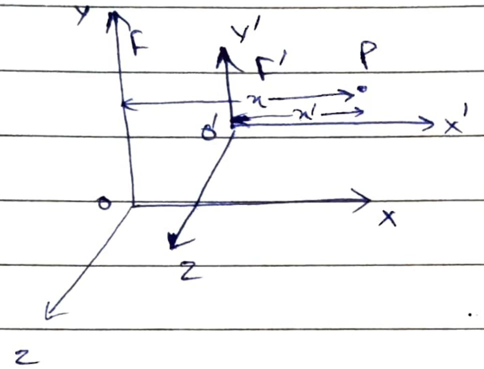
THEORY OF RELATIVITY:

FRAMES OF REFERENCES

1.) Inertial (Newton's laws hold good)

2.) Non-Inertial (Newton's laws are invalid)

GALILEAN TRANSFORMATION



Frame F' moving with $\vec{v}$ w.r.t frame F

$\therefore$ For point P

$$\left.\begin{aligned} x' &= x - v_x t \\ y' &= y - v_y t \\ z' &= z - v_z t \end{aligned}\right\} \text{(Galilean transform for position)}$$

where $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$

Using $\frac{d}{dt} = \frac{d}{dt'}$

$$\left.\begin{aligned} \frac{dx'}{dt'} &= \frac{dx}{dt} - v_x \\ \frac{dy'}{dt'} &= \frac{dy}{dt} - v_y \\ \frac{dz'}{dt'} &= \frac{dz}{dt} - v_z \end{aligned}\right\} = \left.\begin{aligned} u_x' &= u_x - v_x \\ u_y' &= u_y - v_y \\ u_z' &= u_z - v_z \end{aligned}\right\} \text{(Galilean transformation for velocity)}$$

whereas

u_x, u_y, u_z are velocities of particle observed from frame F while u_x', u_y', u_z' from frame F'.

$\therefore \vec{u}' = \vec{u} - \vec{v}$

Similarly Galilean $\vec{a}$ transformation can be represented

However, with further d/dt, we get

$a_x' = a_x \quad a_y' = a_y \quad a_z' = a_z$

Thus $\vec{a}$ is invariant in inertial FOR

LORENTZ TRANSFORMATION

$\vec{v}$ in galilean transf. was not invariant.
However, speed of light turns out to be invariant in all inertial frames.

In this case, when F' moves with $\vec{v}$ w.r.t F

$x' = k(x - vt)$ (here, k is a proportionality const.)

Also, $x = k(x' + vt')$ (t and t' are not equal yet)

$(\vec{v} = v_x\hat{i} + 0 + 0)$

$\therefore x = k(k(x - vt) + vt')$

$\therefore \frac{x}{k} = kx - kvt + vt'$

$$t' = \frac{x}{kv} - \frac{kx}{v} + kt$$

$$\boxed{t' = kt - \frac{kx}{v}\left(1 - \frac{1}{k^2}\right)} \longleftarrow (3)$$

$\because$ Acc to speacial theory c is invariant

$x = ct$

$x' = ct' = k(x - vt) = k(ct - vt)$

$ct' = kt(c - v)$ —— (1)

Also $ct = kt'(c + v)$ —— (2)

Multiplying (1) & (2)

$c^2 tt' = k^2 tt'(c^2 - v^2)$

$$k^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - v^2/c^2}.$$

$$\therefore \boxed{k = \frac{1}{\sqrt{1 - v^2/c^2}}}$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{k^2}$$

$$\boxed{x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}}$$

By ③

$$\boxed{t' = \frac{\left(t - \frac{xv}{c^2}\right)}{\sqrt{1 - v^2/c^2}}}$$

$$\boxed{x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}}, \; y = y', \; z = z', \; \boxed{t = \frac{\left(t' + x'v/c^2\right)}{\sqrt{1 - v^2/c^2}}}$$

↓

Lorentz Transf. Equations.

EFFECTS

#1 Time Dilation

For an observer in F let a time interval b/w the start and stop of a ~~stop~~clock be Δt

$\Delta t = t_2 - t_1$

For the same observer this interval in F' frame be $\Delta t'$

$\Delta t' = t_2' - t_1'$.

$$\because \; t = \frac{\left(t' + \frac{x'v}{c^2}\right)}{\sqrt{1 - v^2/c^2}}$$

$$\therefore \; t_2 - t_1 = \frac{\left(t_2' + x'v/c^2\right)}{\sqrt{1 - v^2/c^2}} - \frac{\left(t_1' + x'v/c^2\right)}{\sqrt{1 - v^2/c^2}}$$

$$\boxed{t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}}}$$

$$\boxed{\Delta t' = \frac{\Delta t}{k}} \quad \text{Also} \quad \boxed{\Delta t = \Delta t' k}$$

$\therefore$ Time is delayed by a factor of k in moving frame

#2 Length Contraction

For an object placed in F', the length can be written in the form of differences in the co-ordinates of its end points w.r.t the two frames

$\therefore L' = x_2' - x_1'$

$L = x_2 - x_1$

$$\because x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$\therefore L' = \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}} - \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}}$$

$$L' = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}}$$

$$\left[L' = \frac{L}{\sqrt{1 - v^2/c^2}}\right] \quad [L' = Lk] \quad \text{Also} \quad \left[L = L'\sqrt{1 - \frac{v^2}{c^2}}\right]$$

#3 Addition of velocities

$$u_x = \frac{dx}{dt} \qquad u_y = \frac{dy}{dt} \qquad u_z = \frac{dz}{dt}$$

$$u_x' = \frac{dx'}{dt'} \qquad u_y' = \frac{dy'}{dt'} \qquad u_z' = \frac{dz'}{dt'}$$

By Lorentz transf. eqn we have,

$$x = \frac{x' + vt}{\sqrt{1 - v^2/c^2}}, \quad y = y', \quad z = z', \quad t = \frac{t' + xv/c^2}{\sqrt{1 - v^2/c^2}}$$

Differentiating ~~by~~ these,

$$dx = \frac{dx' + vdt'}{\sqrt{1 - v^2/c^2}}, \quad dy = dy', \quad dz = dz', \quad dt = \frac{dt' + v/c^2\, dx'}{\sqrt{1 - v^2/c^2}}$$

$$\because u_x = \frac{dx}{dt} = \frac{dx' + vdt'}{\sqrt{1 - v^2/c^2}} \cdot \frac{\sqrt{1 - v^2/c^2}}{dt' + v/c^2\, dx'}$$

$$= \frac{dx' + vdt'}{dt' + v/c^2\, dx'}$$

Dividing num & denom by dt'

$$u_x = \frac{dx'/dt' + v}{1 + v/c^2 \cdot dx'/dt'} = \frac{u_x' + v}{1 + v\,u_x'/c^2}$$

$$\therefore \boxed{u_x = \frac{u_x' + v}{1 + \frac{v}{c^2} u_x'}}$$

~~Also,~~

$$\cancel{u_y = \frac{u_y' + v}{1 + \frac{v}{c^2} u_y'}} \quad \& \quad \cancel{u_z = \frac{u_z' + v}{1 + \frac{v}{c^2} u_z'}}$$

$$u_y = \frac{dy}{dt} = \frac{dy' \cdot \sqrt{1 - v^2/c^2}}{dt' + \frac{v\,dx'}{c^2}} = \frac{dy'/dt' \sqrt{1 - v^2/c^2}}{1 + v/c^2 \, dx'/dt'}$$

$$\boxed{u_y = \frac{u_y' \sqrt{1 - v^2/c^2}}{1 + \frac{v}{c^2} u_x'}}$$

Similiarly

$$\boxed{u_z = \frac{u_z' \sqrt{1 - v^2/c^2}}{1 + \frac{v}{c^2} u_x'}}$$

#5 Einstein's Energy Mass Relation

(By a derivation out of syllabus $m = \frac{+ m_0}{\sqrt{1 - v^2/c^2}}$)

Change in energy = Work done = $F \cdot dx$

$$F = \frac{d}{dt}(mv)$$

$$\therefore = v\frac{dm}{dt} + m\frac{dv}{dt} \quad (\because \text{ mass not const.})$$

$$F \cdot dx = \frac{v\,dm\,dx}{dt} + \frac{m\,dv\,dx}{dt}$$

$$dE = v^2\,dm + mv\,dv \longrightarrow \text{LHS}$$

Now,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \qquad m^2 = \frac{m_0 c^2}{c^2 - v^2}$$

$\therefore m^2c^2 - m^2v^2 = m_0c^2$

Differentiating this we get

$c^2\, 2m\,dm - v^2\, 2m\,dm - m^2\, 2v\,dv = 0$ $\quad (m_0c^2 = \text{const})$

$dm\,c^2 - dm\,v^2 - dv\,mv = 0$

$dm\,c^2 = v^2\,dm + mv\,dv$ — RHS

Equating LHS & RHS we get

$dm\,c^2 = dE$

Integrating both sides

$$c^2\int_{m_0}^{m} dm = \int_{0}^{E_k} dE$$

$c^2(m - m_0) = E_k$

$E_k = mc^2 + m_0c^2$ $\quad$ (m_0c^2 is rest mass energy $\therefore$ Total energy $E = \underset{\text{(kinetic)}}{E_k} + \underset{\text{(restmass)}}{m_0c^2}$)

$$\boxed{E = mc^2}$$