Poisson Distribution Notes

1. Intuition

- Counts how many times an event occurs in a *fixed interval* of **time** or **space**.
- **Assumptions:**
  1. Events occur **independently**.
  2. Constant average rate λ per interval.
  3. No two events occur at the exact same instant.

2. Probability Mass Function (PMF)

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \qquad k = 0, 1, 2, \ldots$$

Derivation sketch (Binomial $\to$ Poisson).

$$\lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k e^{-\lambda}}{k!}.$$

Here n tiny sub-intervals, success-probability λ/n, expected events $np = \lambda$.

3. Moment-Generating Function (MGF)

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} e^{tk} = \exp[\lambda(e^t - 1)].$$

- **Mean:** $M_X'(t) = \lambda e^t \exp[\lambda(e^t - 1)] \implies \mathbb{E}[X] = M_X'(0) = \lambda.$
- **Second moment:** $M_X''(t) = \left(\lambda e^t + (\lambda e^t)^2\right) \exp[\lambda(e^t - 1)] \implies \mathbb{E}[X^2] = M_X''(0) = \lambda + \lambda^2.$
- **Variance:** $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda + \lambda^2 - \lambda^2 = \boxed{\lambda}.$

Neat property: $\mu = \sigma^2 = \lambda.$

4. Shape Insights

- Small λ: sharp, highly skewed to the right.
- Large λ: wider and *approximately* symmetric (tends toward Normal by CLT).

5. Worked Example

Average $\lambda = 3$ calls per hour to a hotline.

$$P(\text{exactly 5 calls}) = \frac{3^5 e^{-3}}{5!} \approx 0.1008.$$

6. Quick Reference Table

Measure	Formula
PMF	$\lambda^k e^{-\lambda}/k!$
Mean	λ
Variance	λ
MGF	$\exp[\lambda(e^t - 1)]$

7. Visualization

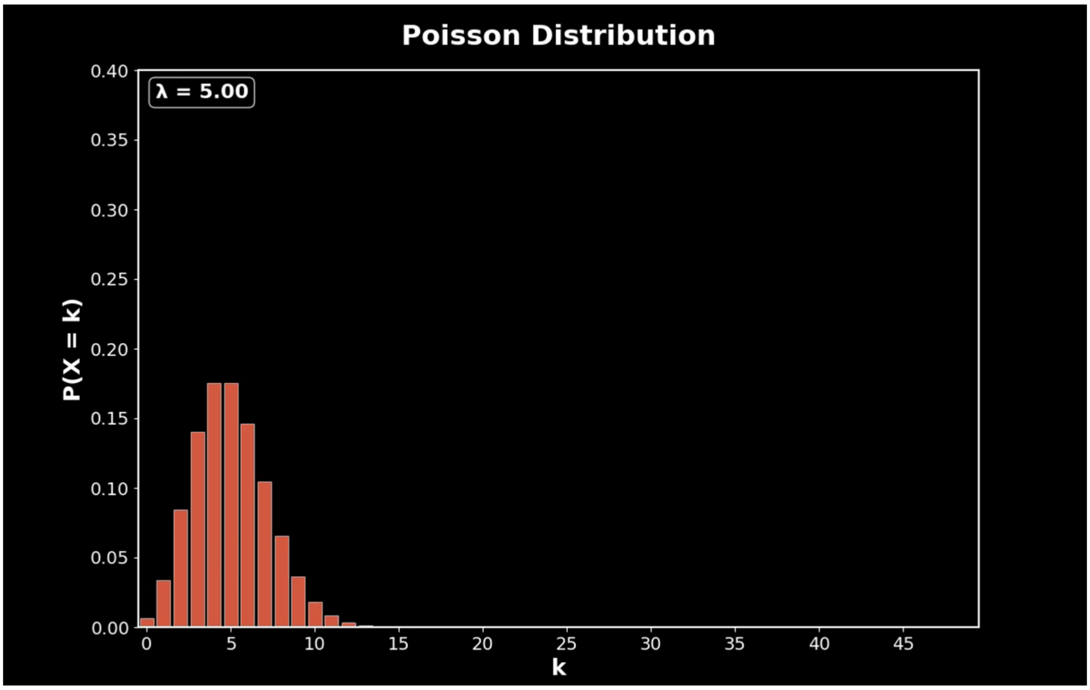

Figure 1: Poisson Distribution - taken from video

7. Typical Applications

- Traffic flow (cars per minute).
- Network packets per second.
- Epidemiology (disease cases per day).