VV186: HW 2

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Exercise 2.2

1. Proof 1 : $m, n \in \mathbb{N}^* \wedge (\dfrac{m^2}{n^2} < 2) \implies 2n^2 > m^2 \implies m^2 + 4mn + 4n^2 > 2m^2 + 4mn + 2n^2 \implies m^2 + 2 \times m \times 2n + 4n^2 > 2(m^2 + 2mn + n^2) \implies (m + 2n)^2 > 2(m + n)^2 \implies \dfrac{(m + 2n)^2}{(m + n)^2} > 2$

   Proof 2 : $\dfrac{m^2}{n^2} < 2 \implies 2n^2 > m^2 \wedge m, n \in \mathbb{N}^* \implies m^3(m + 2n) < 2mn^2(m + 2n) \implies m^2n^2 + 4mn^3 + 4n^4 < 4m^2n^2 + 8mn^3 + 4n^4 - m^4 - 2m^3n - m^2n^2 \wedge (n \neq 0)$
   $\implies m^2 + 4mn + 4n^2 < \frac{(m^2+2mn+n^2)(4n^2-m^2)}{n^2} \wedge (m, n \in \mathbb{N}^*) \implies \frac{(m+2n)^2}{(m+n)^2} < 4 - \frac{m^2}{n^2}$
   $\implies \frac{(m+2n)^2}{(m+n)^2} - 2 < 2 - \frac{m^2}{n^2}$

2. Proof 1 : $m, n \in \mathbb{N}^* \wedge (\dfrac{m^2}{n^2} > 2) \implies 2n^2 < m^2 \implies m^2 + 4mn + 4n^2 < 2m^2 + 4mn + 2n^2 \implies m^2 + 2 \times m \times 2n + 4n^2 < 2(m^2 + 2mn + n^2) \implies (m + 2n)^2 < 2(m + n)^2 \implies \dfrac{(m + 2n)^2}{(m + n)^2} < 2$

   Proof 2 : $\dfrac{m^2}{n^2} > 2 \implies 2n^2 < m^2 \wedge m, n \in \mathbb{N}^* \implies m^3(m + 2n) > 2mn^2(m + 2n) \implies m^2n^2 + 4mn^3 + 4n^4 > 4m^2n^2 + 8mn^3 + 4n^4 - m^4 - 2m^3n - m^2n^2 \wedge (n \neq 0)$
   $\implies m^2 + 4mn + 4n^2 > \frac{(m^2+2mn+n^2)(4n^2-m^2)}{n^2} \wedge (m, n \in \mathbb{N}^*) \implies \frac{(m+2n)^2}{(m+n)^2} > 4 - \frac{m^2}{n^2}$
   $\implies \frac{(m+2n)^2}{(m+n)^2} - 2 > 2 - \frac{m^2}{n^2}$

3. Proof : Since we only need to prove that max $U_1, U_1 = \{a \in \mathbb{Q}\colon a^2 < 2\}$ does not exist in $\mathbb{Q}$, we can assume that $(m, n \in \mathbb{N}^*) \wedge (m', n' \in \mathbb{N}^*)$
   $m, n \in \mathbb{N}^*$ and $\dfrac{m}{n}$ is a rational number, then $\frac{4mn}{m^2+2n^2}$ is also a rational number. Define $\frac{m'}{n'} := \frac{4mn}{m^2+2n^2}$
   $(4mn)^2 - 2(m^2 + 2n^2)^2 = -(m^2 - 2n^2)^2 < 0 \implies (\frac{4mn}{m^2+2n^2})^2 < 2$
   And we also have $\frac{4mn}{m^2+2n^2} > \frac{4mn}{2n^2+2n^2} = \frac{m}{n}$
   Thus, $(\frac{m}{n})^2 < (\frac{4mn}{m^2+2n^2})^2 < 2$, so we have found a bigger rational number $\frac{m'}{n'} = \frac{4mn}{m^2+2n^2}$ which satisfies all the requirements. Thus, max $U_1, U_1 = \{a \in \mathbb{Q}\colon a^2 < 2\}$ does not exist in $\mathbb{Q}$

Exercise 2.3

Proof : From the definition of infimum, we can see that $y \leq x \wedge (\forall \varepsilon > 0, (y + \varepsilon)^2 > x)$ And in this proof, we assume that $y^2 < x$
Let $\varepsilon = \frac{x-y^2}{(x+1)y}$ Then $(\frac{x-y^2}{(x+1)y})^2 = y^2 + 2y\frac{x-y^2}{(x+1)y} + (\frac{x-y^2}{(x+1)y})^2 < y^2 + (x + 1)y\frac{x-y^2}{(x+1)y} = x$ Thus $(y + \varepsilon)^2 < x$ which leads to a contradiction. Thus, the assumption is wrong and there isn't a y , such that $y^2 < x$
Then the original statement that there is only one element in $\{y : y^2 = x \wedge y > 0 \wedge y \in \mathbb{R}\}$ is proved.

Exercise 2.4

1. $maximum = 1.5$
   $supremum = 1.5$
   $infimum = 1$

2. $maximum = 1.25$
   $supremum = 1.25$
   $infimum = -1$

Exercise 2.5

1. a) almost upper bound is $x \in \{x : x > 1\}$ almost lower bound is $x \in \{x : x \le 1\}$
   b) almost upper bound is $x \in \{x : x > 1\}$ almost lower bound is $x \in \{x : x \le -1\}$
   c) almost upper bound is $x \in \{x : x > 0\}$ almost lower bound is $x \in \{x : x < 0\}$
   d) almost upper bound is $x \in \{x : x \ge \sqrt{2}\}$ almost lower bound is $x \in \{x : x \le 0\}$

2. Assume $x \in set\ X$ Since it's bounded , we can have $\forall x \in X, \exists M > 0$ such that $|x| < M$
   For it's in real number set , we can have $x_1 = supX$
   From the definition of almost upper bound , it's clear that x_1 is an almost upper bound . Thus it's nonempty.
   On the other hand, $\forall x \in X, |x| < M \implies -M < x < M$ If the set Y isn't bounded below, then we may have one element that is smaller than $-M$, then all the infinite elements in X is included. It is against the definition of almost upper bound.
   Thus, set Y is bounded below.

3. a) 1
   b) 1
   c) 0
   d) $\sqrt{2}$

4. Y is the set of all almost lower bounds of X If the supremum sup$Y$ exists ,then it's the $limitinferior$

   a) 1
   b) -1
   c) 0
   d) 0

5. (a) Proof: Limit inferior of $A \implies$ There is infinite numbers bigger than this figure.
   Limit superior of $A \implies$ There is infinite numbers smaller than this figure.
   Thus, $\underline{\lim}\, A \le \overline{\lim}\, A$

   (b) Proof:
   If x is the suprenum of A , then $\forall a \in A, x \ge a$ but there can be finite number of a such that $a > \overline{\lim}\, A$
   Thus, $\overline{\lim}\, A \le supA$
   If x is the infimum of A , then $\forall a \in A, x \le a$ but there can be finite number of a such that $a < \overline{\lim}\, A$
   Thus, $\overline{\lim}\, A \ge infA$

Exercise 2.6

1. Proof : $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$
   $z_1 z_2 = a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1)i \implies |z_1z_2| = \sqrt{(a_1a_2 - b_1b_2)^2 + (a_1b_2 + a_2b_1)^2} = \sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2}$
   $|z_1| = \sqrt{a_1^2 + b_1^2}, |z_2| = \sqrt{a_2^2 + b_2^2} \implies |z_1||z_2| = \sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2} = |z_1z_2|$

2. Assume $z = a + bi$ Then $|z + 2| \le |z - 1| \implies |a + 2 + bi| \le |a - 1 + bi| \implies (a + 2)^2 + b^2 \le a^2 - 2a + 1 + b^2 \implies 6a \le -3 \implies a \le -\frac{1}{2}$ Thus, all complex number $z \in \{(x, y) : x, y \in \mathbb{R} \wedge x \le -\frac{1}{2}\}$ satisfies the original inequality.

3. $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$
   $|z_1 + z_2|^2 + |z_1 - z_2|^2 = (a_1 + a_2)^2 + (b_1 + b_2)^2 + (a_1 - a_2)^2 + (b_1 - b_2)^2 = 2(a_1^2 + a_2^2 + b_1^2 + b_2^2)$
   $2(|z_1|^2 + |z_2|^2) = 2(a_1^2 + a_2^2 + b_1^2 + b_2^2)$
   Thus, $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Exercise 2.7

1. i) Proof : Let $\varepsilon > 0$ be fixed , there exists a N such that $N = \lfloor \sqrt{\frac{1}{\varepsilon}} \rfloor$

   Then $\forall n \geq N, \sqrt{\frac{1}{\varepsilon}} < n \implies \frac{1}{n^2} < \varepsilon \implies |\frac{1}{n^2} - 0| < \varepsilon \implies \lim_{n\to\infty} \frac{1}{n^2} = 0$

   ii) Proof : Let $\varepsilon > 0$ be fixed , there exists a N such that $N = \lfloor \frac{5}{\varepsilon} \rfloor$

   Then $\forall n \geq N, \frac{5}{\varepsilon} < n \implies \frac{5}{n} < \varepsilon \implies |\frac{2n-5}{n} - 2| < \varepsilon \implies \lim_{n\to\infty} \frac{2n-5}{n} = 2$