ARGUMENTS AND PROOFS

Definition: An **argument** is a sequence of statements called premises, plus a statement called the conclusion. A **valid** argument is an argument such that the conclusion is true whenever the premises are all true.

Note: An argument has the following form:
$(P_1 \wedge P_2 \wedge ... \wedge P_n) \to C$

Example: Argument:
If it is Saturday, then it will rain.
It is Saturday.
Therefore it will rain.

Is this a valid argument? Let's come up with the general form.
P: It is Saturday.
Q: It will rain.
Premises: $P \to Q$, P
Conclusion: Q
Typically we write the argument like this:

$$\begin{array}{l} P \to Q \\ \underline{P \qquad\quad} \\ \therefore Q \end{array}$$

Note: the symbol $\therefore$ stands for "therefore".

Note: We can see via a truth table that, for propositions P and Q, $((P \to Q) \wedge P) \to Q$ is a tautology:

Truth table (in class):

Modus Ponens

This general form of argument, the rule of inference called **modus ponens**:

$$\begin{array}{l} P \rightarrow Q \\ \underline{\mathrm{P}\qquad\quad} \\ \therefore \mathrm{Q} \end{array}$$

is valid since whenever the premises are true, the conclusion is also true.

Example: If it snows, then the schools will be closed. It snows. Therefore the schools are closed.
P: It snows.
Q: The schools will be closed.

If $P \rightarrow Q$ and P are true, then Q is true, by modus ponens.

Rules of Inference

Rule	Corresponding Tautology	Name
$P \to Q$ $\underline{\text{P}}$ $\therefore Q$	$(P \to Q) \land P \to Q$	modus ponens
$P \to Q$ $\underline{\neg Q}$ $\therefore \neg P$	$(P \to Q) \land \neg Q \to \neg P$	modus tollens
$P \to Q$ $\underline{Q \to R}$ $\therefore P \to R$	$[(P \to Q) \land (Q \to R)] \to (P \to R)$	hypothetical syllogism
$P \lor Q$ $\underline{\neg P}$ $\therefore Q$	$[(P \lor Q) \land \neg P] \to Q$	disjunctive syllogism
$\underline{P}$ $\therefore P \lor Q$	$P \to (P \lor Q)$	addition
$\underline{P \land Q}$ $\therefore P$	$(P \land Q) \to P$	simplification
P $\underline{\text{Q}}$ $\therefore P \land Q$	$[(P) \land (Q)] \to (P \land Q)$	conjunction
$P \lor Q$ $\underline{\neg P \lor R}$ $\therefore Q \lor R$	$[(P \lor Q) \land (\neg P \lor R)] \to (Q \lor R)$	resolution
$(P \to Q) \land (R \to S)$ $\underline{P \lor R}$ $\therefore Q \lor S$		constructive dilemma
$(P \to Q) \land (R \to S)$ $\underline{\neg Q \lor \neg S}$ $\therefore \neg P \lor \neg R$		destructive dilemma

Examples: Rules of Inference

1. If “pigs are pink” is true, then “pigs are pink or gray” is also true. This is an example of **addition**.

2. If “cats meow and hiss” is true, then “cats meow” is also true. This is an example of **simplification**.

3. If the following statements are true:
   “If I am happy, then I smile”
   “I am happy”
   Then the statement “I smile” is also true. This is a **modus ponens** example. This rule is the basis of direct proof.

4. If the following statements are true:
   “If I am happy, then I smile”
   “I am not smiling”
   Then the statement “I am not happy” is also true. This is an example of **modus tollens**.

5. If the statement “It is snowing or raining” is true and it is not snowing, then it must be raining. This is an example of **disjunctive syllogism**.

6. Suppose the statements “if there is a fire in the house, then there will be smoke in the house” and “if there is smoke in the house, then the smoke alarm will beep” are true. Then if there is a fire in the house, the smoke alarm will beep. This is an example of **hypothetical syllogism**.

7. If the statements “it is cold or raining” and “it is not cold or it is snowing” are true, then it is raining or snowing. This is a **resolution** example.

8. If the statements “if it snows, then the schools close” and “if it is sunny, then Sam will go to the park” are true, and if it snows or is sunny, then it is also true that the schools close or Sam will go to the park. This is an example of **constructive dilemma**.

9. If these statements are true:
   If Sam is sick, then Sam will stay home.
   If Vince is awake, then he will want to eat.
   Sam will not stay home or Vince will not want to eat.
   Then it is also true that Sam is not sick or Vince is not awake.
   This is an example of **destructive dilemma**.

Using Truth Tables to Prove Some of These Rules of Inference

Example: Use a truth table to establish the law of disjunctive syllogism.

Example: Prove the law of resolution holds.

Exercise: Prove the law of hypothetical syllogism holds.

Examples - Applying Rules of Inference

Example: Is this argument valid, and if so, which rule(s) of inference are being used?

If there is smoke in here, then there is fire in here.
There is no fire in here.
Therefore there is no smoke.

Example: Is the argument valid? If so, what rules of inference are used?

If I don't study, then I won't understand logic.
If I don't understand logic, then I won't make an A in this class.
Therefore if I don't study, then I won't make an A in this class.

Arguments that use multiple rules of inference

Example: Assign propositional variables to come up with the general argument, and then use a step by step proof to show the argument is valid. That is, show that if we assume the premises are true, then the conclusion must also be true.
F: I made a B on the test.
A: I made an A on the test.
M: I am happy with my test grade.
P: I think I will make a good grade in the course.
Argument:
If I made an B or an A on the test, then I am happy with my test grade.
If I am happy with my test grade, then I think I will make a good grade in the course.
I do not think I will make a good grade in the course.
Therefore I did not make an A on the test.

Argument form?

Proof: (Number your steps, and give a reason or justification for each step - i.e., previous steps, rules of inference, logical equivalences, hypotheses).

More Proofs: Showing an Argument is Valid

Note: Each assertion in the proof is true because it is one of the following:

1. A hypothesis/premise
2. Logically equivalent to a prior assertion in the proof
3. Results from applying a rule of inference to a previous assertion

Example:
If I drink coffee, then I will get a lot of work done.
If I don't drink coffee, then I am sleepy.
If I am sleepy, then I am grumpy.
Therefore if I don't get a lot of work done, then I am grumpy.
P: I drink coffee.
Q: I will get a lot of work done.
R: I am sleepy.
S: I am grumpy.

Argument form:

Proof:

Fallacies

Definition: A **fallacy** is a reasoning error that leads to an invalid argument.

Example: If Otis is a dog, then Otis wags his tail.
Otis wags his tail.
$\therefore$ Otis is a dog.

The general form of this argument is:

$$\begin{array}{l} P \rightarrow Q \\ \underline{\text{Q}\qquad} \\ \therefore \text{P} \end{array}$$

This type of incorrect reasoning is called the **converse error** since if the 1st premise was replaced with its converse, $Q \rightarrow P$, the argument would be valid.

Another Type of Fallacy

Example: If I win the lottery, then I will buy a new car.
I did not win the lottery.
$\therefore$ I will not buy a new car.

What is the argument form?

Note: This is often called an **inverse error**.

Valid and Invalid Arguments

Note: An argument can be valid even if the conclusion is false:

Example:
If 2+3 = 5, then 6 is prime.
2+3 = 5
$\therefore$ 6 is prime.

This is a valid argument of the form
$P \rightarrow Q$
$\underline{\text{P}\qquad}$
$\therefore$ Q

Exercise: Show this argument form is valid with a truth table.

In this example, the first premise is false, so the result is vacuously true.

Note: An argument may be invalid even if the conclusion is true.

Example:
If Socrates was a man, then Socrates was mortal.
Socrates was mortal.
$\therefore$ Socrates was a man.

The premises and conclusion are all true, but the argument is invalid.

Question: What is the general form of the argument? Show the argument form is invalid with a truth table.

The Contradiction Rule

One way to show statement P is true:

Show that assuming P is false leads to a contradiction (something you know is false). Then you know P is true.

Argument Form:

$\underline{\neg P \rightarrow F_0}$ where F_0 is a contradiction.

$\therefore$ P

Proof of validity: (use a truth table)

Note: So if an assumption leads to a contradiction, you know the assumption must be false.

Knights and Knaves - Raymond Smullyan's puzzle

The puzzle:
An island contains two types of people, knights and knaves. Knights always tell the truth, and knaves always lie. You go to the island, and two people approach you.
Person A says: B is a knight.
Person B says: A and I are of opposite type.
What type are A and B?

Solution: (Hint: we will use the contradiction rule, and start by assuming A is a knight.)

Showing an Argument is Invalid

Example: Argument:

P

$P \vee Q$

$Q \rightarrow (R \rightarrow S)$

$\underline{U \rightarrow R \quad}$

$\therefore \neg S \rightarrow \neg U$

To show the argument is invalid: find a way of assigning truth values to the variables so the premises are true and the conclusion is false.

Solution: How should we assign T and F to the propositional variables?

Note: Our assignments to the variables forms a **counterexample** for the argument and shows it is invalid.

Note: To show an argument is valid, we show that for all combinations of proposition truth values that make the premises true, the conclusion is also true.