Normal Probability Distribution

Learning Outcomes

Learning Outcomes

At the end of this lesson, you will be able to:

1. Identify characteristics of a Normal distribution by using its characteristics such as the mean, standard deviation, median and mode.
2. Calculate probabilities, the mean and standard deviation of the Normal distribution using the conversion formula and the Z table.
3. Solve real-life business problems by applying concepts of the Normal distribution.

Introduction to Probability Distribution

- Your company has a social networking site that targets teenagers. To attract and retain visitors, as their web developer, you need to ensure that daily video content can be quickly downloaded and viewed in a user's browser.
- Past data indicate the mean download time is 7 seconds with a standard deviation of 2 seconds. The download times are distributed as a bell-shaped curve.
- How would you use this information to answer questions about download times of the video?

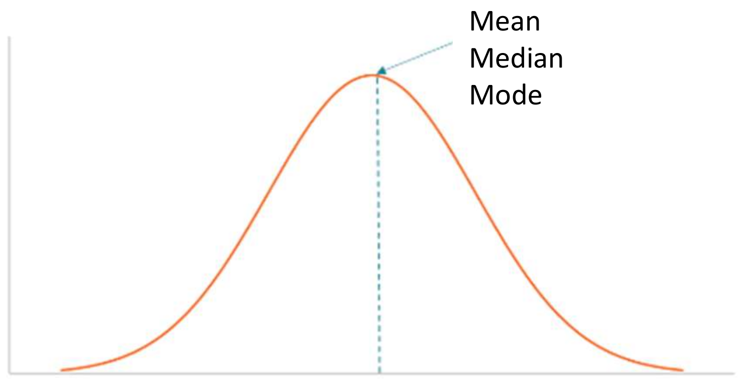
- This case study is an example of a normal probability distribution.
- In this topic, you will apply concepts of normal probability distribution to solve real life problems.

What is a Probability Distribution?

- A **probability distribution** is a listing of all the **possible outcomes** of an experiment and their corresponding **probabilities**.
- As seen in previous topic,
  - A Discrete Random Variable is one that takes on countable values.
  - A **Discrete Probability Distribution** is a table that shows each value of discrete random variable and its probability of occurrence. on Discrete Probability Distribution.
- In this lesson, we will look at **Continuous Probability Distribution**.

Continuous Probability Distribution

- A **Continuous Random Variable** is one that has infinitely many values.
- These values are measured on a continuous scale in such a way that there are no gaps or interruptions. Example: GPA score, weight or height.
- Therefore, we represent a **Continuous Probability Distribution** by a curve.
- The **Normal Distribution** (commonly called 'bell curve') is the most important and most widely used continuous probability distribution in statistics.



Normal Distribution

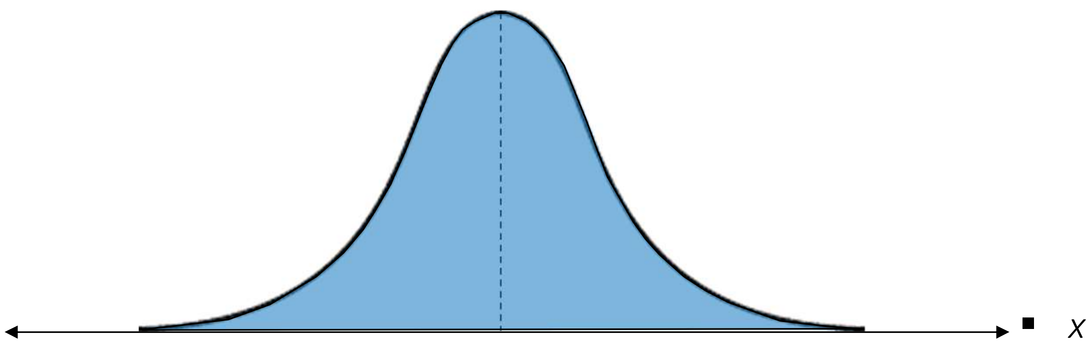
- Examples of Normal Distributions

Birthweight of
new born babies

Height of
12 year old girls

Blood pressure readings

What is a Normal Distribution?

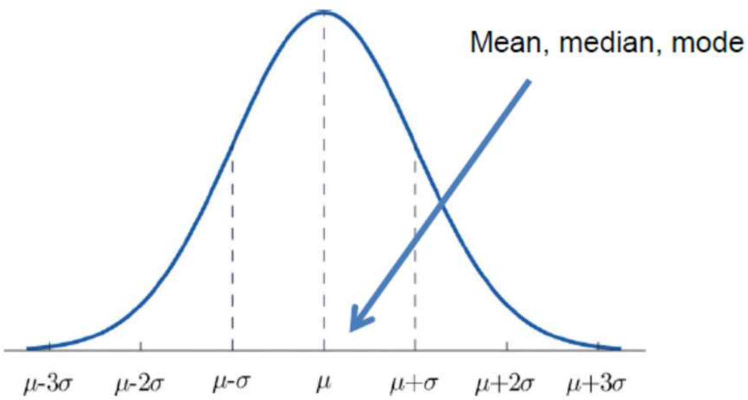


- A normal distribution is a continuous probability distribution for a continuous random variable, x. The graph of a normal distribution is called the normal curve.
- Normal distribution can be described using mean and variance

$$X \sim N(\mu, \sigma^2)$$

Properties of Normal Distribution Curve

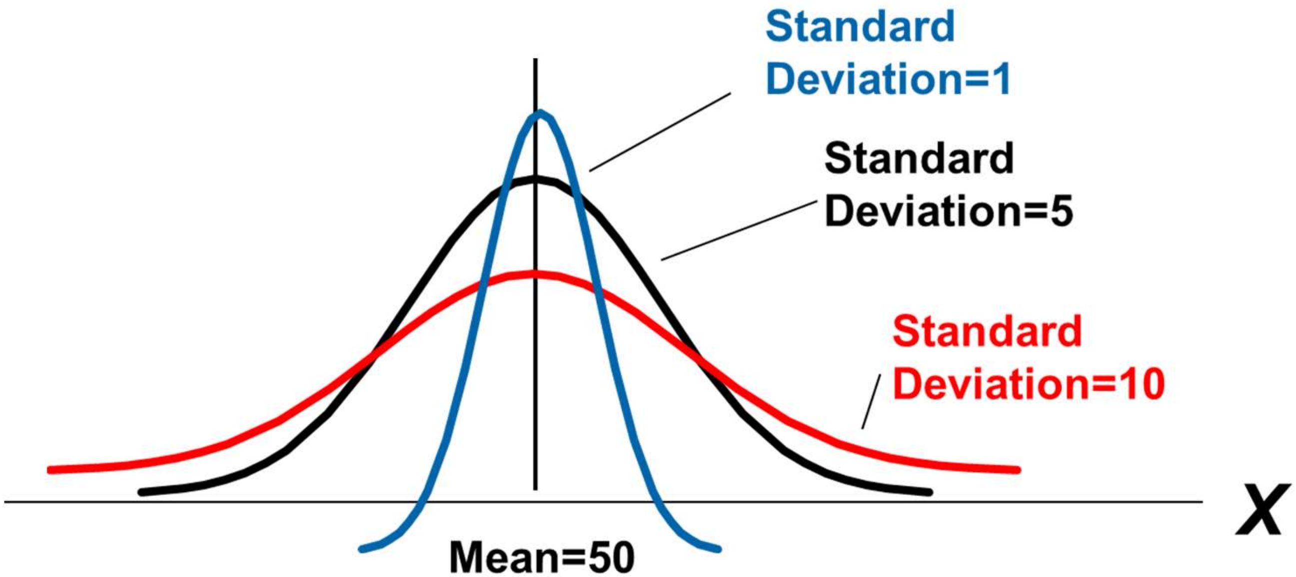
1. The mean, median, and mode are equal.
2. The normal curve is bell-shaped and symmetric about the mean.
3. The total area under the curve is equal to one.



4. Approximately 95% of distribution lies within 2 standard deviation of mean (2 σ rule)

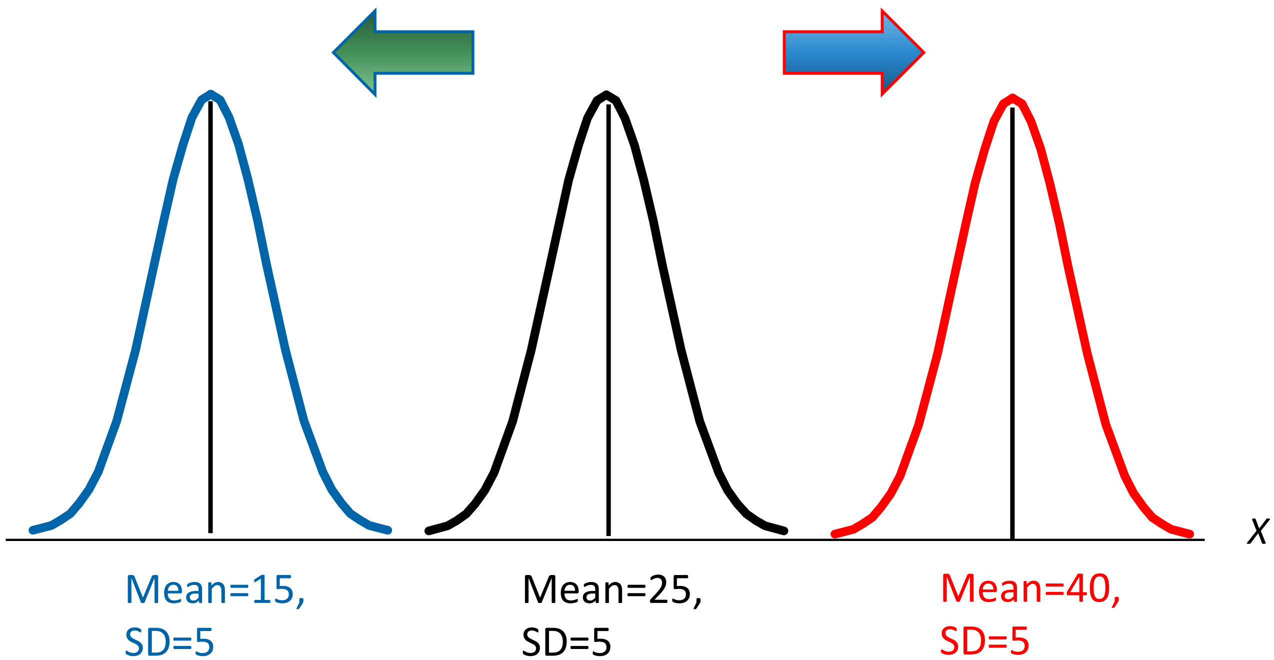
Normal Distribution

- The standard deviation describes the spread of the data.
- Effect of identical Means but **different Standard Deviations** is shown below:



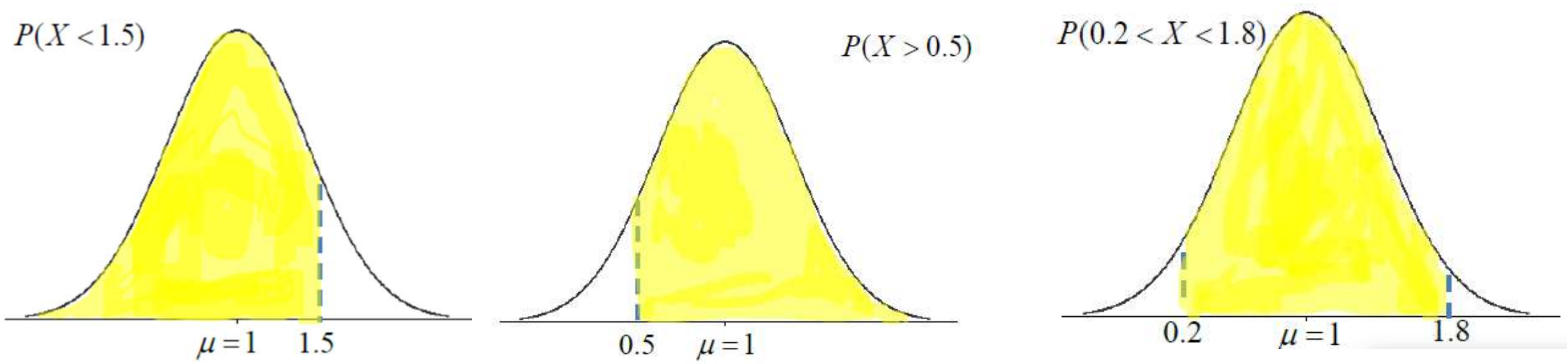
Normal Distribution

- Effect of identical Standard Deviation, but **different Means** is shown below:



Probability for normal distribution

- The probability of a normal distribution curve is interpreted as **area under the curve.**
- For example, for X ~ N(μ, σ^2) where $\mu = 1, \sigma^2 = 2$



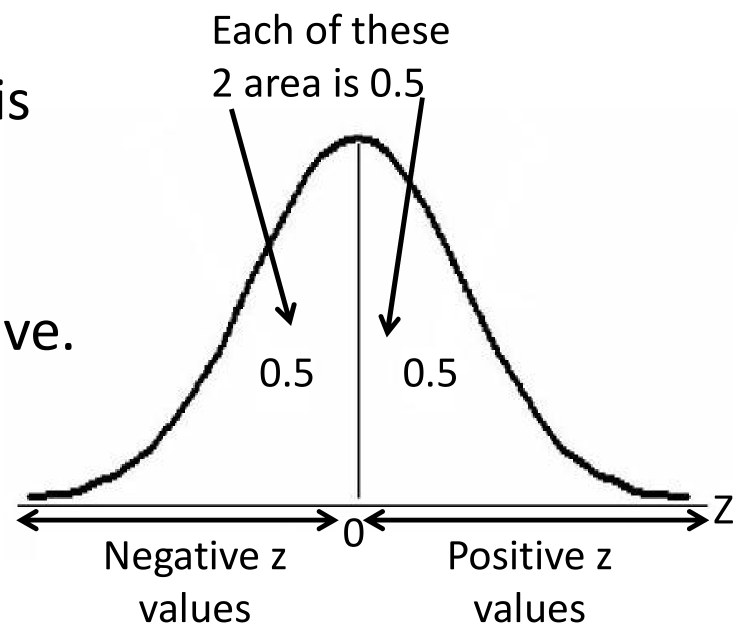
Standard Normal Distribution

- A normal random variable can have many different values of mean and variance.
- When the mean, μ= 0 and variance, σ^2 =1, we call it a **standard normal random variable, denoted as *Z.***
- Notation for Normal and Standard Normal Distribution

Normal Distribution	Standard Normal Distribution
X ~$N(\mu, \sigma^2)$	Z ~$N(0,1)$ since $\mu = 0$ and , σ^2 =1

Probability Values or Areas Under the Standard Normal Curve

- Since total area under the curve is 1.0, then by symmetry, the area on either side of the mean is 0.5.
- Although Z values can be negative, probability values or areas under the Z curve must be positive.



Probability Values or Areas Under the Standard Normal Curve

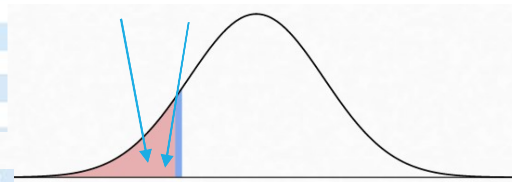
- Z-table lists the areas (probabilities) under the curve.

z	.09	.08	.07	.06	.05	.04
−3.4	.0002	.0003	.0003	.0003	.0003	.0003
−3.3	.0003	.0004	.0004	.0004	.0004	.0004
−3.2	.0005	.0005	.0005	.0006	.0006	.0006
−3.1	.0007	.0007	.0008	.0008	.0008	.0008
−3.0	.0010	.0010	.0011	.0011	.0011	.0012
−2.9	.0014	.0014	.0015	.0015	.0016	.0016
−2.8	.0019	.0020	.0021	.0021	.0022	.0023
−1.7	.0367	.0375	.0384	.0392	.0401	.0409
−1.6	.0455	.0465	.0475	.0485	.0495	.0505
−1.5	.0559	.0571	.0582	.0594	.0606	.0618

- Example 1: Find P(Z ≤ -1.55)

From Z-Table, P(Z ≤ -1.55) = <u>0.0606</u>

Area = 0.0606



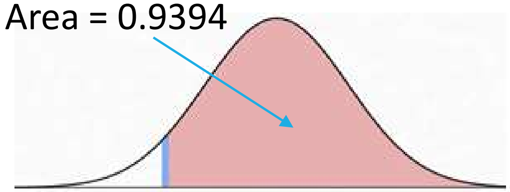
- Example 2: Find P(Z ≥ -1.55)

From previous example, P(Z ≤ -1.55) = 0.0606

Therefore, P(Z ≥ -1.55) = 1 – 0.0606

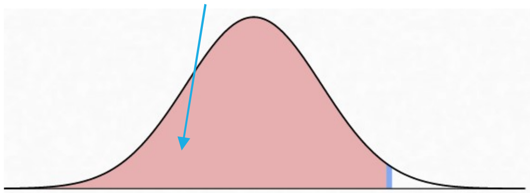
= <u>0.9394</u>

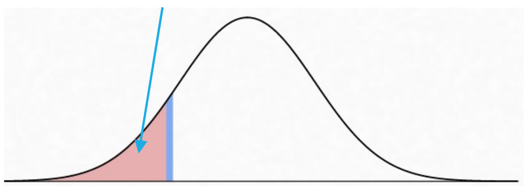
Area = 0.9394



Probability Values or Areas Under the Standard Normal Curve

- Example 3: Find P(-1.55 ≤ Z ≤ 2.00)





z	0.00	0.01	0.02	0.03	0.04
1.7	0.9554	0.9564	0.9573	0.9582	0.9591
1.8	0.9641	0.9649	0.9656	0.9664	0.9671
1.9	0.9713	0.9719	0.9726	0.9732	0.9738
2.0	0.9772	0.9778	0.9783	0.9788	0.9793
2.1	0.9821	0.9826	0.9830	0.9834	0.9838
2.2	0.9861	0.9864	0.9868	0.9871	0.9875

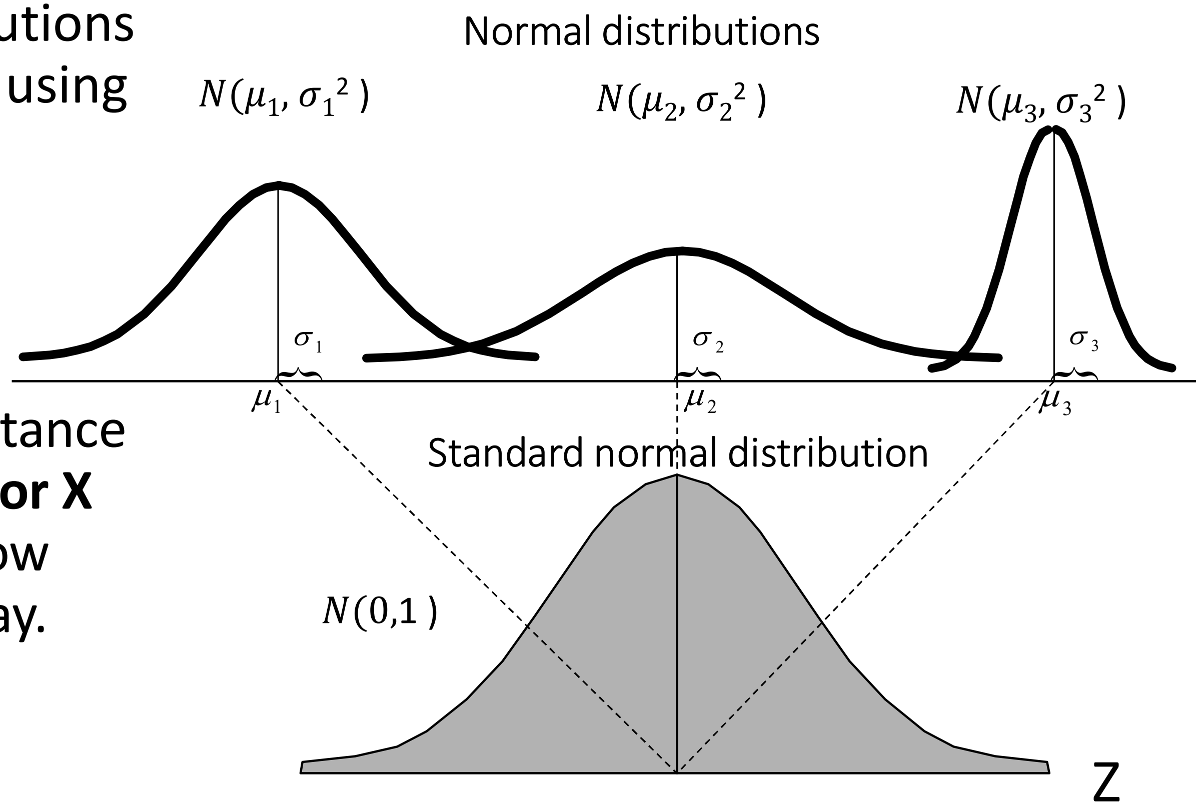
$$
\begin{aligned}
P(-1.55 \le Z \le 2.00) &= P(Z \le 2.00) - P(Z \le -1.55) \\
&= 0.9772 - 0.0606 \\
&= \underline{0.9166}
\end{aligned}
$$

Standard Normal Distribution

- We can convert all Normal Distributions to Standard Normal Distributions using the following equation:

$$z = \frac{\text{Value - Mean}}{\text{Standard deviation}} = \frac{x - \mu}{\sigma}.$$

- The Z-score is a measure of the distance between a **particular observation or X value and the mean** in terms of how many **standard deviations** it is away.

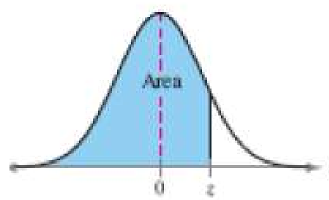


Standard Normal Distribution

Example 4: Given that X ~N(2,5), find P(X≤ 3).

Step 1 : Find the z score

$$z = \frac{x-\mu}{\sigma} = \frac{3-2}{\sqrt{5}} = 0.4472 = 0.45$$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190

Step 2: Express in the form P(z <= k)

P(z ≤ 0.45)

Step 3: Obtain the required probabilities' value from the standard normal table

Hence, P(X ≤ 3) = P(z ≤ 0.45) = <u>0.6736</u>

Standard Normal Distribution

Example 5: Given that $X \sim N(2,5)$, find $P(X > 3)$

$P(X > 3) = 1 - P(X \leq 3)$

$= 1 - P(z \leq 0.45)$ (from previous example)

$= 1 - 0.6736$

$= \underline{0.3263}$

Standard Normal Distribution

Example 6: Given the normally distributed variable *X* with mean 20 and standard deviation 4, find P(X > 28)

$X \sim N(20,4^2)$

Step 1 : Find the z score

$$z = \frac{x-\mu}{\sigma} = \frac{28-2}{4} = 2.0$$

z	.00	.01	.02	.03
1.5	.9332	.9345	.9357	.9370
1.6	.9452	.9463	.9474	.9484
1.7	.9554	.9564	.9573	.9582
1.8	.9641	.9649	.9656	.9664
1.9	.9713	.9719	.9726	.9732
2.0	.9772	.9778	.9783	.9788
2.1	.9821	.9826	.9830	.9834
2.2	.9861	.9864	.9868	.9871
0.8	.7881	.7910	.7939	.7967

Step 2: Express in the form P(z <= k)

$P(X > 28) = 1 - P(x \leq 28) = 1 - P(z \leq 2.0)$

Step 3: Obtain the required probabilities' value from the standard normal table

$P(X > 28) = 1 - 0.9772 = \underline{0.0228}$

Standard Normal Distribution

Example 7: Given the normally distributed variable X with mean 20 and standard deviation 4, find P(17.5 < x <22.5)

$X \sim N(20, 4^2)$

Step 1 : Find the z score

$z_1 = \frac{x-\mu}{\sigma} = \frac{17.5-20}{4} = -0.625 \quad z_2 = \frac{x-\mu}{\sigma} = \frac{22.5-20}{4} = 0.625$

z	.00	.01	.02	.03	.04
0.0	.5000	.5040	.5080	.5120	.5160
0.1	.5398	.5438	.5478	.5517	.5557
0.2	.5793	.5832	.5871	.5910	.5948
0.3	.6179	.6217	.6255	.6293	.6331
0.4	.6554	.6591	.6628	.6664	.6700
0.5	.6915	.6950	.6985	.7019	.7054
0.6	.7257	.7291	.7324	.7357	.7389
0.7	.7580	.7611	.7642	.7673	.7704

Step 2: Express in the form P(z <= k)

P(17.5 < x < 22.5) = P(x < 22.5) - P(x < 17.5)

= P(z < 0.625) - P(z < - 0.625)

z	.09	.08	.07	.06	.05	.04	.03	.02
−0.8	.1867	.1894	.1922	.1949	.1977	.2005	.2033	.2061
−0.7	.2148	.2177	.2206	.2236	.2266	.2296	.2327	.2358
−0.6	.2451	.2483	.2514	.2546	.2578	.2611	.2643	.2676

Step 3: Obtain the required probabilities' value from the standard normal table

P(17.5 <x <22.5) = P(z < 0.625) - P(z < - 0.625)

= 0.7341 - 0.2660 = <u>0.4681</u>

Applications of Standard Normal Distribution

Example 8 : A traffic police officer has clocked many cars in a marked speed zone and has found that the mean speed is 65 km/h. Assume the standard deviation is 3 km/h. If the distribution of auto speeds is normal and if any car exceeding 70 km/h is speeding, what is the probability that cars travelling this stretch of road are speeding?

X = speed of cars, X ~N(65,3^2)

P (X > 70) =?

Step 1 : Find the z score

$$z = \frac{x-\mu}{\sigma} = \frac{70-6}{3} = 1.67$$

Step 2: Express in the form P(z < k)

P(X > 70) = P(Z > 1.67)

P(Z > 1.67) = 1 - P (Z < 1.67)

Step 3: Obtain the required probabilities' value from the standard normal table

P(Z > 1.67) = 1 - P (Z < 1.67)

= 1 – 0.9525

= 0.0475

P(X > 70) = <u>0.0475</u>

Probability cars are speeding = 0.0475

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525

Applications of Standard Normal Distribution

Example 9: Good Drinks produces many types of soft drinks, including Orange Cola. The filling machines are adjusted to pour 12 mg of salt in each can of Orange Cola.

However, the actual amount of salt poured into each can is not exactly 12 mg, it varies from can to can. It is found that the net amount of salt in such a can has a normal distribution with a mean of 12 mg and a standard deviation of 0.015 mg.

Calculate the probability that a randomly selected can of Orange Cola contains 11.97 mg to 11.99 mg of salt.

Applications of Standard Normal Distribution

X = net amount of salt in a can of Orange Cola, X ~N(12,0.015^2)

P (11.97 < X < 11.99) = ?

Step 1 : Find the z score

$z_1 = \frac{x-\mu}{\sigma} = \frac{11.97-1}{0.015}$ = -2

$z_2 = \frac{x-\mu}{\sigma} = \frac{11.99-12}{0.015}$ = -0.67

Step 2:   Express in the form P(z < k)

P(11.97 < X < 11.99)

= P(-2 < Z < -0.67)

Step 3: Obtain the required probabilities' value from the standard normal table

P(11.97 < X < 11.99)

= P(-2 < Z < -0.67)

= P (Z < -0.67) – P(Z < -2)

= 0.2514 - 0.0228

= 0.2286

Probability a selected can of Orange Cola contains 11.97 mg to 11.99 mg of salt = 0.2286