Computer Vision for HCI

Principal Components Analysis (PCA)

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Feature Sub-Spaces

- Many times a high-dimensional feature vector is contained within a lower-dimensional “sub-space”
- When processing the feature data (for modeling and/or recognition), it is beneficial to deal with the lower-dimensional sub-space
- PCA offers **<u>linear</u>** approximation to the sub-space which can be reduced to only the *major* sub-space dimensions

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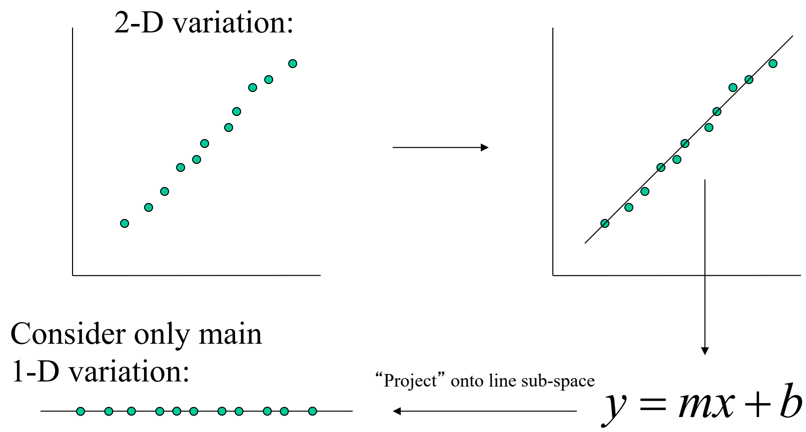
Dimensionality Reduction



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Linear Basis Set

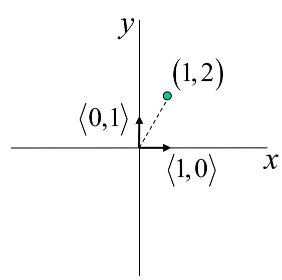
- 2-D basis set

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

x,y coordinates

x-axis

y-axis



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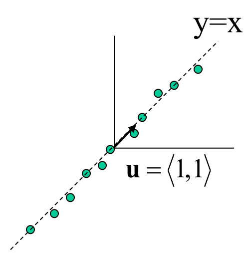
Linear Basis Set

- 1-D sub-space basis set



2-D space:

$$\mathbf{x_i} = a_i \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_i \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1-D sub-space:

$$y_i = \gamma_i \cdot \mathbf{u} = \gamma_i \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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Linear Basis Set

- Generate smaller dimension basis for feature vectors

$$\mathbf{x_i} = \gamma_1 \cdot \mathbf{u_1} + \gamma_2 \cdot \mathbf{u_2} + \cdots + \gamma_m \cdot \mathbf{u_m}$$

$$\text{where} \quad \dim(\mathbf{x}) > m$$

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Principal Components Analysis

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PCA

- The main idea for PCA
  - Fit a multi-dimensional Gaussian around data
    - Use covariance of data to model Gaussian
  - Select only those dimensions capturing most of the variance in data
    - Reduce dimensionality
  - Sub-space is uncorrelated

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PCA Primer: Equation for Eigenvalues

Square matrix

$$\mathbf{Ax} = \lambda \underline{\mathbf{x}}$$

Stretch/Scale **x** (Eigenvalue)

Same direction as **Ax**

Certain vector (Eigenvector)

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

A is required to be square matrix for PCA

Want **x** to be non-zero, thus **A** is not invertible (determinant is zero). Each root leads to **x**.

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PCA Primer: Example for Finding Eigenvalues

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(4-\lambda) - (2)(2) = \lambda^2 - 5\lambda = 0$$

two roots: $\lambda_1 = 0, \quad \lambda_2 = 5$

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PCA Primer: Finding the Eigenvectors

$$(\mathbf{A}-\lambda_i\mathbf{I})\mathbf{x}_i=\mathbf{0}$$

$$(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{e}_1=\mathbf{0}$$

$$\left(\begin{bmatrix}1&2\\2&4\end{bmatrix}-\begin{bmatrix}0&0\\0&0\end{bmatrix}\right)\mathbf{e}_1=\mathbf{0}$$

$$\begin{bmatrix}1&2\\2&4\end{bmatrix}\begin{bmatrix}x_1\\y_1\end{bmatrix}=\mathbf{0}$$

$$\mathbf{e}_1=\begin{bmatrix}x_1\\y_1\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$$

$$(\mathbf{A}-\lambda_2\mathbf{I})\mathbf{e}_2=\mathbf{0}$$

$$\left(\begin{bmatrix}1&2\\2&4\end{bmatrix}-\begin{bmatrix}5&0\\0&5\end{bmatrix}\right)\mathbf{e}_2=\mathbf{0}$$

$$\begin{bmatrix}-4&2\\2&-1\end{bmatrix}\begin{bmatrix}x_2\\y_2\end{bmatrix}=\mathbf{0}$$

$$\mathbf{e}_2=\begin{bmatrix}x_2\\y_2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

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PCA Primer: Test Our Results

$$\mathbf{A}\mathbf{e}_1\overset{?}{=}\lambda_1\mathbf{e}_1$$

$$\begin{bmatrix}1&2\\2&4\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}(1\cdot2-2\cdot1)\\(2\cdot2-4\cdot1)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}=0\begin{bmatrix}2\\-1\end{bmatrix}$$

$$\mathbf{A}\mathbf{e}_2\overset{?}{=}\lambda_2\mathbf{e}_2$$

$$\begin{bmatrix}1&2\\2&4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}(1\cdot1+2\cdot2)\\(2\cdot1+4\cdot2)\end{bmatrix}=\begin{bmatrix}5\\10\end{bmatrix}=5\begin{bmatrix}1\\2\end{bmatrix}$$

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PCA Primer: Symmetric Matrices

- If **A** is <u>symmetric</u> ($\mathbf{A} = \mathbf{A}^T$)
  - Real-valued eigenvalues***
  - Eigenvectors can be chosen orthonormal***
  - Can be factorized as

    $$\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$$

    where **Q** orthonormal ($\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$)

    and Λ is diagonal
  - Eigenvalues go into diagonal entries of Λ
    - **Convention: put <u>largest</u> eigenvalues <u>first</u> (descending values)**
    - **Alternative: put <u>smallest</u> eigenvalues <u>first</u> (ascending values)**
  - Corresponding orthogonal eigenvectors are normalized (to become orthonormal) and go into columns of **Q**

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PCA Primer: Matlab

```
>> A= [ 1 2; 2 4]
A =
    1    2
    2    4

>> [Q, S] = eig(A)
Q =
  -0.8944    0.4472
   0.4472    0.8944
S =
    0    0
    0    5
```

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PCA Primer: Matlab

```
>> A= [ 1 2; 2 4];

>> [Q, S] = eig(A);

>> Q'*Q
ans =
    1.0000         0
         0    1.0000

>> A - Q*S*Q'
ans =
     0     0
     0     0
```

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Positive Definite Matrices

- Matrix A is <u>positive definite</u> for all non-zero **x** if
  - Symmetric <u>and</u> $\mathbf{x}^T \mathbf{A}\mathbf{x} > 0$

Positive <u>semi</u>-definite if
$\mathbf{x}^T \mathbf{A}\mathbf{x} \geq 0$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} > 0$$

$$ax^2 + 2bxy + cy^2 > 0$$

- Recall equation for ellipse

$$ax^2 + by^2 = c$$

Rotated ellipse: $ax^2 + 2bxy + cy^2 = d$

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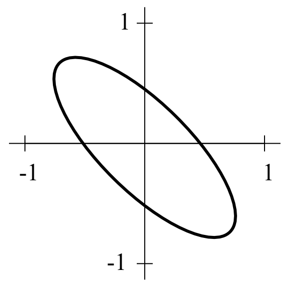
Ellipse Factorization

Consider the rotated ellipse:

$$5x^2 + 8xy + 5y^2 = 1$$



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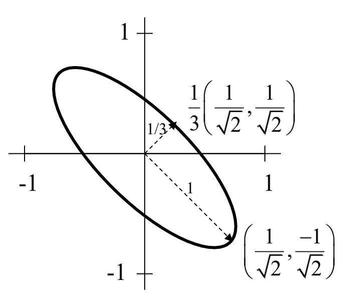
Ellipse Factorization

$$5x^2 + 8xy + 5y^2 = 1 \quad \rightarrow \quad a = 5,\ b = 4,\ c = 5$$

$$\mathbf{x}^T \mathbf{A}\mathbf{x} = 1$$

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \mathbf{Q\Lambda Q^T} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{\sqrt{2}} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{\sqrt{2}}$$



Axes of ellipse point along eigenvectors

Half-lengths of axes are $1/\sqrt{\lambda_i}$ (NOTE: <u>bigger</u> eigenvalues give <u>shorter</u> axes!)

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Eigenvector Projection

- The matrix $\mathbf{Q^T}$ acts as a rotation matrix

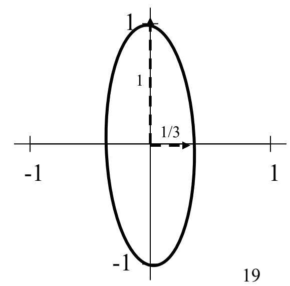
$$\mathbf{x^T A x} = \mathbf{x^T}\left(\mathbf{Q \Lambda Q^T}\right)\mathbf{x} = \left(\mathbf{x^T Q}\right)\mathbf{\Lambda}\left(\mathbf{Q^T x}\right) = \mathbf{X^T \Lambda X}$$

$$\mathbf{X} = \mathbf{Q^T x} = \begin{bmatrix} \hat{\mathbf{e}}_1^T \mathbf{x} \\ \hat{\mathbf{e}}_2^T \mathbf{x} \end{bmatrix}$$

Rotate previous axes:

$$\mathbf{X}_1 = \mathbf{Q^T x}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\sqrt{2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{X}_2 = \mathbf{Q^T x}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\sqrt{2}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



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Gaussian Density

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} |\mathbf{K}|^{\frac{1}{2}}} \cdot e^{\left[-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{K}^{-1}(\mathbf{x}-\mathbf{m})\right]}$$

Remember this???

(squared) Mahalanobis distance: $(\mathbf{x}-\mathbf{m})^T \mathbf{K}^{-1}(\mathbf{x}-\mathbf{m}) = C$

Locus of all points at given distance C from mean (i.e., variance contour with C = # stdev^2)

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Covariance

Covariance of vector **x**:

$$K = E\left[(\mathbf{x}-\mathbf{m})(\mathbf{x}-\mathbf{m})^T\right] = \frac{1}{N}\mathbf{B}\mathbf{B}^\mathbf{T}$$

K is symmetric (and positive [semi-]definite)

Look at MATLAB's function **cov( )**

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Plotting Gaussian Density Function Contours

Describes a (rotated) ellipse (at a *C* variance contour, or squared stdev contour) centered around mean:

$$\left(\mathbf{x}-\mathbf{m}\right)^T \mathbf{K}^{\mathbf{-1}}\left(\mathbf{x}-\mathbf{m}\right) = C$$

Thus we can factorize $\mathbf{K}^{\mathbf{-1}}$ into eigenvectors (axes) and eigenvalues to give direction and half-lengths of ellipse axes

$$\mathbf{K} = \mathbf{Q\Lambda Q^T}$$

$$\mathbf{K^{-1}} = \mathbf{Q\Lambda^{-1}Q^T}$$

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Axes

- To compute the ellipse at Gaussian "variance" contour C (= # stdev^2):
  - **Method #1**: From matrix $\mathbf{K}^{-1}$ (<u>inverse</u> covariance)
    - Axes are columns in **Q**,
    - Half-lengths of axes are $\frac{\sqrt{C}}{\sqrt{\lambda_i}}$, with λ_i from $\mathbf{\Lambda}^{-1}$
  - **Method #2**: From matrix **K** (covariance)
    - Axes are columns in **Q**,
    - Half-lengths of axes are $\sqrt{C\lambda_i}$, with λ_i from $\mathbf{\Lambda}$

Putting it all together…

Ellipse formula:

$$x^T A x = 1$$

Ellipse defined by eigenvectors/values of A

Gaussian & Mahalanobis:

$$(x - m)^T K^{-1} (x - m) = C$$

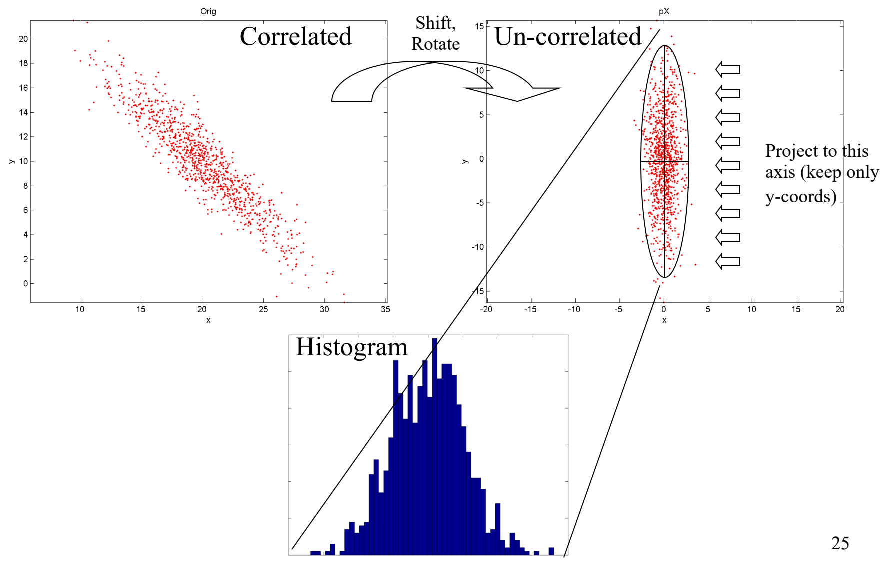
$$A = K^{-1}$$

Ellipse defined by eigenvectors/values of K^{-1}

<u>Larger</u> values of C give <u>bigger</u> variance contour ellipses

Or use K and just "flip" the eigenvalues (saves doing the inverse operation)

Gaussian Spaces



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Face Recognition
(early days…)

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Face Recognition Using EigenFaces (Turk & Pentland 91)

- Treat face recognition as 2-D view-based problem (images) rather than 3-D reconstruction and matching
- Project face images onto Gaussian feature space spanning significant variations among known face images
- Significant features in eigenspace for projection are called "EigenFaces"
  - Those eigenvectors capturing the majority of "variance" in data
    - Largest eigenvalues correspond to largest component variances
  - Project face image (vector) onto selected top eigenvectors

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Using EigenFaces to Classify a Face Image

- Recognition achieved by comparing weights/coefficients of new face (after projection onto eigenspace) to other stored face weights/coefficients
  - Distance calculation more compact and efficient (uses small number of weights/coefficients)
- Calculate distance from face space or individual
  - Does it look like a face?
  - Does it look like "Joe"?

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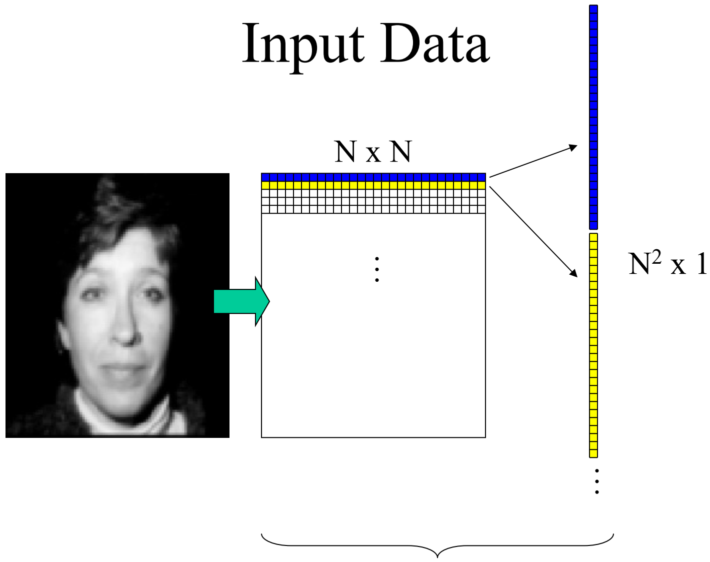
Input Data



Rasterize the image into vector

Input Data

- Compute mean face image $\boldsymbol{\Psi}$
  - From set of rasterized face images $\boldsymbol{\Gamma}_i$ (training images)
- Subtract mean from images
  - Remove the mean using $\boldsymbol{\Phi}_i = \boldsymbol{\Gamma}_i - \boldsymbol{\Psi}$
  - Form matrix of training faces

$$A = [\boldsymbol{\Phi}_1 \ \ \boldsymbol{\Phi}_2 \ \ \boldsymbol{\Phi}_3 \ \ \cdots \ \ \boldsymbol{\Phi}_M]$$

- Compute covariance matrix of A

Extra: Eigen "Trick"

- Compute eigenvectors and eigenvalues of A from its covariance matrix
  - Gaussian sub-space
- Number of face images (M) is much less than the dimension of the space (N^2)
- Thus only M-1 meaningful eigenvectors in matrix A
  - Remaining eigenvectors have eigenvalues = 0
- Can solve using a much smaller MxM matrix and convert back to N^2x1 dimensionality

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Extra: Eigen "Trick"

- Consider the eigenvectors $\boldsymbol{v}_i$ of A^TA
  - Recall AA^T is how to compute covariance matrix K for $\boldsymbol{\Phi}_i$ data

$$(A^TA)\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$$

- Pre-multiplying both sized by A, we get

$$A(A^TA)\boldsymbol{v}_i = \lambda_i A\boldsymbol{v}_i$$

- Thus $\boldsymbol{u}_i=A\boldsymbol{v}_i$ are the eigenvectors of AA^T

$$\text{Covariance matrix} \longrightarrow (AA^T)[A\boldsymbol{v}_i] = \lambda_i [A\boldsymbol{v}_i]$$

- Size comparison
  - Matrix A^TA is size MxM
  - Matrix AA^T (a covariance) is size N^2xN^2
- Hence, compute eigenvectors/eigenvalues from A^TA and use $\boldsymbol{u}_i=A\boldsymbol{v}_i$ to recover the desired dimensionality
  - Make sure to normalize the $\boldsymbol{u}_i$ to make them unit vectors!

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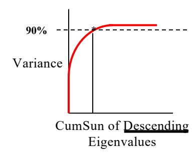
Compute Eigenvectors

- Retain only top *m* eigenvectors ($\boldsymbol{u}_k$)
  - Determine from strength of eigenvalues
  - These eigenvectors are the "EigenFaces"
- Accumulate eigenvalues until reach desired percentage of total sum of eigenvalues
  - Pre-sort eigenvalues from largest to smallest
  - Considered as "% variance captured"
  - Typically use around 90%



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Projection *into* Face Space

- Project each rasterized (mean-subtracted) face image into sub-space (*m* eigenvectors)
  - Project onto each eigenvector ("EigenFace")

$$\omega_k = \boldsymbol{u}_k^T \cdot \boldsymbol{\Phi}_i$$

- Keep projection coefficients as representation of rasterized image

$$\boldsymbol{\Phi}_i \longrightarrow \boldsymbol{\Omega}_i = [\omega_1\ \omega_2\ \omega_3\ \cdots\ \omega_m]^T$$

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Reconstruction *from* Face Space

- Reconstruct face image from sub-space
  - Reconstruct from projection coefficients on each eigenvector

$$\boldsymbol{\Phi}_{recon} = \sum \omega_k \cdot \boldsymbol{u}_k$$

- Add back mean face

$$\boldsymbol{\Gamma}_{recon} = \boldsymbol{\Phi}_{recon} + \boldsymbol{\Psi}$$

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Recognition Pipeline

- Get new face image
  - Rasterize
  - Mean-subtract using $\boldsymbol{\Psi}$
- Compute $\boldsymbol{\Omega}$
- Use Sum-of-Squared-Error (SSE) to other faces in the database
  - Find best match for database items $\boldsymbol{\Omega}_i$ (assumes that new face is in the database)

$$argmin_i \ \mathcal{E} = \|\boldsymbol{\Omega} - \boldsymbol{\Omega}_i\|^2$$

- Make sure "close enough" to face space and to person
  - Reconstruct new face image (from its projection coefficients and the eigenvectors), then threshold error distance from original face image
    - Face space only reconstructs face-like images
  - Make sure best match is within error tolerance to that person
- Other methods exist, including probabilistic methods

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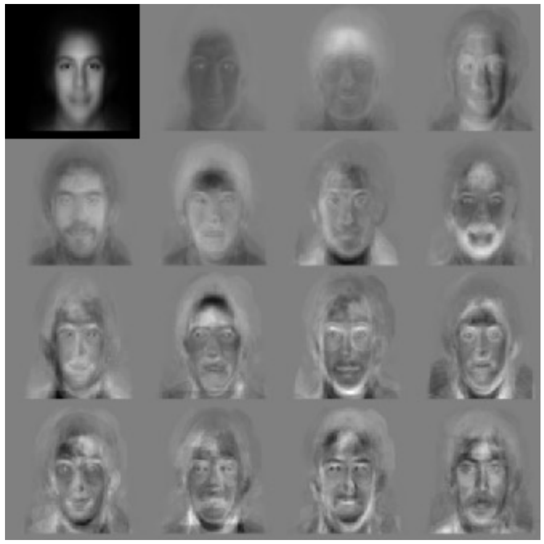
Example EigenFaces

Mean face



40 vectors were sufficient for 115 training face images

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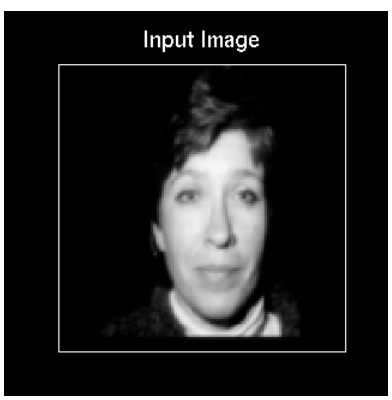
EigenFace Reconstruction



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Face Recognition System

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Summary

- Reduce high-dimensional input to lower-dimensional "sub-space"
  - Beneficial for recognition
- PCA offers <u>linear</u> approximation to the sub-space which can be reduced to only the *major* sub-space dimensions
  - Assumes Gaussian distribution
- Initial face recognition methods based on PCA
  - EigenFaces

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