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Assignment-I Solutions

Mathematics - I (MTH 110) Sections - F&G

1. If

$$V = \ln \sin \frac{\pi(2x^2 + y^2 + xz)^{1/2}}{2(x^2 + xy + 2yz + z^2)^{1/3}}$$

Find the value of $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z}$ when x=0, y=1 and z=2.

Solution: Note $\sin^{-1}(e^V) = \frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}$. So, $\sin^{-1} e^v$ is a homogeneous function of degree $1 - 2/3 = 1/3$. Then, using Euler's theorem

$$x\frac{\partial \sin^{-1} e^V}{\partial x} + y\frac{\partial \sin^{-1} e^V}{\partial y} + z\frac{\partial \sin^{-1} e^V}{\partial z} = \tfrac{1}{3}\sin^{-1} e^V$$

which gives

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = \frac{1}{3}\left(\frac{\sqrt{1-e^{2V}}}{e^V}\right)\sin^{-1} e^V.$$

On substituting $x = 0, y = 1, z = 2$ in above equation gives $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = \frac{\pi}{4\sqrt{2}}$.

2. Find the value of the parameter n so that $V = r^n(3\cos^2\theta - 1)$ satisfies

$$\frac{\partial}{\partial r}\big(r^2\frac{\partial V}{\partial r}\big) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta\frac{\partial V}{\partial\theta}\big) = 0$$

Solution: Here,

$$\frac{\partial}{\partial r}\big(r^2\frac{\partial V}{\partial r}\big) = n(n+1)r^n(3\cos^2\theta - 1) \tag{1}$$

and

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta\frac{\partial V}{\partial\theta}\big) = -6r^n(3\cos^2\theta - 1) \tag{2}$$

Adding Eqs. (1) and (2),

$$\frac{\partial}{\partial r}\big(r^2\frac{\partial V}{\partial r}\big) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta\frac{\partial V}{\partial\theta}\big) = (n(n+1) - 6)((3\cos^2\theta - 1))$$

$$(n(n+1) - 6)((3\cos^2\theta - 1)) = 0$$

$$\implies (n(n+1) - 6) = 0$$

So, $n = 2, -3$.

3. Skipped.

4. Find three positive numbers whose sum is 48 and such that their product is as large as possible.
Solution: Let three number x, y, z such that $x+y+z=48$. Then we have to find value of x, y, x such that xyz is maximum. Consider Lagrangian function

$$\phi(x, y, x, \lambda) = xyz + \lambda(x+y+z-48),$$

where λ is a Lagrange parameter.
Now

$$\phi_x(x, y, z, \lambda) = yz + \lambda \tag{3}$$
$$\phi_y(x, y, z, \lambda) = xz + \lambda \tag{4}$$
$$\phi_z(x, y, z, \lambda) = xy + \lambda \tag{5}$$
$$\phi_\lambda(x, y, z, \lambda) = x+y+z-48 \tag{6}$$

Solving Eqs. (3-6), we get the possible values $x = y = z = 16$.

5. An international airline has a regulation that each passenger can carry a suitcase having the sum of its width, length, and height less than or equal to 129 cm. Find the dimensions of the suitcase of maximum volume that a passenger can carry under this regulation.
Solution: Let width, length and height of the suitcase is x, y, z, respectively, such that $x+y+z=129$. Then, we have to find maximum volume that a passenger can carry under regulation. Volume of suitcase is xyz. Therefore, the Lagrangian function can be constructed as

$$\phi(x, y, x, \lambda) = xyz + \lambda(x+y+z-129)$$

Now,

$$\begin{aligned}
\phi_x(x, y, z, \lambda) &= yz + \lambda \\
\phi_y(x, y, z, \lambda) &= xz + \lambda \\
\phi_z(x, y, z, \lambda) &= xy + \lambda \\
\phi_\lambda(x, y, z, \lambda) &= x+y+z-129
\end{aligned}$$

Solving the above equations, we get the required dimensions $x = y = z = 43$.

6. Discuss the maxima and minima of the function $u(x, y, z) = \sin x \sin y \sin z$, where x, y, and z are the angles of a triangle.
Solution: Here, $u(x, y, z) = \sin x \sin y \sin z$ such that $x+y+z=\pi$. Therefore,

$$\phi(x, y, z, \lambda) = \sin x \sin y \sin z + \lambda(x+y+x-\pi)$$

Now,

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \sin y \sin z + \lambda = 0 \\
\frac{\partial \phi}{\partial y} &= \sin x \sin z + \lambda = 0 \\
\frac{\partial \phi}{\partial z} &= \sin x \sin y + \lambda = 0
\end{aligned}$$

$$\frac{\partial \phi}{\partial \lambda} = x + y + z - \pi = 0$$

Then, we get $x = y = z = \pi/3$.
So $u(\pi/3, \pi/3, \pi/3) = \frac{3}{8}\sqrt{3}$ is maximum at point $(\pi/3, \pi/3, \pi/3)$.

7. Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and also determine their nature.
Solution: Here, $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$. Therefore,
$$f_x(x, y) = 4x^3 - 4x + 4y = 0$$
$$f_y(x, y) = 4y^3 + 4x - 4y = 0$$
Solving the above equations, we get three critical points as $(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$. By using second derivative test $f(x, y)$ attains local minima at point $(\sqrt{2}, -\sqrt{2})$ and local maxima at point $(-\sqrt{2}, \sqrt{2})$ and while test is inconclusive at point $(0, 0)$.

8. Find all the stationary points of the function $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$. Also, examine whether the function is maximum or minimum at these points.
Solution: Here, $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.
Now,
$$f_x(x, y) = 3x^2 + 3y^2 - 30x + 72 = 0,$$
$$f_y(x, y) = 6xy - 30 = 0.$$
There are four critical points as $(5, 1), (5, -1), (6, 0)$, and $(4, 0)$.
Now,
$$f_{xx}(x, y) = 6x - 30,$$
$$f_{yy}(x, y) = 6x - 30,$$
$$f_{xy}(x, y) = 6y.$$
Then, by second derivative test, $(5, 1)$ and $(5, -1)$ are saddle points, $(4, 0)$ is a point of maxima and $(6, 0)$ is a point of minima.

9. Locate the points of the surface $x^2 - yz = 5$ that are closest to the origin.
Solution: To find the points on the surface defined by the equation $x^2 - yz = 5$ that are closest to the origin, we need to minimize the distance function:
$$D = \sqrt{x^2 + y^2 + z^2}$$
i.e.,
$$F = D^2 = x^2 + y^2 + z^2$$
subject to the constraint
$$g(x, y, z) = x^2 - yz - 5.$$
Using Lagrange multipliers,
$$f = F + \lambda g$$
leads to the equations
$$2x = \lambda(2x),$$
$$2y = -\lambda z,$$
$$2z = -\lambda y.$$
On solving these equations, the points closest to the origin are obtained as,
$$(\sqrt{5}, 0, 0) \text{ and } (-\sqrt{5}, 0, 0).$$

10. Find the extreme value of $x^2 + y^2 + z^2 + xy + xz + yz$ subject to the constraints $x + y + z = 1$ and $x + 2y + 3z = 3$.
Solution: Maximize $f(x, y, z) = x^2 + y^2 + z^2 + xy + xz + yz$
subject to the constraints

$$g(x, y, z) = x + y + z - 1,$$

$$h(x, y, z) = x + 2y + 3z - 3.$$

The Lagrangian function is given as,

$$\mathcal{L}(x, y, z, \lambda, \mu) = f(x, y, z) + \lambda g(x, y, z) + \mu h(x, y, z).$$

On differentiating, we obtain

$$2x + y + z + \lambda + \mu = 0,$$

$$2y + x + z + \lambda + 2\mu = 0,$$

$$2z + x + y + \lambda + 3\mu = 0.$$

Therefore, the critical point is obtained $\left(-\frac{1}{6}, \frac{1}{3}, \frac{5}{6}\right)$, the function as extreme value $\frac{11}{12}$.

11. Suppose a function $f(x, y)$ is differentiable at the point $(1, 1)$ with $f_x(1, 1) = 2$ and $f(1, 1) = 3$. Let $L(x, y)$ denote the local linear approximation of f at $(1, 1)$. If $L(1.1, 0.9) = 3.15$, find the value of $f_y(1, 1)$.
Solution:

For a function $f(x, y)$ that is differentiable at the point (a, b), the linear approximation (first-order Taylor expansion) around this point is given by:

$$f(x, y) \approx f(a, b) + (x - a)\frac{\partial f}{\partial x}(a, b) + (y - b)\frac{\partial f}{\partial y}(a, b).$$

Substituting the given values, we get $f_y(1, 1) = 0.5$.

12. Obtain the first-order Taylor series approximation to the function $f(x, y) = e^y \log(x+y)$ about the point $(1, 0)$. Estimate the maximum absolute error over the rectangle $|x - 1| < 0.1$, $|y - 1| < 0.1$.
Solution: First-order Taylor approximation:

$$f(x, y) \approx f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) = (x - 1) + y.$$

Thus, the approximation is given as

$$f(x, y) \approx x + y - 1.$$

Over the rectangle $|x-1| < 0.1$, $|y| < 0.1$, we have $\max(|f_{xx}|) \leq 0.620$, $\max(|f_{yy}|) \leq 4.473$ and $\max(|f_{xx}|) \leq 0.741$.

Now, the maximum absolute error is given to be

$$|M| \leq \frac{B}{2}\left(|x - 1| + |y - 1|\right)^2$$

where $B = \max(|f_{xx}|, |f_{yy}|, |f_{xy}|)$. Therefore, $|M| \leq 0.089$.

13. Expand $f(x, y) = \sin(x + 2y)$ in a Taylor series up to third-order terms about the point $(0, 0)$. Find the maximum error over the rectangle $|x| < 0.1$, $|y| < 0.1$.

Solution: Following the same approach as in the previous question, we get

$$f(x, y) \approx x + 2y - \frac{1}{6}\left((x + 2y)^3\right).$$

and absolute error $|M| \leq 0.0003$.