Dynamic Programming Squared Model of Economy (DRAFT)

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1. INTRODUCTION

This paper is theoretical and computation examination of stochastic economic behavior of principal and identical agents in a one-good, pure exchange economy. Each agent hold shares of assets. An asset(eg. land) empowers the agent to claim the output(eg. rent) generated by it. The output of an asset is affected by investment decision of agents and the principle. For example, localities with better public infrastructure (principal investments) and well furnished houses (agents investments) leads to higher rents generation. However, agents will only cooperate with the principal if they are atleast as well off as they are without the principal involvement. That is, the principal(government) formulates a contract consisting of tax rate, private and public investment decision sequences such that an agent utility level is unchanged with principal involvement but maximizes the welfare of principals. These class of dynamic programming problems are called ***Dynamic Programming Squared Models***.

The motivation for this paper arose from YouTube Talks of Prof. Thomas Sargent on Computational Challenges in Macroeconomics. He highlighted the complex computational aspect of *Dynamic Programming Squared Models*. These types of problems arose from interaction of government (principal) policies with the actions of the private agents. It's basic structure is described as follows:

$$W\big[v(x), x\big] = \max_{d, v(x')} \left\{ R(x,d) + \beta \int W\big[v(x'), x'\big] dF(x'|x) \right\} \tag{1.1}$$

where the maximization is subject to

$$v(x) = \max_{c} \left\{ u(x,c) + \beta \int v(x') dF(x'|x,c) \right\} \tag{1.2}$$

The interior bellman equation $v(x)$ captures the incentive effects of people whose incentives are affect by government polices. The outer bellman equation $W\left[v(x), x\right]$ puts structure on government incentives subject to agent value maximization constraint.

The economy[1]is informally described in the Section 2. In Section 3, the equilibrium is defined in the stochastic environment given the incentive compatibility constraints of agents. In Section 4, the functional equations of governments and agents are derived and studied.

2. DESCRIPTION OF THE ECONOMY

2.1. Agents

There are $i = 1, ..., l$ number of productive assets in the economy. They are in fixed supply such as different types of land plots. The assets produce a random quantities of single consumption good in each period depending upon the choices of public investment, x_t, and private investment k_t in it; we can call it as rents. Thus an asset is a claim to stochastic rents stream given the sequences $\{x_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=0}^{\infty}$. The assets has been normalized such that each consumer holds one unit of each asset. There are large number of consumers (landlords) with identical preferences and with equal endowment of all assets. There is no storage of consumption good.

In each period, there are markets for exchange of consumption goods and shares of assets. Due to identical nature of all agent, competitive equilibrium is trivial, in which in each period each agents hold one unit of each asset ,and consumes and invests all the rents (consumption goods) produced

[1]The structure of this economy closely follows the structure of Lucas Asset Prices Model in an exchange economy and [] One-Sector Model of Optimal Growth.

by these assets. Thus given the competitive markets exist but there is no trade.**?**

The preferences of a representative agent over the random consumption sequences are

$$\mathbb{E}_0 \left\{ \sum_{0}^{\infty} \beta^t U(c_t) \right\} \tag{2.1}$$

where c_t follows stochastic process, depending on investment decisions of the consumers (agents) and government (principal), represent consumption of a single good, β is a discount factor, U is current period utility function, and $\mathbb{E}\{.\}$ is expectation operator. $U : \mathbf{R}_+ \to \mathbf{R}$ is bounded, continously differentiable, strictly increasing, and stricly concave, with $U(0) = 0$ and where $\beta \in (0, 1)$.

Let Y be compact subset of $\mathbf{R}^l_+$, with its Borel subsets $\mathscr{Y}$. Rents in any period are denoted by the vector $y = (y_1, ..., y_l) \in \boldsymbol{Y}$, here y_i denotes the rent by asset i. Similary, let K be compact subset of $\mathbf{R}^l_+$, with its Borel subsets $\mathscr{K}$. Private investment decisions in any period are denoted by the vector $k = (k_1, ..., k_l) \in \boldsymbol{K}$, here k_i denotes the rent by asset i. We will assume that the rents endowments are Markovian, following the exogenous process

$$y^i_{t+1} = G^i(y^i_t, x_t, k^i_t, z_t) \tag{2.2}$$

where G^i is a bounded continuous function and $\{z_t\}$ is an iid shock sequence with known distribution ϕ, x_t denotes the public investment decision at time t, and $x_t, k_t \geq 0$ for all t.

Thus at time period t, given the government contract $G(\tau, x_t)$ and the state of the economy y_t, the agent chooses c_t, k_{t+1} so as to maximize the present discounted expected utility eq(2.1) subject to constraints:

$$c_t + k_t * \mathbf{1} \leq (1 - \tau) * y_t * \mathbf{1}, \text{ all } z^t, \text{ all } t \tag{2.3}$$

2.2. Government

Let the $\{g_t\}_{t=0}^{\infty}$ denotes the consumption sequence of the government (principal). The government problem can also be model as stochastic analogue of one-sector model of optimal growth as follows

$$\sup \, \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \gamma^t W(g_t) \right\} \tag{2.4}$$

such that

$$g_t + x_t \leq \tau * y_t \tag{2.5}$$

where $G = (G^1, ..., G^l)$, G^is are defined in eq(2.2), g_t follows a stochastic process, representing consumption of a single good by principal, γ is a discount factor, W is a current period utility function, $\mathbb{E}_0$ is the expectations operator which indicates the expected value with respect to the probability distribution of the random variables g_t, x_t, z_t over all t based on information available in period $t = 0$. $W : \mathbf{R}_+ \to \mathbf{R}$ is bounded, continuously differentiable, strictly increasing, and strictly concave, with $W(0) = 0$ and where $\gamma \in (0, 1)$.

3. DEFINITION OF EQUILIBRIUM

Let $v^*(y)$ be the value of the objective(2.1) for an agent who begins in state y with current period shock as z and does not participate in government contract and follows an optimum consumption investment policy thereafter.

$$v(y_t) = \max_{c_t,k_t} \left\{ U(c_t) + \beta \int v\big[G_t(y_t, x_t = 0, k_t, z_t)\big] Q(z_{t-1}, dz_t) \right\} \tag{3.1a}$$

$$\text{subject to } c_t + k_t \leq y_t = G_{t-1}(y_{t-1}, x_{t-1}, z_{t-1}) \; ; c_t \geq 0 \; ; k_t \geq 0 \text{ for all } t \tag{3.1b}$$

Let $v^*(y_t)$ satisfy eq(3.1). However eq(3.1) can be simplified by changing the state variable to $y_t = y$ as follows

$$v(y) = \max_{k} \left\{ U(y-k) + \beta \int v(G(y, x=0, k, z)) Q(z', dz) \right\} \tag{3.2a}$$

$$\text{where } y = y_t \; k = k_t \; ; x = x_t \; ; z = z_t \; ; z' = z_{t-1} \; ; y - k \geq 0 \; ; y_{t+1} = G(y, x, k, z) \tag{3.2b}$$

Eq(3.2) can also be written in expectation form, as it will be useful later on to solve the

$$v(y_t) = \max_{c_t,k_t} \mathbb{E}_t \left\{ U(c_t) + \beta v \Big[G(y_t, x_t = 0, k_t, z_t) \Big] \right\} \tag{3.3a}$$

$$v(y_t) = \max_{c_t,k_t} \left\{ U(c_t) + \beta \mathbb{E}_t \big[v(y_{t+1}) \big] \right\} \tag{3.3b}$$

Thus given the state y, principal has to formulate a contract in such a way that the agent is willing to participate. This implies that agent's welfare is atleast $v^*(y)$ from the contract $(\tau, \{x_t\}_{t=0}^{\infty})$ for each $y \in Y$.

DEFINITION: An equilibrium is set of continuous functions $w(y, v^*(y)) : Y \times \mathbf{R}_+ \to \mathbf{R}_+$, $v(y) : Y \to \mathbf{R}_+$ and $v^*(y) : Y \to \mathbf{R}_+$ such that:

$$w(y, v^*(y)) = \max_{x,k} \left\{ W(g) + \gamma \int w(y', v^*(y')) Q(y, dy') \right\} \tag{3.4a}$$

$$\text{subject to } g + x \leq \tau y \; ; y' = G(y, x, k, z) \; ; g \geq 0 \; ; x \geq 0 \tag{3.4b}$$

$$v(y) = \left\{ U[(1-\tau)y - k] + \beta \int v(y') Q(y, dy') \right\} \geq v^*(y) \tag{3.4c}$$

where $v^*(y)$ defines the agent utility without any principal role, as defined in eq(3.2). Equation(3.4a) say that given the state y and $v^*(y)$, the principal allocates resources τy optimally in current consumption g and end-of-period investment x'. Equation(3.4b) states the budget constraint for the government. Equation(3.4c) say that *principal allocates agent's resources* $(1-\tau)y$ *in agent current consumption* c *and agent end-of-period investment* k' *such that it is as well off as it is without any principal role.* Thus equation set(3.4) can stated as problem of dynamic programming squared model.

4. MATHEMATICAL SOLUTION

To study the properties of solution to dynamic programming squared model, let us consider following classes of dynamic programming squared problems.

4.1. Deterministic Dynamic Programming Squared Model

4.1.1. With No Government Contract

To develop some intuition and checking the feasibility of this problems. Lets first consider a finite horizon version of this problem. Lets assume that both the agent and the principal dies in a terminal period T.
The agent utility under a capital accumulation sequence $k = (k_0, k_1, ..., k_T)$ can be written as

$$U(y_0, k) = \sum_{t=0}^{T} \beta^t * h_t(y_t, k_t) + W_{T+1}(y_{T+1}) \tag{4.1}$$

where y_0 is a given initial state, $k_t \in \Gamma_t(y_t)$, y_t evolves according to transition function G_t such that $y_{t+1} = G_t(y_t, x_t, k_t)$ and W_{T+1} is the terminal value.

The Lagrangian for the above optimization problem could be written as

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t h_t(y_t, k_t) + W_{T+1}(y_{T+1}) + \sum_{t=0}^{T} \lambda_t \left[G_t(y_t, x_t, k_t) - y_{t+1} \right] \tag{4.2}$$

The first order conditions for the above optimization problem could be written as

$$\frac{\partial \mathcal{L}}{\partial k_t} = \beta^t \frac{\partial h_t}{\partial k_t} + \lambda_t * \frac{\partial y_{t+1}}{\partial k_t} = 0 \tag{4.3a}$$
$$\frac{\partial \mathcal{L}}{\partial y_t} = \beta^t \frac{\partial h_t}{\partial y_t} + \lambda_t \frac{\partial y_{t+1}}{\partial y_t} - \lambda_{t-1} = 0 \tag{4.3b}$$
$$\frac{\partial \mathcal{L}}{\partial y_{T+1}} = W'_{T+1}(y_{T+1}) - \lambda_T = 0 \tag{4.3c}$$

The above system of equations(4.3) can be further solved for λ_t and λ_{t-1} to obtain an Euler Equation as follows

$$\lambda_t = -\beta^t * \frac{\frac{\partial h_t}{\partial k_t}}{\frac{\partial y_{t+1}}{\partial k_t}} \; ; \lambda_{t-1} = \beta^t \frac{\partial h_t}{\partial y_t} - \beta^t \frac{\frac{\partial h_t}{\partial k_t}}{\frac{\partial y_{t+1}}{\partial k_t}} * \frac{\partial y_{t+1}}{\partial y_t} \tag{4.4a}$$

$$\lambda_t = \beta^{t+1} \frac{\partial h_{t+1}}{\partial y_{t+1}} - \beta^{t+1} \frac{\frac{\partial h_{t+1}}{\partial k_{t+1}}}{\frac{\partial y_{t+2}}{\partial k_{t+1}}} * \frac{\partial y_{t+2}}{\partial y_{t+1}} \tag{4.4b}$$

$$\beta^t \frac{\partial h_t}{\partial k_t} + \left[\beta^{t+1} \frac{\partial h_{t+1}}{\partial y_{t+1}} - \beta^{t+1} \frac{\frac{\partial h_{t+1}}{\partial k_{t+1}}}{\frac{\partial y_{t+2}}{\partial k_{t+1}}} * \frac{\partial y_{t+2}}{\partial y_{t+1}} \right] * \frac{\partial y_{t+1}}{\partial k_t} = 0 \tag{4.4c}$$

$$\frac{\partial h_t}{\partial k_t} = \beta \left[-\frac{\partial h_{t+1}}{\partial y_{t+1}} + \frac{\frac{\partial h_{t+1}}{\partial k_{t+1}}}{\frac{\partial y_{t+2}}{\partial k_{t+1}}} * \frac{\partial y_{t+2}}{\partial y_{t+1}} \right] * \frac{\partial y_{t+1}}{\partial k_t} \tag{4.4d}$$

Thus, the equation(4.4d) could be used to relate k_t and k_{t+1}. Now under the assumption that the agent dies at time T thus $W_{T+1} = 0$ for all y_{T+1}. At time period T, the agent solves $h_T(y_T, k_T)$ such that $k_T \in \Gamma_T(y_T)$, as $\frac{\partial h_t}{\partial k_t} < 0$ thus agent chooses $k_T = 0$. The equation(4.4d) can also be used to do policy iteration using Coleman Operator as shown in .

4.1.2. With Government Contract

The principal design a contract $\mathcal{C} = (k, x, \tau, v^*)$ consisting of public investment sequence $x = (x_0, x_1, ...x_T)$, private investment sequence $k = (k_0, k_1, k_2, ..k_T)$, tax rate τ and utility value function $v^*(y_t)$ such that the agent under acceptance of the contract is not worse off that $v(y_t) \geq v^*(y_t)$ where $v(y_t)$ refers to the future discounted present value of the utility from the consumption streams under the contract given the current state y_t and $v^*(y_t)$ refers to value of same without the contract.
Thus government optimization problem in terms of value function could be written as:

$$w_t(y_t) = \max_{x_t, k_t} \{f_t(y_t, x_t) + \beta w_{t+1}(y_{t+1})\} \tag{4.5a}$$

$$\text{such that } v(y_t) = h(y_t, k_t) + \beta * v_{t+1}(y_{t+1}) \geq v_t^*(y_t) \tag{4.5b}$$

$$y_{t+1} = G_t(y_t, x_t, k_t) \tag{4.5c}$$

It can be argued that equation(4.5b) holds with equality because if $v_t(y_t) > v_t^*(y_t)$ then the government can simply increase k_t so as to decrease $v_t(y_t)$ as $\frac{\partial h_t}{\partial k_t} < 0$ to bring in equality. This marginal increase in k_t increases the utility of the government as $\frac{\partial y_{t+1}}{\partial k_t} > 0$.

The Lagrangian for the above optimization can be written as

$$\mathcal{L} = w_t(y_t) = f_t(y_t, x_t) + \beta w_{t+1}(y_{t+1}) + \lambda_t \left[h_t(y_t, k_t) + \beta * v_{t+1}(y_{t+1}) - v_t^*(y_t)\right] \tag{4.6}$$

The first order conditions for the above optimization could be written as

$$\frac{\partial \mathcal{L}}{\partial x_t} = \frac{\partial f_t}{\partial k_t} + \beta * \frac{\partial w_{t+1}}{\partial y_{t+1}} * \frac{\partial y_{t+1}}{\partial x_t} + \lambda_t * \beta \frac{\partial v_{t+1}}{\partial y_{t+1}} * \frac{\partial y_{t+1}}{\partial x_t} = 0 \tag{4.7a}$$

$$\frac{\partial \mathcal{L}}{\partial k_t} = \beta * \frac{\partial w_{t+1}}{\partial y_{t+1}} * \frac{\partial y_{t+1}}{\partial k_t} + \lambda_t \left[\frac{\partial h_t}{\partial k_t} + \beta * \frac{\partial v_{t+1}}{\partial y_{t+1}} * \frac{\partial y_{t+1}}{\partial k_t} \right] = 0 \tag{4.7b}$$

The equations set(4.7) can be used to solve for λ_t and $\dfrac{\partial w_{t+1}}{\partial y_{t+1}}$ as follows

$$\beta \frac{\partial w_{t+1}}{\partial y_{t+1}} = \frac{-\lambda_t}{\frac{\partial y_{t+1}}{\partial k_t}} \left[\frac{\partial h_t}{\partial k_t} + \beta * \frac{\partial v_{t+1}}{\partial y_{t+1}} * \frac{\partial y_{t+1}}{\partial k_t} \right] \tag{4.8a}$$

$$\frac{\partial f_t}{\partial k_t} - \lambda_t \frac{\frac{\partial y_{t+1}}{\partial x_t}}{\frac{\partial y_{t+1}}{\partial k_t}} \left[\frac{\partial h_t}{\partial k_t} + \beta * \frac{\partial v_{t+1}}{\partial y_{t+1}} * \frac{\partial y_{t+1}}{\partial k_t} \right] + \lambda_t * \beta \frac{\partial v_{t+1}}{\partial y_{t+1}} * \frac{\partial y_{t+1}}{\partial x_t} = 0 \tag{4.8b}$$

$$\lambda_t = \frac{\frac{\partial f_t}{\partial x_t} * \frac{\partial y_{t+1}}{\partial k_t}}{\frac{\partial y_{t+1}}{\partial x_t} * \frac{\partial h_t}{\partial k_t}} ; \beta \frac{\partial w_{t+1}}{\partial y_{t+1}} = \frac{-\frac{\partial f_t}{\partial x_t}}{\frac{\partial y_{t+1}}{\partial x_t} * \frac{\partial h_t}{\partial k_t}} \left[\frac{\partial h_t}{\partial k_t} + \beta \frac{\partial v_{t+1}}{\partial y_{t+1}} * \frac{\partial y_{t+1}}{\partial k_t} \right] \tag{4.8c}$$

The Lagrangian equation(4.6) can be totally differentiated with respect to y to obtain

$$\begin{aligned} w_t'(y_t) = & \frac{\partial f_t}{\partial y_t} + \frac{\partial f_t}{\partial x_t} x_t' + \beta w_{t+1}'(y_{t+1}) \frac{\partial y_{t+1}}{\partial y_t} + \beta w_{t+1}'(y_{t+1}) \frac{\partial y_{t+1}}{\partial x_t} x_t' \\ & + \beta w_{t+1}'(y_{t+1}) \frac{\partial y_{t+1}}{\partial k_t} k_t' + \lambda_t' \left[h(y_t, k_t) + \beta * v_{t+1}(y_{t+1}) - v_t^*(y_t) \right] \\ & \lambda_t \left[\frac{\partial h_t}{\partial y_t} + \frac{\partial h_t}{\partial k_t} k_t' - v'^*(y_t) + \beta v_{t+1}'(y_{t+1}) \frac{\partial y_{t+1}}{\partial y_t} + \beta v_{t+1}'(y_{t+1}) \frac{\partial y_{t+1}}{\partial x_t} x_t' + \beta v_{t+1}'(y_{t+1}) \frac{\partial y_{t+1}}{\partial k_t} k_t' \right] \end{aligned}$$

The above equation can be simplified by first collecting the terms with $x't$ and k_t' and then using first order conditions given in equation set(4.7) to set the coefficients of $x't$ and k_t' to 0. As at the optimum reserve utility constraint holds with equality, thus the coefficient of λ_t' becomes 0. The equation can be derived as

$$w_t'(y_t) = \frac{\partial f_t}{\partial y_t} + \beta w_{t+1}'(y_{t+1}) \frac{\partial y_{t+1}}{\partial y_t} + \lambda_t \left[\frac{\partial h_t}{\partial y_t} + \beta v_{t+1}'(y_{t+1}) \frac{\partial y_{t+1}}{\partial y_t} - v'^*(y_t) \right] \tag{4.9}$$

The above equation can be simplified by substituting for λ_t and $\beta \frac{\partial w_{t+1}}{\partial y_{t+1}}$ from equation(4.8c) as follows

$$w_t'(y_t) = \frac{\partial f_t}{\partial y_t} - \frac{\frac{\partial f_t}{\partial x_t} * \frac{\partial y_{t+1}}{\partial y_t}}{\frac{\partial y_{t+1}}{\partial x_t} * \frac{\partial h_t}{\partial k_t}} \left[\frac{\partial h_t}{\partial k_t} + \beta \frac{\partial v_{t+1}}{\partial y_{t+1}} * \frac{\partial y_{t+1}}{\partial k_t} \right] + \frac{\frac{\partial f_t}{\partial x_t} * \frac{\partial y_{t+1}}{\partial k_t}}{\frac{\partial y_{t+1}}{\partial x_t} * \frac{\partial h_t}{\partial k_t}} \left[\frac{\partial h_t}{\partial y_t} + \beta v_{t+1}'(y_{t+1}) \frac{\partial y_{t+1}}{\partial y_t} - v'^*(y_t) \right]$$

$$w_t'(y_t) = \frac{\partial w_t}{\partial y_t} = \frac{\partial f_t}{\partial y_t} - \frac{\frac{\partial f_t}{\partial x_t} * \frac{\partial y_{t+1}}{\partial y_t}}{\frac{\partial y_{t+1}}{\partial x_t}} + \frac{\frac{\partial f_t}{\partial x_t} * \frac{\partial y_{t+1}}{\partial k_t}}{\frac{\partial y_{t+1}}{\partial x_t} * \frac{\partial h_t}{\partial k_t}} \left[\frac{\partial h_t}{\partial y_t} - v'^*(y_t) \right] \tag{4.10}$$

The expression for $\frac{\partial v_{t+1}}{\partial y_{t+1}}$ can be determined as follows

$$h(y_t, k_t) + \beta * v_{t+1}(y_{t+1}) = v_t^*(y_t) \tag{4.11a}$$

$$\frac{\partial h_t(h_t, k_t)}{\partial y_t} + \beta \frac{\partial v(y_{t+1})}{\partial y_{t+1}} * \frac{\partial y_{t+1}}{\partial y_t} = \frac{\partial v_t^*(y_t)}{\partial y_t} \tag{4.11b}$$

$$\beta \frac{\partial v(y_{t+1})}{\partial y_{t+1}} = \frac{1}{\frac{\partial y_{t+1}}{\partial y_t}} \left[\frac{\partial v_t^*(y_t)}{\partial y_t} - \frac{\partial h_t(y_t, k_t)}{\partial y_t} \right] \tag{4.11c}$$

The value function without the government contract $v_t^*(y_t)$ can be written as $v_t^*(y_t) = h(y_t, k_t) + \beta v_{t+1}^*(y_{t+1})$ where y_t is state variable and k_t is control variable. The first order condition for k_t is

$\frac{\partial h_t}{\partial k_t} + \beta\frac{\partial v_{t+1}}{\partial y_{t+1}}\frac{\partial y_{t+1}}{\partial k_t} = 0$. Taking the derivative of the value function v^* at optimum with respect to state variable yields $\frac{\partial v(y_t)}{\partial y_t} = \frac{\partial h_t}{\partial y_t} + \beta * \frac{\partial v_{t+1}}{\partial y_{t+1}} * \frac{\partial y_{t+1}}{\partial y_t}$. Thus the expression for $\frac{\partial v_t^*}{\partial y_t}$ can be written as

$$\frac{\partial v_t^*}{\partial y_t} = \frac{\partial h_t}{\partial y_t} - \frac{\partial h_t}{\partial k_t} * \frac{\frac{\partial y_{t+1}}{\partial y_t}}{\frac{\partial y_{t+1}}{\partial k_t}} \tag{4.12}$$

Finally the equation system(4.7) can be solved by substituting for $\frac{\partial w_{t+1}}{\partial y_{t+1}}$ from equation(4.10), $\beta\frac{\partial v_{t+1}}{\partial y_{t+1}}$ from equation(4.11c) and finally for $\frac{\partial v_t^*}{\partial y_t}$ from equation(4.12).

4.2. Stochastic Dynamic Programming Squared Model

The mathematical structure for solution to stochastic dynamic programming squared model is almost similar to deterministic case. The equation set(3.4) can be written as

$$w(y_t) = \max_{k_t, x_t} \left\{ f(y_t, x_t) + \beta \int w\big[G(y_t, x_t, k_t)z\big]\phi(dz) \right\} \tag{4.13a}$$

$$\text{such that } v(y_t) = h(y_t, k_t) + \beta \int v\big[G(y_t, x_t, k_t) * z\big]\phi(dz) \geq v^*(y_t) \tag{4.13b}$$

$$\text{where } v^*(y_t) = \max_{k_t} \left\{ h(y_t, k_t) + \beta \int v^*\big[G(y_t, x_t = 0, k_t) * z\big]\phi(dz) \right\} \tag{4.13c}$$

The Lagrangian for the optimization could be written as

$$\begin{aligned}\mathcal{L} = w(y_t) = f(y_t, x_t) + \beta \int w\big[G(y_t, x_t, k_t) * z\big]\phi(dz) \\ + \lambda_t \left[h(y_t, k_t) + \beta \int v\big[G(y_t, x_t, k_t) * z\big]\phi(dz) - v^*(y_t) \right]\end{aligned}$$

The first order conditions can be derived as follows

$$x \ : \frac{\partial f}{\partial x_t} + \beta\frac{\partial G}{\partial x_t}\int \frac{\partial w(y_{t+1})}{\partial y_{t+1}} z\phi(dz) + \lambda_t\beta\frac{\partial G}{\partial x_t}\int \frac{\partial v(y_{t+1})}{\partial y_{t+1}} z\phi(dz) = 0 \tag{4.14a}$$

$$k \ : \beta\frac{\partial G}{\partial k_t}\int \frac{\partial w(y_{t+1})}{\partial y_{t+1}} z\phi(dz) + \lambda_t \left[\frac{\partial h}{\partial k_t} + \beta\frac{\partial G}{\partial k_t}\int \frac{\partial v(y_{t+1})}{\partial y_{t+1}} z\phi(dz) \right] = 0 \tag{4.14b}$$

Thus the equation set(4.14) resembles closely the equation set (4.7), thus following the solution steps in above equation, following solution is obtained for λ_t and $\frac{\partial w(y_{t+1})}{\partial y_{t+1}}$

$$\lambda_t = \frac{\frac{\partial f}{\partial x_t} * \frac{\partial G}{\partial k_t}}{\frac{\partial G}{\partial x_t} * \frac{\partial h}{\partial k_t}} \tag{4.15a}$$

$$\frac{\partial w(y_{t+1})}{\partial y_{t+1}} = \frac{\partial f}{\partial y_{t+1}} - \frac{\frac{\partial f}{\partial x_{t+1}} * \frac{\partial G}{\partial y_{t+1}}}{\frac{\partial G}{\partial x_{t+1}}} + \frac{\frac{\partial f}{\partial x_{t+1}} * \frac{\partial G}{\partial k_{t+1}}}{\frac{\partial G}{\partial x_{t+1}} * \frac{\partial h}{\partial k_{t+1}}} \left[\frac{\partial h}{\partial y_{t+1}} - \frac{\partial v^*}{\partial y_{t+1}} \right] \tag{4.15b}$$

The solution for $\int \frac{\partial v(y_{t+1})}{\partial y_{t+1}} z\phi(dz)$ can be obtained similary as done in equation set(4.11) as follows

$$\int \frac{\partial v(y_{t+1})}{\partial y_{t+1}} z\phi(dz) = \frac{1}{\beta * \dfrac{\partial G}{\partial y_t}} \left[\frac{\partial v^*}{\partial y_t} - \frac{\partial h}{\partial y_t} \right] \tag{4.16}$$

Thus equation set(4.14) can be solved by replacing values of λ_t, $\frac{\partial w(y_{t+1})}{\partial y_{t+1}}$ and $\int \frac{\partial v(y_{t+1})}{\partial y_{t+1}} z\phi(dz)$ from above equations.