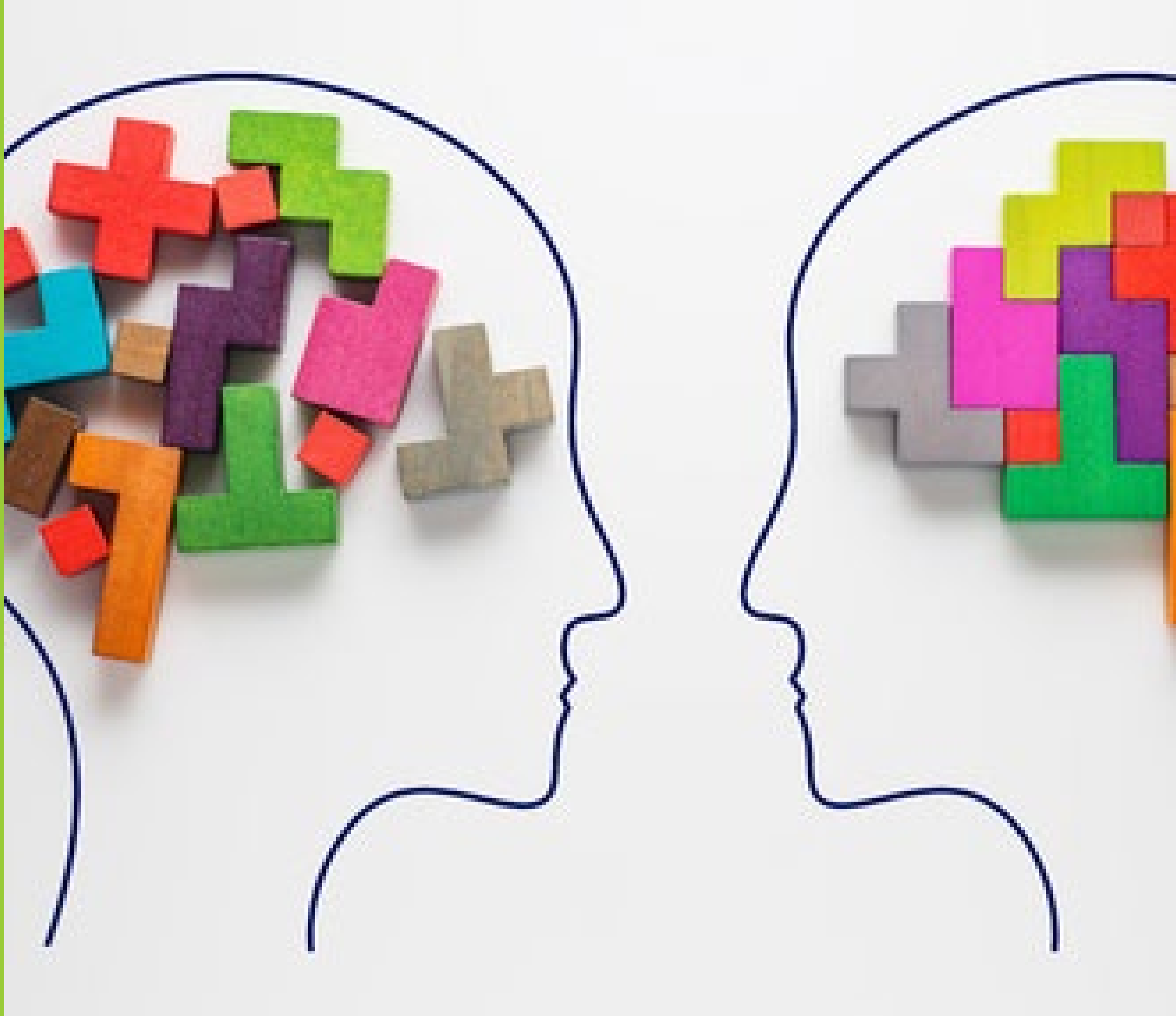
Logic



Learning Outcomes

Learning Outcomes

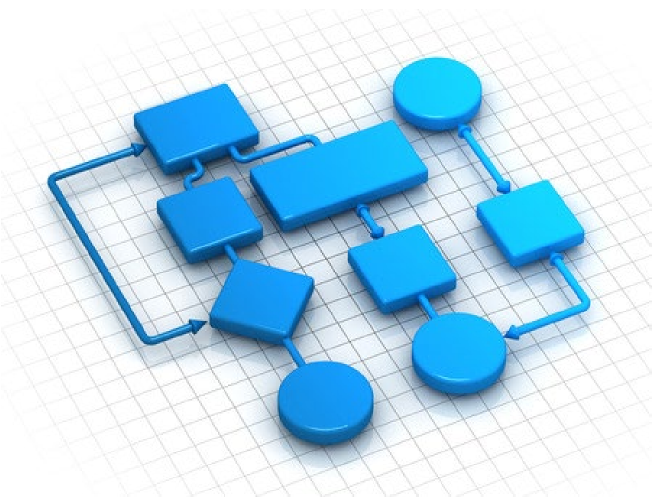
At the end of this lesson, you will be able to:

1. explain and use the different types of Logic and its terminology.

2. deduce the truth table for different logic combinations

3. apply the laws of the algebra of propositions to prove identities.

Introduction to Logic



- Logic has numerous applications in computer science.
- The rules of logic are used in the design of computer circuits, the construction of computer programmes, the verification of the correctness of programme, etc.
- In Logic, a statement or proposition is a declarative sentence that is either true or false, but not both.
- In this topic, we will use the lowercase letters $p, q, r, \ldots.$ to denote statements.

Introduction to Logic

- In Logic, a statement or proposition is a **declarative sentence that is either true or false**, but not both.

Example 1

Which of the following are statements?

a) The sun rises in the east.
b) $1 + 2 = 4$
c) $5 - x = 3$
d) Do you speak French?
e) Do your homework now.
f) The temperature on the Sun is 1000 degrees Celsius.
g) It will rain tomorrow.

a) is a statement, and it happens to be true.
b) is also a statement, but it happens to be false.
c) is not a statement, because it may be true or false depending on the value of *x*.
d) is not a statement because it is a question.
e) is not a statement because it is an order.
f) is a statement and it happens to be true.
g) is a statement and it could be true.

Introduction to Logic

- Simple Statement
  - In the previous example, statements such as those shown below are examples of Simple Statements, as they contain only one piece of information
    - ✓ The sun rises in the east
    - ✓ $1 + 2 = 4$
    - ✓ **The temperature on the Sun is 1000 degrees Celsius.**

- Compound Statement
  - A compound statement is formed by combining two or more simple statements, called components

Example 2 The following is an example of a compound statement:

Beijing is the capital of China and $1 + 2 = 5$

Truth Tables

- Truth tables are used to list the different possible outcomes. For example, a Truth Table with a **single** proposition, p, is shown:

P	Outcome
T	
F	

- A Truth Table with a **two** propositions, p and q, is shown:

p	q	Outcome
T	T	
T	F	
F	T	
F	F	

Truth Tables

- A Truth Table with a **three** propositions, p, q and r, is shown:

p	q	r	Outcome
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Truth Tables

- Simple statements can be combined to form new statements (called compound statements) using words such as **not**, **and**, **or**, **if.. then**, **if and only if** etc.
- We can represent the truth values of these new statements using Truth Tables.
- **Negation**
  - If p is a statement, the negation of p is the statement **not** $\boldsymbol{p}$, denoted by $\sim p$ (read "not p").
  - ~p is the statement "it is not the case that p"

Truth Tables

- **Negation**

<u>Example 3</u>

(1) Write the negation of the statement “Today is Friday”.

The negation is “It is not the case that today is Friday”.

(2) p : All cats can fly.

Then $\sim p$: It is not the case that all cats can fly.

P	~p
T	F
F	T

Truth Table for Negation

Truth Tables

- Conjunction
  - Any two statements p and q can be combined by the word '**and**' to form a compound statement called the **conjunction of the original statements**.
  - The compound statement is denoted by $p \wedge q$ (read as "p **and** q").

Example 4

Let p and q be the following statements:

p : John eats fries.

q : Mary drinks coke.

Then the conjunction of p and q is $p \wedge q$: John eats fries **and** Mary drinks coke

Truth Tables

- **Conjunction**

  A Truth Table for $p \wedge q$ is

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Outcome is True only if both $\boldsymbol{p}$ and $\boldsymbol{q}$ is True. Otherwise it is False.

Truth Tables

- Disjunction
  - Any two statements p and q can be combined by the word ‘**or**’ to form a compound statement called the **disjunction of the original statements**.
  - The compound statement is denoted by $p \vee q$ (read as “p **or** q”).

<u>Example 5</u>

Let p and q be the following statements:

p : I passed my Mathematics exam

q : I passed my Chinese exam.

Then the disjunction of p and q is $p \vee q$:

I passed my Mathematics exam or I passed my Chinese exam.

Truth Tables

- **Disjunction**

  A Truth Table for $p \vee q$ is

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Outcome is True if either p or q is True. Otherwise it is False.

Truth Tables

<u>Example 6(a)</u>

Construct the truth table for the proposition $\sim(p \wedge \sim q)$.

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Truth Tables

<u>Example 6(b)</u>

Construct the truth table for the proposition$(p \vee q) \wedge [(p \vee r) \wedge \sim r]$.

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p	q	r	$p \vee r$	$\sim r$	$(p \vee r) \wedge \sim r$	$p \vee q$	$(p \vee q) \wedge [(p \vee r) \wedge \sim r]$
T	T	T	T	F	F	T	F
T	T	F	T	T	T	T	T
T	F	T	T	F	F	T	F
T	F	F	T	T	T	T	T
F	T	T	T	F	F	T	F
F	T	F	F	T	F	T	F
F	F	T	T	F	F	F	F
F	F	F	F	T	F	F	F

Truth Tables

- ## Tautologies
  - A proposition that is **always true**, no matter what the truth values of the variables . The last column of the Truth Table is always true.

<u>Example 7 (a):</u>

Use the truth table to show that $p \lor \sim p$ is a tautology

p	~p	$p \lor \sim p$
T	F	T
F	T	T

Hence, $p \lor \sim p$ is a tautology.

Truth Tables

- **Contradiction**
  - A proposition that is **always false**, no matter what the truth values of the variables . The last column of the Truth Table is always false.

<u>Example 7 (b):</u> Use the truth table to show that $p \wedge \sim p$ is a contradiction.

p	~p	$p \wedge \sim p$
T	F	F
F	T	F

Hence, $p \wedge \sim p$ is a contradiction.

Truth Tables

Example 8: Use the truth table to show that $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction.

p	q	$p \vee q$	$\sim(p \vee q)$	$p \wedge q$	$(p \wedge q) \wedge \sim(p \vee q)$
T	T	T	F	T	F
T	F	T	F	F	F
F	T	T	F	F	F
F	F	F	T	F	F

Truth Tables

- Conditional Statements
  - Many statements, called **Conditional** or an **Implication statement** (denoted by $p \to q$) are of the form "if p then q".

hypothesis or **antecedent** or the **sufficient condition**

conclusion or **consequent** or the **necessary condition**

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  - A Truth Table for $p \to q$ is

p	q	$\boldsymbol{p \to q}$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Tables

- Conditional Statements

Example 9:

Consider the following statements: p: Rafi has a driving license. q: Rafi is above 17 years old .
Then we have $p \rightarrow q$: If Rafi has a driving license, then Rafi is above 17.

p	q	$p \rightarrow q$	
T	T	T	T: If Rafi has a driving license, he is above 17 years
T	F	F	F: p (Rafi has driving license) is violated
F	T	T	T : so long as p is false
F	F	T	T : so long p is false

Rafi does not have a driving license

The implication is false if and only if the statement is violated.

Truth Tables

- Biconditional Statements
  - The statement of the form "$\boldsymbol{p}$ **if and only if** $\boldsymbol{q}$" is called **Bi-conditional or Equivalent statement** and denoted by $p \leftrightarrow q$.
  - A Truth Table for $p \leftrightarrow q$ is

p	q	$\boldsymbol{p \leftrightarrow q}$
T	T	T
T	F	F
F	T	F
F	F	T

Logical Equivalence: Algebra of Propositions

- Two propositions are said to be **logically equivalent**, or simply equivalent or equal, denoted by ≡ or ⇔, if they have the same truth table.

Example 10

Show using truth table that ~(p ∧ q) and ~p ∨ ~q are logically equivalent

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Since both ~(p ∧ q) and ~p∨~q have the same truth table, then these two propositions are said to be logically equivalent.

Laws of the Algebra of Propositions

1	Idempotent Laws	$p \vee p \equiv p$	$p \wedge p \equiv p$
2	Associative Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
3	Commutative Laws	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
4	Distributive Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
5	Identity Laws	$p \vee F \equiv p$ $p \vee T \equiv T$	$p \wedge T \equiv p$ $p \wedge F \equiv F$
6	Complement Laws	$p \vee \sim p \equiv T$ $\sim T \equiv F$	$p \wedge \sim p \equiv F$ $\sim F \equiv T$
7	Involution Law	$\sim (\sim p) \equiv p$	
8	De Morgan's Laws	$\sim (p \vee q) \equiv \sim p \wedge \sim q$	$\sim (p \wedge q) \equiv \sim p \vee \sim q$

Laws of the Algebra of Propositions

Example 11

(1) $p \wedge (p \vee \sim p)$

$\equiv p \wedge T$ Complement Laws

$\equiv p$ Identity Laws

(2) $\sim (p \vee q) \vee (\sim p \wedge q)$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$ De Morgan's Law

$\equiv \sim p \wedge (\sim q \vee q)$ Distributive Law

$\equiv \sim p \wedge T$ Complement Law

$\equiv \sim p$ Identity Law

End of Lesson