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The shift operators and translations of spherical harmonics

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Abstract

Solid and surface spherical harmonics functions have very simple transformation properties with respect to the gradient and angular momentum operators. These properties can be utilized for the derivation of translation relations of the spherical harmonic functions.

1 Introduction

Already many papers have been published about the transformational properties of the spherical harmonics functions. To cite a few: Hobson (1955), Rose (1957), Aardoom (1969), Giacaglia (1980) and Epton and Dembart (1994). The formulas presented in this paper are not new, but they are derived in a particular straightforward manner which we believe to be much simpler than often found in other literature.

First we give some definitions, then we show the properties of the operators applied and finally we show how they can be used to derive translation relations for spherical harmonic functions and their coefficients.

2 General properties

Since many definitions can be found for the spherical harmonic functions, we first start with the definitions used in this paper. For simplicity, the complex spherical harmonics are used; defined as in e.g. Edmonds (1957). We start with the *associated Legendre function*:

$$P_{\ell,m}(t) = \frac{1}{2^\ell \ell!}(1-t^2)^{m/2}\frac{d^{\ell+m}}{dt^{\ell+m}}(t^2-1)^\ell .$$

It has the following symmetry with respect to order m

$$P_{\ell,-m}(t) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_{\ell,m}(t).$$

The *spherical harmonics* are defined as

$$Y_{\ell,m}(\theta,\lambda) = P_{\ell,m}(\cos\theta)\, \mathrm{e}^{im\lambda}.$$

Often we will write

$$Y_{\ell,m}(\mathbf{x}) = Y_{\ell,m}(\mathbf{x}/\|\mathbf{x}\|) \equiv Y_{\ell,m}(\theta,\lambda);$$

with $\mathbf{x} \in \mathbb{R}^3$ and where $\|\mathbf{x}\|$ is the Eucledian norm of $\mathbf{x}$. For the spherical harmonics a *normalisation*

$$\overline{Y}_{\ell,m} = \beta_{\ell,m} Y_{\ell,m} \quad \text{with} \quad \beta_{\ell,m} = (-1)^m \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} \tag{1}$$

can be used such that the spherical harmonics are ortho-normal:

$$\int\int_\sigma \overline{Y}_{\ell,m}(\mathbf{x})\, \overline{Y}^*_{\ell',m'}(\mathbf{x})\, d\mathbf{x} = \delta_{\ell'\ell}\, \delta_{m'm}.$$

The asterix * denotes the complex conjugate; the integration is taken over the (unit) sphere. For the *regular solid spherical harmonics* $\|\mathbf{x}\|^\ell Y_{\ell,m}(\mathbf{x})$ and the *irregular solid spherical harmonics* $\frac{1}{\|\mathbf{x}\|^{\ell+1}} Y_{\ell,m}(\mathbf{x})$ the following abbreviations are introduced:

$$\begin{aligned} S_{\ell,m}(\mathbf{x}) &= (-1)^m (\ell-m)!\, \frac{1}{\|\mathbf{x}\|^{\ell+1}} Y_{\ell,m}(\mathbf{x}) \\ R_{\ell,m}(\mathbf{x}) &= (-1)^m \frac{1}{(\ell+m)!}\, \|\mathbf{x}\|^\ell Y_{\ell,m}(\mathbf{x}). \end{aligned} \tag{2}$$

With respect to the sphere, they are only orthogonal; but the ’normalisation’ used here will render very simple formulas. From the definitions it is easily derived that the following symmetry relations hold:

$$\begin{aligned} Y^*_{\ell,m} &= (-1)^m \frac{(\ell+m)!}{(\ell-m)!}\, Y_{\ell,-m} & \qquad Y_{\ell,m}(\mathbf{x}) &= (-1)^\ell Y_{\ell,m}(-\mathbf{x}) \\ \overline{Y}^*_{\ell,m} &= (-1)^m\, \overline{Y}_{\ell,-m} & \overline{Y}_{\ell,m}(\mathbf{x}) &= (-1)^\ell \overline{Y}_{\ell,m}(-\mathbf{x}) \\ S^*_{\ell,m} &= (-1)^m S_{\ell,-m} & S_{\ell,m}(\mathbf{x}) &= (-1)^\ell S_{\ell,m}(-\mathbf{x}) \\ R^*_{\ell,m} &= (-1)^m R_{\ell,-m} & R_{\ell,m}(\mathbf{x}) &= (-1)^\ell R_{\ell,m}(-\mathbf{x}). \end{aligned} \tag{3}$$

Apart from the usual geocentric cartesian frame $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ a new frame $\mathbf{e}_-, \mathbf{e}_0, \mathbf{e}_+$ is used (Van Gelderen, 1999) with:

$$\begin{aligned} \mathbf{e}_- &= \frac{1}{\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y) \\ \mathbf{e}_0 &= \mathbf{e}_z \\ \mathbf{e}_+ &= -\frac{1}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y). \end{aligned}$$

All *covariant* vector components v_x, v_y, v_z of the vector $\mathbf{v}$ transform in the same way:

$$\begin{aligned} v_- &= \frac{1}{\sqrt{2}}(v_x - iv_y) \\ v_0 &= v_z \\ v_+ &= -\frac{1}{\sqrt{2}}(v_x + iv_y). \end{aligned} \tag{4}$$

For the radial distance $r \equiv \|\mathbf{x}\|$ we then obtain

$$r^2 = x_x^2 + x_y^2 + x_z^2 = x_0^2 - 2x_+x_-.$$

From the well-known recursion relations of the Legendre functions, see e.g. Ilk (1983), the following relations for the solid spherical harmonics are derived:

$$\begin{aligned}
(\ell+m)x_0 R_{\ell,m} &= r^2 R_{\ell-1,m} + (\ell-m+1)\sqrt{2}x_+ R_{\ell,m-1} \\
(\ell+m)\sqrt{2}x_- R_{\ell,m} &= (\ell-m+1)x_0 R_{\ell,m-1} - r^2 R_{\ell-1,m-1} \\
2mx_0 R_{\ell,m} &= (\ell-m+1)\sqrt{2}x_+ R_{\ell,m-1} - (\ell+m+1)\sqrt{2}x_- R_{\ell,m+1} \\
(\ell-m+1)x_0 S_{\ell,m} &= r^2 S_{\ell+1,m} - (\ell+m)\sqrt{2}x_+ S_{\ell,m-1} \\
(\ell+m+1)x_0 S_{\ell,m} &= r^2 S_{\ell+1,m} - (\ell-m)\sqrt{2}x_- S_{\ell,m+1} \\
2mx_0 S_{\ell,m} &= (\ell+m)\sqrt{2}x_+ S_{\ell,m-1} - (\ell-m)\sqrt{2}x_- S_{\ell,m+1}.
\end{aligned} \tag{5}$$

3 The ladder operators

In this section operators are introduced which change the spherical harmonics by one degree or order. Two differential operators are used: the *gradient operator* $\boldsymbol{\nabla}$ and the *angular momentum operator* $\mathbf{L}$. The gradient operator with respect to the cartesian basis reads:

$$\begin{aligned}
\boldsymbol{\nabla} f &= \mathbf{e}_x \frac{\partial f}{\partial x} + \mathbf{e}_y \frac{\partial f}{\partial y} + \mathbf{e}_z \frac{\partial f}{\partial z} \\
&\equiv (\mathbf{e}_x \nabla_x + \mathbf{e}_y \nabla_y + \mathbf{e}_z \nabla_z) f;
\end{aligned}$$

f is a funtion in $\mathbb{R}^3$. The gradient operator can be split up into a radial and a surface part:

$$\boldsymbol{\nabla} = \mathbf{e}_r \nabla_r + \frac{1}{r} \boldsymbol{\nabla}_{\text{surf}}$$

The operator $\mathbf{L}$ is a tangential vector operator, i.e. the vector $\mathbf{L}f$ is tangential to the sphere, defined as

$$\begin{aligned}
\mathbf{L} &= -i\mathbf{e}_r \times \boldsymbol{\nabla} = -ir\mathbf{e}_r \times \boldsymbol{\nabla}_{\text{surf}} \qquad \Leftrightarrow \\
\boldsymbol{\nabla} &= \mathbf{e}_r \nabla_r + \frac{1}{r}\boldsymbol{\nabla}_{\text{surf}} = \mathbf{e}_r \nabla_r - \frac{i}{r}(\mathbf{e}_r \times \mathbf{L}),
\end{aligned} \tag{6}$$

with $\mathbf{e}_r$ the radial basis vector. The vector $\mathbf{L}f$ is always perpendicular to $\mathbf{e}_r$ and $\boldsymbol{\nabla} f$, which can directly be seen from its definition; see e.g. Jackson (1967) for more definitions and properties of these operators. The components of both operators with respect to the $\{\mathbf{e}_-, \mathbf{e}_0, \mathbf{e}_+\}$ are defined as (4) since always covariant differentiation is used:

$$\begin{aligned}
\nabla_\pm &= \mp \frac{1}{\sqrt{2}}(\nabla_x \pm i\nabla_y), \qquad \nabla_0 = \nabla_z \\
\mathrm{L}_\pm &= \mp \frac{1}{\sqrt{2}}(\mathrm{L}_x \pm i\mathrm{L}_y), \qquad \mathrm{L}_0 = \mathrm{L}_z.
\end{aligned}$$

First the operators $\mathrm{L}_{-,0,+}$ are applied to the spherical harmonics. Their action on the $Y_{\ell,m}$ is straightforward; this is related to the fact that they are the joint eigenfunctions of the operators

$\mathbf{L}^2$ and L_0; see Edmonds (1957). Since $\mathbf{L}$ is a pure tangential operator, also the relations for the solid harmonics are found directly:

$$
\begin{aligned}
\mathrm{L}_{-} Y_{\ell,m} &= -\frac{(\ell+m)(\ell-m+1)}{\sqrt{2}}\, Y_{\ell,m-1} \\
\mathrm{L}_{0} Y_{\ell,m} &= m\, Y_{\ell,m} \\
\mathrm{L}_{+} Y_{\ell,m} &= \frac{1}{\sqrt{2}} Y_{\ell,m+1}
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\mathrm{L}_{-} \overline{Y}_{\ell,m} &= \sqrt{\frac{(\ell+m)(\ell-m+1)}{2}}\, \overline{Y}_{\ell,m-1} \\
\mathrm{L}_{0} \overline{Y}_{\ell,m} &= m\, \overline{Y}_{\ell,m} \\
\mathrm{L}_{+} \overline{Y}_{\ell,m} &= -\sqrt{\frac{(\ell-m)(\ell+m+1)}{2}}\, \overline{Y}_{\ell,m+1}
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\mathrm{L}_{-} S_{\ell,m} &= \frac{\ell+m}{\sqrt{2}}\, S_{\ell,m-1} \\
\mathrm{L}_{0} S_{\ell,m} &= m\, S_{\ell,m} \\
\mathrm{L}_{+} S_{\ell,m} &= -\frac{\ell-m}{\sqrt{2}} S_{\ell,m+1}
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\mathrm{L}_{-} R_{\ell,m} &= \frac{\ell-m+1}{\sqrt{2}}\, R_{\ell,m-1} \\
\mathrm{L}_{0} R_{\ell,m} &= m\, R_{\ell,m} \\
\mathrm{L}_{+} R_{\ell,m} &= -\frac{\ell+m+1}{\sqrt{2}} R_{\ell,m+1}
\end{aligned}
\tag{10}
$$

Now the $\boldsymbol{\nabla}$ operator is applied to the surface spherical harmonics. It is decomposed into a radial and surface component (6). In components, see Van Gelderen (1999),

$$
\begin{pmatrix} \nabla_{-} \\ \nabla_{0} \\ \nabla_{+} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} x_{-} \\ x_{0} \\ x_{+} \end{pmatrix} \nabla_r + \frac{1}{r^2} \begin{pmatrix} x_{-}\mathrm{L}_0 - x_0\mathrm{L}_{-} \\ x_{-}\mathrm{L}_{+} - x_{+}\mathrm{L}_{-} \\ x_0\mathrm{L}_{+} - x_{+}\mathrm{L}_0 \end{pmatrix} .
$$

The action of the first part on spherical harmonics is straighforward; for the second part the equations (9-10) are used. By applying the recurrence relations (5) the outcome can be reduced to very simple expressions:

$$
\begin{aligned}
\nabla_{-} S_{\ell,m} &= -\frac{1}{\sqrt{2}} S_{\ell+1,m-1} \\
\nabla_{0} S_{\ell,m} &= -S_{\ell+1,m} \\
\nabla_{+} S_{\ell,m} &= -\frac{1}{\sqrt{2}} S_{\ell+1,m+1}
\end{aligned}
\tag{11}
$$

and

$$\begin{aligned}
\nabla_{-} R_{\ell,m} &= -\frac{1}{\sqrt{2}} R_{\ell-1,m-1} \\
\nabla_{0} R_{\ell,m} &= R_{\ell-1,m} \\
\nabla_{+} R_{\ell,m} &= -\frac{1}{\sqrt{2}} R_{\ell-1,m+1}
\end{aligned} \tag{12}$$

With (1-2) we then find

$$\begin{aligned}
\nabla_{-} \frac{1}{r^{\ell+1}}\, Y_{\ell,m} &= \frac{(\ell-m+1)(\ell-m+2)}{\sqrt{2}} \frac{1}{r^{\ell+2}}\, Y_{\ell+1,m-1} \\
\nabla_{0} \frac{1}{r^{\ell+1}}\, Y_{\ell,m} &= -(\ell-m+1) \frac{1}{r^{\ell+2}} Y_{\ell+1,m} \\
\nabla_{+} \frac{1}{r^{\ell+1}} Y_{\ell,m} &= \frac{1}{\sqrt{2}} \frac{1}{r^{\ell+2}}\, Y_{\ell+1,m+1}
\end{aligned} \tag{13}$$

$$\begin{aligned}
\nabla_{-} \frac{1}{r^{\ell+1}} \overline{Y}_{\ell,m} &= -\sqrt{\frac{(2\ell+1)(\ell-m+1)(\ell-m+2)}{2(2\ell+3)}} \frac{1}{r^{\ell+2}} \overline{Y}_{\ell+1,m-1} \\
\nabla_{0} \frac{1}{r^{\ell+1}} \overline{Y}_{\ell,m} &= -\sqrt{\frac{(2\ell+1)(\ell+m+1)(\ell-m+1)}{(2\ell+3)}} \frac{1}{r^{\ell+2}} \overline{Y}_{\ell+1,m} \\
\nabla_{+} \frac{1}{r^{\ell+1}} \overline{Y}_{\ell,m} &= -\sqrt{\frac{(2\ell+1)(\ell+m+1)(\ell+m+2)}{2(2\ell+3)}} \frac{1}{r^{\ell+2}} \overline{Y}_{\ell+1,m+1}
\end{aligned} \tag{14}$$

$$\begin{aligned}
\nabla_{-} r^{\ell} Y_{\ell,m} &= \frac{(\ell+m)(\ell+m-1)}{\sqrt{2}}\, r^{\ell-1} Y_{\ell-1,m-1} \\
\nabla_{0} r^{\ell} Y_{\ell,m} &= (\ell+m)\, r^{\ell-1} Y_{\ell-1,m} \\
\nabla_{+} r^{\ell} Y_{\ell,m} &= \frac{1}{\sqrt{2}} r^{\ell-1} Y_{\ell-1,m+1}
\end{aligned} \tag{15}$$

$$\begin{aligned}
\nabla_{-} r^{\ell} \overline{Y}_{\ell,m} &= -\sqrt{\frac{(2\ell+1)(\ell+m)(\ell+m-1)}{2(\ell-1)}}\, r^{\ell-1} \overline{Y}_{\ell-1,m-1} \\
\nabla_{0} r^{\ell} \overline{Y}_{\ell,m} &= \sqrt{\frac{(2\ell+1)(\ell+m)(\ell-m)}{(2\ell-1)}}\, r^{\ell-1} \overline{Y}_{\ell-1,m} \\
\nabla_{+} r^{\ell} \overline{Y}_{\ell,m} &= -\sqrt{\frac{(2\ell+1)(\ell-m)(\ell-m-1)}{2(\ell-1)}}\, r^{\ell-1} \overline{Y}_{\ell-1,m+1}
\end{aligned} \tag{16}$$

The components of the operators $\boldsymbol{\nabla}$ and $\mathbf{L}$ are sometimes called *ladder operators* since they relate a spherical harmonic function to another of one degree and/or order higher or lower. The $L_{\pm}$ are real ladder operators for the surface spherical harmonics in the sense that they only make one step in the m (order) for a fixed degree. This is related to the fact that all the surface

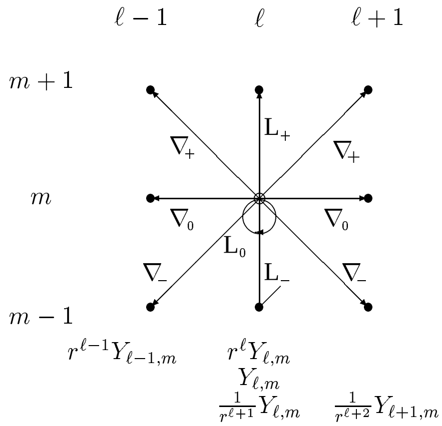

Fig. 1. The ladder operators of spherical harmonics functions

spherical harmonics of fixed degree ℓ form a basis for a $2\ell+1$ dimensional representation of the Lie algebra so3; cf. Hamermesh (1964). The $\nabla_\pm$ also go one step up or down in the m-direction, but always increase (irregular solid spherical harmonics) or decrease (regular solid spherical harmonics) the degree.

The spherical harmonics linked up to eachother by the ladder operators are depicted in Figure 1.

As $S_{0,0} = Y_{0,0} = \frac{1}{r}$, all irregular solid spherical harmonics can be obtained from the iterative use of the ∇-operators using (11,13,14):

$$S_{\ell,m} = (-1)^\ell \, 2^{|m|/2} \, {\nabla_\pm}^{|m|} {\nabla_0}^{\ell-|m|} \frac{1}{r} \tag{17}$$

$$\frac{1}{r^{\ell+1}} Y_{\ell,m} = (-1)^{\ell-m} \, 2^{|m|/2} \, \frac{1}{(\ell-m)!} \, {\nabla_\pm}^{|m|} {\nabla_0}^{\ell-|m|} \frac{1}{r}, \tag{18}$$

$$\frac{1}{r^{\ell+1}} \overline{Y}_{\ell,m} = (-1)^\ell 2^{|m|/2} \sqrt{\frac{(2\ell+1)}{4\pi(\ell+m)!(\ell-m)!}} \, {\nabla_\pm}^{|m|} {\nabla_0}^{\ell-|m|} \frac{1}{r}, \tag{19}$$

where $\nabla_\pm$ denotes ∇_+ for $m \geq 0$ and ∇_- for $m < 0$. Since $S_{\ell,m}$ is a harmonic function we have:

$$\Delta S_{\ell,m} = ({\nabla_0}^2 - 2\nabla_+\nabla_-) S_{\ell,m} = 0 \Leftrightarrow 2\nabla_+\nabla_- S_{\ell,m} = {\nabla_0}^2 S_{\ell,m}.$$

This property can be used to reduce combination of powers of ∇_- and ∇_+:

$$(\sqrt{2}\nabla_+)^m (\sqrt{2}\nabla_-)^{m'} = \begin{cases} (\sqrt{2}\nabla_+)^{m-m'} \, {\nabla_0}^{2m'} & m \geq m' \\ (\sqrt{2}\nabla_-)^{m'-m} \, {\nabla_0}^{2m} & m < m' \end{cases} \tag{20}$$

4 Translation relations

The inverse distance is expanded into spherical harmonics as (Hobson, 1955)

$$\frac{1}{\|\mathbf{x}-\mathbf{y}\|} = \sum_{\ell,m} R^*_{\ell,m}(\mathbf{y}) S_{\ell,m}(\mathbf{x}) = \sum_{\ell,m} R_{\ell,m}(\mathbf{y}) S^*_{\ell,m}(\mathbf{x}) \qquad \|\mathbf{y}\| < \|\mathbf{x}\|. \tag{21}$$

Translated spherical harmonics can be obtained easily from this expansion with (17):

$$\begin{aligned} S_{\ell,m}(\mathbf{x}-\mathbf{y}) &= (-1)^\ell 2^{|m|/2} \nabla_{\pm}{}^{|m|} \nabla_0{}^{\ell-|m|} \frac{1}{\|\mathbf{x}-\mathbf{y}\|} \\ &= \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} R^*_{\ell',m'}(\mathbf{y}) (-1)^\ell 2^{|m|/2}\, \nabla_{\pm}{}^{|m|} \nabla_0{}^{\ell-|m|} S_{\ell',m'}(\mathbf{x}). \end{aligned}$$

With, using (20),

$$\begin{aligned} (-1)^\ell 2^{|m|/2} \nabla_{\pm}{}^{|m|} \nabla_0{}^{\ell-|m|} S_{\ell',m'}(\mathbf{x}) &= (-1)^{\ell+\ell'} 2^{(|m+m'|)/2}\, \nabla_{\pm}{}^{|m+m'|} \nabla_0{}^{\ell+\ell'-|m+m'|} \frac{1}{\|\mathbf{x}\|} \\ &= S_{\ell+\ell',m+m'}(\mathbf{x}) \end{aligned}$$

this can be written as

$$S_{\ell,m}(\mathbf{x}-\mathbf{y}) = \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} R^*_{\ell',m'}(\mathbf{y}) S_{\ell+\ell',m+m'}(\mathbf{x}). \tag{22}$$

This is the translation relation for the irregular solid spherical harmonics. For the translation of the regular spherical harmonics it is less straightforward to find the relation directly; see Rose (1958) or Epton and Dembart (1994). Much easier is to start from the expansion of the inverse distance and apply (22):

$$\begin{aligned} \frac{1}{\|\mathbf{x}-\mathbf{y}\|} &= \sum_{\ell,m} R^*_{\ell,m}(\mathbf{y}) S_{\ell,m}(\mathbf{x}) = \sum_{\ell,m} R^*_{\ell,m}(\mathbf{y}-\boldsymbol{\Delta}) S_{\ell,m}(\mathbf{x}-\boldsymbol{\Delta}) \\ &= \sum_{\ell,m} R^*_{\ell,m}(\mathbf{y}-\boldsymbol{\Delta}) \sum_{\ell',m'} R^*_{\ell'-\ell,m'-m}(\boldsymbol{\Delta}) S_{\ell',m'}(\mathbf{x}) \\ &= \sum_{\ell',m'} S_{\ell',m'}(\mathbf{x}) \sum_{\ell,m} R^*_{\ell',m'}(\mathbf{y}-\boldsymbol{\Delta}) R^*_{\ell-\ell',m-m'}(\boldsymbol{\Delta}). \end{aligned}$$

Since the spherical harmonics are a set of independent basis vectors, the expansion coefficients of a function with respect to them are unique. Confrontation of the last with the first line of the equation above gives:

$$R^*_{\ell,m}(\mathbf{y}) = \sum_{\ell',m'} R^*_{\ell',m'}(\mathbf{y}-\boldsymbol{\Delta}) R^*_{\ell-\ell',m-m'}(\boldsymbol{\Delta}) \Leftrightarrow$$

$$R_{\ell,m}(\mathbf{x}+\mathbf{y}) = \sum_{\ell',m'} R_{\ell',m'}(\mathbf{y}) R_{\ell-\ell',m-m'}(\mathbf{x}). \tag{23}$$

Using the symmetry relations (3), the following set of relations can be derived from (22-23):

$$\begin{aligned}
S_{\ell,m}(\mathbf{x}-\mathbf{y}) &= \sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell} R^*_{\ell',m'}(\mathbf{y})S_{\ell+\ell',m+m'}(\mathbf{x}) \qquad (24)\\
&= \sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell} (-1)^{m'}R_{\ell',m'}(\mathbf{y})S_{\ell+\ell',m-m'}(\mathbf{x})\\
&= \sum_{\ell'=\ell}^{\infty}\sum_{m'=-\ell'}^{\ell} R^*_{\ell'-\ell,m'-m}(\mathbf{y})S_{\ell',m'}(\mathbf{x})\\
&= \sum_{\ell'=\ell}^{\infty}\sum_{m'=-\ell'}^{\ell} (-1)^{m'-m}R_{\ell'-\ell,m-m'}(\mathbf{y})S_{\ell',m'}(\mathbf{x})\\
R_{\ell,m}(\mathbf{x}-\mathbf{y}) &= \sum_{\ell'=0}^{\ell}\sum_{m'=-\ell'}^{\ell} (-1)^{\ell'}R_{\ell',m'}(\mathbf{y})R_{\ell-\ell',m-m'}(\mathbf{x}) \qquad (25)\\
&= \sum_{\ell'=0}^{\ell}\sum_{m'=-\ell'}^{\ell} (-1)^{\ell'+m'}R^*_{\ell',m'}(\mathbf{y})R_{\ell-\ell',m+m'}(\mathbf{x}).
\end{aligned}$$

or

$$\begin{aligned}
S_{\ell,m}(\mathbf{x}+\mathbf{y}) &= \sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell} (-1)^{\ell'}R^*_{\ell',m'}(\mathbf{y})S_{\ell+\ell',m+m'}(\mathbf{x}) \qquad (26)\\
&= \sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell} (-1)^{\ell'+m'}R_{\ell',m'}(\mathbf{y})S_{\ell+\ell',m-m'}(\mathbf{x})\\
&= \sum_{\ell'=\ell}^{\infty}\sum_{m'=-\ell'}^{\ell} (-1)^{\ell'}R^*_{\ell'-\ell,m'-m}(\mathbf{y})S_{\ell',m'}(\mathbf{x})\\
&= \sum_{\ell'=\ell}^{\infty}\sum_{m'=-\ell'}^{\ell} (-1)^{\ell'+m'-m}R_{\ell'-\ell,m-m'}(\mathbf{y})S_{\ell',m'}(\mathbf{x})\\
R_{\ell,m}(\mathbf{x}+\mathbf{y}) &= \sum_{\ell'=0}^{\ell}\sum_{m'=-\ell'}^{\ell} R_{\ell',m'}(\mathbf{y})R_{\ell-\ell',m-m'}(\mathbf{x}) \qquad (27)\\
&= \sum_{\ell'=0}^{\ell}\sum_{m'=-\ell'}^{\ell} (-1)^{m'}R^*_{\ell',m'}(\mathbf{y})R_{\ell-\ell',m+m'}(\mathbf{x}).
\end{aligned}$$

From the relations above, the equivalent relations for the $Y_{\ell,m}$ and the $\overline{Y}_{\ell,m}$ are found directly with (1) and (2). The inverse distance reads

$$\begin{aligned}
\frac{1}{\|\mathbf{x}-\mathbf{y}\|} &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{(\ell-m)!}{(\ell+m)!}\,\|\mathbf{y}\|^{\ell}Y^*_{\ell,m}(\mathbf{y})\,\frac{1}{\|\mathbf{x}\|^{\ell+1}}Y_{\ell,m}(\mathbf{x})\\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\,\|\mathbf{y}\|^{\ell}\overline{Y}^*_{\ell,m}(\mathbf{y})\,\frac{1}{\|\mathbf{x}\|^{\ell+1}}\overline{Y}_{\ell,m}(\mathbf{x}).
\end{aligned}$$

Translation of the irregular solid harmonics:

$$\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{\ell+1}} Y_{\ell,m}(\mathbf{x}-\mathbf{y}) = \sum_{\ell',m'} \frac{(\ell+\ell'-m-m')!}{(\ell-m)!(\ell'+m')!} \|\mathbf{y}\|^{\ell'} Y^*_{\ell',m'}(\mathbf{y}) \frac{1}{\|\mathbf{x}\|^{\ell+\ell'+1}} Y_{\ell+\ell',m+m'}(\mathbf{x}).$$

and for the normalized $\overline{Y}_{\ell,m}$

$$\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{\ell+1}} \overline{Y}_{\ell,m}(\mathbf{x}-\mathbf{y}) = \sum_{\ell',m'} \sqrt{\frac{4\pi(2\ell+1)}{(2\ell+2\ell'+1)(2\ell'+1)}} \cdot$$
$$\sqrt{\frac{(\ell-m)!(\ell'+m')!(\ell+\ell'+m+m')!}{(\ell+m)!(\ell'-m')!(\ell+\ell'-m-m')!}} \|\mathbf{y}\|^{\ell'} \overline{Y}^*_{\ell',m'}(\mathbf{y}) \frac{1}{\|\mathbf{x}\|^{\ell+\ell'+1}} \overline{Y}_{\ell+\ell',m+m'}(\mathbf{x}).$$

And for the regular solid harmonics we have

$$\|\mathbf{x}-\mathbf{y}\|^{\ell} Y_{\ell,m}(\mathbf{x}-\mathbf{y}) = \sum_{\ell',m'} (-1)^{\ell'} \frac{(\ell+m)!}{(\ell'+m')!} \frac{1}{(\ell-\ell'+m-m')!} \cdot$$
$$\|\mathbf{y}\|^{\ell'} Y_{\ell',m'}(\mathbf{y}) \, \|\mathbf{x}\|^{\ell-\ell'} Y_{\ell-\ell',m-m'}(\mathbf{x})$$

and for the normalized $\overline{Y}_{\ell,m}$:

$$\|\mathbf{x}-\mathbf{y}\|^{\ell} \overline{Y}_{\ell,m}(\mathbf{x}-\mathbf{y}) = \sum_{\ell',m'} (-1)^{\ell'} \sqrt{\frac{4\pi(2\ell+1)}{(2\ell-2\ell'+1)(2\ell'+1)}} \cdot$$
$$\frac{1}{\sqrt{(\ell-\ell'-m+m')!(\ell-\ell'+m-m')!}} \sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell'-m')!(\ell'+m')!}} \cdot$$
$$\|\mathbf{y}\|^{\ell'} \overline{Y}_{\ell',m'}(\mathbf{y}) \, \|\mathbf{x}\|^{\ell-\ell'} \overline{Y}_{\ell-\ell',m-m'}(\mathbf{x}).$$

Translation relations for coefficients

Often a spherical harmonic expansion is used to represent a function in 3D space. If the origin of the expansion is shifted, all its coefficients change. This is directly derived from the properties of the spherical harmonics. We take the following example. For a mass density ρ contained in a volume V, the total potential is

$$G \int_V \frac{\rho(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \, d\mathbf{y}.$$

We define the potential function ϕ and apply (21):

$$\begin{aligned} \phi(\mathbf{x}) &= G \int_V \frac{\rho(\mathbf{y}-\mathbf{x}_0)}{\|\mathbf{x}-\mathbf{y}\|} \, d\mathbf{y} \\ &= G \int_V \rho(\mathbf{y}-\mathbf{x}_0) \sum_{\ell,m} R^*_{\ell,m}(\mathbf{y}) S_{\ell,m}(\mathbf{x}) \, d\mathbf{y} \\ &= \sum_{\ell,m} G \int_V \rho(\mathbf{y}-\mathbf{x}_0) R^*_{\ell,m}(\mathbf{y}) \, d\mathbf{y} \; S_{\ell,m}(\mathbf{x}) \\ &= \sum_{\ell,m} \underbrace{G \int_V \rho(\mathbf{y}-\mathbf{x}_0) R^*_{\ell,m}(\mathbf{y}-\mathbf{x}_0) \, d\mathbf{y}}_{\equiv M_{\ell,m}(\mathbf{x}_0)} S_{\ell,m}(\mathbf{x}-\mathbf{x}_0), \end{aligned} \tag{28}$$

where $\mathbf{x}_0$ is a point of reference. The coefficients $M_{\ell,m}(\mathbf{x}_0)$ are the *multipole coefficients* of the potential due to the mass distribution $\rho(\mathbf{x})$ with respect to the origin $\mathbf{x}_0$. It is actually a Laurent series. If there is no mass outside a sphere of radius a around $\mathbf{x}_0$ then convergence is guarenteed for $\|\mathbf{x}\| > a$. Likewise we obtain the Taylor expansion

$$\phi(\mathbf{x}) = \sum_{\ell,m} \underbrace{G \int_V \rho(\mathbf{y}-\mathbf{x}_0) S^*_{\ell,m}(\mathbf{y}-\mathbf{x}_0)\, d\mathbf{y}}_{\equiv L_{\ell,m}(\mathbf{x}_0)} R_{\ell,m}(\mathbf{x}-\mathbf{x}_0);$$

where the $L_{\ell,m}(\mathbf{x}_0)$ are the *local expansion coefficients* of the potential of the mass distribution $\rho(\mathbf{x})$ with respect to the origin $\mathbf{x}_0$. If there is only mass *outside* the sphere of radius b around $\mathbf{x}_0$, then we have convergence for $\|\mathbf{x}\| < b$.

The translation relations for the coefficients are obtained by inserting (from (26))

$$S_{\ell,m}(\mathbf{x}-\mathbf{x}_0) = \sum_{\ell',m'} (-1)^{\ell'} R^*_{\ell'-\ell,m'-m}(\mathbf{\Delta}) S_{\ell',m'}(\mathbf{x}-\mathbf{x}_0-\mathbf{\Delta})$$

into (28)

$$\phi(\mathbf{x}) = \sum_{\ell',m'} S_{\ell',m'}(\mathbf{x}-(\mathbf{x}_0+\mathbf{\Delta})) \underbrace{\sum_{\ell,m} (-1)^{\ell'} M_{\ell,m}(\mathbf{x}_0) R^*_{\ell'-\ell,m'-m}(\mathbf{\Delta})}_{= M_{\ell,m}(\mathbf{x}_0+\mathbf{\Delta})}.$$

Likewise the translation relations for local coefficients and the relation for the multipole to local expansioncoefficients are obtained:

$$\begin{aligned}
M_{\ell,m}(\mathbf{x}_0+\mathbf{\Delta}) &= \sum_{\ell',m'} (-1)^{\ell'} M_{\ell',m'}(\mathbf{x}_0) R^*_{\ell-\ell',m-m'}(\mathbf{\Delta}) && \|\mathbf{x}\| > \|\mathbf{\Delta}\| + a \\
L_{\ell,m}(\mathbf{x}_0+\mathbf{\Delta}) &= \sum_{\ell',m'} L_{\ell',m'}(\mathbf{x}_0) R_{\ell-\ell',m-m'}(\mathbf{\Delta}) && \|\mathbf{x}\| < b - \|\mathbf{\Delta}\| \\
L_{\ell,m}(\mathbf{x}_0+\mathbf{\Delta}) &= \sum_{\ell',m'} (-1)^{\ell'+m'} M_{\ell',m'}(\mathbf{x}_0) S^*_{\ell+\ell',m-m'}(\mathbf{\Delta}) && \|\mathbf{x}\| > \|\mathbf{\Delta}\| - a.
\end{aligned}$$

The translation of the center of expansion is only allowed if the convergence criteria for the new expansion are fulfilled. This leads to the criteria indicated above.

With the translation (26) also a double expansion of the inverse distance can be constructed:

$$\begin{aligned}
\frac{1}{\|\mathbf{x}-\mathbf{y}\|} &= \sum_{\ell,m} R^*_{\ell,m}(\mathbf{y}) S_{\ell,m}(\mathbf{y}) \\
&= \sum_{\ell,m} R^*_{\ell,m}(\mathbf{y}-\mathbf{y}_0) S_{\ell,m}(\mathbf{y}-\mathbf{y}_0) \\
&= \sum_{\ell,m} R^*_{\ell,m}(\mathbf{y}-\mathbf{y}_0) \sum_{\ell',m'} (-1)^{\ell'} R^*_{\ell',m'}(\mathbf{x}-\mathbf{x}_0) S_{\ell+\ell',m+m'}(\mathbf{x}_0-\mathbf{y}_0) \\
&= \sum_{\substack{\ell,m \\ \ell',m'}} \xi_{\ell,\ell',m,m'}(\mathbf{x}_0,\mathbf{y}_0) R^*_{\ell,m}(\mathbf{y}-\mathbf{y}_0) R^*_{\ell',m'}(\mathbf{x}-\mathbf{x}_0)
\end{aligned} \tag{29}$$

with the coefficients

$$\xi_{\ell,\ell',m,m'}(\mathbf{x}_0,\mathbf{y}_0) = (-1)^{\ell'} S_{\ell+\ell',m+m'}(\mathbf{x}_0-\mathbf{y}_0).$$

Where $\mathbf{x}$ is close to the expansion centre $\mathbf{x}_0$ and $\mathbf{y}$ to $\mathbf{y}_0$. More exactly we can state that for the convergence of the expansion it is required $\sup \|\mathbf{x}-\mathbf{x}_0\| + \sup \|\mathbf{y}-\mathbf{y}_0| < \|\mathbf{x}_0-\mathbf{y}_0\|$.

Applying (29) to the potential ϕ, point $\mathbf{x}_0$ can be used as the local expansion centre for the potential and $\mathbf{y}_0$ as the local centre for the multipole coefficients of the mass distribution. Obviously this new expansion directly relates to the multipole and local expansion:

$$\phi(\mathbf{x}) = \sum_{\ell',m'} \underbrace{\sum_{\ell,m} \underbrace{G\int_V R^*_{\ell,m}(\mathbf{y}-\mathbf{y}_0)d\mathbf{y}}_{M_{\ell,m}(\mathbf{y}_0)}\, \xi_{\ell,\ell',m,m'}(\mathbf{x}_0,\mathbf{y}_0)}_{L^*_{\ell',m'}(\mathbf{x}_0)}\, R^*_{\ell',m'}(\mathbf{x}-\mathbf{x}_0).$$

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