JOINT INSTITUTE
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PHYSICS LABORATORY I

VP141

Exercise III

Simple Harmonic Motion Oscillations in Mechanical Systems

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1 Introduction

The objective of this exercise is to study simple harmonic oscillation, including how to find the spring constant, how to analyze the relationship between the oscillation period and the mass of the oscillator, whether the oscillation period depends on the amplitude. Besides, the relationship between the maximum speed and the amplitude will also be examined. Simple harmonic motion is one of the most fundamental periodic motions, where the restoring force is proportional to the displacement from the equilibrium position so that the position of a particle depends on time as a sine (or cosine) function.

1.1 Hooke's law

Within the elastic limit of deformation, the force F_x needed to compress or strech a spring is proportional to the distance, *i.e.*,

$$F_x = kx, \tag{1}$$

where k, the spring constant, characterizing how easy to deform a spring. We use *Jolly balance* in this exercise to find that constant. The relation 1 is known as the *Hooke's Law*. Therefore, the elastic force is found according to Newton's third law of dynamics, whose magnitude is the same but direction is opposite. It's also known as restoring forceas it always tries to restore the system back to the equilibrium.

1.2 Equation of Motion of the Simple Harmonic Oscillator

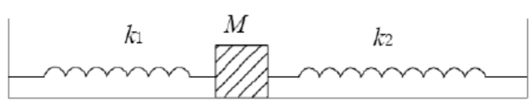


Figure 1: Mass-spring system

As shown in Figure 1.2, the mass with two springs is placed on an air track, which aims at eliminating the frictional forces. Assuming that the restoring force is the only force acting on mass M, the equation of motion od mass M is

$$M\frac{d^2x}{dt^2} + (k_1 + k_2)x = 0. \tag{2}$$

Hence the general solution to Eq. 2 is

$$x(t) = Acos(\omega_0 t + \phi_0), \tag{3}$$

where $\omega_0 = \sqrt{(k_1 + k_2)/M}$ is the natural angular frequency of the oscillations (determined by the parameters of the system), A is the amplitude, and ϕ_0 is the initial phase (determined by initial conditions). The natural period of oscillation is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{M}{k_1 + k_2}}. \tag{4}$$

In this exercise, the relationship mentioned above will be studied.

1.3 Mass of the Spring

We take into the mass of the springs in terms of the so-called *effective mass*, which is the sum of the mass of the object M and the effective mass of springs m_0. The angular frequency can be expressed as

$$\omega_0 = \sqrt{\frac{k_1 + k_2}{M + m_0}} \tag{5}$$

where m_0 is 1/3 of the actual mass of the spring.

1.4 Mechanical Energy in Harmonic Motion

The elastic potential energy for a spring-mass system is $U = kx^2/2$ and the kinetic energy of an oscillating mass is $K = mv^2/2$. At the equilibrium position ($x = 0$), the speed of the mass is maximum $v = v_{max}$. At this point the total mechanical energy is equal to maximum kinetic energy K_{max} . On the other hand, at maximum displacement ($x = \pm A$) the mass is instantaneously at rest, and the contribution to the total mechanical energy is due to the potential energy only, which is at its maximum U_{max}. In the absence of non-conservative forces (such as frictional forces or drag forces), the total mechanical energy is conserved and $K_{max} = U_{max}$, which implies

$$k = \frac{mv_{max}^2}{A^2}. \tag{6}$$

2 Experimental Setup

The measurement equipment consists of the following elements: springs, Jolly balance, air track, electronic timer, electronic balance and masses.

In order to measure the spring constant using the Jolly balance, we need to place the small mirror C (see Figure 2) in the tube D and make three lines coincide: the line on the mirror, the line on the glass tube and its reflection in the mirror. First, without adding any

A: Sliding bar with metric scale;
H: Vernier for reading;
C: Small mirror with a horizontal line in the middle;
D: Fixed glass tube also with a horizontal line in the middle;
G: Knob for ascending and descending the sliding bar;
S: Spring attached to top of the bar A.

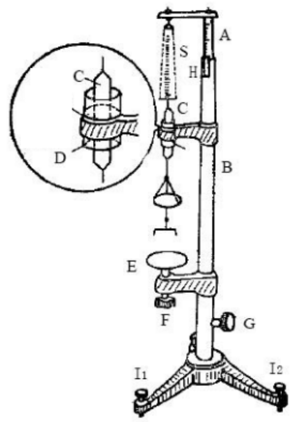


Figure 2: Jolly balance

weight on the bottom end of the spring, adjust the knob G and make the three lines coincide. Then read the scale L_1.

Second, add mass m to the bottom of the spring. The spring is stretched and the three lines no longer coincide. Adjust knob G to make them into one line again and read the corresponding number on scale L_2. The spring constant may be then found as

$$k = \frac{mg}{L_2 - L_1} \tag{7}$$

so that we can estimate the spring constant by finding a linear fit to the data using the least squares method.

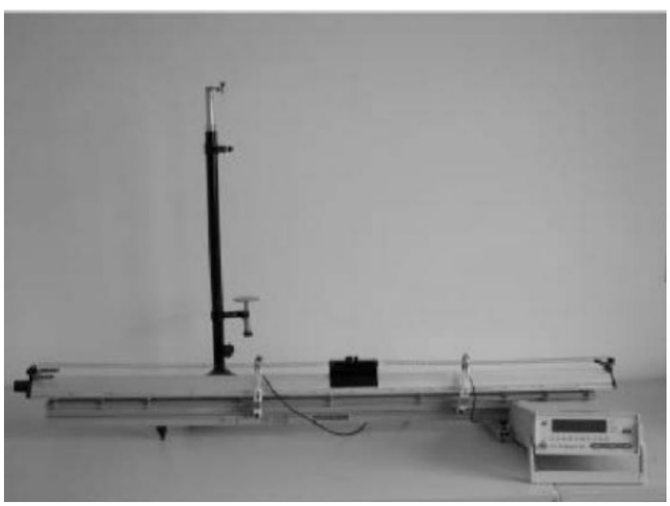


Figure 3: The experimental setup

A photoelectric measuring system consists of two photoelectric gates and an electronic timer. When a shutter on the object blocks the light, the computer will record the time. For period measurements, we use the I-shape shutter.

When measuring the instaneous speed, the U-shape shutter is required. Δt presents the time interval when the object travels a distance of $\Delta x = (x_{in} + x_{out})/2$. Hence we estimate the speed as $v = \Delta x/\Delta t$.

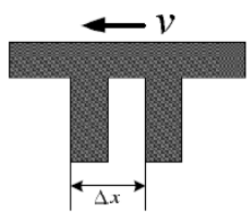


Figure 4: The U-shape shutter

3 Measurement Procedure

3.1 Spring constant

1. Adjust the Jolly balance to vertical and attach the spring Add a 20g preload and adjust knob I_1 and I_2 to make sure the mirror can move freely through the tube. Check if the balance parallel to the spring.

2. Adjust knob G to make the three lines in the tube coincide.

3. Reading the reading on the scale, add mass from 1 to 6 and record L_i in order.

4. Estimate the spring constant k_1 by fitting.

5. Repeat the measurements with spring 2. Calculate k_2.

6. Remove the preload and repeat the measurements for spring 1 and 2 connected. Calculate k_3.

3.2 Relation between oscillation period T and the mass of the oscillator M

(a) Adjust the air track so that it's horizontal.

1. Turn on the air track and check whether there's blocked holes.
2. Place the cart on the track. Adjust the track with the knob on the side with a single one.

(b) Horizontal air track

1. Attach springs to the sides of the cart and set up the I-shape shutter. Make sure that the photoelectric gate is at the equilibrium position.
2. Set the timer into "T" mode. Add weight in order and release it with a caliper. Record the correspoding time intervals for 10 periods.
3. Analyze the relation between M and T by plotting a graph.

(c) Inclined air track

1. Add three plastic plates under the air track everytime. Repeat the steps in (b).
2. Analyze the relation between M and T by plotting a graph.

3.3 Relation between the oscillation period and the amplitude

1. Keep the mass of the cart unchanged and change the amplitude (choose 6 different values). The amplitude is about 5.0/ 10.0/ 15.0/ ... /30.0 cm.

2. Apply linear fit to the data and comment on the relation between the oscillation period T and the amplitude A based on the correlation coefficient γ.

3.4 Relation between the maximum speed and the amplitude

1. Keep the mass of the cart unchanged and change the amplitude from 5.0 *to* 30.0cm with gap of 5.0cm.

2. Measure v_{max} for different amplitudes.

3.5 Mass measurement

1. Adjust the electronic balance every time before you use it. The level bubble should be in the center of the circle.

2. Add weights according to a fixed order. Weigh the cart with the I-shape shutter and with the U-shape shutter. Measure the mass of spring 1 and spring 2.

3. Record the data only after the circular symbol on the scales display disappears.

3.6 Devices Precision

The precisions of the devices are shown in Table 1.

Devices	Precision	Unit
Jolly balance	0.01	[cm]
Ruler	0.1	[cm]
Timer for periods	0.1	[ms]
Timer for maximum speed	0.01	[ms]
Calliper	0.02	[mm]
Electronic scale	0.01	[g]

Table 1: Devices precision

4 Results

4.1 Measurements for spring constant

From the raw spring length measurement data in Table 2, we obtain the change amount of the spring length by $\Delta L_i = L_i - L_0$ in Table 3.

Since the acceleration due to gravity is exactly $9.794 m/s^2$, we obtain the weights of weight stacks from their masses, as shown in Table 4. The relation between the elastic force and the change of spring length is linear.Using the least-squares method, we can calculate the most optimal estimate of the slope and intercept. The errorbars of W and Δ_L are based on their own combined uncertainties, which is just their type B uncertainties Δ_B in this case. Specific error calculaton is included in section 5.

Unit: $[\times 10^{-2} m] \pm 0.01 [\times 10^{-2} m]$					
No.	spring 1	No.	spring 2	No.	series
L_0	2.25	L_0	2.84	L_0	5.06
L_1	4.40	L_1	4.96	L_1	9.25
L_2	6.56	L_2	7.08	L_2	13.38
L_3	8.70	L_3	9.18	L_3	17.60
L_4	10.78	L_4	11.25	L_4	21.98
L_5	12.85	L_5	13.30	L_5	26.09
L_6	15.03	L_6	15.42	L_6	30.31

Table 2: Spring constant measurement data

The errorbar is **not clear enough** because the error is relatively **small**. The sample calculation of k_1 done by Matlab is also shown here. The fitting curve in Figure 5 is based on Table 5.

Unit: $[\times 10^{-2}m] \pm 0.01[\times 10^{-2}m]$					
No.	spring 1	No.	spring 2	No.	series
ΔL_1	2.15	ΔL_1	2.12	ΔL_1	4.19
ΔL_2	4.31	ΔL_2	4.24	ΔL_2	8.32
ΔL_3	6.45	ΔL_3	6.34	ΔL_3	12.54
ΔL_4	8.53	ΔL_4	8.41	ΔL_4	16.92
ΔL_5	10.60	ΔL_5	10.46	ΔL_5	21.03
ΔL_6	12.78	ΔL_6	12.58	ΔL_6	25.25

Table 3: Calculated spring measurement data

	$m[\times 10^{-3}kg] \pm 0.01[\times 10^{-3}kg]$	$W[\times 10^{-3}N] \pm 0.1[\times 10^{-3}N]$
1	4.83	47.3
2	9.65	94.5
3	14.50	142.0
4	19.24	188.4
5	23.99	235.0
6	28.80	282.1

Table 4: Weight measurement data

No.	$\Delta_L[m]$	$u_{\Delta_L}[m]$	$W[N]$	$u_W[N]$
1	0.0215	0.00014	0.04730	0.0001
1	0.0431	0.00014	0.09451	0.0001
1	0.0645	0.00014	0.1420	0.0001
1	0.0853	0.00014	0.1884	0.0001
1	0.1060	0.00014	0.2350	0.0001
1	0.1278	0.00014	0.2821	0.0001

Table 5: Data for $W vs. \Delta L$ of spring 1 in Figure 5

$$
\begin{aligned}
k_1 &= \frac{\overline{\Delta L W} - \overline{\Delta L} \cdot \overline{W}}{\overline{\Delta L^2} - \overline{\Delta L}^2} \\
&\approx \frac{1.52 \times 10^{-2} - 7.47 \times 10^{-2} \times 0.165}{68.9 \times 10^{-4} - 55.8 \times 10^{-4}} \\
&= 2.22 \pm 0.02 kg/s^2.
\end{aligned}
$$

$$u_{r,k_1} = 0.9\%$$

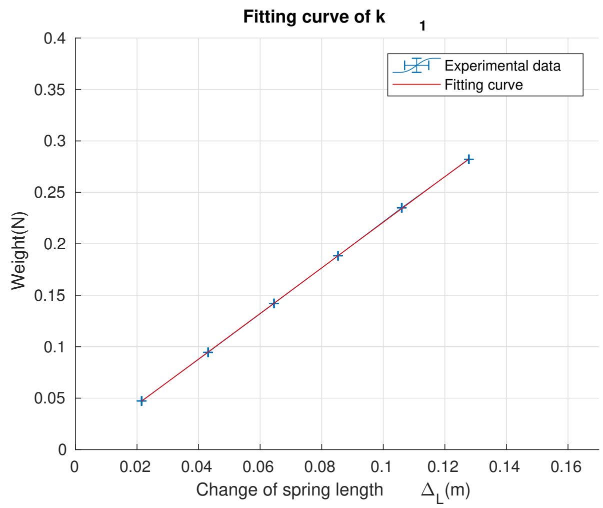
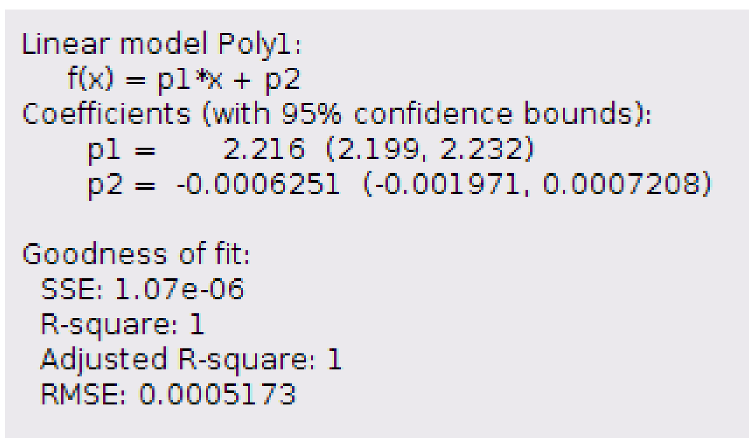


Figure 5: Fitting curve of spring 1



```
Linear model Poly1:
    f(x) = p1*x + p2
Coefficients (with 95% confidence bounds):
      p1 =       2.216  (2.199, 2.232)
      p2 =  -0.0006251  (-0.001971, 0.0007208)

Goodness of fit:
  SSE: 1.07e-06
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.0005173
```

Figure 6: Information of fit in Figure 5

Simliarly, we obtain k_2 and k_3.

No.	$\Delta_L[m]$	$u_{\Delta_L}[m]$	$W[N]$	$u_W[N]$
1	0.0212	0.00014	0.04730	0.0001
1	0.0424	0.00014	0.09451	0.0001
1	0.0634	0.00014	0.1420	0.0001
1	0.0841	0.00014	0.1884	0.0001
1	0.1046	0.00014	0.2350	0.0001
1	0.1258	0.00014	0.2821	0.0001

Table 6: Data for $W vs. \Delta_L$ of spring 2 in Figure 7

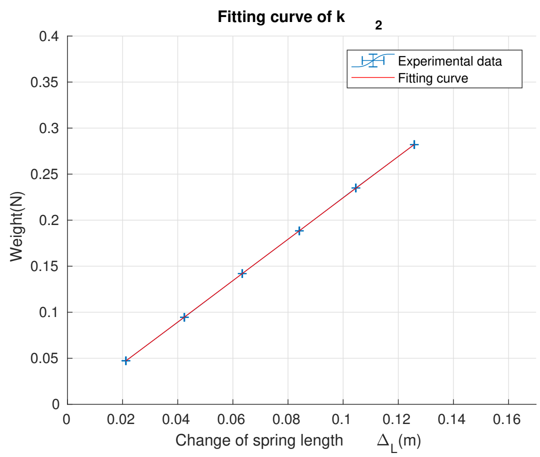


Figure 7: Fitting curve of spring 2

```
Linear model Poly1:
    f(x) = p1*x + p2
Coefficients (with 95% confidence bounds):
      p1 =       2.248  (2.239, 2.257)
      p2 =  -0.0005441  (-0.001258, 0.0001694)

Goodness of fit:
  SSE: 3.011e-07
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.0002744
```

Figure 8: Information of fit in Figure 7

$$k_2 \approx 2.25 \pm 0.010 kg/s^2, \quad u_{r,k_2} = 0.4\%$$

No.	$\Delta_L[m]$	$u_{\Delta_L}[m]$	$W[N]$	$u_W[N]$
1	0.0419	0.00014	0.04730	0.0001
2	0.0832	0.00014	0.09451	0.0001
3	0.1254	0.00014	0.1420	0.0001
4	0.1692	0.00014	0.1884	0.0001
5	0.2103	0.00014	0.2350	0.0001
6	0.2525	0.00014	0.2821	0.0001

Table 7: Data for $W vs. \Delta_L$ of spring series in Figure 9

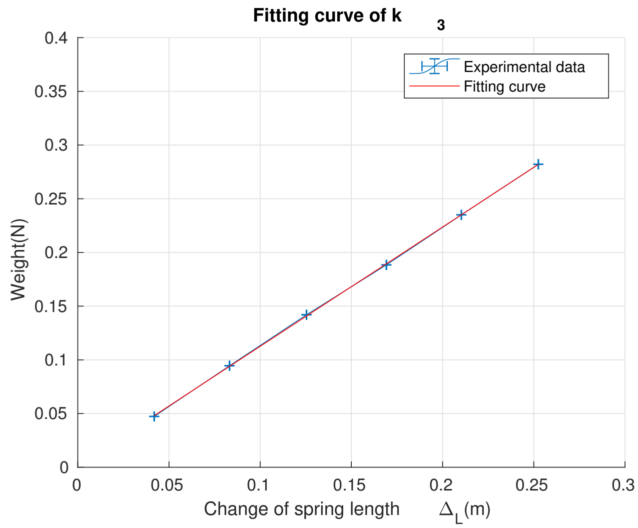


Figure 9: Fitting curve of series of spring 1 and spring 2

```
Linear model Poly1:
    f(x) = p1*x + p2
Coefficients (with 95% confidence bounds):
      p1 =       1.111  (1.096, 1.125)
      p2 =    0.001516  (-0.0008839, 0.003916)

Goodness of fit:
  SSE: 3.477e-06
  R-square: 0.9999
  Adjusted R-square: 0.9999
  RMSE: 0.0009323
```

Figure 10: Information of fit in Figure 9

$$k_3 \approx 1.11 \pm 0.017 kg/s^2, \quad u_{r,k_3} = 1.5\%$$

However, k_3 could be theoretically calculated by k_1 and k_2. This will be further discussed in section 5.

4.2 Relation between the period T and the mass M

In this experiment we use I-shape shutter to block the photoelectric gate. The mass data are listed in Table 8 and. The effective mass in Table 8 is the sum of m_{objI} and the mass of weights in Table 4.

object with I-shape	$m_{objI} = 176.55 \pm 0.01[\times 10^{-3} kg]$
object with U-shape	$m_{objU} = 186.75 \pm 0.01[\times 10^{-3} kg]$
mass of spring 1	$m_{spr1} = 10.74 \pm 0.01[\times 10^{-3} kg]$
mass of spring 2	$m_{spr2} = 10.77 \pm 0.01[\times 10^{-3} kg]$
equivalent mass	$M_I = m_{objI} + \frac{1}{3}m_{spr1} + \frac{1}{3}m_{spr2} = 183.72 \pm 0.015[\times 10^{-3} kg]$
equivalent mass	$M_U = m_{objU} + \frac{1}{3}m_{spr1} + \frac{1}{3}m_{spr2} = 193.92 \pm 0.015[\times 10^{-3} kg]$

Table 8: Mass measurement data

Again using the least-squares method we can plot the fitting curve of $T^2 vs.M$ (see Figure 11). The errorbar is still **not clear** though.

No.	$M[kg]$	$u_M[kg]$	$T^2[s^2]$	$u_{T^2}[s^2]$
1	0.18855	0.000015	1.63812	0.00003
2	0.19337	0.000015	1.68089	0.00003
3	0.19822	0.000015	1.72323	0.00003
4	0.20296	0.000015	1.76427	0.00003
5	0.20771	0.000015	1.80534	0.00003
6	0.21252	0.000015	1.84753	0.00003

Table 9: Data for $T^2 vs.M$ for the horizontal track in Figure 11

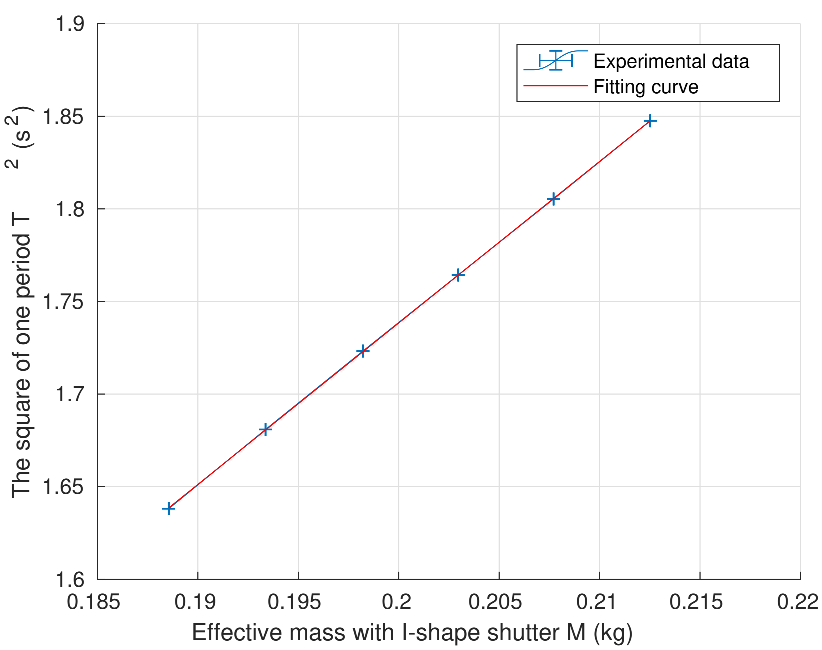


Figure 11: $T^2 vs.M$ for the horizontal track. $slope_{hor} = 8.72 \pm 0.05 s^2/kg$, $u_{r,hor} = 0.6\%$.

```
Linear model Poly1:
   f(x) = p1*x + p2
Coefficients (with 95% confidence bounds):
     p1 =        8.72  (8.671, 8.768)
     p2 =   -0.005537  (-0.01534, 0.004267)

Goodness of fit:
  SSE: 4.969e-07
  R-square: 1
  Adjusted R-square: 1
  RMSE: 0.0003524
```

Figure 12: Information of fit in Figure 11

No.	$M[kg]$	$u_M[kg]$	$T^2[s^2]$	$u_{T^2}[s^2]$
1	0.18855	0.000015	1.63945	0.00003
2	0.19337	0.000015	1.67884	0.00003
3	0.19822	0.000015	1.72232	0.00003
4	0.20296	0.000015	1.76430	0.00003
5	0.20771	0.000015	1.80615	0.00003
6	0.21252	0.000015	1.84732	0.00003

Table 10: Data for $T^2 vs.M$ for incline 1 in Figure 13

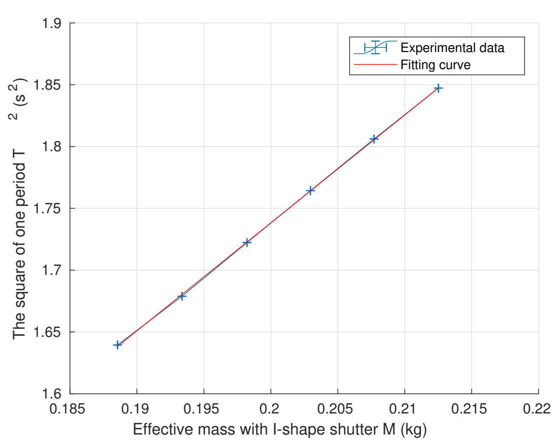


Figure 13: $T^2 vs.M$ for incline 1. $slope_{inc1} = 8.73 \pm 0.14 s^2/kg$, $u_{r,inc1} = 1.6\%$.

From the results above, we can conclude that therelation between T^2 and M is that $T^2 \propto M$. Whether it's incline or horizontal will **not** affect the relation.

```
Linear model Poly1:
    f(x) = p1*x + p2
Coefficients (with 95% confidence bounds):
      p1 =        8.73  (8.586, 8.873)
      p2 =   -0.007759  (-0.03655, 0.02103)

Goodness of fit:
  SSE: 4.283e-06
  R-square: 0.9999
  Adjusted R-square: 0.9998
  RMSE: 0.001035
```

Figure 14: Information of fit in Figure 13

No.	$M[kg]$	$u_M[kg]$	$T^2[s^2]$	$u_{T^2}[s^2]$
1	0.18855	0.000015	1.63843	0.00003
2	0.19337	0.000015	1.67842	0.00003
3	0.19822	0.000015	1.72318	0.00003
4	0.20296	0.000015	1.76242	0.00003
5	0.20771	0.000015	1.80534	0.00003
6	0.21252	0.000015	1.84821	0.00003

Table 11: Data for $T^2 vs. M$ for incline 2 in Figure 15

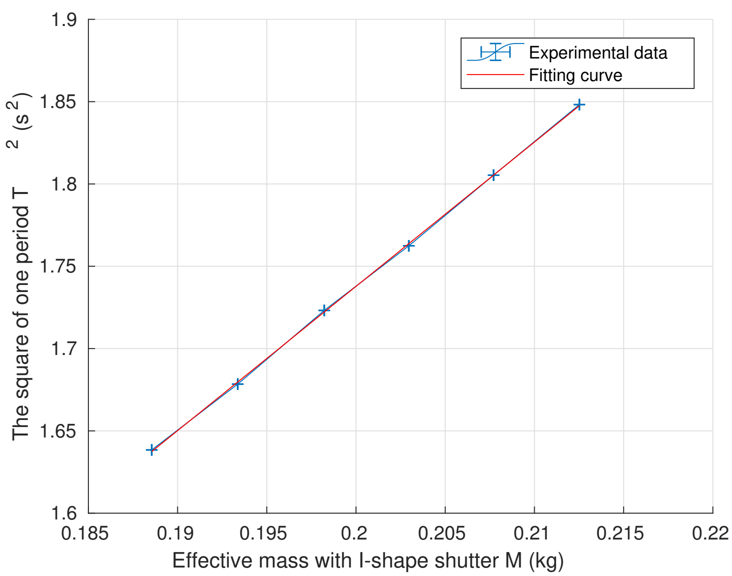


Figure 15: $T^2 vs. M$ for incline 2. $slope_{inc2} = 8.76 \pm 0.17 s^2/kg$, $u_{r,inc2} = 1.9\%$.

```
Linear model Poly1:
    f(x) = p1*x + p2
Coefficients (with 95% confidence bounds):
     p1 =       8.764  (8.598, 8.93)
     p2 =    -0.01499  (-0.04836, 0.01838)

Goodness of fit:
  SSE: 5.757e-06
  R-square: 0.9998
  Adjusted R-square: 0.9998
  RMSE: 0.0012
```

Figure 16: Information of fit in Figure 15

4.3 Relation between period T and amplitude A

No.	$A[m]$	$u_A[m]$	$T[s]$	$u_T[s]$
1	0.05	0.001	1.26451	0.0001
2	0.10	0.001	1.26361	0.0001
3	0.15	0.001	1.26287	0.0001
4	0.20	0.001	1.26358	0.0001
5	0.25	0.001	1.26362	0.0001
6	0.30	0.001	1.26360	0.0001

Table 12: Data for A $vs.$ T in Figure 17

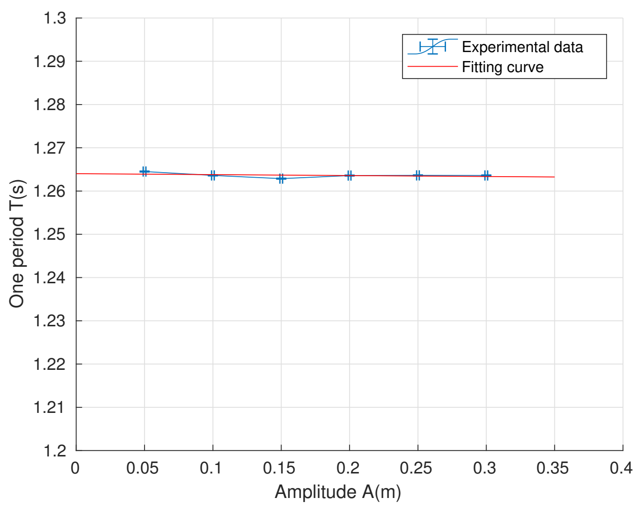


Figure 17: Fitting curve of T $vs.$ A, $\quad slope = (-2.18 \pm 7) \times 10^{-3} s/m, \quad u_r = 300\%$

```
Linear model Poly1:
    f(x) = p1*x + p2
Coefficients (with 95% confidence bounds):
      p1 =   -0.002177  (-0.00929, 0.004936)
      p2 =       1.264  (1.263, 1.265)

Goodness of fit:
  SSE: 1.149e-06
  R-square: 0.1529
  Adjusted R-square: -0.05882
  RMSE: 0.0005358
```

Figure 18: Information of fit in Figure 17

To figure out whether T depends on A, we need to calculate the correlation coefficient γ. The result generated by Matlab shows that

$$\gamma_{A,T} = \frac{cov(A,T)}{s_A \cdot s_T} = \frac{\sum(A-\bar{A})(T-\bar{T})}{ns_A \cdot s_T} \approx -0.391.$$

In fact we know that the period T does not depend on A, which means the theoretical value of γ should be 0. Nevertheless, the experimental value shows that T and A is weakly correlated. This will be discussed in section 6.

4.4 Relation between the maximum speed and the amplitude

First, we need the average values of x_{in} and x_{out} using the data from Table 13.

No.	$x_{in}[\times 10^{-3}m] \pm 0.02[\times 10^{-3}m]$	$x_{out}[\times 10^{-3}m] \pm 0.02[\times 10^{-3}m]$
1	4.48	15.40
2	4.52	15.40
3	4.46	15.42

Table 13: Data for the calculation of Δx

$$\begin{aligned} x_{in} &= \frac{1}{3}\sum_{i=1}^{3} x_{in,i} = \frac{0.00448+0.00452+0.00446}{3} \\ &= (4.49 \pm 0.08) \times 10^{-3}m, \quad u_{r,x_{in}} = 1.8\%. \end{aligned}$$

$$\begin{aligned} x_{out} &= \frac{1}{3}\sum_{i=1}^{3} x_{out,i} = \frac{0.01540+0.01540+0.01542}{3} \\ &= (15.41 \pm 0.03) \times 10^{-3}m, \quad u_{r,x_{out}} = 0.2\%. \end{aligned}$$

$$
\begin{aligned}
\Delta x &= (x_{in} + x_{out})/2 \\
&= (9.95 \pm 0.04) \times 10^{-3} m, \quad u_{r,\Delta x} = 0.4\%.
\end{aligned}
$$

No.	$A^2[m^2]$	$u_{A^2}[m^2]$	$v_{max}^2[m^2/s^2]$	$u_{v_{max}^2}[m^2/s^2]$
1	0.0025	0.0001	0.0570	0.0005
2	0.0100	0.0002	0.233	0.002
3	0.0225	0.0003	0.518	0.004
4	0.0400	0.0004	0.907	0.007
5	0.0625	0.0005	1.41	0.012
6	0.0900	0.0006	2.00	0.017

Table 14: Data for the fitting curve in Figure 19

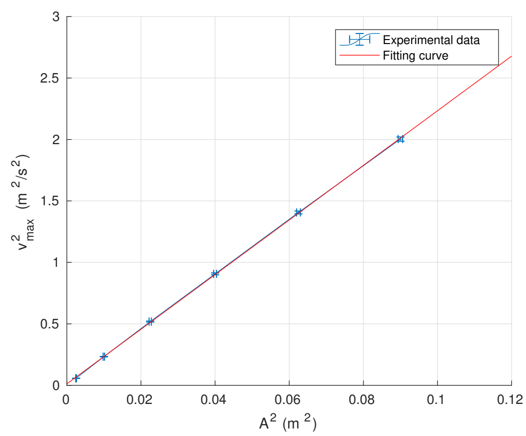


Figure 19: Fitting curve of v_{max}^2 $vs.$ A^2, $\quad slope = 22.22 \pm 0.31 s^{-2}, \quad u_r = 1.4\%$

```
Linear model Poly1:
    f(x) = p1*x + p2
Coefficients (with 95% confidence bounds):
      p1 =       22.22  (21.91, 22.54)
      p2 =     0.01151  (-0.003949, 0.02696)

Goodness of fit:
  SSE: 0.0002926
  R-square: 0.9999
  Adjusted R-square: 0.9999
  RMSE: 0.008552
```

Figure 20: Information of fit in Figure 19

5 Measurement Uncertainty Analysis

5.1 Uncertainty of spring constants

To determine the uncertainty of $\Delta L = L_i - L_0$, we need that

$$\frac{\partial \Delta L}{\partial L_i} = 1, \quad \frac{\partial \Delta L}{\partial L_0} = -1,$$
$$u_{L_i} = u_{L_0} = \Delta_{dev} = 1 \times 10^{-4} m,$$
$$u_{\Delta L} = \sqrt{(\frac{\partial \Delta L}{\partial L_i})^2 \cdot (u_{L_i})^2 + (\frac{\partial \Delta L}{\partial L_0})^2 \cdot (u_{L_0})^2} = \sqrt{2} \times 1 \times 10^{-4} \approx 1.4 \times 10^{-4} m.$$

. Similarly, u_W of $W = mg$ is

$$\frac{\partial W}{\partial m} = g = 9.794 kg \cdot m/s^2,$$
$$u_m = \Delta_{dev} = 1 \times 10^{-5} kg,$$
$$u_W = \sqrt{(\frac{\partial W}{\partial m})^2 \cdot (u_m)^2} = \sqrt{(9.794)^2 \times (1 \times 10^{-5})^2} \approx 0.1 \times 10^{-3} N.$$

. The errors of ΔL and W have been shown in Figure 5, 7 and 9.

5.1.1 Uncertainty of fitting spring constants

The standard deviation of least-squares method is calculated by

$$\sigma_W = \sqrt{\frac{1}{k-n} \sum_{i=1}^{k} \varepsilon_i^2},$$

where k is the number of measurements and n is the number of unknown quantity. In this experiment, we need to figure out both k and b so n should be 2 here. Here's the **sample calculation**. Considering spring 1, we plug in the data and obtain

$$\sigma_W \approx \sqrt{\frac{1}{6-2} \times [(0.288)^2 + (-0.360)^2 + (-0.285)^2 + (0.0290)^2 + (0.765)^2 + (-0.437)^2] \times 10^{-6}}$$
$$= 5.17 \times 10^{-4} N$$

Therefore, the standard deviation of the slop estimate

$$\begin{aligned}
\overline{\Delta L^2} &= \frac{1}{6}[(0.0215)^2 + (0.0431)^2 + (0.0645)^2 + (0.0853)^2 + (0.1060)^2 + (0.1278)^2] \\
&\approx 0.00689 m^2, \\
\overline{\Delta L}^2 &= [\frac{1}{6}[0.0215 + 0.0431 + 0.0645 + 0.0853 + 0.1060 + 0.1278]^2 \\
&\approx 0.00558 m^2, \\
\sigma_{k_1} &= \frac{\sigma_W}{\sqrt{\overline{\Delta L^2} - \overline{\Delta L}^2}} = \frac{5.17 \times 10^{-4}}{\sqrt{0.00689 - 0.00558}} \\
&\approx 14.3 \times 10^{-3} kg/s^2.
\end{aligned}$$

Thus, the 0.95-confidence deviation is

$$u_{k_1} = \frac{t_{0.95}}{\sqrt{n-2}} \sigma_{k_1} = \frac{2.57}{\sqrt{6-2}} \times 14.3 \times 10^{-3} = 0.0184 \approx 0.02 kg/s^2.$$

Similarly, we obtain u_{k_2} and u_{k_3}.

$$\sigma_{k_2} = 0.008 kg/s^2, \quad u_{k_2} = \frac{t_{0.95}}{\sqrt{n-2}} \sigma_{k_2} = 0.01 kg/s^2.$$
$$\sigma_{k_3} = 0.013 kg/s^2, \quad u_{k_3} = \frac{t_{0.95}}{\sqrt{n-2}} \sigma_{k_3} = 0.017 kg/s^2.$$

The relative uncertainty can be calculated by $u_r = u/\bar{X} \times 100\%$.
For example,

$$u_{r,k_1} = \frac{0.02}{2.22} \times 100\% = 0.9\%.$$

Finally, the experimentally found k_1, k_2 and k_3 is

$$k_1 = 2.22 \pm 0.02 kg/s^2, \quad u_{r,k_1} = 0.9\%$$
$$k_2 = 2.25 \pm 0.01 kg/s^2, \quad u_{r,k_2} = 0.4\%$$
$$k_3 = 1.11 \pm 0.017 kg/s^2, \quad u_{r,k_3} = 1.5\%$$

5.1.2 Uncertainty of spring series' constants

To determine the theoretical value of k_3, we have the equations

$$F = k_1 \Delta L_1,$$
$$F = k_2 \Delta L_2,$$
$$F = k_3^{'}(\Delta L_1 + \Delta L_2),$$

whose solution is $k_3^{'} = \frac{k_1 k_2}{k_1+k_2}$. The theoretical value of k_3 is

$$k_3^{'} = \frac{k_1 k_2}{k_1 + k_2} = \frac{2.22 \times 2.25}{2.22 + 2.25} = 1.12 kg/s^2.$$

The propagated uncertainty of k_3 is estimated by the formula

$$\begin{aligned} u_{k_3^{'}} &= \sqrt{(\frac{\partial k_3^{'}}{\partial k_1})^2 (u_{k_1})^2 + (\frac{\partial k_3^{'}}{\partial k_2})^2 (u_{k_2})^2} \\ &= \sqrt{((\frac{k_2}{k_1 + k_2})^2)^2 (u_{k_1})^2 + ((\frac{k_1}{k_1 + k_2})^2)^2 (u_{k_2})^2} \\ &= \sqrt{(\frac{2.25}{2.22 + 2.25})^4 (0.02)^2 + (\frac{2.22}{2.22 + 2.25})^4 (0.01)^2} \\ &= 0.006 kg/s^2 \end{aligned}$$

and the relative uncertainty is

$$u_{rk_3^{'}} = \frac{u_{k_3^{'}}}{k_3^{'}} = \frac{0.006}{1.12} = 0.5\%.$$

Hence the theoretical value of $k_3^{'}$ is

$$k_3^{'} = 1.12 + 0.006 kg/s^2, \quad u_{rk_3^{'}} = 0.5\%.$$

Compared with k_3 we obtain from curve fitting, we can calculate the deviation Δk_3 between k_3 and $k_3^{'}$ and the relative deviation $\Delta_r k_3$.

$$\begin{aligned} \Delta k_3 &= k_3 - k_3^{'} = 1.11 - 1.12 = -0.01 kg/s^2, \\ \Delta_r k_3 &= \frac{k_3 - k_3^{'}}{k_3^{'}} = \frac{1.11 - 1.12}{1.12} = -0.9\%. \end{aligned}$$

Recall that the 0.95-confidence interval of k_3 is $(1.093, 1.127)$. The theoretical value $k_3^{'} = 1.12 + 0.006 kg/s^2$ is indeed in this interval, which means the uncertainty analysis of k_3 is reasonable.

5.2 Uncertainty of the slope of T^2 $vs.$ M

The uncertainty of one period is $u_T/10 = 1 \times 10^{-5}s$ and the uncertainty of mass is

$$
\begin{aligned}
M &= m_{objI} + \frac{1}{3}m_{spr1} + \frac{1}{3}m_{spr2} + m_i, \\
u_M &= \sqrt{(\frac{\partial M}{\partial m_{objI}})^2(u_{m_{objI}})^2 + (\frac{\partial M}{\partial m_{spr1}})^2(u_{m_{spr1}})^2 + (\frac{\partial M}{\partial m_{spr2}})^2(u_{m_{spr2}})^2 + (\frac{\partial M}{\partial m_i})^2(u_{m_i})^2} \\
&= \sqrt{(1)^2(u_{m_{objI}})^2 + (\frac{1}{3})^2(u_{m_{spr1}})^2 + (\frac{1}{3})^2(u_{m_{spr2}})^2 + (1)^2(u_{m_i})^2} \\
&= \sqrt{(u_{m_{objI}})^2 + (u_{m_{spr1}})^2/9 + (u_{m_{spr2}})^2/9 + (u_{m_i})^2}.
\end{aligned}
$$

For example, when $m_{objI} = (176.55 \pm 0.01) \times 10^{-3}kg$, $m_{spr1} = (10.74 \pm 0.01) \times 10^{-3}kg$, $m_{spr2} = (10.77 \pm 0.01) \times 10^{-3}kg$ and $m_1 = (4.83 \pm 0.01) \times 10^{-3}kg$,

$$
\begin{aligned}
u_M &= \sqrt{0.00001^2 + 0.00001^2/9 + 0.00001^2/9 + 0.00001^2} = 0.0000149 \\
&\approx 1.5 \times 10^{-5}kg.
\end{aligned}
$$

The uncertainty of T^2 is propagated uncertainty, which is calculated as

$$
\begin{aligned}
&\frac{\partial T^2}{\partial T} = 2T, \\
&u_{T^2} = \sqrt{(2T)^2(u_T)^2} = 2Tu_T.
\end{aligned}
$$

For example, for $T = 1.27989s$, $u_{T^2} = 2 \times 1.27989 \times 1 \times 10^{-5} = 2.55978 \times 10^{-5} \approx 3 \times 10^{-5}s^2$. The complete results are listed in Table 9, 10 and 11.

From the formula of period $T = 2\pi\sqrt{\frac{M}{k}}$ we know that the slope of T^2 $vs.$ M is

$$slope = \frac{T^2}{M} = \frac{4\pi^2}{k}, \tag{8}$$

where k is the effective spring constant. In this experiment, the effective spring constant can be calculated by

$$
\begin{aligned}
F &= k_1\Delta x + k_2\Delta x, \\
F &= k_{eff}\Delta x.
\end{aligned}
$$

Thus we get $k_{eff} = k_1 + k_2$. We take the reults of k_1 and k_2 as the theoretical value and ignore its uncertainty, then

$$k_{eff} = k_1 + k_2 = 2.22 + 2.25 = 4.47kg/s^2. \tag{9}$$

Plugging it into Eq. 8, we get the theoretical slope $slope_t = 4 \times 3.14^2/4.47 = 8.82s^2/kg$ By curve fitting, we get the slopes

$$slope_{hor} = 8.72 \pm 0.05s^2/kg, \quad u_{r,hor} = 0.6\%,$$
$$slope_{inc1} = 8.73 \pm 0.14s^2/kg, \quad u_{r,inc1} = 1.6\%,$$
$$slope_{inc2} = 8.76 \pm 0.17s^2/kg, \quad u_{r,inc2} = 1.9\%,$$

where the uncertainty is calcualted by $t_0.95/\sqrt{6-2} \times \sigma$. Hence the experimental value of the slope is $slope_e = \overline{slope} = 8.74s^/kg$. Similarly, we get the propagated and relative uncertainty of slope, which is

$$\begin{aligned} u_{slope} &= \sqrt{(\frac{\partial slope}{\partial slope_{hor}})^2 \cdot (u_{slope_{hor}})^2 + (\frac{\partial slope}{\partial slope_{inc1}})^2 \cdot (u_{slope_{inc1}})^2 + (\frac{\partial slope}{\partial slope_{inc2}})^2 \cdot (u_{slope_{inc2}})^2} \\ &= \sqrt{(\frac{1}{3})^2 \cdot (0.05)^2 + (\frac{1}{3})^2 \cdot (0.14)^2 + (\frac{1}{3})^2 \cdot (0.17)^2} = 0.08s^2/kg. \\ u_{r,slope} &= \frac{u_{slope}}{\overline{slope}} \times 100\% = \frac{0.08}{8.74} \times 100\% = 0.9\%. \end{aligned}$$

Hence, $slope_e = \overline{slope} = 8.74 \pm 0.08s^/kg, \quad u_{r,slope} = 0.9\%$. Compared with the "theoretical" value of $slope_t = 8.82s^2/kg$, we calculate the deviation $\Delta slope$ and relative deviation $\Delta_r slope$

$$\Delta slope = 8.74 - 8.82 = -0.08s^2/kg,$$
$$\Delta_r slope = \frac{8.74 - 8.82}{8.82} \times 100\% = -0.9\%.$$

The theoretical value is just on the boundary of uncertainty interval. It shows that our experimental values are relatively reliable.

5.3 Uncertainty in the v_{max}^2 $vs.$ A^2 relation

5.3.1 Uncertainty of Δx

First, we need to determine the uncertainty of x_{in} and x_{out}. The uncertainty of type-B of a calliper is $\Delta_{x,B} = \Delta dev = 0.02 \times 10^{-3}m$. The distance is found by the average value of 3 measurements. To estimate type-A uncertainty, the standard deviation of the average value is

$$s_{\overline{x_{in}}} = \sqrt{\frac{1}{n(n-1)} \sum_{i=1}^{n} (x_{in,i} - \overline{x_{in}})^2}.$$

Using the data from Table 13 we find that $s_{\overline{x_{in}}} \approx 0.0176 \times 10^{-3}m$. Considering $t_{0.95} = 4.30$ for $n = 3$, the type-A uncertainty is estimated as $\Delta_{x_{in},A} = 4.30 \times 0.0176 \times 10^{-3}m \approx$

$0.0757 \times 10^{-3} m$.
Hence the combined uncertainty is

$$u_{x_{in}} = \sqrt{\Delta^2_{x_{in},A} + \Delta^2_{x_{in},B}} = \sqrt{(0.02 \times 10^{-3})^2 + (0.0757 \times 10^{-3})^2} \approx 0.08 \times 10^{-3} m$$

and the corresponding relative uncertainty is

$$u_{r,x_{in}} = \frac{u_{x_{in}}}{\overline{x_{in}}} \times 100\% = 1.8\%.$$

The experimentally found x_{in} is

$$x_{in} = (4.49 \pm 0.08) \times 10^{-3} m, \quad u_{r,x_{in}} = 1.8\%.$$

Similarly, we know about x_{out} that

$$\begin{aligned}
&\Delta_{x_{out},A} = t_{0.95} \cdot s_{\overline{x_{out}}} = 0.0287 \times 10^{-3} m \\
&\Delta_{x,B} = 0.02 \times 10^{-3} m \\
&u_{x_{out}} = 0.03 \times 10^{-3} m, \quad u_{r,x_{out}} = 0.2\%
\end{aligned}$$

$$x_{out} = (15.41 \pm 0.03) \times 10^{-3} m, \quad u_{r,x_{out}} = 0.2\%.$$

Then we can calculate the propagated uncertainty of Δx

$$\begin{aligned}
\frac{\partial \Delta x}{\partial x_{in}} &= \frac{\partial \Delta x}{\partial x_{out}} = \frac{1}{2}, \\
u_{\Delta x} &= \sqrt{(\frac{\partial \Delta x}{\partial x_{in}})^2 (u_{x_{in}})^2 + (\frac{\partial \Delta x}{\partial x_{out}})^2 (u_{x_{out}})^2} \\
&= \sqrt{(\frac{1}{2})^2 (0.00008)^2 + (\frac{!}{2})^2 (0.00003)^2} \\
&= 0.04 \times 10^{-3} m, \\
u_{r,\Delta x} &= \frac{u_{\Delta x}}{\Delta x} \times 100\% = 0.4\%
\end{aligned}$$

$$\Delta x = (9.95 \pm 0.04) \times 10^{-3} m, \quad u_{r,\Delta x} = 0.4\%.$$

5.3.2 Uncertainty of the maximum speed v_{max}

Then we can calculate the propagated uncertainty of $v_{max} = \Delta x / \Delta t$. The partial derivatives are

$$\frac{\partial v_{max}}{\partial \Delta x} = \frac{1}{\Delta t}.$$

$$\frac{\partial v_{max}}{\partial \Delta t} = -\frac{\Delta x}{(\Delta t)^2}.$$

Hence,

$$u_{v_{max}} = \sqrt{(\frac{\partial v_{max}}{\partial \Delta x})^2(u_{\Delta x})^2 + (\frac{\partial v_{max}}{\partial \Delta t})^2(u_{\Delta t})^2}$$

$$= \sqrt{(\frac{1}{\Delta t})^2(u_{\Delta x})^2 + (-\frac{\Delta x}{(\Delta t)^2})^2(u_{\Delta t})^2}$$

For example, in the case that $\Delta x = 0.00995 \pm 0.00004m$ and $\Delta t = 0.04169 \pm 0.00001s$,

$u_{v_{max}} = \sqrt{(0.00004)^2/(0.04169)^2 + (0.00995)^2(0.00001)^2/(0.04169)^4} \approx 0.001m/s$,

$u_{r,v_{max}} = \frac{u_{v_{max}}}{v_{max}} \times 100\% \approx 0.4\%$

No.	$\Delta x[m]$	$u_{\Delta x}[m]$	$\Delta t[s]$	$u_{\Delta t}[s]$
1	0.00995	0.00004	0.04169	0.00001
2	0.00995	0.00004	0.02061	0.00001
3	0.00995	0.00004	0.01382	0.00001
4	0.00995	0.00004	0.01045	0.00001
5	0.00995	0.00004	0.00839	0.00001
6	0.00995	0.00004	0.00703	0.00001

Table 15: Data for the calculation of v_{max}

Since the calculations are repeated and too complicated, the results of v_{max} are listed in Table 16.

No.	$v_{max}[m/s]$	$u_{v_{max}}[m/s]$	$u_{r,v_{max}}[\%]$
1	0.239	0.001	0.4
2	0.483	0.002	0.4
3	0.720	0.003	0.4
4	0.952	0.004	0.4
5	1.19	0.005	0.4
6	1.42	0.006	0.4

Table 16: Results of v_{max}

Then we can calculate the propagated uncertainty of v_{max}^2. The partial derivative is

$$\frac{\partial v_{max}^2}{\partial v_{max}} = 2v_{max}.$$

Hence,

$$u_{v_{max}^2} = \sqrt{(\frac{\partial v_{max}^2}{\partial v_{max}})^2(u_{v_{max}})^2} = 2v_{max}u_{v_{max}}$$

For example, in the case that $v_{max} = 0.239 \pm 0.001 m/s$,

$$u_{v^2_{max}} = 2 \times 0.239 \times 0.001 \approx 0.0005 m/s,$$
$$u_{r,v^2_{max}} = \frac{u_{v^2_{max}}}{v^2_{max}} \times 100\% \approx 0.8\%$$

The results of v^2_{max} are shown in Table 17.

No.	$v^2_{max}[m^2/s^2]$	$u_{v^2_{max}}[m^2/s^2]$	$u_{r,v^2_{max}}[\%]$
1	0.0570	0.0005	0.8
2	0.233	0.002	0.8
3	0.518	0.004	0.8
4	0.907	0.007	0.8
5	1.41	0.012	0.8
6	2.00	0.017	0.9

Table 17: Results of v_{max}

Now we will calculate the square of amplitudes A^2. The uncertainty of A is $u_A = \Delta_{dev} = 0.001m$.

$$\frac{\partial A^2}{\partial A} = 2A.$$
$$u_{A^2} = \sqrt{(\frac{\partial A^2}{\partial A})^2 (u_A)^2} = 2Au_A$$

For example, in the case that $A = 0.050 \pm 0.001m$,

$$u_{A^2} = 2 \times 0.050 \times 0.001 \approx 0.0001 m/s,$$
$$u_{r,A^2} = \frac{u_{A^2}}{A^2} \times 100\% \approx 4\%$$

The results of A^2 are shown in Table 18.

No.	$A^2[m^2]$	$u_{A^2}[m^2]$	$u_{r,A^2}[\%]$
1	0.0025	0.0001	4
2	0.0100	0.0002	2
3	0.0225	0.0003	1.3
4	0.0400	0.0004	1.0
5	0.0625	0.0005	0.8
6	0.0900	0.0006	0.6

Table 18: Results of A^2

Finally, by curve fitting we find the relation is that $v^2_{max} \propto A^2$. The slope is

$$slope = 22.22 \pm 0.31 s^{-2}, \quad u_r = 1.4\%. \tag{10}$$

According to the formula of energy conservation in simple harmonic motion $\frac{1}{2}mv_{max}^2 = \frac{1}{2}kA^2$, we obtain the theoretical value of the slope k/m from the experimentally found k (from Eq. 9) and m (from Table 8), which will be regarded as theoretical value. Here m is the mass of the object with U-shape shutter and springs.

$$k = k_{eff} = 4.47kg/s^2$$
$$m = m_{objU} = 0.19392kg$$
$$\frac{k}{m} = \frac{4.47}{0.19392} = 23.1s^{-2}.$$

Compared with the fitting slope in Eq. 10, we can calculate the deviation $\Delta slope$ and relative deviation $\Delta_r slope$

$$\Delta slope = 22.2 - 23.1 = -0.9s^2/kg,$$
$$\Delta_r slope = \frac{22.2 - 23.1}{23.1} \times 100\% = -4\%.$$

We see that the theoretical value is beyond the boundary of uncertainty interval. It shows that our experimental values are too small and not relatively reliable. The reason will be further discussed in section 6.

6 Conclusion

In this experiment, we study the simple harmonic motion.

Firstly, using measurements of *Jolly balance* we get the spring constants:

$$k_1 = 2.22 \pm 0.02kg/s^2, \quad u_{r,k_1} = 0.9\%,$$
$$k_2 = 2.25 \pm 0.01kg/s^2, \quad u_{r,k_2} = 0.4\%,$$
$$k_3 = 1.11 \pm 0.017kg/s^2, \quad u_{r,k_3} = 1.5\%.$$

We discuss the theoretical value of k_3 and find that it's within the confidence interval, which means the experiment process is precise.

Secondly, we study the relation between T^2 and M and find that $T^2 \propto M$ and it does not depend on whether it's horizontal or not. Then we compare the slope of T^2 *vs.* M with "theoretical value". It's also within the confidence interval, confirming the experiment precision.

$$slope_e = 8.74 \pm 0.08s^/kg$$
$$slope_t = 8.82s^2/kg$$

After that, we discuss the relation between T and A. The correlation coefficient is -0.39. Based on the interpretation of correlation coefficient(see Table 19), we find that

Absolute value of correlation	Interpretation
(0.90, 1.00)	very high correlation
(0.70, 0.90)	high correlation
(0.50, 0.70)	moderate correlation
(0.30, 0.50)	low correlation
(0.00, 0.30)	negligible correlation

Table 19: Rule of interpretation of correlation coefficient [1]

-0.39 is within the range of "low correlation". I suppose that it's because the period of one of the points in Figure 17 is too small. Thus the correlation coefficient is negative. Considering experimental procedure, I think it's just random error and can be eliminated if we have more times of experiments.

Finally, we discuss the relation between v_{max} and A. It's found that $v_{max}^2 \propto A^2$. We compare the experimental slope $slope_e$ and the "theoretical slope" $slope_t$.

$$slope_e = 22.22 \pm 0.31 s^{-2}$$
$$slope_t = 23.1 s^{-2}.$$

It shows that the theoretical slope is out of the confidence interval. The experimental value is too small. It reminds me that during the experiment for each time I released the object, the distance it reached after a period was always less than the initial amplitude, which causes the kinetic energy is smaller than we expect. The possible reason might be the frictional force or the spring constants. The fiction between the springs and the hooks on the track ends might affect that, too. Besides, at the beginning the object needs more force to be at rest than that during the oscillation.

I suggest that the U-shape shutter can be designed to be more narrow so that v_{max} can be more precise; the program in the timer should be optimized to automatically record the proper average time in which the shutter travels, for example, recording the time intervals if the gaps between three consecutive readings are smaller than $0.5ms$.

References

[1] Hinkle DE, Wiersma W, Jurs SG (2003). Applied Statistics for the Behavioral Sciences 5th ed. Boston: Houghton Mifflin.

[2] Qin Tian, Zeng Ming, Cao Jianjun, Han Xugen, Feng Yaming, Mateusz Krzyzosiak, Physics Laboratory (Vp141/Vp241) Student Handbook Introduction to Measurement Data Analysis.

[3] Qin Tian, Zheng Huan, Li Yingyu, Mateusz Krzyzosiak Physics Laboratory Vp141 Exercise 3 Simple Harmonic Motion: Oscillations in Mechanical Systems.