Problem Set 1: Solutions

Econ 8861: Monetary Economics I

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1 Steady State of the Standard Incomplete Markets Model

1.1 Model Setup

The standard incomplete markets model is solved with the following parametrization:

- Utility function: $u(c) = \log c$
- Income process: $\log s_{it} = \rho \log s_{it-1} + \epsilon_{it}$ with $\epsilon_{it} \sim \mathcal{N}(0, \sigma_\epsilon \sqrt{1-\rho^2})$
- Parameters: $\rho = 0.975$, $\sigma_\epsilon = 0.7$, $Y = 1$, $r = 0.01/4$, $\underline{a} = 0$, $\beta = 0.988$

1.2 Grid Setup (Questions 1a–1b)

Income Grid: Using Rouwenhorst's method with 7 grid points, the income grid is:

$$s \in \{0.14, 0.25, 0.44, 0.79, 1.39, 2.46, 4.36\}$$

with stationary distribution:

$$\pi_s = \{0.0156, 0.0938, 0.2344, 0.3125, 0.2344, 0.0938, 0.0156\}$$

Asset Grid: The asset grid uses 500 points between $\underline{a} = 0$ and $a^{\max} = 200$, constructed using the double-exponential grid method to provide finer resolution near the borrowing constraint.

1.3 Backward Iteration (Questions 2–4)

The `backward_step` function implements the endogenous gridpoint method. Given $\partial V/\partial a$, it:

1. Computes expected marginal value: $\mathbb{E}_s[\partial V/\partial a] = \Pi \cdot (\partial V/\partial a)$

2. Uses the Euler equation to find current consumption: $c = 1/(\beta \cdot \mathbb{E}_s[\partial V/\partial a])$

3. Applies the budget constraint to find current assets: $a_- = (c + a' - sY)/(1 + r)$

4. Interpolates to obtain $a'(s, a_-)$ on the original grid

5. Enforces the borrowing constraint: $a'(s, a_-) = \max\{a'(s, a_-), 0\}$

6. Computes the consumption policy: $c(s, a_-) = (1 + r)a_- + sY - a'(s, a_-)$

7. Updates using the envelope condition: $\partial V/\partial a_- = (1 + r)/c(s, a_-)$

The initial guess uses $c(s, a_-) = 0.1 \cdot ((1+r)a_- + sY)$. The backward iteration converged after approximately 220 iterations.

1.4 Policy Functions (Question 5)

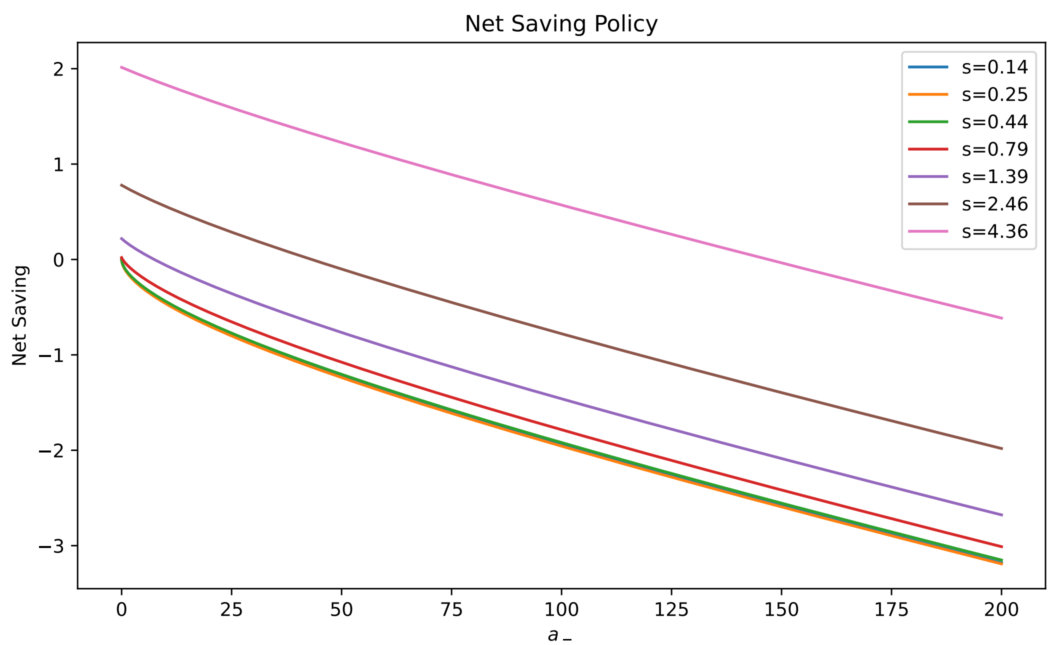

Figure 1: Net saving policy $a'(s, a_-) - a_-$ as a function of beginning-of-period assets a_-, for each income level s. Higher income households save more at any given asset level. At the borrowing constraint ($a_- = 0$), all but the highest income households dissave.

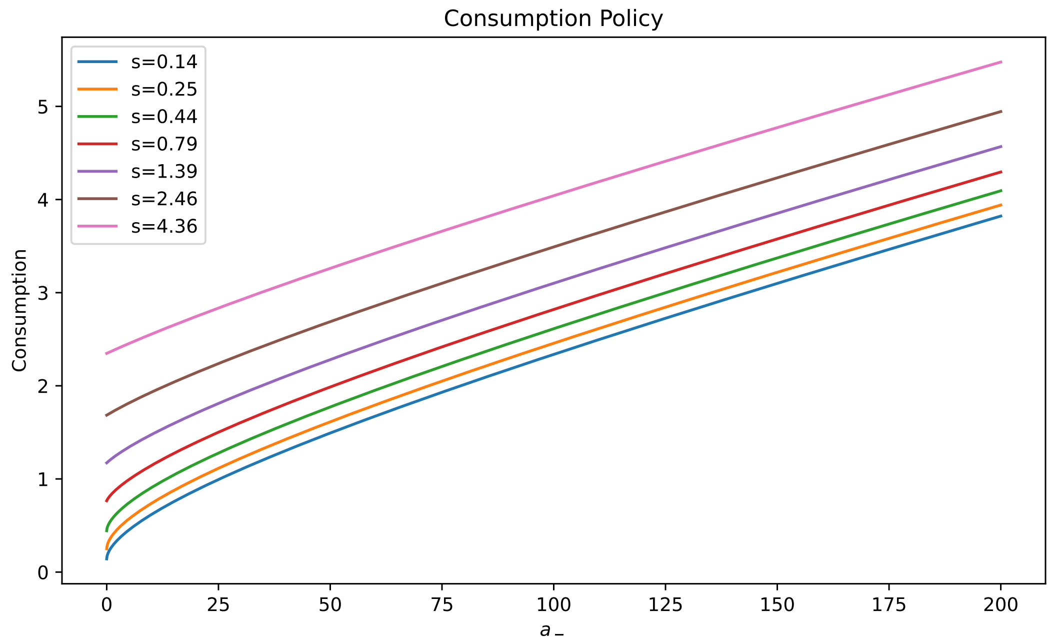

Figure 2: Consumption policy $c(s, a_-)$ as a function of beginning-of-period assets a_-. Consumption increases in both assets and income, with the income effect being particularly strong for low-asset households.

1.5 Forward Iteration (Questions 6–8)

Young's Method (Question 6): The savings policy is discretized by finding the bracketing grid points for each $a'(s, a_-)$:

$$a'(s, a_-) = \pi_a \cdot a_{\text{grid}}[i_a] + (1 - \pi_a) \cdot a_{\text{grid}}[i_a + 1]$$

Forward Step (Question 7): The `forward_step` function:

1. Initializes $D^{\text{end}} = 0$
2. For each state (s, a_-), distributes mass $D(s, a_-)$ across the bracketing asset grid points using weights π_a and $(1 - \pi_a)$
3. Updates the income distribution: $D_{t+1} = \Pi^\top D^{\text{end}}$

Convergence (Question 8): Starting from the initial distribution where a is uniform and s follows its stationary distribution, the forward iteration converged after approximately 450 iterations.

1.6 Aggregate Statistics (Question 9)

Using the invariant distribution $D(s, a_-)$, I compute:

Statistic	Value
Aggregate Assets (normalized by annual earnings)	$A = \mathbf{1.4735}$
Average Marginal Propensity to Consume	$\overline{MPC} = \mathbf{0.2426}$

Table 1: Key aggregate statistics for baseline calibration ($\beta = 0.988$)

The aggregate asset stock is computed as:

$$A = \sum_{s,a_-} D(s, a_-) \cdot \frac{a_-}{4Y} = 1.4735$$

The average MPC is computed using forward differences:

$$\overline{MPC} = \frac{1}{1+r} \sum_{s,a_-} D(s, a_-) \cdot \frac{\partial c(s, a_-)}{\partial a_-} = 0.2426$$

1.7 Comparative Statics with β (Question 10)

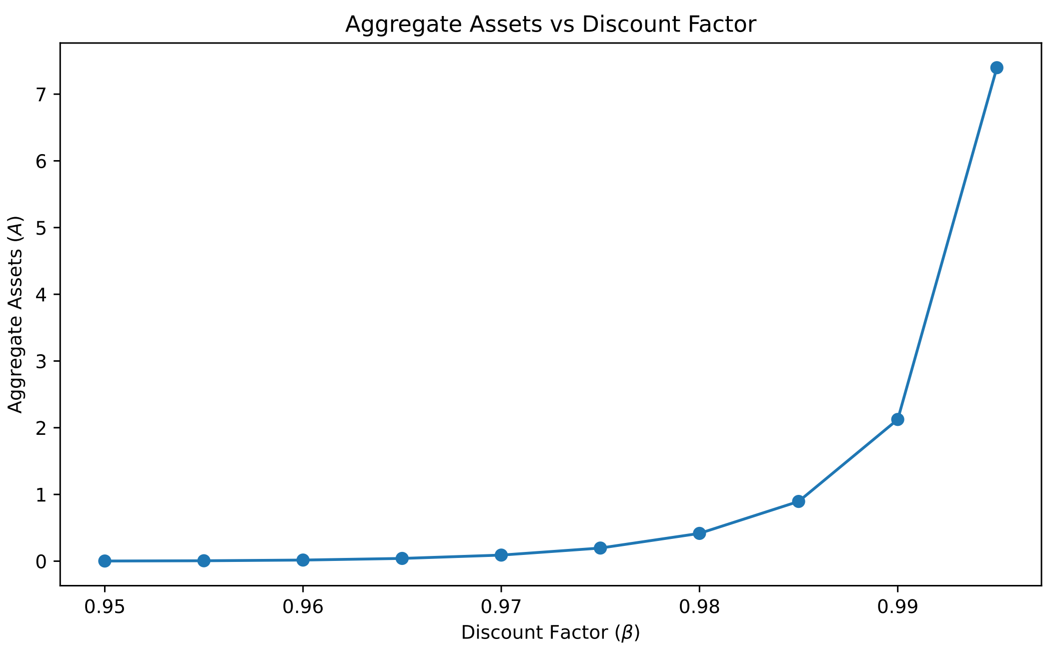

Figure 3: Aggregate assets as a function of the discount factor β. More patient households (higher β) accumulate significantly more assets. The relationship is highly nonlinear, with assets increasing steeply as $\beta \to 1$.

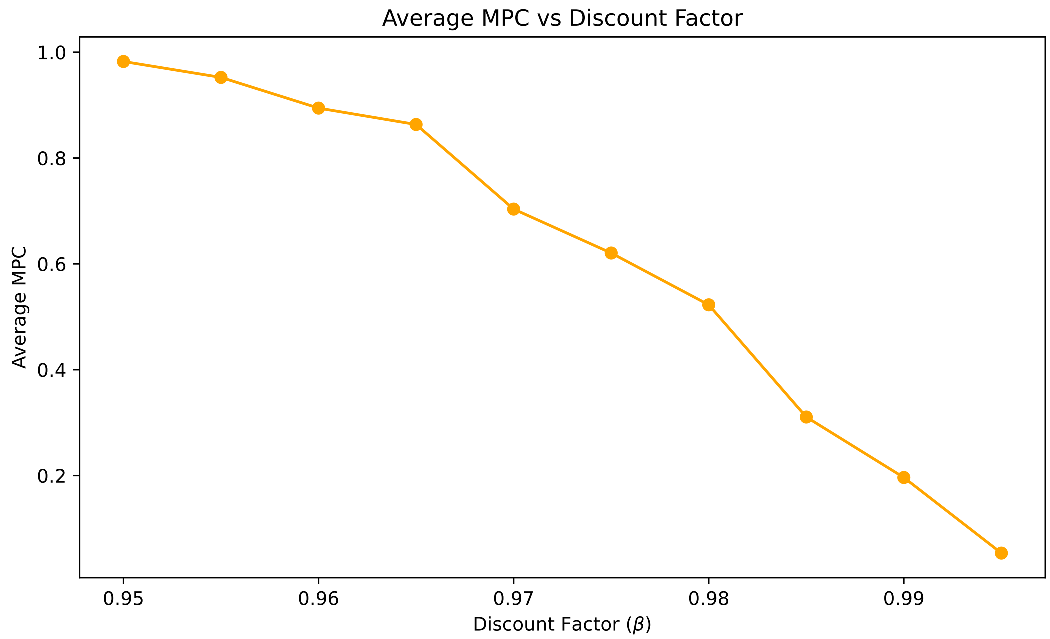

Figure 4: Average MPC as a function of the discount factor β. Less patient households (lower β) have higher MPCs because they hold fewer assets and are more likely to be constrained. As β increases, households accumulate more buffer stock savings, reducing their average MPC.

β	Aggregate Assets (A)	Average MPC
0.950	0.0020	0.9824
0.955	0.0059	0.9523
0.960	0.0164	0.8943
0.965	0.0409	0.8634
0.970	0.0908	0.7035
0.975	0.1963	0.6205
0.980	0.4161	0.5226
0.985	0.8951	0.3106
0.990	2.1238	0.1963
0.995	7.3966	0.0535

Table 2: Aggregate assets and average MPC for different values of β

Key observations:

- Aggregate assets increase sharply with patience (β), rising from essentially zero at $\beta = 0.95$ to over 7 times annual income at $\beta = 0.995$.
- The average MPC decreases monotonically with β, falling from near 1 (hand-to-mouth) at $\beta = 0.95$ to about 0.05 at $\beta = 0.995$.

- These patterns reflect the precautionary savings motive: more patient households build larger buffer stocks, making them less sensitive to transitory income shocks.

1.8 Asset Distribution (Question 11)

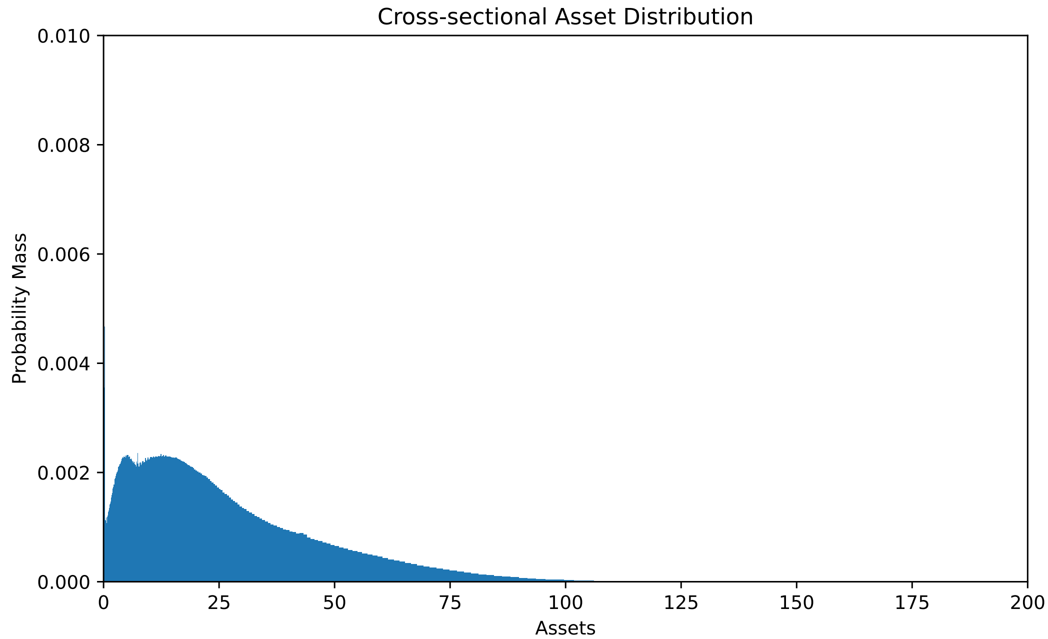

Figure 5: Cross-sectional distribution of assets. The distribution is highly skewed with significant mass near the borrowing constraint and a long right tail. This reflects the combination of impatient households who remain near the constraint and patient/lucky households who accumulate substantial wealth.

2 Homogeneity of the Aggregate Savings Function

Proposition 1 (Homogeneity of Aggregate Assets)**.** *For the utility function* $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$ *with* $\gamma > 0$*, the aggregate asset function*

$$A(r, \beta, Y, \underline{a}, \sigma_\epsilon) = \sum_s \pi_s \int a_- D(s, da_-)$$

is homogeneous of degree 1 in $(Y, \underline{a})$*. That is, for any* $\lambda > 0$*:*

$$A(r, \beta, \lambda Y, \lambda \underline{a}, \sigma_\epsilon) = \lambda A(r, \beta, Y, \underline{a}, \sigma_\epsilon)$$

Proof. I will prove this by showing that the optimal policies and distribution in the scaled economy transform in a way that preserves homogeneity.

Step 1: Define the Original Economy

The household problem in the original economy with income Y and borrowing limit $\underline{a}$ is:

$$V(s, a_-) = \max_{a \geq \underline{a}} \{u((1+r)a_- + sY - a) + \beta \mathbb{E}_s[V(s', a)]\}$$

The first-order condition (when interior) is:

$$u'(c^*(s, a_-)) = \beta(1+r)\mathbb{E}_s[u'(c^*(s', a^*(s, a_-)))] \tag{1}$$

The budget constraint is:

$$c^*(s, a_-) = (1+r)a_- + sY - a^*(s, a_-) \tag{2}$$

The stationary distribution $D(s, A)$ satisfies:

$$D(s', A) = \sum_s \Pi_{ss'} D(s, (a^*(s, \cdot))^{-1}(A)) \tag{3}$$

where $\Pi_{ss'}$ is the income transition matrix with stationary distribution π_s.

Step 2: Define the Scaled Economy

Consider the scaled economy with income λY and borrowing limit $\lambda \underline{a}$:

$$V_\lambda(s, a_-) = \max_{a \geq \lambda \underline{a}} \{u((1+r)a_- + s\lambda Y - a) + \beta \mathbb{E}_s[V_\lambda(s', a)]\}$$

Step 3: Guess the Policy Function Transformation

I conjecture that the optimal policies in the scaled economy are:

$$c^*_\lambda(s, a_-) = \lambda c^*(s, a_-/\lambda) \tag{4}$$
$$a^*_\lambda(s, a_-) = \lambda a^*(s, a_-/\lambda) \tag{5}$$

where $c^*(\cdot)$ and $a^*(\cdot)$ are the policies from the original economy.

Step 4: Verify the Budget Constraint

Substituting (4) and (5) into the budget constraint of the scaled economy:

$$c^*_\lambda(s, a_-) = (1 + r)a_- + s\lambda Y - a^*_\lambda(s, a_-)$$
$$\lambda c^*(s, a_-/\lambda) = (1 + r)a_- + s\lambda Y - \lambda a^*(s, a_-/\lambda)$$

Dividing both sides by λ:

$$c^*(s, a_-/\lambda) = (1 + r)(a_-/\lambda) + sY - a^*(s, a_-/\lambda)$$

This is exactly equation (2) evaluated at a_-/λ. ✓

Step 5: Key CRRA Property

Lemma 1. *For CRRA utility $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, we have:*

$$u'(\lambda c) = \lambda^{-\gamma} u'(c)$$

Proof of Lemma. Since $u'(c) = c^{-\gamma}$, we have:

$$u'(\lambda c) = (\lambda c)^{-\gamma} = \lambda^{-\gamma} c^{-\gamma} = \lambda^{-\gamma} u'(c) \qquad \square$$

Step 6: Verify the First-Order Condition

The FOC for the scaled economy requires:

$$u'(c^*_\lambda(s, a_-)) = \beta(1 + r)\mathbb{E}_s[u'(c^*_\lambda(s', a^*_\lambda(s, a_-)))] \tag{6}$$

Substituting our conjectured policies (4) and (5):

$$u'(\lambda c^*(s, a_-/\lambda)) = \beta(1 + r)\mathbb{E}_s[u'(\lambda c^*(s', \lambda a^*(s, a_-/\lambda)))]$$

Applying Lemma 1:

$$\lambda^{-\gamma} u'(c^*(s, a_-/\lambda)) = \beta(1 + r)\mathbb{E}_s[\lambda^{-\gamma} u'(c^*(s', a^*(s, a_-/\lambda)))]$$

Canceling $\lambda^{-\gamma}$ from both sides:

$$u'(c^*(s, a_-/\lambda)) = \beta(1 + r)\mathbb{E}_s[u'(c^*(s', a^*(s, a_-/\lambda)))]$$

This is exactly equation (1) evaluated at a_-/λ. ✓

Since the conjectured policies satisfy both the FOC and the budget constraint, and the problem is strictly concave, these are the unique optimal policies in the scaled economy.

Step 7: Distribution Transformation

Since the optimal policy in the scaled economy is $a^*_\lambda(s, a_-) = \lambda a^*(s, a_-/\lambda)$, an agent at state (s, a_-) in the scaled economy behaves identically to an agent at state $(s, a_-/\lambda)$ in the original economy, scaled by factor λ.

Define the distribution in the scaled economy as:

$$D_\lambda(s, A) = D(s, A/\lambda) \tag{7}$$

for measurable sets A.

To verify this is consistent with the law of motion, note that $a^*_\lambda(s, a_-) \in A$ if and only if $\lambda a^*(s, a_-/\lambda) \in A$, which occurs if and only if $a^*(s, a_-/\lambda) \in A/\lambda$.

Therefore:

$$(a^*_\lambda(s, \cdot))^{-1}(A) = \lambda(a^*(s, \cdot))^{-1}(A/\lambda)$$

The distribution D_λ satisfies:

$$\begin{aligned} D_\lambda(s', A) &= \sum_s \Pi_{ss'} D_\lambda(s, (a^*_\lambda(s, \cdot))^{-1}(A)) \\ &= \sum_s \Pi_{ss'} D_\lambda(s, \lambda(a^*(s, \cdot))^{-1}(A/\lambda)) \\ &= \sum_s \Pi_{ss'} D(s, (a^*(s, \cdot))^{-1}(A/\lambda)) \\ &= D(s', A/\lambda) = D_\lambda(s', A) \end{aligned}$$

Thus, the distribution transformation (7) is consistent with the equilibrium law of motion. ✓

Step 8: Compute Aggregate Assets

In the scaled economy, aggregate assets are:

$$A_\lambda = \sum_s \pi_s \int a_- D_\lambda(s, da_-)$$

Substituting $D_\lambda(s, da_-) = D(s, d(a_-/\lambda))$ from (7), and using the change of variables $\tilde{a}_- = a_-/\lambda$ (so $a_- = \lambda\tilde{a}_-$ and $da_- = \lambda d\tilde{a}_-$):

$$\begin{aligned} A_\lambda &= \sum_s \pi_s \int \lambda\tilde{a}_- \, D(s, d\tilde{a}_-) \\ &= \lambda \sum_s \pi_s \int \tilde{a}_- \, D(s, d\tilde{a}_-) \\ &= \lambda A \end{aligned}$$

Therefore, I have established that:

$$\boxed{A(r, \beta, \lambda Y, \lambda \underline{a}, \sigma_\epsilon) = \lambda A(r, \beta, Y, \underline{a}, \sigma_\epsilon)}$$

for all $\lambda > 0$, completing the proof. □