Binomial Distribution

1. Introduction & Intuition

- **Definition:** Counts the number of *successes* in n *independent* Bernoulli trials, each with success-probability p.
- **Notation:** $X \sim \text{Binomial}(n, p)$.
- **Typical scenarios:**
  - Flip a coin n times $\to$ number of heads.
  - n website visits $\to$ number of ad clicks.
  - Inspect n items $\to$ number of defectives.

2. Probability Mass Function (PMF)

$$\boxed{P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}}, \qquad k = 0, 1, \ldots, n.$$

Why this formula? — Three building blocks

1. *Independence:* a specific sequence with k successes and $n-k$ failures has probability $p^k\,(1-p)^{n-k}$.
2. *Ordering:* there are $\binom{n}{k}$ ways to place those k successes among the n trials.
3. Multiply the two parts $\Rightarrow$ full PMF.

3. Expectation and Variance

Expectation

$$\mathbb{E}[X] = \underbrace{np}_{\substack{\text{sum of}\\ \text{independent}\\ \text{Bernoulli means}}}.$$

Variance

$$\operatorname{Var}(X) = \underbrace{np(1-p)}_{\substack{\text{independent variances}\\ \text{add up}}} = npq, \quad q = 1 - p.$$

4. Shape, Symmetry & Skewness

- **Symmetric** when $p = 0.5$. Then $P(X = k) = P(X = n - k)$.
- **Left-skewed** when $p > 0.5$ (tail on the left). **Right-skewed** when $p < 0.5$ (tail on the right).

Insert animation or trio of PMF plots here showing $p = 0.2, 0.5, 0.8$.

5. Mode (Most likely value)

$$k_{\text{mode}} = \lfloor (n+1)p \rfloor .$$

Two adjacent modes appear if $(n+1)p$ is an integer; they are k and $k-1$.

6. Worked Examples

1. **Fair coin,** $n = 10$. $P(X = 0) = P(X = 10) = \binom{10}{0} 0.5^{10} = \binom{10}{10} 0.5^{10}$.
2. **Ad clicks.** $n = 100,\ p = 0.2$. $\mathbb{E}[X] = 100 \times 0.2 = 20, \quad \mathrm{Var}(X) = 100 \times 0.2 \times 0.8 = 16$.

7. Quick Facts Table

Measure	Formula	Interpretation
Mean	$\mu = np$	Expected number of successes
Mode	$\lfloor (n+1)p \rfloor$	Most probable count
Variance	$\sigma^2 = np(1-p)$	Spread around the mean

8. Where is it used?

- Neural-network *dropout*: each neuron kept (success) with prob. p.
- *Binary classification* accuracy counts.
- *A/B testing*: number of conversions in n trials.

9. Summary

- Built from n independent Bernoulli(p) trials.
- PMF: $\binom{n}{k}p^k(1-p)^{n-k}$.
- Mean np, variance $np(1-p)$, mode $\lfloor (n+1)p \rfloor$.
- Shape flips from right-skewed $\to$ symmetric $\to$ left-skewed as p moves from 0 to 1.