← Partial Differentiation →

1. Function

$f: A \to B$ → single variable → A has the element of type 1 tuple
→ Multivariable → A has the element of n tuple.

1 tuple $\mathbb{R} = [1, 2, 0.5]$
2 tuple $\mathbb{R}^2 = [(a, b) : a, b \in \mathbb{R}]$
3 tuple $R \times R \times R$ or $R^3 = \{a, b, c : a, b, c \in R, (a, b, c)\}$

2. Limit & continuity

$\left[\lim_{x \to a} f(x)\right.$ if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-}$ exist and are equal.

$\left[\lim_{x \to a} f(x) = f(a)\right]$

A function is continuous at c when the following 3 conditions are met

(1) $f(c)$ is defined
(2) $\lim_{x \to c} f(x)$ exists
(3) $\lim_{x \to c} f(x) = fc$

*indefinite tangents pass through a curve in 3D space, gives a tangent to the curve in a specific direction

3. differentiability

differentiability: unique tangent exists $\left.\frac{dy}{dx}\right|_{x=a}$ represent the slope of tangent at point.

or

Rate of change of dependent variable wrt to independent variable

1 tuple: ∞ type of tangent at a space so partial differentiation at particular direction needed.

2 tuple: → at (a, b) $\lim_{\substack{x \to a \\ y \to b}} f(x, y)$, exists if this gave a unique value along every possible path.

4. partial differentiation.

Let $z = f(x, y)$ be a function of two variables then the partial diff. of f with respect to x is the derivative wrt x treating the variable y as constant

← 2 variable →

$f(x)_{\to (a,b)} = \lim_{h \to 0} \dfrac{f(a+h, b) - f(a, b)}{h}$

$f(y) \to (a, b) = \lim_{k \to 0} \dfrac{f(a, b+k) - f(a, b)}{k}$

← in case of more than 2 variable →

$f: R^n \to R$

$\lim \dfrac{f(x+h, y, z) - f(x, y, z)}{h}$

similarly for y & z.

5. High order

- $f_{xx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$
- $f_{yx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$ ← In general the first order partial derivatives are the functions of x & y so these can be diff. further partially wrt x & y both →
- $f_{yx^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 f}{\partial x \partial y}\right) = \dfrac{\partial^3 f}{\partial x^2 \partial y}$

$f_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$

Q1: $x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$

Ans. $u_{yx} = u_{xy} = \frac{x^2-y^2}{x^2+y^2}$.

$$\frac{\partial u}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) + x^2 \times \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(-\frac{y}{x^2}\right) - y^2 \times \frac{1}{y} \times \frac{1}{1+\frac{x^2}{y^2}}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - y\left[\frac{x^2+y^2}{x^2+y^2}\right]$$

$$= \boxed{2x\tan^{-1}\left(\frac{y}{x}\right) - y.}$$

$$\frac{\partial u}{\partial y\,\partial x} = 2x\frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x} - 1$$

$$= \frac{2x^2}{x^2+y^2} - 1 = \boxed{\frac{x^2-y^2}{x^2+y}}$$

$$\frac{\partial u}{\partial y} = x^2 \times \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x} - \left[2y\tan^{-1}\left(\frac{x}{y}\right) + y^2\left[\frac{1}{1+\left(\frac{x}{y}\right)^2}\right]\left[-\frac{x}{y^2}\right]\right]$$

$$\frac{x^3}{x^2+y^2} - \left[2y\tan^{-1}\left(\frac{x}{y}\right) + \left(-\frac{y^2 x}{x^2+y^2}\right)\right]$$

$$\boxed{x - 2y\tan^{-1}\left(\frac{x}{y}\right)}$$

$$\frac{\partial u}{\partial x\,\partial y} = 1 - 2y \times \frac{1}{1+\left(\frac{x}{y}\right)^2} \times \frac{1}{y}$$

$$= 1 - \frac{2y^2}{y^2+x^2}.$$

$$= \boxed{\frac{x^2-y^2}{x^2+y^2}}$$ Answer

Q2] $f(x+ct)+\phi(x-ct)=z$ $\boxed{\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}}$

Here

$$\frac{dz}{dt} = f'(a+ct)$$

$$\frac{dz}{dt} = f'(x+ct)c + \phi'(x-ct)(-c)$$

$$\frac{d^2z}{dt^2} = f''(x+ct)c^2 + \phi''(x-ct)(c^2)$$

$$= c^2[f''(x+ct) + \phi''(x-ct)]$$

$$\frac{\partial z}{\partial x} = f'(x+ct) + \phi'(x-ct)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct)$$

$$\frac{d^2z}{dt^2} = c^2[f''(x+ct) + \phi''(x-ct)]$$

$$\frac{d^2z}{dx^2} = f''(x+ct) + \phi''(x-ct)$$

hence proved $\boxed{\frac{d^2z}{dt^2} = c^2\frac{d^2z}{dx^2}}$

Q3] $\theta = t^n e^{-r^2/4t}$ $\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{d\theta}{dr}\right) = \frac{\partial\theta}{dt}}$ find (n)

new

$$\Rightarrow \frac{d\theta}{dt} = nt^{n-1}e^{\frac{-r^2}{4t}} + t^n e^{-\frac{r^2}{4t}}\left[-\frac{r^2}{4}\right]\times\left(-\frac{1}{t^2}\right) = t^n e^{-\frac{r^2}{4t}}\left[\frac{n}{t}+\frac{r^2}{4t^2}\right]$$

$n = -3/2$

$$\Rightarrow \frac{d\theta}{dt} = \theta\left[\frac{n}{t}+\frac{r^2}{4t}\right] \quad \text{—(1)}$$

$$\Rightarrow \frac{d\theta}{dr} = t^n e^{-\frac{r^2}{4t}}\left[-\frac{1}{4t}\right]\times[2r] = -\frac{\theta}{2}\left[\frac{r}{t}\right]$$

$$r^2\times\frac{d\theta}{dr} = \frac{r^3\theta}{2t} \quad \text{—(2)}$$

diff (2) wrt r.

$$-\frac{1}{2t}\left[3r^2\theta + r^3\frac{d\theta}{dr}\right]$$

$$\frac{-1}{2t}\left[3r^2\theta + r^3\left(-\frac{\theta r}{2t}\right)\right]$$

$$r^2\theta\left[-\frac{3}{2t}+\frac{r^2}{4t}\right] = \frac{\partial}{\partial r}\left[r^2\left(\frac{d\theta}{dr}\right)\right] \quad \text{(3)}$$

$3\times\frac{1}{(r^2)}$ $\Rightarrow \theta\left[-\frac{3}{2t}+\frac{r^2}{4t}\right] = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\times\left(r^2\frac{d\theta}{dr}\right)\right]$ —(4)

(4)=(1) $\theta\left[\frac{-3}{2t}+\frac{r^2}{4t}\right] = \theta\left[\frac{n}{t}+\frac{r^2}{4t}\right]$ so $\boxed{n = -3/2}$ Ans.

Q.4. x^3+y^3-3axy.

$\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$. $\longrightarrow$ $\frac{\partial f}{\partial y} = x^3+3y^2-3ax$

$\boxed{\frac{\partial f}{\partial x \partial y} = 3x^2-3a = 3(x^2-a)}$

$\frac{df}{dx} \to 3x^2+y^3-3ay$

$\boxed{\frac{\partial^2 f}{\partial x^2} \to 6x+y^3+0}$

6. Clairaut theorum

$\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist and are continuous then, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

HW Q5. $u = f(r)$ $\quad r = x\sec\theta$, $x = r\cos\theta$, $y = r\sin\theta$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r) + \frac{1}{r^2}\frac{d^2u}{d\theta^2}$

$\frac{d\theta}{dx} = \frac{1}{dx/d\theta} = \frac{d^2\theta}{dx^2} = \frac{d(d\theta/dx)}{dx}$.

$\to \frac{d\left(\frac{1}{dx/d\theta}\right)}{dx}$

similarly for du/dz

$\frac{d^2\theta}{dx^2} \to \frac{-1}{\left(\frac{dx}{d\theta}\right)^2} \frac{d(dx/d\theta)}{dx}$.

* $\frac{du}{dx} = \frac{du}{dr} \times \frac{dr}{dx} \Big| \frac{dy}{dy} = \frac{du}{dr} \times \frac{dr}{dy}$.

$=$ ~~$\frac{du}{d\theta} \times \sec\theta \times \frac{d\theta}{dx}$~~

$\boxed{\frac{dr}{dx} = \frac{dx}{dr} = \frac{x}{r}}$

$r^2 = x^2+y^2$ D wrt x.

$\cancel{2}r\frac{dr}{dx} = \cancel{2}x$ $\quad$ $\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right) \text{ or } \frac{y}{x} = \tan\theta}$ $\quad$ $\boxed{\frac{d\theta}{dx} = -\frac{y}{r^2}}$

$\frac{dr}{dx} = \frac{x}{r}$ $\quad$ $\boxed{\frac{du}{dx} = f'(r) \times \frac{x}{r}}$ $\boxed{\frac{du}{dy} f'(r) \times \frac{y}{r}}$

$\cancel{2}r\frac{dr}{dy} = 0+\cancel{2}y$

$\frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{f''(r)}{r}$

$x = r\cos\theta$

$\frac{dx}{d\theta} = -r\sin\theta$

$\frac{dx}{r\,d\theta} = \frac{-y}{r}$

So $\boxed{r\frac{d\theta}{dx} = \frac{-y}{r}} \boxed{= \frac{r}{r}\frac{dx}{d\theta}}$.

Q6. $(x^2+y^2+z^2)^{-1/2}$

$x\frac{du}{dx} + y\frac{du}{dy} + z\frac{du}{dz} = -u$

1. $\frac{du}{dx} = \frac{\partial}{\partial x}[x^2+y^2+z^2]^{-1/2}$

$= \boxed{-\frac{1}{\cancel{2}}[x^2+y^2+z^2]^{-3/2} \times \cancel{2}x}$

2. $\frac{du}{dy} = \boxed{-\frac{1}{2}[x^2+y^2+z^2] \times 2y}$

3. $x\frac{du}{dx} + y\frac{du}{dy} + z\frac{du}{dz}$. $\longrightarrow$

$-x \times x[x^2+y^2+z^2]^{-3/2} - y \times y[x^2+y^2+z^2]^{-3/2}$

$-z^2 \times [\quad]^{-3/2}$

$(x^2+y^2+z^2)^{-3/2} \times [-x^2-y^2-z^2]$

$\Rightarrow -[x^2+y^2+z^2]^{-1/2}$

$\Rightarrow \boxed{-u}$

← hence proved →

$x = r\cos\theta$
$y = r\sin\theta$

$x^2+y^2=r^2$

$2x = 2r\frac{dr}{dx}$

1. $\frac{x}{r} = \frac{dr}{\partial x} = \frac{r\cos\theta}{r} = \cos\theta$.

2. $\frac{y}{r} = \frac{dr}{dy} = \sin\theta$

3. $\frac{y}{x} = \tan\theta$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\frac{d\theta}{dx} = \frac{d}{\partial x}(\ ) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$

4. $\frac{d\theta}{dx} = \frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}$

5. $\frac{d\theta}{dy} = +\frac{\cos\theta}{r}$ similarly

$\frac{du}{\partial x} = \frac{du}{dr}\times\frac{dr}{dx} + \frac{du}{d\theta} * \frac{d\theta}{dx}$

$$\boxed{\frac{du}{\partial x} = \frac{du}{dr}\times\cos\theta + \frac{du}{d\theta}\times\left(\frac{-\sin\theta}{r}\right)}$$

$$\boxed{\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}}$$

$\frac{d^2u}{dx^2} = \frac{d}{\partial x}\left(\frac{du}{dx}\right) = \left(\cos\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\left(\frac{\partial u}{\partial\theta}\right)\right)$

$\Rightarrow \cos\theta\left(\frac{\partial}{\partial r}\right)\left(\cos\theta\frac{du}{dr} - \frac{\sin\theta}{r}\frac{du}{d\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{du}{dr} - \frac{\sin\theta}{r}\left(\frac{du}{\partial\theta}\right)\right)$

$\Rightarrow \cos\theta\left[\frac{\partial}{\partial r}\left(\cos\theta\frac{du}{dr} - \frac{\sin\theta}{r}\left(\frac{\partial u}{\partial\theta}\right)\right) - \frac{\sin\theta}{r}\left(\frac{\partial}{\partial\theta}\right)(\cos\theta\ du/dr - \sin\theta/r\ du/d\theta\right.$

$\rightarrow \cos\theta\left[\cos\theta\frac{d^2u}{dr^2} - \sin\theta\frac{d}{\partial r}\left(\frac{1}{r}\frac{du}{d\theta}\right)\right] - \frac{\sin\theta}{r}$

$\left[\frac{\partial}{\partial\theta}\left(\cos\theta\frac{du}{dr}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\sin\theta\frac{du}{d\theta}\right]$

$\rightarrow \cos\theta\left[\cos\theta\frac{\partial^2u}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial^2u}{\partial r\partial\theta} - \sin\theta\frac{\partial u}{\partial\theta}(-1)r^{-2}\right] - \frac{\sin\theta}{r}$

$\left[\cos\theta\frac{\partial^2u}{\partial\theta\partial r} - \sin\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial^2u}{\partial\theta^2} - \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right]$

$\rightarrow \cos^2\theta\frac{\partial^2u}{\partial r^2} - \frac{\sin\theta\cos\theta}{r}\frac{d^2u}{\partial r\, d\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{du}{d\theta}$

$-\frac{\sin\theta\cos\theta}{r}\frac{d^2u}{d\theta\, dr} + \frac{\sin^2\theta}{r}\frac{du}{dr} +$

$\frac{\sin^2\theta}{r^2}\frac{d^2u}{d\theta^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{du}{d\theta}$.

$\rightarrow \cos^2\theta\frac{\partial^2u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2u}{\partial r\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{du}{d\theta}$

$+ \frac{\sin^2\theta}{r}\frac{du}{dr} + \frac{\sin^2\theta}{r^2}\frac{d^2u}{d\theta^2}$

$y\frac{\partial z}{\partial y} = y x^{n-1}\phi'\left(\frac{y}{x}\right)$

$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n\phi\left(\frac{y}{x}\right) = \boxed{nz}$

hence proved.

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} +$$

$$\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$\left[\cos^2\theta\frac{\partial^2 u}{\partial r^2} - \cancel{\frac{2\sin\cos}{r}}\frac{\partial^2 u}{\partial r\partial\theta} + \cancel{\frac{2\sin\theta\cos\theta}{r^2}}\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$+\left[\sin^2\theta\frac{\partial^2 u}{\partial r^2} + \cancel{\frac{2\sin\theta\cos\theta}{r}}\frac{\partial^2 u}{\partial r\partial\theta} - \cancel{\frac{2\sin\theta\cos\theta}{r^2}}\frac{\partial u}{\partial\theta}\right.$$

$$\left.+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$(\cos^2+\sin^2)\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}(\sin^2+\cos^2)\frac{\partial u}{\partial r} + \frac{1}{r^2}(\quad)$$

$\phi\left(\frac{y}{x}\right)$

$$\ldots\left(\frac{y}{x}\right) - x \quad y\,\phi'\left(\frac{y}{x}\right)$$

$$\frac{dz}{dy} = x^n\phi'\frac{y}{x}\cdot\frac{1}{x}$$

$$y\frac{\partial z}{\partial y} = y x^{n-1}\phi'\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n x^n\phi\left(\frac{y}{x}\right) = \boxed{nz}$$

hence proved.

example: $f(x,y) = \frac{x^2y + xy^2}{x^2+y^2}$ or $\frac{x^2y+x}{x^2+y^2}$

$\downarrow$ homogenous $\quad$ $\downarrow$ not homogenous.

7. **Homogenous: same**

a function $f(x,y)$ of two independent variables x and y is stb homog. function of degree n if it is of either of the following forms.

$$f(x,y) = x^n \phi\left(\frac{y}{x}\right) \qquad t^n f(x,y)$$

eg: $\frac{x^2y + xy^2}{x^2+y^2}$ $\qquad$ put $x \to tx$, $y \to ty$

$$\frac{x^3\left[\frac{y}{x} + \left(\frac{y}{x}\right)^2\right]}{x^2\left[1+\frac{y}{x}\right]}$$

$$x^1 \phi\left(\frac{y}{x}\right)$$

8. **Euler (oiler: pronunciation)**

let $z = f(x,y)$ be a homogen. function of degree n defined on $E \leq \mathbb{R}^2$ then if $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ then

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$

eg: $\frac{\tan^{-1}\frac{y}{x} + \frac{y}{x}}{\sin^{-1}\frac{y}{x}} \to n = 0$ degree.

Proof: $z = f(x,y)$ a homogenous eq^n function of degree n, $\boxed{z = x^n \phi\left(\frac{y}{x}\right)}$

$z = x^n \phi\left(\frac{y}{x}\right)$: $\quad \frac{dz}{dx} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi' \frac{y}{x} \times \left(\frac{-y}{x^2}\right)$

$$x \cdot \frac{dz}{dx} = nx^n \phi\left(\frac{y}{x}\right) - x^{n-1} y \phi'\left(\frac{y}{x}\right)$$

$$\frac{dz}{dy} = x^n \phi' \frac{y}{x} \cdot \frac{1}{x}$$

$$y\frac{\partial z}{\partial y} = y x^{n-1} \phi'\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n \phi\left(\frac{y}{x}\right) = \boxed{nz}$$

hence proved.

9. Extention of euler's theorem for Higher order P. differentiation

Let $f(x,y)$ be a real valued homog. eq^n function of degree n defined on $E \leq \mathbb{R}^2$ If the second order partial derivatis exist and are continuous then;

$$\boxed{x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z}$$

Proof: $z = f(x,y)$ a homogenous fun^n of degre n by euler's theorum then we have:

$$* \quad \boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$

* diff wrt x

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x\partial y} = n\frac{\partial z}{\partial x} \quad \text{--- (1)}$$

(1) $\times x$

$$x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x\partial y}xy = nx\frac{\partial z}{\partial x} \quad \text{--- (3)}$$

$$\boxed{x^2\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial x\partial y} = nx\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial x}} \quad \text{--- (3')}$$

* diff wrt y

$$x\frac{\partial^2 z}{\partial y\partial x} + \frac{\partial z}{\partial y} + y\frac{\partial^2 z}{\partial y^2} = n\frac{\partial z}{\partial y}. \quad \text{--- (2)}$$

(2) $\times y$

$$xy\frac{\partial^2 z}{\partial y\partial x} + y\frac{\partial z}{\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = ny\frac{\partial z}{\partial y}. \quad \text{--- (4)}$$

$$\boxed{xy\frac{\partial^2 z}{\partial y\partial x} + y^2\frac{\partial^2 z}{\partial y^2} = ny\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial y}} \quad \text{--- (4')}$$

.adding 3 & 4;

$$LHS_{3'+4'} = RHS_{3'+4'}$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = nx\frac{\partial z}{\partial x} + ny\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$$= (n-1)x\frac{\partial z}{\partial x} + (n-1)y\frac{\partial z}{\partial y}$$

$$= (n-1)\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right]$$

hence proved

$(n-1) \times nz$

Euler's application:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$$

Q1. $\log u = \dfrac{x^3+y^3}{3x+4y}$ ↪ degree $= 2 \rightarrow \dfrac{x^{\cancel{3}\,2}\left(1+\left(\frac{y}{x}\right)^3\right)}{\cancel{x}\left(3+4\left(\frac{y}{x}\right)\right)}$

Sol^n

Let $z = \log u$

$$x\frac{du}{dx} + y\frac{\partial u}{\partial y} = 2z = nz$$

$$x\frac{d\log u}{dx} + y\frac{\partial \log u}{\partial y} = 2\log u$$

$$\frac{x}{u}\frac{du}{dx} + \frac{y}{u}\frac{du}{dy} = 2\log u$$

$$\rightarrow \boxed{x\frac{du}{dx} + y\frac{du}{dy} = 2\log u \times u}$$

Q2. $u = \tan^{-1}\left(\dfrac{x^3+y^3}{ax+yb}\right) \rightarrow \dfrac{x^3+y^3}{ax+yb} = \tan u$

degree $=2$.
by putting $x \to tx$
$y \to ty$.
$\rightarrow t\ \phi\ f(x)$.

$$x\frac{du}{dx} + y\frac{du}{dy} = 2\sin 2u$$

Sol^n Let $z = \tan u$

$$x\frac{\partial \tan u}{dx} + y\frac{\partial \tan u}{\partial y} = 2\tan u$$

$$x\sec^2 u\frac{du}{dx} + y\sec^2 u\frac{du}{dy} = 2\tan u$$

$$x\frac{du}{dy} + y\frac{du}{dy} = \frac{2\tan u}{\sec^2 u} = 2\frac{\sin u\cos^2 u}{\cos u}$$

$$\rightarrow = \boxed{\sin 2u}$$

Q3. $u = \log_e\left(\dfrac{x^4+y^4}{x+y}\right)$

$$x\frac{du}{dx} + y\frac{du}{dy} = ?$$

Sol^n.

$e^u = \cancel{\ln}\left(\dfrac{x^4+y^4}{x+y}\right) \rightarrow$ degree $=3=n$

$$x\frac{de^u}{dx} + y\frac{de^{u}}{dy} = 3e^u$$

$$x e^u \frac{du}{dx} + y e^u \frac{du}{dy} = 3e^u$$

$$\rightarrow \boxed{x\frac{du}{dx} + y\frac{du}{dy} = 3}$$

Q4.

$$u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$$

degree : $\dfrac{tx+2ty+3tz}{\sqrt{t^8x^8+t^8y^8+t^8z^8}}$

$\dfrac{t}{t^4}[\ ] = t^{-3}$

[degree $= -3$]

$$x\frac{d\sin u}{dx} + y\frac{\sin u}{dy} = -3\sin u = nz$$

$$x\frac{du}{dx}\times\cos u + y\frac{du}{dy}\times\cos u = -3\sin u$$

$$\rightarrow \boxed{x\frac{du}{dx} + y\frac{du}{dy} = -3\tan u}$$

Q5.

$$u = \tan^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$$

$$\tan u = \frac{x^3+y^3}{\sqrt{x}+\sqrt{y}} \rightarrow \frac{(tx)^3 + (ty)^3}{\sqrt{tx}+\sqrt{ty}} = \left[3-\frac{1}{2} = \frac{5}{2}\right]$$

degree.

$$x\frac{d\tan u}{dx} + y\frac{d\tan u}{dy} = \frac{5}{2}\tan u$$

$$x\sec^2 u\frac{du}{dx} + y\sec^2\frac{du}{dy} = \frac{5}{2}\tan u$$

$$\rightarrow \boxed{\begin{aligned} x\frac{du}{dx} + y\frac{du}{dy} &= \frac{5}{2}\sin\cos \\ &= \frac{5}{4}\sin 2u \end{aligned}}$$

: $\dfrac{tx + 2ty + 3tz}{\sqrt{t^2x^2 + t^2y^2 + t^2z^2}}$

$\dfrac{t}{t^4}[\ \ ] = t^{-3}$

[degree $= -3$]

$= [3 - \frac{1}{2} = \frac{5}{2}]$ degree.

verify Euler's theorem : Q$_1$ $x^n \log\left(\frac{y}{x}\right) \longrightarrow$ degree $= n$

Q$_2$ $z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) \longrightarrow$ degree $= 0$.

1 $\Rightarrow x\dfrac{du}{dx} + y\dfrac{du}{dy} = nz$

$\Rightarrow \dfrac{du}{dx} = nx^{n-1}\log\left(\frac{y}{x}\right) + \dfrac{1}{y/x} \times -\dfrac{y}{x^2} \times x^n$

$= nx^{n-1}\log\left(\frac{y}{x}\right) + \dfrac{-y x^n}{xy}$

$\dfrac{du}{dx}$ $= \boxed{nx^{n-1}\log\left(\frac{y}{x}\right) - \dfrac{\cancel{y}x^{n-1}}{\cancel{y}}}$

$\dfrac{du}{dy} \Rightarrow \dfrac{du}{dy} = x^n \dfrac{1}{y/\cancel{x}} \times \dfrac{1}{\cancel{x}} = \boxed{\dfrac{x^n}{y}}$

$\Rightarrow$

RHS $\longrightarrow \dfrac{du}{dx} \times x + y\dfrac{du}{dy} = nz = \boxed{n \times x^n \log\left(\frac{y}{x}\right)}$

LHS

$x\dfrac{du}{dx} + y\dfrac{du}{dy} \longrightarrow$

$\Rightarrow \cancel{x} \times n\dfrac{x^n}{\cancel{x}}\log\left(\frac{y}{x}\right) - \dfrac{x^{n-\cancel{1}}}{y} \times \cancel{x} + x^n \Rightarrow nx^n\log\left(\frac{y}{x}\right)$

$\Rightarrow nx^n\log\left(\frac{y}{x}\right) - \dfrac{x^n}{\cancel{y}} + x^n$.

$\Rightarrow \boxed{nx^n\log\left(\frac{y}{x}\right)}$

Hence proved.

LHS = RHS.

Q. $\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) = 0$.

LHS : $\frac{du}{dx} \Rightarrow \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \times \frac{1}{y} + \frac{-1}{1+\left(\frac{y}{x}\right)^2} \times \frac{y}{x^2}$

$$\frac{\cancel{y}}{\cancel{y}\sqrt{y^2-x^2}} - \frac{\cancel{x^2}}{x^2+y^2} \times \frac{y}{\cancel{x^2}}$$

$x \frac{du}{dx} \Rightarrow \boxed{\frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}} \times x$.

$$\rightarrow \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y} : x\frac{du}{dx}$$

$\frac{du}{dy} \Rightarrow -\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \times \frac{x}{y^2} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x}$.

$$\Rightarrow \frac{-x\cancel{y}}{y^{\cancel{2}}\sqrt{y^2-x^2}} + \frac{\cancel{x^2}}{\cancel{x}[x^2+y^2]}$$

$$\Rightarrow \boxed{\frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}} \times y$$

$$= \frac{-xy}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} : y\frac{du}{dy}.$$

Since $\underbrace{x\frac{du}{dx} + y\frac{du}{dy}}_{\text{LHS}} = \underbrace{0}_{\text{RHS}}$

03 as n=0 so LHS = RHS

proved.

10. Composite function

→ If $u = f(x, y)$, where $x = \phi(t)$ $y = \psi(t)$ then u is called a composite function of (the single variable) t and we can find $\frac{du}{dt}$

→ If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$ then z is called a composite function of two variable u and v so that we can find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

differentiation of composite function

If u is component function of t defined by relations $u = f(x, y)$; $x = \phi(t)$ and $\psi(t) = y$ then

$$\boxed{\frac{du}{\partial t} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \times \frac{dy}{\partial t}}$$

Cor (1) theorum: corollary.

If, $u = f(x, y, z)$ and x, y, z are function of t then 'u' is composite function of t and

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{\delta t} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{dt}}$$

cor (2) If $z = f(x, y)$ and x, y are function of u and v then

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{dx}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{dy}{\delta u}}$$

and

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{dx}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{dy}{\delta v}}$$

Ques $z = xy$ where $x = \cos t$ $y = \sin t$ at $\boxed{t = \pi/2}$ derivates

$\cancel{\frac{dz}{dy}} \Rightarrow \cancel{\frac{dz}{\cdot}}$ ~~-tant~~ $\Rightarrow$ ~~-1~~ ~~at $\frac{\pi}{2}$~~

Soln 1. $\frac{dx}{dt} = -\sin t$

$\frac{dy}{dt} = \cos t.$

$$\boxed{\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \times \frac{dy}{dt}}.$$

$$= \frac{d(xy)}{dx} \frac{d(\cos t)}{dt} + \frac{d(xy)}{dy} \cdot \frac{d \sin t}{dt}$$

$$= -y \sin t + x \cos t.$$

$$= -\sin^2 t + \cos^2 t \quad \text{at } \frac{\pi}{2} = t.$$

$$= \cos 2t \quad \text{at } \pi/2 = t$$

$$= \cos \pi \ / \ \cos 180^\circ = -1$$

Answer.

Quest.

$\frac{dw}{dt}$ if $w = xy + z$ $x = \cos t$

$y = \sin t$

$z = t.$

$$\rightarrow \boxed{\frac{d(xy+z)}{dx} \times \frac{dx}{dt} + \frac{d(xy+z)}{dy} \frac{dy}{dt} + \frac{d(xy+z)}{dz} \frac{dz}{dt}}$$

$$\rightarrow y \times -\sin t + x \times \cos t + 1 \times 1$$

$$\rightarrow -\sin^2 t + \cos^2 t + 1$$

$$\rightarrow \boxed{\cos 2t + 1}$$

Answer

Q. $z = f(x, y)$, $x = e^u(\cos v)$ & $y = e^u \sin v$ prove that

$$x\frac{dz}{dv} + \frac{dz}{du} xy = e^{2u} \times \frac{dz}{dy}.$$

$$\frac{dz}{du} = \frac{dz}{dx} \times \frac{dx}{du} + \frac{dz}{dy} \times \frac{dy}{du}$$

$$\frac{dz}{dv} = \frac{dz}{dx} \times \frac{dx}{dv} + \frac{dz}{dy} \times \frac{dy}{dv}.$$

$$x\frac{dz}{dv} + y\frac{dz}{du} = e^u \frac{dz}{dx}[e^u \sin v \cos v - e^u \cos v \sin v) +$$

$$e^u \frac{dz}{dy}[e^u \sin v \sin v + e^u \cos v \cos v)$$ we get rearranging (1)

$$= e^{2u}\frac{dz}{dy}[\sin^2 v + \cos^2 v]$$

$x = e^u \cos v$

$y = e^u \sin v$ $\rightarrow$ $= e^{2u}\frac{dz}{dy}[1]$ Answer.

$\frac{dx}{dv} = e^u(-\sin v)$ $\quad \frac{dx}{du} = e^u(\cos v)$

$\frac{dy}{dv} = e^u(\cos v)$ $\quad \frac{dy}{du} = e^u(\cos v)$

$$x\frac{dz}{dv} + y\frac{dz}{du} \Rightarrow$$

(1) — $e^u \sin v\left[\frac{dz}{dx} \times e^u \cos v + \frac{dz}{dy} \times e^u \cos v\right]$

$e^u \cos v\left[\frac{dz}{dx} \times e^u(-\sin v) + \frac{dz}{dy} \times e^u \cos v\right]$