# Binomial Distribution Notes

## 1. Introduction & Intuition

- **Definition:** Counts the number of *successes* in $n$ *independent* Bernoulli trials, each with success-probability $p$.
- **Notation:** $X \sim \text{Binomial}(n, p)$.
- **Typical scenarios:**
  - Flip a coin $n$ times $\to$ number of heads.
  - $n$ website visits $\to$ number of ad clicks.
  - Inspect $n$ items $\to$ number of defectives.

## 2. Probability Mass Function (PMF)

$$\boxed{P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}}, \qquad k = 0, 1, \ldots, n.$$

### Why this formula? — Three building blocks

1. *Independence:* a specific sequence with $k$ successes and $n-k$ failures has probability $p^k\,(1-p)^{n-k}$.
2. *Ordering:* there are $\binom{n}{k}$ ways to place those $k$ successes among the $n$ trials.
3. Multiply the two parts $\Rightarrow$ full PMF.

## 3. Expectation and Variance

### Expectation

$$\mathbb{E}[X] = \underbrace{np}_{\substack{\text{sum of}\\ \text{independent}\\ \text{Bernoulli means}}}.$$

### Variance

$$\mathrm{Var}(X) = \underbrace{np(1-p)}_{\substack{\text{independent variances}\\ \text{add up}}} = npq, \quad q = 1 - p.$$

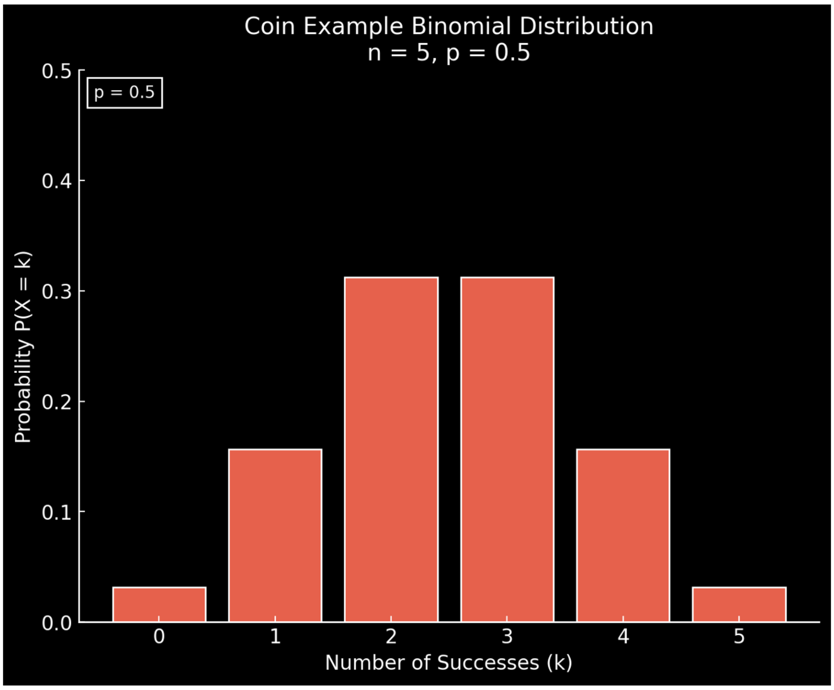


Figure 1: Binomial Distribution

## 4. Shape, Symmetry & Skewness

- **Symmetric** when $p = 0.5$. Then $P(X = k) = P(X = n - k)$.
- **Left-skewed** when $p > 0.5$ (tail on the left). **Right-skewed** when $p < 0.5$ (tail on the right).

## 5. Mode (Most likely value)

$$k_{\text{mode}} = \lfloor (n+1)p \rfloor .$$

*Two adjacent modes* appear if $(n + 1)p$ is an integer; they are $k$ and $k - 1$.

## 6. Worked Examples

1. **Fair coin,** $n = 10$. $P(X = 0) = P(X = 10) = \binom{10}{0} 0.5^{10} = \binom{10}{10} 0.5^{10}$.
2. **Ad clicks.** $n = 100,\ p = 0.2$. $\mathbb{E}[X] = 100 \times 0.2 = 20, \quad \mathrm{Var}(X) = 100 \times 0.2 \times 0.8 = 16.$

## 7. Quick Facts Table

| Measure | Formula | Interpretation |
|---|---|---|
| Mean | $\mu = np$ | Expected number of successes |
| Mode | $\lfloor (n+1)p \rfloor$ | Most probable count |
| Variance | $\sigma^2 = np(1-p)$ | Spread around the mean |

## 8. Where is it used?

- Neural-network *dropout*: each neuron kept (success) with prob. $p$.
- *Binary classification* accuracy counts.
- *A/B testing*: number of conversions in $n$ trials.

## 9. Summary

- Built from $n$ independent Bernoulli($p$) trials.
- PMF: $\binom{n}{k} p^k (1-p)^{n-k}$.
- Mean $np$, variance $np(1-p)$, mode $\lfloor (n+1)p \rfloor$.
- Shape flips from right-skewed $\to$ symmetric $\to$ left-skewed as $p$ moves from 0 to 1.