Probability Concepts

PART I

- BASIC PROBABILITY CONCEPTS
- RULES FOR CALCULATING PROBABILITIES

Learning Outcomes

Learning Outcomes

At the end of this lesson, you will be able to:

1. Describe a probability event by listing all the possible outcomes in that event
2. Calculate probabilities using probability concepts such as rules of addition and multiplication
3. Apply the number of outcomes using principles of counting.

Introduction to Probability Concepts

- You work for Premium Chocolates Ltd. Your customers have complained that the boxes of chocolate they purchased have variations in quantity of chocolate inside.
- A check of 4000 boxes of chocolates packaged in past month is shown below:

Weight	Number of boxes	Probability of occurrence
Underweight	100	0.025
Satisfactory	3600	0.900
Overweight	300	0.075

- What is the probability of a box being overweight or underweight?

- This case study requires you to apply the Rules of Addition for computing probabilities.
- In this topic, you will apply concepts of probability to solve real life problems.

Terms used in Probability

Event	Each possible outcome of a variable. A simple event is described by a single characteristics
Joint Event	An event with 2 or more characteristics
Complement	A complement of event A includes all events which are not part of A
Sample Space	Collection of all possible events

Terms used in Probability

Example 1: An unbiased coin is tossed three times, list out the sample space and the event in which there is exactly one head.

Sample space = {HHH, THH, HTH, HHT, THT, TTH, HTT, TTT}

Event (exactly 1 head) = {THT, TTH, HTT}

Terms used in Probability

Example 2: Super Electronics Ltd presented a survey of 1000 households on their purchasing decision of large screen TV. The results is presented in a Contingency Table* below:

Plan to purchase	Actually Purchased		Total
	Yes	No	
Yes	200	50	250
No	100	650	750
Total	300	700	1000

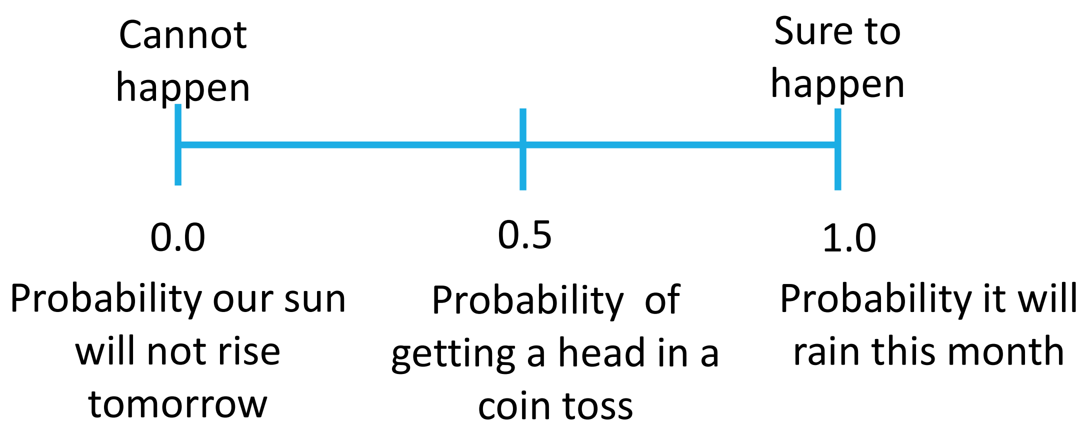
- ✓ **Sample space** comprises 1000 respondents
- ✓ Example of **Simple events** are "Plan to purchase", "Did not plan to purchase", "Actually Purchased", "Actually did not purchased"
- ✓ Example of a **complement** is, if event is "Plan to purchase", then its complement is "Did not plan to purchase"
- ✓ Example of **Joint Event** is "Plan to purchase" and "Actually purchased"

*Contingency Table is used to classify sample observations based on 2 or more identifiable characteristics

Probability Concepts

Probability is a measure of how likely an event will happen in an experiment.

- A probability is often expressed as a decimal (e.g. 0.70), fraction (e.g. ¾) or percentage (e.g. 75%)



Probability Concepts

- Probability of an Event E is between 0 and 1, inclusive

$$0 \leq P(E) \leq 1$$

- Sum of all Probabilities with simple events for an experiment is always 1.

$$\text{Sum of } P(E) = P(E1) + P(E2) \ldots. = 1$$

Probability Concepts

In an experiment, if each outcome in the sample space is equally likely to happen, for an event E:

$$P(E) = \frac{\text{Number of outcomes in Event E}}{\text{Number of outcomes in sample space}}$$

Probability Concepts

Example 3: An unbiased coin is tossed three times, list out the sample space and the event in which there is exactly one head. What is probability of exactly 1 head?

Sample space = {HHH, THH, HTH, HHT, THT, TTH, HTT, TTT}

Event (exactly 1 head) = {THT, TTH, HTT}

$$P(E) = \frac{Number\ outcomes\ in\ Event\ E}{Number\ of\ outcomes\ in\ sample\ space}$$

$$P(\text{Exactly 1 head}) = \frac{3}{8}$$

Probability Concepts

Example 4: A dice is rolled. Find the probability of each event:

a) Rolling a 4

b) Rolling a 7

c) Rolling a number less than 5

a) P(Rolling a 4) = $\frac{1}{6}$

b) P(Rolling a 7) = 0

c) P(Rolling a number less than 5) = $\frac{4}{6} = \frac{2}{3}$

Rules for calculating Probabilities

General Rule of Addition

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- "A and B" (notation $A \cap B$) is the intersection of A and B. It is set of outcomes that is common to **both** *A* **and** *B*.
- "A or B" (notation $A \cup B$) is the union of A and B. It is set of outcomes that is either *A* ***or*** *B*.

General Rule of Addition

Example 5: A sample of 200 tourists shows 120 went to the Zoo, 100 went to Sentosa and 60 visited both. What is the probability they visited either of attractions?

P(Zoo) = 120/200 = 0.60

P(Sentosa) = 100/200 = 0.50

P(Zoo and Sentosa) = 60/200 = 0.30

P(Zoo or Sentosa) = P(Zoo) + P(Sentosa) – P(Zoo and Sentosa)

= 0.6 + 0.50 - 0.30

= 0.80

General Rule of Addition

Example 6: In a hospital, there are 8 nurses and 5 pharmacists. 7 nurses and 3 pharmacists are female. If a person is selected, find the probability that the person is a nurse or a male.

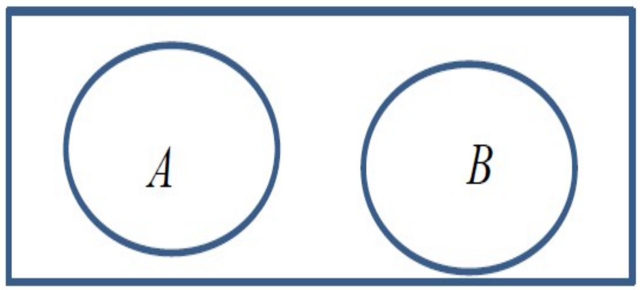
Staff	Female	Male	Total
Nurse	7	1	8
Pharmacist	3	2	5
Total	10	3	13

P (Nurse or Male) = P(Nurse) + P (Male) – P (Nurse and Male)

$$= \frac{8}{13} + \frac{3}{13} - \frac{1}{13}$$

$$= \frac{10}{13}$$

Mutually exclusive events



- Two events *A* and *B* are **mutually exclusive** if they share no common outcome. i.e. **$P(A \cap B) = 0$**

> General Rule of Addition (for mutually exclusive event)
> $P(A \cup B) = P(A) + P(B)$

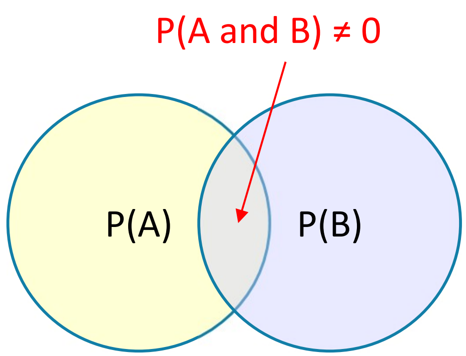
- Example: In tossing a coin, you either get a Head or Tail but not both

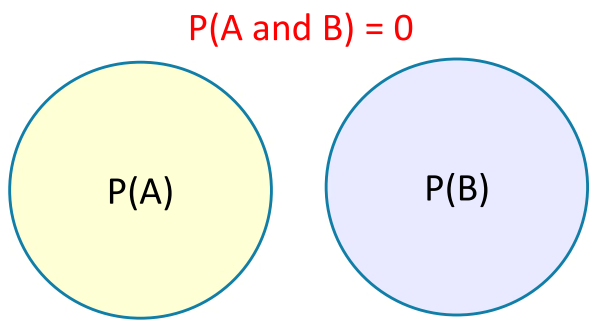
Addition Rules of Probability: Summary

- In general,

If 2 events are NOT **mutually exclusive**

P(A or B) = P(A) + P(B) – P(A and B)





If 2 events are **mutually exclusive**

P(A or B) = P(A) + P(B)   i.e P(A and B) = 0

Mutually exclusive events

Example 7: A check of 4000 boxes of chocolates packaged in past month is shown. What the probability of a box being overweight or underweight?

Weight	Number of boxes	Probability of occurrence
Underweight	100	0.025
Satisfactory	3600	0.900
Overweight	300	0.075

P(box overweight) = 0.075
P(box underweight) = 0.025
P(box overweight or underweight) = P(box overweight) + P(box underweight)
= 0.075 + 0.025
= 0.1

Independent Events

- Two events *A* and *B* are **independent** if the probability of one event occurring does not affect the probability of the other event occurring.

- Example: In tossing a coin and rolling dice, Events A (tossing a coin)and B (rolling a dice) are said to be independent since outcome of A does NOT affect probability of B

Independent Events

- Multiplication rules can be used to find probability of two or more events that occur in sequence.

Multiplication Rule 1 (for Independent Events A and B)

$$P(A \cap B) = P(A) \times P(B)$$

Example: A coin is tossed and a dice is rolled. Find the probability of getting a head on the coin and a 4 on the dice.

P(getting head and 4) = P(getting head) x P(getting 4)

$$= \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

Independent Events

Example 8: A survey by the Raffles Club revealed 60% of its members made airline reservations last year. Two members, R1 and R2 are selected at random. What is the probability both made airline reservations last year?

$P(R_1 \text{ and } R_2) = P(R_1) \times P(R_2) = (0.60) \times (0.60) = 0.36$

Example 9: A box contains several balls : 3 red, 2 blue and 5 white. A ball is selected and put back into the box. A second ball is then selected. Find the probability in each case:

a) Selecting 2 blue balls

b) Selecting 1 blue ball and then 1 white ball

a) P(blue and blue) = P(blue) x P(blue) $= \left(\frac{2}{10}\right)\left(\frac{2}{10}\right) = \frac{1}{25}$

b) P(blue and white) = P(blue) x P(white) $= \left(\frac{2}{10}\right)\left(\frac{5}{10}\right) = \frac{1}{10}$

Dependent Events

- Two events *A* and *B* are **dependent** if the probability of first event A affects probability of second event B

Multiplication Rule 2 (for Dependent Events A and B)

$$P(A \cap B) = P(A) \times P(B|A)$$

means probability of event B given A has already occurred

- Example of a Dependent Event is drawing two cards from a deck of 52 poker cards without replacement. Let B be the event that the 2nd card is an ace of heart, A be the event that the 1st card drawn is an ace.

Dependent Events

Example 10: A golfer has 12 golf shirts in his closet. Suppose 9 of these shirts are white and the others are blue. He gets dressed in the dark, so he just grabs a shirt and puts in on. He plays golf two days in a row and does not return the shirts to the closet. What is the probability both shirts are white?

$$P(W_1 \cap W_2) = P(W_1) \text{ x } P(W_2|W_1) = \left(\frac{9}{12}\right)\left(\frac{8}{11}\right) = 0.55$$

Dependent Events

- Multiplication Rule 2 (for Dependent Events A and B) can be rearranged as follows:

$$P(A \cap B) = P(A) \times P(B|A)$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Dependent Events

- Example 11: Two ordinary dice are thrown. Let A be the event that numbers shown on both dice are equal, B be the event that the total sum of the two numbers is 8. Calculate P(B|A).

A = { (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) }

B = { (2, 6), (6, 2), (3, 5), (5, 3), (4, 4) }

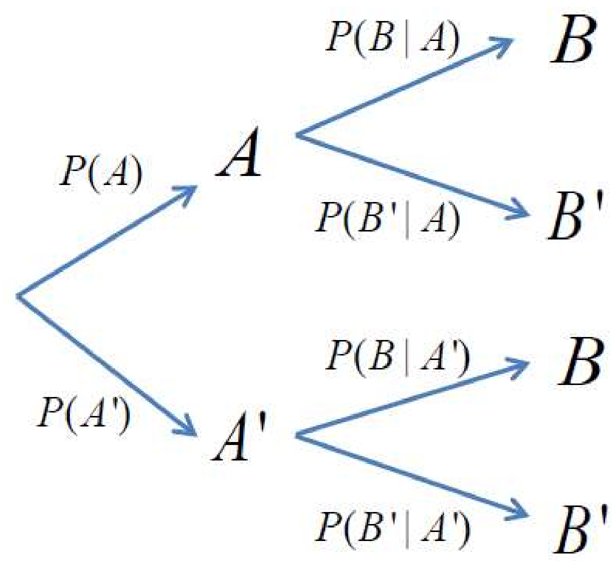
$A \cap B = \{(4, 4)\}$

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

$$= \frac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}$$

Tree diagram and multiplication rule

- A tree diagram is useful to calculate probabilities involving experiments happening in stages with multiple events.
- Using the tree diagram, we can calculate probabilities such as $P(A \cap B)=P(A)P(B|A)$



Tree diagram and multiplication rule

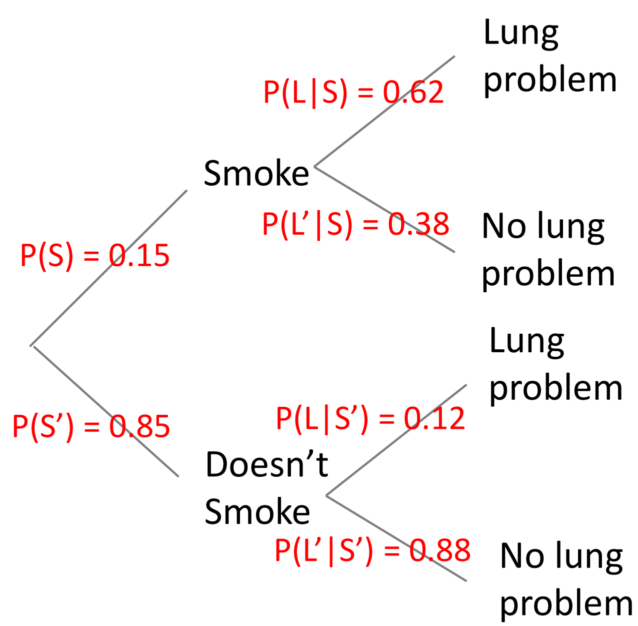
Example 12: 15% of Singaporean adult smokes cigarettes. It is found that 62% of the smokers and 12% of non-smokers develop lung problem by age 60.

a) Find the probability that a randomly selected 60-year adult has lung problem.
b) Given that a randomly selected 60-year adult has lung problem, what is the probability that he smokes?

Tree diagram and multiplication rule

S = Adults who smokes
L = Adults with lung problem

a) $P(L) = P(S) \times P(L \mid S) + P(S') \times P(L \mid S')$
$= 0.15 \times 0.62 \quad + 0.85 \times 0.12$
$= 0.195$
P(has lung problem) = 0.195



b) $P\ (S \text{ and } L) = P(S) \times P(L \mid S)$
$= 0.15 \times 0.62 = 0.093$

$$P(S \mid L) = \frac{P(S \text{ and } L)}{P(L)}$$
$$= \frac{0.093}{0.195} = 0.477$$

P(smoke given adult has lung problem) = 0.477

Probability Concepts

PART II

- PRINCIPLES OF COUNTING

Multiplication Principle

- In a counting event whereby it can be broken down into n stages, and if there is m_1 ways for step 1, m_2 ways for step 2, … m_n ways for step n, then by multiplication principle, there are m_1 x m_2 …… m_n ways

Multiplication Principle : Total number of ways = $m_1 \times m_2 \ldots\ldots \times m_n$

Multiplication Principle

Example 13: A female student has the following in her wardrobe: 4 blouses, 7 skirts, 6 pairs of shoes, 3 sets of jewellery and 5 handbags.

Assuming that all the items can be matched in terms of colours and styles, how many possible ways could she dress herself up?

Possible ways to dress up = 4 x 7 x 6 x 3 x 5 = 2520 ways

Multiplication Principle

Example 14: A PIN number consists of 4 letters and 2 digits. How many PINS are possible if

(a) Repetition of the letters & digits are allowed

(b) Repetition of letters and digits are NOT allowed

26	26	26	26	10	10
L	L	L	L	D	D

(a) $(26^4)(10^2) = 45,697,600$

26	25	24	23	10	9
L	L	L	L	D	D

(b) $(26)(25)(24)(23)(10)(9) = 32,292,000$

Addition Principle

- In a counting event whereby it can be broken down into *n* **non – overlapping** cases, and there are m_1 for case 1, and m_2 ways for case 2… m_n ways for case n, then by **addition principle**, there are a total $m_1 + m_2 \dots m_n$ ways

Addition Principle : The total number of ways $= m_1 + m_2 \ \ldots\ldots + m_n$

Addition Principle

Example 16: You intend to buy a laptop from three brands. Acer has 3 models, Lenovo has 5 models and Dell has 2 models. How many choices can you have to buy your laptop?

Total number of choices of laptop = 3 + 5 + 2 = <u>10 choices</u>

Factorial notation

The factorial notation uses the exclamation point

Example : $5! = 5 \times 4 \times 3 \times 2 \times 1$

$3! = 3 \times 2 \times 1$

Note : $0! = 1$

Permutation

- A permutation is an arrangement of N objects in a **specific order.**
- For example, for the combination lock, the number sequence 1-2-3 has a different order from, say, 2-1-3.

Permutations

Example 15: How many ways can the word TOY be arranged?

TOY, TYO, OTY, OYT, YTO, YOT
Number of ways to be arranged in 3! = 3 x 2 x 1 = <u>6 ways</u>

Example 16: James has shortlisted 5 games he intends to buy. He decides to rank each game by its popularity. How many ways can he rank?

Number of ways of ranking = 5! = 5 x 4 x 3 x 2 x 1 = <u>120 ways</u>

Permutations

In general, the arrangements of n objects in a specific order using r objects is written as ${}_nP_r$. The formula is given by

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Permutations

Example 17: A biologist has decided to use colours to label the collection of cell specimens in the laboratory. If he has 5 colours (red, blue, green, yellow and pink) to choose from, how many 3-colour codes can he make with no repetitions of each colour selected?

$${}_{5}P_{3} = \frac{5!}{(5-3)!} = 5 \text{x} 4 \text{x} 3 = 60$$

Example 18: A class of 10 students consist of 6 men and 4 women.

a) How many ways can all of them arrange themselves in a row?

b) How many ways can we arrange 6 of them in a row?

a) $${}_{10}P_{10} = \frac{10!}{(10-10)!} = 10! = 3{,}628{,}800$$

b) $${}_{10}P_{6} = \frac{10!}{(10-6)!} = \frac{10\text{x}9\text{x}8\text{x}7\text{x}6\text{x}5\text{x}4\text{x}3\text{x}2\text{x}1}{(4\text{x}3\text{x}2\quad)} = 10\text{x}9\text{x}8\text{x}7\text{x}6\text{x}5 = 151{,}200$$

Permutations where not all objects are *distinct*

- If n objects **not all distinct, and** k_1 objects identical of type 1, k_2 objects identical of type 2,…, k_m objects identical of type m, then

$$\frac{n!}{k_1!\,k_2!\,\ldots.\,k_m!}$$

Example 19: How many ways can ENGINEERING be arranged?

$$\frac{11!}{3!\,2!\,2!\,3!} = 277{,}200 \text{ ways}$$

Combination

- In permutation, arrangement of N objects in a specific order matters.
- In combination, **selection** of objects' positions does not matter. Order of arrangement is NOT important
- If there n distinct objects, and r to be selected, then

$$_nC_r = \frac{n!}{(n-r)!r!}$$

- Examples of combination:
  a) Forming a team of 5 people out of 10 people,
  b) Choosing 5 balls from a box full of balls in various colours.

Combination

Example 20: 3 different species of orchid are to be selected from 20 unique species for cross-breeding. How many possible selections can be made?

$$ {}_{20}C_3 = \frac{20!}{(20-3)!3!} = \frac{20!}{17!3!} = \frac{20\text{x}19\text{x}18}{3\text{x}2\text{x}1} = 1{,}140 $$

Example 21: The manager of a marketing department wants to form a 4 person committee from the 15 employees in the department. In how many ways can the manager form this committee?

$$ {}_{15}C_4 = \frac{15!}{(15-4)!4!} = \frac{15!}{11!4!} = \frac{15\text{x}14\text{x}13\text{x}12}{4\text{x}3\text{x}2\text{x}1} = 1{,}365 $$