
Cumulative Reading Assignments:

- Read Boyd & Vandenberghe, Chapter 2, Sections 2.1-2.3
- Read Boyd & Vandenberghe, Chapter 3, Sections 3.1-3.2, and 3.5.
- Read Boyd & Vandenberghe, Chapter 4, Sections 4.1-4.2
- Read Boyd & Vandenberghe, Chapter 5, Sections 5.1-5.6
- Read Boyd & Vandenberghe, Chapter 9, Sections 9.1-9.4

- (1) (Strong Duality) Prove part (ii) of the Strong Duality Theorem for Inequality Constraints that was left out in Lecture 15.

(Hint: Consider the non-vertical hyperplane from the proof of part (i) with normal $(\tilde{\mu}, 1)^t$ that achieves q^* , and show that

$$\sum_{j=1}^m \tilde{\mu}_j \leq \frac{f(\mathbf{y}) - q^*}{\min_j |g_j(\mathbf{y})|} < \infty,$$

where \mathbf{y} is the vector achieving the Slater's Condition.)

Remark: It turns out that, conversely, Slater's condition must hold for a problem with *bounded* set of dual optimal solutions.

- (2) (Optimal Primal-Dual Pairs) Prove the primal-dual optimality theorem that is left out on Lecture 16.

(Hint: Since we have already proven a very similar theorem stated on Lecture 15, all you need to prove is that condition (iv) of the previous theorem is true if and only if the complementary slackness condition of this theorem holds.)

- (3) (Lagrangian Duality) Problem 5.1 from Boyd & Vandenberghe.

- (4) (Multiple Duals of the same Primal) Consider the following optimization problem in \mathcal{R}^2 :

$$\begin{array}{ll} \text{minimize} & e^{-x_2} \\ \text{subject to} & \|x\| \leq x_1, \ x_2 \geq 0. \end{array}$$

- (a) What is the constraint set? What is the optimal value of the problem?
- (b) Consider a dual problem obtained by relaxing **only** the constraint $x_2 \geq 0$. What is the dual optimal value? Is there a duality gap?
- (c) Consider a dual problem obtained by relaxing **only** the constraint $\|x\| \leq x_1$. What is the dual optimal value in this case? Is there a duality gap?
- (d) What is your interpretation of the meaning of the results obtained in (b) and (c)?
- (5) (Convexity and Duality) Problem 5.22 parts (a), (b), and (d) from Boyd & Vandenberghe.
- (6) (KKT Conditions) Problem 5.29 from Boyd & Vandenberghe.

- (7) (Dual Formulation and Solution) Derive the dual of the following problem as an unconstrained maximization over a k dimensional space, comment on the existence of primal dual solutions, and clearly relate (if they exist) the dual optimal to the primal optimal solutions.

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n \left[x_i \log \left(\frac{x_i}{c_i} \right) - x_i \right] \\ & \text{subject to} && \mathbf{x} \succeq \mathbf{0}, \quad A\mathbf{x} = \mathbf{b}, \end{aligned}$$

where $\mathbf{c} \succeq \mathbf{0}$, $\mathbf{b} \in \mathcal{R}^k$, and $A \in \mathcal{R}^{k \times n}$ is a $k \times n$ matrix with i^{th} column denoted as $A_i \in \mathcal{R}^k$.

- (8) (Gradient-Projection Method Implementation) Write the code (submit along with your solution) for the projection gradient method that solves:

$$\begin{aligned} & \text{minimize} && x_1^2 + 9x_2^2 \\ & \text{subject to} && \mathbf{x} \succeq \mathbf{0}, \\ & && 2x_1 + x_2 \geq 1, \\ & && x_1 + 3x_2 \geq 1. \end{aligned}$$

Choose different stepsizes and demonstrate how the algorithm converges and/or diverges, and provide a justified suggestion of an optimal stepsize choice.