4 = 3\sigma^4$$

The first relation shows that σ^2 is the variance of normal distribution, i.e., the parameter σ is the s.d. From the second relation

$$3\sigma^4 = 48 \text{ (given)}; \quad \text{i.e.} \quad \sigma = 2$$

Ans. Mean = 50, S.D. = 2

12.14 NORMAL APPROXIMATION TO BINOMIAL (POISSON)

In the binomial distribution (12.5.1), if neither p or q is very small, but n is large, then the binomial distribution can be closely approximated by the normal distribution having the same mean and standard deviation as those of the binomial,

i.e. $\mu = np$ and $\sigma = \sqrt{npq}$

In order to find the probability $P(a \le x \le b)$ of the discrete binomial distribution by the continuous normal distribution, some corrections to the values of a and b are necessary. The number of successes is always a whole number, and hence the number of successes a corresponds to an interval $a - \frac{1}{2}$ to $a + \frac{1}{2}$ in the continuous scale (e.g. '47 successes' refers to any value between 46.5 and 47.5). Therefore, the number of successes a actually starts from $a - \frac{1}{2}$, and the number of successes b ends at $b + \frac{1}{2}$ in the normal distribution. The probability $P(a \le x \le b)$ that the number of successes lies between a and b (both inclusive) therefore, corresponds to $P(a - \frac{1}{2} \le x \le b + \frac{1}{2})$ of the normal distribution. Using (12.11.3), this probability is given by the area under standard normal curve between the vertical lines at the standardized values.

$P(a \le x \le b)$ of binomial distribution

$= P\left(a - \frac{1}{2} \le x \le b + \frac{1}{2}\right)$ of the approximating normal distribution.

= Area under standard normal curve between the vertical lines at a' and b'. (12.14.1)

where $$a' = \frac{\left(a - \frac{1}{2}\right) - np}{\sqrt{npq}} \quad \text{and} \quad b' = \frac{\left(b + \frac{1}{2}\right) - np}{\sqrt{npq}}$$

The larger the value of n, the better is the approximation. But, for practical purposes, the approximation is generally used, if both p and q are larger than 0.1 and n is not less than 30.

When the parameter m is large. Poisson distribution can be approximated by the normal distribution with the same mean and standard deviation as those of Poisson,

i.e. mean = m and s.d. = $\sqrt{m}$

In practice, the approximation is used, when $m \ge 10$.

Example 12.42 *A fair coin is tossed 400 times. Using normal approximation, find the probability of obtaining (i) exactly 200 heads, (ii) less than 210 heads, (iii) between 190 and 210 heads, both inclusive. Given that the area under standard normal curve*

between $z = 0$ and $z = .05$ is $.0199$
between $z = 0$ and $z = .95$ is $.3289$
between $z = 0$ and $z = 1.05$ is $.3531$

Solution Let us denote the occurrence of a head as "success". The number of heads (x) follows binomial distribution with $n = 400$, and $p = \frac{1}{2}$. Since n is large, we use the normal approximation to the binomial with

$$\mu = np = 400 \times \frac{1}{2} = 200$$

$$\sigma = \sqrt{npq} = \sqrt{400 \times \frac{1}{2} \times \frac{1}{2}} = 10$$

(i) Probability of obtaining exactly 200 heads

$= P(x = 200)$ of binomial distribution

$= P(199.5 \leq x \leq 200.5)$ of normal distribution

= Area under standard normal curve between the standardized values z

$= \frac{(199.5 - 200)}{10} = -.05$ and $z = \frac{(200.5 - 200)}{10} = +.05$

$= 2 \times$ (Area between $z = 0$ and $z = +.05$)

$= 2 \times 0.199 = 0.4$ (approx.)

(ii) Probability of obtaining less than 210 heads

$= P(x < 210)$ of binomial distribution

$= P(x \leq 209.5)$ of normal distribution

= Area under the standard normal curve to the *left* of

$$z = \frac{(209.5 - 200)}{10} = .95$$

= (Area to the left of $z = 0$) + (Area between $z = 0$ and $z = .95$)

$= 0.5 + 0.3289 = 0.83$ (approx.)

(iii) Probability of obtaining between 190 and 210 heads

$= P(190 \leq x \leq 210)$ of binomial distribution

$= P(189.5 \leq x \leq 210.5)$ of normal distribution

= Area under standard normal curve between $z = \frac{189.5 - 200}{10}$

$= -1.05$ and $z = \frac{210.5 - 200}{10} = +1.05$

$= 2 \times$ (Area between $x = 0$ and $z = 1.05$)

$= 2 \times .3531 = 0.71$ (approx.)

Ans. 0.04, 0.83, 0.71

Example 12.43 *A variable x follows Poisson distribution with mean 16. Find the probability $P(x \geq 20)$. Given $F(0.875) = .8092$ where $F(x)$ is the standards normal distribution function.*

Solution Using normal approximation to Poisson distribution,

Mean $(\mu) = m = 16$, S.D. $(\sigma) = \sqrt{m} = \sqrt{16} = 4$

$P(x \geq 20)$ of Poisson distribution

$= P(x \geq 19.5)$ of normal distribution

= Area under standard normal curve to the *right* of

$$z = \frac{(19.5 - 16)}{4} = 0.875$$

= 1 – (Area to the *left* of $z = 0.875$)

= $1 - F(0.875)$

= $1 - 0.8092 = 0.19$ (approx.)

12.15 CONTROL LIMIT THEOREM

The normal distribution is used as an approximation to the binomial and Poisson distributions, because the former can be obtained as a limiting case under certain conditions. The *Central Limit Theorem*, stated below, shows that a large class of distributions can in fact be approximated by the normal distribution.

Central Limit Theorem—Let $x_1, x_2, \ldots, x_n, \ldots$ be independent random variables which are identically distributed (i.e. all have the same probability distribution) with mean μ and variance σ^2. If $\bar{x}_n = \dfrac{(x_1 + x_2 + \ldots x_n)}{n}$ denotes the mean of the first n variables, then the distribution of

$$z_n = \frac{(\bar{x}_n - \mu)}{\dfrac{\sigma}{\sqrt{n}}}$$

approaches the Standard Normal Distribution, as n becomes larger and larger.

This shows that the mean of n independently and identically distributed random variables with mean μ and standard deviation σ can be approximated by the normal distribution with

$$\text{Mean} = \mu, \quad \text{S.D.} = \frac{\sigma}{\sqrt{n}}$$

whatever by the probability distribution of the variables, provided n is sufficiently large. (Note that the distribution of the random variables may be discrete or continuous and of any form). For practical purposes $n \geq 30$ gives a sufficiently good approximation, the accuracy increasing with larger values of n.

EXERCISES

1. Explain the concept of 'theoretical distribution' with reference to a discrete random variable.
2. What is meant by the probability distribution of a discrete random variable? *(W.B.H.S. '78; C.U., B.Sc., '79)*
3. What is meant by Probability Mean Function of a discrete random variable? Write down these functions for the Binomial as well as for Poisson distributions. *[I.C.W.A., June '79]*

4. What do you understand by 'expectation' of a random variable? State how you will find the 'mean' and 'standard deviation' of a discrete probability distribution with p.m.f. $f(x)$.

5. State whether each of the following is *True* or *False*:

   (a) The probability mass function of a random variable may be zero.

   (b) The expected number of points to be obtained from a throw of a perfect die is 3.

   (c) The expectation of a random variable cannot be negative.

   (d) The square of $E(x)$ is larger than $E(x^2)$.

   (e) If Var $(x) = 0$, then all values of x are equal.

6. A discrete r.v. assumes the values 1, 2, 3, ... 10 with probabilities proportional to 1, 2, 3, ... , 10 respectively. Calculate the mean and variance.

7. Let X be a random variable with p.m.f. $f(x) = \left(\frac{1}{32}\right) {}^5C_x$ where $x = 0, 1, 2, ... 5$. Find the mean and standard deviation.

8. A discrete random variable x follows uniform distribution and takes only the values 6, 8, 11, 12, 17. Find the probabilities

$$P(x = 8), P(x = 10), P(x \leq 12), P\ (x < 12)\ P(x > 10).$$

9. Define 'Binomial distribution' and state the conditions under which the distribution holds. *[I.C.W.A., June '74, Dec. '79]*

10. Four coins are tossed simultaneously. What is the probability of getting 2 heads and 2 tails? *[C.A., May '77]*

11. Eight coins are thrown simultaneously. Show that the probability of obtaining at least 6 heads is $\frac{37}{256}$. *[I.C.W.A. June, '74]*

12. Find the probability that in a family of 5 children there will be (i) at least one boy, (ii) at least one boy and one girl. (Assume that the probability of a female birth is 1/2). *[I.C.W.A., Dec. '76]*

13. If a sample of 5 items is drawn randomly form a lot containing 10% defective items, what is the probability of getting not more than one defective item? *[M.B.A. '77, Dip. Mgmt. '77]*

14. In a shooting competition, the probability of a man hitting a target is $\frac{1}{5}$. If he fires 5 times, what is the probability of hitting the target at least twice? *[C.U., M.Com., '75]*

15. Assume that on the average 30% of the candidates appearing in an examination from a certain college get First Division. What is the probability that out of a group of 4 such candidates not more than two will fail to get a First Division?

16. Let the probability of a patient recovering from a certain disease by 0.75. Find the distribution of the number of recoveries among 4 patients. Hence compute the mean and standard deviation. *[W.B.H.S. '80]*

17. The probability of a bomb hitting a target is $\frac{2}{5}$. Four direct hits are necessary to destroy a bridge completely. If 6 bombs are aimed at the bridge, what is the probability that the bridge will be destroyed?

18. Two dice are thrown n times in succession. What is the probability of obtaining double-six at least once? *[C.U., B.Sc., '75]*

19. (a) What is a Bernoullian series of trials? Give an example.
    (b) Find the probability of at least one success in a Bernoullian series of n trials with a probability of success p in each trial.

20. In an infinite series of Bernoulli trials with a probability p of success in a trial, find the probability of x failures preceding the first success, and hence find $E(x)$. *[C.U., B.Sc. '81]*

21. In 8 independent trials of a random experiment with a constant probability of success in each trial, it is known that the probability of 5 successes is exactly equal to the probability of 3 successes. If the experiment is repeated 6 times, what is the probability of obtaining 4 successes?

22. Suppose that half the population of a town are consumers of rice, 100 investigators are appointed to find out its truth. Each investigator interviews 10 individuals. How many investigators do you expect to report that three or less of the people interviewed are consumers of rice? *[I.C.W.A., June '79]*

23. (a) What is the probability of guessing correctly at least six of ten answers in True-False objective test?
    (b) The following statement cannot be true—why? "The mean of a Binomial distribution is 4 and the standard deviation is 3". *[I.C.W.A., Dec. '77]*

24. For a binomial distribution, the mean is 3 and the variance is 2. Find the values of n and p. Hence find the probability that X (the variable value) is 5. *[W.B.H.S. '78; M.B.A. '77]*

25. For a binomial distribution, the mean and S.D. are respectively 4 and $\sqrt{3}$. Calculate the probability of getting a non-zero value from this distribution. *[Dip. Mgmt. '78]*

26. Let x be a binomial distributed random variable with parameters n and p. For what value of p is Var (x) a maximum, if you assume that n is fixed? *[W.B.H.S. '80]*

27. If y is the proportion of success in a series of Bernoullian trials, then show that the standard deviation of y cannot be greater than $\frac{1}{(2\sqrt{n})}$.

28. Show that the binomial distribution is symmetric when $p = \frac{1}{2}$. *[W.B.H.S. '79]*

29. Find the mode of binomial distribution with parameters (i) $n = 11$, $p = 0.8$; (ii) $n = 8$, $p = \frac{1}{3}$.

30. Find the maximum term in the expansion of $\left(\frac{1}{3}+\frac{2}{3}\right)^6$. [W.B.H.S. '80]

31. (a) Find the most probable number of heads in $2n$ tosses of a fair coin.
    (b) If a fair die is tossed 100 times how many "sixes" are most likely to appear?

32. If the probability of a defective bolt is $\frac{1}{10}$, find (i) mean, (ii) variance, (iii) moment coefficient of skewness and (iv) kurtosis, for the distribution of defective bolts in a total of 400. [I.C.W.A., June, '77, '80]

33. Write down the expressions which define Binomial, Poisson and Normal probability distributions. Give 3 physical situations illustrating a Poisson random variable. [I.C.W.A. June, '79]

34. A random variable x follows Poisson distribution with parameter $m = 2$. Find the probabilities $P(x = 1)$, $P(x \leq 1)$, $P(x < 1)$, $P(x > 1)$, $P(1 \leq x \leq 3)$. Given $e^{-2} = 0.1353$.

35. The standard deviation of a Poisson distribution is 2. Find the probability that $x = 3$. (Given $e^{-4} = .0183$). [I.C.W.A., Dec., '81]

36. Is it possible that a Poisson distribution has the same mean and standard deviation? If so, what is the probability that the variable takes the value zero?

37. For a Poisson distribution, $P(x = 0) = Pr\,(x = 1)$. Find $Pr\,(x > 0)$. [M.B.A., '79]

38. A discrete random variable x follows Poisson distribution such that $P\,(x = 1) = P\,(x = 2)$. Find the mean and variance of the distribution. [C.U., B.Com(Hons) '82]

39. The probability that a Poisson variate X takes a positive value is $(1 - e^{-2})$. Find the (i) Mean, (ii) Mode, (iii) probability that X lies between -1 and 1.5.

40. If 3% of the bolts manufactured by a company are defective, what is the probability that in a sample of 200 bolts, 5 will be defective? (Given $\frac{1}{e^6} = 0.00248$). [I.C.W.A., Dec., '75]

41. The average number of misprints per page of a book is 2. Assuming Poisson distribution, what is the probability that a particular page is free from misprints? If the book contains 1000 pages, how many of the pages contain more than 2 misprints?

42. Suppose that the number of telephone calls an operator receives from 11.00 a.m. to 11.05 a.m. follows a Poisson distribution with $m = 3$. (i) Find the probabilities that the operator will receive no calls in that time interval tomorrow. (ii) Find the probability that in the next 3 days the operator will receive a total of 1 call in that time interval. ($e = 2.718$). [C.U., M.Com. '78]

43. The average number of defects per yard on a piece of cloth is 0.9. What is the probability that a one-yard piece chosen at random contains less than 2 defects? (Given $e^{0.9} = 2.46$).

44. A system contains 1000 components. Each component fails independently of the others and the probability of its failure in one months is $\frac{1}{1000}$. What is the probability that the system will function (i.e. no component fails) at the end of the month?

45. If 5% of the electrical bulbs manufactured by a company are defective, use Poisson distribution to find the probability that in a sample of 100 bulbs (i) none is defective, (ii) 5 bulbs will be defective. (Given $e^{-5} = .007$).

[I.C.W.A., Dec. '79]

46. Find the probability that at most 5 defective bolts will be found in a box of 200 bolts, if it is known that 2 per cent of such bolts are expected to be defective. You may take the distribution to be Poisson). (Given $e^{-4} = 0.0183$).

[I.C.W.A., June '80]

47. The manufacturer of a certain electronic component knows that 3% of his product is defective. He sells the components in boxes of 100 and guarantees that not more than 3 in any box will be defective. What is the probability that a box will fail to meet the guarantee? (Given $e^3 = 20.1$).

48. In a certain factory, blades are manufactured in packets of 10. There is a 0.2% probability for any blade to be defective. Using Poisson distribution calculate approximately the number of packets containing two defective blades in a consignment of 20,000 packets. (Given that $e^{-.02} = .9802$).

[I.C.W.A., June '81]

49. A car hire firm has 2 cars which it hires out day by day. The number of demands for a car on each day follows Poisson distribution with mean 1.5. Calculate the proportion of days on which (i) neither car is used, (ii) some demand is refused. (Given $e^{-1.5} = 0.2231$).

50. What is the probability of getting 3 white balls in a draw of 5 balls from a box containing 5 white and 4 black balls?

51. 10 cards are drawn at random one by one without replacement from a full pack of 52 playing cards. Find the mean and variance of the number of red cards obtained.

52. State the conditions under which a hypergeometric distribution can be approximated by a binomial distribution, which again can be approximated by a normal distribution.

53. An unbiased dice is thrown 7 times. Find the probability that the face 1 turns up twice, face 2 once, face 4 twice and face 6 twice.

54. A box contains 5 white, 2 blue and 3 red balls. If 6 balls are drawn with replacement, find the probability that there are (i) 2 white, 3 blue and 1 red balls; (ii) 4 white and 2 red balls; (iii) 2 balls of each colour.

55. Find the mean and variance of the rectangular distribution given by the p.d.f. $f(x) = 1, (0 \leq x \leq 1)$.

56. Find the mean and the standard deviation of the continuous probability distribution given by the p.d.f.

$$f(x) = y_0 x^2 (1 - x); \quad (0 \le x \le 1)$$

where y_0 is a constant.

57. A continuous variate X has the p.d.f.

$$\begin{aligned} f(x) &= \frac{x}{2}, & 0 \le x \le 1 \\ &= \frac{1}{2} & 1 < x \le 2 \\ &= \frac{(3-x)}{2} & 2 < x \le 3. \end{aligned}$$

Find the mean of the distribution.

58. Explain clearly what you understand by the following statement—"A random variable x follows Normal distribution". Write down the important properties of this distribution.

59. (a) Explain the concept of probability density function (p.d.f.).

[I.C.W.A., Dec. '75]

(b) Write down the probability density function of a normal variate. What is the ordinate of the normal curve at its mean? State its important properties.

[C.U., M.Com. '78; B.Com. (Hons) 82; I.C.W.A., Dec. '80]

60. Comment on the accuracy of the following results:

(i) For a binomial distribution, mean = 4 and variance = 3.

(ii) For a Poisson distribution, mean = 10 and s.d. = 5.

(iii) For a normal distribution, mean = 50, median = 52.

[I.C.W.A., Dec. '80]

61. Comment on the following statements:

(a) For a binomial distribution, mean = 16, s.d. = 4.

(b) For a normal distribution, mean = 50, mode = 90.

(c) Any measure of skewness for Normal distribution is + 5.

(d) For Normal distribution, the value of the fourth central moment per unit of square of second central moment is + 3. *[C.U., M.Com. '78]*

62. State whether each of the following is *true* or *false*:

(a) The variance of a binomial distribution can never exceed the mean.

(b) The expected value of Poisson variate must be a positive integer.

(c) The mean of normal distribution cannot be negative.

(d) The Poisson distribution may have one or two modes.

63. The height distribution of a group of 2989 individuals is known to be normal distribution with mean 65″ and standard deviation 2.1″. Find the number of individuals whose heights lie between 60.8 and 67.1″. Find also the number of individuals whose height are above 67.1″

64. What is a 'Standard Normal distribution'? State some of its important properties. As a result of test on electric light bulbs, it was found that the lifetime of a particular make was distributed normally with an average life of 1000 burning hours and standard deviation of 200 hours. Out of 10,000 bulbs produced by the company how many bulbs are expected to fail (i) in the first 800 burning hours, (ii) between 800 and 1,200 burning hours. (Given $\Phi(1) = 0.84134$).
[C.U., M.Com. '76]

65. Assume the mean height of soldiers to be 68.22 inches with a variance of 10.8 sq. inches. How many soldiers in a regiment of 1000 would you expect to be over 6 ft. tall? (Given that the area under the standard normal curve between $x = 0$ and $x = 0.35$ is 0.1368 and between $x = 0$ and $x = 1.15$ is 0.3749).

66. The mean I.Q. of a group of children is 90 with a standard deviation of 20. Assuming that I.Q. is normally distributed, find the percentage of children with I.Q. over 100. (Given $\Phi(0.5) = 0.6915$, where $\Phi(x)$ is the cumulative distribution function of standard normal distribution.)

67. 5000 candidates appeared at an examination, in which the minimum for a pass is 40 and the minimum for a distinction is 50. If it is known that the average mark obtained by the candidates is 43 and the S.D. is 7. Find how many of the candidates expect to get simply 'pass' and the number obtaining distinction. Assume normal distribution. (Given $\Phi(0.43) = 0.6664$, $\Phi(1) = 0.8413$).

68. In a sample of 120 workers in a factory, the mean and s.d. of wages were Rs 11.35 and Rs 3.03 respectively. Find the percentage of workers getting wages between Rs 9 and Rs 17 in the whole factor, assuming that the wages are normally distributed. (Given, Area under standard normal curve from $z = 0$ to $z = 0.78$ is 0.2823 and to $z = 1.86$ is 0.4686).
[C.A.. May '81]

69. The mean of the inner diameters (in inch) of a sample of 200 tubes produced by a machine is 0.502 and the standard deviation is 0.005. The purpose for which these tubes are intended allows a maximum tolerance in the diameter of 0.496 to 0.508 (i.e. otherwise the tubes are considered defective). What percentage of the tubes produced by the machine is defective, if the diameters are found to be normally distributed? (Area under the standard normal curve between $z = 0$ and $z = 1.2$ is 0.3849).
[I.C.W.A., June '80]

70. In a certain city, the daily supply of electric power (in mega watt) can be treated as a random variable having a normal distribution with mean 300 m.w. and s.d. 50 m.w. Since the supply is not a constant, the local authorities have imposed a system of rationing to deal with the problem. It is known that to ensure proper rationing a minimum of 250 m.w. is required, otherwise load-shedding has to be imposed. There is no need of rationing whenever supply exceed 350 m.w. On the other hand, the maximum consumption in the city can never exceed 425 m.w. Find the percentage of days in which the city experiences load-shedding, the percentage of days on which proper rationing is implemented, and the percentage of days when there is an excess of power. (Given, $\Phi(1) = .8413$, $\Phi(2.5) = .9983$).
[C.U. M.Com. '77]

71. X follows a normal distribution whose mean is 12 and standard deviation is 4.

Find $P(X \geq 20)$. Given $\int_{-\infty}^{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 0.9772499.$ [W.B.H.S. '81]

72. In a normal distribution, 8% of the items are under 50 and 10% are over 60.

Find the mean and s.d. Given $\int_{\alpha}^{\infty} \frac{1}{\sqrt{2x}} e^{-\frac{z^2}{2}} dz = 0.08$ or 0.10, according as

$x = 1.4$ or 1.28.

73. A die is tossed 1200 times. Find the probability that the number of 'sixes' lies between 190 and 210. (Given that the area under the standard normal curve between $z = 0$ and $z = 0.78$ is .2823 and between $z = 0$ and $z = 0.81$ is .2910).

74. Show that the probability that the number of heads in 400 tosses of a fair coin lies between 180 and 220 is approximately 2, $\Phi(2) - 1$, where $\Phi(x)$ denotes the standard normal distribution functions.

75. A normal distribution has man 10.4 and s.d. 1.2. Find the limits within which an observations chosen at random almost certainly lies.

76. Player A tosses 5 coins and B tosses 8 coins. If the coins are unbiased, what is the probability of obtaining a total of 6 heads by the two players?

77. The number of deaths per day in a city due to road accidents and due to other causes independently follow Poisson distributions with parameters 2 and 6 respectively. Find the probability that the total number of deaths on a particular day is 2 or fewer. (Given $e^{-4} = .018$).

78. In an examination, marks obtained by the students in Mathematics, Physics and Chemistry are independently and normally distributed with means 50, 52, 48 and s.ds. 15, 12, 16 respectively. Find the probability that the total marks of a student are (i) 180 or more, (ii) 90 or less. [Given $\Phi(1.2) = 0.8849$, $\Phi(2.4) = 0.9918$].

79. When are two random variables said to 'independent'? Show that the covariance of two independent random variables is zero.

[C.U., B.A. (Econ) '81]

80. The joint probability distribution of the random variables x and y is shown below:

x \ y	0	1	2
2	.05	.10	.25
4	.15	.05	.15
6	.10	.10	.05

Find (a) the marginal distribution of x and y.

(b) the conditional distribution of x, given $y = 1$.

(c) the conditional distribution of y, given $x = 2$.

(d) the probability $P(x + y \geq 6)$.

Are x and y independent?

81. *Joint Distribution of x and y*

x \ y	2	3	4	Total
1	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$
2	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{2}$
Total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Find the correlation coefficient between x and y.

82. Two random variables x and y are jointly distributed so that

$$P(x = 1) = \frac{1}{3}, \qquad P(x = 2) = \frac{2}{3}$$

$$P(y = 0) = \frac{2}{3}, \qquad P(y = 1) = \frac{1}{3}$$

$$P(x = 2, y = 1) = p$$

where $0 \le p \le \frac{1}{3}$. Find out the correlation coefficient between x and y. For what value of p are x and y uncorrelated? For this value of p, will x and y be independent, too? *[W.B.H.S. '79]*

83. Two discrete random variables x and y are jointly distributed with

$$P(x = 2) = P(y = 3) = P(y = 4) = \frac{1}{3};$$

$$P(x = 1, y = 2) = \frac{1}{4}; \qquad P(x = 2, y = 3) = \frac{1}{12}$$

Find (i) the conditional mean of y, given $x = 2$

(ii) the correlation coefficient between x and y.

(iii) the probability distribution of $z = x + y$.

84. Following is the joint distribution of two random variables x and y:

x \ y	1	2	3
0	0.1	0.3	0.1
2	0.2	0.1	0.2

(i) Compute $P(x = 2)$, $P(y = 3)$, $P(x = 2/y = 3)$, μ_x, μ_y, σ_x, σ_y and ρ_{xy}. *[W.B.H.S. '80]*

(ii) Are x and y independent?

85. If X and Y are jointly distributed random variables, and a, b, c, d are arbitrary constants, prove that

$$\text{Cov}(aX + b, cY + d) = ac\ \text{Cov}(X, Y).$$

86. A fair die is rolled twice independent. Let X and Y denote the number of points showing on the first roll and the second roll respectively. Define $U = X + Y$ and $V = X - Y$. Find the correlation coefficient between U and V.
[C.U., B.Sc. '80]

87. State whether *True* or *False*:
   (a) If x and y are independent, then cov $(x, y) = 0$.
   (b) If cov $(x, y) = 0$, then x and y are independent.
   (c) If x and y are independent, then $E(xy) = E(x).E(y)$.
   (d) If $E(xy) = E(x).E(y)$, then x and y are independent.
   (e) If $\text{Var}(x + y) = \text{Var}(x) + \text{Var}(y)$, then $\rho_{xy} = 0$.
   (f) If the correlation coefficient between x and y is zero, then $\text{Var}(x + y) = \text{Var}(x) + \text{Var}(y)$.
   (g) If the random variables x and y are independent, then $\text{Var}(x - y) = \text{Var}(x) - \text{Var}(y)$.
   (h) If x and y are uncorrelated, then $(x + y)$ and $(x - y)$ are also uncorrelated.
   (i) Two independent variables must be uncorrelated.
   (j) Two uncorrelated variables must be independent.

88. If x and y are two indepenaent Poisson variates with parameters m and n respectively, show that their sum $(x + y)$ follows Poisson distribution with parameter $(m + n)$.

89. If ρ denotes the correlation coefficient between x and y, show that the variance of $(x/\sigma_s + y/\sigma_y)$ is $2(1 + \rho)$.

90. x and y are jointly distributed random variables with standard deviations σ_x, σ_y and correlation coefficient ρ (positive). Find the value of k so that $x + ky$ and $x + (\sigma_x/\sigma_y)y$ are uncorrelated,

ANSWERS

5. True, False False, False, True
6. 7, 6
7. 5/2, $\sqrt{5}/2$
8. 1/5, 0, 4/5, 3/5, 3/5
10. 3/8
12. 31/32, 15/16
13. $1.4(0.9)^4$
14. 821/3125
15. 0.3483
16.

x :	0	1	2	3	4	Total	Mean = 3
Prob :	$\frac{1}{256}$	$\frac{3}{64}$	$\frac{27}{128}$	$\frac{27}{64}$	$\frac{81}{256}$	1	S. D. = $\sqrt{3}/2$

17. 112/625
18. $1 - (35/36)^n$
19. $1 - (1 - p)^n$
20. pq^x, p/q $(p + q = 1)$
21. 15/64
22. 17
23. (a) 193/512; (b) Mean of binomial cannot be less than $(\text{S.D.})^2$
24. 9, 1/3 ; 224/2187
25. $1 - (0.75)^{16}$
26. 1/2
29. 9; 2 and 3

30. 80/243
31. n; 16
32. 40, 36, 2/15, 23/1800
34. 0.2706, .4059, .1353, .5941, .7216
35. 0.1952
36. yes; e^{-1}
37. $1 - e^{-1}$
38. 2, 2
39. 2; 1 and 2; $3e^{-2}$
40. 0.16
41. e^{-2}; $1000(1 - 5e^{-2})$
42. $e^{-3} = .050$; $9e^{-9} = .0011$
43. 0.772
44. $e^{-1} = 0.37$
45. .007, 0.182
46. 0.78
47. $1 - 13\,e^{-3} = 0.35$
48. 4
49. 0.2231, 0.1913
50. 10/21
51. 5, 35/17
53. $105/6^6$
54. 9/250, 27/320, 81/1000
55. 1/2, 1/12
56. 3/5, 1/5
57. 3/2
59. $1/\sigma\sqrt{2\pi}$
60. (i) may be correct; for binomial $n = 16$, $p = 1/4$.
   (ii) Incorrect; for Poisson, mean = $(\text{S.D.})^2$.
   (iii) Incorrect; for normal, Mean = Median.