Physics Laboratory I

VP141

Exercise III

Simple Harmonic Motion Oscillations Mechanical Systems

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1 Introduction

The objective of this exercise is to study simple harmonic oscillation, including how to find the spring constant, how to analyze the relationship between the oscillation period and the mass of the oscillator, whether the oscillation period depends on the amplitude. Besides, the relationship between the maximum speed and the amplitude will also be examined.
Simple harmonic motion is one of the most fundamental periodic motions, where the restoring force is proportional to the displacement from the equilibrium position so that the position of a particle depends on time as a sine (or cosine) function.

1.1 Hooke's law

Within the elastic limit of deformation, the force F_x needed to compress or strech a spring is proportional to the distance, *i.e.*,

$$F_x = kx, \tag{1}$$

where k, the spring constant, characterizing how easy to deform a spring. We use *Jolly balance* in this exercise to find that constant. The relation 1 is known as the *Hooke's Law*. Therefore, the elastic force is found according to Newton's third law of dynamics, whose magnitude is the same but direction is opposite. It's also known as restoring forceas it always tries to restore the system back to the equilibrium.

1.2 Equation of Motion of the Simple Harmonic Oscillator

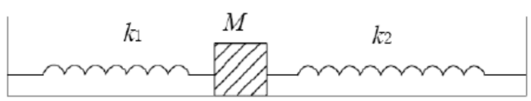

Figure 1: Mass-spring system

As shown in Figure 1.2, the mass with two springs is placed on an air track, which aims at eliminating the frictional forces. Assuming that the restoring force is the only force acting on mass M, the equation of motion od mass M is

$$M\frac{d^2x}{dt^2} + (k_1 + k_2)x = 0. \tag{2}$$

Hence the general solution to Eq. 2 is

$$x(t) = Acos(\omega_0 t + \phi_0), \tag{3}$$

where $\omega_0 = \sqrt{(k_1 + k_2)/M}$ is the natural angular frequency of the oscillations (determined by the parameters of the system), A is the amplitude, and ϕ_0 is the initial phase (determined by initial conditions). The natural period of oscillation is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{M}{k_1 + k_2}}. \tag{4}$$

In this exercise, the relationship mentioned above will be studied.

1.3 Mass of the Spring

We take into the mass of the springs in terms of the so-called *effective mass*, which is the sum of the mass of the object M and the effective mass of springs m_0. The angular frequency can be expressed as

$$\omega_0 = \sqrt{\frac{k_1 + k_2}{M + m_0}} \tag{5}$$

where m_0 is 1/3 of the actual mass of the spring.

1.4 Mechanical Energy in Harmonic Motion

The elastic potential energy for a spring-mass system is $U = kx^2/2$ and the kinetic energy of an oscillating mass is $K = mv^2/2$. At the equilibrium position ($x = 0$), the speed of the mass is maximum $v = v_{max}$. At this point the total mechanical energy is equal to maximum kinetic energy K_{max} . On the other hand, at maximum displacement ($x = \pm A$) the mass is instantaneously at rest, and the contribution to the total mechanical energy is due to the potential energy only, which is at its maximum U_{max}. In the absence of non-conservative forces (such as frictional forces or drag forces), the total mechanical energy is conserved and $K_{max} = U_{max}$, which implies

$$k = \frac{mv_{max}^2}{A^2}. \tag{6}$$

2 Experimental setup

The measurement equipment consists of the following elements: springs, Jolly balance, air track, electronic timer, electronic balance and masses.

In order to measure the spring constant using the Jolly balance, we need to place the small mirror C (see Figure 2) in the tube D and make three lines coincide: the line on the mirror, the line on the glass tube and its reflection in the mirror. First, without adding any

A: Sliding bar with metric scale;
H: Vernier for reading;
C: Small mirror with a horizontal line in the middle;
D: Fixed glass tube also with a horizontal line in the middle;
G: Knob for ascending and descending the sliding bar;
S: Spring attached to top of the bar A.

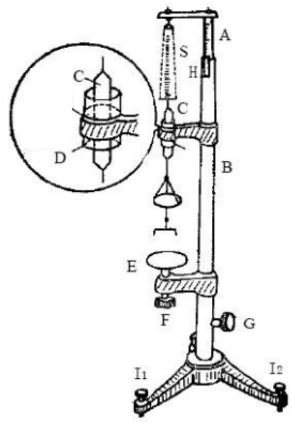

Figure 2: Jolly balance

weight on the bottom end of the spring, adjust the knob G and make the three lines coincide. Then read the scale L_1.

Second, add mass m to the bottom of the spring. The spring is stretched and the three lines no longer coincide. Adjust knob G to make them into one line again and read the corresponding number on scale L_2. The spring constant may be then found as

$$k = \frac{mg}{L_2 - L_1} \tag{7}$$

so that we can estimate the spring constant by finding a linear fit to the data using the least squares method.

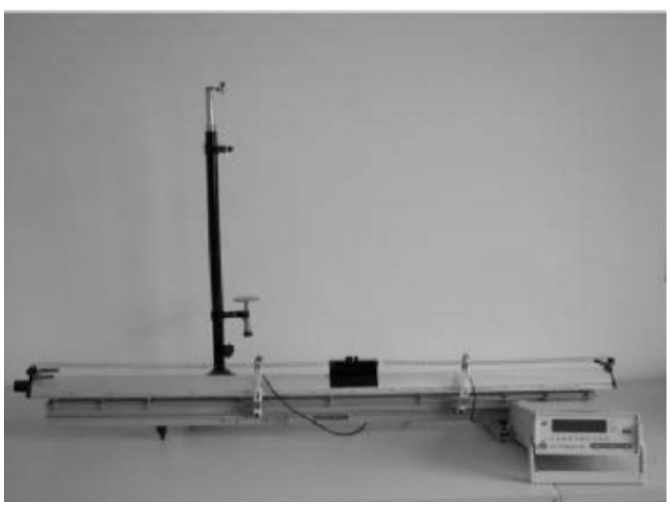

Figure 3: The experimental setup

A photoelectric measuring system consists of two photoelectric gates and an electronic timer. When a shutter on the object blocks the light, the computer will record the time. For period measurements, we use the I-shape shutter.

When measuring the instaneous speed, the U-shape shutter is required. Δt presents the time interval when the object travels a distance of $\Delta x = (x_{in} + x_{out})/2$. Hence we estimate the speed as $v = \Delta x/\Delta t$.

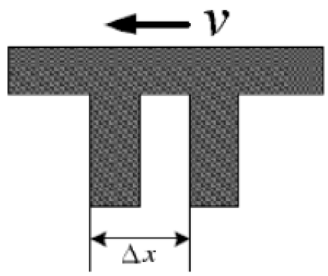

Figure 4: The U-shape shutter

3 Measurement Procedure

3.1 Spring constant

1. Adjust the Jolly balance to vertical and attach the spring Add a 20g preload and adjust knob I_1 and I_2 to make sure the mirror can move freely through the tube. Check if the balance parallel to the spring.

2. Adjust knob G to make the three lines in the tube coincide.

3. Reading the reading on the scale, add mass from 1 to 6 and record L_i in order.

4. Estimate the spring constant k_1 by fitting.

5. Repeat the measurements with spring 2. Calculate k_2.

6. Remove the preload and repeat the measurements for spring 1 and 2 connected. Calculate k_3.

3.2 Oscillation period T versus the mass of the oscillator M

(a) Adjust the air track so that it's horizontal.

1. Turn on the air track and check whether there's blocked holes.

2. Place the cart on the track. Adjust the track with the knob on the side with a single one.

(b) Horizontal air track

1. Attach springs to the sides of the cart and set up the I-shape shutter. Make sure that the photoelectric gate is at the equilibrium position.
2. Set the timer into "T" mode. Add weight in order and release it with a caliper. Record the correspoding time intervals for 10 periods.
3. Analyze the relation between M and T by plotting a graph.

(c) Inclined air track

1. Add three plastic plates under the air track everytime. Repeat the steps in (b).
2. Analyze the relation between M and T by plotting a graph.

3.3 The maximum speed versus the amplitude

1. Keep the mass of the cart unchanged and change the amplitude from $5.0 to 30.0 cm$ with gap of $5.0cm$.
2. Measure v_{max} for different amplitudes.

3.4 Mass measurement

1. Adjust the electronic balance every time before you use it. The level bubble should be in the center of the circle.
2. Add weights according to a fixed order. Weigh the cart with the I-shape shutter and with the U-shape shutter. Measure the mass of spring 1 and spring 2.
3. Record the data only after the circular symbol on the scales display disappears.

4 Results

4.1 Measurements for spring constant

From the raw spring length measurement data in Table 1, we obtain the change amount of the spring length by $\Delta_{L_i} = L_i - L_0$ in Table 2.

Since the acceleration due to gravity is exactly $9.794m/s^2$, we obtain the weights of weight stacks from their masses, as shown in Table 3.

Unit: $[\times 10^{-2}m] \pm 0.01[\times 10^{-2}m]$					
No.	spring 1	No.	spring 2	No.	series
L_0	2.25	L_0	2.84	L_0	5.06
L_1	4.40	L_1	4.96	L_1	9.25
L_2	6.56	L_2	7.08	L_2	13.38
L_3	8.70	L_3	9.18	L_3	17.60
L_4	10.78	L_4	11.25	L_4	21.98
L_5	12.85	L_5	13.30	L_5	26.09
L_6	15.03	L_6	15.42	L_6	30.31

Table 1: Spring constant measurement data

Unit: $[\times 10^{-2}m] \pm 0.01[\times 10^{-2}m]$					
No.	spring 1	No.	spring 2	No.	series
Δ_{L_1}	2.15	Δ_{L_1}	2.12	Δ_{L_1}	4.19
Δ_{L_2}	4.31	Δ_{L_2}	4.24	Δ_{L_2}	8.32
Δ_{L_3}	6.45	Δ_{L_3}	6.34	Δ_{L_3}	12.54
Δ_{L_4}	8.53	Δ_{L_4}	8.41	Δ_{L_4}	16.92
Δ_{L_5}	10.60	Δ_{L_5}	10.46	Δ_{L_5}	21.03
Δ_{L_6}	12.78	Δ_{L_6}	12.58	Δ_{L_6}	25.25

Table 2: Calculated spring measurement data

	$m[\times 10^{-3}kg] \pm 0.01[\times 10^{-3}kg]$	$W[\times 10^{-3}N] \pm 0.01[\times 10^{-3}N]$
1	4.83	47.3
2	9.65	94.5
3	14.50	142.0
4	19.24	188.4
5	23.99	235.0
6	28.80	282.1

Table 3: Weight measurement data

We have known that the relation between the elastic force and the change of spring length is linear. Therefore, applying the least-squares method, we can calculate the most optimal estimate of the slope and intercept of the regression line. The errorbars of W and Δ_L are based on their own combined uncertainties, which is just their type B uncertainties Δ_B in this case.

$$k_1 = \frac{\overline{\Delta_L W} - \overline{\Delta_L} \cdot \overline{W}}{\overline{\Delta_L^2} - \overline{\Delta_L}^2}$$
$$\approx \frac{1.52 \times 10^{-2} - 7.47 \times 10^{-2} \times 0.165}{68.9 \times 10^{-4} - 55.8 \times 10^{-4}}$$
$$= 2.22 \pm 0.02 kg/s^2,$$

$$b_1 = \overline{W} - k_1 \overline{\Delta_L}$$
$$\approx 0.1649 \times 2.22 \times 0.07470$$
$$= (-6.25 \pm 12.5) \times 10^{-4} N.$$

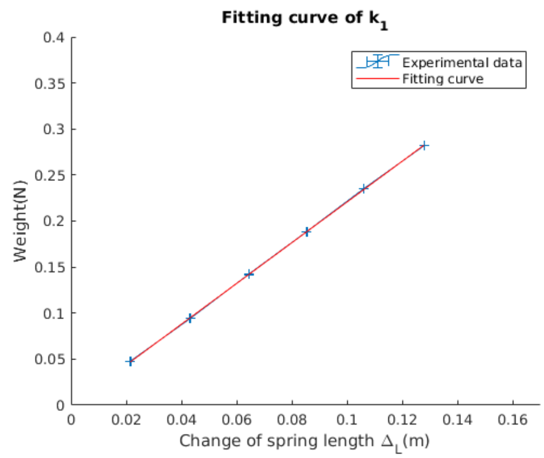

Figure 5: Fitting curve of spring 1

Simliarly, we obtain k_2 and k_3.

$$k_2 \approx 2.25 \pm 0.01 kg/s^2,$$
$$b_2 \approx (-5.44 \pm 70) \times 10^{-4} N.$$

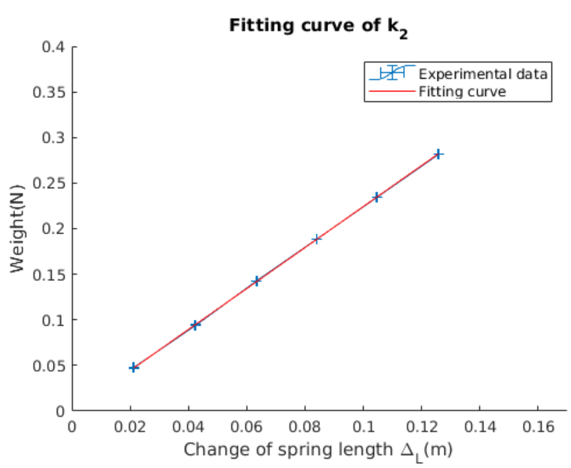

Figure 6: Fitting curve of spring 2

$$k_3 \approx 1.11 \pm 0.017 kg/s^2,$$
$$b_3 \approx (15.1 \pm 20) \times 10^{-4} N.$$

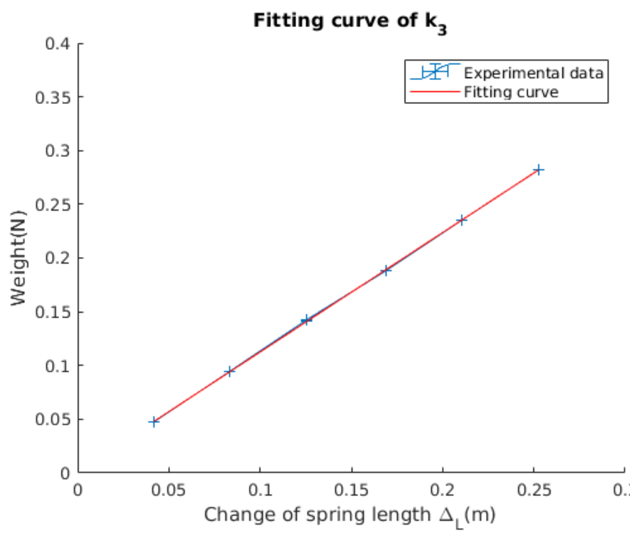

Figure 7: Fitting curve of series of spring 1 and spring 2

However, k_3 could be theoretically calculated by k_1 and k_2, which will be discussed in the section 5.

5 Measurement Uncertainty Analysis

5.1 Uncertainty of spring constants

The uncertainty of $\Delta_L = L_i - L_0$ is

$$u_{\Delta_L} = \frac{d\Delta_L}{dL_i} \cdot u_{L_i} = 1 \cdot 1 \times 10^{-4} = 1 \times 10^{-4} m$$

. Since W is calculated by $W = mg$, then the uncertainty of W is $u_W = \frac{dW}{dm} \cdot u_m = 9.794 u_m = 9.794 \times 10^{-5} N$, which is shown in Figure 5, 6 and 7.

5.1.1 Uncertainty of fitting spring constants

The standard deviation of least-squares method is calculated by

$$\sigma_W = \sqrt{\frac{1}{k-n}\sum_{i=1}^{k}\varepsilon_i^2}$$

, where k is the number of measurements and n is the number of unknown quantity. In this experiment, we need to figure out both k and b so n should be 2 here. Considering spring 1, we plug in the data and obtain

$$\begin{aligned}\sigma_W &\approx \sqrt{\frac{1}{6-2}\times[(0.288)^2+(-0.360)^2+(-0.285)^2+(0.0290)^2+(0.765)^2+(-0.437)^2]\times 10^{-6}}\\ &= 5.17\times 10^{-4}N\end{aligned}$$

Therefore, the standard deviation of the slop estimate

$$\begin{aligned}\overline{\Delta_L^2} &= \frac{1}{6}[(0.0215)^2+(0.0431)^2+(0.0645)^2+(0.0853)^2+(0.1060)^2+(0.1278)^2]\\ &\approx 0.00689m^2,\end{aligned}$$

$$\begin{aligned}\overline{\Delta_L}^2 &= [\frac{1}{6}[0.0215+0.0431+0.0645+0.0853+0.1060+0.1278]^2\\ &\approx 0.00558m^2,\end{aligned}$$

$$\begin{aligned}\sigma_{k_1} &= \frac{\sigma_W}{\sqrt{\overline{\Delta_L^2}-\overline{\Delta_L}^2}} = \frac{5.17\times 10^{-4}}{\sqrt{0.00689-0.00558}}\\ &\approx 14.3\times 10^{-3}kg/s^2,\end{aligned}$$

and the standard deviation of the intercept estimate

$$\sigma_{b_1} = \sqrt{\overline{\Delta_L^2}}\sigma_{k_1} = \sqrt{0.00689}\times 14.3\times 10^{-3} \approx 1.19\times 10^{-3}N$$

Thus, the 0.95-confidence deviation is

$$u_{k_1} = \frac{t_{0.95}}{\sqrt{n-2}}\sigma_{k_1} = \frac{2.57}{\sqrt{6-2}}\times 14.3\times 10^{-3} = 0.0184 \approx 0.02kg/s^2.$$

$$u_{b_1} = \frac{t_0.95}{\sqrt{n}}\sigma_{b_1} = \frac{2.57}{\sqrt{6}}\times 1.19\times 10^{-3} \approx 12.5\times 10^{-4}N.$$

Similarly, we obtain u_{k_2}, u_{b_2}, u_{k_3} and u_{b_3}.

$$\begin{aligned}
\sigma_{k_2} &= 0.008kg/s^2, \quad & u_{k_2} &= \frac{t_{0.95}}{\sqrt{n-2}}\sigma_{k_2} = 0.01kg/s^2. \\
\sigma_{b_2} &= 0.006N, & u_{b_2} &= \frac{t_{0.95}}{\sqrt{n}}\sigma_{b_2} = 0.007N. \\
\sigma_{k_3} &= 0.013kg/s^2, \quad & u_{k_3} &= \frac{t_{0.95}}{\sqrt{n-2}}\sigma_{k_3} = 0.017kg/s^2. \\
\sigma_{b_3} &= 0.002N, & u_{b_3} &= \frac{t_{0.95}}{\sqrt{n}}\sigma_{b_3} = 0.002N.
\end{aligned}$$

The relative uncertainty can be calculated by $u_r = u/\bar{X}$.
Finally, the experimentally found k_1, k_2 and k_3 is

$$\begin{aligned}
k_1 &= 2.22 \pm 0.02kg/s^2, \quad u_{rk_1} = 0.9\% \\
k_2 &= 2.25 \pm 0.01kg/s^2, \quad u_{rk_2} = 0.4\% \\
k_3 &= 1.11 \pm 0.017kg/s^2, \quad u_{rk_3} = 1.5\%
\end{aligned}$$

5.1.2 Uncertainty of spring series' constants

To determine the theoretical value of k_3, we have the equations

$$\begin{aligned}
F &= k_1\Delta_{L_1}, \\
F &= k_2\Delta_{L_2}, \\
F &= k_3^{'}(\Delta_{L_1} + \Delta_{L_2}),
\end{aligned}$$

whose solution is $k_3^{'} = \frac{k_1 k_2}{k_1+k_2}$. The theoretical value of k_3 is

$$k_3^{'} = \frac{k_1 k_2}{k_1 + k_2} = \frac{2.22 \times 2.25}{2.22 + 2.25} = 1.12kg/s^2.$$

The propagated uncertainty of k_3 is estimated by the formula

$$\begin{aligned}
u_{k_3'} &= \sqrt{(\frac{\partial k_3'}{\partial k_1})^2 (u_{k_1})^2 + (\frac{\partial k_3'}{\partial k_2})^2 (u_{k_2})^2} \\
&= \sqrt{((\frac{k_2}{k_1+k_2})^2)^2 (u_{k_1})^2 + ((\frac{k_1}{k_1+k_2})^2)^2 (u_{k_2})^2} \\
&= \sqrt{(\frac{2.25}{2.22+2.25})^4 (0.02)^2 + (\frac{2.22}{2.22+2.25})^4 (0.01)^2} \\
&= 0.006kg/s^2
\end{aligned}$$

and the relative uncertainty is

$$u_{rk_3'} = \frac{u_{k_3'}}{\overline{k_3'}} = \frac{0.006}{1.12} = 0.5\%.$$

Hence the theoretical value of $k_3^{'}$ is

$$k_3^{'} = 1.12 + 0.006 kg/s^2, \quad u_{rk_3'} = 0.5\%.$$

Compared with k_3 we obtain from curve fitting, we can calculate the deviation $u^{'}$ between k_3 and $k_3^{'}$ and the relative deviation $u_r^{'}$.

$$u^{'} = k_3 - k_3^{'} = 1.11 - 1.12 = -0.01 kg/s^2,$$
$$u_r^{'} = \frac{k_3 - k_3^{'}}{k_3^{'}} = \frac{1.11 - 1.12}{1.12} = -0.9\%.$$

Recall that the 0.95-confidence interval of k_3 is (1.093,1.127). The theoretical value $k_3^{'}$ is indeed in this interval, which means the uncertainty analysis of k_3 is reasonable.

5.2 Uncertainty of