Fourier Series

A series expansion in terms of a trignometric function $\cos nx$ & $\sin nx$ is called fourier series

∴ Let $f(x)$ be periodic function of period '$2L$' defined as $[-l, l]$ & can be expanded in orthogonal series in terms of trignometric function then fourier series and $f(x)$ is.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

where

Euler's formula

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Orthogonal function :- A set of function $\{\phi_i(x)\}$ is said to be orthogonal on an interval $[a,b]$ if

$$\int_a^b \phi_i(x)\,dx \cdot \phi_j(x)\,dx = 0, \quad \boxed{i \neq j}$$

Orthonormal function

a set of function $\{\phi_i(x)\}$ is said to be orthonormal if they are orthogonal on $[a,b]$ and have normal is 1 ie

$\|\phi_i(x)\| = 1$ for all (i)

where Normal of $\|\phi_i(x)\| = 1$ for all i's

norm of $\phi_{(i)}(x)$: $\boxed{\|\phi_i(x)\| = \sqrt{\int_a^b \phi_{ix}^2 \, \phi_{ix}^2 \, dx}}$

Question : $f(x)$

$$f(x+T) = f(x)$$

find the fourier series of function :

$$f(x) = x$$

$$-\pi \leq x \leq \pi$$

$$f(x+2\pi) = f(x)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx.$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx.$$

$$b_n = \frac{1}{l} \int_{l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0$$

$$\rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0$$

odd function

even function

$$\rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx.$$

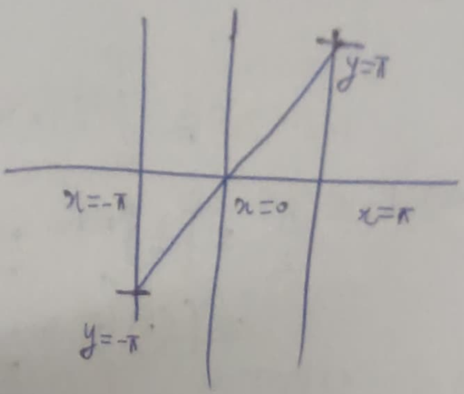
$$\frac{2}{\pi}\left[-x\left(\frac{\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\frac{-\pi}{n}\ \cancel{\sin n\pi}\ \cos n\pi\right]$$

$$= \frac{2}{\pi}\ \boxed{\frac{2}{n}(-1)^{n+1}}$$

$$x = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$$

$$\boxed{x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)}$$



$f(x) = x$

$-\pi \le x \le \pi$

$f(x) = f(x+2\pi)$

$-\pi < x+2\pi \le \pi$

$-3\pi < x \le \pi$

Question: $f(x) = \begin{cases} \pi+x, & \pi < x < 0 \\ 0 & 0 \le x < \pi \end{cases}$.

$f(x+2\pi) = f(x)$

$\frac{1}{\pi}\int_0^{\pi} (\pi+x)\,dx.$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi+x)\,dx = \frac{\pi}{2}$

$2\left[\frac{1}{\pi}\int_0^{\pi}\pi\,dx + \int_0^{\pi}\frac{x\,dx}{\pi}\right]$

$2\left[(\pi-0) + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}\right]$

$\frac{\pi}{2}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{l}\,dx$

$=$

$a_n. \quad \frac{1}{\pi}\int_{-\pi}^{0} (\pi+x)\cos nx\,dx = \frac{1}{\pi n^2}\left[1-(-1)^n\right].$

$= \begin{cases} 0 & \text{for } n \text{ is even} \\ \frac{2}{\pi n^2} & n \text{ is odd} \end{cases}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (\pi+x)\sin nx\,dx$

$= \frac{-1}{n}$

$$\pi + x = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1}{\pi n^2}(1-(-1)^n)\cos nx \right.$$

$$\frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} \cdots\right] - \left[\frac{\sin x}{1} + \right.$$

Question

$2+x \quad -2 \le x \le 0$ $\longrightarrow$ neither even/odd

$2-x \quad 0 < x \le 2$ where $T=4$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \Big]$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx.$$

$$a_0 = \frac{1}{2}\int_{-2}^{0}(2+x)\,dx + \frac{1}{2}\int_{0}^{2}(2-x)\,dx$$

$$\frac{1}{2}\left[\left[2x + \frac{x^2}{2}\right]_{-2}^{0} + \left[2x - \frac{x^2}{2}\right]_{0}^{2}\right] = 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx.$$

$$a_n = \frac{1}{2}\int_{-2}^{0}(2+x)\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2}(2-x)\cos\frac{n\pi x}{2}\,dx$$

$$\frac{4}{n^2\pi^2}\left[1-(-1)^n\right] \text{ or } \begin{Bmatrix} 0 & \text{even} \\ \frac{8}{n^2\pi} & \text{odd} \end{Bmatrix}$$

$b_n = 0$

$$f(x) = 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)\frac{\pi x}{2}$$

Fourier Series of even & odd

$f(x) = x$

$f(x) = x^2$

$f(x) = \sin x / \cos x$

$f(x)$ even : $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$ —— (1).

$$a_0 = \frac{2}{l} \int_0^l f(x)\, dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

$f(x)$ odd.

$f(x) : \sum_{n=1}^{n} b_n \sin \frac{n\pi x}{l}$, $b_n = \frac{2}{l}\int_0^l \sin \frac{n\pi x}{l}$