Physics Laboratory I

VP141

Exercise I

Measurements of the Moment of Inertia

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Contents

1 Introduction

The objectives of this exercise include measuring the moment of inertia of a rigid body with the constant-torque method. The dependence of the moment of inertia on mass distribution and on the choice of the rotation axis will be studied. The parallel axis (Steiner's) theorem can be verified. This exercise also involves the operation of a photo-gate electronic timer.

Moment of inertia of a rigid body about an axis characterizes the body's resistance (inertia) to change of angular velocity in rotation about that axis. It's determined by both its mass and mass distribution. If the rigid body obtains an irregular shape or non-uniformly distributed mass, we can easily use experimental methods to calculate it.

1.1 Second law of dynamics for rotational motion

The second law of dynamics for rotational motion about a fixed axis is

$$\tau_z = I\beta_z, \tag{1}$$

relates the component of the torque tau_z about the axis of rotation with the moment of inertia about this axis, and the angular acceleration component β_c. Therefore, the moment of inertia I can be found with these two quantities measured.

Also we know that the moment of inertia is additive. The moment of inertia of a combined rigid body AB composed of A and B, about the same axis of rotation, is

$$I_{AB} = I_A + I_B.$$

1.2 Parallel axis theorem

If the moment of inertia of a rigid body about the axis through the body's center of mass is I_0, the nfor any axis parallel to that, the moment of inertia is

$$I = I_0 + md^2, \tag{2}$$

where d is the distance between the two axes. This theorem is known as Parallel axis theorem or Steiner's theorem.

2 Experiment Setup

2.1 Measurement setup and measurement method

The measurement setup is shown in Figure 1. It consists of a turntable with an integrated photo-gate system used for time measurements.

Empty turntable is initially rotating and its moment of inertia with respect to the rotation axis is I_1. Since the bearings of the turntable are not frictionless, there will be a non-zero frictional torque M_μ causing the turntable to decelerate with angular acceleration β_1, which can be presented by the following equation,

$$M_\mu = -I_1\beta_1. \tag{3}$$

Figure 1: Measurement setup

Below the turntable, there is a conical pulley of radius R, with a light and inextensible string wound on it. The axis of the cone coincides with the axis of rotation. Attached to the other end of the string passing through a disk pulley, there is a weight with mass m, free to move downwards after it is released. If the mass moves downwards with constant acceleration a, the tension in the string T is constant and $T = m(g - a)$. If the turntable rotates with angular acceleration β_2, then $a = R\beta_2$ (we assume that the string does not slip on the pulleys). Hence, the torque is $\tau = TR = m(g - R\beta_2)R$. Consequently, taking into account the frictional torque, the net torque on the turntable is $TR - M_\mu$, and the equation of motion for the turntable is

$$m(g - R\beta_2)R - M_\mu = I_1\beta_2. \tag{4}$$

Plugging in the Eq.3 to Eq.4, we find

$$I_1 = \frac{mR(g - R\beta_2)}{\beta_2 - \beta_1}. \tag{5}$$

Similarly, we may find the moment of inertia of a rigid body on the turntable by

$$I_2 = \frac{mR(g - R\beta_4)}{\beta_4 - \beta_3}. \tag{6}$$

where β_3 is the magnitude of the angular deceleration of the turntable with the body, and β_4 is its angular acceleration, when the mass m is released and moves downwards.

Using the fact that the moment of inertia is an additvie quantity, the moment of inertia of the rigid object placed on the turntable, with repect to the axis of the rotation, may be found as the difference

$$I_3 = I_2 - I_1. \tag{7}$$

2.2 Measurement of angular acceleration

At the edge of the turtable two shielding pins are fixed. The pins will send signals to the photo-gate with the phase interval of π. An integrated counter-type electronic timer is used to measure the consecutive number k and the time of the photo-gate signal.

If (k, t) is a set of the measurement data, the corresponding angular position is

$$\theta = k\pi = \omega_0 t + \frac{1}{2}\beta t^2$$

where ω_0 is the initial angular speed. Now, by performing a quadratic fit to the measurement data, we can find the angular acceleration.

2.3 Devices precision

The precisions of the devices are shown in Table 1.

Devices	Precision	Unit
Calliper	2×10^{-5}	[m]
Electronic balance	1×10^{-4}	[kg]
Timer	$1 \times 10^{-4} \pm 0.004\%$	[s]

Table 1: Precision of the measurement instruments

3 Measurement Procedure

1. Measure the mass of the weight, the hoop, the disk and the cylinder, as well as the radius of the cone pulley and the cylinder (following the instructor's requirements). Calculate the moment of inertia of the hoop and the disk analytically.
   Use a reliable source to find the local value of the acceleration due to gravity in Shanghai.

2. Turn the electronic timer on and switch it to mode 1-2 (single gate, multiple pulses).

3. Place the instrument close to the edge of the desk and stretch the disk pulley arm outside so that the weight can move downwards unobstructed.

4. Level the turntable with bubble level.

5. Make the turntable rotating and press the start button on the timer. After at least 8 signals are recorded, stop the turntable and record the data in your data sheet.

6. Attach the weight to one end of the string. Place the string on the disk pulley, thread through the hole in the arm, and wind the string around the third ring of the cone pulley. Adjust the arm holder so that the string goes through the center of the hole.

7. Release the weight and the start the timer. Stop the turntable when the weight hits the floor. Write down the recorded data.

8. The angular acceleration can be found by plotting $\theta = k\pi$ against t and performing a quadratic fit using data processing software. (The magnitude of the angular acceleration is equal to the coefficient next to t^2 multiplied by two. The uncertainty of the angular acceleration can be read directly from the fitting result.)

4 Results

4.1 Measurements of distances

	1	2	3	4
Disk $\varnothing$ [$\times 10^{-2}$m] $\pm$ 0.002[$\times 10^{-2}$m]	23.994	23.990	23.992	23.992
Hoop $\varnothing_1$ [$\times 10^{-2}$m] $\pm$ 0.002[$\times 10^{-2}$m]	24.010	24.000	24.016	24.012
Hoop $\varnothing_2$ [$\times 10^{-2}$m] $\pm$ 0.002[$\times 10^{-2}$m]	20.988	21.000	20.980	21.000
Cylinder A $\varnothing$ [$\times 10^{-2}$m] $\pm$ 0.002[$\times 10^{-2}$m]	3.026	3.000	2.998	3.000
Cylinder B $\varnothing$ [$\times 10^{-2}$m] $\pm$ 0.002[$\times 10^{-2}$m]	2.998	3.000	3.000	2.990
Cone pulley $\varnothing$ [$\times 10^{-2}$m] $\pm$ 0.002[$\times 10^{-2}$m]	5.020	5.020	5.012	5.014

Table 2: Calliper measurements

	Inner distance	outer distance
Hole 1 d [$\times 10^{-2}$m] $\pm$ 0.002[$\times 10^{-2}$m]	3.982	5.028
Hole 2 d [$\times 10^{-2}$m] $\pm$ 0.002[$\times 10^{-2}$m]	4.010	5.022
Hole 3 d [$\times 10^{-2}$m] $\pm$ 0.002[$\times 10^{-2}$m]	7.010	8.028
Hole 4 d [$\times 10^{-2}$m] $\pm$ 0.002[$\times 10^{-2}$m]	6.994	8.026

Table 3: Calliper measurements for holes

The mean values can be calculated by

$$\varnothing = \frac{1}{n}\sum_{i=1}^{n}\varnothing_i.$$

For example, for the Disk $\varnothing$ in Table 2,

$$\bar{\varnothing} = \frac{23.994 + 23.990 + 23.992 + 23.992}{4} \times 10^{-2} = 0.23992 \pm 0.00003m, \quad u_{\varnothing,r} = 0.014\%.$$

Similarly, from the raw data in Table 2, we obtain the different sets of radii.

	mean value [m]	uncertainty [m]	relative uncertainty [%]
Disk $\varnothing$	0.23992	0.00003	0.014
Hoop $\varnothing_1$	0.24010	0.00011	0.05
Hoop $\varnothing_2$	0.20992	0.00016	0.07
Cylinder A $\varnothing_A$	0.0301	0.0002	0.7
Cylinder B $\varnothing_B$	0.02997	0.00001	0.3
Cone pulley $\varnothing_C$	0.05017	0.00007	0.14

Table 4: Average results of diameters

We know that the distances from the rotating axis to the holes are calculated by the average value of inner distance and outer distances (see Table 3).

Here's the **sample calculation** for hole 1.

$$d_1 = \frac{d_{1,in} + d_{1,out}}{2} = \frac{3.982 + 5.028}{2} \times 10^{-2} = 0.04505 \pm 0.00001 m, \quad u_{d_1,r} = 0.03\%$$

SImilarly, we obtain the data as shown in Table 5.

	mean value [m]	uncertainty [m]	relative uncertainty [%]
Hole 1 d_1	0.04505	0.00001	0.03
Hole 2 d_2	0.04516	0.00001	0.03
Hole 3 d_3	0.07519	0.00001	0.02
Hole 4 d_4	0.07510	0.00001	0.02

Table 5: Average results of calliper measurements

4.2 Measurements of mass

The following table shows the mass measuremnts.

	mean value [kg]	uncertainty [kg]	relative uncertainty [%]
Disk	0.4881	0.0001	0.02
Hoop	0.3998	0.0001	0.03
Cylinder A	0.1660	0.0001	0.06
Cylinder B	0.1660	0.0001	0.06
Weight	0.0545	0.0001	0.18

Table 6: Mass measurements

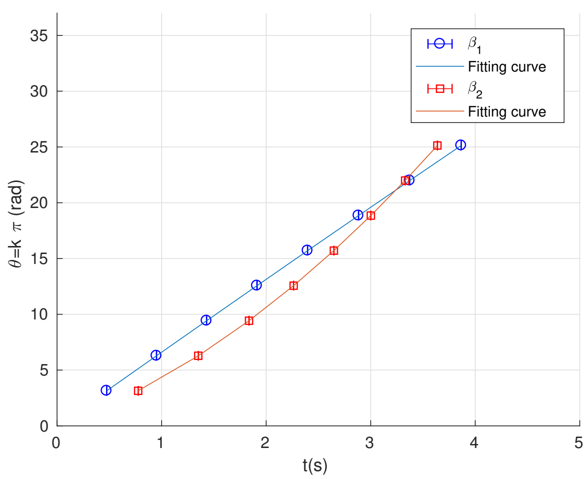
4.3 Measurements of the acceleration

In this exercise, we get the value of angular acceleration β by θ vs. t quadratic fitting. The data for curve fitting are shown in the corresponding table.

4.3.1 Measurements for empty turntable

	Deceleration							
θ	3.142	6.283	9.425	12.57	15.71	18.85	21.99	25.13
t[s]	0.4748	0.9520	1.4311	1.9131	2.3975	2.8843	3.3738	3.8665
$u_t[\times 10^{-3}\text{s}]$	0.12	0.14	0.2	0.2	0.2	0.2	0.2	0.3
	Acceleration							
θ	3.142	6.283	9.425	12.57	15.71	18.85	21.99	25.13
t[s]	0.7761	1.3508	1.8366	2.2633	2.6480	3.0014	3.3299	3.6383
$u_t[\times 10^{-3}\text{s}]$	0.13	0.2	0.2	0.2	0.2	0.2	0.2	0.3

Table 7: Measurements for empty turntable



$$\beta_1 = -0.071 \pm 0.002 rad/s^2,$$
$$\beta_2 = 1.956 \pm 0.012 rad/s^2.$$

Figure 2: Quadratic fitting for empty turntable

```
Linear model Poly2:
    f(x) = p1*x^2 + p2*x + p3
Coefficients (with 95% confidence bounds):
      p1 =    -0.03547  (-0.03647, -0.03448)
      p2 =       6.638  (6.634, 6.643)
      p3 =   -0.003207  (-0.007384, 0.0009704)

Goodness of fit:
  SSE: 6.941e-06
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.001178
```

Figure 3: Deceleration of empty turntable

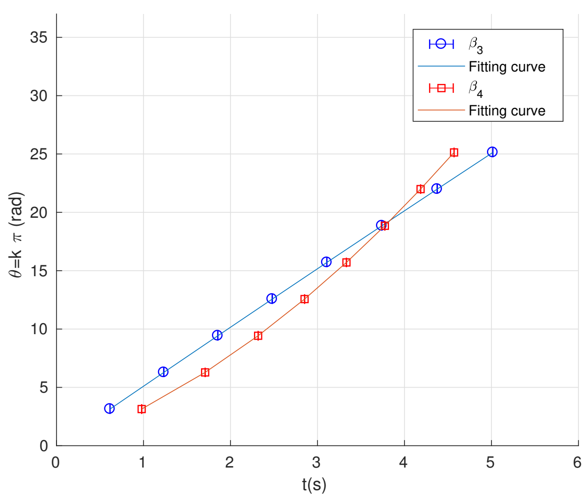
```
Linear model Poly2:
    f(x) = p1*x^2 + p2*x + p3
Coefficients (with 95% confidence bounds):
      p1 =       0.978  (0.9721, 0.984)
      p2 =       3.363  (3.336, 3.39)
      p3 =     -0.0527  (-0.08003, -0.02536)

Goodness of fit:
  SSE: 0.0001269
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.005038
```

Figure 4: Acceleration of empty turntable

4.3.2 Measurements for turntable with disk

	Deceleration							
θ	3.142	6.283	9.425	12.57	15.71	18.85	21.99	25.13
t[s]	0.6150	1.2334	1.8548	2.4803	3.1085	3.7404	4.3765	5.0164
$u_t[\times 10^{-3}\text{s}]$	0.12	0.15	0.2	0.2	0.2	0.2	0.3	0.3
	Acceleration							
θ	3.142	6.283	9.425	12.57	15.71	18.85	21.99	25.13
t[s]	0.9789	1.7085	2.3176	2.8516	3.3332	3.7747	4.1853	4.5701
$u_t[\times 10^{-3}\text{s}]$	0.14	0.2	0.2	0.2	0.2	0.3	0.3	0.3

Table 8: Measurements for turntable with disk



$$\beta_3 = -0.0452 \pm 0.0013 rad/s^2,$$

$$\beta_4 = 1.2700 \pm 0.0012 rad/s^2.$$

Figure 5: Quadratic fitting for turntable with disk

```
Linear model Poly2:
    f(x) = p1*x^2 + p2*x + p3
Coefficients (with 95% confidence bounds):
      p1 =    -0.02258  (-0.02323, -0.02193)
      p2 =       5.124  (5.12, 5.128)
      p3 =   -0.001759  (-0.006335, 0.002817)

Goodness of fit:
  SSE: 8.347e-06
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.001292
```

Figure 6: Deceleration of turntable with disk

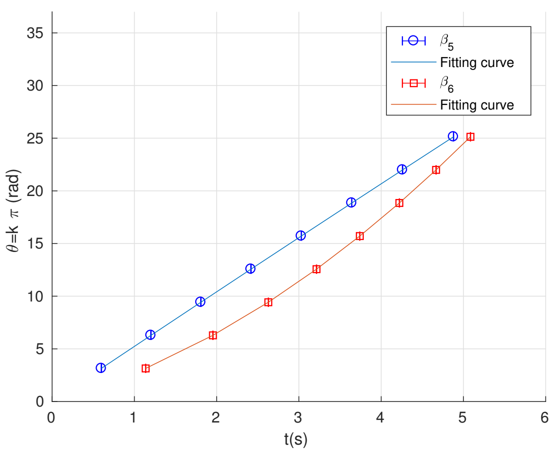
```
Linear model Poly2:
    f(x) = p1*x^2 + p2*x + p3
Coefficients (with 95% confidence bounds):
      p1 =       0.635  (0.6343, 0.6356)
      p2 =         2.6  (2.596, 2.604)
      p3 =    -0.01195  (-0.01677, -0.007128)

Goodness of fit:
  SSE: 3.91e-06
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.0008843
```

Figure 7: Acceleration of turntable with disk

4.3.3 Measurements for turntable with hoop

	Deceleration							
θ	3.142	6.283	9.425	12.57	15.71	18.85	21.99	25.13
t[s]	0.5987	1.2010	1.8081	2.4173	3.0291	3.6429	4.2600	4.8792
$u_t[\times 10^{-3}\text{s}]$	0.12	0.15	0.2	0.2	0.2	0.2	0.3	0.3
	Acceleration							
θ	3.142	6.283	9.425	12.57	15.71	18.85	21.99	25.13
t[s]	1.1370	1.9548	2.6283	3.2142	3.7402	4.2213	4.6676	5.0855
$u_t[\times 10^{-3}\text{s}]$	0.15	0.2	0.2	0.2	0.2	0.3	0.3	0.3

Table 9: Measurements for turntable with hoop



$$\beta_5 = -0.036 \pm 0.003 rad/s^2,$$
$$\beta_6 = 1.1034 \pm 0.0008 rad/s^2.$$

Figure 8: Quadratic fitting for turntable with hoop

```
Linear model Poly2:
    f(x) = p1*x^2 + p2*x + p3
Coefficients (with 95% confidence bounds):
      p1 =    -0.01809  (-0.01955, -0.01662)
      p2 =       5.236  (5.227, 5.244)
      p3 =     0.01674  (0.006932, 0.02654)

Goodness of fit:
  SSE: 3.826e-05
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.002766
```

Figure 9: Deceleration of turntable with hoop

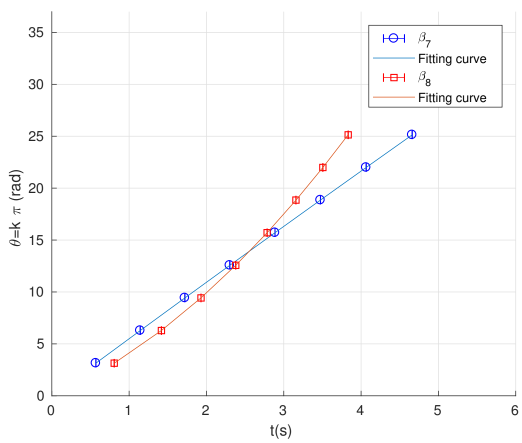
```
Linear model Poly2:
    f(x) = p1*x^2 + p2*x + p3
Coefficients (with 95% confidence bounds):
      p1 =      0.5517  (0.5514, 0.5521)
      p2 =       2.136  (2.134, 2.138)
      p3 =   -0.001116  (-0.003984, 0.001752)

Goodness of fit:
  SSE: 1.274e-06
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.0005047
```

Figure 10: Acceleration of turntable with hoop

4.3.4 Measurements for turntable with cylinder A in hole 1 and B in 2

	Deceleration							
θ	3.142	6.283	9.425	12.57	15.71	18.85	21.99	25.13
t[s]	0.5702	1.1439	1.7209	2.3022	2.8866	3.4748	4.0660	4.6609
$u_t[\times 10^{-3}\text{s}]$	0.12	0.15	0.2	0.2	0.2	0.2	0.3	0.3
	Acceleration							
θ	3.142	6.283	9.425	12.57	15.71	18.85	21.99	25.13
t[s]	0.8071	1.4176	1.9295	2.3788	2.7848	3.1574	3.5055	3.8331
$u_t[\times 10^{-3}\text{s}]$	0.13	0.2	0.2	0.2	0.2	0.2	0.2	0.3

Table 10: Measurements for turntable with cylinder A in 1 and B in 2



$$\beta_7 = -0.056 \pm 0.002 rad/s^2,$$
$$\beta_8 = 1.748 \pm 0.016 rad/s^2.$$

Figure 11: Quadratic fitting for turntable with cylinder A in 1 and B in 2

```
Linear model Poly2:
    f(x) = p1*x^2 + p2*x + p3
Coefficients (with 95% confidence bounds):
      p1 =    -0.02777  (-0.02877, -0.02678)
      p2 =       5.521  (5.515, 5.526)
      p3 =    0.004172  (-0.001865, 0.01021)

Goodness of fit:
  SSE: 1.457e-05
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.001707
```

Figure 12: Deceleration of turntable with cylinder A in 1 and B in 2

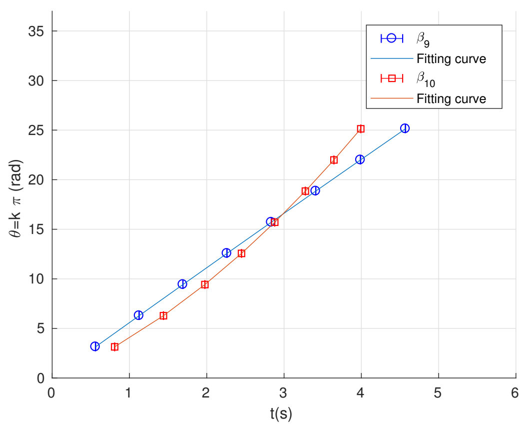
```
Linear model Poly2:
    f(x) = p1*x^2 + p2*x + p3
Coefficients (with 95% confidence bounds):
      p1 =      0.8741  (0.8662, 0.8821)
      p2 =       3.216  (3.178, 3.253)
      p3 =    -0.02794  (-0.06812, 0.01223)

Goodness of fit:
  SSE: 0.0002808
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.007494
```

Figure 13: Acceleration of turntable with cylinder A in 1 and B in 2

4.3.5 Measurements for turntable with cylinder A in hole 3 and B in 4

	Deceleration							
θ	3.142	6.283	9.425	12.57	15.71	18.85	21.99	25.13
t[s]	0.5611	1.1250	1.6916	2.2618	2.8345	3.4103	3.9888	4.5704
$u_t[\times 10^{-3}\text{s}]$	0.12	0.15	0.2	0.2	0.2	0.2	0.3	0.3
	Acceleration							
θ	3.142	6.283	9.425	12.57	15.71	18.85	21.99	25.13
t[s]	0.8086	1.4397	1.9758	2.4497	2.8797	3.2757	3.6453	3.9925
$u_t[\times 10^{-3}\text{s}]$	0.13	0.2	0.2	0.2	0.2	0.2	0.2	0.3

Table 11: Measurements for turntable with cylinder A in 3 and B in 4



$$\beta_9 = -0.0492 \pm 0.0006 rad/s^2,$$

$$\beta_{10} = 1.509 \pm 0.001 rad/s^2.$$

Figure 14: Quadratic fitting for turntable with cylinder A in 3 and B in 4

```
Linear model Poly2:
    f(x) = p1*x^2 + p2*x + p3
Coefficients (with 95% confidence bounds):
      p1 =    -0.02459  (-0.02526, -0.02392)
      p2 =       5.611  (5.607, 5.614)
      p3 =    0.001923  (-0.002016, 0.005863)

Goodness of fit:
  SSE: 6.174e-06
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.001111
```

Figure 15: Deceleration of turntable with cylinder A in 3 and B in 4

```
Linear model Poly2:
    f(x) = p1*x^2 + p2*x + p3
Coefficients (with 95% confidence bounds):
      p1 =      0.7547  (0.7533, 0.7561)
      p2 =       3.284  (3.277, 3.291)
      p3 =   -0.008402  (-0.01604, -0.0007673)

Goodness of fit:
  SSE: 1.089e-05
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.001476
```

Figure 16: Acceleration of turntable with cylinder A in 3 and B in 4

4.4 Calculations for the moment of inertia

First, according to United States Department of Commerce, the acceleration due to gravity in Shanghai is $9.794 m/s^2$.

Here we only have the **sample calculation** of the moment of inertia of the empty turntable. As stated in Eq. 7, we can derive I_1 from the data in Table 4, 5 and 6.

$\beta_1 = -0.071 \pm 0.002 rad/s^2$,

$\beta_2 = 1.956 \pm 0.012 rad/s^2$,

$R_{cone} = 0.02509 \pm 0.00003 m$,

$m_{weight} = 0.0545 \pm 0.0001 kg$,

$g = 9.794 m/s^2$,

$$I_1 = \frac{mR(g - R\beta_2)}{\beta_2 - \beta_1} = \frac{0.0545 \times 0.02509 \times (9.794 - 0.02509 \times 1.956)}{1.956 - (-0.071)} = (0.00707 \pm 0.00005) kg \cdot m^2.$$

Hence, $I_1 = (7.07 \pm 0.05) \times 10^{-3} kg \cdot m^2, \quad u_{I_1,r} = 0.7\%$.

Simlarly, we can calculate the other moments of inertia (see Table 12).

	I [$\times 10^{-3} kg \cdot m^2$]	u_I [$\times 10^{-3} kg \cdot m^2$]	$u_{I,r}$ [%]
Empty turntable I_1	7.07	0.05	0.7
Turntable with disk I_2	10.90	0.03	0.3
Turntable with hoop I_3	12.51	0.05	0.4
A in 1 & B in 2 I_4	7.88	0.08	1.0
A in 3 & B in 4 I_5	9.14	0.02	0.2

Table 12: Results for the moments of inertia (combined)

Finally, due to the additivity of the moment of inertia, we calculate the difference.

For example, for the turntable with disk, we know that

$$I_{disk} = I_2 - I_1 = (3.83 \pm 0.06) \times 10^{-3} kg \cdot m^2.$$

	I [$\times 10^{-3} kg \cdot m^2$]	u_I [$\times 10^{-3} kg \cdot m^2$]	$u_{I,r}$ [%]
I_{empty}	7.07	0.05	0.7
I_{disk}	3.83	0.06	1.5
I_{hoop}	5.44	0.07	1.2
$I_{1,2}$	0.81	0.09	11
$I_{3,4}$	2.07	0.05	3

Table 13: Results for the moments of inertia

4.5 Calculations for theoretical value of moment of inertia

Since it's the theoretical value of moment of inertia, then we just consider it as the precise value without uncertainty. The following table shows the conslusion of theoretical values and the specific calculations are presented respectively.

	I_t $[\times 10^{-3} kg \cdot m^2]$
I_{disk}	3.51
I_{hoop}	5.08
$I_{1,2}$	0.72
$I_{3,4}$	1.91

Table 14: Results for theoretical moments of inertia

4.5.1 Theoretical Moment inertia of disk

Recall the measurements for the disk (see Table 4 and 6).

$$m_{disk} = 0.4881 kg.$$
$$R_{disk} = 0.23992/2 = 0.11996 m.$$

According to the formula of moment of inertia, we obtain that

$$I_{disk} = \frac{1}{2} m_{disk} R_{disk}^2 = \frac{1}{2} \times 0.4881 \times 0.11996^2 = 3.51 \times 10^{-3} kg \cdot m^2.$$

4.5.2 Theoretical Moment inertia of hoop

Recall the measurements for the hoop (see Table 4 and 6).

$$m_{hoop} = 0.3998 kg.$$
$$R_{hoop,in} = 0.20992/2 = 0.10496 m.$$
$$R_{hoop,out} = 0.24010/2 = 0.12005 m.$$

According to the formula of moment of inertia, we obtain that

$$\begin{aligned} I_{hoop} &= \frac{1}{2} m_{hoop} (R_{hoop,in}^2 + R_{hoop,out}^2) = \frac{1}{2} \times 0.3998 \times (0.10496^2 + 0.12005^2) \\ &= 5.08 \times 10^{-3} kg \cdot m^2. \end{aligned}$$

4.5.3 Theoretical Moment inertia of cylinder condition

Recall the measurements for the cylinder A in hole 1 and B in 2 (see Table 4 and 6).

$$
\begin{aligned}
m_A &= 0.1660kg, \\
m_B &= 0.1660kg, \\
R_A &= 0.0301/2 = 0.01505m, \\
R_B &= 0.02997/2 = 0.014985m, \\
d_1 &= 0.04505m, \\
d_2 &= 0.04516m.
\end{aligned}
$$

According to the formula of moment of inertia, we obtain that

$$
\begin{aligned}
I_A &= \frac{1}{2}m_A R_A^2 = \frac{1}{2} \times 0.1660 \times 0.01505^2 = 1.88 \times 10^{-5} kg \cdot m^2. \\
I_B &= 1.86 \times 10^{-5} kg \cdot m^2.
\end{aligned}
$$

Accoring to the parallel axis theorem, we know that

$$
\begin{aligned}
I_{1,2} &= I_A + m_A d_1^2 + I_B + m_B d_2^2 \\
&= 1.88 \times 10^{-5} + 0.1660 \cdot 0.04505^2 + 1.86 \times 10^{-5} + 0.1660 \cdot 0.04516^2 = 0.72 \times 10^{-3} kg \cdot m^2.
\end{aligned}
$$

Similarly, we can derive $I_{3,4}$,

$$
I_{3,4} = 1.91 \times 10^{-3} kg \cdot m^2.
$$

5 Measurement Uncetainty Analysis

5.1 Uncertainty for distances

For the calliper measurements except that diameters of holes, we have four measurements for each quantity. To demonstrate the uncertainty, the **sample calculation** for diameters of the disk will be presented. (raw data see Table 2)

$$
\begin{aligned}
&\bar{\varnothing} = 0.23992m \\
&\overline{s_{\varnothing}} = \sqrt{\frac{1}{4(4-1)}\sum_{i=1}^{4}(\varnothing_i - \bar{\varnothing})^2} \\
&= \sqrt{\frac{(0.23994 - 0.23992)^2 + (0.23990 - 0.23992)^2 + (0.23992 - 0.23992)^2 + (0.23992 - 0.23992)^2}{12}} \\
&= 8.1650 \times 10^{-6} m.
\end{aligned}
$$

Then, we obtain the type-A uncertainty and type-B uncertainty respectively. Their square root of sum of squre contribute to the combined uncertainty.

$$\begin{aligned}
&t_{0.95} = 3.18, \quad n = 4, \\
&\Delta_{A,\varnothing} = t_{0.95} \times \overline{s_{\varnothing}} \approx 3 \times 10^{-5} m, \\
&\Delta_{B,\varnothing} = 2 \times 10^{-5} m, \\
&u_{\varnothing} = \sqrt{\Delta_{A,\varnothing}^2 + \Delta_{B,\varnothing}^2} = \sqrt{(3 \times 10^{-5})^2 + (2 \times 10^{-5})^2} = 3 \times 10^{-5} m, \\
&u_{\varnothing,r} = \frac{u_{\varnothing}}{\varnothing} \times 100\% = \frac{3 \times 10^{-5}}{0.23992} \times 100\% = 0.014\%
\end{aligned}$$

Hence,

$$\varnothing = 0.23992 \pm 0.00003 m, \quad u_{\varnothing,r} = 0.014\%.$$

Similarly, we are capable of calculating all the uncertainties (see Table 4).

Then, for the uncertainty of distance from the rotating axis to the holes, we need to calcualte the propagated uncertainty. Here's the **sample calculation** for hole 1.

$$\begin{aligned}
&R_1 = \frac{d_{1,in} + d_{1,out}}{2}, \\
&\frac{\partial R_1}{\partial d_{1,in}} = \frac{\partial R_1}{\partial d_{1,out}} = \frac{1}{2}, \\
&u_{d_{1,in}} = u_{d_{1,out}} = \Delta_{dev} = 2 \times 10^{-5} m \\
&u_{R_1} = \sqrt{(\frac{\partial R_1}{\partial d_{1,in}})^2 (u_{d_{1,in}})^2 + (\frac{\partial R_1}{\partial d_{1,out}})^2 (u_{d_{1,out}})^2} \\
&= \sqrt{\frac{1}{4}(2 \times 10^{-5})^2 + \frac{1}{4}(2 \times 10^{-5})^2} = 1.4 \times 10^{-5} m, \\
&u_{R_1,r} = \frac{u_{R_1}}{R_1} \times 100\% = \frac{1.4 \times 10^{-5}}{0.04505} = 0.03\%
\end{aligned}$$

Hence,

$$R_1 = 0.04505 \pm 0.00001 m, \quad u_{R_1,r} = 0.03\%.$$

Similarly, we obtain all the uncertainties of distances between holes (see Table 5).

5.2 Uncertainty of mass measurements

For example, for the mass of the disk, its uncertainty is merely its type-B uncertainty.

$$\begin{aligned}
&u_{m_{disk}} = \Delta_B = 1 \times 10^{-4} kg, \\
&u_{m_{disk},r} = \frac{u_{m_{disk}}}{m_{disk}} \times 100\% = 0.02\%
\end{aligned}$$

All the uncertainties of mass between holes (see Table 6).

5.3 Uncertainty of angular acceleration

We will calculate the uncertainty of quadratic fitting by $u = t_{0.95} \cdot std/\sqrt{8-2}$, where $t_{0.95} = 2.36$ for n=8. Hoever, the angular acceleration is twice of coefficient of quadratic item. **For example**, when $p_1 = -0.03547 \pm 0.00099 rad/s^2$ (p_1 is the coefficient of quadratic item),

$$\begin{aligned}
\beta &= 2p_1 = 2 \times (-0.03547) = -0.071 rad/s^2, \\
u_\beta &= \sqrt{(\frac{\partial \beta}{\partial p_1})^2 (u_{p_1})^2} = 2u_{p_1} = 0.002 rad/s^2, \\
u_{\beta,r} &= \frac{u_\beta}{\beta} \times 100\% = \frac{0.002}{0.071} = 3\%.
\end{aligned}$$

Similar calculation results are listed in Table 7, 8, 9, 10, 11.

5.4 Uncertainty of the moment of inertia

According to Eq. 7, we know that the uncertainty of the moment of inertia obtains propagated uncertainty.

Here we have the **sample calcultion** for empty turntable.

$$\begin{aligned}
\frac{\partial I_1}{\partial m} &= \frac{R(g - R\beta_2)}{\beta_2 - \beta_1}, \\
\frac{\partial I_1}{\partial R} &= \frac{mg - 2mR\beta_2}{\beta_2 - \beta_1}, \\
\frac{\partial I_1}{\partial \beta_1} &= \frac{mR(g - R\beta_2)}{(\beta_2 - \beta_1)^2}, \\
\frac{\partial I_1}{\partial \beta_2} &= -\frac{mR(R\beta_1 + g)}{(\beta_2 - \beta_1)^2},
\end{aligned}$$

$$\begin{aligned}
u_{I_1} &= \sqrt{(\frac{\partial I_1}{\partial m})^2 (u_m)^2 + (\frac{\partial I_1}{\partial R})^2 (u_R)^2 + (\frac{\partial I_1}{\partial \beta_1})^2 (u_{\beta_1})^2 + (\frac{\partial I_1}{\partial \beta_2})^2 (u_{\beta_2})^2} \\
&= \sqrt{(\frac{R(g - R\beta_2)}{\beta_2 - \beta_1})^2 (u_m)^2 + (\frac{mg - 2mR\beta_2}{\beta_2 - \beta_1})^2 (u_R)^2 + (\frac{mR(g - R\beta_2)}{(\beta_2 - \beta_1)^2})^2 (u_{\beta_1})^2 + (-\frac{mR(R\beta_1 + g)}{(\beta_2 - \beta_1)^2})^2 (u_{\beta_2})^2} \\
&= \sqrt{(0.130)^2 (0.0001)^2 + (0.280)^2 (0.00003)^2 + (0.004)^2 (0.002)^2 + (-0.004)^2 (0.012)^2} \\
&\approx 5 \times 10^{-5} kg \cdot m^2. \\
u_{I_1,r} &= \frac{u_{I_1}}{I_1} \times 100\% = \frac{0.00005}{0.00707} = 0.7\%.
\end{aligned}$$

Hence,

$$I_1 = (7.07 \pm 0.05) \times 10^{-3} kg \cdot m^2, \quad u_{I_1,r} = 0.7\%.$$

Simlarly, we can calculate uncertainties for the other combined moments of inertia (see Table 12).

Finally, due to the additivity of the moment of inertia, we calculate the difference. We take the disk as the **example**.

$$
\begin{aligned}
&I_{disk} = I_2 - I_1, \\
&\frac{\partial I_{disk}}{\partial I_2} = 1, \\
&\frac{\partial I_{disk}}{\partial I_1} = -1, \\
&u_I = \sqrt{(\frac{\partial I_{disk}}{\partial I_2})^2 (u_{I_2})^2 + (\frac{\partial I_{disk}}{\partial I_1})^2 (u_{I_1})^2} \\
&= \sqrt{(3 \times 10^{-5})^2 + (5 \times 10^{-5})^2} \approx 0.06 \times 10^{-3} kg \cdot m^2.
\end{aligned}
$$

Simlarly, we can calculate uncertainties for the other moments of inertia (see Table 15).

6 Conclusion and Discussion

In this exercise, we measure the corresponding physics quantities of the moment of inertia and calculate it by experiment. After that, we calculate the theoretical values of I according to the moment of inertia formula we derive by integral. By comparison, we try to verify the parallel axis (Steiner's) theorem.

Here list our final experimental and theoretical data again.

	Theoretical value $[\times 10^{-3} kg \cdot m^2]$	Experimental value $[\times 10^{-3} kg \cdot m^2]$	u_I $[\times 10^{-3} kg \cdot m^2]$	$u_{I,r}$ [%]
I_{disk}	3.51	3.83	0.06	1.5
I_{hoop}	5.08	5.44	0.07	1.2
$I_{1,2}$	0.72	0.81	0.09	11
$I_{3,4}$	1.91	2.07	0.05	3

Table 15: Results for the moments of inertia

For the disk, the relative deviation is $(3.83 - 3.51)/3.51 \times 100\% = 9\%$, which is out of its 0.95 confidence bound. However, we can basically know the moment of inertia through this experiment.

For the hoop, the relative deviation is

$$(5.44 - 5.08)/5.08 \times 100\% = 7\%$$

, which is also out of its 0.95 confidence bound, but still acceptable.

For the cylinder case, we can verify the parallel axis theorem. In the first case, the relative deviation is

$$(0.81 - 0.72)/0.72 \times 100\% = 11\%$$

, which is just on 0.95 confidence bound point. In fact it shows that in this case the angular acceleration obtains larger uncertainty. In the second case, the relative deviation is

$$(2.07 - 1.91)/1.91 \times 100\% = 8\%$$

, which is out of the 0.95 confidence bound. The possible reasons include

1. Shaking and displacement of the cylinders.
2. Displacement of rotating axis.
3. Change of frictional torque.

However, basically, the small deviation generally prove the parallel axis theorem.

I find it interesting that all the experimental values are larger than the theoretical values. Based on the formula, the possible reason include the gravity and the frictional torque.

The exact value of acceleration due to gravity might be smaller than the standard value I reference, which seems to strongly explain the phenomenon.

On the other hand, during my experiment I find that the string we use is fairly thick compared to the uncertainty. When it's winded, the radius of cone pulley rally changes greatly, let alone its friction.

For improvements, I suppose that we'd better obtain a more accurate value of g and use a kind of fishing line, which is thin, strong and smooth, relatively. Also, we should eliminate the shaking of cylinders. For example, we can decrease the initial angular velocity so that the cylinders won't move under high angular velocity.

Finally, we find that in the cylinder case, the moment of inertia is quite small. Therefore, in order to lower the relative uncertainty, we need to choose cylinders with larger mass. When the mass of cylinders are larger, the deviation will be correspondingly smaller.

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