STAT3006 Assignment 2

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Contents

1 Question 1

For $n = 500$ i.i.d. r.v.s which follow Poisson(λ), unobserved variables are $\lambda, y_1, y_2, ..., y_{68}$, in which y's denote the r.v.s which are larger than or equal to 4.

$$\text{Prior: } \pi(\lambda) \propto \frac{1}{\lambda}$$

$$P(X, Y|\lambda) = \prod_{i=1}^{432} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \prod_{j=1}^{68} \frac{e^{-\lambda}\lambda^{y_j}}{y_j!} I(y_j \geq 4)$$

$$P(\lambda|X, Y) \propto P(X, Y|\lambda)\pi(\lambda) \propto e^{-n\lambda}\lambda^{\sum_i x_i + \sum_j y_j - 1}$$

$$P(y_j|X, \lambda) = \frac{e^{-\lambda}\lambda^{y_j}}{y_j!} I(y_j \geq 4) \propto \frac{\lambda^{y_j}}{y_j!} I(y_j \geq 4)$$

$$\lambda|X, Y \propto Gamma(\sum_i x_i + \sum_j y_j, n)$$

The MH-Step to sample 68 unobserved y' is $y_j^* = \begin{cases} y_j^{(t)} - 1, & \text{with probability } \frac{1}{3} \\ y_j^{(t)}, & \text{with probability } \frac{1}{3} \\ y_j^{(t)} + 1, & \text{with probability } \frac{1}{3} \end{cases}$

$$r = \min \left\{ \frac{\left[\lambda^{(t+1)}\right]^{y_j^*} / y_j^*!}{\left[\lambda^{(t+1)}\right]^{y_j^{(t)}} / y_j^{(t)}!} I(y_j \geq 4), 1 \right\}, \text{ which is the accept-reject ratio.}$$

The estimated λ is `1.45744`.

2 Question 2

The complete-data likelihood function is:

$$
\begin{aligned}
f(X,Z|\Pi,\Theta) = L(\Pi,\Theta|X,Z) &= \prod_{i=1}^{1000}\prod_{k=1}^{3}\left[P(Z_i=k|\Pi,\Theta)P(X_{ij},j=1,2|Z_j,\Pi,\Theta)\right]^{I(Z_j=k)} \\
&= \prod_{i=1}^{1000}\prod_{k=1}^{3}\left[\pi_k\prod_{j=1}^{2}P(X_{ij}|Z_j,\Theta)\right]^{I(Z_j=k)} \\
&= \prod_{i=1}^{1000}\prod_{k=1}^{3}\left[\pi_k\prod_{j=1}^{2}\binom{10\times j}{X_{ij}}\theta_{jk}^{X_{ij}}(1-\theta_{jk})^{10\times j-X_{ij}}\right]^{I(Z_j=k)}
\end{aligned}
$$

Given $Z_i = k$, we have each sample i, $X_{ij} \sim Bino(10\times j, \theta_{jk})$, $P(\theta_{jk}) \propto Beta(a,b)$ with prior $P(\theta_jk) \propto Beta(1,1) \propto 1$ and $(\pi_1,\pi_2,\pi_3) \sim Dirichlet(\alpha_1,\alpha_2,\alpha_3)$.

$$
\begin{aligned}
P(\Pi,\Theta|X,Z) \propto & P(\Pi)P(\Theta)P(X,Z|\Pi,\Theta) \\
\propto & \prod_{k=1}^{3}\pi_k^{\alpha_k-1}\prod_{j=1}^{2}\prod_{k=1}^{3}\theta_{jk}^{a-1}(1-\theta_{jk})^{b-1}\prod_{i=1}^{1000}\prod_{k=1}^{3}\left[\pi_k\prod_{j=1}^{2}\binom{10\times j}{X_{ij}}\theta_{jk}^{X_{ij}}(1-\theta_{jk})^{10\times j-X_{ij}}\right]^{I(Z_j=k)} \\
\text{For } \pi\text{: } f(\Pi|\Theta,Z) \propto & \prod_{k=1}^{3}\pi_k^{\alpha_k-1}\prod_{i=1}^{1000}\prod_{k=1}^{3}\pi_k^{I(Z_i=k)} \\
\propto & \prod_{k=1}^{3}\pi_k^{\alpha_k-1}\prod_{k=1}^{3}\pi_k^{\sum_{i=1}^{1000}I(Z_i=k)} \\
\propto & \prod_{k=1}^{3}\pi_k^{\alpha_k-1+\sum_{i=1}^{1000}I(Z_i=k)} \\
\Pi|\Theta,Z \sim & Dirichlet(\alpha_1+\sum_{i=1}^{1000}I(Z_i=1),\alpha_2+\sum_{i=1}^{1000}I(Z_i=2),\alpha_3+\sum_{i=1}^{1000}I(Z_i=3)) \\
\text{For } \theta\text{: } \theta_{jk}|- \propto & \prod_{j=1}^{2}\prod_{k=1}^{3}\theta_{jk}^{a-1}(1-\theta_{jk})^{b-1}\prod_{i=1}^{1000}\prod_{k=1}^{3}\left[\prod_{j=1}^{2}\binom{10\times j}{X_{ij}}\theta_{jk}^{X_{ij}}(1-\theta_{jk})^{10\times j-X_{ij}}\right]^{I(Z_j=k)} \\
\propto & \theta_{jk}^{a-1}(1-\theta_{jk})^{b-1}\prod_{i=1}^{1000}\left[\theta_{jk}^{X_{ij}}(1-\theta_{jk})^{10\times j-X_{ij}}\right]^{I(Z_j=k)} \\
\propto & \theta_{jk}^{a-1}(1-\theta_{jk})^{b-1}\prod_{i=1}^{1000}\left[\theta_{jk}^{X_{ij}}(1-\theta_{jk})^{10\times j-X_{ij}}\right]^{I(Z_j=k)} \\
\propto & \theta_{jk}^{\sum_{i=1}^{1000}X_{ij}I(Z_i=k)+a-1}(1-\theta_{jk})^{\sum_{i=1}^{1000}(10\times j-X_{ij})I(Z_i=k)+b-1} \\
\theta_{jk}|- \sim & Beta\left(a+\sum_{i=1}^{1000}X_{ij}I(Z_i=k),b+\sum_{i=1}^{1000}(10\times j-X_{ij})I(Z_i=k)\right) \\
\text{For } Z_i\text{: } Z_i \propto & \prod_{i=1}^{1000}\prod_{k=1}^{3}\pi_k^{I(Z_i=k)}\prod_{i=1}^{1000}\prod_{k=1}^{3}\prod_{j=1}^{2}\binom{10\times j}{X_{ij}}^{I(Z_i=k)}\theta_{jk}^{X_{ij}I(Z_i=k)}\left[(1-\theta_{jk})^{10\times j-X_{ij}}\right]^{I(Z_i=k)} \\
\propto & \prod_{k=1}^{3}\pi_k^{I(Z_i=k)}\prod_{k=1}^{3}\prod_{j=1}^{2}\binom{10\times j}{X_{ij}}^{I(Z_i=k)}\theta_{jk}^{X_{ij}I(Z_i=k)}\left[(1-\theta_{jk})^{10\times j-X_{ij}}\right]^{I(Z_i=k)} \\
\propto & \prod_{k=1}^{3}\left[\pi_k\prod_{j=1}^{2}\binom{10\times j}{X_{ij}}\theta_{jk}^{X_{ij}}(1-\theta_{jk})^{10\times j-X_{ij}}\right]^{I(Z_i=k)}
\end{aligned}
$$

For Gibbs Sampler algorithm, given $\Pi^{(t)},\Theta^{(t)},Z^{(t)}$, the updates are as follows.

$$
\begin{aligned}
\Pi|\Theta,Z \sim & Dirichlet(\alpha_1+\sum_{i=1}^{1000}I(Z_i^{(t)}=1),\alpha_2+\sum_{i=1}^{1000}I(Z_i^{(t)}=2),\alpha_3+\sum_{i=1}^{1000}I(Z_i^{(t)}=3)) \\
\theta_{jk}|- \sim & Beta\left(a+\sum_{i=1}^{1000}X_{ij}I(Z_i^{(t)}=k),b+\sum_{i=1}^{1000}(10\times j-X_{ij})I(Z_i^{(t)}=k)\right) \\
P(Z_i^{(t+1)}=k|-) = & \frac{\pi_k^{(t+1)}\prod_{j=1}^{2}\binom{10\times j}{X_{ij}}\theta_{jk}^{X_{ij}}(1-\theta_{jk})^{10\times j-X_{ij}}}{\sum_{l=1}^{3}\pi_l^{(t+1)}\prod_{j=1}^{2}\binom{10\times j}{X_{ij}}\theta_{jl}^{X_{ik}}(1-\theta_{jl})^{10\times j-X_{ij}}}
\end{aligned}
$$

The estimated $\Pi = (\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3) = (0.3047501, 0.1008389, 0.5944110)$.
The estimated $\Theta = (\hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{21}, \hat{\theta}_{22}, \hat{\theta}_{23}) = (0.5093235, 0.1909221, 0.8023880, 0.4920218, 0.7818400, 0.1984175)$.
The estimated Z are as follows.

```
> which(estimated_z==1)
  [1]    1    3    5   10   11   16   17   21   22   29   30   31   34   37   43   53   56
 [18]   58   64   75   77   81   83   85   86   88   91   97   98  101  102  105  107  109
 [35]  114  119  121  124  126  127  130  132  133  136  142  144  145  148  149  151  158
 [52]  159  161  167  176  178  182  183  190  194  197  201  203  205  206  208  211  218
 [69]  227  229  231  234  246  249  254  255  261  264  265  266  271  272  275  276  277
 [86]  287  288  289  291  292  294  298  299  303  304  306  313  314  316  317  319  332
[103]  341  342  348  349  354  355  356  360  363  364  367  372  375  376  378  380  384  386
[120]  390  392  398  403  410  411  415  422  426  433  437  438  439  441  443  447  448
[137]  449  450  451  454  457  463  464  466  467  476  477  480  482  485  486  493  495
[154]  496  497  503  504  505  507  508  518  519  522  524  531  532  536  539  542  543
[171]  546  547  560  561  562  565  568  569  570  574  584  585  586  589  593  595  600
[188]  603  604  605  606  609  610  612  621  624  627  630  631  633  636  644  646  653
[205]  654  656  659  670  677  684  685  689  693  694  701  702  705  711  712  714  719
[222]  722  723  724  729  733  736  740  747  750  756  757  762  763  765  766  767  768
[239]  769  770  772  775  781  785  789  791  793  795  796  800  803  806  813  821  823
[256]  827  828  830  831  837  847  857  858  859  860  864  867  872  873  874  876  890
[273]  898  899  900  903  905  907  912  921  928  929  932  933  937  943  944  945  947
[290]  948  950  958  969  970  974  975  978  980  983  987  993 1000
> which(estimated_z==2)
 [1]   9  13  35  41  62  87  95 103 110 111 117 125 168 177 179 193 207 214 222 228 244 247
[23] 248 250 256 257 297 327 333 338 340 343 362 377 385 399 400 408 421 442 452 458 459 465
[45] 468 469 470 484 510 514 533 538 544 551 556 559 566 572 576 582 583 607 611 616 628 635
[67] 645 648 655 662 671 691 698 731 732 751 776 778 794 799 801 804 807 819 822 851 877 891
[89] 901 902 909 916 923 939 956 959 971 986 991
> which(estimated_z==3)
  [1]   2   4   6   7   8  12  14  15  18  19  20  23  24  25  26  27  28  32  33  36  38
 [22]  39  40  42  44  45  46  47  48  49  50  51  52  54  55  57  59  60  61  63  65  66
 [43]  67  68  69  70  71  72  73  74  76  78  79  80  82  84  89  90  92  93  94  96  99
 [64] 100 104 106 108 112 113 115 116 118 120 122 123 128 129 131 134 135 137 138 139 140
 [85] 141 143 146 147 150 152 153 154 155 156 157 160 162 163 164 165 166 169 170 171 172
[106] 173 174 175 180 181 184 185 186 187 188 189 191 192 195 196 198 199 200 202 204 209
[127] 210 212 213 215 216 217 219 220 221 223 224 225 226 230 232 233 235 236 237 238 239
[148] 240 241 242 243 245 251 252 253 258 259 260 262 263 267 268 269 270 273 274 278 279
[169] 280 281 282 283 284 285 286 290 293 295 296 300 301 302 305 307 308 309 310 311 312
[190] 315 318 320 321 322 323 324 325 326 328 329 330 331 334 335 336 337 339 344 345 346
[211] 347 350 351 352 353 357 358 359 361 365 366 368 369 370 371 373 374 379 381 382
[232] 383 387 388 389 391 393 394 395 396 397 401 402 404 405 406 407 409 412 413 414 416
[253] 417 418 419 420 423 424 425 427 428 429 430 431 432 434 435 436 440 444 445 446 453
[274] 455 456 460 461 462 471 472 473 474 475 478 479 481 483 487 488 489 490 491 492 494
[295] 498 499 500 501 502 506 509 511 512 513 515 516 517 520 521 523 525 526 527 528 529
[316] 530 534 535 537 540 541 545 548 549 550 552 553 554 555 557 558 563 564 567 571 573
[337] 575 577 578 579 580 581 587 588 590 591 592 594 596 597 598 599 601 602 608 613 614
[358] 615 617 618 619 620 622 623 625 626 629 632 634 637 638 639 640 641 642 643 647 649
[379] 650 651 652 657 658 660 661 663 664 665 666 667 668 669 672 673 674 675 676 678 679
[400] 680 681 682 683 686 687 688 690 692 695 696 697 699 700 703 704 706 707 708 709 710
[421] 713 715 716 717 718 720 721 725 726 727 728 730 734 735 737 738 739 741 742 743 744
[442] 745 746 748 749 752 753 754 755 758 759 760 761 764 771 773 774 777 779 780 782 783
[463] 784 786 787 788 790 792 797 798 802 805 808 809 810 811 812 814 815 816 817 818 820
[484] 824 825 826 829 832 833 834 835 836 838 839 840 841 842 843 844 845 846 848 849 850
[505] 852 853 854 855 856 861 862 863 865 866 868 869 870 871 875 878 879 880 881 882 883
[526] 884 885 886 887 888 889 892 893 894 895 896 897 904 906 908 910 911 913 914 915 917
[547] 918 919 920 922 924 925 926 927 930 931 934 935 936 938 940 941 942 946 949 951 952
[568] 953 954 955 957 960 961 962 963 964 965 966 967 968 972 973 976 977 979 981 982 984
[589] 985 988 989 990 992 994 995 996 997 998 999
```

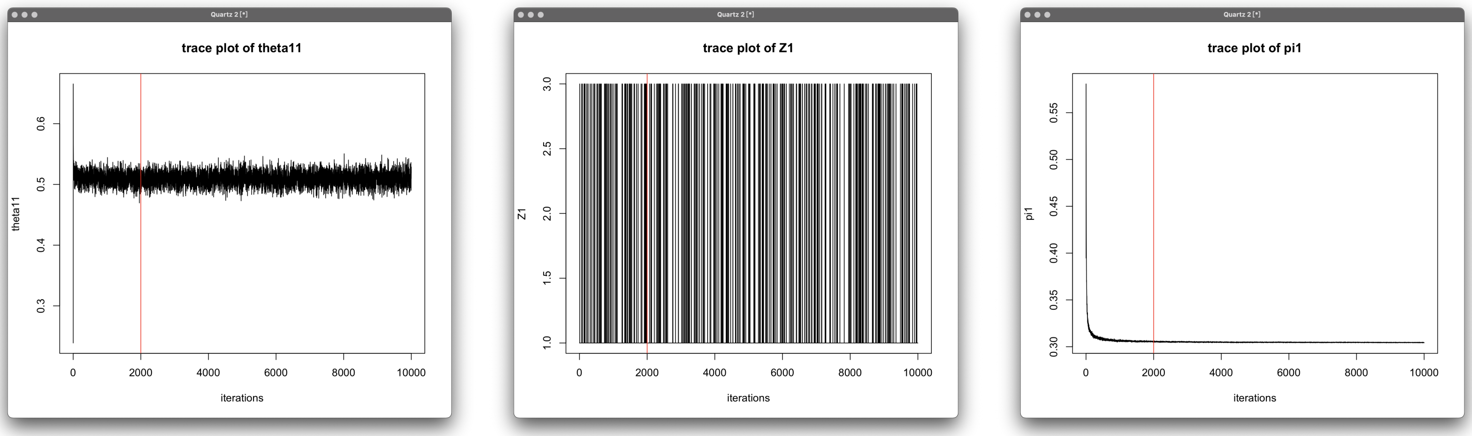

Figure 1: Trace Plot of θ_{11} (left), Z_1 (Middle) and π_1 (Right)

3 Question 3

For $i = 1, 2$, $\boldsymbol{Y}_i \sim multinomial(100, p_1, p_2, p_3, p_4)$.

$$
\begin{aligned}
\text{Prior: } \pi(\boldsymbol{p}) &\propto Dirchlet(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \propto Dirchlet(2, 2, 2, 2) \propto p_1 p_2 p_3 p_4 \\
P(y_{i1}, y_{i2}, y_{i3}, y_{i4} | p_1, p_2, p_3, p_4) &= \frac{100!}{y_{i1}! y_{i2}! y_{i3}! y_{i4}!} p_1^y{}_{i1} p_2^y{}_{i2} p_3^y{}_{i3} p_4^y{}_{i4} \\
P(p_1, p_2, p_3, p_4 | y_{i1}, y_{i2}, y_{i3}, y_{i4}) &\propto P(y_{i1}, y_{i2}, y_{i3}, y_{i4} | p_1, p_2, p_3, p_4) \cdot f(p_1, p_2, p_3, p_4 | y_{i1}, y_{i2}, y_{i3}, y_{i4}) \\
&\propto \frac{100!}{y_{i1}! y_{i2}! y_{i3}! y_{i4}!} p_1^{y_{i1}+\alpha_1-1} p_2^{y_{i2}+\alpha_2-1} p_3^{y_{i3}+\alpha_3-1} p_4^{y_{i4}+\alpha_4-1} \\
p_1, p_2, p_3, p_4 | y_{i1}, y_{i2}, y_{i3}, y_{i4} &\sim Dirchlet(y_{i1} + \alpha_1, y_{i2} + \alpha_2, y_{i3} + \alpha_3, y_{i4} + \alpha_4) \\
P(y_{12} | \boldsymbol{Y}, \boldsymbol{P}) &\propto \frac{p_2^{y_{12}}}{y_{12}!} \cdot \frac{p_3^{y_{13}}}{y_{13}!} = \frac{p_2^{y_{12}}}{y_{12}!} \cdot \frac{p_3^{44-y_{12}}}{(44 - y_{12})!} \\
P(y_{22} | \boldsymbol{Y}, \boldsymbol{P}) &\propto \frac{p_2^{y_{22}}}{y_{22}!} \cdot \frac{p_4^{y_{24}}}{y_{24}!} = \frac{p_2^{y_{22}}}{y_{22}!} \cdot \frac{p_4^{48-y_{22}}}{(48 - y_{22})!}
\end{aligned}
$$

Denote the upper bound of y_{i2} by c, $c = \begin{cases} 35, \text{ for } i = 1 \\ 30, \text{ for } i = 2 \end{cases}$

Then MH-Step to update y_{i2} for $i = 1, 2$ is as follows.

$$
\begin{aligned}
y_{i2}^{(t+1)} &= \begin{cases} y_{i2}^{(t)} + 1 \text{ with probability 0.5 or if } y_{i2}^{(t)} = 15 \\ y_{i2}^{(t)} - 1 \text{ with probability 0.5 or if } y_{i2}^{(t)} = c \end{cases} \\
r &= \begin{cases} \min\{2 \times \frac{P(y_{i2}^{(t+1)} | \boldsymbol{Y}, \boldsymbol{P})}{P(y_{i2}^{(t)} | \boldsymbol{Y}, \boldsymbol{P})}, 1\} & \text{,if } y_{i2}^{(t+1)} = c \text{ or } 15 \\ \min\{\frac{1}{2} \times \frac{P(y_{i2}^{(t+1)} | \boldsymbol{Y}, \boldsymbol{P})}{P(y_{i2}^{(t)} | \boldsymbol{Y}, \boldsymbol{P})}, 1\} & \text{,if } y_{i2}^{(t)} = c \text{ or } 15 \\ \min\{\frac{P(y_{i2}^{(t+1)} | \boldsymbol{Y}, \boldsymbol{P})}{P(y_{i2}^{(t)} | \boldsymbol{Y}, \boldsymbol{P})}, 1\} & \text{, otherwise} \end{cases}
\end{aligned}
$$

where r is the accept-reject ratio. For y_{13} and y_{24},

$$
\begin{aligned}
y_{13}^{(t+1)} &= 100 - y_{11} - y_{14} - y_{12}^{(t+1)} = 44 - y_{12}^{(t+1)} \\
y_{24}^{(t+1)} &= 100 - y_{21} - y_{23} - y_{22}^{(t+1)} = 48 - y_{22}^{(t+1)}
\end{aligned}
$$

The estimated $\boldsymbol{P} = (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4) = (0.4561099, 0.2653649, 0.1546046, 0.1239205)$