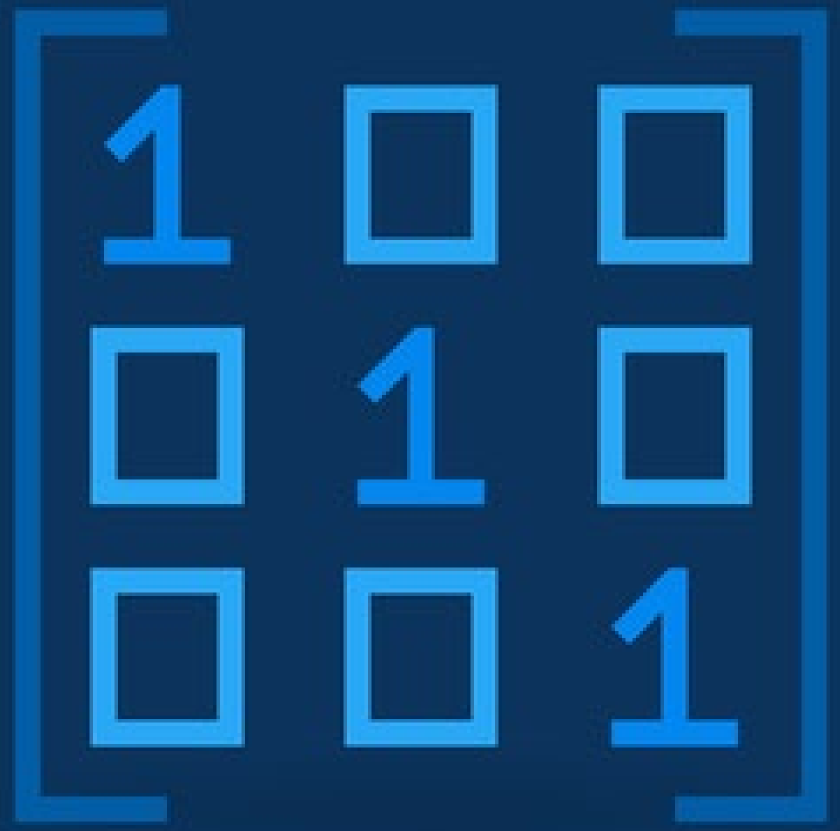
Matrices

PART 1: TYPES OF MATRICES, MATRIX ARITHMETIC AND MULTIPLICATION



Learning Outcomes

Learning Outcomes

At the end of this lesson, you will be able to:

1. identify and use the various type of matrices
2. perform arithmetic operation on matrices

Introduction to Matrix

- An $m \times n$ matrix A is a rectangular array of numbers in the form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- The pair of numbers $m \times n$ is called the size of the matrix where m represents the number of rows and n the number of columns.

Example 1

The rectangular array $\begin{bmatrix} 1 & -3 & 4 \\ 0 & 5 & -2 \end{bmatrix}$ is a 2×3 matrix.

Introduction to Matrix

- Row and Column vector
  - A matrix with only **one row** is called a **row vector** and a matrix with only **one column** is called a **column vector**.

Example 2

Row vector: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is a $1\ X\ 3$ matrix. Column vector: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a $3\ X\ 1$ matrix.

- Zero Matrix
  - A matrix whose entries are all zero is called a zero matrix.

Example 3

The 2×4 zero matrix is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Introduction to Matrix

- Equality of two matrices
  - Two matrices A and B are equal, denoted by $A = B$, if they have the **same size** and the **same corresponding element**.

Matrix Addition/Subtraction

- Two matrices A and B of the same size can be added (or subtracted) by adding (or subtracting) their corresponding elements:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \dots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \dots & a_{2n} \pm b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \dots & a_{mn} \pm b_{mn} \end{bmatrix}$$

<u>Example 4</u>

Evaluate $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & -6 \\ 2 & -3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+3 & -2+0 & 3-6 \\ 0+2 & 4-3 & 5+1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 & -3 \\ 2 & 1 & 6 \end{bmatrix}$$

Matrix Addition/Subtraction

- As with ordinary addition, the communicative law and the associative laws apply to matrix addition.
  - Commutative Law: $A + B \quad = \quad B + A$
  - Associative Law : $A + (B + C) \quad = \quad (A + B) + C$

- A matrix defined as $-A$ is called the negative of A.

Matrix Scalar Multiplication

- The product of a scalar k and a matrix A, written as kA or Ak, is obtained by multiplying each element of A by k:

$$k\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

Example 5

Evaluate $3\begin{bmatrix} 1 & -2 & 0 \\ 4 & 3 & -5 \end{bmatrix}$

$$= \begin{bmatrix} 3 & -6 & 0 \\ 12 & 9 & -15 \end{bmatrix}$$

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Matrix Multiplication

- Two matrices A and B can be multiplied together if and only if the number of columns in A is the same as the number of rows in B.

$A_{\boxed{m}} \times p$ $\qquad\qquad$ $B_p \times \boxed{n} \leftarrow$ no. of cols in product AB

$\downarrow$ $\qquad\qquad$ $\uparrow$ $\qquad\qquad\qquad\qquad$ $\uparrow$

$\downarrow$ $\qquad$ no. of cols in A $\;=\;$ no. of rows in B

no. of rows in AB

- Thus, $A_{(m\times p)}$ and $B_{(p\times n)}$ can be multiplied together to give a new matrix C of size m x n.

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Matrix Multiplication

- Example 6

$$A = \begin{pmatrix} -1 & 3 \\ 6 & 0 \\ 5 & 1 \\ -2 & -4 \end{pmatrix}_{4\times 2} \quad \text{and} \quad B = \begin{pmatrix} 2 & -3 & 7 \\ 0 & 8 & -6 \end{pmatrix}_{2\times 3}$$

We can find the product *AB*, because the 2 columns in *A* equals the 2 rows in *B.* The product is a *4×3* matrix and it is formed as follows:

$$\begin{pmatrix} -1 & 3 \\ 6 & 0 \\ 5 & 1 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 2 & -3 & 7 \\ 0 & 8 & -6 \end{pmatrix} = \begin{pmatrix} \boxed{(-1)(2)+(3)(0)} & (-1)(-3)+(3)(8) & (-1)(7)+(3)(-6) \\ (6)(2)+(0)(0) & (6)(-3)+(0)(8) & (6)(7)+(0)(-6) \\ (5)(2)+(1)(0) & (5)(-3)+(1)(8) & (5)(7)+(1)(-6) \\ (-2)(2)+(-4)(0) & \boxed{(-2)(-3)+(-4)(8)} & (-2)(7)+(-4)(-6) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 27 & -25 \\ 12 & -18 & 42 \\ 10 & -7 & 29 \\ -4 & -26 & 10 \end{pmatrix}$$

Properties of matrix multiplication

1. Matrix multiplication is associative:

   $A(BC) = (AB)C$ and $(kA)B = k(AB) = A(kB)$

2. Matrix multiplication is distributive over addition:

   $A(B+C) = AB + AC$ *and* $(B+C)A = BA + CA$

3. Matrix multiplication is not communicative. In general, $AB \neq BA$

4. $AB = 0$ does not necessarily imply $A = 0$ or $B = 0$

Transpose matrix

- The transpose matrix of A , denoted by A^T is obtained by interchanging the rows and columns of A.

<u>Example 7</u> Given that $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then, $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

- Properties of Transpose Matrix, A^T

$$(A+B)^T = A^T + B^T$$

$$(A^T)^T = A$$

$$(kA)^T = kA^T \quad , \quad (k \text{ is a scalar})$$

$$(AB)^T = B^T A^T$$

Square matrix

- A matrix with the **same number of rows and columns** is called a square matrix.

Example 8 The following matrix is a square matrix of order 3:

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & -1 \\ 5 & 3 & 2 \end{bmatrix}$$

Special types of Square Matrices

- Identity Matrix
  - If the main diagonal of a square matrix is filled with 1's and 0 's elsewhere, it is called an identity matrix, denoted by I.

Example 9

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2nd order identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3rd order identity matrix

Special types of Square Matrices

- Identity Matrix
  - For any square matrix A, $AI = IA = A$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Special types of Square Matrices

- Diagonal matrix
  - A diagonal matrix is a square matrix with all its non-diagonal elements being zero.

    Example 10

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Special types of Square Matrices

- Upper triangular and lower triangular matrix
  - An **upper triangular** matrix is a square matrix with all its elements **below the main diagonal being zero**. A **lower triangular** matrix is a square matrix with all its elements **above the main diagonal being zero**.

Example 11

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

Upper Triangular matrix　　Lower Triangular matrix

Special types of Square Matrices

- Symmetric matrix : $A^T = A$.

<u>Example 12a</u>

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 5 & -7 \\ 5 & -7 & 3 \end{bmatrix} \quad \mathrm{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 5 & -7 \\ 5 & -7 & 3 \end{bmatrix}$$

- Skewed-symmetric matrix: $A^T = -A$.

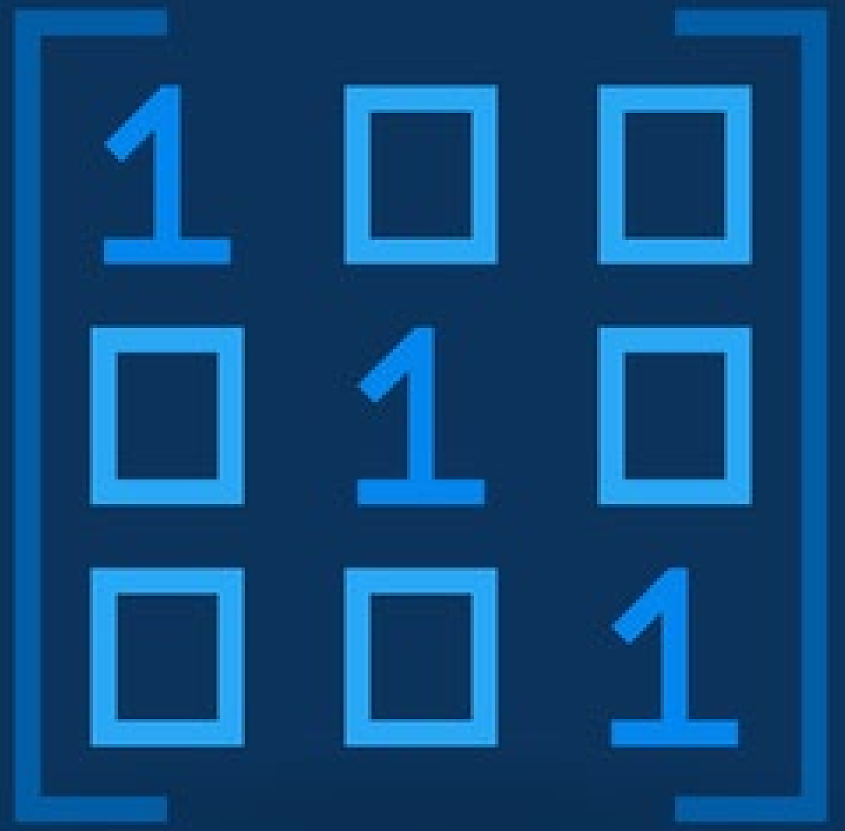
<u>Example 12b</u>

$$B = \begin{bmatrix} 0 & -5 & 4 \\ 5 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix} \quad \mathrm{B}^{\mathrm{T}} = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix}$$

- Note that all the main diagonal elements in skew-symmetric matrix are zero.

Matrices

PART II: DETERMINANTS & SOLVING SIMULTANEOUS EQUATION



Learning Outcomes

Learning Outcomes

At the end of this lesson, you will be able to:

1. compute determinant of a matrix
2. use inverse matrix to solve simultaneous equations
3. use determinants to solve simultaneous equations

Solving simultaneous using matrix inversion

- Inverse Matrix
  - A square matrix A is said to be invertible if there exists a matrix B with the property that

$$AB = BA = I$$

  - If A is invertible, then B is called the inverse of A and is denoted by A^{-1}.
  - For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is given by $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Solving simultaneous using matrix inversion

- Inverse Matrix

Example 13 Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find its inverse matrix.

$$A^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

Note that $AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Solving simultaneous using matrix inversion

Example 14 Use inverse matrix to solve the following equations

$$2x + 3y = 5$$
$$3x + 5y = 9$$

- The above equations can be rewritten as

$$AX = K \quad \text{where } A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \; X = \begin{bmatrix} x \\ y \end{bmatrix} \;\; K = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Since $A^{-1}A = I$,

$$AX = K$$
$$A^{-1}(AX) = A^{-1}K$$
$$X = A^{-1}K$$

Solving simultaneous using matrix inversion

$X = A^{-1}K$

$A^{-1} = \frac{1}{10-9}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$

$X = A^{-1}K = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$x = -2, y = 3$

Determinant

- We can also use Determinants to solve simultaneous equations.
- The determinant of n x n square matrix A is denoted by det(A) or |A| and is known as the determinant of order *n*.
- Determinant of order 2 is given by: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Determinant

- Minor of a Determinant
  - For a determinant | A| of order n, we can form a new determinant of order (n-1) from the remaining elements, after we delete the *i*th row and the *j*th column. The new determinant if called minor of element a_{ij}

<u>Example 15</u>

Let $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ be a determinant of order 3, then the minor of a_{12} is $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Determinant

- Cofactor a Determinant
  - If we multiply the minor by its sign according to the positional pattern, the result so obtained is called the cofactor. For a determinant of order 3, the pattern is as shown on right:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

<u>Example 16</u>

Given $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ the cofactor of a_{12}, denoted by written as A_{12} = - $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Determinant

- Expansion and evaluation of determinants
  - We can evaluate a determinant of any order by expanding it along any one row or any one column. For example, by expanding along row 1, a determinant of order 3 is given by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant

Example 17

(1) Evaluate $\begin{vmatrix} 0 & 3 \\ 5 & 2 \end{vmatrix} = (0)2 - 5(3) = -15$

(2) Evaluate $\begin{vmatrix} 2x+5 & 4-3x \\ 3-4x & 2+3x \end{vmatrix} = (2x+5)(2+3x) - (4-3x)(3-4x)$

$$= (6x^2 + 4x + 15x + 10) - (12x^2 - 9x - 16x + 12)$$

$$= -6x^2 + 44x - 2$$

(3) Evaluate $\begin{vmatrix} 1 & -1 & 3 \\ 0 & 2 & 5 \\ -2 & 1 & 6 \end{vmatrix} = 1\begin{vmatrix} 2 & 5 \\ 1 & 6 \end{vmatrix} - (-1)\begin{vmatrix} 0 & 5 \\ -2 & 6 \end{vmatrix} + 3\begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix}$

$$= (12 - 5) + 10 + 12$$

$$= 29$$

Determinant

- Sarrus' Rule for evaluation of 3 x 3 determinants
  - This is a simple rule for evaluating 3 x 3 determinants using the following steps:
    - Repeat the first two columns and multiply the diagonals with 3 elements.
    - Arrows pointing down give positive multiplication and arrows pointing up give negative multiplication

Determinant

Example18 Using Sarrus's Rule find the determinant $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 6 & -1 \\ 3 & 5 & 4 \end{vmatrix}$

Step 1: Replicate column 1 and 2 to the right

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & 6 & -1 \\ 3 & 5 & 4 \end{vmatrix}\begin{matrix} 1 & 2 \\ 2 & 6 \\ 3 & 5 \end{matrix}$$

Step 2 : Circle the diagonals as shown

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & 6 & -1 \\ 3 & 5 & 4 \end{vmatrix}\begin{matrix} 1 & 2 \\ 2 & 6 \\ 3 & 5 \end{matrix} \qquad \begin{vmatrix} 1 & 2 & 0 \\ 2 & 6 & -1 \\ 3 & 5 & 4 \end{vmatrix}\begin{matrix} 1 & 2 \\ 2 & 6 \\ 3 & 5 \end{matrix}$$

Determinant

Step 3 : Add the totals of the multiplication of the numbers in the circled diagonals for both the left and right separately

$$1(6)(4) + (2)(-1)(3)$$
$$+ (0)(2)(5)$$
$$= 24 - 6 + 0 = 18$$

(3)(6)(0)+(5)(-1)(1)+(4)(2)(2)
=0 – 5 + 16 =11

Step 4 : Minus the right total with the one on the left

Hence, determinant = 18-11 =7

Using Cramer's Rule to solve simultaneous equations

- 2 x2 simultaneous equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Solution is given by $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$ where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, $D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Using Cramer's Rule to solve simultaneous equations

Example 19 Solve the system of equations using Cramer's rule:

$$3x - 2y = 7$$
$$4x + 5y = 2$$

$$D = \begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 15 + 8 = 23$$

$$D_x = \begin{vmatrix} 7 & -2 \\ 2 & 5 \end{vmatrix} = 35 + 4 = 39$$

$$D_y = \begin{vmatrix} 3 & 7 \\ 4 & 2 \end{vmatrix} = 6 - 28 = -22$$

$$x = \frac{D_x}{D} = \frac{39}{23} \qquad y = \frac{D_y}{D} = \frac{-22}{23}$$

Using Cramer's Rule to solve simultaneous equations

- 3 x 3 simultaneous equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Solution is given by $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$ where

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Using Cramer's Rule to solve simultaneous equations

<u>Example 20</u>: Solve the system of equations using Cramer's rule:

$$x + 2y + 3z = 14$$
$$2x + y + 2z = 10$$
$$3x + 4y - 3z = 2$$

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 4 & -3 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 4 & -3 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & -3 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = -11 + 24 + 15 = 28$$

$$D_x = \begin{vmatrix} 14 & 2 & 3 \\ 10 & 1 & 2 \\ 2 & 4 & -3 \end{vmatrix} = 14\begin{vmatrix} 1 & 2 \\ 4 & -3 \end{vmatrix} - 2\begin{vmatrix} 10 & 2 \\ 2 & -3 \end{vmatrix} + 3\begin{vmatrix} 10 & 1 \\ 2 & 4 \end{vmatrix} = -154 + 68 + 114 = 28$$

Using Cramer's Rule to solve simultaneous equations

$$D_y = \begin{vmatrix} 1 & 14 & 3 \\ 2 & 10 & 2 \\ 3 & 2 & -3 \end{vmatrix} = 1\begin{vmatrix} 10 & 2 \\ 2 & -3 \end{vmatrix} - 14\begin{vmatrix} 2 & 2 \\ 3 & -3 \end{vmatrix} + 3\begin{vmatrix} 2 & 10 \\ 3 & 2 \end{vmatrix} = -34 + 168 - 78 = 56$$

$$D_z = \begin{vmatrix} 1 & 2 & 14 \\ 2 & 1 & 10 \\ 3 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 10 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 10 \\ 3 & 2 \end{vmatrix} + 14\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = -38 + 52 + 70 = 84$$

$$x = \frac{D_x}{D} = \frac{28}{28} = 1$$

$$y = \frac{D_y}{D} = \frac{56}{28} = 2$$

$$z = \frac{D_z}{D} = \frac{84}{28} = 3$$

End of Lesson