Moments & the MGF Notes

1. What are Moments?

- Numerical summaries that describe the *shape* of a distribution.
- First raw moment $\mathbb{E}[X]$: **centre**. Second raw moment $\mathbb{E}[X^2]$: part of **spread**. Higher moments capture skewness, kurtosis, and so on.

2. The Moment-Generating Function (MGF)

$$M_X(t) \; = \; \mathbb{E}[e^{tX}], \qquad t \in \mathbb{R} \text{ (where the expectation exists).}$$

- Expand e^{tX} via its **Taylor series**:

$$e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!} = 1 + tX + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \dots$$

- Taking expectation term-by-term gives

$$M_X(t) = 1 + \mu_1\, t + \frac{\mu_2\, t^2}{2!} + \frac{\mu_3\, t^3}{3!} + \dots,$$

where $\mu_n = \mathbb{E}[X^n]$ is the n-th *raw* moment.

$$\mu_n \; = \; \left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0}$$

The above equation is the Key Result.

3. Why evaluate at $t = 0$?

$$\left.\frac{d^n}{dt^n}\, e^{tX}\right|_{t=0} = X^n,$$

so the exponential term *vanishes*, leaving only the desired power X^n. Evaluating at other $t \neq 0$ multiplies by e^{tX}, producing a *tilted* distribution.

4. Mini-Review: Taylor Series of e^t

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

Adding more terms rapidly improves the approximation around $t = 0$. (See animated plots in video for visual intuition.)

5. Worked Example

Discrete r.v. X takes

$$P(X = 1) = 0.5, \quad P(X = 3) = 0.5.$$

MGF.

$$M_X(t) = 0.5\, e^t + 0.5\, e^{3t}.$$

1. First derivative:

$$M_X'(t) = 0.5\, e^t + 1.5\, e^{3t} \Longrightarrow \mathbb{E}[X] = M_X'(0) = 0.5 + 1.5 = 2.$$

2. Second derivative:

$$M_X''(t) = 0.5\, e^t + 4.5\, e^{3t} \Longrightarrow \mathbb{E}[X^2] = M_X''(0) = 0.5 + 4.5 = 5.$$

3. Variance:

$$\mathrm{Var}(X) = 5 - 2^2 = 1.$$

4. Third moment (sketch):

$$M_X'''(0) = 0.5 + 13.5 = 14 \quad \Longrightarrow \quad \mathbb{E}[X^3] = 14.$$

n	$\mu_n = \mathbb{E}[X^n]$	Obtained via
1	2	$M_X'(0)$
2	5	$M_X''(0)$
3	14	$M_X'''(0)$

6. Why exponentials?

- The exponential's series contains *all* powers X^n. Differentiation plucks out each moment cleanly.
- Many nice properties (non-negativity, closure under sums of independent variables) make e^{tX} analytically convenient.

7. Visualization

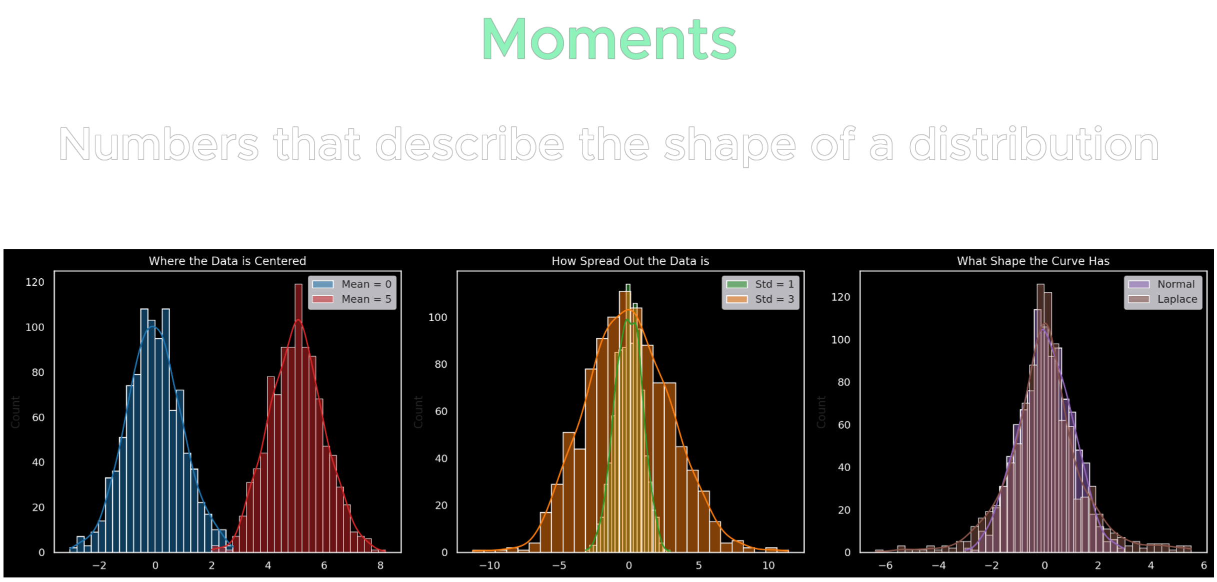

Figure 1: Moments - taken from video

8. Recap

- **Moments** capture centre, spread, skewness, and richer shape information.
- **MGF** packages all raw moments into one function: $M_X(t) = \mathbb{E}[e^{tX}]$.
- n-th moment $=$ n-th derivative of MGF at 0.
- Variance $= \mu_2 - \mu_1^2$; higher moments via further derivatives.