Problem Set #2 Solutions

Heterogeneous-Agent Search Model (Borovičková and Shimer, 2024)

Suraj Kumar

Model Setup and Notation

Key Variables

- $x \in \mathcal{X}$: Set of characteristics of the worker
- $y \in \mathcal{Y}$: Firm characteristics
- No on-the-job search
- $u(x)$: Unemployment rate of worker with characteristic x
- $v(y)$: Vacancy rate of firm with characteristic y

Surplus and Value Functions

The surplus of a match between x and y is:

$$S(x, y, z) = zf(x, y) - \rho V_u(x) - \rho V_v(y) \tag{1}$$

where:

- $V_u(x)$: Value of a type-x worker being unmatched
- $V_v(y)$: Value of a type-y firm being unmatched

A match is only formed if $z \geq \bar{z}(x, y)$ such that:

$$S(x, y, \bar{z}(x, y)) = 0 \tag{2}$$

Conditional Expected Value and Survivor Functions

Conditional expected value ω_z:

$$\omega_z(k) = \frac{\int_k^\infty z\,dG(z)}{1 - G(k)} \quad \text{if } G(k) < 1, \quad \omega_z(k) = k \text{ otherwise} \tag{3}$$

Survivor function p:

$$p(k) = 1 - G(k) \quad \text{(Probability of } z \text{ exceeding threshold } k\text{)} \tag{4}$$

1 Question 1: HJB Equation for Vacant Firm

Derivation

The total surplus is:

$$V^s_{\bar{z}}(y,z) = \max_z \left[z f(x,y) - \rho V_u(x) - \rho V_v(y)\right] \tag{5}$$

The value function for the searching firm is:

$$\rho V_v(y) = \lambda \int_{\mathcal{X}} \int_{z \geq \bar{z}(x,y)} \frac{1-\beta}{\rho+\delta} \left[z f(x,y) - \rho V_u(x) - \rho V_v(y)\right] dG(z)\, u(x)\, dx \tag{6}$$

Simplifying using the conditional expectation and survivor function:

$$\rho V_v(y) = \frac{\lambda(1-\beta)}{\rho+\delta} \int_{\mathcal{X}} p(\bar{z}(x,y)) \times \left[\omega(\bar{z}(x,y)) f(x,y) - \rho V_u(x) - \rho V_v(y)\right] u(x)\, dx \tag{7}$$

Non-Uniform Distributions

If the distribution of the firm and worker is not uniform, then:

Equation (1) of PS2 becomes:

$$\rho V_u(x) = \lambda \int_{\mathcal{Y}} \int_{z \geq \bar{z}(x,y)} \frac{\beta}{\rho+\delta} \left[z f(x,y) - \rho V_u(x) - \rho V_v(y)\right] dG(z)\, v(y)\, dF_y(y) \tag{8}$$

Or equivalently:

$$\rho V_u(x) = \frac{\lambda\beta}{\rho+\delta} \int_{\mathcal{Y}} p(\bar{z}(x,y)) \times \left[\omega(\bar{z}(x,y)) f(x,y) - \rho V_u(x) - \rho V_v(y)\right] v(y)\, dF_y(y) \tag{9}$$

Similarly, for the firm, the equation transforms to:

$$\rho V_v(y) = \frac{\lambda(1-\beta)}{\rho+\delta} \int_{\mathcal{X}} p(\bar{z}(x,y)) \times \left[\omega(\bar{z}(x,y)) f(x,y) - \rho V_u(x) - \rho V_v(y)\right] u(x)\, dF_x(x) \tag{10}$$

Flow Equations

In the uniform distribution case, we have:

$$\delta[1-u(x)] = \lambda u(x) \int_{\mathcal{Y}} p(\bar{z}(x,y))\, v(y)\, dy \tag{11}$$

$$\delta[1-v(y)] = \lambda v(y) \int_{\mathcal{X}} p(\bar{z}(x,y))\, u(x)\, dx \tag{12}$$

Interpretation:

- LHS: Number of matched workers/firms breaking up
- RHS: New matches being made

In the case of non-uniform distribution, **equations (3) and (4) from the PS2** are transformed as:

$$\delta[1-u(x)] = \lambda u(x) \int_{\mathcal{Y}} p(\bar{z}(x,y))\, v(y)\, dF_y(y) \tag{13}$$

$$\delta[1-v(y)] = \lambda v(y) \int_{\mathcal{X}} p(\bar{z}(x,y))\, u(x)\, dF_x(x) \tag{14}$$

2 Question 2: Assumptions for Threshold Rule

The match decision is described by a threshold rule under the following assumptions:

1. $f(x, y)$ is **strictly positive** for all x and y
2. $f(x, y)$ is **strictly increasing** in both arguments

Explanation:
These assumptions ensure that:

- The surplus function $S(x, y, z)$ is monotone in z
- There exists a unique threshold $\bar{z}(x, y)$ such that $S(x, y, \bar{z}(x, y)) = 0$
- Matches form if and only if $z \geq \bar{z}(x, y)$
- The threshold is well-defined because production is always positive and increasing in both worker and firm quality

Since $f(x, y)$ is strictly increasing in both arguments, higher quality workers and firms produce more output, making matches more valuable. The strict positivity ensures that production always occurs when a match is formed, and the strict monotonicity guarantees a clear cutoff rule for the productivity shock z.

3 Question 3: Numerical Solution

The stationary equilibrium was solved numerically using Julia with the following approach:

Parameters

- Economic parameters: $\rho = 0.05$, $\beta = 0.5$, $\lambda = 10$, $\delta = 0.25$, $\zeta = 0.5$
- Pareto distribution: $z_{\min} = 1.0$, $\sigma_z = 0.1 \Rightarrow \alpha \approx 11.958$
- Discretization: 10 equally spaced grid points in $[0.001, 1]$ for both x and y
- Convergence tolerances: $\epsilon_{\text{outer}} = \epsilon_{\text{inner}} = 10^{-6}$

Computational Algorithm

The equilibrium is computed using a nested fixed-point iteration algorithm:

Algorithm 1 Nested Fixed Point Algorithm for Stationary Equilibrium

Input: Grids $\{x_i\}_{i=1}^{N_x}$, $\{y_j\}_{j=1}^{N_y}$, parameters $(\rho, \beta, \lambda, \delta, \zeta, z_{\min}, \alpha)$
Output: Equilibrium distributions $u(x)$, $v(y)$ and value functions $V_u(x)$, $V_v(y)$

Initialize: $u(x) \leftarrow 0.1$, $v(y) \leftarrow 0.1$, $V_u(x) \leftarrow 2$, $V_v(y) \leftarrow 2\ \forall x, y$
Compute production function: $f(x,y) = \left(0.5x^{\frac{\zeta-1}{\zeta}} + 0.5y^{\frac{\zeta-1}{\zeta}}\right)^{\frac{\zeta}{\zeta-1}}$

repeat ▷ **Outer Loop**: Iterate on distributions
  $u^{\text{old}} \leftarrow u(x)$, $v^{\text{old}} \leftarrow v(y)$

  repeat ▷ **Inner Loop**: Solve for value functions
    $V_u^{\text{old}} \leftarrow V_u(x)$, $V_v^{\text{old}} \leftarrow V_v(y)$

    Step 1: Compute threshold productivity for all (x, y) pairs:
      $\bar{z}(x,y) = \rho \cdot \frac{V_u(x)+V_v(y)}{f(x,y)}$
      $\bar{z}(x,y) \leftarrow \max\{\bar{z}(x,y), z_{\min} \cdot 1.001\}$ ▷ **Clipping to ensure valid Pareto**

    Step 2: Compute matching probabilities (Pareto survivor function):
      $p(\bar{z}(x,y)) = \left(\frac{z_{\min}}{\bar{z}(x,y)}\right)^{\alpha}$

    Step 3: Compute conditional expectations (Pareto conditional mean):
      $\omega_z(\bar{z}(x,y)) = \frac{\alpha}{\alpha-1}\bar{z}(x,y)$

    Step 4: Update value functions using HJB equations:
      For unemployed workers:
        $V_u(x) = \frac{\lambda\beta}{\rho+\delta}\int_{\mathcal{Y}} p(\bar{z}(x,y))\,[\omega_z(\bar{z}(x,y))f(x,y) - \rho(V_u(x)+V_v(y))]\,v(y)\,dy$

      For vacant firms:
        $V_v(y) = \frac{\lambda(1-\beta)}{\rho+\delta}\int_{\mathcal{X}} p(\bar{z}(x,y))\,[\omega_z(\bar{z}(x,y))f(x,y) - \rho(V_u(x)+V_v(y))]\,u(x)\,dx$

  until $\max\{\|V_u - V_u^{\text{old}}\|_\infty, \|V_v - V_v^{\text{old}}\|_\infty\} < \epsilon_{\text{inner}}$

  Step 5: Update distributions using flow equations:
    $u(x) = \frac{\delta}{\delta+\lambda\int_{\mathcal{Y}} p(\bar{z}(x,y))v(y)\,dy}$ ▷ Unemployment rate

    $v(y) = \frac{\delta}{\delta+\lambda\int_{\mathcal{X}} p(\bar{z}(x,y))u(x)\,dx}$ ▷ Vacancy rate

until $\max\{\|u - u^{\text{old}}\|_\infty, \|v - v^{\text{old}}\|_\infty\} < \epsilon_{\text{outer}}$

return $u(x)$, $v(y)$, $V_u(x)$, $V_v(y)$

Key Implementation Notes:

- The **clipping step** in line 11 ensures $\bar{z}(x, y) \geq z_{\text{min}}$ to maintain a valid Pareto distribution, as the survivor function $p(\bar{z})$ is only defined for $\bar{z} \geq z_{\text{min}}$
- Integrals are approximated using the trapezoidal rule on the discretized grids
- The algorithm uses vectorized operations for all (x, y) pairs to improve computational efficiency
- Convergence typically occurs in 30-40 outer iterations

Results

The algorithm converged in 36 outer iterations. High-resolution plots were generated showing:

- Unemployment rate $u(x)$ by worker type
- Vacancy rate $v(y)$ by firm type
- Value of unemployment $V_u(x)$
- Value of vacancy $V_v(y)$
- Matching probability heatmap $p(\bar{z}(x, y))$
- Threshold productivity heatmap $\bar{z}(x, y)$

Implementation Details

The CES production function used was:

$$f(x, y) = \left(0.5x^{\frac{\xi-1}{\xi}} + 0.5y^{\frac{\xi-1}{\xi}}\right)^{\frac{\xi}{\xi-1}} \tag{15}$$

The Pareto shape parameter α was solved to match the target standard deviation:

$$\sigma_z = z_{\text{min}} \sqrt{\frac{\alpha}{(\alpha - 1)^2(\alpha - 2)}} = 0.1 \tag{16}$$

This yielded $\alpha \approx 11.958$.

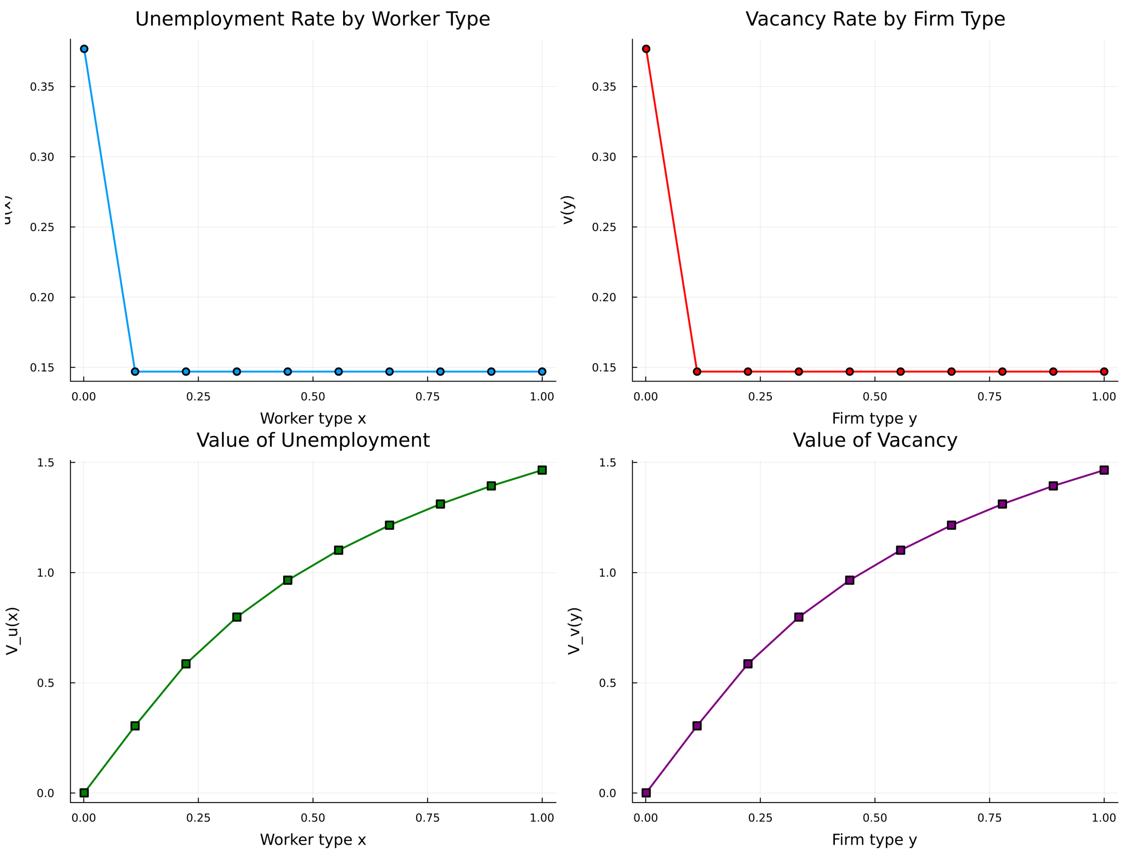

Figure 1: Equilibrium summary showing unemployment rates, vacancy rates, and value functions.

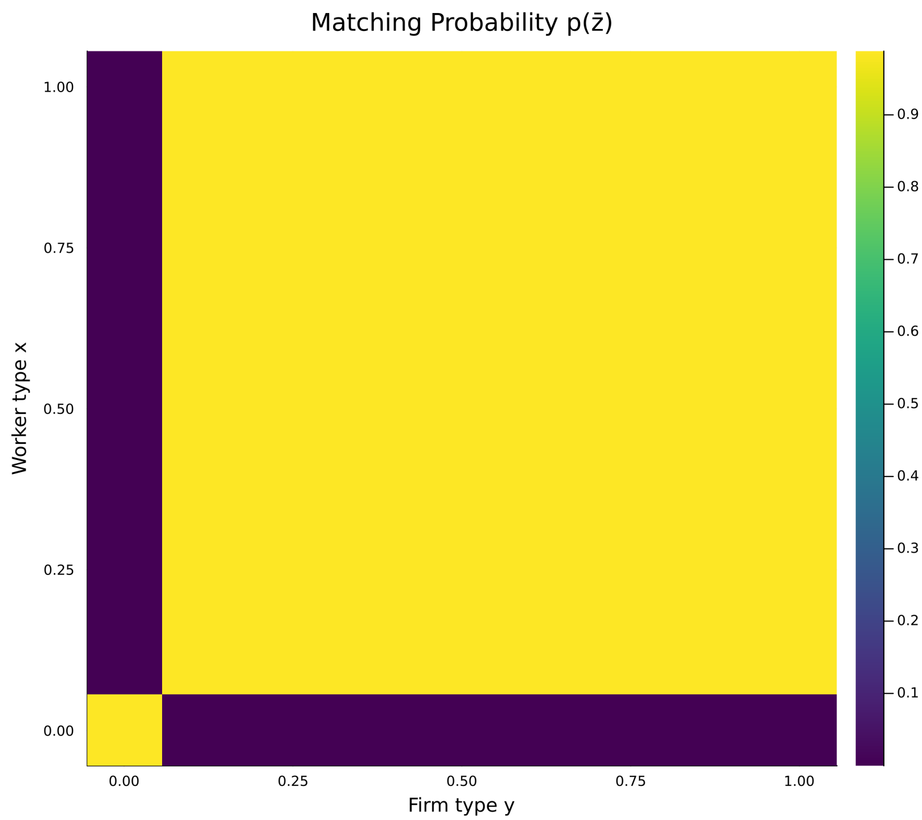

Figure 2: Matching probability heatmap $p(\bar{z}(x, y))$ across worker and firm types.

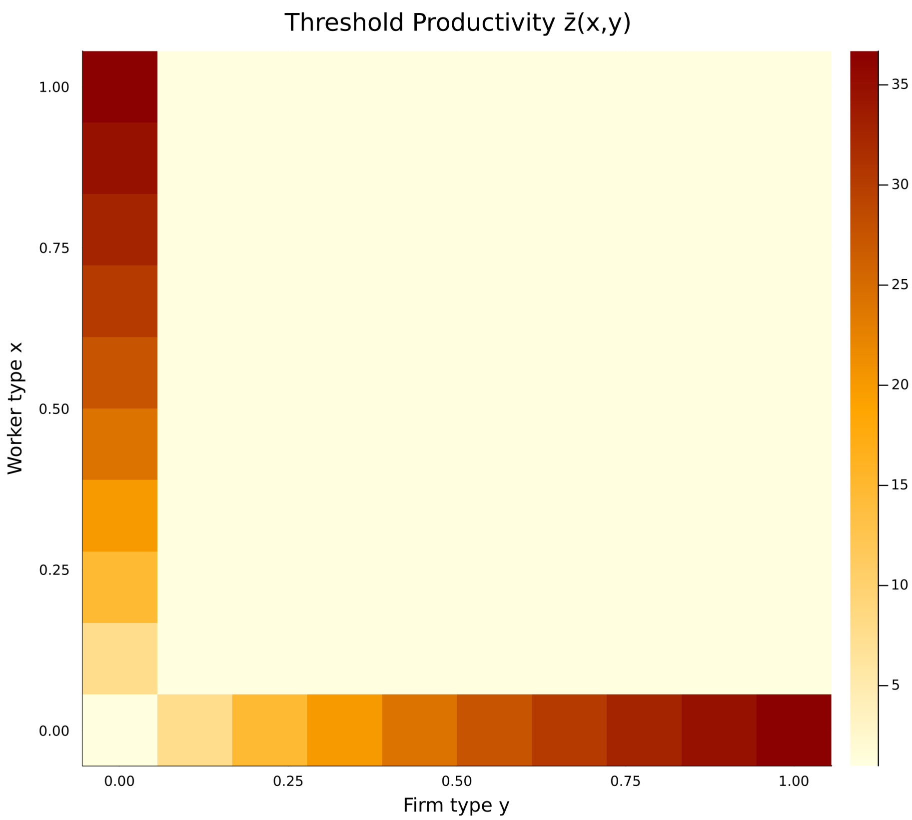

Figure 3: Threshold productivity $\bar{z}(x, y)$ across worker and firm types.

4 Question 4:

In my code, as shown in Figure: 2, not all the matching configurations are formed with strictly positive probability. For instance, worker/firm with lowest type never pair up firms/workers of type higher than the lowest type. Or in other word, lowest type firms only pairup with the lowest type employees.

If there is no dispersion in the match-specific producivity shocks, then the productivity draw during a match is perfectly known. This drawn z will eithr be larger or smaller than the threshold productivity $\bar{z}(x, y)$. This will lead to more match configurations not forming. For instance, if the productivity draw for the lowest type worker and lowest type firm is lower than the threshold productivity $\bar{z}(x_{min}, y_{min})$, then this match will never form. This will lead to a situation where some worker/firm types will never be matched with certain firm/worker types, leading to zero matching probabilities for those configurations.

5 Question 5: Effect of Meeting Rate and Complementarity on Sorting Correlation

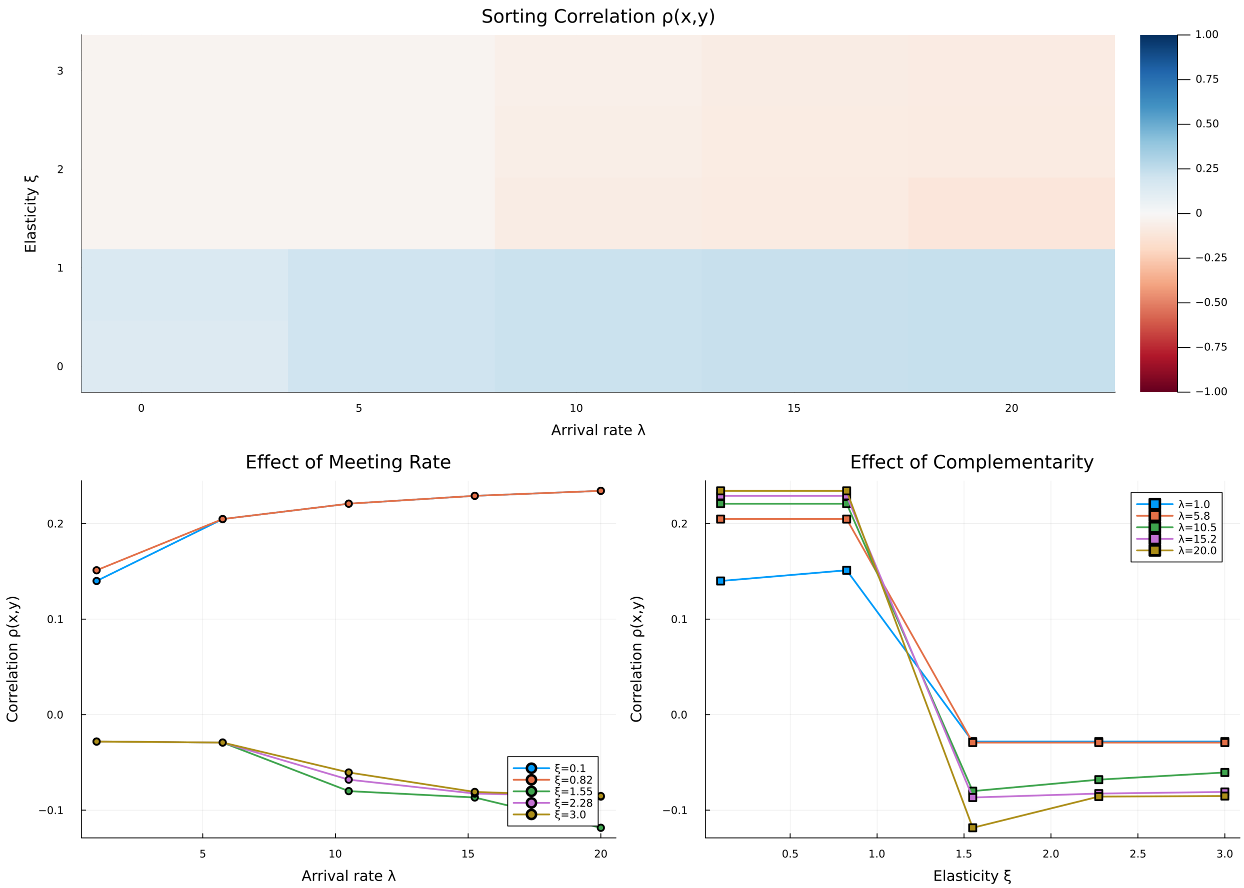

Figure 4: Sorting Correlation across the different meeting rates and complementarity

We observe from Figure: 4 the following patterns:

- **Effect of Complementarity (ξ):** As the elasticity of substitution ξ increases (meaning lower complementarity), the sorting correlation decreases. This is because when goods are more substitutable, the incentive for high-type workers to match with high-type firms diminishes, leading to less assortative matching.
- **Effect of Arrival Rate(λ):** As the arrival rate λ increases, I would expect the sorting correlation to increase. A higher arrival rate means more frequent opportunities for matches, which can enhance the likelihood of high-type workers and firms finding each other, thus promoting assortative matching. However in the plot, we see two patterns:
  - For low complementarity (high ξ), increasing arrival rate λ seems to decrease sorting correlation. This could be because when goods are more substitutable, the increased meeting opportunities allow for more diverse matches, reducing the assortative matching effect.

- For high complementarity (low ξ), increasing arrival rate λ seems to increase sorting correlation. This aligns with the expectation that more frequent meetings help high-type agents find each other in a complementary setting.

6 Question 6:

If the agents' types is not observeble, then to recover the worker type from the panel data on individuals' employment states and wages, we can use the following approach:

- **Unemployment rate:** . The model steady state unmeployment rate is:

$$u(x) = \frac{\delta}{\delta + \lambda \int_{\mathcal{Y}} p(\bar{z}(x,y))v(y)\,dy}$$

  Thus, the high type workers are more productive, so more firms would like to match with them. Thus, they will reduce technology threshold $\bar{z}$ for matching with high type workers. This will increase the matching probability $p(\bar{z}(x,y))$ for high type workers. Thus, the denominator of the unempolyment rate equation will be higher for high type workers, leading to lower unemployment rate for high type workers. Thus, by observing the unemployment rate of individuals over time, we can infer their types - lower unemployment rates indicate higher worker types.

- **Wage observations:** Wages in the model are determined by the surplus sharing rule:

$$w(x,y,z) \propto \beta S(x,y,z)$$

  Since high type workers generate higher surplus in matches, they will have higher wages on average. Thus, by analyzing the wage distribution across individuals, we can identify higher type workers as those with consistently higher wages.

References

Borovičková, K. and Shimer, R. (2024). Assortative matching and wages: The role of selection. Technical report, National Bureau of Economic Research.