STAT3006 Assignment 1

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1 Question 1

$f(x) = x^3 + 5.9x^2 - 32.46x - 29.97$ is a cubic polynomial function and hence there are 3 roots.
By method of bisection, the roots are 4.018555, -9.106445, -0.8154297.

2 Question 2

2.1 Part i

$$\begin{aligned}
log(\lambda_i) &= \alpha + \beta x_i \\
E(Y|x) &= \exp{(\alpha + \beta x_i)} = \lambda_i \\
L(\alpha, \beta|\vec{x}, \vec{y}) = p(y_1, ..., y_{15}|x_1, ..., x_{15}, \alpha, \beta) &= \prod_{i=1}^{15} \frac{\exp{(y_i(\alpha + \beta x_i))} \exp{(-\exp{(\alpha + \beta x_i)})}}{y_i!}
\end{aligned}$$

2.2 Part ii

Consider log-likelihood function.

$$\begin{aligned}
L(\alpha, \beta|\vec{x}, \vec{y}) &= \prod_{i=1}^{15} \frac{\exp{(y_i(\alpha + \beta x_i))} \exp{(-\exp{(\alpha + \beta x_i)})}}{y_i!} \\
l(\alpha, \beta|\vec{x}, \vec{y}) &= \sum_{i=1}^{15} (y_i(\alpha + \beta x_i) - \exp{(\alpha + \beta x_i)} - log(y_i!))
\end{aligned}$$

Maximise of $l(\alpha, \beta|\vec{x}, \vec{y})$.

$$\begin{aligned}
\frac{\partial l}{\partial \alpha} &= -\sum_{i=1}^{15} \exp{(\alpha + \beta x_i)} + \sum_{i=1}^{15} y_i \\
\frac{\partial l}{\partial \beta} &= -\sum_{i=1}^{15} x_i \exp{(\alpha + \beta x_i)} + \sum_{i=1}^{15} x_i y_i \\
\frac{\partial^2 l}{\partial \alpha^2} &= -\sum_{i=1}^{15} \exp{(\alpha + \beta x_i)} \\
\frac{\partial^2 l}{\partial \beta^2} &= -\sum_{i=1}^{15} x_i^2 \exp{(\alpha + \beta x_i)} \\
\frac{\partial^2 l}{\partial \alpha \beta} &= -\sum_{i=1}^{15} x_i \exp{(\alpha + \beta x_i)}
\end{aligned}$$

Hence, the Newton step is as follows.

$$\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} \alpha_{k-1} \\ \beta_{k-1} \end{bmatrix} - \begin{bmatrix} -\sum_{i=1}^{15} e^{(\alpha_{k-1} + \beta_{k-1} x_i)} & -\sum_{i=1}^{15} x_i e^{(\alpha_{k-1} + \beta_{k-1} x_i)} \\ -\sum_{i=1}^{15} x_i e^{(\alpha_{k-1} + \beta_{k-1} x_i)} & -\sum_{i=1}^{15} x_i^2 e^{(\alpha_{k-1} + \beta_{k-1} x_i)} \end{bmatrix}^{-1} \cdot \begin{bmatrix} -\sum_{i=1}^{15} e^{(\alpha_{k-1} + \beta_{k-1} x_i)} + \sum_{i=1}^{15} y_i \\ -\sum_{i=1}^{15} x_i e^{(\alpha_{k-1} + \beta_{k-1} x_i)} + \sum_{i=1}^{15} x_i y_i \end{bmatrix}$$

3 Question 3

3.1 Part i

$$
\begin{aligned}
logit(p_i) &= \alpha + \beta x_i \\
\frac{p_i}{1-p_i} &= \exp{(\alpha + \beta x_i)} \\
p_i &= \frac{\exp{(\alpha + \beta x_i)}}{1+\exp{(\alpha + \beta x_i)}} \\
L(\alpha, \beta | \vec{x}, \vec{y}) = p(y_1, ..., y_{15} | x_1, ..., x_{15}, \alpha, \beta) &= \prod_{i=1}^{15} (\frac{\exp{(\alpha + \beta x_i)}}{1+\exp{(\alpha + \beta x_i)}})^{y_i} (1 - \frac{\exp{(\alpha + \beta x_i)}}{1+\exp{(\alpha + \beta x_i)}})^{1-y_i}
\end{aligned}
$$

3.2 Part ii

Consider log-likelihood function.

$$
\begin{aligned}
L(\alpha, \beta | \vec{x}, \vec{y}) &= \prod_{i=1}^{15} (\frac{\exp{(\alpha + \beta x_i)}}{1+\exp{(\alpha + \beta x_i)}})^{y_i} (1 - \frac{\exp{(\alpha + \beta x_i)}}{1+\exp{(\alpha + \beta x_i)}})^{1-y_i} \\
&= \prod_{i=1}^{15} (\frac{\exp{(\alpha + \beta x_i)}}{1+\exp{(\alpha + \beta x_i)}})^{y_i} (\frac{1}{1+\exp{(\alpha + \beta x_i)}})^{1-y_i} \\
l(\alpha, \beta | \vec{x}, \vec{y}) &= \sum_{i=1}^{15} y_i(\alpha + \beta x_i) - y_i log(1 + \exp(\alpha + \beta x_i)) + (y_i - 1) log(1 + \exp(\alpha + \beta x_i)) \\
&= \sum_{i=1}^{15} \alpha y_i + \beta x_i y_i - y_i log(1 + \exp(\alpha + \beta x_i)) + y_i log(1 + \exp(\alpha + \beta x_i)) - log(1 + \exp(\alpha + \beta x_i)) \\
&= \sum_{i=1}^{15} \alpha y_i + \beta x_i y_i - log(1 + \exp(\alpha + \beta x_i))
\end{aligned}
$$

Maximise of $l(\alpha, \beta | \vec{x}, \vec{y})$.

$$
\begin{aligned}
\frac{\partial l}{\partial \alpha} &= \sum_{i=1}^{15} (y_i - \frac{\exp(\alpha + \beta x_i)}{1 + \exp(\alpha + \beta x_i)}) \\
\frac{\partial l}{\partial \beta} &= \sum_{i=1}^{15} (x_i y_i - \frac{x_i \exp(\alpha + \beta x_i)}{1 + \exp(\alpha + \beta x_i)}) \\
\frac{\partial^2 l}{\partial \alpha^2} &= -\sum_{i=1}^{15} \frac{\exp(\alpha + \beta x_i)}{(1 + \exp(\alpha + \beta x_i))^2} \\
\frac{\partial^2 l}{\partial \beta^2} &= -\sum_{i=1}^{15} \frac{x_i^2 \exp(\alpha + \beta x_i)}{(1 + \exp(\alpha + \beta x_i))^2} \\
\frac{\partial^2 l}{\partial \alpha \beta} &= -\sum_{i=1}^{15} \frac{x_i \exp(\alpha + \beta x_i)}{(1 + \exp(\alpha + \beta x_i))^2}
\end{aligned}
$$

Hence, the Newton step is as follows.

$$
\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} \alpha_{k-1} \\ \beta_{k-1} \end{bmatrix} - \begin{bmatrix} -\sum_{i=1}^{15} \frac{\exp(\alpha+\beta x_i)}{(1+\exp(\alpha+\beta x_i))^2} & -\sum_{i=1}^{15} \frac{x_i \exp(\alpha+\beta x_i)}{(1+\exp(\alpha+\beta x_i))^2} \\ -\sum_{i=1}^{15} \frac{x_i \exp(\alpha+\beta x_i)}{(1+\exp(\alpha+\beta x_i))^2} & -\sum_{i=1}^{15} \frac{x_i^2 \exp(\alpha+\beta x_i)}{(1+\exp(\alpha+\beta x_i))^2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \sum_{i=1}^{15} (y_i - \frac{\exp(\alpha+\beta x_i)}{1+\exp(\alpha+\beta x_i)}) \\ \sum_{i=1}^{15} (x_i y_i - \frac{x_i \exp(\alpha+\beta x_i)}{1+\exp(\alpha+\beta x_i)}) \end{bmatrix}
$$

4 Question 4

4.1 Part i

$$L(\boldsymbol{\pi}, \mu_1, \mu_2, \sigma_1, \sigma_2 | \mathbf{Y}, \mathbf{Z}) = \prod_{i=1}^{8000} \left[\boldsymbol{\pi} \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(\frac{-(y_i - \mu_1)^2}{2\sigma_1^2}\right) \right]^{(2-z_i)} \left[(1-\boldsymbol{\pi}) \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(\frac{-(y_i - \mu_2)^2}{2\sigma_2^2}\right) \right]^{z_i - 1}$$

4.2 Part ii

$$\begin{aligned} l(\boldsymbol{\pi}, \mu_1, \mu_2, \sigma_1, \sigma_2 | \mathbf{Y}, \mathbf{Z}) = \sum_{i=1}^{8000} \{ &(2 - z_i) \left[-\frac{1}{2} log(2\pi) - \frac{1}{2} log(\sigma_1^2) + log(\boldsymbol{\pi}) - \frac{(y_i - \mu_1)^2}{2\sigma_1^2} \right] \\ &+ (z_i - 1) \left[-\frac{1}{2} log(2\pi) - \frac{1}{2} log(\sigma_2^2) + log(1 - \boldsymbol{\pi}) - \frac{(y_i - \mu_2)^2}{2\sigma_2^2} \right] \} \end{aligned}$$

Put $A_i = 2 - Z_i$ and $B_i = 1 - A_i = Z_i - 1$.
If $Z_i = 1$, $A_i = 1$ and $B_i = 0$ which indicate females.
If $Z_i = 2$, $A_i = 0$ and $B_i = 1$ which indicate males.

$$\begin{aligned} l(\boldsymbol{\pi}, \mu_1, \mu_2, \sigma_1, \sigma_2 | \mathbf{Y}, \mathbf{Z}) = \sum_{i=1}^{8000} \{ &a_i \left[-\frac{1}{2} log(2\pi) - \frac{1}{2} log(\sigma_1^2) + log(\boldsymbol{\pi}) - \frac{(y_i - \mu_1)^2}{2\sigma_1^2} \right] \\ &+ b_i \left[-\frac{1}{2} log(2\pi) - \frac{1}{2} log(\sigma_2^2) + log(1 - \boldsymbol{\pi}) - \frac{(y_i - \mu_2)^2}{2\sigma_2^2} \right] \} \end{aligned}$$

$$\frac{\partial l}{\partial \mu_1} = 0$$

$$\hat{\mu}_1 = \frac{\sum_{i=1}^{8000} a_i y_i}{\sum_{i=1}^{8000} a_i} = \frac{\sum_{i=1}^{8000} (2 - z_i) y_i}{\sum_{i=1}^{8000} (2 - z_i)}$$

Similarly,

$$\hat{\mu}_2 = \frac{\sum_{i=1}^{8000} b_i y_i}{\sum_{i=1}^{8000} b_i} + \frac{\sum_{i=1}^{8000} (z_i - 1) y_i}{\sum_{i=1}^{8000} (z_i - 1)}$$

$$\frac{\partial l}{\partial \boldsymbol{\pi}} = 0$$

$$\sum_{i=1}^{8000} \left(\frac{a_i}{\boldsymbol{\pi}} - \frac{b_i}{1 - \boldsymbol{\pi}}\right) = 0$$

$$\hat{\boldsymbol{\pi}} = \frac{\sum_{i=1}^{8000} a_i}{\sum_{i=1}^{8000} (a_i + b_i)} = \frac{\sum_{i=1}^{8000} (2 - z_i)}{8000}$$

$$\frac{\partial l}{\partial \sigma_1} = 0$$

$$\sum_{i=1}^{8000} \left(-\frac{a_i}{\sigma_1} + \frac{a_i (y_i - \hat{\mu}_1)^2}{\sigma_1^3}\right) = 0$$

$$\sum_{i=1}^{8000} \frac{a_i (y_i - \hat{\mu}_1)^2}{\sigma_1^3} = \sum_{i=1}^{8000} \frac{a_i}{\sigma_1}$$

$$\hat{\sigma}_1 = \sqrt{\frac{\sum_{i=1}^{8000} a_i (y_i - \hat{\mu}_1)^2}{\sum_{i=1}^{8000} a_i}} = \sqrt{\frac{\sum_{i=1}^{8000} (2 - z_i)(y_i - \hat{\mu}_1)^2}{\sum_{i=1}^{8000} (2 - z_i)}}$$

Similarly,

$$\hat{\sigma}_2 = \sqrt{\frac{\sum_{i=1}^{8000} b_i (y_i - \hat{\mu}_2)^2}{\sum_{i=1}^{8000} b_i}} = \sqrt{\frac{\sum_{i=1}^{8000} (z_i - 1)(y_i - \hat{\mu}_2)^2}{\sum_{i=1}^{8000} (z_i - 1)}}$$

E step:

$$
\begin{aligned}
w_{it} &= \mathbf{E}(\mathbf{Z}|\boldsymbol{\pi}^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \sigma_1^{(t)}, \sigma_2^{(t)}, \mathbf{Y}) \\
&= P(\mathbf{Z}=1|\boldsymbol{\pi}^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \sigma_1^{(t)}, \sigma_2^{(t)}, \mathbf{Y}) + 2 \cdot P(\mathbf{Z}=2|\boldsymbol{\pi}^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \sigma_1^{(t)}, \sigma_2^{(t)}, \mathbf{Y}) \\
&= \frac{P(\mathbf{Z}=1, \mathbf{Y}|\boldsymbol{\pi}^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \sigma_1^{(t)}, \sigma_2^{(t)}) + 2 \cdot P(\mathbf{Z}=2, \mathbf{Y}|\boldsymbol{\pi}^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \sigma_1^{(t)}, \sigma_2^{(t)})}{P(\mathbf{Z}=1, \mathbf{Y}|\boldsymbol{\pi}^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \sigma_1^{(t)}, \sigma_2^{(t)}) + P(\mathbf{Z}=2, \mathbf{Y}|\boldsymbol{\pi}^{(t)}, \mu_1^{(t)}, \mu_2^{(t)}, \sigma_1^{(t)}, \sigma_2^{(t)})} \\
&= \frac{\boldsymbol{\pi}^{(t)} \frac{1}{\sqrt{2\pi}\sigma_1^{(t)}} \exp\left(\frac{-(y_i - \mu_1^{(t)})^2}{2\sigma_1^{2\,(t)}}\right)}{\boldsymbol{\pi}^{(t)} \frac{1}{\sqrt{2\pi}\sigma_1^{(t)}} \exp\left(\frac{-(y_i - \mu_1^{(t)})^2}{2\sigma_1^{2\,(t)}}\right) + (1 - \boldsymbol{\pi}^{(t)}) \frac{1}{\sqrt{2\pi}\sigma_2^{(t)}} \exp\left(\frac{-(y_i - \mu_2^{(t)})^2}{2\sigma_2^{2\,(t)}}\right)}
\end{aligned}
$$

M steps:

$$
\begin{aligned}
\hat{\boldsymbol{\pi}}^{(t)} &= \frac{\sum_{i=1}^{8000}(2 - w_{it})}{8000} \\
\hat{\mu}_1^{(t)} &= \frac{\sum_{i=1}^{8000}(2 - w_{it})y_i}{\sum_{i=1}^{8000}(2 - w_{it})} \\
\hat{\mu}_2^{(t)} &= \frac{\sum_{i=1}^{8000}(w_{it} - 1)y_i}{\sum_{i=1}^{8000}(w_{it} - 1)} \\
\hat{\sigma}_1^{(t)} &= \sqrt{\frac{\sum_{i=1}^{8000}(2 - w_{it})(y_i - \hat{\mu}_1)^2}{\sum_{i=1}^{8000}(2 - w_{it})}} \\
\hat{\sigma}_2^{(t)} &= \sqrt{\frac{\sum_{i=1}^{8000}(w_{it} - 1)(y_i - \hat{\mu}_2)^2}{\sum_{i=1}^{8000}(w_{it} - 1)}}
\end{aligned}
$$