Problem Set #2

Heterogeneous-agent search model

In this assignment you will numerically solve and analyze a heterogeneous-agent search model with heterogeneous workers and heterogeneous firms and match-specific productivity shocks, i.e., a model building on Shimer-Smith (2000, Econometrica) and more specifically following Borovickova-Shimer (2025).

Model sketch

Time is continuous. All agents are risk neutral and discount the future at a common rate ρ. There is a unit measure of workers with characteristics $x \in \mathcal{X}$ and a unit measure of firms with characteristics $y \in \mathcal{Y}$. The distribution of both x and y is uniform in the population (i.e., distribution functions F_x and F_y are both continuous uniform). Matching of workers and firms is 1-to-1, occurs via random search, and only unmatched agents can search (i.e., no OJS).

Let $u(x)$ be the unemployment rate among workers with characteristic x and $v(y)$ the vacancy rate among firms with characteristics y. An unmatched worker meets a vacancy at Poisson rate λ; the type y is randomly drawn from the pool of unmatched firms. Upon meeting, their pair draws an i.i.d. match-specific productivity $z \in \mathcal{Z} \subset \mathbb{R}_{++}$ from a cumulative distribution function G, so that if they match, they produce flow $zf(x, y)$ where $f(x, y)$ is strictly positive for all x and y and strictly increasing in both arguments. Note that z is fixed for the duration of the match. If a match is formed, the surplus is split according to Nash bargaining, with worker bargaining power $\beta \in (0, 1)$. Matches end randomly at a Poisson rate $\delta > 0$, leaving the worker unemployed and the job vacant.

The surplus of a match between x and y given shock z is

$$S(x, y, z) = zf(x, y) - \rho V_u(x) - \rho V_v(y) = 0,$$

where $V_u(x)$ is the value of a type-x worker being unmatched and $V_v(y)$ is the value of a type-y firm being unmatched. The decision to match is described by a threshold rule, where a match is formed if $z \geq \bar{z}(x, y)$, where the threshold satisfies

$$S(x, y, \bar{z}(x, y)) = 0.$$

It is useful to define the conditional expected value, ωz, and survivor functions, p, as

$$\omega_z(k) = \frac{\int_k^\infty z dG(z)}{1 - G(k)} \quad \text{if} \quad G(k) < 1,\ \omega_z(k) = k \text{ otherwise}$$
$$p(k) = 1 - G(k).$$

The value of a type-x worker being unemployed then satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\rho V_u(x) = \lambda \int_{\mathcal{Y}} \int_{z \geq \bar{z}(x,y)} \frac{\beta}{\rho + \delta} \left(zf(x, y) - \rho V_u(x) - \rho V_v(y) \right) dG(z) v(y) \tag{1}$$
$$= \frac{\lambda \beta}{\rho + \delta} \int_{\mathcal{Y}} p(\bar{z}(x, y)) \times \left(\omega(\bar{z}(x, y) f(x, y)) - \rho V_u(x) - \rho V_v(y) \right) v(y) \tag{2}$$

You will derive the HJB equation for $V_v(y)$ below.

Finally, in the stationary equilibrium, the following balanced-flow conditions hold:

$$\delta\left(1-u(x)\right)=\lambda u(x)\int_{\mathcal{Y}} p(\bar{z}(x,y))v(y), \tag{3}$$

$$\delta\left(1-v(y)\right)=\lambda v(y)\int_{\mathcal{X}} p(\bar{z}(x,y))u(x). \tag{4}$$

Tasks

1. Derive the HJB equation for the value function of a vacant firm of type y, $V_v(y)$. If the distribution of worker and firm types was not uniform, how would you need to amend the model and the HJB equation? What about the flow equations?

2. Explain what assumption ensures the match decision is described by a threshold rule.

3. Solve for the stationary equilibrium of the model using your programming language of choice.

   Parameter values. Assume the production function is CES with elasticity of substitution parameter ξ:

   $$f(x,y)=\left(0.5x^{\frac{\xi-1}{\xi}}+0.5y^{\frac{\xi-1}{\xi}}\right)^{\frac{\xi}{\xi-1}}.$$

   Use the following parameter values: $\rho=0.05$, $\beta=0.5$, $\lambda=10$, $\delta=0.25$ and the production elasticity of substitution is $\xi=0.5$. Let the distribution of match-specific productivity shocks be Pareto, with $\min_z=1$ and standard deviation $\sigma_z=0.1$.

   Guidance.

   - Set up parameters, including functional forms for production and shock distribution; discretize the type space into 10 equally spaced bins
   - Initialize densities of unmatched, matching probabilities, and value functions for unmatched agents
   - Solve for a nested fixed point by iteratively updating (a) the densities of matched agents (e.g. using a simple iterative fixed point or a non-linear equation solver), (b) the value functions of unmatched agents, and (c) the matching probabilities from the unmatched values.
     – For (b), one approach: recover the cutoff value of the shock from the guess on matching probabilities, compute the conditional expected value by match pair, then update the values given these expected values),
     – For (c), one approach: find the cutoff values for which the surplus is zero, from that recover the match probabilities as the survival probabilities

4. Are all possible match configurations (x,y) formed with strictly positive probability? How would your answer change if the dispersion of match-specific productivity shocks were zero? Provide an explanation.

5. Suppose the agents' types were observable. What is the degree of sorting in the economy, as judged by the correlation between worker and firm types in the model, ρ_{xy}? Using the code you developed in the previous step, loop over a range of parameter values for the elasticity of substitution ξ and for the arrival rate λ. Illustrate your answer visually and explain the patterns.

6. Suppose that agents' types were not observable, but you had panel data on individuals' employment states and wages. Through the lens of the model, how could you recover workers' types? *Hint:* Try to focus on observable statistics that are monotone in the worker type / firm type.