Dynamic Programming Squared Model of Economy

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1. INTRODUCTION

This paper is theoretical and computation examination of stochastic economic behavior of principal and identical agents in a one-good, pure exchange economy. Each agent hold shares of assets. An asset(eg. land) empowers the agent to claim the output(eg. rent) generated by it. The output of an asset is affected by investment decision of agents and the principle. For example, localities with better public infrastructure (principal investments) and well furnished houses (agents investments) leads to higher rents generation. However, agents will only cooperate with the principal if they are atleast as well off as they are without the principal involvement. That is, the principal(government) formulates a contract consisting of tax rate, private and public investment decision sequences such that an agent utility level is unchanged with principal involvement but maximizes the welfare of principals. These class of dynamic programming problems are called ***Dynamic Programming Squared Models***.

The motivation for this paper arose from YouTube Talks of Prof. Thomas Sargent on Computational Challenges in Macroeconomics. He highlighted the complex computational aspect of *Dynamic Programming Squared Models*. These types of problems arose from interaction of government (principal) policies with the actions of the private agents. It's basic structure is described as follows:

$$W\big[v(x), x\big] = \max_{d, v(x')} \left\{ R(x, d) + \beta \int W\big[v(x'), x'\big] dF(x'|x) \right\} \tag{1.1}$$

where the maximization is subject to

$$v(x) = \max_{c} \left\{ u(x, c) + \beta \int v(x') dF(x'|x, c) \right\} \tag{1.2}$$

The interior bellman equation $v(x)$ captures the incentive effects of people whose incentives are affect by government polices. The outer bellman equation $W\left[v(x), x\right]$ puts structure on government incentives subject to agent value maximization constraint.

The economy[1]is informally described in the Section 2. In Section 3, the equilibrium is defined in the stochastic environment given the incentive compatibility constraints of agents. In Section 4, the functional equations of governments and agents are derived and studied. Section 5, the computational algorithm to solve dynamic programming squared problems will be studied along with some *'comparative static'* exercises. Section 6 concludes the papers.

2. DESCRIPTION OF THE ECONOMY

2.1. Agents

There are $i = 1, ..., l$ number of productive assets in the economy. They are in fixed supply such as different types of land plots. The assets produce a random quantities of single consumption good in each period depending upon the choices of public investment, x_t, and private investment k_t in it; we can call it as rents. Thus an asset is a claim to stochastic rents stream given the sequences $\{x_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=0}^{\infty}$. The assets has been normalized such that each consumer holds one unit of each asset. There are large number of consumers (landlords) with identical preferences and with equal endowment of all assets. There is no storage of consumption good.

In each period, there are markets for exchange of consumption goods and shares of assets. Due to identical nature of all agent, competitive equilibrium is trivial, in which in each period each agents

[1]The structure of this economy closely follows the structure of Lucas Asset Prices Model in an exchange economy and [] One-Sector Model of Optimal Growth.

hold one unit of each asset ,and consumes and invests all the rents (consumption goods) produced by these assets. Thus given the competitive markets exist but there is no trade.**?**

The preferences of a representative agent over the random consumption sequences are

$$\mathbb{E}_0 \left\{ \sum_{0}^{\infty} \beta^t U(c_t) \right\} \tag{2.1}$$

where c_t follows stochastic process, depending on investment decisions of the consumers (agents) and government (principal), represent consumption of a single good, β is a discount factor, U is current period utility function, and $\mathbb{E}\{.\}$ is expectation operator. $U : \mathbf{R}_+ \to \mathbf{R}$ is bounded, continously differentiable, strictly increasing, and stricly concave, with $U(0) = 0$ and where $\beta \in (0, 1)$.

Let Y be compact subset of $\mathbf{R}^l_+$, with its Borel subsets $\mathscr{Y}$. Rents in any period are denoted by the vector $y = (y_1, ..., y_l) \in \boldsymbol{Y}$, here y_i denotes the rent by asset i. Similary, let K be compact subset of $\mathbf{R}^l_+$, with its Borel subsets $\mathscr{K}$. Private investment decisions in any period are denoted by the vector $k = (k_1, ..., k_l) \in \boldsymbol{K}$, here k_i denotes the rent by asset i. We will assume that the rents endowments are Markovian, following the exogenous process

$$y^i_t = G(y^i_{t-1}, x_t, k^i_t, z_t) \tag{2.2}$$

where G^i is a bounded continuous function and $\{z_t\}$ is an iid shock sequence with known distribution ϕ, x_t denotes the public investment decision at time t, and $x_t, k_t \geq 0$ for all t.

Thus at time period t, given the government contract $G(\tau, x_t)$ and the state of the economy y_t, the agent chooses c_t, k_{t+1} so as to maximize the present discounted expected utility eq(2.1) subject to constraints:

$$c_t + k_{t+1} * \mathbf{1} \leq (1 - \tau) * y_t * \mathbf{1}, \text{ all } z^t, \text{ all } t \tag{2.3}$$

2.2. Government

Let the $\{g_t\}_{t=0}^{\infty}$ denotes the consumption sequence of the government (principal). The government problem can also be model as stochastic analogue of one-sector model of optimal growth as follows

$$\sup\ \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \gamma^t W(g_t) \right\} \tag{2.4}$$

such that

$$g_t + x_{t+1} \leq \tau * y_t = \tau * G(y_{t-1}, x_t, k_t, z_t) \tag{2.5}$$

where $G = (G^1, ..., G^l)$, G^is are defined in eq(2.2), g_t follows a stochastic process, representing consumption of a single good by principal, γ is a discount factor, W is a current period utility function, $\mathbb{E}_0$ is the expectations operator which indicates the expected value with respect to the probability distribution of the random variables g_t, x_{t+1}, z_t over all t based on information available in period $t = 0$. $W : \mathbf{R}_+ \to \mathbf{R}$ is bounded, continuously differentiable, strictly increasing, and strictly concave, with $W(0) = 0$ and where $\gamma \in (0, 1)$.

3. DEFINITION OF EQUILIBRIUM

Let $v^*(y,z)$ be the value of the objective(2.1) for an agent who begins in state y with current period shock as z and does not participate in government contract and follows an optimum consumption investment policy thereafter.

$$v(y_{t-1},z_t) = \max_{c_t,k_{t+1}} \left\{ U(c_t) + \beta \int v(y_t,z_{t+1})Q(z_t,dz_{t+1}) \right\} \tag{3.1a}$$

$$\text{subject to } c_t + k_{t+1} \leq G(y_{t-1},x_t=0,k_t,z_t) = y_t\,, c_t \geq 0\,, k_{t+1} \geq 0 \text{ for all } t \tag{3.1b}$$

Let $v^*(y_{t-1},z_t)$ satisfy eq(3.1). However eq(3.1) can be simplified by changing the state variable to $y_t = y$ as follows

$$v(y) = \max_{k'} \left\{ U(y-k') + \beta \int v(G(y,x'=0,k',z'))Q(z,dz') \right\} \tag{3.2a}$$

$$\text{where } k' = k_{t+1}\,, x' = x_{t+1}\,, z' = z_{t+1}\,, y - k' \geq 0\,, y' = G(y,x',k',z') \tag{3.2b}$$

Eq(3.2) can also be written in expectation form, as it will be useful later on to solve the

$$v(y_t) = \max_{c_t,k_{t+1}} \mathbb{E}_t \left\{ U(c_t) + \beta v \Big[G(y_t,x_{t+1}=0,k_{t+1},z_{t+1}) \Big] \right\} \tag{3.3a}$$

$$v(y_t) = \max_{c_t,k_{t+1}} \left\{ U(c_t) + \beta \mathbb{E}_t \big[v(y_{t+1}) \big] \right\} \tag{3.3b}$$

Thus given the state y, principal has to formulate a contract in such a way that the agent is willing to participate. This implies that agent's welfare is atleast $v^*(y)$ from the contract $(\tau, {x_t}_{t=0}^{\infty})$ for each $y \in Y$.

DEFINITION: An equilibrium is set of continuous functions $w(y,v^*(y)) : Y \times \mathbf{R}_+ \to \mathbf{R}_+$, $v(y) : Y \to \mathbf{R}_+$ and $v^*(y) : Y \to \mathbf{R}_+$ such that:

$$w(y,v^*(y)) = \max_{g,x'} \left\{ W(g) + \gamma \int w(y',v^*(y'))Q(y,dy') \right\} \tag{3.4a}$$

$$\text{subject to } g + x' \leq \tau y\,, y' = G(y,x',k',z')\,, g \geq 0\,, x' \geq 0 \tag{3.4b}$$

$$v(y) = \max_{k'} \left\{ U[(1-\tau)y - k'] + \beta \int v(y')Q(y,dy') \right\} \geq v^*(y) \tag{3.4c}$$

where $v^*(y)$ defines the agent utility without any principal role, as defined in eq(3.2). Equation(3.4a) say that given the state y and $v^*(y)$, the principal allocates resources τy optimally in current consumption g and end-of-period investment x'. Equation(3.4b) states the budget constraint for the government. Equation(3.4c) say that *principal allocates agent's resources* $(1-\tau)y$ *in agent current consumption* c *and agent end-of-period investment* k' *such that it is as well off as it is without any principal role.* Thus equation set(3.4) can stated as problem of dynamic programming squared model.

4. CONSTRUCTION OF THE EQUILIBRIUM

5. COMPUTATION

6. CONCLUSION

7. APPENDIX-A: MATHEMATICAL PRELIMINARIES

8. APPENDIX-B: ALGORITHM