VECTOR CALCULUS

VECTOR DIFFERENTIATION

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

f = scalar

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$\downarrow$

$\boxed{\text{grad } f}$

f = vector

$$\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

(i) $\vec{\nabla}\cdot\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$\downarrow$

div f

(ii) $\vec{\nabla}\times\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

$\downarrow$

curl f

Normal Vector

$$\overrightarrow{N} = \text{grad } f$$

Unit Normal Vector

$$\hat{N} = \frac{\text{grad } f}{|\text{grad } f|}$$

Directional Derivative in $\vec{A}$ dir

$$= (\text{grad } f)\cdot\hat{A}$$

Angle

$$\cos\theta = \frac{(\text{grad } f)\cdot(\text{grad } g)}{|\text{grad } f|\,|\text{grad } g|}$$

Q. A particle moves on the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 4$, where 't' is time find component of velocity & acceleration at $t=1$ in $\hat{i} - 3\hat{j} + 2\hat{k}$ dir^n.

A. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = (2t^2)\hat{i} + (t^2 - 4t)\hat{j} + (3t - 4)\hat{k}$

$\vec{v} = \frac{d\vec{r}}{dt} = (4t)\hat{i} + (2t - 4)\hat{j} + (3)\hat{k}$

$\vec{v} = \left.\frac{d\vec{r}}{dt}\right|_{t=1} = (4\hat{i} - 2\hat{j} + 3\hat{k})$

and $\vec{a} = \left.\frac{d^2\vec{r}}{dt^2}\right|_{t=1} = \left.4\hat{i} + 2\hat{j} + 0\hat{k}\right|_{t=1} = (4\hat{i} + 2\hat{j})$

Component of $\vec{v}$ & $\vec{a}$ in $\vec{A} = \hat{i} - 3\hat{j} + 2\hat{k}$ dirⁿ

$$\text{Comp. of } \vec{v} \text{ in } \vec{A} \text{ dir} = \vec{v} \cdot \hat{A}$$

$$= (4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{1+9+4}}$$

$$= \frac{16}{\sqrt{14}}$$

Similarly, comp. of $\vec{a}$ in $\vec{A}$ dirⁿ $= \vec{a} \cdot \hat{A}$

$$= (4\hat{i} + 2\hat{j}) \cdot \hat{A}$$

$$= \frac{-2}{\sqrt{14}}$$

Q. Find Unit Tangent Vector at any point on curve $x = t^2+1$, $y = 4t-3$, $z = 2t^2-6t$, also find unit Tangent vector at $t = 2$.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = (t^2+1)\hat{i} + (4t-3)\hat{j} + (2t^2-6t)\hat{k}$$

$$\frac{d\vec{r}}{dt} = (2t)\hat{i} + (4)\hat{j} + (4t-6)\hat{k} = \vec{T} \text{ (Tangent Vector)}$$

$$\text{Unit Tangent Vector} = \frac{\vec{T}}{|\vec{T}|} = \frac{(2t)\hat{i} + (4)\hat{j} + (4t-6)\hat{k}}{\sqrt{(2t)^2 + (4)^2 + (4t-6)^2}} = \hat{T}$$

$$(\hat{T})_{at\ t=2} = \frac{4\hat{i} + 4\hat{j} + 2\hat{k}}{6}$$

Q. If $f = x^3 y z^2$, find grad$f$ at $(1, 1, 1)$.

$$\text{grad} f = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}(3x^2yz^2) + \hat{j}(x^3z^2) + \hat{k}(2x^3yz)$$

$$\text{grad} f\Big|_{(1,1,1)} = (3\hat{i} + \hat{j} + 2\hat{k})$$

Q. find D.D. of $f = x^2 - 2y^2 + 4z^2$ at $(1,1,-1)$ in $2\hat{i} + \hat{j} - \hat{k} \longrightarrow \vec{A}$.

Also, find dir^n of max. D.D. at $(1,1,-1)$ & its max. value.

$$\nabla f = \text{grad} f = \left(\frac{\partial f}{\partial x}\right)\hat{i} + \left(\frac{\partial f}{\partial y}\right)\hat{j} + \left(\frac{\partial f}{\partial z}\right)\hat{k}$$

$$= (2x)\hat{i} + (-4y)\hat{j} + (8z)\hat{k}$$

$$(\text{grad} f)_{(1,1,-1)} = (2\hat{i} - 4\hat{j} - 8\hat{k})$$

$$\text{D.D.} = (\text{grad} f)\cdot\hat{A}$$

$$= (2\hat{i} - 4\hat{j} - 8\hat{k})\cdot\left(\frac{2\hat{i} + \hat{j} - \hat{k}}{\sqrt{4+1+1}}\right)$$

$$= \cancel{\frac{4}{\sqrt{6}}} - \cancel{\frac{4}{\sqrt{6}}} + \frac{8}{\sqrt{6}} = \frac{8}{\sqrt{6}}$$

Max. D.D.

$\downarrow$

(in its own dir^n)

$$\text{max DD} = |\text{grad} f| = \sqrt{4 + 16 + 64} = \sqrt{84}$$

Q. find Angle b/w surfaces $x^2 + y^2 + z^2 = 4$ and $z = x^2 + y^2 - 3$ at $(2,-1,2)$.

$$f = x^2 + y^2 + z^2 - 4$$

$$g = x^2 + y^2 - z - 3$$

$$\nabla f = \left(\frac{\partial f}{\partial x}\right)\hat{i} + \left(\frac{\partial f}{\partial y}\right)\hat{j} + \left(\frac{\partial f}{\partial z}\right)\hat{k}$$

$$= (2x)\hat{i} + (2y)\hat{j} + (2z)\hat{k}$$

$$\nabla f\Big|_{(2,-1,2)} = (4\hat{i} - 2\hat{j} + 4\hat{k})$$

$$\nabla g = \left(\frac{\partial g}{\partial x}\right)\hat{i} + \left(\frac{\partial g}{\partial y}\right)\hat{j} + \left(\frac{\partial g}{\partial z}\right)\hat{k}$$

$$= (2x)\hat{i} + (2y)\hat{j} + (1)\hat{k}$$

$$\nabla g\Big|_{(2,-1,2)} = (4\hat{i} - 2\hat{j} + \hat{k})$$

Now, $$\cos\theta = \frac{(\text{grad} f)\cdot(\text{grad} g)}{|\text{grad} f|\,|\text{grad} g|} = \frac{(4\hat{i} - 2\hat{j} + 4\hat{k})\cdot(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{16+4+16}\cdot\sqrt{16+4+1}}$$

$$= \frac{8 + 4 + 4}{\sqrt{36}\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

Q. Verify Stoke Theorem for Vector field $\vec{f} = (2x-y)\hat{i} - (yz^2)\hat{j} - (y^2z)\hat{k}$ over upper half surface of $x^2+y^2+z^2=1$ bounded by its projection on the XY plane.

Soln:

By Stoke Thm:

$$\vec{f} = (2x-y)\hat{i} - (yz^2)\hat{j} - (y^2z)\hat{k}$$

$$\text{curl}\,\vec{f} = \vec{\nabla}\times\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \hat{i}(-2yz + 2yz) - \hat{j}(0-0) + \hat{k}(0+1)$$

$$\text{curl}\,\vec{f} = \hat{k}$$

XY-plane : $\vec{n} = \hat{k}$

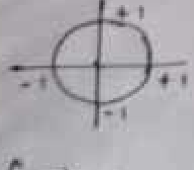
$$\int_c \vec{f}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{f}\cdot\vec{n}\,dS = \iint_R \hat{k}\cdot\hat{k}\,dx\,dy = \iint_R dx\,dy$$

$$= \pi(1)^2 = \pi$$

By line integral

in xy plane, $z=0$, $dz=0$

$\therefore x^2+y^2=1$



$x = \cos\theta$

$y = \sin\theta$

$$\int_c \vec{f}\cdot d\vec{r} = \int_c (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$

$$= \int_c (2x-y)\,dx - 0 - 0$$

$$= -\int_0^{2\pi} (2\cos\theta - \sin\theta)(\sin\theta\,d\theta) = -\int_0^{2\pi} (\sin 2\theta - \sin^2\theta)\,d\theta$$

$$= \pi$$

Q. find unit Normal Vector at surface $x^2y + 2xz = 4$ at $(2,-2,3)$.

$$f = x^2y + 2xz - 4$$

$$\vec{N} = \text{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}\right)\hat{i} + \left(\frac{\partial f}{\partial y}\right)\hat{j} + \left(\frac{\partial f}{\partial z}\right)\hat{k}$$

$$= (2xy + 2z)\hat{i} + (x^2)\hat{j} + (2x)\hat{k}$$

$$\hat{N} = \frac{\vec{N}}{|\vec{N}|}$$

$$= \frac{(2xy+2z)\hat{i} + (x^2)\hat{j} + (2x)\hat{k}}{\sqrt{(2xy+2z)^2 + (x^2)^2 + (2x)^2}}$$

$$\hat{N}\Big|_{(2,-2,3)} = \left(\frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{6}\right)$$

Q. find D.D. of $f = x^2yz + 4xz^2$ at $(1,-2,-1)$ in $\underbrace{(2\hat{i} - \hat{j} - 2\hat{k})}_{(\vec{A})}$ dir^n.

$$\text{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}\right)\hat{i} + \left(\frac{\partial f}{\partial y}\right)\hat{j} + \left(\frac{\partial f}{\partial z}\right)\hat{k}$$

$$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$$\text{grad} f\Big|_{(1,-2,-1)} = (8\hat{i} - \hat{j} - 10\hat{k})$$

$$D.D. = (\text{grad} f) \cdot \hat{A}$$

$$= (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \frac{(2\hat{i} - \hat{j} - 2\hat{k})}{\sqrt{4+1+4}}$$

$$= \left(\frac{16}{3} + \frac{1}{3} + \frac{20}{3}\right)$$

$$= \frac{37}{3}$$

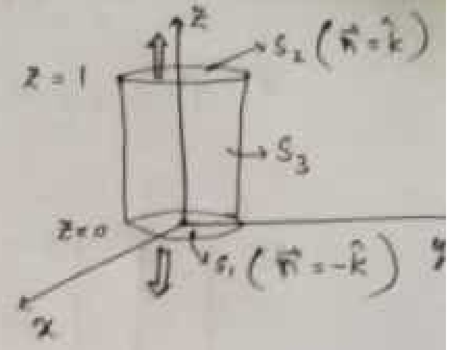
(By Surface integral)

Clearly, surface consists of 3 parts —

(i) Base, S_1, i.e., $x^2+y^2=4$, $z=0$

(ii) Upper face, S_2, $x^2+y^2=4$, $z=1$

(iii) Curved surface, S_3, $x^2+y^2=4$



$$\iint_S \vec{f}\cdot\vec{n}\,dS = \iint_{S_1} \vec{f}\cdot\vec{n}\,dS + \iint_{S_2} \vec{f}\cdot\vec{n}\,dS + \iint_{S_3} \vec{f}\cdot\vec{n}\,dS$$

$$\iint_{S_1} \vec{f}\cdot\vec{n}\,dS = \iint_{S_1} (x\hat{i}-y\hat{j}-\hat{k})\cdot(-\hat{k})\,dx\,dy$$

$$\left\{ \begin{array}{l} x^2+y^2=4 \\ z=0 \\ \vec{n}=-\hat{k} \\ \vec{f}=x\hat{i}-y\hat{j}-\hat{k} \end{array} \right\}$$

$$= \iint_{S_1} dx\,dy = \pi(2)^2 = 4\pi$$

$$\iint_{S_2} \vec{f}\cdot\vec{n}\,dS = \iint_{S_2} (x\hat{i}-y\hat{j}+0\hat{k})\cdot(\hat{k})\,dx\,dy$$

$$\left\{ \begin{array}{l} x^2+y^2=4 \\ z=1 \\ \vec{n}=\hat{k} \\ \vec{f}=x\hat{i}-y\hat{j}+0\hat{k} \end{array} \right\}$$

$$= 0$$

$$\iint_{S_3} \vec{f}\cdot\vec{n}\,dS = \iint_{S_3} \left(x\hat{i}-y\hat{j}+(z^2-1)\hat{k}\right)\cdot\left(\frac{x\hat{i}+y\hat{j}}{2}\right) dS$$

$$= \iint_{S_3} \left(\frac{x^2-y^2}{2}\right) dS$$

$$\left\{ \begin{array}{l} \phi = x^2+y^2 \\ \hat{n} = \dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \dfrac{2x\hat{i}+2y\hat{j}}{2(x^2+y^2)} = \dfrac{x\hat{i}+y\hat{j}}{2} \\ \hline x = 2\cos\theta \\ y = 2\sin\theta \end{array} \right.$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{1} (4\cos^2\theta - 4\sin^2\theta)\,d\theta\,dz$$

$$= 2\int_{0}^{2\pi}\int_{z=0}^{1} (\cos 2\theta)\,d\theta\,dz = 0$$

$$\therefore \iint_S \vec{f}\cdot\hat{n}\,dS = (4\pi)+(0)+(0) = 4\pi$$

Q. find $div\vec{f}$ & $curl\vec{f}$ where $\vec{f} = grad(x^3+y^3+z^3-3xyz)$

$$\vec{f} = (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$$

$$div\vec{f} = \vec{\nabla}\cdot\vec{f}$$

$$= (6x + 6y + 6z)$$

$$curl\vec{f} = \vec{\nabla}\times\vec{f}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \hat{i}(-3x+3x) - \hat{j}(-3y+3y) + \hat{k}(-3z+3z) = 0$$

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line integral

↳ integral which to be evaluated along a curve.

Q. Evaluate $\int_c \vec{f}\cdot d\vec{r}$; $\vec{f} = x^2y^2\hat{i} + y\hat{j}$ & 'c' is curve $y^2=4x$ in XY plane from (0,0) to (4,4).

Soln:

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\int_c \vec{f}\cdot d\vec{r} = \int_c (x^2y^2\hat{i} + y\hat{j} + 0\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_c (x^2y^2dx + ydy)$$

$y^2 = 4x$
$\Rightarrow 2ydy = 4dx$
$\Rightarrow ydy = 2dx$

$$= \int_0^4 x^2(4x)dx + 2dx$$

$$= \cancel{4}\left(\frac{x^4}{\cancel{4}}\right)_0^4 + 2(x)_0^4 = 256 + 8 = 264$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V 3\,dx\,dy\,dz$$

$$= 3\iiint_V dx\,dy\,dz$$

$$= 3\left(\frac{4}{3}\pi(3)^3\right) = 108\pi$$

Q. Evaluate $\iint_S \vec{F}\cdot\vec{n}\,dS$, where $\vec{F} = (x)\hat{i} - (y)\hat{j} + (z^2-1)\hat{k}$ & 'S' is closed surface bounded by plane $z=0$ & $z=1$ & cylinder $x^2+y^2=4$. Also, verify Gauss divergence Theorem by volume integral $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \operatorname{div}\vec{F}\,dV$

Soln:

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2-1)$$

$$= \cancel{(1)} - \cancel{(1)} + (2z) = 2z$$

By Gauss divergence Thm.

$$\iiint_V \operatorname{div}\vec{F}\,dV = \iiint_V (2z)\,dx\,dy\,dz$$

$$= 2\int_{-2}^{+2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} dy \int_0^1 z\,dz$$

$$= \cancel{2}\int_{-2}^{+2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} dy \left[\frac{z^2}{\cancel{2}}\right]_0^1$$

$$= \int_{-2}^{+2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} dy = 4\int_0^2 dx \int_0^{\sqrt{4-x^2}} dy$$

$$= 4\int_0^2 \sqrt{4-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 4\left[0 + 2\sin^{-1}(1)\right] = 4\pi$$

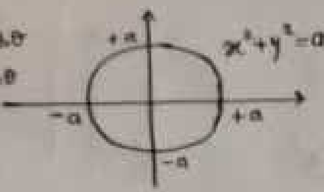
$$\therefore \int_C \vec{F}\cdot d\vec{r} = \left(\tfrac{8}{3}\right) + \left(\tfrac{-2}{3}\right) + \left(\tfrac{-8}{3}\right) + \left(\tfrac{2}{3}\right) = 0$$

Q. A Vector field is given by $\vec{F} = (\sin y)\,\hat{i} + x(1+\cos y)\,\hat{j}$.

Evaluate $\int_C \vec{F}\cdot d\vec{r}$ over circular path C given by $x^2+y^2=a^2$, $z=0$.

Soln:

$x = a\cos\theta$
$y = a\sin\theta$



$$\int_C \vec{F}\cdot d\vec{r} = \int_C \sin y\,dx + x(1+\cos y)\,dy$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \sin y\,dy + x\cos y\,dy + \int_C x\,dy$$

$$= \int_C d(x\sin y) + \int_C x\,dy$$

$$= \int_0^{2\pi} d\left(a\cos\theta\cdot\sin(a\sin\theta)\right) + \int_0^{2\pi} a\cos\theta\,\overbrace{(a\cos\theta)\,d\theta}^{dy}$$

$$= \left[a\cos\theta\cdot\sin(a\sin\theta)\right]_0^{2\pi} + a^2\int_0^{2\pi}\cos^2\theta\,d\theta$$

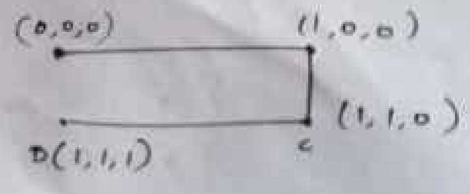
$$= 0 + \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \pi a^2$$

Q. Evaluate the line integral from (0,0,0) to (1,1,1) along path (0,0,0) to (1,0,0), then to (1,1,0) & then to (1,1,1).

$$\vec{F} = (3x^2+6y)\,\hat{i} - (14yz)\,\hat{j} + (20xz^2)\,\hat{k}$$

Soln:

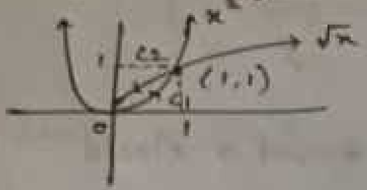
$$\int_C \vec{F}\cdot d\vec{r} = \int_C (3x^2+6y)\,dx - (14yz)\,dy + (20xz^2)\,dz$$



Q. Verify Green Theorem in the plane for $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where 'c' is bounded by the region defined by $y = \sqrt{x}$ & $y = x^2$.

Soln:

By line integral $\left\{ \begin{array}{l} C_1 \to y = x^2 \\ C_2 \to y = \sqrt{x} \end{array} \right\}$



$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \qquad \left\{ y = x^2,\ dy = 2x\,dx \right.$$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)(2x\,dx)$$

$$= -1$$

and $\int_{C_2} \vec{F}\cdot d\vec{r} = \int_1^0 (3y^4 - 8y^2)(2y\,dy) + (4y - 6y^3)\,dy$ $\left\{ \begin{array}{l} y = \sqrt{x} \Rightarrow y^2 = x \\ \Downarrow \\ 2y\,dy = dx \end{array} \right.$

$$= 5/2$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = (-1) + (5/2) = 3/2$$

By Green Theorem

here, $P = 3x^2 - 8y^2 \Rightarrow \frac{\partial P}{\partial y} = -16y$

$ \varphi = 4y - 6xy \Rightarrow \frac{\partial \varphi}{\partial x} = -6y$

$$\therefore \int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \iint_R \left(\frac{\partial \varphi}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$= \iint_R (-6y + 16y)\,dx\,dy = 10 \iint_R y\,dx\,dy$$

$$= 10\int_0^1 dx \int_{x^2}^{\sqrt{x}} y\,dy$$

$$= \frac{10}{2}\int_0^1 dx \left[y^2\right]_{x^2}^{\sqrt{x}} = 5\int_0^1 dx\left(x - x^4\right)$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

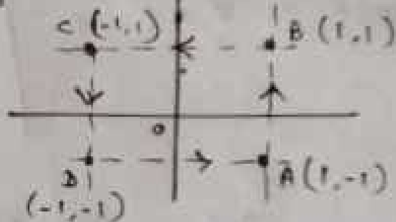
$$= 5\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{5}{2} - 1 = 3/2$$

Q. Verify Green Theorem in $\int_C (x^2+xy)dx + (x^2+y^2)dy$, where 'C' is square formed by lines $x = \pm 1$, $y = \pm 1$.

Soln: By line integral

$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r}$$

$$= \underline{0}$$



By Green Theorem

here, $P = x^2 + xy \Rightarrow \frac{\partial P}{\partial y} = x$

$\phi = x^2 + y^2 \Rightarrow \frac{\partial \phi}{\partial x} = 2x$

$$\therefore \int_C (x^2+xy)dx + (x^2+y^2)dy = \iint_R \left(\frac{\partial \phi}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R (2x - x)\,dx\,dy$$

$$= \int_{-1}^{+1} x\,dx \int_{-1}^{+1} dy = 2\int_{-1}^{+1} x\,dx = 2\left[\frac{x^2}{2}\right]_{-1}^{+1}$$

$$= (1-1) = \underline{0}$$

Q. Show that integral $\int_{(1,2)}^{(3,4)} (xy^2 + y^3)\,dx + (x^2y + 3xy^2)\,dy$ is independent of path joining $(1,2)$ & $(3,4)$. Hence evaluate integral.

Soln:-

$$\vec{f} = (xy^2 + y^3)\hat{i} + (x^2y + 3xy^2)\hat{j} + (0)\hat{k}$$

$$\text{Curl } \vec{f} = \vec{\nabla} \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2+y^3 & x^2y+3xy^2 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2xy + 3y^2 - 2xy - 3y^2)$$

$$= 0$$

$\because$ curl $\vec{f} = 0$

$\therefore$ $\vec{f}$ is conservative vector field.

Let 'φ' be scalar potential of $\vec{f}$.

, i.e., $\vec{f} = \text{grad } \varphi$

$$\Rightarrow d\varphi = (xy^2 + y^3)\,dx + (x^2y + 3xy^2)\,dy$$

$$\Rightarrow d\varphi = \frac{1}{2}(2xy^2 dx + 2x^2 y\,dy) + (y^3 dx + 3xy^2 dy)$$

$$\Rightarrow \int d\varphi = \frac{1}{2}\int d(x^2y^2) + \int d(xy^3)$$

$$\Rightarrow \boxed{\varphi = \frac{1}{2}(x^2y^2) + (xy^3) + c}$$

$\therefore$ given integral

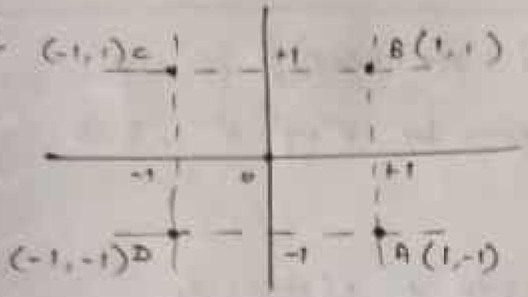
$$= \left[\varphi\right]_{(1,2)}^{(3,4)}$$

$$= \left[\frac{1}{2}(x^2y^2) + (xy^3)\right]_{(1,2)}^{(3,4)}$$

$$= 254$$

Q. Evaluate the line integral $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$, where, 'C' is the square formed by lines $x = \pm 1$, $y = \pm 1$.

Soln:



$$\int_C \vec{f}\cdot d\vec{r} = \int_{AB} \vec{f}\cdot d\vec{r} + \int_{BC} \vec{f}\cdot d\vec{r} + \int_{CD} \vec{f}\cdot d\vec{r} + \int_{DA} \vec{f}\cdot d\vec{r}$$

$$\vec{f} = (x^2+xy)\,\hat{i} + (x^2+y^2)\,\hat{j}$$

Now,

$$\int_{AB} \vec{f}\cdot d\vec{r} = \int_{-1}^{+1} 0 + (1^2+y^2)\,dy \qquad \left\{ \begin{array}{l} x = 1,\ dx = 0 \\ y = -1 \text{ to } +1 \end{array} \right.$$

$$= 2\int_0^1 (y^2+1)\,dy = 2\left(\frac{y^3}{3} + y\right)_0^1$$

$$= \frac{8}{3}$$

$$\int_{BC} \vec{f}\cdot d\vec{r} = \int_{+1}^{-1} (x^2+x)\,dx + 0 \qquad \left\{ \begin{array}{l} y = 1,\ dy = 0 \\ x = +1 \text{ to } -1 \end{array} \right.$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{+1}^{-1}$$

$$= \left(-\frac{1}{3}+\frac{1}{2}\right) - \left(\frac{1}{3}+\frac{1}{2}\right) = -\frac{2}{3}$$

$$\int_{CD} \vec{f}\cdot d\vec{r} = \int_{+1}^{-1} 0 + (1+y^2)\,dy \qquad \left\{ \begin{array}{l} x = -1,\ dx = 0 \\ y = +1 \text{ to } -1 \end{array} \right.$$

$$= \left[y + \frac{y^3}{3}\right]_{+1}^{-1} = \left(-1-\frac{1}{3}\right) - \left(1+\frac{1}{3}\right) = -2 - \frac{2}{3} = -\frac{8}{3}$$

$$\int_{DA} \vec{f}\cdot d\vec{r} = \int_{-1}^{+1} (x^2+x)\,dx + 0 \qquad \left\{ \begin{array}{l} y = 1,\ dy = 0 \\ x = -1 \text{ to } +1 \end{array} \right.$$

$$= \left(\frac{x^3}{3} + \frac{x^2}{2}\right)_{-1}^{+1} = \left(\frac{1}{3}+\frac{1}{2}\right) - \left(-\frac{1}{3}+\frac{1}{2}\right) = \frac{2}{3}$$

Q. find ~~components~~ a & b, so that surface $ax^2 - bxy = (a+2)x$ will be orthogonal to surface $4x^2y + z^3 = 4$ at $(1,-1,2)$.

$$f = 4x^2y + z^3 - 4$$

$$g = ax^2 - bxy - (a+2)x \longrightarrow a + b - (a+2) = 0 \Rightarrow b = 2$$

$$\nabla f = (8xy)\hat{i} + (4x^2)\hat{j} + (3z^2)\hat{k}$$

$$\nabla f\Big|_{at\,(1,-1,2)} = (-8\hat{i} + 4\hat{j} + 12\hat{k})$$

and $\nabla g = \big(2ax - by - (a+2)\big)\hat{i} + (-bx)\hat{j} + (0)\hat{k}$

$$\nabla g\Big|_{at\,(1,-1,2)} = (2a - b - a - 2)\hat{i} + (-b)\hat{j}$$

$$= (a - b - 2)\hat{i} - (b)\hat{j}$$

Now, $(\nabla f)\cdot(\nabla g) = 0 \quad (\because \theta = 90^\circ)$

$\Rightarrow -8(a-b-2) - 4b = 0$

$\Rightarrow -8a + 8b + 16 - 4b = 0$

$\Rightarrow 4b - 8a + 16 = 0$

$\Rightarrow b - 2a + 4 = 0 \Rightarrow (2) - 2a + 4 = 0 \Rightarrow a = 3$

$\text{div} f = 0 \implies f$ is Solenoidal Vector

$\text{curl} f = 0 \implies f$ is Irrotational Vector

Q. find $\text{div}\vec{f}$ & $\text{curl}\vec{f}$, where $\vec{f} = xy^2\hat{i} + 2xy^2z\,\hat{j} - 3yz^2\hat{k}$ at $(1,-1,1)$.

$$\text{div}\vec{f} = \vec{\nabla}\cdot\vec{f} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2xy^2z) + \frac{\partial}{\partial z}(-3yz^2)$$

$$= y^2 + 4xyz - 6yz \Rightarrow \text{div}\vec{f}\Big|_{(1,-1,1)} = 1 - 4 + 6 = 3$$

$$\text{curl}\vec{f} = \vec{\nabla}\times\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2xy^2z & -3yz^2 \end{vmatrix}$$

$$= \hat{i}(-3z^2 - 2xy^2) - \hat{j}(0 - 0) + \hat{k}(2y^2z - 2xy)$$

$$\text{curl}\vec{f}\Big|_{(1,-1,1)} = \hat{i}(-3-2) + \hat{k}(2+2) = (-5\hat{i} + 4\hat{k})$$

Green Thearem

If $p(x,y)$ & $\varphi(x,y)$ are Continuous function of x & y having continuous first order partial derivative in a Region R of XY plane bounded by a closed curve C, then

$$\oint_c (p\,dx + \varphi\,dy) = \iint_R \left(\frac{\partial \varphi}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy$$

Green Thearem is useful for evaluating of line integral around a Closed curve 'C'.

Q. A vector field $\vec{f}$ is given by

$$\vec{f} = (\sin y)\,\hat{i} + x(1+\cos y)\,\hat{j}$$

Evaluate the line integral $\int_c \vec{f}\cdot d\vec{r}$, where 'C' is a Circular path given by $x^2+y^2=a^2$.

Soln:

$$\int_c \vec{f}\cdot d\vec{r} = \int_c (\sin y)dx + x(1+\cos y)dy$$

here, $p = \sin y$

$\varphi = x(1+\cos y)$

$$\int_c \vec{f}\cdot d\vec{r} = \int_c (\sin y)\,dx + x(1+\cos y)\,dy$$

$$= \iint_R \left((1+\cancel{\cos y}) - (\cancel{\cos y})\right) dx\,dy$$

$$= \iint_R dx\,dy = \underline{\underline{\pi a^2}}$$

Q. find constant 'a' so that 'A' is a conservative field, where,

$$\vec{A} = (axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)xz^2\hat{k}.$$

Calculate its Potential & work done in moving a particle from $(1,2,-3)$ to $(1,-4,2)$ in the field.

Soln:- $\text{Curl } \vec{A} = \vec{\nabla} \times \vec{A} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$$

$$\hat{i}(0-0) - \hat{j}\left((1-a)z^2 + 3z^2\right) + \hat{k}\left(2(a-2) - ax\right) = 0$$

$$0\hat{i} - z^2(4-a)\hat{j} + \hat{k}\left(2a - 4 - ax\right) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$\therefore$ $4 - a = 0 \Rightarrow a = 4$

$\therefore$ $\vec{A} = (4xy - z^3)\hat{i} + (2x^2)\hat{j} + (-3xz^2)\hat{k}$

Let 'φ' be Scalar potential of $\vec{A}$.

,i.e., $\vec{A} = \text{grad } \varphi$

$$\Rightarrow d\varphi = (4xy - z^3)dx + (2x^2)dy + (-3xz^2)dz$$

$$\Rightarrow d\varphi = (4xy\,dx + 2x^2dy) - (z^3dx + 3xz^2dz)$$

$$\Rightarrow \int d\varphi = \int d(2x^2y) - \int d(xz^3)$$

$$\Rightarrow \boxed{\varphi = 2x^2y - xz^3 + c}$$

Now, Work done $= \left[\varphi\right]_{(1,2,-3)}^{(1,-4,2)}$

$= -47$ unit

Gauss's Divergence Theorem

The Normal Surface integral of a vector function f over the boundary of a closed region is equal to the Volume integral of "div f" taken throughout the region.

$$\iint_S \vec{f}\cdot\vec{n}\,ds = \iiint \operatorname{div} f \, dV$$

$$\iint_S \left(f_1\,dy\,dz + f_2\,dz\,dx + f_3\,dx\,dy \right) = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx\,dy\,dz$$

Q. Show that $\iint_S \left(x\,dy\,dz + y\,dx\,dz + z\,dx\,dy \right) = 4\pi a^3$, where the 'S' is sphere, $x^2+y^2+z^2=a^2$

Soln: here, $f_1 = x$
$f_2 = y$
$f_3 = z$

$$\iint_S \left(x\,dy\,dz + y\,dx\,dz + z\,dx\,dy \right)$$

$$= \iiint_V \left(\frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial z} f_3 \right) dx\,dy\,dz$$

$$= 3 \iiint_V dx\,dy\,dz = 3\left(\frac{4}{3}\pi a^3 \right) = \underline{4\pi a^3}$$

Q. Find $\iint_S \vec{f}\cdot\vec{n}\,ds$, where, $\vec{f} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$, where, 'S' is surface of ~~surface of~~ sphere having $(3,-1,2)$ & radius 3.

Soln:

$$\iint_S \vec{f}\cdot\vec{n}\,dS = \iiint \operatorname{div}\vec{f}\,dV$$

$$\operatorname{div}\vec{f} = \vec{\nabla}\cdot\vec{f} = \frac{\partial}{\partial x}(2x+3z) - \frac{\partial}{\partial y}(xz+y) + \frac{\partial}{\partial z}(y^2+2z)$$

$$= (2) - (1) + (2) = 3$$

Q. Show that $\vec{F} = (2xy + z^3)\hat{i} + (x^2)\hat{j} + (3z^2x)\hat{k}$ is Conservative field. find its Scalar potential and also the Work done in moving particle from $(1,-2,1)$ to $(3,1,4)$.

Soln:

$\boxed{\text{Curl}(\text{grad } f) = 0}$

$$\text{Curl}\vec{f} = \vec{\nabla} \times \vec{f}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3z^2x \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x) = 0$$

hence, f is Conservative Vector field.

Let 'φ' is Scalar potential of $\vec{f}$.

i.e., $\vec{f} = \text{grad}\,\varphi$

$$\Rightarrow d\varphi = (2xy + z^3)\,dx + (x^2)dy + (3z^2x)\,dz$$

$$\Rightarrow d\varphi = (2xy\,dx + x^2dy) + (z^3dx + 3z^2x\,dz)$$

$$\Rightarrow \int d\varphi = \int d(x^2y) + \int d(xz^3)$$

$$\Rightarrow \boxed{\varphi = x^2y + xz^3 + c}$$

$$\text{Work done} = (\varphi)_{(1,-2,1)}^{(3,1,4)}$$

$$= (x^2y + xz^3)_{(1,-2,1)}^{(3,1,4)}$$

$$= 202 \text{ unit}$$

Stoke Theorem

Surface integral of the component of $\vec{f}$ along the Normal to the surface 'S' taken over the surface 'S' bounded by curve 'C' is equal to the line integral of vector point function $\vec{f}$ taken along the closed curve 'C'.

$$\oint_C \vec{f}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{f}\cdot\vec{n}\,dS$$

Q. Using Stoke Theorem, evaluate $\int_C xy\,dx + xy^2\,dy$, where 'C' is square in XY plane with $(1,0), (-1,0), (0,1)$ & $(0,-1)$.

Soln:

$$\vec{f} = (xy)\hat{i} + (xy^2)\hat{j}$$

$$\text{curl}\,\vec{f} = \vec{\nabla}\times\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(y^2 - x)$$

$$\text{curl}\,\vec{f} = (y^2-x)\hat{k}$$

XY-plane : $\vec{n} = \hat{k}$

$$\oint_C \vec{f}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{f}\cdot\vec{n}\,dS$$

$$= \int_{-1}^{+1}\int_{-1}^{+1} \hat{k}(y^2-x)\cdot\hat{k}\,dx\,dy$$

$$= \int_{-1}^{+1} dx \int_{-1}^{+1} (y^2-x)\,dy = \int_{-1}^{+1} dx\left(\frac{y^3}{3} - xy\right)_{-1}^{+1}$$

$$= \int_{-1}^{+1} dx\left(\frac{1}{3} - x + \frac{1}{3} - x\right)$$

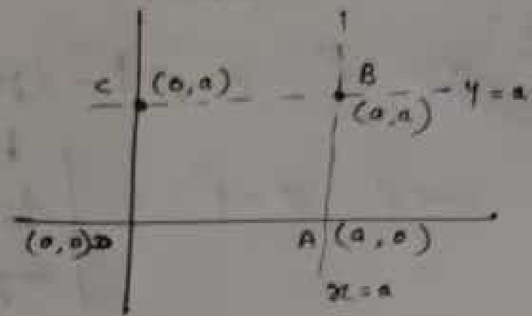
$$= \int_{-1}^{+1}\left(\frac{2}{3} - 2x\right)dx = \frac{2}{3}(x)_{-1}^{+1} - (x^2)_{-1}^{+1}$$

$$= \left(\frac{4}{3}\right) - (0) = \frac{4}{3}$$

4. Verify Stoke Thm. for the function $\vec{f} = (x^2)\hat{i} + (xy)\hat{j}$ integrated round the square in plane $z=0$ whose sides are $x=y=0$ & $x=y=a$.

Soln:-

By line integral



$$\int_C \vec{f}\cdot d\vec{r}$$

$$= \int_{AB} \vec{f}\cdot d\vec{r} + \int_{BC} \vec{f}\cdot d\vec{r} + \int_{CD} \vec{f}\cdot d\vec{r} + \int_{DA} \vec{f}\cdot d\vec{r}$$

$$= \int_0^a (0 + ay\,dy) + \cancel{\int_a^0 (x^2 dx + 0)} + \int_a^0 (0+0) + \cancel{\int_0^a (x^2 dx + 0)}$$

$$= a\left[\frac{y^2}{2}\right]_0^a = \frac{a}{2}(a^2) = \frac{a^3}{2}$$

By Stoke Thm.

$$\vec{f} = (x^2)\hat{i} + (xy)\hat{j}$$

$$\text{curl}\,\vec{f} = \vec{\nabla}\times\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(y-0) = (y)\hat{k}$$

xy plane : $\vec{n} = \hat{k}$

$$\therefore \int_C \vec{f}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{f}\cdot\vec{n}\,dS = \iint y\hat{k}\cdot\hat{k}\,dS$$

$$= \int_0^a\int_0^a y\,dx\,dy$$

$$= \int_0^a dx \int_0^a y\,dy = \frac{1}{2}\int_0^a dx\,(y^2)_0^a$$

$$= \frac{a^2}{2}\int_0^a dx = \frac{a^3}{2}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r}$$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^1 (3x^2+0)\,dx + 0 + 0 \qquad \left\{\begin{array}{l} x = 0 \text{ to } 1 \\ y = 0,\ dy = 0 \\ z = 0,\ dz = 0 \end{array}\right.$$

$$= \left(x^3\right)_0^1 = 1$$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_0^1 0 + (-14y\times 0)\,dy + 0 = 0 \qquad \left\{\begin{array}{l} y = 0 \text{ to } 1 \\ x = 1,\ dx = 0 \\ z = 0,\ dz = 0 \end{array}\right.$$

$$\int_{CD} \vec{F}\cdot d\vec{r} = \int_0^1 0 + 0 + 20(1)(z^2)\,dz \qquad \left\{\begin{array}{l} z = 0 \text{ to } 1 \\ y = 1,\ dy = 0 \\ x = 1,\ dx = 0 \end{array}\right.$$

$$= 20\int_0^1 z^2\,dz = \frac{20}{3}\left[z^3\right]_0^1 = \frac{20}{3}$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = (1) + (0) + \left(\frac{20}{3}\right) = \frac{23}{3}$$

Application of line integral :—

① Work done by a Force, $\vec{F}$ in the displacement $\vec{r}(t)$ along curve C from point A to B,

$$\text{Work done} = \int_A^B \vec{F}\cdot d\vec{r}$$

If $\vec{F}$ denotes Velocity of Fluid, then $\oint_C \vec{F}\cdot d\vec{r}$ is called Circulation of $\vec{F}$ round closed curve C.

② Independent of path — Conservative field and Scalar Potential !!