ECE6500 **Assignment # 4**
Issued: March 28, 2023 Due: April 14, 2023

Cumulative Reading Assignments:

- Read Boyd & Vandenberghe, Sections 2.1-2.3, 2.5-2.6
- Read Boyd & Vandenberghe, Sections 3.1-3.2, 3.4
- Read Boyd & Vandenberghe, Sections 4.1-4.2
- Read Boyd & Vandenberghe, Sections 5.1-5.6
- Read Boyd & Vandenberghe, Sections 9.1-9.5
- Read Bertsekas & Tsitsiklis, Sections 3.1.1, 3.2.1-3.2.3, 3.3.1-3.3.3

(1) Consider a communication network with L links that is operating in a time-slotted manner. Each link l has an associated set of interfering neighbors given by $\mathcal{N}_l$. A constant probability vector $\mathbf{p} := (p_l)_l$ is to be assigned to the links so that in each slot link l will attempt a packet transmission with its assigned probability $p_l \in [0, 1]$ independently. The transmission of link l succeeds in a slot only if it transmits and none of its interfering neighbor does.

(a) Express the average rate of success of link l transmissions, denoted as $x_l(\mathbf{p})$, as a function of $\mathbf{p}$.

(b) Suppose the utility that link l receives from a successful transmission rate of x_l is given by $\omega_l \log(x_l)$ for a given constant $\omega_l > 0$. Then, express the total utility maximization problem as a convex problem, justifying its convexity.

(c) Solve the problem of part (b) to express the optimal choice of $\mathbf{p}^*$ as a function of the weight vector $(\omega_l)_l$. Comment on the result.

(2) Consider a network represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is the set of nodes, $\mathcal{L}$ is the set of directed links. Let $\mathbf{x} = (x_s^d)$ be the rate of flow from source node s to destination d, and $U_{sd}(\cdot)$ be the utility function associated with that flow. There are no apriori set routes for the communication.

Each node $i \in \mathcal{N}$ has a maximum power constraint P_i^{max} that is available for transmission over its outgoing links. For each link (i, j), there is a noise level given by N_{ij} (which also accounts for the channel gain between nodes i and j). Then, the rate achieved by transmitting data at rate P_{ij} over link $(i, j) \in \mathcal{L}$ is given by

$$R_{ij}(P_{ij}) = \log(1 + P_{ij}/N_{ij}).$$

(Note that the formulation is free from interference).

(a) Formulate the joint congestion control for utility maximization and link rate allocation for optimal routing problem under the power constrained network setting described above. Clearly describe each new term and indicate what each constraint is due to.

(b) Write the Lagrangian function and develop the gradient-based algorithm for the problem designed in part (a) (You can leave some components of the algorithm description as maximizations/minimizations).

(3) In PS2, Problem 6 (referring to Boyd & Vandenberghe 4.62), you have confirmed the convexity of the optimization problem (with known positive constants $(\alpha_i)_i$ and $(\beta_i)_i$):

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} \alpha_i W_i \log\left(1+\frac{\beta_i P_i}{W_i}\right) \\ \text{subject to} & P_i \geq 0, W_i \geq 0, \quad \forall i \\ & \sum_{i=1}^{n} P_i \leq P_{tot} \\ & \sum_{i=1}^{n} W_i = W_{tot} \end{array}$$

Now, associate a Lagrange multiplier μ only with the inequality constraint $\sum_{i=1}^{n} P_i \leq P_{tot}$ and develop the associated gradient method that clearly describes the update rule for $\{\mu^{(t)}\}_t$ as well as the power and bandwidth allocations in each step.

(4) Consider a single discrete-time queue with a service process $S(t) = Bernoulli(\mu)$ for some $\mu \leq 1$, i,e, $P(S(t) = 1) = 1 - P(S(t) = 0) = \mu$, and an arrival process $A(t)$ that follows the following congestion control rule:

$$P(A(t) = 1 |\, Q(t) = q) = 1 - P(A(t) = 0 |\, Q(t) = q) = \frac{1}{(1+q)}, \quad \forall q \in \{0, 1, 2, \cdots\}$$

where $Q(t)$ is the queue-length at time t that evolves as

$$Q(t+1) = (Q(t) + A(t) - S(t))^{+}.$$

(Note that a packet can arrive and leave the queue in the same time slot).

(a) Draw the Markov Chain diagram of $Q(t)$ showing the transition probabilities.

(b) Using Foster-Lyapunov criterion, find the range of μ for which the Markov Chain is positive recurrent (with proof).

(c) For any μ in the range found in (b), find the stationary distribution $\pi^* = (\pi_i^*)_{i=0}^{\infty}$. Do not leave terms in infinite sums. (Hints: Write global balance equations for sets of states $\{0\}, \{0, 1\}, \{0, 1, 2\}$, so on; and recall that $\sum_{k=0}^{\infty} \frac{\rho^k}{k!} = e^{\rho}$).

(5) (Programming Exercise) In this problem, we will consider fair allocation of a shared resource among users in the context of telecommunication networks. Consider a network consisting of N users.

User i derives a utility of $U_i(x_i)$ by sending data at rate x_i. We assume that U_i is an increasing, continuously differentiable function. The service rate to user i is $r_i \geq 0$. In order for stability, each user must send data at a rate less than its service rate, i.e., $x_i \leq r_i$. Due to interference between links and limited capacity of each link, the service rate vector r must satisfy the following constraint:

$$\sum_{i=1}^{N} \frac{r_i}{c_i} \leq 1 \tag{1}$$

for given constants $(c_i)_i$, which are defined by the network. The objective in this problem is to maximize utility under these constraints:

$$\begin{aligned}
\max_{x} \quad & \sum_{i=1}^{N} U_i(x_i) \\
\text{subject to} \quad & x_i \le r_i, \ i = 1, \dots, N, \\
& \sum_{i=1}^{N} \frac{r_i}{c_i} \le 1
\end{aligned} \tag{2}$$

Assume that $U_i(x) = \log(x)$ for all i.

1. (Dual Algorithm) Associate each constraint (2) with Lagrange multiplier $q_i \ge 0$ and write the Lagrangian of the primal problem $L(x, r, q)$. Then, for given q, find the optimum x and r by solving the following problems:

$$x^* = \arg\max_{x} L(x, r, q) \tag{3}$$

$$r^* = \arg\max_{r: \sum_i \frac{r_i}{c_i} \le 1} L(x, r, q) \tag{4}$$

   Note that maximization over x and r are decoupled from each other, and the dual function is $D(q) = L(x^*, r^*, q)$.We will use gradient descent to minimize $D(q)$:

$$q_i(t) = (q_i(t-1) - \gamma \frac{d}{dq_i} D(q(t)))^+ \tag{5}$$

   In summary, at stage t of the Dual Algorithm, for given $q = q(t-1)$, you will first find $x(t) = x^*$ and $r(t) = r^*$ as the optimizers of problems (3) and (4), respectively. Then you will update the dual variable as in (5).

   (a) Solve the problems (3), (4), (5), and find the explicit rules for $x(t)$, $r(t)$ and $q(t)$. (Hint: Exploit the additive structure of U_i and thus decoupling of $(x_i)_i$ and r. Also, for finding r^*, observe that only one user receives a non-zero service rate at each time slot. Which one?)

   (b) Implement the Dual Algorithm for $N = 2$ and $U_i(x) = \log(x), \forall i$. Let the initial queue-lengths be $q_1(0) = q_2(0) = 1$ and $(c_1, c_2) = (1, 2)$. Show the evolution of $(x_i(t))_i$ and $(q_i(t))_i$. In a two-dimensional plot, show the trajectory of $(x_1(t), x_2(t))$.

   (c) How can we interpret the dual variable $q_i(t)$? How does its increase affect $x_i(t)$ and $r_i(t)$? (Hint: Identify an arrival and a service process in the explicit form of (5).)

2. (Primal-Dual Algorithm) In Primal-Dual Algorithm, instead of finding $x_i(t)$ and at one step as in (3), we will use gradient ascent. So, for given $q(t-1)$,

$$x_i(t) = (x_i(t-1) + \alpha \frac{d}{dx_i} L(x, r, q(t-1)))^+ \tag{6}$$

for all $i = 1, 2, \ldots, N$. The rest of the algorithm is identical to Dual Algorithm.

(a) Find the update rules for $(x_i(t))_i$.

(b) Implement the Primal-Dual Algorithm for $N = 2$ and $U_i(x) = \log(x), \forall i$. Let the initial queue-lengths be $q_1(0) = q_2(0) = 1$ and $(c_1, c_2) = (1, 2)$. Show the evolution of $(x_i(t))_i$ and $(q_i(t))_i$. In a two-dimensional plot, show the trajectory of $(x_1(t), x_2(t))$.

(c) Compare the performances of Dual and Primal-Dual Algorithm. Which one provides faster convergence? Explain why.

3. (Heavy-Ball Method) For $(c_1, c_2) = (1, 2)$, $N = 2$ and $U_i(x) = \log(x)$, Heavy-Ball Method works as follows:

   - Choose parameters $K > 0$ and $\beta \in [0, 1)$. Set $t = 0$.
   - Let $q_1(0) = q_2(0) = 0$. Defined the weights for each queue $w_i(0) = w_i(-1) = 0,\ i = 1, 2$.
   - For $t \geq 1$, do the following at each iteration:
     * $r(t) = \underset{r:\sum_i \frac{r_i}{c_i} \leq 1}{\arg\max} \sum_i w_i(t) r_i$. (Hint: Only one user receives a non-zero service rate at each time slot. Which one?)
     * $x_i(t) = \min\{U_i'^{-1}(\frac{w_i(t)}{K}), x^{max}\}$.
     * $q_i(t+1) = (q_i(t) - r_i(t) + x_i(t))^+$.
     * Let $\Delta q_i(t) = q_i(t+1) - q_i(t)$. Then, update the weights as follows:

$$w_i(t+1) = \Big(w_i(t) + \Delta q_i(t) + \beta(w_i(t) - w_i(t-1))\Big)^+ \tag{7}$$

   where x^{max} is a sufficiently large constant.

   (a) Fix $\beta = 0.5$ and $K = 100$, and implement the Heavy-Ball Method described above. Show the evolution of $(x_i(t))_i$ and $(q_i(t))_i$. In a two-dimensional plot, show the trajectory of $(x_1(t), x_2(t))$.

   (b) For fixed $\beta = 0.5$, increase K and observe how the total utility changes after a sufficiently long time for increasing K.

   (c) Compare the performance of Heavy-Ball Method with the algorithms implemented previously. Which one converges fastest? Explain why.