# Problem Set-4

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**Q1]** Let the probability of a successful transmission for link $l$

a) in a given time slot as $S_l(P)$, which is equal to the probability that link $l$ attempts a transmission and none of its interfering neighbours do so. Thus,

$$S_e(P) = P_e \cdot \prod (1-P_i) \quad \forall\, i \in N_e$$

where $\prod(1-P_i)$ denotes the product of $(1-P_i)$ over all interfering neighbours $i$ of link $l$.

Now, average rate of success of link $l$ transmissions, denoted as $\mathcal{X}_e(P)$, is given by the expectation of $S_e(P)$ over all time slots. That is,

$$\mathcal{X}_e(P) = E[S_e(P)]$$

To compute $S_l(P)$, we can use the fact that the transmission attempts of different links are independent. Thus, the probability that a particular subset $S$ of links transmits in a given slot is:

$$P(S) = \prod (P_e \text{ if } l \in S;\ 1-P_e \text{ if } l \notin S)$$

Probability that link $l$ successfully transmits, given that its interfering neighbours do not transmit is:

$$s_l = P(\{l\} \cup N_l^c) \Big/ P(N_l^c)$$

where $N_l^c$ denotes the complement of $N_l$ i.e. the set of links that do not interfere with link $l$. Here, $P(\{l\} \cup N_l^c)$ is the probability that only link $l$ transmits, whereas $P(N_l^c)$ is the probability that none of the interfering neighbours of link $l$ transmit.

Using $s_l$ and $P(S)$, we can express $x_l(p)$ as:

$$x_l(p) = E[s_l(p)]$$

$$= \sum\{S \subseteq \{1, \ldots L\}, l \in S\} P(S) * s_l$$

$$= \sum\{S \subseteq \{1, \ldots L\}, l \in S\} P(S) * P(\{l\} \cup N_l^c) \Big/ P(N_l^c)$$

$$= p_l * \sum\{S \subseteq N_l^c\} P(\{l\} \cup S) \Big/ P(N_l^c)$$

$$= \sum_{S \in 2^{N_l}} \Big( \prod_{j \in S} 1 - p$$

(c) Total utility maximization problem can be expressed as:

$$\max \sum_{l} W_l \log(x_l(P))$$

subject to $0 \leq P_l \leq 1 \quad \forall\, l$

where $W_l > 0$ is the utility parameter for link $l$, $\log(x_l(P))$ is the logarithmic of the success rate of link $l$ transmission, and $P$ is the vector of link probabilities.

To show that this problem is convex, we can take the second derivative of the objective funct^n with respect to the probability vector $p$:

$$\frac{\partial^2}{\partial p^2}\left(\sum_{l} W_l \log(x_l(P))\right) = -\sum_{l}\left(\frac{W_l}{x_l(P)}\right)^2 \cdot \frac{\partial}{\partial p^2}(x_l(P))$$

Here, the objective function is the sum of logarithmic terms, & second derivative is $\leq 0$

$\therefore$ Given objective funct^n is concave

c) $\max \sum_{\ell} w_\ell \log(x_\ell(p))$

subject to $0 \leq p_\ell \leq 1 \quad \forall \ell$

$$\frac{\partial}{\partial p} \sum_{\ell} w_\ell \log(x_\ell(p)) = \frac{w_\ell}{x_\ell(p)} * \frac{\partial}{\partial p}(x_\ell(p)) * \frac{1}{p_\ell}$$

Setting this to zero, we get:

$$\frac{\partial}{\partial p}(x_\ell(p)) * \frac{1}{p_\ell} = 0$$

$\because \frac{1}{p_\ell} > 0 \quad \forall p_\ell \in [0, 1]$

$\therefore \frac{\partial}{\partial p}(x_\ell(p)) = 0$

$\therefore p^*$ is given by:

$$p_\ell = \frac{x_\ell(p^*)}{\sum\{q \in N_\ell^c\} x_\ell(p^*)}$$

where $x_\ell(p^*)$ is the optimal success rate of link $\ell$ transmission, Q3] $\max \sum_{i=1}^{n} \alpha_i W_i \log\left(1+\frac{\beta_i P_i}{W_i}\right)$

subject to $P_i \geq 0, \; W_i \geq 0 \quad \forall i$

$$\sum_{i=1}^{n} P_i \leq P_{tot}$$

$$\sum_{i=1}^{n} W_i = W_{tot}$$

Associate a Lagrangian multiplier $\mu$ with inequality constraint $\sum_{i=1}^{n} P_i \leq P_{tot}$, we get

$$L(p,\mu) = \sum_{i=1}^{n} \alpha_i W_i \log\left(1+\frac{\beta_i P_i}{W_i}\right) - \mu\left(\sum_{i=1}^{n} P_i - P_{tot}\right)$$

Negative sign is due to maximization instead of minimization.

$$= \sum_{i=1}^{n}\left(\alpha_i W_i \log\left(1+\frac{\beta_i P_i}{W_i}\right) - \mu P_i\right) + \mu P_{tot}$$

$$q(\mu) = \max_{P \geq 0} L(p,\mu) = \max \sum_{i=1}^{n}\left[\alpha_i W_i \log\left(1+\frac{\beta_i P_i}{W_i}\right) - \mu P_i\right] + \mu P_{tot}$$

To find $P_i^*$:

$$\frac{\partial q}{\partial P_i} = 0 \Rightarrow \frac{\alpha_i W_i}{1+\frac{\beta_i P_i^*}{W_i}} \cdot \frac{\beta_i}{W_i} - \mu = 0$$

$$\Rightarrow \frac{\alpha_i \beta_i}{\dfrac{\omega_i + \beta_i p_i^*}{\omega_i}} = \mu$$

$$\Rightarrow \frac{\alpha_i \beta_i \omega_i}{\mu} = \omega_i + \beta_i p_i^*$$

$$\Rightarrow \beta_i p_i^* = \frac{\alpha_i \beta_i \omega_i - \mu \omega_i}{\mu}$$

$$\Rightarrow \boxed{p_i^* = \frac{\alpha_i \omega_i}{\mu} - \frac{\omega_i}{\beta_i}}$$

GPM for $\min_{\mu \geq 0} q(\mu)$ is

$$\mu^{(t+1)} = \left[\mu^{(t)} - \gamma^{(t)} \nabla q(\mu)\right]^+$$

$$\boxed{\mu^{(t+1)} = \mu^{(t)} - \gamma^{(t)} \left( \sum_{i=1}^{n} \left[\alpha_i \omega_i \log\left(1 + \frac{\beta_i p_i^*}{\omega_i}\right) - \mu p_i^*\right] + \mu p_{tot} \right)}$$

Q2) a) Joint Congestion Control problem for utility maximization and link rate allocation for optimal routing in the power constrained network setting can be formulated as an optimization problem.

Let $x\_ds$ be the rate of flow from source node to destination, $U_{sd}(x\_ds)$ be utility function associated with that flow and the rate achieved by transmitting data at rate $P_{ij}$ over link $(i,j)$ as $R_{ij}(P_{ij})$

Optimization problem can be formulated as:

$$\max. \sum_{s \in S} \sum_{d \in D} U_{sd}(x\_ds)$$

subject to
$$\sum_{i \in N} x\_ids - \sum_{j \in N} x\_sjd = 0 \quad \forall s \in S, d \in D, s \neq d$$

$$R_{ij}(P_{ij}) \leq \log(1 + P_{max_i}/N_{ij}) \quad \forall (i,j) \in L$$

$$x_{ij} \geq 0 \quad \forall (i,j) \in L$$

(b) Lagrangian funct$^n$ for joint congestion control problem for utility maximization and link rate allocation in power constrained n/w

$$L(x, \lambda, \mu, \vartheta) = \sum_{s\in S} \sum_{d\in D} U_{sd}(x_{ds}) + \sum_{(i,j)\in L} \lambda_{ij} R_{ij}(P_{ij})$$

$$- \log\left(1 + P_{max,i} / N_{ij}\right) + \mu_{ij} R_{ij}(P_{ij}) - \log\left(1 + P_{max,j} / N_{ij}\right)$$

$$+ \sum_{(s,d)\in(S,D)} \vartheta_{sd}\left(\sum_{i\in N} x_{ds} - \sum_{j\in N} x_{-dj d}\right)$$

where $d$, $\lambda_{ij}$, $\mu_{ij}$, $\vartheta_{sd}$ are Lagrangian multiplier associated with power constraint, link capacity constraint, flow conservation constraint.

To develop GD, we use Lagrange multiplier:

1. Initialize flow rates $x_{ds}$, Lagr. multipliers $\lambda_{ij}$, $\mu_{ij}$, $\vartheta_{sd}$.

2. Update flow rates $x_{ds}$ by solving optimization problem $s\in S$, $d\in D$

$$x_{ds} = \underset{x_{ds}}{\arg\max} \left(\vartheta_{sd}(x_{ds})\right) + \sum_{i\in N} \lambda_{ij} \left(\log\left(1 + P_{ij}/N_{ij}\right)\right.$$

$$- \log\left(1 + \frac{P_{max\,i}}{N_{ij}}\right) + \mu_{ij}\left(\log 1 + \frac{P_{ij}}{N_{ij}}\right) - \log\left(1 + \frac{P_{max\,j}}{N_{ij}}\right)$$

$$+ \vartheta_{sd}\left(\sum_{i\in N} x_{ids} - \sum_{j\in N} x_{sjd}\right)$$

3. Update:

$$\lambda_{ij} = \lambda_{ij} + \alpha\left(R_{ij}(P_{ij}) - \log\left(1 + \frac{P_{max\,i}}{N_{ij}}\right)\right)$$

$$\mu_{ij} = \mu_{ij} + \alpha\left(R_{ij}(P_{ij}) - \log\left(1 + \frac{P_{max\,j}}{N_{ij}}\right)\right)$$

$$\vartheta_{sd} = \vartheta_{sd} + \beta\left(\sum_{i\in N} x_{ids} - \sum_{j\in N} x_{sjd}\right)$$

4. Check for convergence criteria

5. Once it converges, $x_{ds}$ can be obtained as sol$^n$ of optimization problem, Lagrange multiplier $\lambda_{ij}$, $\mu_{ij}$ & $\vartheta_{sd}$ determine power allocation and satisfy the constraints.

Q4] For a single discrete time queue.

Service process $S(t) = \text{Bernoulli}(\mu)$, $\mu \leq 1$

$$P(S(t)=1) = 1 - P(S(t)=0) = \mu$$

Arrival process, $A(t)$

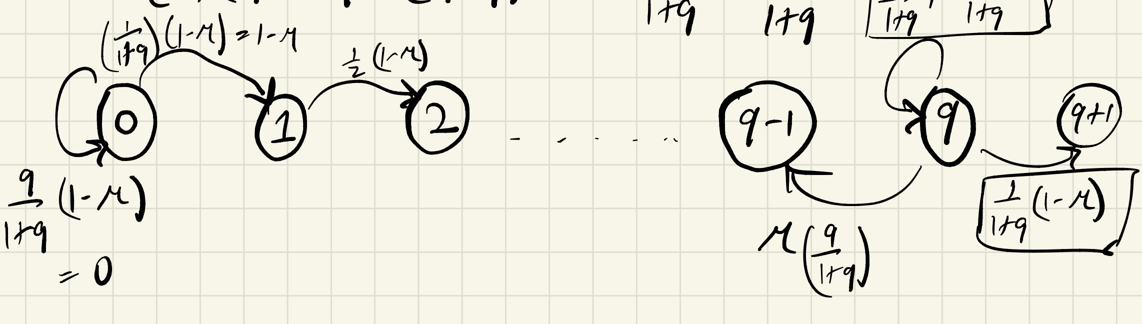
$$P(A(t)=1 \mid Q(t)=q) = 1 - P(A(t)=0 \mid Q(t)=q)$$
$$= \frac{1}{1+q},$$
$$\forall q \in \{0, 1, 2, \dots\}$$

$Q(t)$ is the queue length at time $t$

$$Q(t+1) = \left(Q(t) + A(t) - S(t)\right)^{+}$$

a) Markov Chain diagram of $Q(t)$

$$P(A(t)=0 \mid Q(t)=q) = 1 - \frac{1}{1+q} = \frac{q}{1+q}$$



c) To apply Foster-Lyapunov criterion,

$$\nabla_t V(q) = E[V(Q(t+1)) - V(Q(t)) \mid Q(t) = q]$$
$$\leq -\varepsilon + (M+\varepsilon) \cdot 1(q \in B)$$

where $\varepsilon > 0$ and $M < \infty$ and $B$ is a finite set $\exists$ [illegible]

for $q > 0$, we have

1. $A(t) = 1$, $S(t) = 1$, with probability $\mu\left(\frac{1}{1+q}\right)$
   Queue length remains same so $V(Q(t+1)) - V(Q(t)) = 0$

2. $A(t) = 1$, $S(t) = 0$, with probability $(1-\mu)\left(\frac{1}{1+q}\right)$
   Queue length increases by 1 so $V(Q(t+1)) - V(Q(t)) = 1$

3. $A(t) = 0$, $S(t) = 1$, with probability $\mu\left(\frac{q}{1+q}\right)$
   Queue length decreases by 1 so $V(Q(t+1)) - V(Q(t)) = -1$

4. $A(t) = 0$, $S(t) = 0$, with probability $(1-\mu)\left(\frac{q}{1+q}\right)$
   Queue length remains same so $V(Q(t+1)) - V(Q(t)) = 0$

$$\therefore \nabla_t V(q) = 0 \cdot \mu\left(\frac{1}{1+q}\right) + 1(1-\mu)\left(\frac{1}{1+q}\right) - 1 \cdot \mu\left(\frac{q}{1+q}\right) + 0(1-\mu)\left(\frac{q}{1+q}\right)$$
$$= (1-\mu)\left(\frac{1}{1+q}\right) - \mu\left(\frac{q}{1+q}\right)$$

so, $(1-r)\left(\frac{1}{1+q}\right) - r\left(\frac{q}{1+q}\right) \leq -\varepsilon + (M+\varepsilon)q$

for $\lim_{q\to\infty} q(1-r) - rq \leq -\varepsilon + (M+\varepsilon)q$

This is only possible if $r > 1-\varepsilon$, for $\varepsilon > 0$

Thus Markov Chain is recurrent for $r \in (1-\varepsilon, 1]$.