Geometric Distribution Notes

1. Motivation

- Previous distributions counted *how many* successes occur in a fixed number of trials.
- **Geometric question:** *"How many trials until the **first** success?"*
- Repeated Bernoulli(p) trials until a success appears.

Let
$$X \sim \text{Geo}(p), \qquad X \in \{1, 2, 3, \dots\},$$
where X counts the trial number of the first success.

2. Probability Mass Function (PMF)

$$\boxed{P(X = k) = (1-p)^{k-1} p, \qquad k = 1, 2, \dots}$$

Why? Need $k-1$ consecutive failures ($(1-p)^{k-1}$) followed by one success (p).

3. Example (Biased coin)

Heads $= 0.3$ ($p = 0.3$):
$$P(X = 3) = (1-p)^2 p = 0.7^2(0.3) = 0.147, \qquad P(X = 7) = 0.7^6(0.3) = 0.0353.$$

4. Cumulative Distribution Function (CDF)

$$\boxed{F(k) = P(X \le k) = \sum_{j=1}^{k} (1-p)^{j-1} p = p\frac{1-(1-p)^k}{1-(1-p)} = 1-(1-p)^k.}$$

Tail probability.
$$P(X > k) = 1 - F(k) = (1-p)^k.$$

5. Memoryless Property

$$\boxed{P(X > m + n \mid X > m) = P(X > n).}$$

Interpretation: After waiting m failures, the process "restarts"—future waiting time is independent of the past.

6. Expectation (Mean)

$$\mu = \mathbb{E}[X].$$

$$\begin{aligned}\mu &= p\cdot 1 + (1-p)(1+\mu)\\ &= p + (1-p) + (1-p)\mu\\ &\implies \mu(1-(1-p)) = 1 \implies \boxed{\mu = \frac{1}{p}}.\end{aligned}$$

7. Variance

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mu^2.$$

Compute $\mathbb{E}[X^2]$ via recursion.

$$\mathbb{E}[X^2] = p\cdot 1^2 + (1-p)(1+\mu)^2 = p + (1-p)(1+2\mu+\mu^2).$$

Substitute $\mu = \frac{1}{p}$ and simplify:

$$\mathbb{E}[X^2] = \frac{2-p}{p^2}.$$

Hence

$$\mathrm{Var}(X) = \frac{2-p}{p^2} - \frac{1}{p^2} = \boxed{\frac{1-p}{p^2}}.$$

8. Shape Insights

- PMF decays exponentially—often called a *discrete exponential.*
- Larger $p \Rightarrow$ steeper drop (success likely early). Smaller $p \Rightarrow$ flatter curve (longer wait).

9. Quick Reference Table

Measure	Formula
PMF	$(1-p)^{k-1}p$
CDF	$1-(1-p)^k$
Mean	$1/p$
Variance	$(1-p)/p^2$
Memoryless	$P(X > m+n \mid X > m) = (1-p)^n$

10. Summary

- Counts trials until the first success in repeated Bernoulli(p) experiments.
- PMF $\left(1-p\right)^{k-1}p$; CDF $1-\left(1-p\right)^{k}$.
- Only discrete distribution with the *memoryless* property.
- Mean $1/p$, variance $(1-p)/p^2$.