# Structural stiffness of round bar

## Introduction

In this project, we will calculate stiffness of a steel round bar under mechanical loading. By leveraging COMSOL for 3D visualization, we will study how stiffness is computed and interpreted across different modeling dimensions, with a particular emphasis on 3D analysis. Furthermore, we will analyze the key factors, such as material properties and geometry, that impact the bar's stiffness to develop a thorough understanding of its structural behavior.

## Definition of terms

When an external force acts on a round bar, the object resists being deformed, and this is called stiffness. While stiffness is often viewed as a material attribute, it is in fact influenced by both the geometry and material composition of the object. This research will delve into these factors comprehensively.

To proceed, we need to define stiffness mathematically. Suppose a force $F$ is applied to a body, causing a deformation $u$. If a small force $\Delta F$ leads to an infinitesimally small deformation $\Delta u$, the ratio of these two values represents the stiffness at that particular operating point, defined by the state variables $F$ and $u$.

This concept, known as linearized stiffness, can be applied to both linear and nonlinear force-displacement relationships, which are mathematically expressed as follows:

$$F(u) = F_0 + k\left(F_0, u_0\right)\left(u - u_0\right)$$

$$k\left(F_0, u_0\right) = \lim_{\Delta u \to 0} \frac{\Delta F}{\Delta u} = \left.\frac{\partial F}{\partial u}\right|_{F=F_0, u=u_0}$$

## Problem

In our project, which focuses on the structural stiffness of a round steel bar, one key objective is to determine an accurate stiffness value and understand its significance based on how it is calculated from the given structural scenario. Stiffness, in general, is influenced by factors such as material properties, geometric dimensions, material orientation, the direction of applied loads, the type of constraints used, and the specific regions where forces and constraints are applied.

For this analysis, we will consider a steel round bar with a length of 1 meter and a radius of 0.1 meters. One end of the bar will be rigidly fixed, meaning there will be no displacement in the x-, y-, or z-directions, while a uniformly distributed force will be applied to the other end. The remaining surfaces of the bar will be free to deform as they are unconstrained and not subjected to any additional loads.

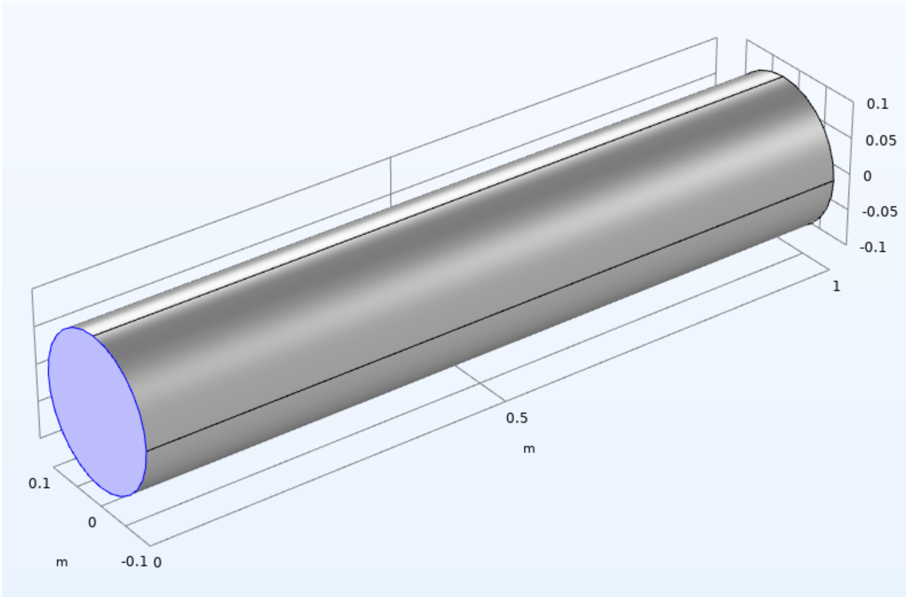


Figure 1. The image of solid round bar made of steel with parameters: Length = 1 meter and radius = 0.1 meters

## Structural stiffness from single point

Before analyzing all three dimension we will start from zero dimension, therefore we attach the load to one single point and our round bar acts like a regular spring.

Assuming steel follows Hooke's law, meaning stress remains directly proportional to strain as long as it stays within the elastic limit, the stress-strain relationship can be formulated using Young's modulus, $E$, represented as:

$$\sigma = E\epsilon$$

To simplify, stress is defined as the applied force over the cross-sectional area, $\sigma = \frac{F}{A}$, where $A = \pi r^2$, and strain is the ratio of deformation to the original length, $\epsilon = \frac{u}{L}$, By combining these equations, we derive:

$$F = \left(\frac{EA}{L}\right) u$$

This equation represents a linear force-displacement relationship, showing that stiffness is independent of the operating point and spatial variation in force, displacement, and material properties.

Thus, the axial stiffness of round bar for the 0D model can be determined as:

$$k = \frac{EA}{L}$$

With a Young's modulus of 200 GPa, for steel, the calculated axial stiffness for the round bar equals k = $628.32 * 10^7 N/m$

Parameters

| N | Expression | Value | Description |
|---|---|---|---|
| L | 1[m] | 1 m | Length |
| r | 0.1[m] | 0.1 m | Radius |
| A | pi*r*r | 0.031416 m² | Cross section area |
| E | 200[GPa] | 2E11 Pa | Young's Modulus |
| k | E*A/L | 6.2832E9 N/... | Lumped stiffness |
| F0 | 1e4[N] | 10000 N | Axial Load |

Figure 2. By setting up the Parameters in COMSOL, we can model 0D scenario

## Structural stiffness for 1D

Deformation for round bar is different along the each cross section, therefore we use 1D model to compute this limitation.

For our project, utilizing a 1D model will involve solving the axial force balance equation along a 1D domain representing the round bar. This allows us to determine the axial displacement $u$ as a function of the x-coordinate, which defines the 1D space. We can express the axial force balance equation in this form:

$$-E\frac{d^2u}{dx^2} = 0$$

If we consider $u = 0$ at $x = 0$ and $E\frac{du}{dx} = \frac{F}{A}$ at $x = L$.

Eventually it will be $u(x) = \frac{Fx}{EA}$, where u(x) quantifies the deformation of the round bar in the longitudinal direction.

The stiffness of every n-spring ($\mathbf{k}_{\mathrm{i}}$) will be $k_i = nEA/L$.

The maximum displacement occurs at the end of the bar x = L, and its value is given by $u_{\max} = FL/EA$. This equation allows us to calculate the equivalent single spring constant, k, using the formula

$$k = \frac{EA}{L}$$

The 1D model's spring constant at the end point ($x = L$) is similar to the 0D model's for these parameters and equals to $k = 628.32 \times 10^7$ N/m

## Bending Stiffness for 1D

For a round bar, where the y and z perspectives are similar, it is sufficient to measure from one position. An additional benefit of using a simplified 1D model is that we can still analyze the effect of the loading direction. Even though the analysis is in 1D, we can calculate the out-of-plane displacements in either the y or z direction. Based on the clamped-free boundary conditions, the displacement will vary as a function of the x-coordinate.

For a round bar, investigating this scenario would require introducing additional stiffness terms that relate the bending force to the out-of-plane displacements. This involves solving the following moment-balance equation:

$$-\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)=0$$

with the boundary conditions: at $x=0; w=0$ and $\frac{dw}{dx}=0$ and at $x=L; \frac{d^2w}{dx^2}=0$ and $-EI\frac{d^3w}{dx^3}=F$

For a round bar, we have used the displacement (w) along the z-direction for illustrative purposes, but the same concept applies to the displacement (v) along the y-direction. Since the bar's deformation is assumed to be much smaller than its size, these expressions can be interpreted as follows.

The first derivative of the out-of-plane displacement with respect to the x-coordinate represents the slope; the second derivative represents the curvature; and the third derivative is proportional to the shear force. In these equations, the term $I$ represents the second moment of inertia, which depends on the direction of bending. For bending about the y-axis (i.e., force acting along the z-direction), it can be expressed as:

$$I_b=\frac{1}{4}\pi r^4$$

In our case, we have cantilever beam loaded at end which impact to round bar's deflection and spring rate both same for y and z perspective. The spring rate formula will be:

$$k_b=\frac{3EI}{L^3}=\frac{3E\pi r^4}{4L^3}$$

For our round bar's parameters r = 0.1 m and L = 1 m, $\mathrm{k_b} = 4.71239\times10^7$ N/m

## Computing k in COMSOL

We have simple interface to see what happens to material if we add 3 load groups assigned to "Point Load". We will add 3 variables: $k_{xx}, k_{yy}, k_{zz}$.

Variables

| Name | Expression | Unit |
|---|---|---|
| kxx | F0/with(1, aveop1(u)) | N/m |
| kyy | F0/with(2, aveop1(u)) | N/m |
| kzz | F0/with(3, aveop1(u)) | N/m |

Figure 3. Here Average Coupling operator assigned to free end and it is necessary to compute displacement at point x = L. The operator with() is useful to analyze solutions from 3 load cases.

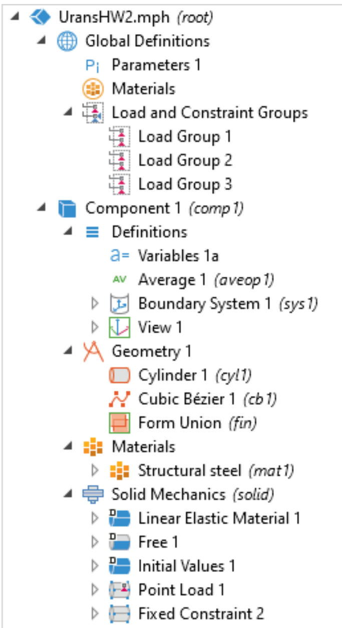


Figure 4. A screenshot of the 3D model of round bar interface

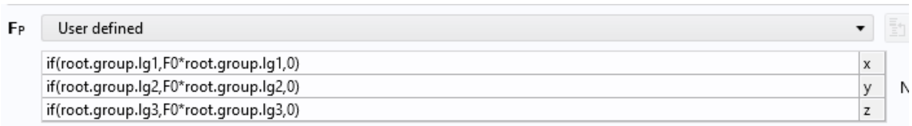


Figure 5. Point load parameters for load groups, 3 points where L=1 was picked to evaluate best result

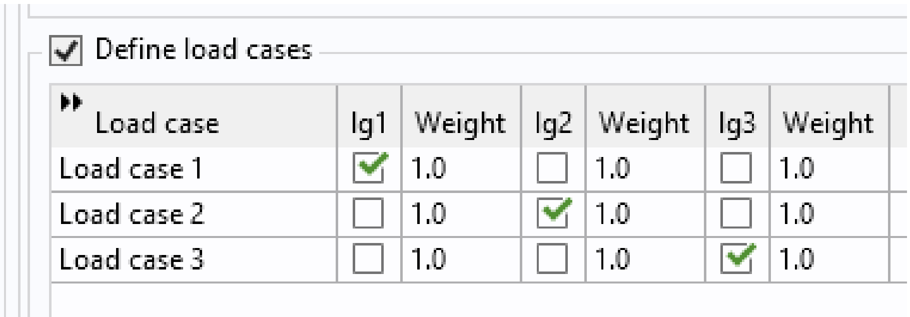


Figure 6. Three load cases for stationary study with enabled parameters fro lg1, lg2 and lg3

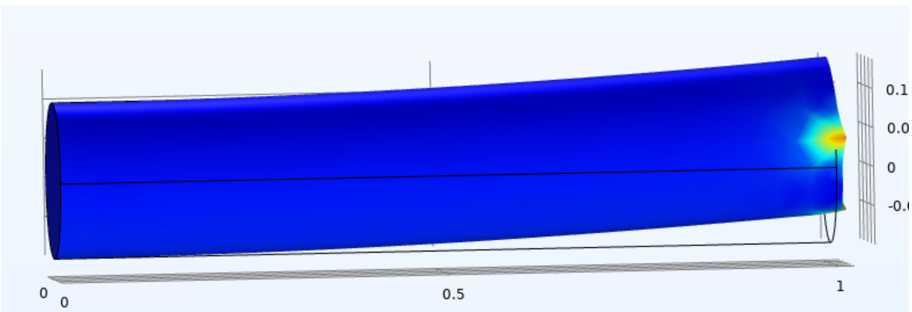


Figure 7. Deformed round bar after point load. The result from this study shows that stiffness from simulation where $k_b = 4.46 \times 10^7$ N/m

## Stiffness for 2D

When modeling a round bar, we don't deal with typical 2D plane stress or plane strain assumptions, as the symmetry of the bar simplifies the analysis. However, similar considerations can be applied when using 2D modeling in COMSOL Multiphysics by selecting the 2D space dimension under the Solid Mechanics interface. You can choose between "plane stress" and "plane strain" conditions from the drop-down menu, but for a round bar, these options may need to be interpreted based on cylindrical symmetry.

**Plane Strain Option**

The "plane strain" option assumes zero out-of-plane strain, which can be useful when there are only in-plane forces and the structure is constrained in the out-of-plane direction. This assumption, however, is not typically suitable for a round bar, as its deformation and stress distribution are not confined to a single plane. In this formulation, COMSOL solves for in-plane displacements, but it wouldn't capture the full 3D behavior of a round bar.

**Plane Stress Option**

The "plane stress" option assumes zero out-of-plane stress and is better suited for thin structures where out-of-plane stresses are negligible. In this case, COMSOL solves for in-plane displacements and can include the out-of-plane strain. For a round bar under symmetric loading conditions, this option may still miss important aspects of its behavior, but if you're modeling a simplified 2D section, it can provide approximate results. The choice depends on the boundary conditions, and for certain cases, such as axial or torsional loads, the cylindrical symmetry should be accounted for.

Figure 8. Perfect stress condition for round bar with axial load

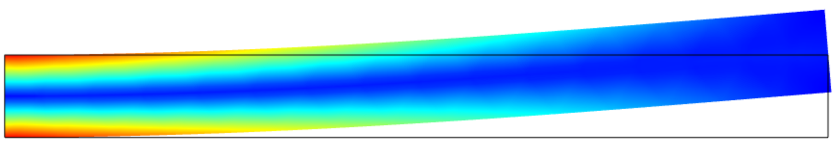

Figure 9. Perfect stress condition for round bar with transverse load

These illustrations show that axial load cause consistent stress across the cylinder, therefore stiffness for this deformation for given object is constant. On the other hand, Figure 9 indicating that stiffness near fixed constraint is higher due to concentrated stiffening near the fixed end of the round bar. In conclusion, the average stiffness for axial and transverse deformation is $k = 4.71239 \times 10^7$ N/m same as results from 1D deformation.

## Stiffness for 3D

The 2D modeling technique works well for a round bar as long as there are no forces acting perpendicular to the plane and the forces within the plane remain uniform along the bar's length. However, for more complex loading scenarios and constraints, a 3D model would deliver more precise results, though it would demand more computational resources.

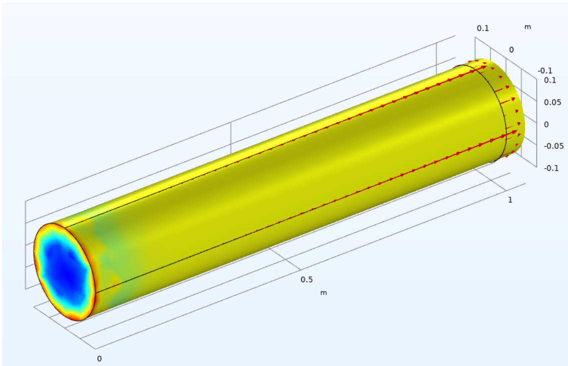

Figure 10. Axial loading for cylinder using $\sigma_{xx}$ component of stress

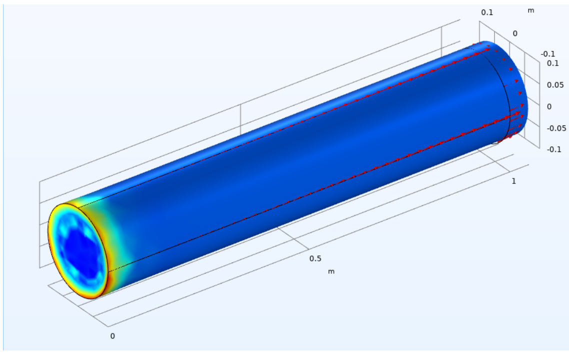

Figure 11. Axial loading for cylinder using $\sigma_{yy}$ component of stress

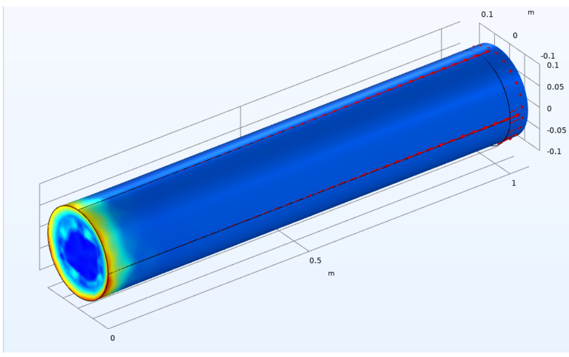

Figure 12. Axial loading for cylinder using $\sigma_{zz}$ component of stress

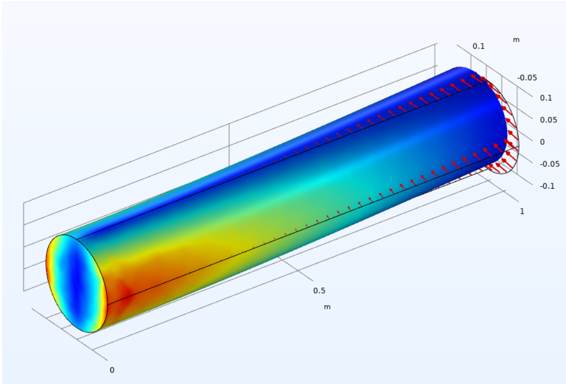


Figure 13. Transverse loading in y-axis direction for cylinder using $\sigma_{xx}$ component of stress

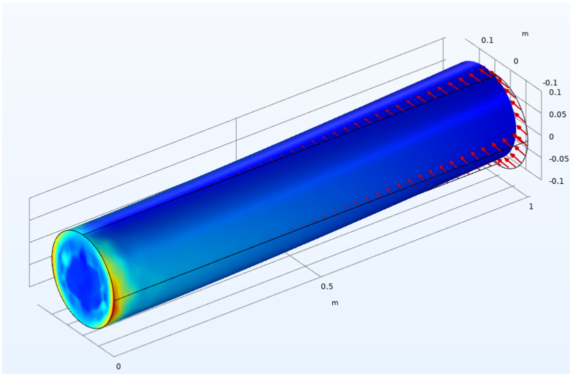


Figure 14. Transverse loading in y-axis direction for cylinder using $\sigma_{yy}$ component of stress

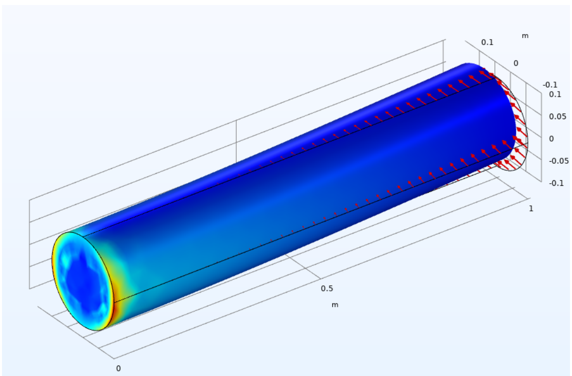


Figure 15. Transverse loading in y-axis direction for cylinder using $\sigma_{zz}$ component of stress

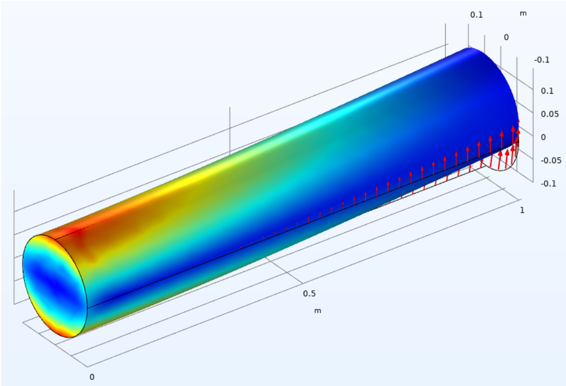


Figure 16. Transverse loading in z-axis direction for cylinder using $\sigma_{xx}$ component of stress

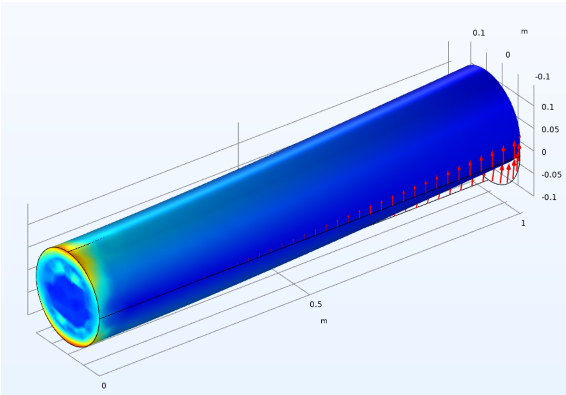


Figure 17. Transverse loading in z-axis direction for cylinder using $\sigma_{yy}$ component of stress

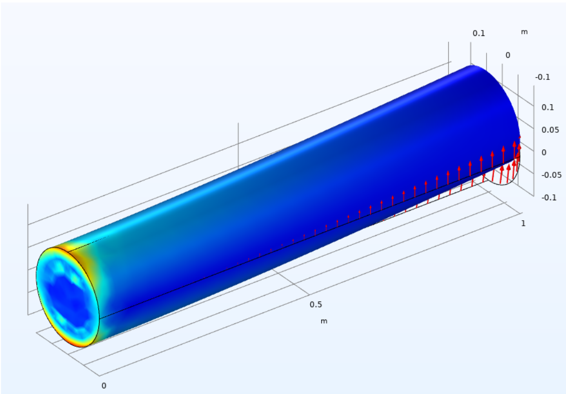


Figure 18. Transverse loading in z-axis direction for cylinder using $\sigma_{zz}$ component of stress

## Results

The results from the 3D structural analysis revealed that the stiffness of the round steel bar under transverse loading in both the y-axis and z-axis directions was found to be consistent. Specifically, the stiffness in these directions was calculated to be $k_{yy} = k_{zz} = 4.7 \times 10^7$ N/m. This uniform stiffness is due to the symmetric nature of the bar's geometry and the boundary conditions applied, as demonstrated in Figures 13-18, which produce similar deformation behavior under transverse loads in the orthogonal directions.

In contrast, the stiffness along the x-axis, where axial loading was applied, was significantly higher. The axial stiffness was determined to be $k_{xx} = 6.2 \times 10^9$ N/m. Looking for the Figures 10-12, we can say that normal stress for three components were quite similar. This increased stiffness is attributable to the bar's resistance to deformation along its longitudinal axis, as axial stiffness is primarily influenced by the material's Young's modulus and the cross-sectional area of the bar.

These findings underscore the importance of considering directional stiffness variations in structural mechanics, especially when dealing with symmetric geometries like round bars. The differences in stiffness between axial and transverse directions directly reflect the mechanical properties and geometric configuration of the structure, which are crucial for accurately predicting its response under varying load conditions.