Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

Site search: Go

RSS What's this?

Nick's Mathematical Puzzles: 1 to 10

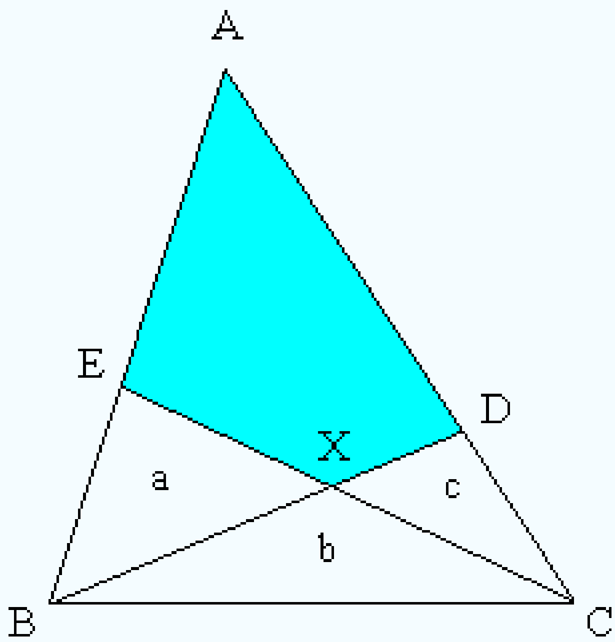
1. Folded sheet of paper ☆

A rectangular sheet of paper is folded so that two diagonally opposite corners come together. If the crease formed is the same length as the longer side of the sheet, what is the ratio of the longer side of the sheet to the shorter side?

Hint - Answer - Solution

2. Triangular area ☆☆

In △ ABC, produce a line from B to AC, meeting at D, and from C to AB, meeting at E. Let BD and CE meet at X.
Let △ BXE have area a, △ BXC have area b, and △ CXD have area c. Find the area of quadrilateral AEXD in terms of a, b, and c.



Hint - Answer - Solution

3. Two logicians ☆☆☆☆

Two perfect logicians, S and P, are told that integers x and y have been chosen such that $1 < x < y$ and $x + y < 100$. S is given the value $x+y$ and P is given the value xy. They then have the following conversation.

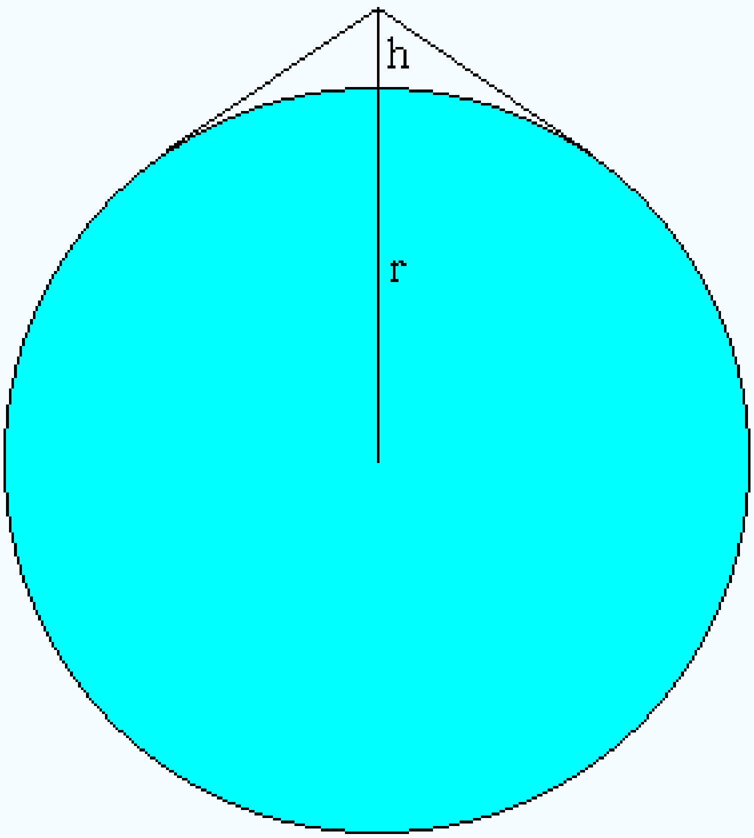
> **P: I cannot determine the two numbers.**
> **S: I knew that.**
> **P: Now I can determine them.**
> **S: So can I.**

Given that the above statements are true, what are the two numbers? (Computer assistance allowed.)

Answer - Solution

4. Equatorial belt ☆☆

A snug-fitting belt is placed around the Earth's equator. Suppose you added an extra 1 meter of length to the belt, held it at a point, and lifted until all the slack was gone. How high above the Earth's surface would you then be? That is, find h in the diagram below.



Assume that the Earth is a perfect sphere of radius 6400 km, and that the belt material does not stretch. An approximate solution is acceptable.

Hint - Answer - Solution

5. Confused bank teller

A confused bank teller transposed the dollars and cents when he cashed a check for Ms Smith, giving her dollars instead of cents and cents instead of dollars. After buying a newspaper for 50 cents, Ms Smith noticed that she had left exactly three times as much as the original check. What was the amount of the check? (Note: 1 dollar = 100 cents.)

Hint - Answer - Solution

6. Ant on a box ☆

A 12 by 25 by 36 cm cereal box is lying on the floor on one of its 25 by 36 cm faces. An ant, located at one of the bottom corners of the box, must crawl along the outside of the box to reach the opposite bottom corner. What is the length of the shortest such path?

Note: The ant can walk on any of the five faces of the box, except for the bottom face, which is flush in contact with the floor. It can crawl along any of the edges. It cannot crawl under the box.

Hint - Answer - Solution

7. Five men, a monkey, and some coconuts ☆☆

Five men crash-land their airplane on a deserted island in the South Pacific. On their first day they gather as many coconuts as they can find into one big pile. They decide that, since it is getting dark, they will wait until the next day to divide the coconuts.

That night each man took a turn watching for rescue searchers while the others slept. The first watcher got bored so he decided to divide the coconuts into five equal piles. When he did this, he found he had one remaining coconut. He gave this coconut to a monkey, took one of the piles, and hid it for himself. Then he jumbled up the four other piles into one big pile again.

To cut a long story short, each of the five men ended up doing exactly the same thing. They each divided the coconuts into five equal piles and had one extra coconut left over, which they gave to the monkey. They each took one of the five piles and hid those coconuts. They each came back and jumbled up the remaining four piles into one big pile.

What is the smallest number of coconuts there could have been in the original pile?

Hint - Answer - Solution

8. 271 ☆☆☆

Write 271 as the sum of positive real numbers so as to maximize their product.

Hint - Answer - Solution

9. Reciprocals and cubes

The sum of the reciprocals of two real numbers is –1, and the sum of their cubes is 4. What are the numbers?

Hint - Answer - Solution

10. Farmer's enclosure

A farmer has four straight pieces of fencing: 1, 2, 3, and 4 yards in length. What is the maximum area he can enclose by connecting the pieces? Assume the land is flat.

Answer - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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Nick's Mathematical Puzzles: 11 to 20

11. Dice game ☆☆

Two students play a game based on the total roll of two standard dice. Student A says that a 12 will be rolled first. Student B says that two consecutive 7s will be rolled first. The students keep rolling until one of them wins. What is the probability that A will win?

Hint - Answer - Solution

12. Making $1 ☆

You are given n > 0 of each of the standard denomination US coins: 1¢, 5¢, 10¢, 25¢, 50¢, $1. What is the smallest n such that it is impossible to select n coins that make exactly a dollar?

Answer - Solution

13. Coin triplets ☆☆☆

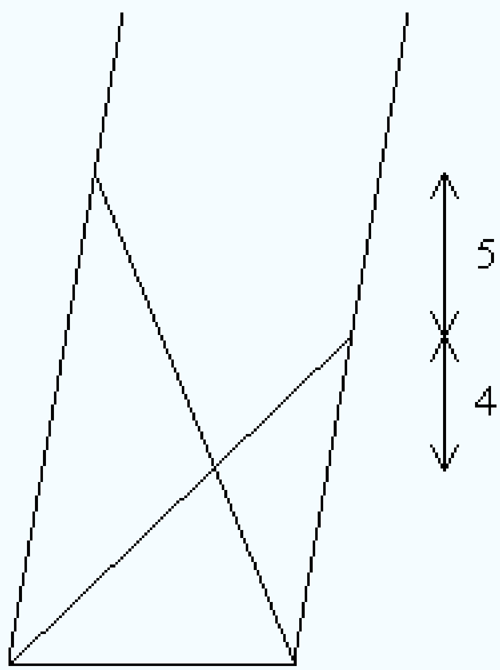
Two players play the following game with a fair coin. Player 1 chooses (and announces) a triplet (HHH, HHT, HTH, HTT, THH, THT, TTH, or TTT) that might result from three successive tosses of the coin. Player 2 then chooses a different triplet. The players toss the coin until one of the two named triplets appears. The triplets may appear in any three consecutive tosses: (1st, 2nd, 3rd), (2nd, 3rd, 4th), and so on. The winner is the player whose triplet appears first.

1. What is the optimal strategy for each player? With best play, who is most likely to win?
2. Suppose the triplets were chosen in secret? What then would be the optimal strategy?
3. What would be the optimal strategy against a randomly selected triplet?

Answer - Solution

14. Two ladders in an alley ☆☆

Two ladders are placed cross-wise in an alley to form a lopsided X-shape. The walls of the alley are not quite vertical, but are parallel to each other. The ground is flat and horizontal. The bottom of each ladder is placed against the opposite wall. The top of the longer ladder touches the alley wall 5 feet vertically higher than the top of the shorter ladder touches the opposite wall, which in turn is 4 feet vertically higher than the intersection of the two ladders. How high vertically above the ground is that intersection?



Hint - Answer - Solution

15. Infinite product ☆☆

Find the value of the infinite product

$$P = \frac{7}{9} \times \frac{26}{28} \times \frac{63}{65} \times \ldots \times \frac{k^3 - 1}{k^3 + 1} \times \ldots$$

Hint - Answer - Solution

16. Zero-sum game ☆

Two players take turns choosing one number at a time (without replacement) from the set {–4, –3, –2, –1, 0, 1, 2, 3, 4}. The first player to obtain three numbers (out of three, four, or five) which sum to 0 wins.

Does either player have a forced win?

Hint - Answer - Solution

17. Three children ☆

On the first day of a new job, a colleague invites you around for a barbecue. As the two of you arrive at his home, a young boy throws open the door to welcome his father. “My other two kids will be home soon!” remarks your colleague.

Waiting in the kitchen while your colleague gets some drinks from the basement, you notice a letter from the principal of the local school tacked to the noticeboard. “Dear Parents,” it begins, “This is the time of year when I write to all parents, such as yourselves, who have a girl or girls in the school, asking you to volunteer your time to help the girls' soccer team.” “Hmmm,” you think to yourself, “clearly they have at least one of each!”

This, of course, leaves two possibilities: two boys and a girl, or two girls and a boy. Are these two possibilities equally likely, or is one more likely than the other?

Note: This is not a trick puzzle. You should assume all things that it seems you're meant to assume, and not assume things that you aren't told to assume. If things can easily be imagined in either of two ways, you should assume that they are equally likely. For example, you may be able to imagine a reason that a colleague with two boys and a girl would be more likely to have invited you to dinner than one with two girls and a boy. If so, this would affect the probabilities of the two possibilities. But if your imagination is that good, you can probably imagine the opposite as well. You should assume that any such extra information not mentioned in the story is not available.

Answer - Solution

18. One extra coin ☆☆☆

Player A has one more coin than player B. Both players throw all of their coins simultaneously and observe the number that come up heads. Assuming all the coins are fair, what is the probability that A obtains more heads than B?

Hint - Answer - Solution

19. Five card trick ☆☆☆

Two information theoreticians, A and B, perform a trick with a shuffled deck of cards, jokers removed. A asks a member of the audience to select five cards at random from the deck. The audience member passes the five cards to A, who examines them, and hands one back. A then arranges the remaining four cards in some way and places them face down, in a neat pile.

B, who has not witnessed these proceedings, then enters the room, looks at the four cards, and determines the missing fifth card, held by the audience member. How is this trick done?

Note: The only communication between A and B is via the arrangement of the four cards. There is no encoded speech or hand signals or ESP, no bent or marked cards, no clue in the orientation of the pile of four cards...

Hint 1 - Hint 2 - Solution

20. Five card trick, part 2 ☆☆☆☆

The two information theoreticians from puzzle 19 now attempt an even more ambitious trick. It is in fact the same trick, but performed this time with a pack of 124 cards! How does *this* trick work?

(Note: 124 cards is the maximum number of cards for which this trick can be performed. The cards may be thought of as four suits with 31 cards each, or perhaps as days from a calendar, using the months January, March, May, and July. Or they may be thought of as a double deck, with 20 extra cards from a third deck thrown in, bearing in mind that the magicians must be able to tell, perhaps from the design on the back of the cards, from which pack a given card is taken. Or they may simply be numbered from 1 to 124.)

Solution

Back to top

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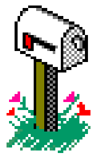


Last updated: June 9, 2004

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

Site search: Go

RSS What's this?

Nick's Mathematical Puzzles: 21 to 30

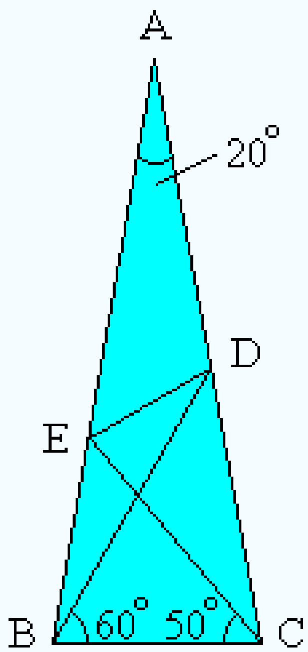
21. Birthday line ☆☆

At a movie theater, the manager announces that a free ticket will be given to the first person in line whose birthday is the same as someone in line who has already bought a ticket. You have the option of getting in line at any time. Assuming that you don't know anyone else's birthday, and that birthdays are uniformly distributed throughout a 365 day year, what position in line gives you the best chance of being the first duplicate birthday?

Hint - Answer - Solution

22. Isosceles angle ☆☆☆☆

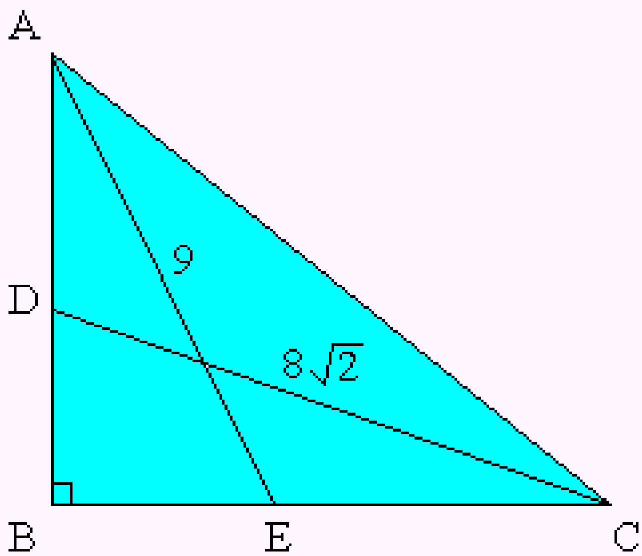
Let ABC be an isosceles triangle (AB = AC) with $\angle$ BAC = 20°. Point D is on side AC such that $\angle$ DBC = 60°. Point E is on side AB such that $\angle$ ECB = 50°. Find, with proof, the measure of $\angle$ EDB.



Answer - Solution

23. Length of hypotenuse ☆☆☆☆

Triangle ABC is right-angled at B. D is a point on AB such that $\angle$BCD = $\angle$DCA. E is a point on BC such that $\angle$BAE = $\angle$EAC. If AE = 9 inches and CD = $8\sqrt{2}$ inches, find AC.



Answer - Solution

24. Die: median throws ☆☆

What is the minimum number of times a fair die must be thrown for there to be at least an even chance that all scores appear at least once? (Computer assistance advisable.)

Hint - Answer - Solution

25. Die: mean throws ☆☆

What is the expected number of times a fair die must be thrown until all scores appear at least once?

Hint - Answer - Solution

26. Two packs of cards ☆☆☆☆

Players A and B each have a well shuffled standard pack of cards, with no jokers. The players deal their cards one at a time, from the top of the deck, checking for an exact match. Player A wins if, once the packs are fully dealt, no matches are found. Player B wins if at least one match occurs. What is the probability that player A wins?

Answer - Solution

27. 1000th digit ☆☆☆

What is the 1000th digit to the right of the decimal point in the decimal representation of $(1 + \sqrt{2})^{3000}$?

Hint - Answer - Solution

28. Making 24 ☆☆

Using only the numbers 1, 3, 4, and 6, together with the operations +, –, ×, and ÷, and unlimited use of brackets, make the number 24. Each number must be used precisely once. Each operation may be used zero or more times. Decimal points are not allowed, nor is implicit use of base 10 by concatenating digits, as in 3 × (14 – 6).

As an example, one way to make 25 is: 4 × (6 + 1) – 3.

Hint 1 - Hint 2 - Answer

29. x^x ☆☆☆

If x is a positive rational number, show that x^x is irrational unless x is an integer.

Hint - Solution

30. Two pool balls ☆

A cloth bag contains a pool ball, which is known to be a solid ball. A second pool ball is chosen at random in such a way that it is equally likely to be a solid or a stripe ball. The ball is added to the bag, the bag is shaken, and a ball is drawn at random. This ball proves to be a solid. What is the probability that the ball remaining in the bag is also a solid?

Answer - Solution

Back to top

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Puzzles: 1-10 11-20 21-30 **31-40** 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

Site search: Go

RSS What's this?

Nick's Mathematical Puzzles: 31 to 40

31. Area of a rhombus

A rhombus, ABCD, has sides of length 10. A circle with center A passes through C (the opposite vertex.) Likewise, a circle with center B passes through D. If the two circles are tangent to each other, what is the area of the rhombus?

Hint - Answer - Solution

32. Differentiation conundrum

The derivative of x^2, with respect to x, is 2x. However, suppose we write x^2 as the sum of x x's, and then take the derivative:

Let f(x) = x + x + ... + x (x times)

Then f'(x) = d/dx[x + x + ... + x] (x times)
= d/dx[x] + d/dx[x] + ... + d/dx[x] (x times)
= 1 + 1 + ... + 1 (x times)
= x

This argument appears to show that the derivative of x^2, with respect to x, is actually x. Where is the fallacy?

Hint - Solution

33. Harmonic sum

Let $H_0 = 0$ and $H_n = 1/1 + 1/2 + ... + 1/n$.

Show that, for $n > 0$, $H_n = 1 + (H_0 + H_1 + ... + H_{n-1})/n$.

(That is, show that H_n is one greater than the arithmetic mean of the n preceding values, H_0 to H_{n-1}.)

Hint - Solution

34. Harmonic sum 2

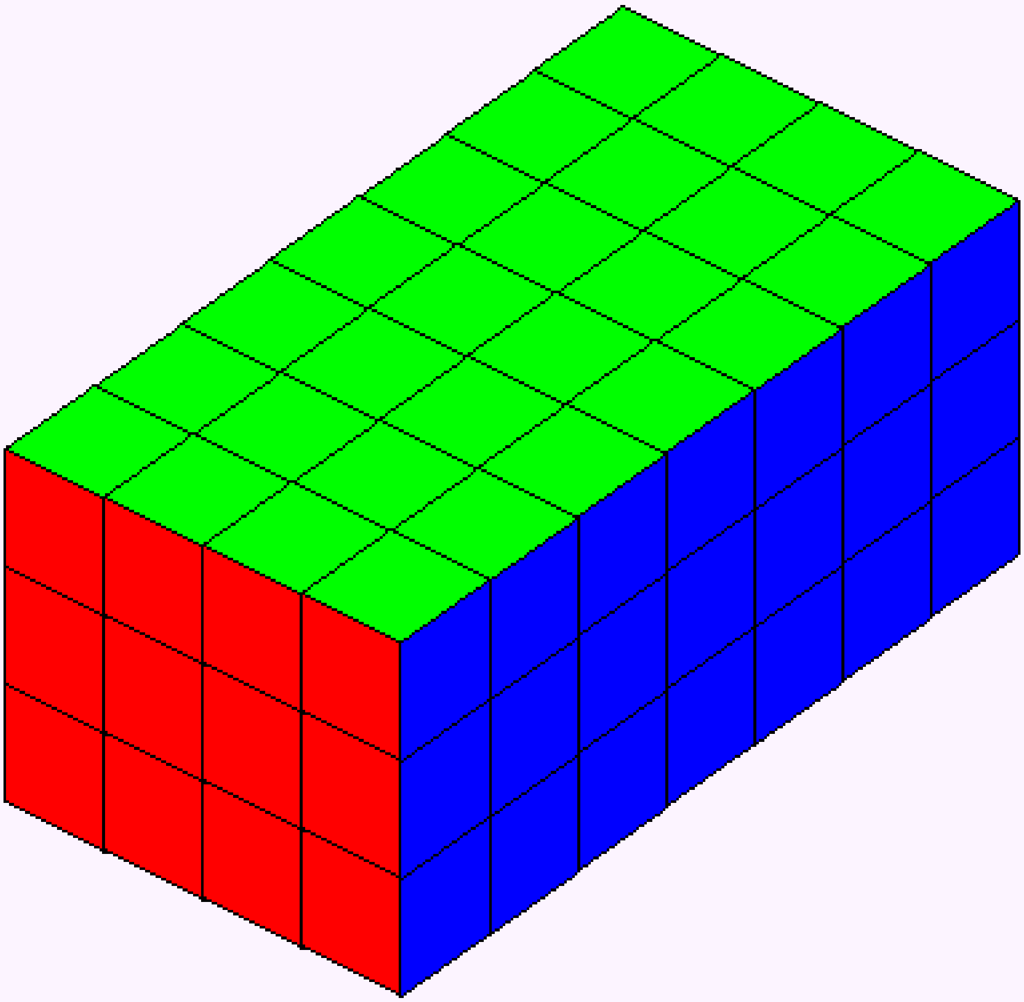
Let $H_n = 1/1 + 1/2 + ... + 1/n$.

Show that, for $n > 1$, H_n is not an integer.

Hint - Solution

35. Cuboids

An a × b × c cuboid is constructed out of abc identical unit cubes -- a la Rubik's Cube. Divide the cubes into two mutually exclusive types. External cubes are those that constitute the faces of the cuboid; internal cubes are completely enclosed. For example, the cuboid below has 74 external and 10 internal cubes.



Find all cuboids such that the number of external cubes equals the number of internal cubes. (That is, give the dimensions of all such cuboids.)

Hint - Answer - Solution

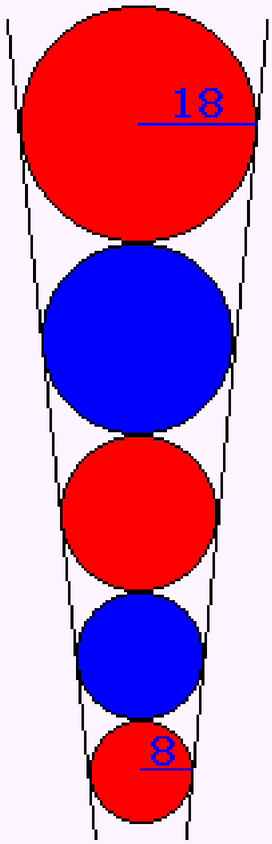
36. Composite numbers ★★★

Take any positive composite integer, m.
We have m = ab = cd, where ab and cd are distinct factorizations, and a, b, c, d $\geq$ 1.
Show that $a^n + b^n + c^n + d^n$ is composite, for all integers n $\geq$ 0.

Hint - Solution

37. Five marbles ☆☆

Five marbles of various sizes are placed in a conical funnel. Each marble is in contact with the adjacent marble(s). Also, each marble is in contact all around the funnel wall.



The smallest marble has a radius of 8mm. The largest marble has a radius of 18mm. What is the radius of the middle marble?

Hint - Answer - Solution

38. Twelve marbles ☆☆☆

A boy has four red marbles and eight blue marbles. He arranges his twelve marbles randomly, in a ring. What is the probability that no two red marbles are adjacent?

Answer - Solution

39. Prime or composite?

Is the number 2438100000001 prime or composite? No calculators or computers allowed!

Hint - Answer - Solution

40. No consecutive heads

A fair coin is tossed n times. What is the probability that no two consecutive heads appear?

Hint - Answer - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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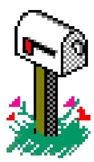

Last updated: April 30, 2005

Nick's Mathematical Puzzles: 41 to 50

41. Crazy dice ☆☆☆☆

Roll a standard pair of six-sided dice, and note the sum. There is one way of obtaining a 2, two ways of obtaining a 3, and so on, up to one way of obtaining a 12. Find all other pairs of six-sided dice such that:

1. The set of dots on each die is not the standard {1,2,3,4,5,6}.
2. Each face has at least one dot.
3. The number of ways of obtaining each sum is the same as for the standard dice.

Hint - Answer - Solution

42. Multiplicative sequence ☆☆☆

Let $\{a_n\}$ be a strictly increasing sequence of positive integers such that:

- $a_2 = 2$
- $a_{mn} = a_m a_n$ for m, n relatively prime (multiplicative property)

Show that $a_n = n$, for every positive integer, n.

Hint - Solution

43. Sum of two powers ☆☆

Show that $n^4 + 4^n$ is composite for all integers $n > 1$.

Hint - Solution

44. Sum of two powers 2 ☆☆☆☆

If x and y are positive real numbers, show that $x^y + y^x > 1$.

Hint - Solution

45. Area of regular 2^n-gon ☆☆

Show that the area of a regular polygon with 2^n sides and unit perimeter is $2^{-(n+2)} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}$, where there are $n - 1$ twos under both sets of nested radical signs.

Hint - Solution

46. Consecutive subsequence ☆☆☆

Given any sequence of n integers, show that there exists a consecutive subsequence the sum of whose elements is a multiple of n.
For example, in sequence {1,5,1,2} a consecutive subsequence with this property is the last three elements; in {1,–3,–7} it is simply the second element.

Hint - Solution

47. 1000 divisors ☆

Find the smallest natural number greater than 1 billion (10^9) that has exactly 1000 positive divisors.
(The term *divisor* includes 1 and the number itself. So, for example, 9 has three positive divisors.)

Hint - Answer - Solution

48. Exponential equation ☆☆☆

Suppose $x^y = y^x$, where x and y are positive real numbers, with $x < y$. Show that $x = 2$, $y = 4$ is the only integer solution. Are there further rational solutions? (That is, with x and y rational.) For what values of x do real solutions exist?

Hint - Answer - Solution

49. An odd polynomial ☆☆

Let p(x) be a polynomial with integer coefficients. Show that, if the constant term is odd, and the sum of all the coefficients is odd, then p has no integer roots. (That is, if $p(x) = a_0 + a_1x + ... + a_nx^n$, a_0 is odd, and $a_0 + a_1 + ... + a_n$ is odd, then there is no integer k such that $p(k) = 0$.)

Hint - Solution

50. Highest score ☆☆

Suppose n fair 6-sided dice are rolled simultaneously. What is the expected value of the score on the highest valued die?

Hint - Answer - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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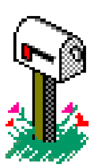


Last updated: March 15, 2003

Puzzles: 1-10 11-20 21-30 31-40 41-50 **51-60** 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

Site search: Go RSS What's this?

Nick's Mathematical Puzzles: 51 to 60

51. Greatest common divisor ☆☆☆

Let a, m, and n be positive integers, with $a > 1$, and m odd.
What is the greatest common divisor of $a^m - 1$ and $a^n + 1$?

Hint - Answer - Solution

52. Floor function sum ☆☆

Let x be a real number and n be a positive integer.
Show that $[x] + [x + 1/n] + ... + [x + (n-1)/n] = [nx]$, where $[x]$ is the greatest integer less than or equal to x.

Hint - Solution

53. The absentminded professor ☆☆☆☆

An absentminded professor buys two boxes of matches and puts them in his pocket. Every time he needs a match, he selects at random (with equal probability) from one or other of the boxes. One day the professor opens a matchbox and finds that it is empty. (He must have absentmindedly put the empty box back in his pocket when he took the last match from it.) If each box originally contained *n* matches, what is the probability that the other box currently contains *k* matches? (Where $0 \le k \le n$.)

Hint - Answer - Solution

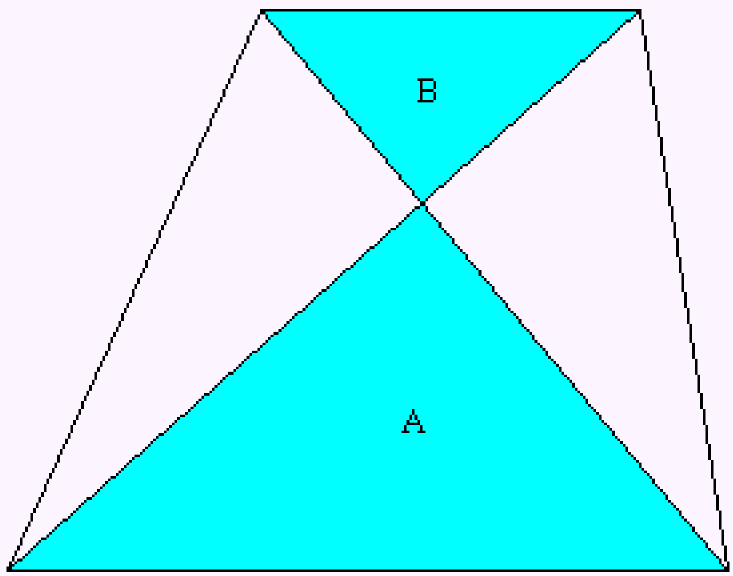
54. Diophantine squares ☆☆☆

Find all solutions to $c^2 + 1 = (a^2 - 1)(b^2 - 1)$, in integers a, b, and c.

Hint - Answer - Solution

55. Area of a trapezoid ★

A trapezoid[1] is divided into four triangles by its diagonals. Let the triangles adjacent to the parallel sides have areas A and B. Find the area of the trapezoid in terms of A and B.



(1) A *trapezoid* is a quadrilateral with at least one pair of parallel sides. In some countries, such a quadrilateral is known as a *trapezium*.

Hint - Answer - Solution

56. Partition identity ★★★★

A *partition* of a positive integer *n* is a way if writing *n* as a sum of positive integers, ignoring the order of the summands. For example, a partition of 7 is 3 + 2 + 1 + 1.

The table below shows all partitions of 5. The *number of 1s* column shows how many times the number 1 occurs in each partition. The *number of distinct parts* column shows how many distinct numbers occur in each partition. The sum for each column, over all the partitions of 5, is shown at the foot of the table.

Partition	Number of 1s	Number of distinct parts
5	0	1
4 + 1	1	2
3 + 2	0	2
3 + 1 + 1	2	2
2 + 2 + 1	1	2
2 + 1 + 1 + 1	3	2
1 + 1 + 1 + 1 + 1	5	1
Total:	12	12

Let a(n) be the number of 1s in all the partitions of n. Let b(n) be the sum, over all partitions of n, of the number of distinct parts. The above table demonstrates that a(5) = b(5).
Show that, for all n, a(n) = b(n).

Hint - Solution

57. Binomial coefficient divisibility ☆☆☆

Show that, for $n > 0$, the binomial coefficient $\binom{2n}{n} = \frac{(2n)!}{n!\ n!}$ is divisible by $n + 1$ and by $4n - 2$.

Hint - Solution

58. Fifth power plus five ☆☆☆☆

Consecutive fifth powers (or, indeed, any powers) of positive integers are always relatively prime. That is, for all $n > 0$, n^5 and $(n + 1)^5$ are relatively prime. Are $n^5 + 5$ and $(n + 1)^5 + 5$ always relatively prime? If not, for what values of n do they have a common factor, and what is that factor?

Hint - Answer - Solution

59. Triangle inequality ☆☆☆

A triangle has sides of length a, b, and c. Show that $\frac{3}{2} \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$

Hint - Solution

60. Sum of reciprocals ☆☆☆

Find $\lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} \right)$

Hint - Answer - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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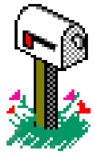


Last updated: May 6, 2003

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

Site search: Go

RSS What's this?

Nick's Mathematical Puzzles: 61 to 70

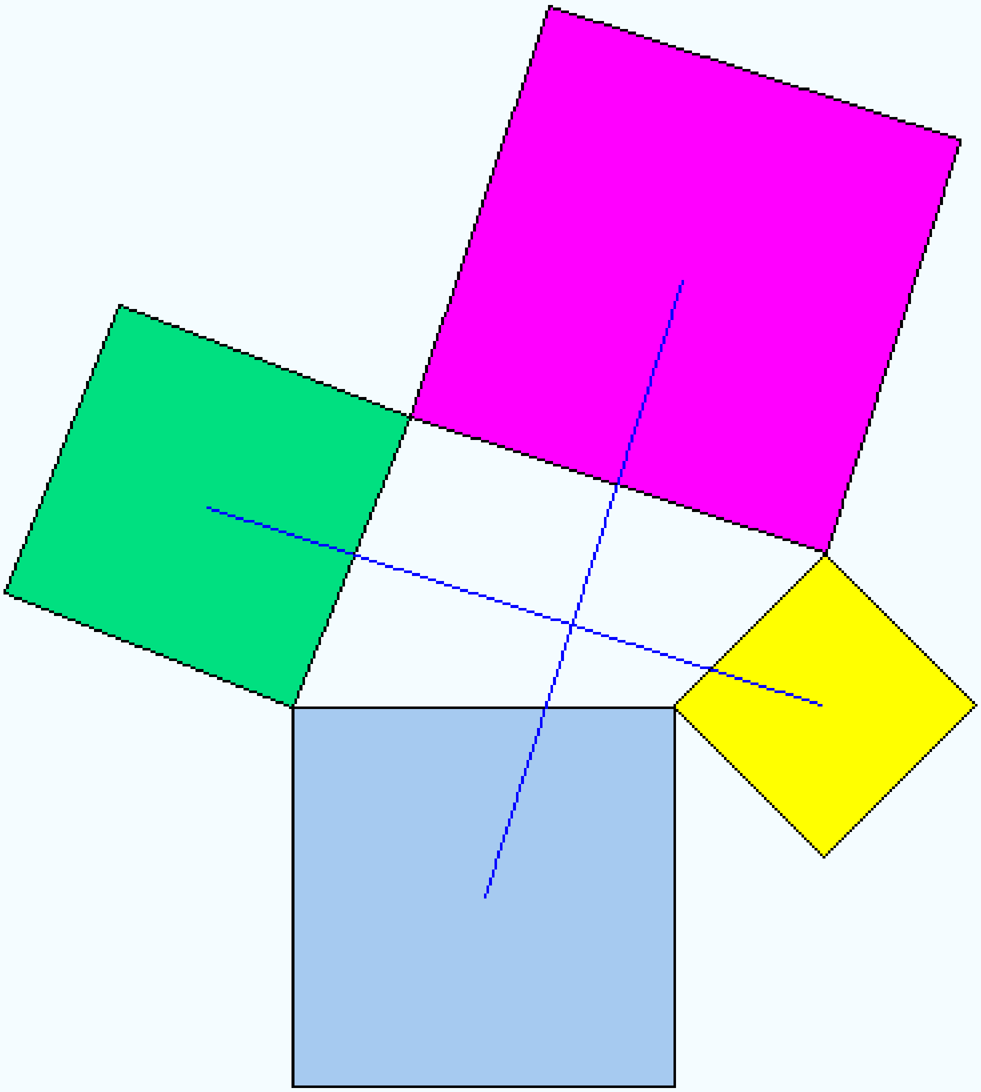
61. Two cubes? ⭐⭐

Let n be an integer. Can both $n + 3$ and $n^2 + 3$ be perfect cubes?

Hint - Answer - Solution

62. Four squares on a quadrilateral ⭐⭐⭐⭐

Squares are constructed externally on the sides of an arbitrary quadrilateral.

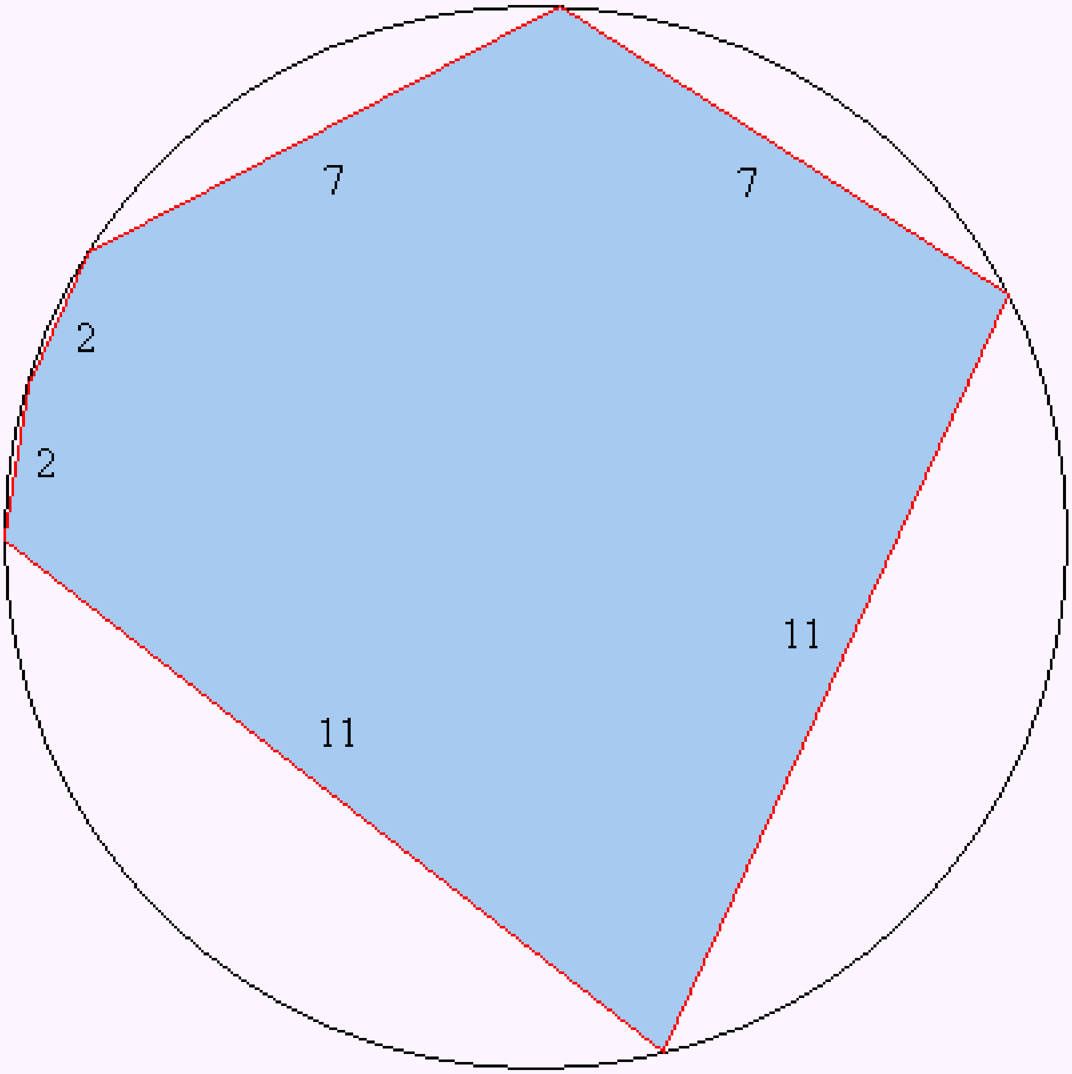


Show that the line segments joining the centers of opposite squares lie on perpendicular lines and are of equal length.

Hint - Solution

63. Cyclic hexagon ★★★★

A hexagon with consecutive sides of lengths 2, 2, 7, 7, 11, and 11 is inscribed in a circle. Find the radius of the circle.



[Hint 1](#) - [Hint 2](#) - [Answer](#) - [Solution](#)

64. Balls in an urn ☆

An urn contains a number of colored balls, with equal numbers of each color. Adding 20 balls of a new color to the urn would not change the probability of drawing (without replacement) two balls of the same color.

How many balls are in the urn? (Before the extra balls are added.)

[Hint](#) - [Answer](#) - [Solution](#)

65. Consecutive integer products ☆☆

Show that each of the following equations has no solution in integers $x > 0$, $y > 0$, $n > 1$.

1. $x(x + 1) = y^n$
2. $x(x + 1)(x + 2) = y^n$

Hint - Solution

66. Quadratic divisibility ☆☆☆

Show that, if n is an integer, $n^2 + 11n + 2$ is not divisible by 12769.

Hint - Solution

67. Random number generator ☆☆☆

A random number generator generates integers in the range 1...n, where n is a parameter passed into the generator. The output from the generator is repeatedly passed back in as the input. If the initial input parameter is one googol (10^{100}), find, to the nearest integer, the expected value of the number of iterations by which the generator first outputs the number 1. That is, what is the expected value of x, after running the following pseudo-code?

```
n = 10^100
x = 0
do while (n > 1)
n = random(n) // Generates random integer in the range 1...n
x = x + 1
end-do
```

Hint 1 - Hint 2 - Answer - Solution

68. Difference of powers ☆☆☆

Find all ordered pairs (a,b) of positive integers such that $|3^a - 2^b| = 1$.

Hint - Answer - Solution

69. Combinatorial sum ★★★

Find a closed form expression for $\sum_{k=1}^{n} \binom{n}{k} k^5$

Hint - Answer - Solution

70. One degree ★★

Show that cos 1°, sin 1°, and tan 1° are irrational numbers.

Hint - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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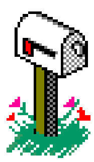


Last updated: September 17, 2003

Nick's Mathematical Puzzles: 71 to 80

71. Consecutive cubes and squares

Show that if the difference of the cubes of two consecutive integers is the square of an integer, then this integer is the sum of the squares of two consecutive integers.

(The smallest non-trivial example is: $8^3 - 7^3 = 169$. This is the square of an integer, namely 13, which can be expressed as $2^2 + 3^2$.)

Hint - Solution

72. Depleted harmonic series

It is well known that the harmonic series, 1/1 + 1/2 + 1/3 + 1/4 + ... , diverges. Consider a *depleted* harmonic series; see below; which contains only terms whose denominator does not contain a 9. (In decimal representation.) Does this series diverge or converge?

S = 1/1 + 1/2 + ... + 1/8 + 1/10 + ... + 1/18 + 1/20 + ... + 1/88 + 1/100 + 1/101 + ...

Hint - Answer - Solution

73. Unobtuse triangle

A triangle has internal angles A, B, and C, none of which exceeds 90°. Show that

- sin A + sin B + sin C > 2
- cos A + cos B + cos C > 1
- tan (A/2) + tan (B/2) + tan (C/2) < 2

Hint - Solution

74. Sum of 9999 consecutive squares ☆☆

Show that the sum of 9999 consecutive squares cannot be a perfect power.

That is, show that $(n + 1)^2 + ... + (n + 9999)^2 = m^r$ has no solution in integers n, m, r > 1.

Hint - Solution

75. Car journey ☆

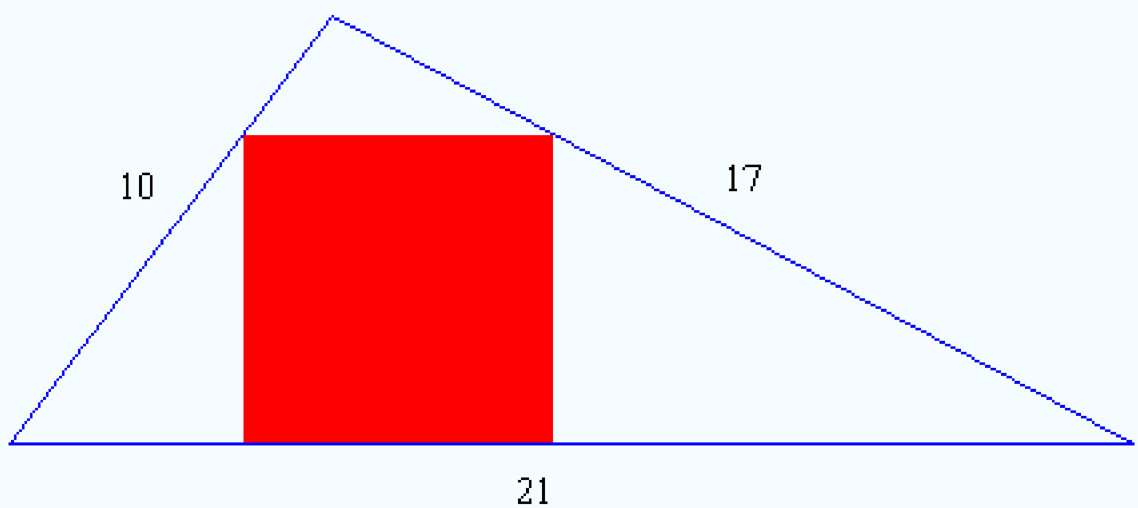
A car travels downhill at 72 mph (miles per hour), on the level at 63 mph, and uphill at only 56 mph The car takes 4 hours to travel from town A to town B. The return trip takes 4 hours and 40 minutes.

Find the distance between the two towns.

Hint - Answer - Solution

76. Square inscribed in a triangle ☆

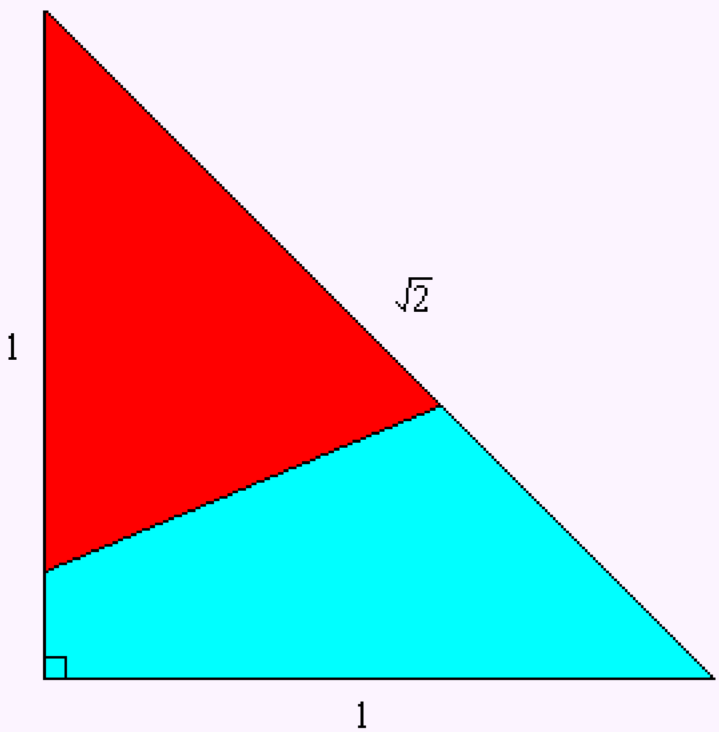
A triangle has sides 10, 17, and 21. A square is inscribed in the triangle. One side of the square lies on the longest side of the triangle. The other two vertices of the square touch the two shorter sides of the triangle. What is the length of the side of the square?



Hint - Answer - Solution

77. Minimal straight cut ☆☆

A piece of wooden board in the shape of an isosceles right triangle, with sides 1, 1, $\sqrt{2}$, is to be sawn into two pieces. Find the length and location of the shortest straight cut which divides the board into two parts of equal area.



Hint - Answer - Solution

78. Perfect square ☆☆☆☆

Find all integer solutions of $y^2 = x^3 - 432$.

Hint - Answer - Solution

79. Sum of fourth powers ★★★

The sum of three numbers is 6, the sum of their squares is 8, and the sum of their cubes is 5. What is the sum of their fourth powers?

Hint - Answer - Solution

80. Sixes and sevens ★★★

Does there exist a (base 10) 67-digit multiple of 2^{67}, written exclusively with the digits 6 and 7?

Hint - Answer - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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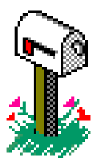


Last updated: March 17, 2004

Nick's Mathematical Puzzles: 81 to 90

81. Digit transfer ☆☆☆

Find the smallest positive integer such that when its last digit is moved to the start of the number (example: 1234 becomes 4123) the resulting number is larger than and is an integral multiple of the original number. Numbers are written in standard decimal notation, with no leading zeroes.

Hint - Answer - Solution

82. Consecutive heads ☆☆

A fair coin is tossed repeatedly until n consecutive heads occur. What is the expected number of times the coin is tossed?

For example, two consecutive heads could be obtained as follows:

- HH (two tosses)
- THH (three tosses)
- HTHH or TTHH (four tosses)

... and so on.

Hint - Answer - Solution

83. Divisibility ☆☆☆☆

Find all integers n such that $2^n - 1$ is divisible by n.

Hint - Answer - Solution

84. Missing digits ☆☆☆

Given that 37! = 13763753091226345046315979581abcdefgh0000000, determine, with a minimum of arithmetical effort, the digits a, b, c, d, e, f, g, and h. No calculators or computers allowed!

Hint - Answer - Solution

85. Fibonacci nines ☆☆

Does there exist a Fibonacci number whose decimal representation ends in nine nines?

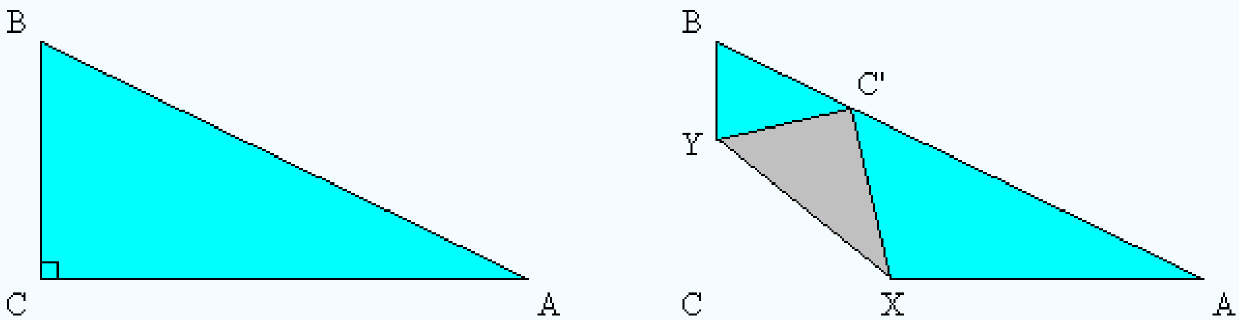
(The Fibonacci numbers are defined by the recurrence equation $F_1 = 1$, $F_2 = 1$, with $F_n = F_{n-1} + F_{n-2}$, for n > 2.)

Hint - Answer - Solution

86. Folded card

A piece of card has the shape of a triangle, ABC, with $\angle$BCA a right angle. It is folded once so that:

- C coincides with C', which lies on AB; and
- the crease extends from Y on BC to X on AC.



If BC = 115 and AC = 236, find the minimum possible value of the area of $\triangle$YXC'.

Hint - Answer - Solution

87. 2004

Evaluate 2^{2004} (modulo 2004).

Hint - Answer - Solution

88. Nested radicals

Solve the equation $\sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4-x}}}} = x$.

(All square roots are to be taken as positive.)

Hint 1 - Hint 2 - Answer - Solution

89. Square digits

A perfect square has *length* n if its last n (decimal) digits are the same and non-zero. What is the maximum possible length? Find all squares that achieve this length.

Hint - Answer - Solution

90. Powers of 2: rearranged digits ☆☆

Does there exist an integral power of 2 such that it is possible to rearrange the digits giving another power of 2? Numbers are written in standard decimal notation, with no leading zeroes.

Hint - Answer - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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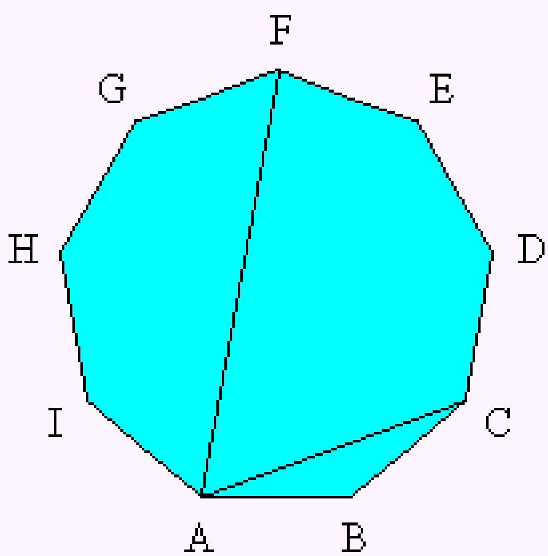
Last updated: August 31, 2004

Nick's Mathematical Puzzles: 91 to 100

91. Nonagon diagonals

In regular nonagon ABCDEFGHI, show that AF = AB + AC.



Hint - Solution

92. Consecutive integer sums

In how many ways can 50! be expressed as a sum of two or more consecutive positive integers?

Hint - Answer - Solution

93. Pascal's triangle

Show that any two elements (both greater than one) drawn from the same row of Pascal's triangle have greatest common divisor greater than one. For example, the greatest common divisor of 28 and 70 is 14.

```
                    1
                  1   1
                1   2   1
              1   3   3   1
            1   4   6   4   1
          1   5  10  10   5   1
        1   6  15  20  15   6   1
      1   7  21  35  35  21   7   1
    1   8  28  56  70  56  28   8   1
```

Hint - Solution

94. Square endings ☆☆☆☆

Find all 8-digit natural numbers n such that n^2 ends in the same 8 digits as n. Numbers are written in standard decimal notation, with no leading zeroes.

Hint - Answer - Solution

95. Integer polynomial ☆

Let P be a polynomial with integer coefficients. If a, b, c are distinct integers, show that

- P(a) = b,
- P(b) = c,
- P(c) = a,

cannot be satisfied simultaneously.

Hint - Solution

96. Five real numbers ☆☆☆

The sum of five real numbers is 7; the sum of their squares is 10. Find the minimum and maximum possible values of any one of the numbers.

Hint - Answer - Solution

97. Two squares ☆☆

Find all pairs of positive integers, x, y, such that $x^2 + 3y$ and $y^2 + 3x$ are both perfect squares.

Hint - Answer - Solution

98. Three powers ☆☆☆

Find all solutions of $3^x + 4^y = 5^z$, for integers x, y, and z.

Hint - Answer - Solution

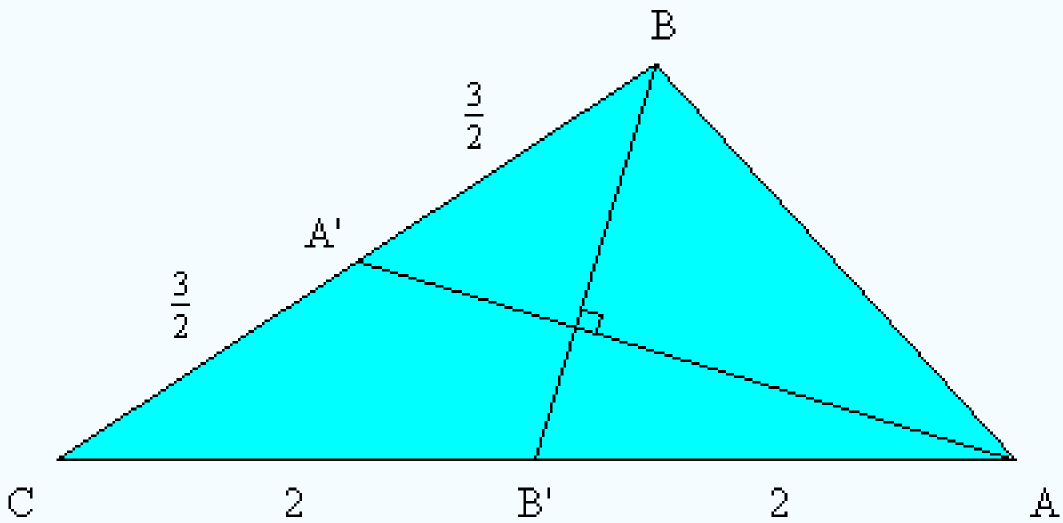
99. Two similar triangles ☆☆☆

Two similar triangles with integral sides have two of their sides the same. If the third sides differ by 20141, find all of the sides.

Hint - Answer - Solution

100. Perpendicular medians ☆

Suppose the medians AA' and BB' of triangle ABC intersect at right angles. If BC = 3 and AC = 4, what is the length of side AB?



Hint - Answer - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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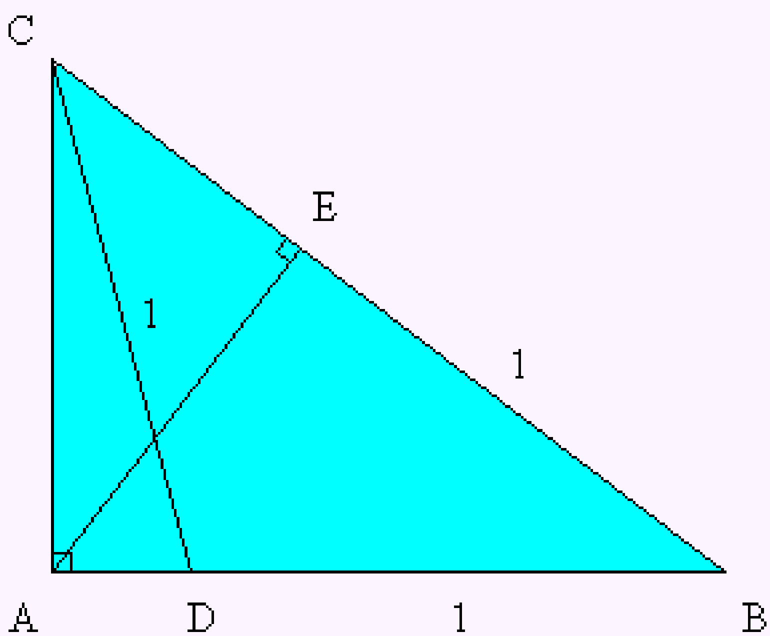
Nick's Mathematical Puzzles: 101 to 110

101. Right triangles

△ ABC is right-angled at A. D is a point on AB such that CD = 1. AE is the altitude from A to BC. If BD = BE = 1, what is the length of AD?



Hint 1 - Hint 2 - Answer - Solution

102. Almost exponential

Show that $1 + x + x^2/2! + x^3/3! + ... + x^{2n}/(2n)!$ is positive for all real values of x.

Hint - Solution

103. Root sums ☆☆

Let a, b, c be rational numbers. Show that each of the following equations can be satisfied only if a = b = c = 0.

- $a + b\sqrt[3]{2} + c\sqrt{2} = 0$.
- $a + b\sqrt[3]{2} + c\sqrt[3]{3} = 0$.
- $a + b\sqrt[3]{2} + c\sqrt[3]{4} = 0$.

Hint - Solution

104. An arbitrary sum ☆☆☆

The first 2n positive integers are arbitrarily divided into two groups of n numbers each. The numbers in the first group are sorted in ascending order: $a_1 < a_2 < ... < a_n$; the numbers in the second group are sorted in descending order: $b_1 > b_2 > ... > b_n$.

Find, with proof, the value of the sum $|a_1 - b_1| + |a_2 - b_2| + ... + |a_n - b_n|$.

Hint - Answer - Solution

105. Difference of nth powers ☆☆☆☆

Let x, y, n be positive integers, with n > 1. How many solutions are there to the equation $x^n - y^n = 2^{100}$?

Hint - Answer - Solution

106. Flying cards ☆☆☆☆

A standard pack of cards is thrown into the air in such a way that each card, independently, is equally likely to land face up or face down. The total value of the cards which landed face up is then calculated. (Card values are assigned as follows: Ace=1, 2=2, ... , 10=10, Jack=11, Queen=12, King=13. There are no jokers.)

What is the probability that the total value is divisible by 13?

Hint 1 - Hint 2 - Answer - Solution

107. A mysterious sequence ☆☆

A sequence of positive real numbers is defined by

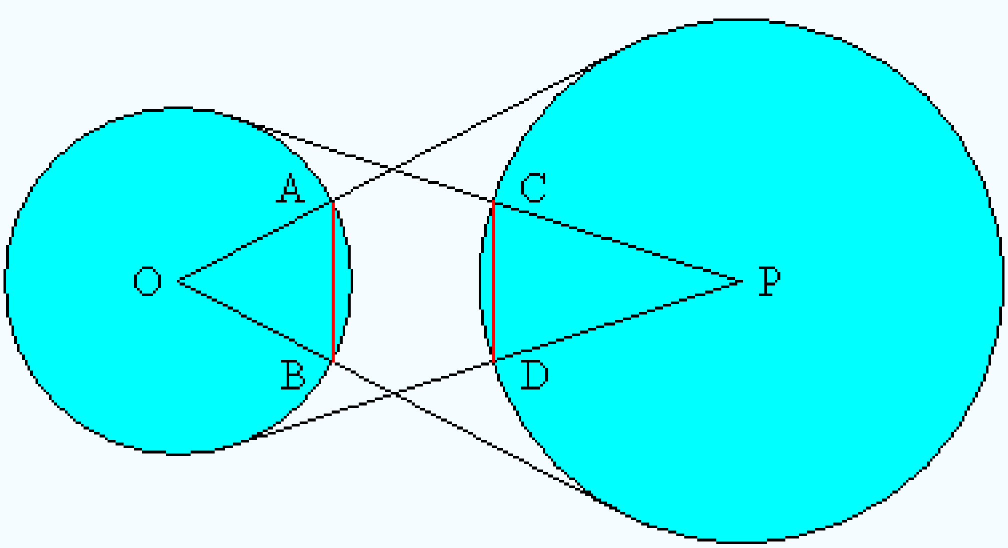
- $a_0 = 1$,
- $a_{n+2} = 2a_n - a_{n+1}$, for n = 0, 1, 2, ...

Find a_{2005}.

Hint - Answer - Solution

108. Eyeball to eyeball ☆

Take two circles, with centers O and P. From the center of each circle, draw two tangents to the circumference of the other circle. Let the tangents from O intersect that circle at A and B, and the tangents from P intersect that circle at C and D. Show that chords AB and CD are of equal length.



Hint - Solution

109. Nested circular functions ★★★

Let x be a real number. Which is greater, sin(cos x) or cos(sin x)?

Hint - Answer - Solution

110. Pairwise products ★★★★

Let n be a positive integer, and let $S_n = \{n^2 + 1, n^2 + 2, \ldots , (n + 1)^2\}$. Find, in terms of n, the cardinality of the set of pairwise products of distinct elements of S_n.

For example, $S_2 = \{5, 6, 7, 8, 9\}$,

$5 \times 6 = 6 \times 5 = 30$,
$5 \times 7 = 7 \times 5 = 35$,
$5 \times 8 = 8 \times 5 = 40$,
$5 \times 9 = 9 \times 5 = 45$,
$6 \times 7 = 7 \times 6 = 42$,
$6 \times 8 = 8 \times 6 = 48$,
$6 \times 9 = 9 \times 6 = 54$,
$7 \times 8 = 8 \times 7 = 56$,
$7 \times 9 = 9 \times 7 = 63$,
$8 \times 9 = 9 \times 8 = 72$,

and the required cardinality is 10.

Hint - Answer - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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Last updated: May 17, 2005

Nick's Mathematical Puzzles: 111 to 120

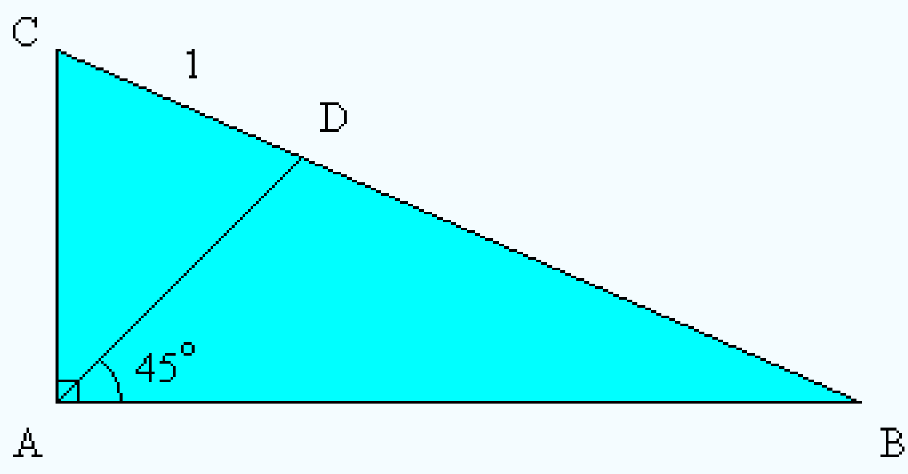
111. Trigonometric progression

Show that $\tan[(n+1)a/2] = \dfrac{\sin a + \sin 2a + \ldots + \sin na}{\cos a + \cos 2a + \ldots + \cos na}$

Hint - Solution

112. Angle bisector

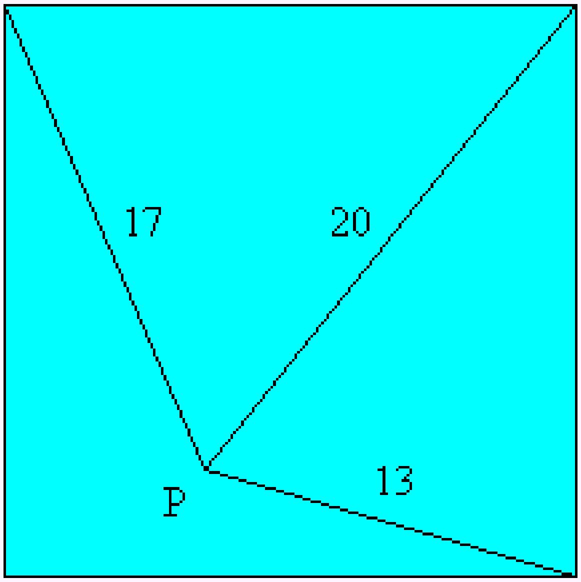
$\triangle$ ABC is right-angled at A. The angle bisector from A meets BC at D, so that $\angle$ DAB = 45°. If CD = 1 and BD = AD + 1, find the lengths of AC and AD.



Hint - Answer - Solution

113. Ant in a field

An ant, located in a square field, is 13 meters from one of the corner posts of the field, 17 meters from the corner post diagonally opposite that one, and 20 meters from a third corner post. Find the area of the field. Assume the land is flat.



Hint - Answer - Solution

114. Sums of squares and cubes ☆☆☆☆

Let a, b, and c be positive real numbers such that abc = 1. Show that $a^2 + b^2 + c^2 \le a^3 + b^3 + c^3$.

Hint - Solution

115. Sum of sines ☆

Let $f(x) = \sin(x) + \sin(x°)$, with domain the real numbers. Is f a periodic function?

(Note: sin(x) is the sine of a real number, x, (or, equivalently, the sine of x radians), while sin(x°) is the sine of x degrees.)

Hint - Answer - Solution

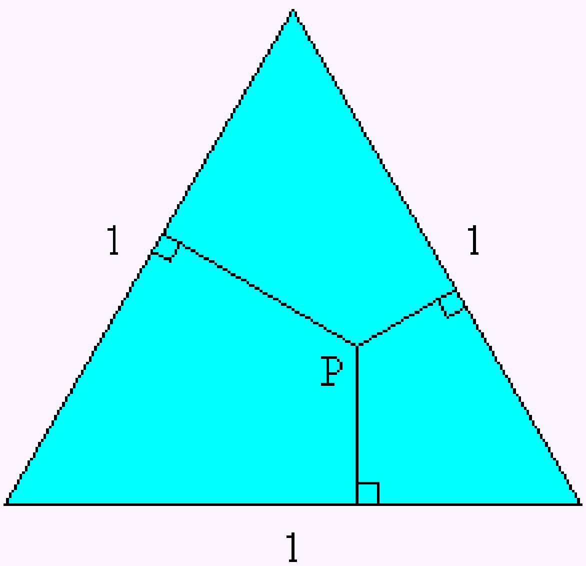
116. Factorial divisors ☆☆

Show that, for each $n \geq 3$, $n!$ can be represented as the sum of n distinct divisors of itself. (For example, $3! = 1 + 2 + 3$.)

Hint - Solution

117. Random point in an equilateral triangle ☆☆

A point P is chosen at random inside an equilateral triangle of side length 1. Find the expected value of the sum of the (perpendicular) distances from P to the three sides of the triangle.



Hint - Answer - Solution

118. Powers of 2: deleted digit ☆☆☆

Find all powers of 2 such that, after deleting the first digit, another power of 2 remains. (For example, $2^5 = 32$. On deleting the initial 3, we are left with $2 = 2^1$.) Numbers are written in standard decimal notation, with no leading zeroes.

Hint - Answer - Solution

119. Three sines ☆☆☆

A triangle has two acute angles, A and B. Show that the triangle is right-angled if, and only if, $\sin^2 A + \sin^2 B = \sin(A + B)$.

Hint - Solution

120. Factorial plus one ☆☆

Let n be a positive integer. Prove that $n! + 1$ is composite for infinitely many values of n.

Hint 1 - Hint 2 - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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Last updated: September 19, 2005

Nick's Mathematical Puzzles: 121 to 130

121. Integer sequence ☆☆

The terms of a sequence of positive integers satisfy $a_{n+3} = a_{n+2}(a_{n+1} + a_n)$, for n = 1, 2, 3, ...

If $a_6 = 8820$, what is a_7?

Hint - Answer - Solution

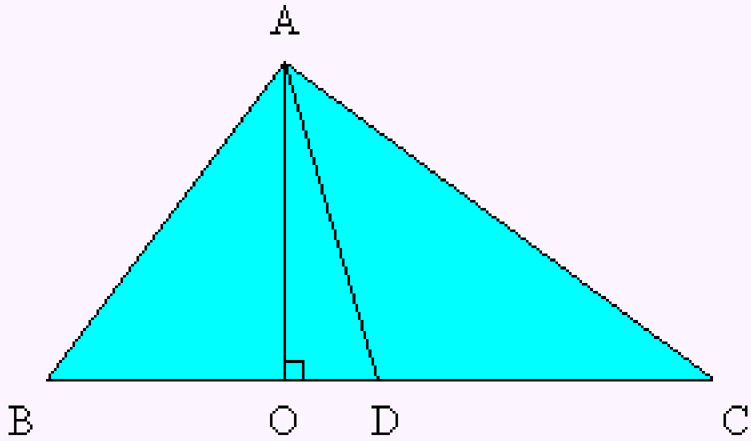
122. Powers of 2 and 5 ☆☆

If the numbers 2^n and 5^n (where n is a positive integer) start with the same digit, what is this digit? The numbers are written in decimal notation, with no leading zeroes.

Hint - Answer - Solution

123. Right angle and median ☆☆☆

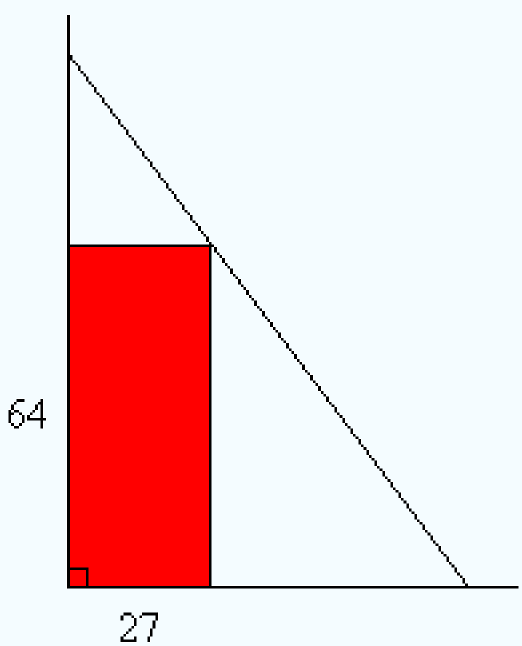
Let ABC be a triangle, with AB ≠ AC. Drop a perpendicular from A to BC, meeting at O. Let AD be the median joining A to BC. If ∠OAB = ∠CAD, show that ∠CAB is a right angle.



Hint - Solution

124. The ladder ☆☆

A ladder, leaning against a building, rests upon the ground and just touches a box, which is flush against the wall and the ground. The box has a height of 64 units and a width of 27 units.



Find the length of the ladder so that there is only one position in which it can touch the ground, the box, and the wall.

Hint - Answer - Solution

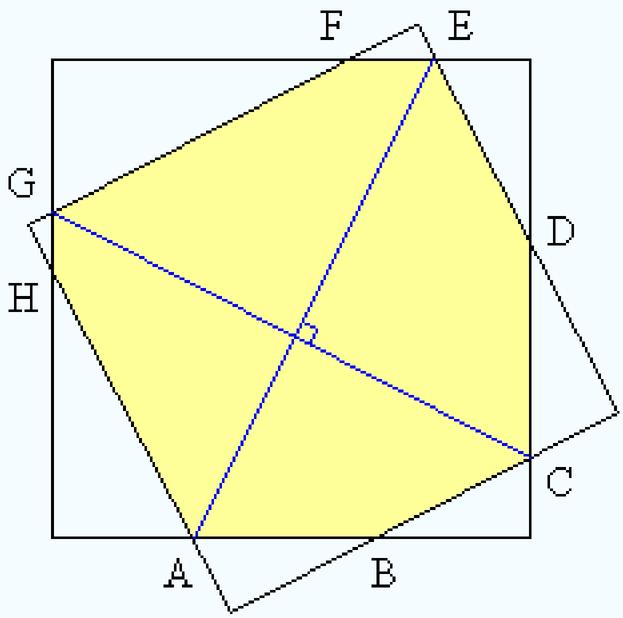
125. Divisibility by 446617991732222310 ☆☆

Show that, for all integers m and n, $mn(m^{420} - n^{420})$ is divisible by 446617991732222310.

Hint - Solution

126. Intersecting squares ☆☆☆

The sides of two squares (not necessarily of the same size) intersect in eight distinct points: A, B, C, D, E, F, G, and H. These eight points form an octagon. Join opposite pairs of vertices to form two non-adjacent diagonals. (For example, diagonals AE and CG.) Show that these two diagonals are perpendicular.



Hint - Solution

127. Prime number generator ☆

Let P = $\{p_1, \ldots, p_n\}$ be the set of the first n prime numbers. Let S be an arbitrary (possibly empty) subset of P. Let A be the product of the elements of S, and B the product of the elements of S', the complement of S. (An empty product is assigned the value of 1.)

Prove that each of A + B and |A – B| is prime, provided that it is less than p_{n+1}^2 and greater than 1.

For example, if P = {2, 3, 5, 7}, the table below shows all the distinct possibilities for A + B and |A – B|. Values of A + B and |A – B| that are less than p_5^2 = 121 and greater than 1, shown in bold, are all prime.

Prime number generator example

S	S'	A	B	A + B	|A – B|
Empty set	{2, 3, 5, 7}	1	210	211	209
{2}	{3, 5, 7}	2	105	**107**	**103**
{3}	{2, 5, 7}	3	70	**73**	**67**
{5}	{2, 3, 7}	5	42	**47**	**37**
{7}	{2, 3, 5}	7	30	**37**	**23**
{2, 3}	{5, 7}	6	35	**41**	**29**
{2, 5}	{3, 7}	10	21	**31**	**11**
{2, 7}	{3, 5}	14	15	**29**	1

Hint - Solution

128. Modular equation ☆☆☆☆

For how many integers n > 1 is $x^{49} \equiv x$ (modulo n) true for all integers x?

Hint - Answer - Solution

129. Abelian group ☆☆☆☆

Let G be a group with the following two properties:

1. (i) For all x, y in G, $(xy)^2 = (yx)^2$,
2. (ii) G has no element of order 2.

Prove that G is abelian.

Hint - Solution

130. Reciprocal polynomial? ☆☆

Let p be a polynomial of degree n with complex coefficients. Is there a value of n such that the equations

- $p(1) = 1/1$,
- $p(2) = 1/2$,

  ...
- $p(n) = 1/n$,
- $p(n + 1) = 1/(n + 1)$,
- $p(n + 2) = 1/(n + 2)$,

can be satisfied simultaneously?

Hint - Answer - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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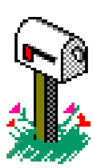


Last updated: January 23, 2006

Nick's Mathematical Puzzles: 131 to 140

131. Horse race ☆☆☆

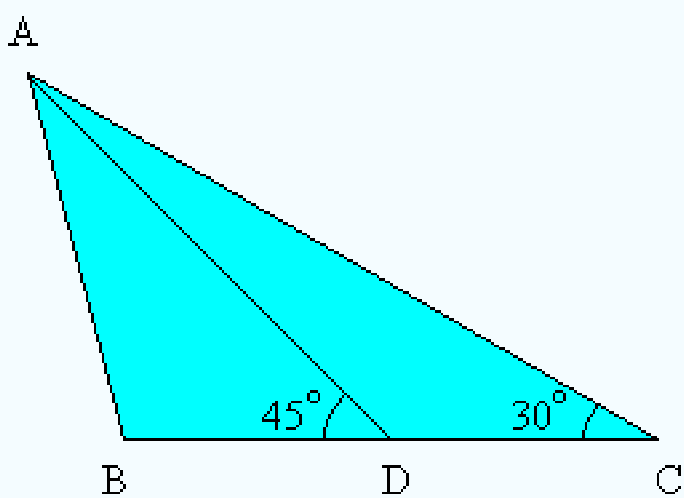
In how many ways, counting ties, can eight horses cross the finishing line?

(For example, two horses, A and B, can finish in three ways: A wins, B wins, A and B tie.)

Hint - Answer - Solution

132. Triangular angle ☆

In $\triangle$ ABC, draw AD, where D is the midpoint of BC.



If $\angle$ ACB = 30° and $\angle$ ADB = 45°, find $\angle$ ABC.

Hint - Answer - Solution

133. Prime sequence? ☆☆

A sequence of integers is defined by

- $a_0 = p$, where $p > 0$ is a prime number,
- $a_{n+1} = 2a_n + 1$, for $n = 0, 1, 2, \ldots$.

Is there a value of p such that the sequence consists entirely of prime numbers?

Hint - Answer - Solution

134. Sum of reciprocal roots ☆☆☆

If the equation $x^4 - x^3 + x + 1 = 0$ has roots a, b, c, d, show that $1/a + 1/b$ is a root of $x^6 + 3x^5 + 3x^4 + x^3 - 5x^2 - 5x - 2 = 0$.

Hint - Solution

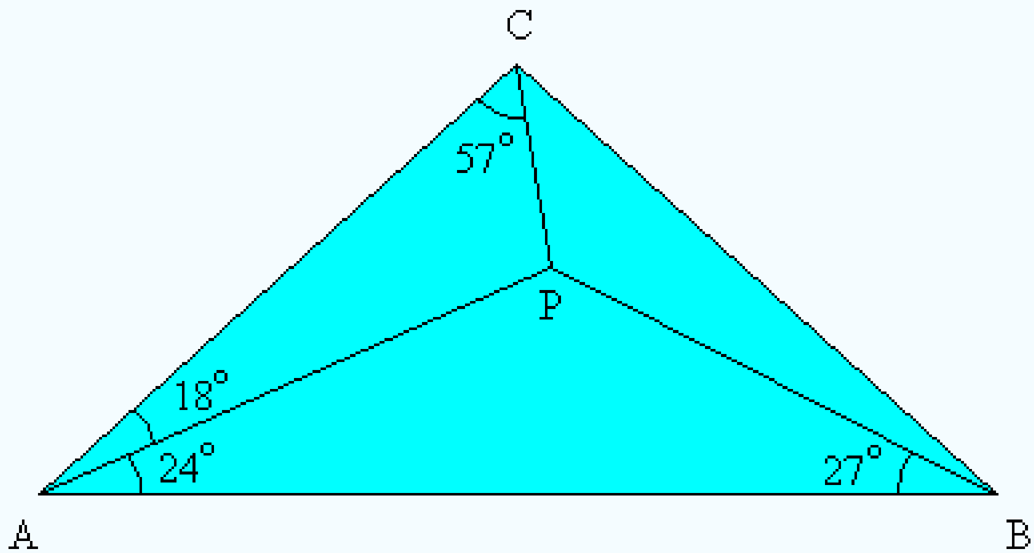
135. Clock hands ☆☆

The minute hand of a clock is twice as long as the hour hand. At what time, between 00:00 and when the hands are next aligned (just after 01:05), is the distance between the tips of the hands increasing at its greatest rate?

Hint - Answer - Solution

136. Point in a triangle ☆☆☆

Point P lies inside $\triangle$ ABC, and is such that $\angle$ PAC = 18°, $\angle$ PCA = 57°, $\angle$ PAB = 24°, and $\angle$ PBA = 27°.



Show that △ ABC is isosceles.

Hint - Solution

137. Factorial plus one equals prime power? ★★★★

Observe that

- $(2-1)! + 1 = 2^1$,
- $(3-1)! + 1 = 3^1$,
- $(5-1)! + 1 = 5^2$.

Are there any other primes p such that $(p-1)! + 1$ is a power of p?

Hint 1 - Hint 2 - Answer - Solution

138. Integer sum of roots ★★

Find all positive real numbers x such that both $\sqrt{x} + 1/\sqrt{x}$ and $\sqrt[3]{x} + 1/\sqrt[3]{x}$ are integers.

Hint - Answer - Solution

139. Three towns ★★

The towns of Alpha, Beta, and Gamma are equidistant from each other. If a car is three miles from Alpha and four miles from Beta, what is the maximum possible distance of the car from Gamma? Assume the land is flat.

Hint - Answer - Solution

140. Six towns ★★★★

The smallest distance between any two of six towns is m miles. The largest distance between any two of the towns is M miles. Show that $M/m \geq \sqrt{3}$. Assume the land is flat.

Hint - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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Nick's Mathematical Puzzles: 141 to 150

141. Alternating series ★★★★

Consider the alternating series $f(x) = x - x^2 + x^4 - x^8 + \ldots + (-1)^n x^{(2^n)} + \ldots$, which converges for $|x| < 1$. Does the limit of f(x) as x approaches 1 from below exist, and if so what is it?

Hint - Answer - Solution

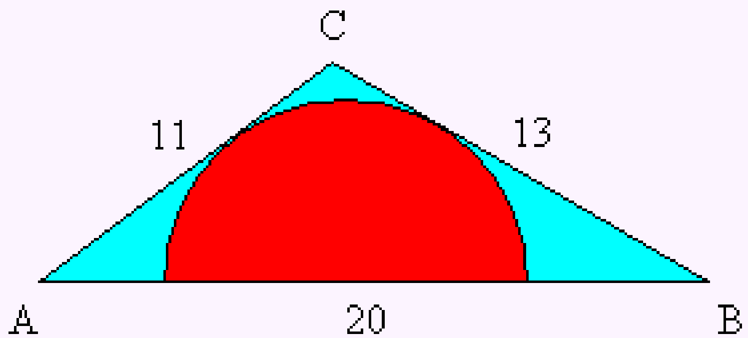
142. Sum of two nth powers ★★

Let a, b, n, and m be positive integers, with n > 1. Show that $a^n + b^n = 2^m \Rightarrow a = b$.

Hint - Solution

143. Semicircle in a triangle ★★

In $\triangle$ ABC, side AB = 20, AC = 11, and BC = 13. Find the diameter of the semicircle inscribed in ABC, whose diameter lies on AB, and that is tangent to AC and BC.



Hint - Answer - Solution

144. Difference of two nth powers ☆☆☆

Let a, b, and n be positive integers, with $a \neq b$. Show that n divides $a^n - b^n$ $\Rightarrow$ n divides $(a^n - b^n)/(a - b)$.

Hint - Solution

145. Heads and tails ☆☆☆

A fair coin is tossed n times and the outcome of each toss is recorded. Find the probability that in the resulting sequence of tosses a head immediately follows a head exactly h times and a tail immediately follows a tail exactly t times. (For example, for the sequence HHHTTHTHH, we have n = 9, h = 3, and t = 1.)

Hint - Answer - Solution

146. Odds and evens ☆☆☆

A and B play a game in which they alternate calling out positive integers less than or equal to n, according to the following rules:

- A goes first and always calls out an odd number.
- B always calls out an even number.
- Each player must call out a number which is greater than the previous number. (Except for A's first turn.)
- The game ends when one player cannot call out a number.

Some example games (for n = 8):

- 1, 8
- 3, 4, 5, 8
- 1, 2, 3, 4, 5, 6, 7, 8

The *length* of a game is defined as the number of numbers called out. For example, the game 1, 8, above, has length 2.

1. How many different possible games are there?
2. How many different possible games of length k are there?

Hint - Answer - Solution

147. Prime or composite 2 ☆☆

Is the number $\frac{2^{58}+1}{5}$ prime or composite?

Hint - Answer - Solution

148. Power series ☆☆

Find the power series (expanded about x = 0) for $\sqrt{\frac{1+x}{1-x}}$.

Hint - Answer - Solution

149. Ones and nines ☆☆☆

Show that all the divisors of any number of the form 19...9 (with an odd number of nines) end in 1 or 9. For example, the numbers 19, 1999, 199999, and 19999999 are prime (so clearly the property holds), and the (positive) divisors of 1999999999 are 1, 31, 64516129 and 1999999999 itself.

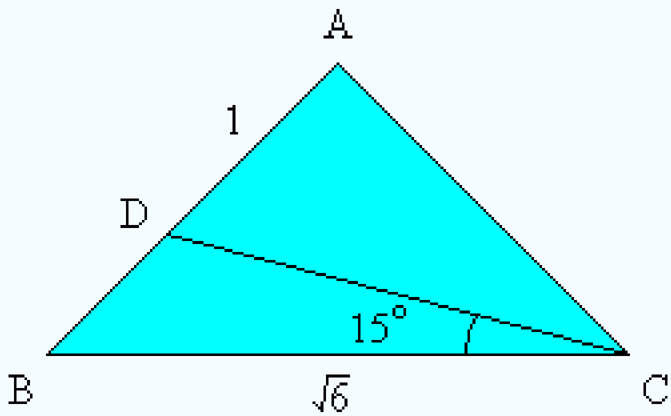
Show further that this property continues to hold if we insert an equal number of zeroes before the nines. For example, the numbers 109, 1000999, 10000099999, 100000009999999, and 1000000000999999999 are prime, and the (positive) divisors of 10000000000099999999999 are 1, 19, 62655709, 1190458471, 8400125030569, 159602375580811, 526315789478947368421, and 10000000000099999999999 itself.

(Dario Alpern's Java applet Factorization using the Elliptic Curve Method may be useful in obtaining divisors of large numbers.)

Hint - Solution

150. Isosceles apex angle ★★★

Triangle ABC is isosceles with AB = AC. Point D on AB is such that ∠BCD = 15° and BC = $\sqrt{6}$AD. Find, with proof, the measure of ∠CAB.



Hint - Answer - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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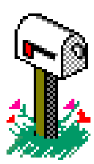

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Nick's Mathematical Puzzles: 151 to 160

151. Painted cubes ★★

Twenty-seven identical white cubes are assembled into a single cube, the outside of which is painted black. The cube is then disassembled and the smaller cubes thoroughly shuffled in a bag. A blindfolded man (who cannot feel the paint) reassembles the pieces into a cube. What is the probability that the outside of this cube is completely black?

Hint - Answer - Solution

152. Totient valence ★★★★

Euler's totient function $\phi(n)$ is defined as the number of positive integers not exceeding n that are relatively prime to n, where 1 is counted as being relatively prime to all numbers. So, for example, $\phi(20) = 8$ because the eight integers 1, 3, 7, 9, 11, 13, 17, and 19 are relatively prime to 20. The table below shows values of $\phi(n)$ for $n \le 20$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
ϕ(n)	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8	16	6	18	8

Euler's totient valence function v(n) is defined as the number of positive integers k such that $\phi(k) = n$. For instance, v(8) = 5 because only the five integers k = 15, 16, 20, 24, and 30 are such that $\phi(k) = 8$. The table below shows values of v(n) for $n \le 16$. (For n not in the table, v(n) = 0.)

n	**v(n)**	**k such that ϕ(k) = n**
1	2	1, 2
2	3	3, 4, 6
4	4	5, 8, 10, 12
6	4	7, 9, 14, 18
8	5	15, 16, 20, 24, 30
10	2	11, 22
12	6	13, 21, 26, 28, 36, 42
16	6	17, 32, 34, 40, 48, 60

Evaluate $v(2^{1000})$.

Hint 1 - Hint 2 - Answer - Solution

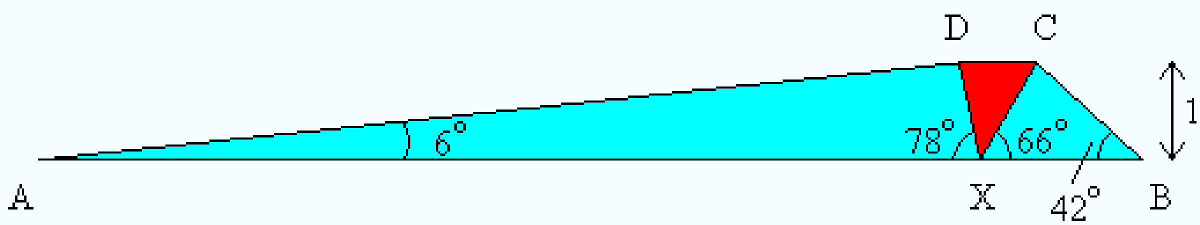
153. Semicircle in a square ☆

Find the area of the largest semicircle that can be inscribed in the unit square.

Hint - Answer - Solution

154. Triangle in a trapezoid ☆☆☆

In trapezoid[1] ABCD, with sides AB and CD parallel, ∠DAB = 6° and ∠ABC = 42°. Point X on side AB is such that ∠AXD = 78° and ∠CXB = 66°. If AB and CD are 1 inch apart, prove that AD + DX − (BC + CX) = 8 inches.



(1) A *trapezoid* is a quadrilateral with at least one pair of parallel sides. In some countries, such a quadrilateral is known as a *trapezium*.

Hint - Solution

155. Sum of two powers is a square? ☆☆

Is $2^n + 3^n$ (where n is an integer) ever the square of a rational number?

Hint - Answer - Solution

156. Three simultaneous equations ☆☆☆

Find all positive real solutions of the simultaneous equations:

- $x + y^2 + z^3 = 3$
- $y + z^2 + x^3 = 3$
- $z + x^2 + y^3 = 3$

Hint - Answer - Solution

157. Trigonometric product

Compute the infinite product

$$[\sin(x)\cos(x/2)]^{1/2} \cdot [\sin(x/2)\cos(x/4)]^{1/4} \cdot [\sin(x/4)\cos(x/8)]^{1/8} \cdot \ldots,$$

where $0 \leq x \leq 2\pi$.

Hint - Answer - Solution

158. Fermat squares

By Fermat's Little Theorem, the number $x = (2^{p-1} - 1)/p$ is always an integer if p is an odd prime. For what values of p is x a perfect square?

Hint - Answer - Solution

159. Eight odd squares

Lagrange's Four-Square Theorem states that every positive integer can be written as the sum of at most four squares. For example, $6 = 2^2 + 1^2 + 1^2$ is the sum of three squares. Given this theorem, prove that any positive multiple of 8 can be written as the sum of eight odd squares.

Hint - Solution

Back to top

Puzzles: 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home About Accessibility

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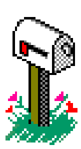

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