Solving linear recurrences (Homogeneous)

Ex $f(N) = f(N-1) + f(N-2)$
(fibo(N))

Form:

$$f(x) = a_1 f(x-1) + a_2 f(x-2) + \text{etc}$$
$$\text{---} \; a_n f(x-n)$$

$$\Rightarrow \boxed{f(x) = \sum_{i=1}^{n} a_i f(x-i)}$$

for a_i, n is fixed
n = order of recurrence

Solution for fibonacci no.

$$f(n) = f(n-1) + f(n-2)$$

Step

① Put $f(n) = \alpha^n$ for some constant α

$$\Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2}$$
$$\alpha^n - \alpha^{n-1} - \alpha^{n-2} = 0$$

divide by α^{n-2}

$$\alpha^2 - \alpha - 1 = 0$$

② roots of $\alpha^2-\alpha-1=0$

$$\boxed{\alpha = \frac{1\pm\sqrt{5}}{2}}$$

$\alpha_1 = \frac{1+\sqrt{5}}{2}$ $\quad \alpha_2 = \frac{1-\sqrt{5}}{2}$

③ $f(n) = c_1\alpha_1^n + c_2\alpha_2^n$

is a solution for fibonacci

$= f(n-1) + f(n-2)$

$\Rightarrow f(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$

④ No. of roots = no of answers ~~~~ we have already

Here we have 2 roots, hence we should have 2 answers already

ie true for ~~~~ fibonacci, we have 0,1 as initial answers

$f(0)=1$ $\quad f(1)=1$

~~[illegible]~~ in original eqⁿ

$$\boxed{f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

↳ formula for n^{th} fibonacci no.

on $n \to \infty$ — This will be close to zero hence less dominating term ignored for time complexity

Time Complexity = $O\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$ → golden ratio

$$\boxed{O\left((1.6180)^n\right)}$$

Que Equal roots

$$f(n) = 2f(n-1) + f(n-2)$$

① $f(n) = \alpha^n$

$$\alpha^n = 2\alpha^{n-1} + \alpha^{n-2}$$

② divide by α^{n-2}

$\alpha^2 - 2\alpha - 1 = 0$

$\alpha = 1$ double root

If α is repeated r times

then $\alpha^n, n\alpha^n, n^2\alpha^n, n^3\alpha^n \ldots n^{r-1}\alpha^n$
are all solution to recurrence.

$\Rightarrow f(n) = c_1(\alpha)^n + c_2 n \alpha^n$
$= c_1 + c_2 n$

$f(0) = 0$ & $f(1) = 1$

$f(0) = 0 = c_1$
$f(1) = 1 = c_1 + c_2$
$c_2 = 1$

Ans = $f(N) = N$ $\Rightarrow$ Time Complexity
$\Rightarrow O(N)$

Non-homogeneous - Linear recurrences

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + a_3 f(n-3) + \dots + a_d f(n-d) = g(n)$$

When this extra function is there it is non homogeneous

How to solve

① replace $g(n)$ by 0 & solve usually

$f(n) - 4f(n-1) - 3 = 0$

$\alpha = 4\alpha^{n-1}$

$\alpha - 4\alpha^{n-1} = 0$

$\alpha - 4 = 0$

$\alpha = 4$

Homogeneous solution

$f(n) = C_1 \alpha^n$

$f(n) = C_1 4^n$

② Take $g(n)$ on one side and particular solution.

$\Rightarrow f(n) - 4f(n-1) = 3^n$

Guess something that is similar to $g(n)$
If $g(n) = n^2$, guess a polynomial of degree 2

guess
$f(n) = C3^n$

put C here

$C3^n - 4C3^{n-1} = 3^n$

$C = -3$

particular solution $= f(n) = -3^{n+1}$

③ add both sol^n together
$f(n) = C_1 4^n + (-3^{n+1})$

$f(1) = 1 \Rightarrow C_1 4 - 3^2 = 1$

$C_1 = \frac{5}{2}$

$\boxed{f(n) = \frac{5}{2} 4^n - 3^{n+1}}$ Ans

How do we guess particular sol^n?

If $g(n)$ is exponential, guess of same type

Ex $g(n) = 2^n + 3^n$

guess $= f(n) = a2^n + b3^n$

If $g(n)$ is polynomial guess of same degree

Ex: $g(n) = n^2 - 1 \Rightarrow$ guess of same degree

$an^2 + bn + c = f(n)$

If $g(n) = 2^n + n$

guess $f(n) = a2^n + bn + c$

If guess fails, then try $(an+b)2^n$, if this also fails increase the degree

$(a^2n + bn + c)2^n$

Eu = $f(n) = 2f(n-1) + 2^n$, $f(0)=1$

① $2^n = 0$

$f(n) = 2f(n-1) =$

$f(n) = \alpha^n$

$\alpha^n = 2\alpha^{n-1}$

$\alpha - 2 = 0$

$\alpha = 2$

② guess particular solution

$g(n) = 2^n$

$f(n) = a\,2^n$

$a2^n = 2a2^{n-1} + 2^n$

$a2^n = a2^n + 2^n$

$a = a+1$ ✗ wrong

hence guess another from our ~~rules~~ rules

$f(n) = (an+b)\, 2^n$

$(an+b)\, 2^n = 2\,(a(n-1)+b)\, 2^{n-1} + 2^n$

$an+b = an - a + b + 1$

$\boxed{a = 1}$

$f(n) = n\, 2^n$ // Particular solution

(3) General solution

$f(n) = C_1\, 2^n + n 2^n$

$f(0) = C_1 = 1$

$C_1 = 1$

$\boxed{f(n) = 2^n + n 2^n}$

$\boxed{\text{Complexity} = n 2^n}$