Recurrence Relation

Theoretical questions

1. Define a recurrence relation. What do you mean by it's solution? **[CU 2011]**
2. State the general form of a linear recurrence relation with constants.
3. Define a Homogeneous linear recurrence relation (HDE) and Non-homogeneous linear recurrence relation. **[CU 2010]**

Problems

On HDE

1. Determine whether the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, ...$ where $a_n = 3n$ for every non-negative integer n. What happens when $a_n = 2^n$. **[CU 2010]**
2. $a_n = a_{n-1} + 6a_{n-1}$; $n \geq 2$ with $a_0 = 1, a_1 = 1$
3. $a_n = 3a_{n-1} - 3a_{n-1} + a_{n-3}$; $a_0 = 0, a_3 = 3, a_5 = 10$
4. $a_n = 7a_{n-1} - 10a_{n-2}$; $a_0 = 4, a_1 = 17$
5. $a_n - 8a_{n-1} + 16a_{n-2} = 0$; $a_2 = 16, a_3 = 80$
6. $a_n = a_{n-1} + 2a_{n-2}$; $a_0 = 2, a_1 = 7$
7. $a_{n+2} + 4a_{n+1} + 5a_n$; $n \geq 0$ with $a_0 = 2, a_1 = 8$
8. $a_n = 6a_{n-1} - 8a_{n-2}$; $n \geq 2$ with $a_0 = 4, a_1 = 10$
9. $a_n = 2an - 1 - a_{n-2}$; $n \geq 2$, with $a_0 = 3, a_1 = 1$
10. $a_n = 6a_{n-1} - 9a_{n-2}$; $n \geq 2$ with $a_0 = 1, a_1 = 6$ **[CU 2014]**
11. $a_n = -4a_{n-1} - 4a_{n-2}$; $n \geq 2$ with $a_0 = 6, a_1 = 8$
12. $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$; $n \geq 3$ with $a_0 = 3, a_1 = 6, a_2 = 0$
13. $a_n = 7a_{n-2} + 6a_{n-3}$; $a_0 = 9, a_1 = 10, a_2 = 32$
14. $a_n = 5a_{n-2} - 4a_{n-4}$; $a_0 = 3, a_1 = 2, a_2 = 6, a_3 = 8$
15. $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$; $a_0 = 7, a_1 = -4, a_2 = 8$
16. $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$; $a_0 = -5, a_1 = 4, a_2 = 88$
17. $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$; $a_0 = 5, a_1 = -9, a_2 = 15$
18. **Fibonacci Sequence** $a_n = a_{n-1} + a_{n-2}$; $a_0 = 0, a_1 = 1$ **[CU 2013]**
19. Show that Fibonacci numbers satisfy the recurrence relation $f_n = 5f_{n-4} + 3f_{n-5}$ for $n = 5, 6, 7, ...$ together with initial conditions, $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3$. Use this relation to show that $5f_n$ is divisible by 5, for $n = 1, 2, 3, ...$ **[CU 2012]**
20. The Lucas numbers satisfy the recurrence relation
    $L_n = L_{n-1} + L_{n-2}$; $L_0 = 2, L_1 = 1$

(a) Show that, $L_n = f_{n-1} + f_{n+1}$ for $n = 2, 3, ...$ where f_n is the nth Fibonacci number.
(b) Find an explicit formula for Lucas numbers.

On IDE

1. $a_n - 5a_{n-1} + 6a_{n-2} = 1$
2. $a_n - 6a_{n-1} + 8a_{n-2} = 3$
3. $a_{n+2} - 2a_{n+1} + a_n = 3n + 5$
4. $a_{n+2} - 5a_{n+1} + 6a_n = n^2$
5. $a_n + 4a_{n-1} + 4a_{n-2} = n^2 - 3n + 5$
6. $2a_n - 7a_{n-1} + 3a_{n-2} = 2^n$
7. $a_n - 4a_{n-1} + 4a_{n-2} = (n+1)2^n$
8. $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = F(n)$ where
   (a) $F(n) = n^2$
   (b) $F(n) = 2^n$
   (c) $F(n) = n2^n$
   (d) $F(n) = (-2)^n$
   (e) $F(n) = n^2 \cdot 2^n$
   (f) $F(n) = n^3 \cdot (-2)^n$
   (g) $F(n) = 3$
9. $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ where $F(n)\ =$
   (a) n^3
   (b) $(-2)^n$
   (c) $n \cdot 2^n$
   (d) $n^2 \cdot 4^n$
   (e) $(n^2 - 2) \cdot (-2)^n$
   (f) $n^4 \cdot 2^n$
   (g) 2
10. $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$ [*Hint:* $a_n^{(p)} = An2^n + A_0 + A_1n$]
11. $a_n = 2a_{n-1} + 3 \cdot 2^n$
12. $a_n = 4a_{n-1} - 4a_{n-2} + (n+1) \cdot 2^n$
13. Let a_n be the sum of the first n perfect squares, i.e. $a_n = \sum_{k=1}^{n} k^2$. Show that the sequence a_n satisfies the linear non-homogeneous recurrence relation $a_n = a_{n-1} + n^2$ and the initial condition $a_1 = 1$.

On IDE with initial conditions

1. $a_n - a_{n-1} = n \ ; \ a_0 = 1$ **[CU 2003]**

2. $a_n - 2a_{n-1} = 6 \ ; a_1 = 2$

3. $a_n - a_{n+1} = 3^n \ ; a_0 = 1$

4. $a_n - a_{n-1} - 2a_{n-2} = 1 \ ; \ a_1 = 1, a_2 = 3$

5. $a_{n+2} - 4a_{n+1} + 4a_n = 2^n \ ; \ a_0 = -2, a_1 = 0$

6. $2a_{n+1} - a_n = (\frac{1}{2})^n \ ; \ a_1 = 2$

7. $a_{n+2} - 6a_{n+1} + 9a_n = 3 \cdot 2^n + 7 \cdot 3^n$ with $a_0 = 1$ and $a_1 = 4$ **[CU 2011]**

8. Determine the values of the constants A and B such that $a_n = An + B$ is a solution of the recurrence relation $a_n = 2a_{n-1} + n + 5 \ ; \ a_0 = 4$

9. Solve the simultaneous recurrence relations

   (a) $a_n = 3a_{n-1} + 2b_{n-1}$

   (b) $b_n = a_{n-1} + 2b_{n-1} \ ; \ a_0 = 1, b_0 = 2$

10. Show that if $a_n = a_{n-1} + a_{n-2} \ ; \ a_0 = s$ and $a_1 = t$ where s and t are constants then $a_n = sf_{n-1} + tf_n$ for all positive initegers n .

11. Express solution of the linear non-homogeneous recurrence relation $a_n = a_{n_1} + a_{n-2} + 1 \ ; \ n \geq 2$ where $a_0 = 0$ and $a_1 = 1$ in terms of Fibonacci numbers.

    [*Hint:* Let, $b_n = a_n + 1$ and apply previous question the sequence for b_n]

12. $a_n = 2a_{n-1}$ if n is even with $a_0 = 0$
    $= 2a_{n_1} + 1$ if n is odd

Miscelleneous

1. Suppose a person deposit Rs. 10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in account after 30 years?

2. Find a recurrence relation for the no of bit strings that contain the string 01.

3. Find a recurrence relation and give initial condition for the number of bit strings of length n that does not have two consequtive 0s. How many such bit strings are there of length 5?

4. A computer system considers a string od decimal digits a valid codeword if it contains an even no of 0 digits. For example, 1230407869 is valid. Let a_n be the no of valid n digit codeword. Find a recurrence relation for a_n.