BETA function & Gamma function

Gamma function

① $\int_0^\infty x^{n-1} \cdot e^{-x} dx = \Gamma n$

② $\int_0^\infty x^{n-1} \cdot e^{-ax} dx = \frac{\Gamma n}{a^n}$

Q. Prove that : $\boxed{\Gamma n+1 = n\Gamma n}$

Soln:

$\because \Gamma n+1 = \int_0^\infty x^n \cdot e^{-x} dx$

$\therefore \Gamma n+1 = \left[x^n \int e^{-x} dx - \int \left(\frac{d}{dx}(x^n) \cdot \int e^{-x} dx\right) dx\right]_0^\infty$

$= (-e^{-x} \cdot x^n)_0^\infty - \int_0^\infty n x^{n-1} \cdot (-e^{-x}) dx$

$= (0-0) + n \int_0^\infty x^{n-1} e^{-x} dx$

$= n\Gamma n$

Q. Prove that : $\int_0^1 \left(\log \frac{1}{y}\right)^{n-1} \cdot dy = \Gamma n$

Soln: Let $\log\left(\frac{1}{y}\right) = x$.

Then, $y = e^{-x}$

$dy = -e^{-x} dx$

$\therefore \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} \cdot dy = \int_\infty^0 (x)^{n-1} \cdot (-e^{-x}) dx$

$= \int_0^\infty x^{n-1} \cdot e^{-x} dx$

$= \Gamma n$

Q. Prove that: $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$

Soln: Let $x^2 = t$

$2x\,dx = dt$

$dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

$$\therefore \int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-t}\left(\frac{dt}{2\sqrt{t}}\right)$$

$$= \frac{1}{2}\int_0^\infty t^{-1/2}\cdot e^{-t}dt$$

$$= \frac{1}{2}\Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{2} \qquad \left[\because \Gamma\tfrac{1}{2} = \sqrt{\pi}\right]$$

Q. ~~Prove that~~ find: $\int_0^\infty \sqrt{x}\cdot e^{-3\sqrt{x}}dx$

Soln: Let $3\sqrt{x} = t$.

Then, $9x = t^2$

$dx = \frac{2t\,dt}{9}$

$$\therefore \int_0^\infty \sqrt{x}\cdot e^{-3\sqrt{x}}dx = \int_0^\infty \left(\frac{t}{3}\right)\cdot e^{-t}\cdot\left(\frac{2t\,dt}{9}\right)$$

$$= \frac{2}{27}\int_0^\infty t^2\cdot e^{-t}\cdot dt$$

$$= \frac{2}{27}\Gamma 3 = \frac{2}{27}\Gamma(2+1) = \frac{2}{27}(2)\cdot\Gamma 2$$

$$= \frac{4}{27}\cdot\Gamma(1+1)$$

$$= \frac{4}{27}(1)(\Gamma 1) = \frac{4}{27}$$

Q. Prove that : $\displaystyle\int_0^\infty x^n \cdot e^{-k^2x^2}\,dx = \frac{1}{2k^{n+1}}\Gamma\frac{n+1}{2}$

Soln: Let $k^2x^2 = t \Rightarrow x^2 = \frac{t}{k^2} \Rightarrow x = \frac{\sqrt{t}}{k}$

Then,

$$2k^2x\,dx = dt$$

$$\Rightarrow dx = \frac{dt}{2k^2x} = \frac{dt}{2k^2\left(\frac{\sqrt{t}}{k}\right)} = \frac{dt}{2\sqrt{t}\cdot k}$$

$$\therefore \int_0^\infty x^n \cdot e^{-k^2x^2}\,dx = \int_0^\infty \left(\frac{\sqrt{t}}{k}\right)^n \cdot e^{-t} \cdot \left(\frac{dt}{2\sqrt{t}\cdot k}\right)$$

$$= \frac{1}{2k^{n+1}}\int_0^\infty t^{n/2-1/2} \cdot e^{-t}\,dt$$

$$= \frac{1}{2k^{n+1}}\int_0^\infty t^{\frac{n-1}{2}} \cdot e^{-t}\,dt$$

$$= \frac{1}{2k^{n+1}} \cdot \Gamma\left(\frac{n-1}{2}+1\right) = \boxed{\frac{1}{2k^{n+1}}\Gamma\frac{n+1}{2}}$$

Beta function

$$\beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1}\,dx$$, where, m, n are integers.

Some other forms of β-function :

① $\displaystyle\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

② $\displaystyle\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\,d\theta$

Relation b/w Gamma & β-function :

$$\boxed{\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}}$$

Proof:
$$\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} \cdot x^{n-1} dx$$

$$\Rightarrow \quad \Gamma n = \int_0^\infty e^{-zx} \cdot x^{n-1} \cdot z^n dx$$

multiply both sides by 'e^{-z}' and 'z^{m-1}' on both sides of Eq^n

$$\Gamma n \cdot e^{-z} \cdot z^{m-1} = \int_0^\infty e^{-zx} \cdot e^{-z} \cdot z^{m-1} \cdot z^n \cdot x^{n-1} \cdot dx$$

integrate w.r.t. 'z' on both sides b/w 0 to ∞,

$$\Gamma n \cdot \int_0^\infty z^{m-1} \cdot e^{-z} dz = \int_0^\infty x^{n-1} dx \cdot \int_0^\infty e^{-z(x+1)} \cdot z^{m+n-1} \cdot dz$$

$$\Gamma n \cdot \Gamma m = \int_0^\infty x^{n-1} dx \left(\frac{\Gamma m+n}{(x+1)^{m+n}} \right)$$

$$\frac{\Gamma m \cdot \Gamma n}{\Gamma m+n} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\Rightarrow \left[\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n} \right]$$

$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Proof: Let $x = \frac{1}{1+y}$,

Then, $dx = \frac{-1}{(1+y)^2} dy$

$$\therefore \beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1} \cdot \left(1 - \frac{1}{1+y}\right)^{n-1} \cdot \left(\frac{-1}{(1+y)^2}\right) dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2} dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Q. Evaluate : $\int_0^\infty \frac{x^2(1+x^4)}{(1+x)^{10}}\,dx$

Soln:

$$\int_0^\infty \frac{x^2(1+x^4)}{(1+x)^{10}}\,dx$$

$$= \int_0^\infty \frac{x^2}{(1+x)^{10}}\,dx + \int_0^\infty \frac{x^6}{(1+x)^{10}}\,dx$$

$$= \beta(3,7) + \beta(7,3)$$

$$= 2\beta(3,7) = 2\cdot\left(\frac{\Gamma 3\cdot\Gamma 7}{\Gamma 10}\right)$$

Q. Evaluate : $\int_0^\infty \frac{x^2}{(1+x^2)^{7/2}}\,dx$

Soln:

Let $x^2 = t$,

Then, $2x\,dx = dt$

$$dx = \frac{dt}{2\sqrt{t}}$$

$$\therefore \int_0^\infty \frac{x^2}{(1+x^2)^{7/2}}\,dx = \int_0^\infty \frac{t}{(1+t)^{7/2}}\cdot\left(\frac{dt}{2\sqrt{t}}\right)$$

$$= \frac{1}{2}\int_0^\infty \frac{t^{1/2}}{(1+t)^{7/2}}\,dt$$

$$= \frac{1}{2}\cdot\beta\left(\frac{3}{2},2\right)$$

$$= \frac{1}{2}\cdot\frac{\Gamma 3/2\cdot\Gamma 2}{\Gamma 7/2}$$

$\boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\,d\theta}$, Hence prove, $\Gamma\frac{1}{2} = \sqrt{\pi}$

Proof: We know that, $\beta(m,n) = \int_0^1 x^{m-1}\cdot(1-x)^{n-1}\,dx$

Let $x = \sin^2\theta$,

Then, $dx = 2\sin\theta\cos\theta\,d\theta$

$$\therefore \beta(m,n) = \int_0^1 x^{m-1}\cdot(1-x)^{n-1}dx$$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\cdot(2\sin\theta\cos\theta)d\theta$$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-2}\theta\cdot\cos^{2n-2}\theta\cdot\sin\theta\cos\theta d\theta$$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cdot\cos^{2n-1}\cdot d\theta$$

Let $2m-1 = p$ & $2n-1 = q$.

Then,

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta\cdot\cos^q\theta\cdot d\theta = \frac{\Gamma\frac{p+1}{2}\cdot\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q}{2}+(1)}$$

Put $p = q = 0$,

$$\therefore \beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^0\theta\cdot\cos^0\theta\cdot d\theta = \frac{\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}}{\Gamma 1}$$

$$\Rightarrow 2\int_0^{\pi/2} d\theta = \left(\Gamma\frac{1}{2}\right)^2$$

$$\Rightarrow \pi = \left(\Gamma\frac{1}{2}\right)^2$$

$$\Rightarrow \boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$$

Q. Evaluate : $\int_0^{\pi/2} \sin^5\theta d\theta$ (OR) $\int_0^{\pi/2} \cos^5\theta d\theta$

Soln: We know that,

$$\int_0^{\pi/2} \sin^p\theta\cdot\cos^q\theta d\theta = \frac{1}{2}\cdot\frac{\Gamma\frac{p+1}{2}\cdot\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$$

$$\therefore \int_0^{\pi/2} \sin^5\theta d\theta = \int_0^{\pi/2} \sin^5\theta\cdot\cos^0\theta d\theta = \frac{1}{2}\cdot\frac{\Gamma\frac{5+1}{2}\cdot\Gamma\frac{0+1}{2}}{\Gamma\frac{5+0+2}{2}} = \frac{1}{2}\cdot\frac{\Gamma 3\cdot\Gamma\frac{1}{2}}{\Gamma\frac{7}{2}}$$

$(p=5, q=0)$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma 3}{\Gamma\frac{7}{2}}$$ Ans.

Q. Evaluate : $\displaystyle\int_0^{\pi/6} \cos^4 3\theta \cdot \sin^2 6\theta \, d\theta$

Soln: Let $3\theta = t$.
Then, $3d\theta = dt \Rightarrow d\theta = \dfrac{dt}{3}$

$$\therefore \int_0^{\pi/6} \cos^4 3\theta \cdot \sin^2 6\theta \, d\theta = \frac{1}{3}\int_0^{\pi/2} \cos^4 t \cdot \sin^2 2t \, (dt)$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^4 t \cdot (2\sin t \cos t)^2 dt$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2 t \cdot \cos^6 t \, d\theta \qquad (p = 2, q = 6)$$

$$= \frac{4}{3} \times \frac{1}{2} \cdot \frac{\Gamma\frac{2+1}{2} \cdot \Gamma\frac{6+1}{2}}{\Gamma\frac{2+6+2}{2}}$$

$$= \frac{2}{3} \cdot \frac{\Gamma\frac{3}{2} \cdot \Gamma\frac{7}{2}}{\Gamma 5} \quad \underline{\text{Ans.}}$$

$$\Gamma n \cdot \Gamma 1-n = \frac{\pi}{\sin(n\pi)}$$

Q. Evaluate : $\displaystyle\int_0^{\infty} \frac{dx}{1+x^4}$.

Soln: Let $x^2 = \tan\theta$
Then, $2x\,dx = \sec^2\theta \, d\theta$

$$dx = \frac{\sec^2\theta \, d\theta}{2x}$$

$$dx = \frac{\sec^2\theta}{2\sqrt{\tan\theta}} d\theta$$

$$\therefore \int_0^{\infty} \frac{dx}{1+x^4} = \int_0^{\pi/2} \frac{1}{\cancel{(1+\tan^2\theta)}} \cdot \left(\frac{\cancel{\sec^2\theta}}{2\sqrt{\tan\theta}} d\theta\right)$$

$$= \frac{1}{2}\int_0^{\pi/2} (\tan\theta)^{-\frac{1}{2}} d\theta = \frac{1}{2}\int_0^{\pi/2} (\sin\theta)^{-\frac{1}{2}} \cdot (\cos\theta)^{\frac{1}{2}} d\theta \quad \left(p = -\frac{1}{2}, q = \frac{1}{2}\right)$$

$$= \frac{1}{2} \times \left[\frac{1}{2} \cdot \frac{\Gamma\frac{-\frac{1}{2}+1}{2} \cdot \Gamma\frac{\frac{1}{2}+1}{2}}{\Gamma\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}}\right] = \left(\Gamma\frac{1}{4} \cdot \Gamma\frac{3}{4}\right) \times \frac{1}{4}$$

$\because \quad \Gamma n \cdot \Gamma 1-n = \dfrac{\pi}{\sin(n\pi)}$

$\therefore$ for $n = \frac{1}{4}$,

$$\Gamma \tfrac{1}{4} \cdot \Gamma \tfrac{3}{4} = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}\cdot\pi$$

Hence, $\displaystyle\int_0^\infty \frac{dx}{1+x^4} = \frac{1}{4}\left(\Gamma \tfrac{1}{4} \cdot \Gamma \tfrac{3}{4}\right)$

$$= \frac{1}{4}\left(\sqrt{2}\cdot\pi\right) = \frac{\sqrt{2}\,\pi}{4}$$

Duplication formula (Legendre's duplication formula)

$$\boxed{\Gamma m \cdot \Gamma m+\tfrac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}} \cdot \Gamma 2m} \qquad (m > 0)$$

Proof: We know that, $\displaystyle 2\cdot\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$

Put $2n-1 = 0$.

$$\therefore \quad 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\Gamma m \cdot \Gamma \frac{1}{2}}{\Gamma m+\frac{1}{2}}$$

$$2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\Gamma m \,\sqrt{\pi}}{\Gamma m+\frac{1}{2}}$$

Now, Put $m = n$.

$$\therefore \quad 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2m-1}\theta \, d\theta = \frac{\Gamma m \cdot \Gamma m}{\Gamma 2m}$$

$$\frac{2}{2^{2m-1}}\int_0^{\pi/2} (2\sin\theta\cos\theta)^{2m-1} d\theta = \frac{\Gamma m \cdot \Gamma m}{\Gamma 2m}$$

$$\frac{2}{2^{2m-1}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1} \cdot d\theta = \frac{(\Gamma m)^2}{\Gamma 2m}$$

Let $2\theta = \varphi$.

Then, $d\theta = \dfrac{d\varphi}{2}$.

$$\frac{\cancel{2}}{2^{2m-1}} \int_0^{\pi} (\sin\phi)^{2m-1} \left(\frac{d\phi}{\cancel{2}}\right) = \frac{(\Gamma m)^2}{\Gamma 2m}$$

$$\frac{2}{2^{2m-1}} \left\{ \int_0^{\pi/2} (\sin\phi)^{2m-1} \cdot d\phi \right\} = \frac{(\Gamma m)^2}{\Gamma 2m}$$

$$\frac{\cancel{2}}{2^{2m-1}} \left\{ \frac{1}{\cancel{2}} \cdot \frac{\cancel{\Gamma m} \cdot \sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} \right\} = \frac{(\Gamma m)^{\cancel{2}}}{\Gamma 2m} \qquad \left[\because 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{(\Gamma m)(\sqrt{\pi})}{\Gamma\left(m+\frac{1}{2}\right)} \right]$$

$$\Rightarrow \quad \frac{\Gamma 2m \cdot \sqrt{\pi}}{2^{2m-1}} = \Gamma m \cdot \Gamma\left(m+\frac{1}{2}\right)$$

$$\Rightarrow \quad \boxed{\Gamma m \cdot \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \cdot \Gamma 2m}$$

MULTIPLE INTEGRALS (UNIT – 03)

→ DOUBLE INTEGRATION

Q. find $\displaystyle\int_{x=0}^{1}\int_{y=0}^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$.

Soln:

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1+x^2}} \frac{dx}{(\sqrt{1+x^2})^2+y^2}\,dy$$

$$= \int_{x=0}^{1}\left(\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right)_0^{\sqrt{1+x^2}} dx$$

$$= \int_{x=0}^{1}\left(\frac{\pi}{4\sqrt{1+x^2}}\right)dx = \frac{\pi}{4}\int_{x=0}^{1}\frac{dx}{\sqrt{1+x^2}} = \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1$$

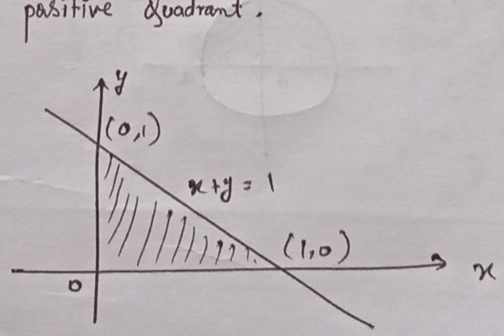
$$= \frac{\pi}{4}\log\left(1+\sqrt{2}\right)$$

Q. Evaluate : $\iint xy\,dx\,dy$, where, the following region of integration is

① $x+y \leq 1$ in positive Quadrant.

② $x^2+y^2=a^2$ in positive Quadrant.

Soln: ① $x+y \leq 1$



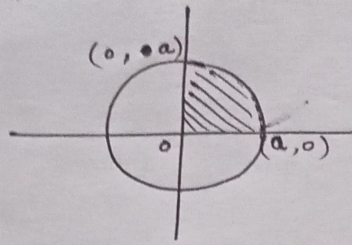
$$\int_{x=0}^{1}\int_{y=0}^{1-x} xy\,dx\,dy = \int_{x=0}^{1} x\left(\frac{y^2}{2}\right)_0^{1-x} dx = \int_{x=0}^{1} x\left(\frac{(1-x)^2}{2}\right)dx$$

$$= \frac{1}{2}\int_{x=0}^{1} x(1-x)^2\,dx$$

$$= \frac{1}{2}\int_0^1 x(1+x^2-2x)\,dx = \frac{1}{2}\int_0^1 (x+x^3-2x^2)\,dx$$

$$= \frac{1}{24}$$

② $x^2+y^2=a^2$



$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dx\,dy = \int_{x=0}^{a} x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{a} x(a^2-x^2)\,dx = \frac{1}{2}\int_{x=0}^{a}(a^2x-x^3)\,dx$$

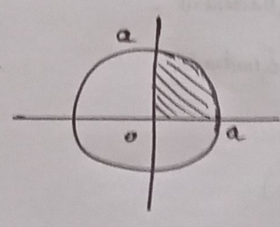
$$= \frac{1}{2}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left(\frac{a^4}{2}-\frac{a^4}{4}\right) = \frac{1}{2}\left(\frac{a^4}{4}\right) = \frac{a^4}{8}$$

→ **Area By Double Integration**

$$\boxed{A = \iint dx\,dy}$$

E.g. find area of $x^2+y^2=a^2$.



$$A = 4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} dx\,dy$$

$$= 4\int_{x=0}^{a}\left(\sqrt{a^2-x^2}\right)dx$$

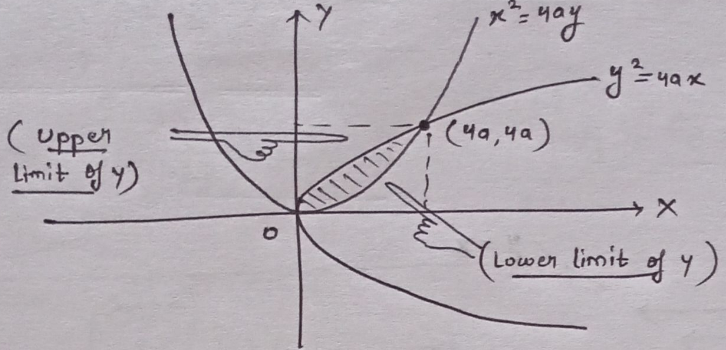
$$= 4\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4\left[\left(0+\frac{a^2}{2}\sin^{-1}(1)\right)-(0)\right]$$

$$= 4\left(\frac{a^2}{2}\cdot\frac{\pi}{2}\right) = \boxed{\pi a^2}$$

Q. find Area of region by double integration $y^2 = 4ax$ & $x^2 = 4ay$.

Soln:



$$\because \quad A = \iint dx\,dy$$

$$\therefore \quad A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} dx\,dy$$

$$A = \int_{x=0}^{4a} \left[y \right]_{\frac{x^2}{4a}}^{2\sqrt{ax}} \cdot dx = \int_{x=0}^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a} \right) dx$$

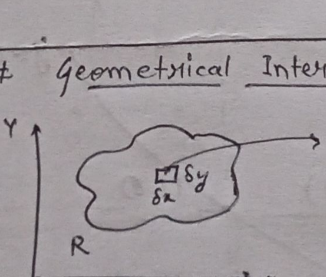
$$= \frac{2}{3} \times 2\sqrt{a} \left[x^{3/2} \right]_0^{4a} - \frac{1}{4a} \times \frac{1}{3} \left[x^3 \right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a} \left((4a)^{3/2} \right) - \frac{1}{12a} (4a)^3$$

$$= \frac{4}{3} \times (2)^3 \cdot a^2 - \frac{1}{3\cancel{12}a} \times (\cancel{64}^{16} a^3)$$

$$= \frac{32}{3} a^2 - \frac{16a^2}{3} = \frac{32a^2 - 16a^2}{3} = \frac{16}{3} a^2$$

Geometrical Interpretation



n sub-divisions

$\delta r = \delta x\, \delta y$

$A = xy$

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\, dx\, dy = \lim_{\substack{n \to \infty \\ \delta r \to 0}} \sum_{r=1}^{n} f(x_r, y_r) \quad \left\{ \begin{matrix} x_1 \leq x \leq x_2 \\ y_1 \leq y \leq y_2 \end{matrix} \right\}$$

Suppose $f(x,y)$ be a function of variable x and y bounded by the closed region R in XY Plane, then the Double Integration defined by

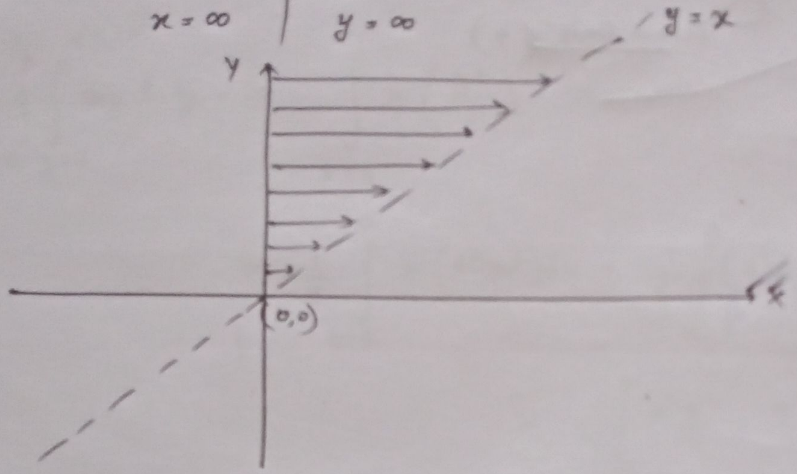
→ Change of Order of Integration

Q. $\int_{0}^{\infty}\int_{x}^{\infty} \frac{e^{-y}}{y}\,dx\,dy = ??$

Soln:

$x = 0 \;\big|\; y = x$

$x = \infty \;\big|\; y = \infty$



$$\int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\,dx\,dy = \int_{y=0}^{\infty} \frac{e^{-y}}{y}(y-0)\,dy$$

$$= \int_{y=0}^{\infty} e^{-y}\,dy = -\left[e^{-y}\right]_{0}^{\infty}$$

$$= -\left[\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right]$$

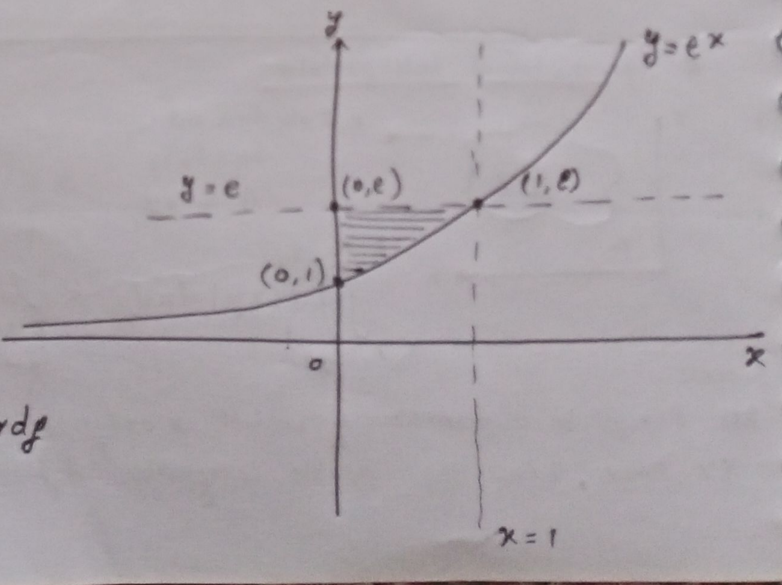
$$= 1$$

Q. $\int_{0}^{1}\int_{e^x}^{e} \left(\frac{1}{\log y}\right) dx\,dy = ??$

Soln:

$x = 0 \;\big|\; y = e^x$

$x = 1 \;\big|\; y = e$



Now,

$$\int_{0}^{1}\int_{e^x}^{e} \left(\frac{1}{\log y}\right) dx\,dy = \int_{y=1}^{e}\int_{x=0}^{\log y} \left(\frac{1}{\log y}\right) dx\,dy$$

$$= \int_{y=1}^{e} \left(\frac{1}{\log y}\right) dy \,(x)_0^{\log y}$$

$$= \int_{y=1}^{e} \frac{dy}{(\log y)} \times (\log y) = \int_{y=1}^{e} dy = \boxed{e-1}$$

Q. $\displaystyle\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy = ??$

Soln:

$x = 0$ | $y = x\tan\alpha$

$x = a\cos\alpha$ | $y = \sqrt{a^2-x^2} \Rightarrow x^2+y^2 = a^2$

$x^2+y^2 = a^2$

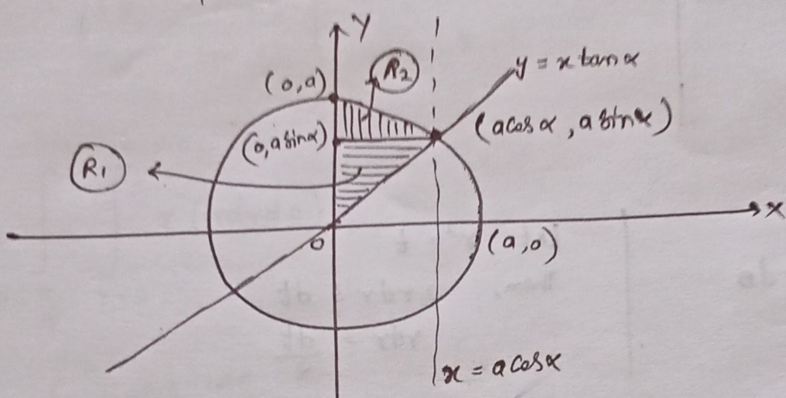
$x^2 + x^2\tan^2\alpha = a^2$

$x^2(1+\tan^2\alpha) = a^2$

$x^2 = \dfrac{a^2}{\sec^2\alpha}$

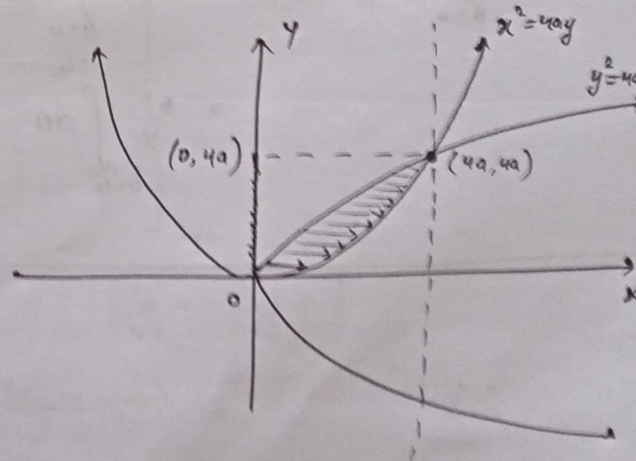
$x^2 = a^2\cos^2\alpha$

$x = a\cos\alpha$



$$\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy = \underbrace{\int_{y=0}^{a\sin\alpha} \int_{x=0}^{y/\tan\alpha} f(x,y)\,dx\,dy}_{R_1} + \underbrace{\int_{y=a\sin\alpha}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy}_{R_2}$$

Q. $\displaystyle\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} f(x,y)\,dx\,dy. = ??$



Soln:

$x = 0$ | $y = x^2/4a \Rightarrow x^2 = 4ay$

$x = 4a$ | $y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$

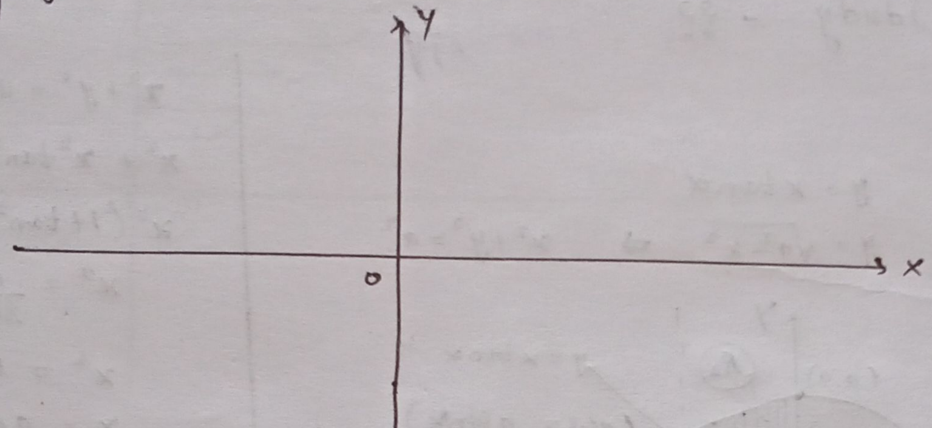
$$I = \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} f(x,y)\,dx\,dy$$

Q. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = ??$ (Change the ORDER in Polar form !!)

Soln:

$x = r\cos\theta$
$y = r\sin\theta$ $\Bigg] \longrightarrow x^2+y^2 = r^2$
$dx\,dy = r\,dr\,d\theta$

$x = 0$	$y = 0$
$x = \infty$	$y = \infty$



$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{r=\infty} e^{-r^2} \cdot r\,dr\,d\theta$$

Let $r^2 = t$

Then, $2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

$$= \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} \frac{e^{-t}}{2}\,dt\,d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{\pi/2} \left(e^{-t}\right)_0^\infty d\theta = -\frac{1}{2} \int_{\theta=0}^{\pi/2} \left(e^{-\infty} - e^{-0}\right) d\theta$$

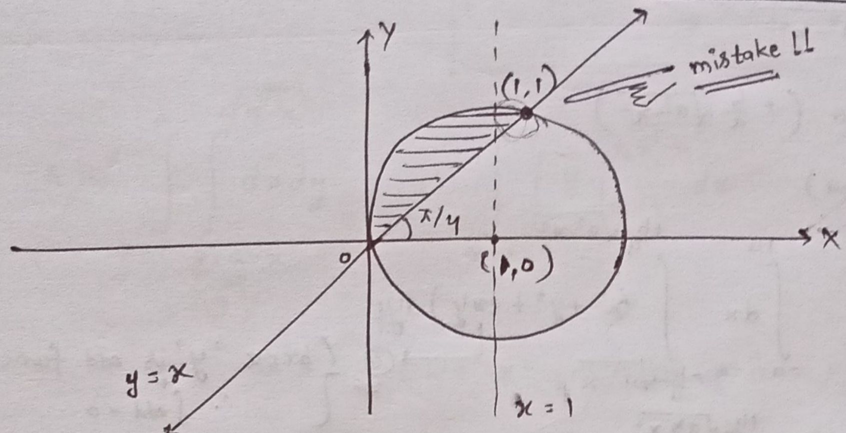
$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} d\theta = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

Q. $\displaystyle\int_0^1 \int_x^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\; dx\,dy = ??$

Soln:

$x = 0$ $\quad$ $x = 1$ | $y = x$

$y = \sqrt{2x-x^2} \Rightarrow \boxed{y^2+x^2-2x=0}$ → $C \equiv (1,0)$, radius $= 1$



$$\left\{ \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ dx\,dy &= r\,dr\,d\theta \end{aligned} \right\}$$

$r^2 - 2r\cos\theta = 0$

$r(r - 2\cos\theta) = 0$

$\Rightarrow r = 0,\ 2\cos\theta$

$\because y = x$

$\therefore r\cos\theta = r\sin\theta$

$\Rightarrow \theta = \frac{\pi}{4}$

$$I = \int_{\theta=\frac{\pi}{4}}^{\pi/2} \int_{r=0}^{2\cos\theta} r\,(r\,dr\,d\theta)$$

$$= \int_{\theta=\frac{\pi}{4}}^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{8}{3}\int_{\theta=\frac{\pi}{4}}^{\pi/2} \cos^3\theta\, d\theta = \frac{8}{3}\int_{\theta=\frac{\pi}{4}}^{\pi/2} \cos^2\theta\cos\theta\, d\theta$$

$$= \frac{8}{3}\int_{\pi/4}^{\pi/2} (1-\sin^2\theta)\cos\theta\, d\theta$$

Put $\sin\theta = t$

$\cos\theta\, d\theta = dt$

$$= \frac{8}{3}\int_{1/\sqrt{2}}^{1} (1-t^2)\,dt = \frac{8}{3}\left[t - \frac{t^3}{3}\right]_{1/\sqrt{2}}^{1}$$

$$= \frac{8}{3}\left[\left(1-\frac{1}{3}\right) - \left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right)\right]$$

$$= \frac{8}{3}\left(\frac{2}{3} - \frac{5}{6\sqrt{2}}\right). \quad \text{Ans.}$$

Q. $\iint (x+y)^2\,dx\,dy$ over the area bounded by Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Soln: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\Rightarrow \quad y = \pm \frac{b}{a}\sqrt{\frac{a^2-x^2}{1}}$$

$$\Rightarrow \quad y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

$y \mapsto \left(-\frac{b}{a}\sqrt{a^2-x^2}\right)$ to $\left(+\frac{b}{a}\sqrt{a^2-x^2}\right)$

and $x \to (-a)$ to $(+a)$

$$\therefore \int_{-a}^{+a} dx \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{+\frac{b}{a}\sqrt{a^2-x^2}} (x+y)^2\,dy = \int_{-a}^{+a} dx \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{+\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2+\underbrace{2xy}_{0})\,dy$$

{b'coz 'y' is odd func. $\therefore \int_{-}^{+} \text{odd} = 0$ }

$$= 2\int_{-a}^{a} dx \int_{0}^{+\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dy$$

$$= 2\int_{-a}^{+a} \left[x^2 y + \frac{y^3}{3}\right]_0^{+\frac{b}{a}\sqrt{a^2-x^2}} dx$$

Let $x = a\sin\theta$
Then, $dx = a\cos\theta\,d\theta$

$$= 2\int_{-a}^{+a} \left[x^2\left(+\frac{b}{a}\sqrt{a^2-x^2}\right) + \frac{b^3}{a^3}(a^2-x^2)^{3/2}\cdot\frac{1}{3}\right] dx$$

$$= 2\int_{-\pi/2}^{+\pi/2} \left[a^2\sin^2\theta\left(\frac{b}{\cancel{a}}\right)(a\cos\theta) + \frac{1}{3}\cdot\frac{b^3}{a^3}(a^2)^{3/2}\cdot(\cos^2\theta)^{3/2}\right] a\cos\theta\,d\theta$$

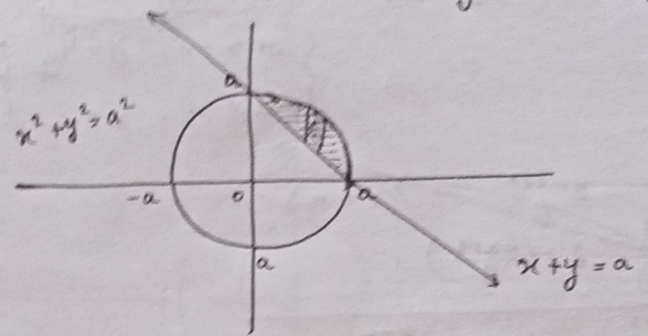
$$= 2\int_{-\pi/2}^{+\pi/2} \left[a^3\sin^2\theta\cos^2\theta\cdot b + \frac{ab^3}{3}\cos^4\theta\cdot\sin^0\theta\right] d\theta$$

$$= 4\int_{0}^{+\pi/2} \left[a^3\sin^2\theta\cos^2\theta\cdot b + \frac{ab^3}{3}\cdot\cos^4\theta\cdot\sin^0\theta\right] d\theta$$

=

Q. Find the area enclosed b/w $x^2+y^2=a^2$ and $x+y=a$. (First Quadrant)

Soln:



$$A = \int_{x=0}^{a}\int_{y=a-x}^{\sqrt{a^2-x^2}} dx\,dy = \int_{x=0}^{a} \left[y\right]_{a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{x=0}^{a} \left(\sqrt{a^2-x^2} - (a-x)\right) dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a - a\left[x\right]_0^a + \left[x^2\right]_0^a \cdot \frac{1}{2}$$

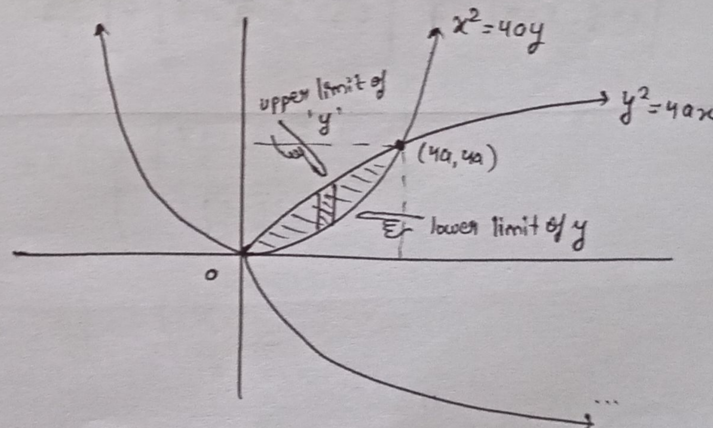
$$= \left[\frac{a}{2}(0) + \frac{a^2}{2}\left(\frac{\pi}{2}\right) - 0\right] - a^2 + \frac{a^2}{2} = \frac{a^2\pi}{4} - \frac{a^2}{2}$$

$$= \boxed{a^2\left(\frac{\pi}{4} - \frac{1}{2}\right)} = \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$$

Q. Find out the Area b/w parabola $y^2 = 4ax$ and $x^2 = 4ay$.

Soln:

$$A = \int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dx\,dy$$



$$= \int_{x=0}^{4a} \left[y\right]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx$$

$$= \int_{x=0}^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx = 2\sqrt{a}\cdot\frac{2}{3}\left[(x)^{3/2}\right]_0^{4a} - \frac{1}{4a}\cdot\frac{1}{3}\left[x^3\right]_0^{4a}$$

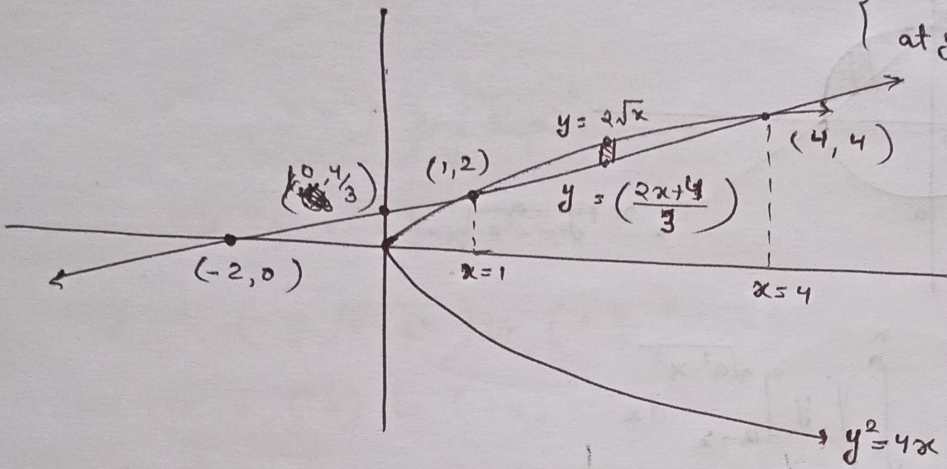
$$= \frac{4}{3}\sqrt{a}\,(4a)^{3/2} - \frac{1}{12a}(64a^3)$$

$$= \frac{32}{3}a^2 - \frac{32}{6}a^2 = 32a^2\left(\frac{6-3}{18}\right) = \frac{32}{6}a^2 = \frac{16a^2}{3}$$

Q. Find the Area bounded by $y^2 = 4x$ and $\underbrace{2x - 3y + 4 = 0}$.

Soln:

$$\left\{\begin{array}{ll} \text{at } x = 0, & y = 4/3 \\ \text{at } y = 0, & x = -2 \end{array}\right\}$$



$$A = \int_1^4 dx \int_{\left(\frac{2x+4}{3}\right)}^{2\sqrt{x}} dy$$

$$= \int_1^4 \left(2\sqrt{x} - \frac{2x}{3} - \frac{4}{3}\right) dx$$

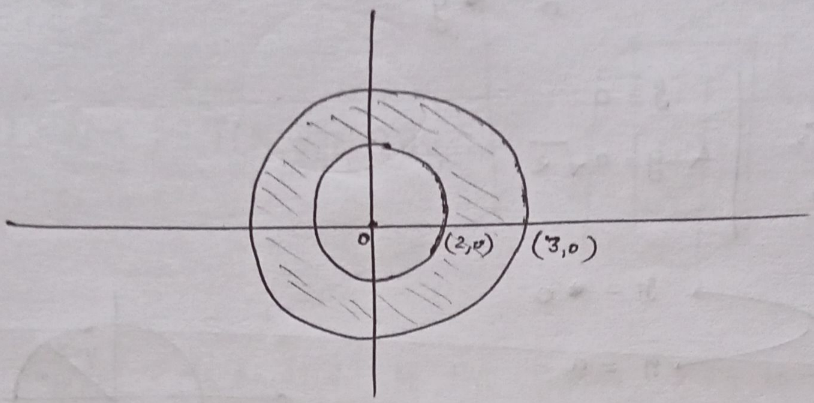
$$= 2 \times \frac{2}{3}\left[(x)^{3/2}\right]_1^4 - \frac{\cancel{2}}{3}\left(\frac{x^2}{\cancel{2}}\right)_1^4 - \frac{4}{3}(x)_1^4$$

$$= \frac{4}{3}(8-1) - \left(\frac{15}{3}\right) - \left(\frac{12}{3}\right)$$

$$= \frac{28}{3} - \frac{15}{3} - \frac{12}{3} = \frac{28-27}{3} = \frac{1}{3}$$

Q Evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ b/w $x^2+y^2=4$ and $x^2+y^2=9$.

Soln:



$$A = \int_{\theta=0}^{2\pi}\int_{r=2}^{3} \frac{r^4\cos^2\theta\sin^2\theta}{r^2}\cdot r\,dr\,d\theta \qquad \left[\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta, \quad x^2+y^2=r^2 \end{array}\right]$$

$$= \int_{\theta=0}^{2\pi}\int_{r=2}^{3} r^3.\sin^2\theta\cdot\cos^2\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[\frac{r^4}{4}\right]_2^3.\sin^2\theta\cos^2\theta\,d\theta$$

$$= \left(\frac{3^4-2^4}{4}\right)\int_{\theta=0}^{2\pi}\sin^2\theta\cos^2\theta\,d\theta = \frac{65}{4}\times(2)(2)\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$= \frac{65}{2}\left[2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta\right] \qquad [p=2,\ q=2]$$

$$= \frac{65}{2}\cdot\left\{\frac{\Gamma\frac{2+1}{2}\cdot\Gamma\frac{2+1}{2}}{\Gamma\frac{2+2+2}{2}}\right\}$$

$$= \frac{65}{2}\cdot\frac{\left(\Gamma\frac{3}{2}\right)^2}{\Gamma 3}$$

$$= \frac{65}{2}\cdot\frac{(\pi/4)}{2} = \frac{65\pi}{16}$$

Q. Find the Area & Evaluate $\displaystyle\int_0^{a\sqrt{2}}\int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2)\,dx\,dy.$

Soln:

$x = y$ | $y = 0$

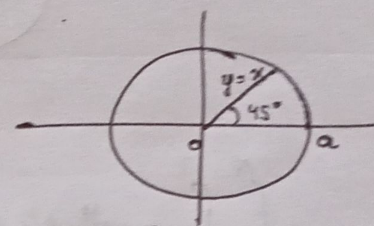
& $x = \sqrt{a^2-y^2}$ | & $y = a\sqrt{2}$

$\Rightarrow x^2+y^2 = a^2$

Put $x^2+y^2 = r^2$ → $r = 0$, $r = a$

$x = r\cos\theta$

$y = r\sin\theta$



$$\therefore\ A = \int_0^{\pi/4}\int_0^{a} \log(r^2)\, r\,dr\,d\theta$$

$$= 2\int_0^{\pi/4} d\theta \int_0^{a} \underbrace{\log(r)}_{u}\cdot \underbrace{r}_{v}\,dr$$

$$= 2\int_0^{\pi/4} d\theta\cdot\left[(\log r)\left(\frac{r^2}{2}\right) - \frac{1}{2}\int \frac{r}{1}\,dr\right]_0^a$$

$$= 2\int_0^{\pi/4} d\theta\left\{(\log a)\frac{a^2}{2} - \left(\frac{a^2}{4}\right)\right\}$$

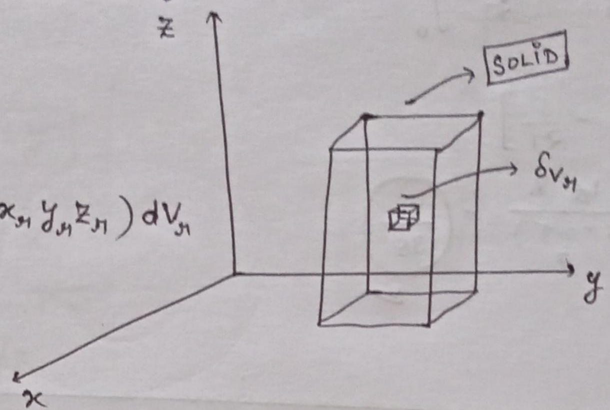
$$= \cancel{2}\left(\frac{\pi}{\cancel{4}\,2}\right)\left[\frac{a^2}{2}\cdot\log a - \frac{a^2}{4}\right]$$

$$= \frac{\pi}{2}\cdot\frac{a^2}{2}\left(\log a - \frac{1}{2}\right) = \frac{\pi a^2}{4}\left(\log a - \frac{1}{2}\right)$$

VOLUME by TRIPLE Integral

$\delta v_r = \delta x\, \delta y\, \delta z$

$$\iiint_V f(x,y,z) = \lim_{\substack{n\to\infty \\ \delta v_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\, dV_r$$



Suppose the region V is bounded by the curves $x=a$ to $x=b$, $y=\phi_1(x)$ to $y=\phi_2(x)$ and $z=f_1(x,y)$ to $z=f_2(x,y)$, then the VOLUME b/w the curves is defined by

$$\int_{x=a}^{b} dx \int_{y=\phi_1(x)}^{\phi_2(x)} dy \int_{z=f_1(x,y)}^{f_2(x,y)} dz$$

Q. Evaluate : $\iiint (x-2y+z)\,dx\,dy\,dz$ $\quad \left\{ \begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq x^2 \\ 0 \leq z \leq x+y \end{array} \right.$

Solve

$$\int_0^1 \int_0^{x^2} \int_0^{x+y} (x-2y+z)\,dx\,dy\,dz$$

$$= \int_0^1 dx \int_0^{x^2} dy \left[xz - 2yz + \frac{z^2}{2} \right]_0^{x+y}$$

$$= \int_0^1 dx \int_0^{x^2} \left(x(x+y) - 2y(x+y) + \frac{(x+y)^2}{2} \right) dy$$

$$= \int_0^1 dx \int_0^{x^2} \left(x^2 + xy - 2xy - 2y^2 + \frac{(x+y)^2}{2} \right) dy$$

$$= \int_0^1 dx \int_0^{x^2} \left(x^2 - 2y^2 - xy + \frac{(x+y)^2}{2} \right) dy$$

$$= \int_0^1 dx \int_0^{x^2} \left(x^2 - 2y^2 - \cancel{xy} + \frac{x^2}{2} + \frac{y^2}{2} + \cancel{xy} \right) dy$$

$$= \int_0^1 dx \int_0^{x^2} \left(\frac{3x^2}{2} - \frac{3y^2}{2} \right) dy = \frac{3}{2} \int_0^1 dx \int_0^{x^2} (x^2 - y^2)\, dy$$

$$= \frac{3}{2} \int_0^1 \left(x^2 y - \frac{y^3}{3} \right)_0^{x^2} dx = \frac{3}{2} \int_0^1 \left(x^4 - \frac{x^6}{3} \right) dx \,.$$

$$= \frac{3}{2}\left[\frac{x^5}{5} - \frac{x^7}{21}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{1}{5} - \frac{1}{21}\right]$$

$$= \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{16}\,8}{(5 \times \cancel{21}_{7})} = \frac{8}{35}$$

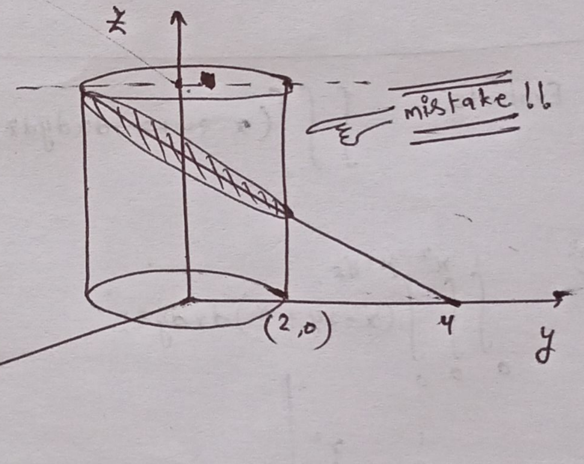
Q. find the volume bounded by the cylinder $\underbrace{x^2+y^2=4}_{\downarrow}$ and the plane $y+z=4$. and $z=0$

$$y = \pm\sqrt{4-x^2} \begin{cases} +\sqrt{4-x^2} \\ -\sqrt{4-x^2} \end{cases}$$

$$\left\{ x \begin{cases} 2 \\ -2 \end{cases} \right\}$$

Soln:

$$V = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} dy \int_{0}^{4-y} dz$$



$$= \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} (4-y)\,dy$$

$$= \int_{-2}^{2} dx \left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}$$

$$= \int_{-2}^{2} dx \left[\left\{4\sqrt{4-x^2} - \cancel{\left(\frac{4-x^2}{2}\right)}\right\} - \left\{-4\sqrt{4-x^2} - \cancel{\left(\frac{4-x^2}{2}\right)}\right\}\right]$$

$$= \int_{-2}^{2} dx \left[8\sqrt{4-x^2}\right] = 8\int_{-2}^{2} \sqrt{4-x^2}\,dx = 16\int_{0}^{2} \sqrt{4-x^2}\,dx$$

$$= 16\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 16\left[\cancel{\frac{2}{2}\sqrt{4-4}}^{\,0} + \frac{4}{2}\sin^{-1}\left(\frac{2}{2}\right) - 0\right]$$

$$= 16\left[\frac{\cancel{4}}{\cancel{2}} \times \frac{\pi}{\cancel{2}}\right] = 16\pi$$

Q. find the Volume ~~common~~ b/w the cylinder $x^2+y^2=a^2$, $x^2+z^2=a^2$

$x = \pm a$

$z^2 = y^2 \Rightarrow \boxed{z = \pm y}$

and $y = \pm\sqrt{a^2-x^2}$

Soln:

$$V = \int_{-a}^{+a} dx \int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} dy \int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} dz$$

$$= 8\int_0^a dx \int_0^{\sqrt{a^2-x^2}} dy \int_0^{\sqrt{a^2-x^2}} dz$$

$$= 8\int_0^a dx \int_0^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2}\right) dy = 8\int_0^a dx \cdot \sqrt{a^2-x^2} \cdot \left[y\right]_0^{\sqrt{a^2-x^2}}$$

$$= 8\int_0^a (a^2-x^2)\, dx = 8\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$V = 8\left[a^3 - \frac{a^3}{3}\right]$$

$$\boxed{V = \frac{16a^3}{3}}$$

Q. find the Volume b/w the cylinder $x^2+y^2=a^2$, $x^2+z^2=a^2$

$x = \pm a$

$z^2 = y^2 \Rightarrow \boxed{z = \pm y}$

and $y = \pm\sqrt{a^2-x^2}$

Soln:

$$V = \int_{-a}^{+a} dx \int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} dy \int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} dz$$

$$= 8\int_0^a dx \int_0^{\sqrt{a^2-x^2}} dy \int_0^{\sqrt{a^2-x^2}} dz$$

$$= 8\int_0^a dx \int_0^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2}\right) dy = 8\int_0^a dx \cdot \sqrt{a^2-x^2} \cdot [y]_0^{\sqrt{a^2-x^2}}$$

$$= 8\int_0^a (a^2-x^2)\, dx = 8\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$V = 8\left[a^3 - \frac{a^3}{3}\right]$$

$$\boxed{V = \frac{16a^3}{3}}$$

Q. Find out the Volume of Sphere : $x^2+y^2+z^2=a^2$.

Soln:

$$V = \int_{-a}^{+a} dx \int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} dy \int_{-\sqrt{a^2-x^2-y^2}}^{+\sqrt{a^2-x^2-y^2}} dz$$

Q. Using Cylindrical Coordinates, Evaluate : $\displaystyle\int_{-2}^{+2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} dy \int_{0}^{4-x^2-y^2} dz$.

(DOUBT) $\Rightarrow$ function NOT given

Soln:

$z=0$ to $z = 4-x^2-y^2 \,] \rightarrow x^2+y^2 = 4-z$

$y = -\sqrt{4-x^2}$ to $y = +\sqrt{4-x^2} \,] \rightarrow x^2+y^2 = 4$

$x=-2$ to $x=+2$

General Eqn of PARABOLOID

$ax^2+by^2 = 2cz$

$$I = \int_0^{2\pi} d\theta \int_{-2}^{+2} r\,dr \int_0^{4-r^2} dz \qquad [\because x^2+y^2=r^2]$$

$dx\,dy\,dz \rightarrow r\,dr\,d\theta\,dz$

$$= \int_0^{2\pi} d\theta \int_{-2}^{+2} r\,dr\,(4-r^2)$$

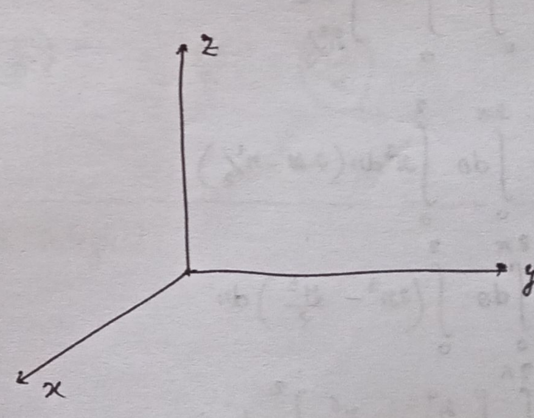
$$= \int_0^{2\pi} d\theta \int_{-2}^{+2} (4r-r^3)\,dr = \int_0^{2\pi} d\theta \left(2r^2 - \frac{r^4}{4}\right)_{-2}^{+2}$$

$$= \int_0^{2\pi} [(8-4)-(8-4)]\,d\theta = 0 \quad (??)$$

Q. Evaluate Triple integration $\iiint_V xyz\, dV \rightarrow dx\,dy\,dz$ bounded by the 3 coordinates planes and $x+y+z=1$

$$\left\{\begin{matrix} z=0 \\ y=0 \\ x=0 \end{matrix}\right\}$$

$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (xyz)\,dx\,dy\,dz$$



$$= \int_0^1 dx \int_0^{1-x} dy \left[xy\cdot\frac{z^2}{2}\right]_0^{1-x-y} = \int_0^1 dx \int_0^{1-x} \frac{xy}{2}(1-x-y)^2\,dy$$

$$= \frac{1}{2}\int_0^1 dx \int_0^{1-x} xy\,(1+x^2+y^2-2x+2xy-2y)\,dy$$

$$= \frac{1}{2}\int_0^1 dx \int_0^{1-x} (xy + x^3y + xy^3 - 2x^2y + 2x^2y^2 - 2xy^2)\,dy$$

$$= \frac{1}{2}\int_0^1 dx \left[\frac{xy^2}{2} + \frac{x^3y^2}{2} + \frac{xy^4}{4} - 2x^2y^2 + \frac{2x^2y^3}{3} - \frac{2xy^3}{3}\right]_0^{1-x}$$

Q. Using Cylindrical Coordinates,

$$\iiint (x^2+y^2)\,dV \text{ , when 'S' is bounded by the Surfaces } \overbrace{(x^2+y^2=2z)}^{\text{Paraboloid}} \text{ and plane } z=2.$$

Soln:

$$x^2+y^2 = 2z$$

Let $x^2+y^2 = r^2$

Then, $r^2 = 2z \Rightarrow z = \frac{r^2}{2}$

$$x^2+y^2 = 2z$$

$$x^2+y^2 = 4 \quad (\because z=2)$$

$$\therefore \quad r = 2 \quad (\because r > 0)$$

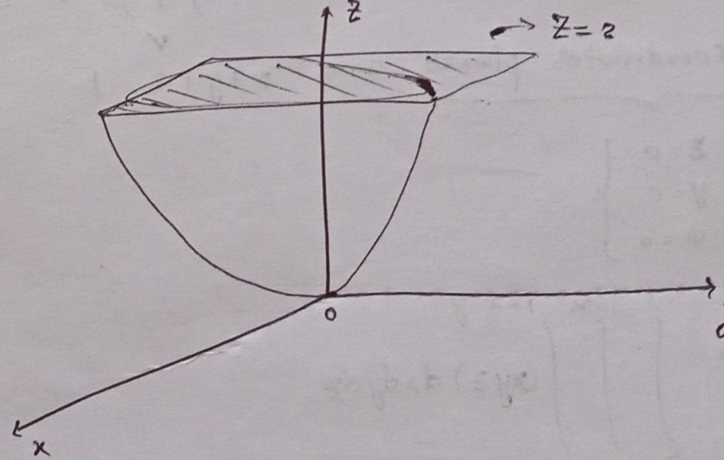
$$V = \int_0^{2\pi} d\theta \int_0^2 r^3\,dr \int_{r^2/2}^{2} dz$$

$$= \int_0^{2\pi} d\theta \int_0^2 r^3\,dr \left(2 - \frac{r^2}{2}\right)$$

$$= \int_0^{2\pi} d\theta \int_0^2 \left(2r^3 - \frac{r^5}{2}\right) dr$$

$$= \int_0^{2\pi} \left[\frac{r^4}{2} - \frac{r^6}{12}\right]_0^2 d\theta$$

$$= \left(8 - \left(\frac{16\times 4}{12_3}\right)\right)(2\pi) = \left(\frac{24-16}{3}\right)(2\pi) = \frac{16\pi}{3}$$



Q. Using Cylindrical Coordinates, Evaluate $\int_{-1}^{+1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{1}(x+y)^2\,dz\,dx\,dy$

Soln:

$z = 0$ to $z = +1$

$x = 0$ to $x = \sqrt{1-y^2} \longrightarrow x^2+y^2=1$

$y = -1$ to $y = +1$

$$\begin{cases} x^2+y^2 = r^2 \\ x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Now, $(x+y)^2 = (x^2+y^2) + 2xy \longrightarrow r^2 + 2r^2\sin\theta\cos\theta$

$$V = \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{0}^{1} r^3(1+\sin2\theta)\,dr\,d\theta\,dz$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \int_0^1 r^3(1+\sin2\theta)\,dr = \int_{-\pi/2}^{\pi/2} d\theta \int_0^1 (r^3 + r^3\cdot\sin2\theta)\,dr$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \left[\frac{r^4}{4} + \frac{r^4}{4}\sin2\theta\right]_0^1$$

$$= \int_{-\pi/2}^{\pi/2}\left(\frac{1}{4} + \frac{\sin2\theta}{4}\right)d\theta$$

$$= \left[\frac{\theta}{4} - \frac{1}{8}(\cos2\theta)\right]_{-\pi/2}^{\pi/2} = \left[\frac{\pi}{8} + \frac{1}{8}\right] - \left[-\frac{\pi}{8} + \frac{1}{8}\right]$$

$$= \frac{2\pi}{8} = \frac{\pi}{4}$$

Q. Find Volume of sphere $x^2+y^2+z^2=a^2$ by Triple integral.

Soln:

$z = \pm\sqrt{a^2-x^2-y^2}$

$y = \pm\sqrt{a^2-x^2}$

$x = \pm a$

$$\therefore\quad V = \int_{-a}^{+a} dx \int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} dy \int_{-\sqrt{a^2-x^2-y^2}}^{+\sqrt{a^2-x^2-y^2}} dz = 8\int_0^{+a} dx \int_0^{+\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\,dy$$

$$= 8\int_0^a dx\left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \left(\frac{a^2-x^2}{2}\right)\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}}$$

$$V = 8\int_0^a \left(\frac{a^2-x^2}{2}\right)\left(\frac{\pi}{2}\right)dx$$

$$= 2\pi \int_0^a (a^2-x^2)\,dx = 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi\left(a^3 - \frac{a^3}{3}\right) = \boxed{\frac{4\pi a^3}{3}}$$

Q. Find the value of $\iiint_V \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$, where 'V' is the volume of sphere $x^2+y^2+z^2=1$.

Soln:

$\because$ $x^2+y^2+z^2=1$

$\therefore$ $z = \pm\sqrt{1-x^2-y^2}$

$y = \pm\sqrt{1-x^2}$

$x = \pm 1$

Now,

$$\int_{-1}^{+1}\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{+\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = 8\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{\left(\sqrt{1-x^2-y^2}\right)^2 - z^2}}$$

$$= 8\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \left(\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}}$$

$$= 8\int_0^1 dx \int_0^{\sqrt{1-x^2}} \left(\frac{\pi}{2}\right) dy$$

$$= 4\pi \int_0^1 dx\,[y]_0^{\sqrt{1-x^2}} = 4\pi\int_0^1 \sqrt{1-x^2}\,dx$$

$$= 4\pi\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_0^1$$

$$= 4\pi\left[\frac{1}{2}\left(\frac{\pi}{2}\right)\right] = \pi^2$$

MAXIMA & MINIMA (to be CONTINUED.....)

Q. $u = \sin x + \sin y + \sin(x+y)$

$$\frac{\partial u}{\partial x} = \cos x + \cos(x+y) = 0 \Rightarrow \cos x = -\cos(x+y)$$

$$\frac{\partial u}{\partial y} = \cos y + \cos(x+y) = 0 \Rightarrow \cos y = -\cos(x+y)$$

$\longrightarrow \cos x = \cos y \Rightarrow \boxed{x = y}$

and, $\cos x = \cos[\pi - (x+y)]$

$\Rightarrow x = \pi - x - y$

$\Rightarrow x = \pi - 2x \quad (\because x = y)$

$\Rightarrow \boxed{x = \pi/3}$ and $y = \pi/3$

$(\frac{\pi}{3}, \pi/3)$

$$r = \frac{\partial^2 u}{\partial x^2} = -\sin x - \sin(x+y)$$

$$t = \frac{\partial^2 u}{\partial y^2} = -\sin y - \sin(x+y)$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = -\sin(x+y)$$

Now,

$$r = \left[-\sin x - \sin(x+y)\right]_{x=\frac{\pi}{3}, y=\pi/3}$$

$$\Rightarrow r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = (-\sqrt{3})$$

$$t = \left[-\sin y - \sin(x+y)\right]_{(\pi/3, \pi/3)} = (-\sqrt{3})$$

and $s = \left[-\sin(x+y)\right]_{(\pi/3, \pi/3)} = -\frac{\sqrt{3}}{2}$

$$\therefore (rt - s^2) = (-\sqrt{3})(-\sqrt{3}) - \left(-\sqrt{3}/2\right)^2$$

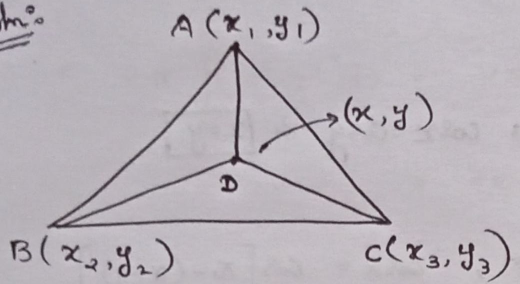
$$= 3 - \frac{3}{4} = \frac{9}{4} > 0$$

and $r = -\sqrt{3} < 0 \Rightarrow (\frac{\pi}{3}, \pi/3)$ is point of MAXIMA.

also, $\boxed{u_{max}} = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

Q. Find a point within a Δ such that the sum of square of its distance from the 3 vertices is minimum.

Soln:



$$(AD^2 + BD^2 + DC^2) = \left[(x-x_1)^2+(y-y_1)^2\right] + \left[(x-x_2)^2+(y-y_2)^2\right] + \left[(x-x_3)^2+(y-y_3)^2\right]$$

OR

$$u = d^2 = \sum_{r=1}^{3} \left[(x-x_r)^2 + (y-y_r)^2\right]$$

$$\frac{\partial u}{\partial x} = \sum_{r=1}^{3} 2(x-x_r) = 2\left[(x-x_1)+(x-x_2)+(x-x_3)\right] = 0 \Rightarrow x = \frac{x_1+x_2+x_3}{3}$$

$$\frac{\partial u}{\partial y} = \sum_{r=1}^{3} 2(y-y_r) = 2\left[(y-y_1)+(y-y_2)+(y-y_3)\right] = 0 \Rightarrow y = \frac{y_1+y_2+y_3}{3}$$

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= r = 6 \\ \frac{\partial^2 u}{\partial y^2} &= t = 6 \\ \frac{\partial^2 u}{\partial x \partial y} &= s = 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} rt - s^2 &= 36 > 0 \\ \text{and } r &= 6 > 0 \end{aligned}\right\} \rightarrow (x,y) \text{ is point of minima !!}$$

$$\downarrow$$

$$\underbrace{\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)}$$

$\boxed{\text{CENTROID}}$

Q. $f(x,y,z) = x^2+y^2+z^2$ $\longrightarrow$ $\frac{\partial f}{\partial x} = 2x$

$\frac{\partial f}{\partial y} = 2y$

$\frac{\partial f}{\partial z} = 2z$

$z^2 = xy + 1$

$\phi = z^2 - xy - 1$

$\frac{\partial \phi}{\partial x} = -y$

$\frac{\partial \phi}{\partial y} = -x$

$\frac{\partial \phi}{\partial z} = 2z$

$\therefore \left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) = 0 \Rightarrow 2x - \lambda y = 0 \Rightarrow \lambda = \frac{2x}{y}$

$\left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) = 0 \Rightarrow 2y - \lambda x = 0 \Rightarrow \lambda = 2y/x$

$\left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) = 0 \Rightarrow 2z + \lambda(2z) = 0 \Rightarrow \lambda = -1$

$\therefore$ $2x - \lambda y = 0 \Rightarrow 2x + y = 0$ $(\because \lambda = -1)$

and $2y - \lambda x = 0 \Rightarrow \underbrace{2y + x = 0}$

$(x = 0, y = 0)$

Stationary points

$\underline{(0,0,1)}$ & $\underline{(0,0,-1)}$

$\because z^2 = xy + 1$

$\therefore z^2 = 0 + 1 \Rightarrow z^2 = 1 \Rightarrow z = \pm 1$

Q. Find out the max. & min. distance of the point $(3,4,12)$ from the sphere $x^2+y^2+z^2=4$.

$$f = d^2 = (x-3)^2 + (y-4)^2 + (z-12)^2$$

$\frac{\partial f}{\partial x} = 2(x-3)$ | $\phi = x^2+y^2+z^2-4=0$

$\frac{\partial f}{\partial y} = 2(y-4)$ | $\frac{\partial \phi}{\partial x} = 2x$

$\frac{\partial f}{\partial z} = 2(z-12)$ | $\frac{\partial \phi}{\partial y} = 2y$

$\frac{\partial \phi}{\partial z} = 2z$

$\therefore \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow 2(x-3) + (2x)\lambda = 0 \Rightarrow x = \frac{3}{(\lambda+1)}$

$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 2(y-4) + \lambda(2y) = 0 \Rightarrow y = \frac{4}{\lambda+1}$

$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow 2(z-12) + \lambda(2z) = 0 \Rightarrow z = \frac{12}{\lambda+1}$

$\because x^2+y^2+z^2 = 4$

$\therefore \left(\frac{3}{\lambda+1}\right)^2 + \left(\frac{4}{\lambda+1}\right)^2 + \left(\frac{12}{\lambda+1}\right)^2 = 4$

$\frac{9+16+144}{4} = (\lambda+1)^2 \Rightarrow (\lambda+1) = \pm 13/2$

$\Rightarrow \boxed{\lambda = \frac{11}{2}, -\frac{15}{2}}$

$\Rightarrow$

For $\lambda = +\frac{11}{2}$,

$x = \frac{6}{13}$

$y = \frac{8}{13}$

$z = \frac{24}{13}$

~~For $\lambda = -$,~~

~~$x = \frac{3}{(-9)} \times 2 = -\frac{2}{3}$~~

~~$y = \frac{4}{(-9)} \times 2 = -8/9$~~

~~$z = \frac{12}{(-9)} \times 2 = -8/3$~~

For $\lambda = -\frac{15}{2}$,

$x = \frac{3}{-13} \times 2 = -6/13$

$y = \frac{4}{-13} \times 2 = -8/13$

$z = \frac{12}{-13} \times 2 = -\frac{24}{13}$

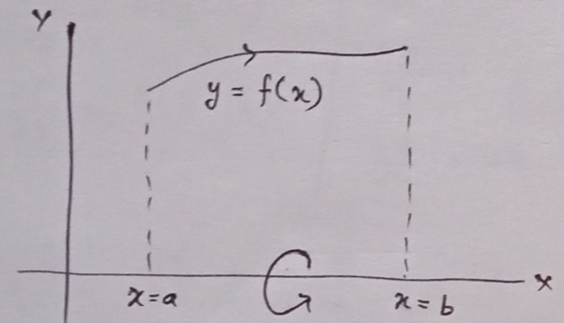
$d_{max} =$

Q. ~~T~~ find out the volume of largest rectangular parallelopiped that can be inscribed in the Ellipsoid $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right)$.

VOLUME of SOLID generated by the Revolution about X-axis and AREA bounded by $y = f(x)$, $x = a$ to $x = b$.

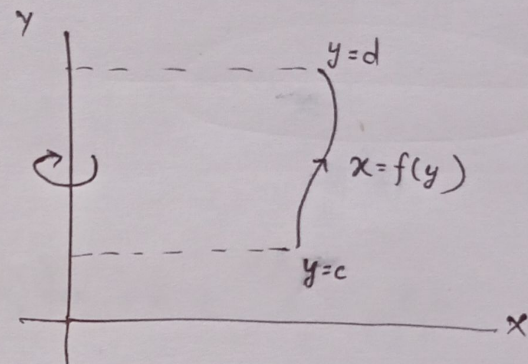
about X-axis :

$$V = \int_{x=a}^{x=b} \pi y^2 \, dx$$



about Y-axis :

$$V = \int_{y=c}^{y=d} \pi x^2 \, dy$$



$x = f(t)$

$y = f(t)$

$$V = \int_{t=t_1}^{t=t_2} \pi (y)^2 \left[\frac{d}{dt}(x)\right] dt$$

$$V = \int_{t=t_1}^{t=t_2} \pi (x)^2 \left[\frac{d}{dt}(y)\right] dt$$

about intial Line ($\theta = 0°$)

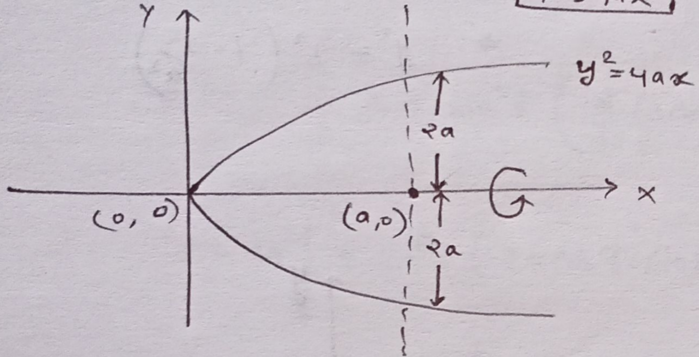
$$V = \int_{\theta=\theta_1}^{\theta=\theta_2} \frac{2}{3} \pi r^3 \sin\theta \, d\theta$$

about $\frac{\pi}{2}$ line

$$V = \int_{\theta=\theta_1}^{\theta=\theta_2} \frac{2}{3} \pi r^3 \cos\theta \, d\theta$$

Q. Find out the volume of parabola generated by the revolution about X-axis from $x=0$ to 'h'.

$$\boxed{y^2 = 4ax}$$

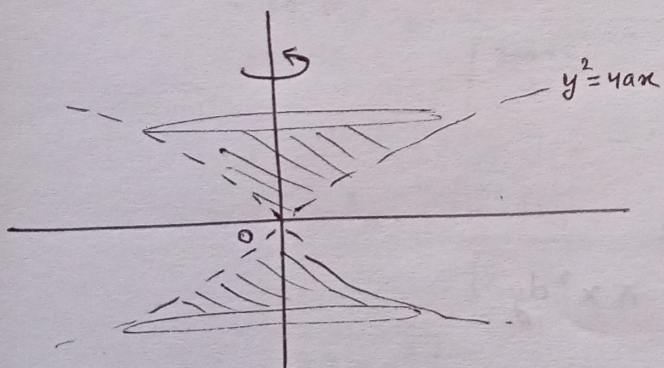


$$V = \int_{x=0}^{h} \pi (\overset{y^2}{4ax})\, dx$$

$$= 4a\pi \int_0^h x\,dx = \frac{4a\pi}{2}\left[x^2\right]_0^h$$

$$= 2a\pi h^2$$

and about Y-axis



$$V = \int_0$$

Q.

$$V = \int_{x_1}^{x_2} \pi y^2 \, dx$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

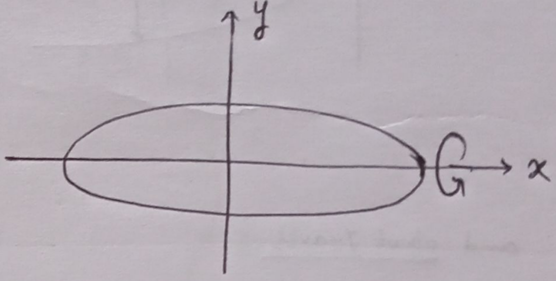
$$\Rightarrow \quad y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

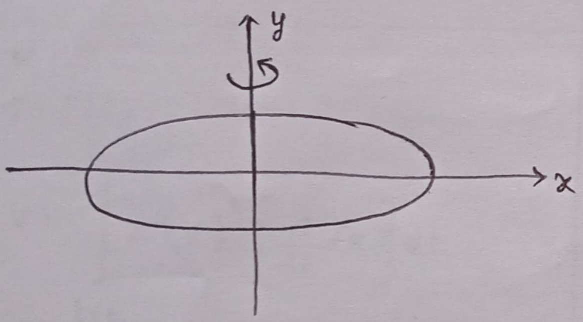
$$= 2\int_0^a \pi \frac{b^2}{a^2}(a^2 - x^2)\, dx$$

$$= \frac{2\pi b^2}{a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{2\pi b^2}{a^2}\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{4\pi b^2 a}{3}$$





$$V = \int_{y=0}^{b} \pi x^2 \, dy$$

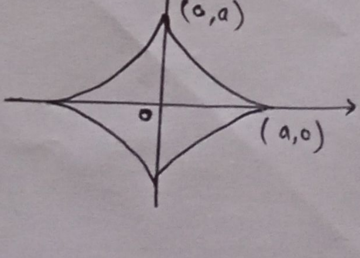
$$= \int_0^b \pi \frac{a^2}{b^2}(b^2 - y^2)\, dy$$

Q. find out the volume of

<u>Asteroid Eqⁿ</u> :

$$x^{2/3} + y^{2/3} = a^{2/3}$$

<u>parametric form</u> : $x = a\cos^3 t \;,\; y = a\sin^3 t$



$$V = \int \pi y^2 \left(\frac{dx}{dt}\right) dt$$

$$= 2\int_0^{\pi/2} \pi a^2 \sin^2 t \left(\frac{d}{dt}(a\cos^3 t)\right) dt$$

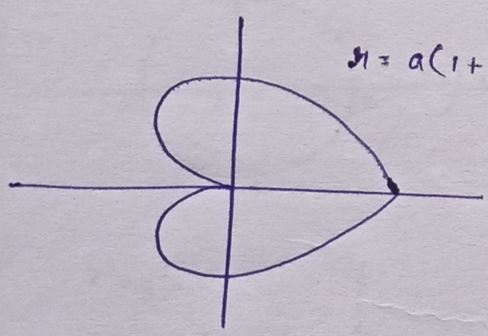
$$= -2a^2 \int_0^{\pi/2} \pi \sin^6 t \,(3a\cos^2 t \cdot \sin t)\, dt$$

$$= -3a^3\pi \cdot 2\int_0^{\pi/2} \sin^7 t \cdot \cos^2 t \, dt$$

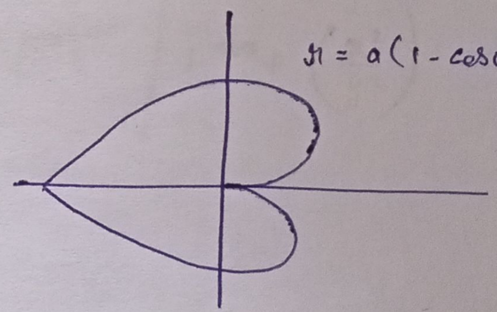
$$= -3a^3\pi \cdot \frac{\Gamma\frac{7+1}{2} \cdot \Gamma\frac{2+1}{2}}{\Gamma\frac{7+2+2}{2}}$$

$$= -3a^3\pi \cdot \frac{\Gamma 4 \cdot \Gamma 3/2}{\Gamma 11/2}$$

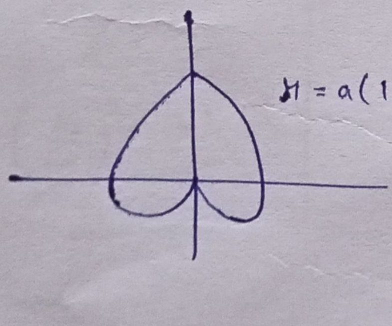
$$= \boxed{\frac{32\pi a^3}{105}}$$



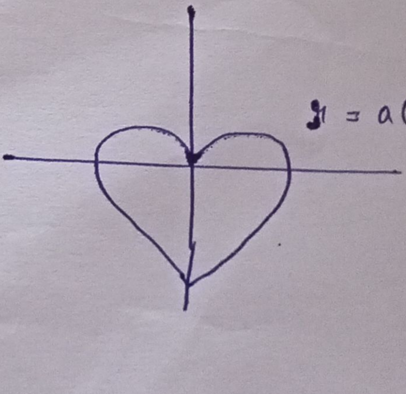
$r = a(1+\cos\theta)$



$r = a(1-\cos\theta)$

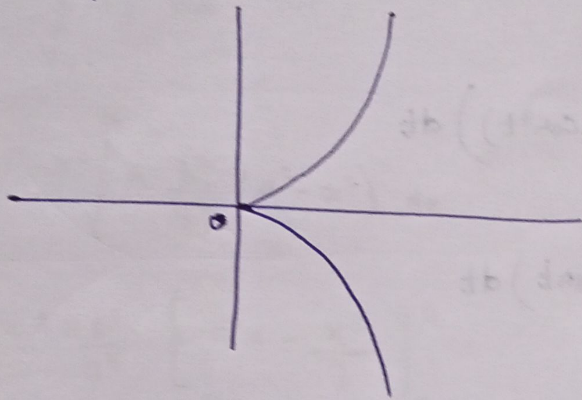


$r = a(1+\sin\theta)$



$r = a(1-\sin\theta)$

Cissoid of Diocles



Cartesian form : $y^2 = \dfrac{x^3}{2a-x}$

Polar form : ~~[illegible]~~

$r = 2a\tan\theta . \sin\theta$

Q. $V = \displaystyle\int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta\, d\theta = \frac{2}{3}\pi \int_0^{\pi} a^3(1+\cos\theta)^3 \sin\theta\, d\theta$

$$S = \int_{x=x_1}^{x=x_2} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{(about x-axis)}$$

$$S = \int_{y=y_1}^{y=y_2} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{(about y-axis)}$$

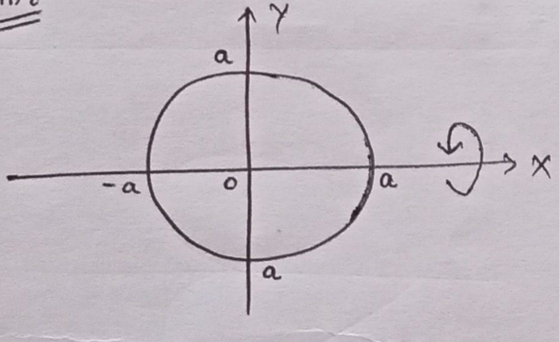
$$S = \int_{\theta_1}^{\theta_2} 2\pi r \sin\theta \left(\frac{ds}{d\theta}\right) d\theta \quad ; \quad \frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \qquad \text{(about initial line)}$$

$$S = \int_{\theta_1}^{\theta_2} 2\pi r \cos\theta \left(\frac{ds}{d\theta}\right) d\theta \qquad \left(\text{about } \frac{\pi}{2} \text{ line}\right)$$

SURFACE AREA of SOLID REVOLUTION

Q. find out the area when we rotate $x^2+y^2=a^2$ about x-axis.

Soln.



$$S = 2 \cdot \int_{x=0}^{x=a} 2\pi\sqrt{a^2-x^2} \cdot \sqrt{1 \cancel{dx} + \left(\frac{x^2}{y^2}\right)} \cdot dx$$

$x^2+y^2=a^2$

$2x + 2y \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{y}$

$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{y^2}$

$$= 2\int_0^a 2\pi y \cdot \sqrt{\frac{a^2}{y^2}}\, dx$$

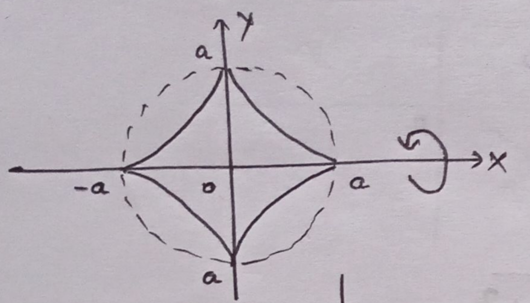
$$= 2\int_0^a 2\pi y \left(\frac{a}{y}\right) dx$$

$$= 4\pi a \int_0^a dx = \underline{4\pi a^2}$$

Q. Find out the Surface Area of ~~Spindle Shaped~~ Asteroid (whole area) about x-axis.
(Spindle Shaped Solid)

Soln:

$$x^{2/3} + y^{2/3} = a^{2/3}$$



$$S = 2 \cdot \int_{\theta=0}^{\theta=\pi/2} 2\pi y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$x = a\cos^3\theta$

$y = a\sin^3\theta$

$$S = 2 \int_0^{\pi/2} 2\pi (a\sin^3\theta) \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta$$

$$S = 2 \int_0^{\pi/2} 2\pi (a\sin^3\theta) \cdot \sqrt{(3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2}\, d\theta$$

$$= 4\pi \int_0^{\pi/2} a\sin^3\theta \cdot \sqrt{9a^2(\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta)}\, d\theta$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin^3\theta \sqrt{\sin^2\theta\cos^2\theta\,(1)}\, d\theta$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin^4\theta \cdot \cos\theta\, d\theta$$

$$= 6\pi a^2 \cdot \left[2\int_0^{\pi/2} \sin^4\theta \cdot \cos\theta\, d\theta\right]$$

$$= 6\pi a^2 \cdot \frac{\Gamma\frac{4+1}{2} \cdot \Gamma\frac{1+1}{2}}{\Gamma\frac{4+1+2}{2}}$$

$$= 6\pi a^2 \cdot \left(\frac{\Gamma\frac{5}{2} \cdot \Gamma 1}{\Gamma\frac{7}{2}}\right) = 6\pi a^2 \cdot \left\{\frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \cdot 1}{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}\right\}$$

$$= \frac{12}{5}\pi a^2$$

Q. find out the surface Area of solid revolution by the curve "$r = 2a\cos\theta$" about initial line

Soln:

$$S = \int_{\theta=0}^{\theta=\pi/2} 2\pi r \sin\theta \left(\frac{dS}{d\theta}\right) d\theta$$

θ	0	$\pi/2$		2π
r	$2a$	0		$2a$

$\Rightarrow$ Symm. about initial line

$$\frac{dS}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$\Rightarrow \frac{dS}{d\theta} = \sqrt{(4a^2\cos^2\theta) + (4a^2\sin^2\theta)}$$

$$\Rightarrow \frac{dS}{d\theta} = 2a$$

$$\therefore \quad S = \int_{\theta=0}^{\pi/2} 2\pi r \sin\theta\,(2a)\, d\theta$$

$$= 4\pi a \int_{\theta=0}^{\pi/2} (2a\cos\theta)\sin\theta\, d\theta = 8\pi a^2 \int_{\theta=0}^{\pi/2} \sin\theta\cos\theta\, d\theta$$

$$= 4\pi a^2 \cdot \left[2\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \right]$$

$$= 4\pi a^2 \left[\frac{\Gamma\frac{1+1}{2} \cdot \Gamma\frac{1+1}{2}}{\Gamma\frac{1+1+2}{2}} \right] = 4\pi a^2 \left\{ \frac{1}{\Gamma 2} \right\}$$

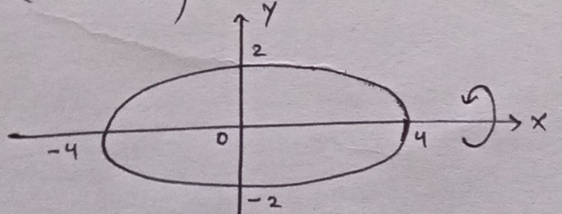
$$= 4\pi a^2 \left\{ \frac{1}{1} \right\} = \underline{\underline{4\pi a^2}}.$$

Q. find the surface Area of rotation of Ellipse $x^2 + 4y^2 = 16$ about major axis.

$$\frac{x^2}{(4)^2} + \frac{y^2}{(2)^2} = 1$$

Soln

$$\left\{ \begin{array}{l} a = 4 \\ b = 2 \end{array} \right\}$$



$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\frac{x}{8} + \frac{y}{2} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = (-x/8) \times \frac{2}{y} = -\frac{x}{4y}$$

$$S = 2\int_{x=0}^{4} 2\pi y \sqrt{1 + \frac{x^2}{16y^2}} \cdot dx$$

$$= 4\pi \int_{x=0}^{4} y \cdot \frac{\sqrt{x^2 + 16y^2}}{4y}\, dx$$

$$= \pi \int_0^4 \sqrt{x^2 + 16\left(\frac{16 - x^2}{4}\right)}\, dx$$

$$S = \pi \int_0^4 \sqrt{x^2 + 64 - 4x^2}\, dx$$

$$= \pi \int_0^4 \sqrt{(8)^2 - (\sqrt{3}x)^2}\, dx$$

$$= \pi \left[\frac{\sqrt{3}x}{2} \sqrt{(8)^2 - (\sqrt{3}x)^2} + \frac{64}{2} \sin^{-1}\left(\frac{\sqrt{3}x}{8}\right) \right]_0^4 \times \frac{1}{\sqrt{3}}$$

$$= \pi \left[2\sqrt{3} \cdot \sqrt{64 - 48} + 32 \times \frac{\pi}{3} \right] \times \frac{1}{\sqrt{3}}$$

$$= \cancel{\pi}$$