III.

Differential eq.

ordinary | Partial de (PDE)

$z(x,0)$

$z$ is a function of $(x,y)$

← 1st order → | ← second order →

$\frac{dz}{dx}$ $\frac{dz}{dy}$ → $\frac{\partial z}{\partial x}$ $\frac{\partial z}{\partial y}$ → $z_x$, $z_y$

$\frac{d^2z}{dx^2}$ → $z_{xx}$ → $r$

$\frac{d^2z}{dy^2}$ → $z_{yy}$ → $t$

$\frac{d^2z}{dxdy}$, $\frac{d^2z}{dydx}$ → $z_{xy} = z_{yx}$ → $s$

← First order PDE →

first order defined as →

$f(x, y, z, p, q) \to 0$

solution will be

$F(x, y, z, a, b) = 0$

$f(x, y, y', y'') = 0$
$f(x, y, a, b) = 0$

← Second order PDE →

$p^2 + q^2 - 2px - 2qy + 2xy = 0$

$r - 2s + t = \sin(2x+3y)$

$\frac{d^2z}{dx^2} - 2\frac{d^2z}{dxdy} + \frac{dz}{dy^2} = \sin(2x+2y)$

$\left[px - qz = z^2 + (x+y)^2\right]$

→ $p\tan x + q\tan y = \tan z$

$f(x, y, a, b) = 0$

→ Elimination of arbitary constant
→ Elimatition of arbitay function

$r(u, v) = 0$

where $u = \phi(x, y, z)$
$v = h(x, y, z)$

$f(x, y, a, b) = 0$

diff + Partially wrt

$$\boxed{\frac{df}{dx} + \frac{df}{dz} \cdot \frac{dz}{dx} = 0} \qquad \boxed{\frac{df}{dx} + p\frac{df}{dz} = 0}$$

similarly diff wrt to y

$$\boxed{\frac{df}{dy} + q\frac{df}{dz} = 0}$$

PDE of function: $z = (x-\alpha)^2 + (y-\beta)^2$

Now diff partially wrt to y

$$\frac{dz}{dy} = 2(y-\beta)$$

$$q = 2(y-\beta).$$

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$\boxed{4z = p^2 + q^2}$$

Question: $Z \Rightarrow ax + by + a^2b^2$

diff wrt $x$ $\quad p = a$

wrt $y$ $\quad q = b$

$$z = px + qy + p^2q^2$$

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Question: $ce^{\omega t} \cos \omega x$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

---

PDE by elimination of arbitary function

wrt to $x$ $\quad p = e^{xy}\phi(x-y$

wrt to $y$ $\quad q = he^{xy}\phi(x-y) + e^{xy}\phi(x-y) \times (-1)$

$\quad q = he^{xy}\phi(x-y) - e^{xy}\phi(x-y)$

using $p \cdot q = x$ wrt $q$

$$q = xz - p$$

$$\boxed{xz = p+q}$$

$$z = f(x+iy) + f(x-iy)$$

$p = f(x+iy)(i) + F'(x-iy)(-i)$

$q = iF(x+iy) - iF'(x-iy)$

Second order derivative

$\frac{d^2z}{dx^2} \Rightarrow f''(x+iy) + F''(x-iy)$

$\frac{d^2z}{dy^2} \Rightarrow -F''(x+iy) - F''(x+iy)$ $\quad \boxed{r+t=0}$

## Solution of First order PDE :-

$f(x, y, z, p, q) = 0$

solution of $F(x, y, z, a, b) = 0$

this is known as complete solution

### Particulate solⁿ

the solution obtained by giving the particular value at a & b is known as particular solⁿ.

### Singular solⁿ.

by eliminating a, b.

$\frac{df}{da} = 0$ & $\frac{df}{db} = 0.$

### general solⁿ

$\frac{dF}{da} = 0.$

Solution of linear partial diff eqⁿ

← Langrage's eqⁿ /method →

$P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$

in lagrange in 2 variables $x, y$ ..

solution.

$\phi(u, v) = 0$

$\phi(u) = v$

$v = \phi(u).$

$u(x,y,z) = c_1 + \phi(c_1, c_2) = 0$

$v[x,y,z] = c_2$

linearly independent solution of ① associated with auxillary eqn.

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} = k$$

$dx = k P(x,y,z)$
$dy = k Q(x,y,z)$.
$dz = k R(x,y,z)$

Question: find complete integral of lag eqn.

$2yzp + 2xq = 3xy$.

$$\frac{dx}{2yz} = \frac{dy}{3x} = \frac{dz}{3xy}.$$

$$\frac{dx}{\cancel{2}z} = \frac{dy}{\cancel{3x}}$$

$x\,dx = 2y\,dy$

$\frac{x^2}{2} - y^2 = c$

$x^2 - 2y^2 = c_1$

$$\frac{dy}{3x} = \frac{dz}{3xy}$$

$$\frac{dy}{3} = \frac{dz}{3y}.$$

$3y\,dy - 3\,dz = 0.$

$3y^2 - z^2 = c_2.$

$$v(x,y,z) = 3y^2 - z^2 = c_3$$

$$\phi(u,v) = \phi(x^2 - 2y^2, 3y^2 - z^2) = 0$$

or

$$\phi(3y^2 - z^2, x^2 - 2y^2) = 0$$

or

$$\phi(x^2 - 2y^2) = 3y^2 - z^2$$

or

$$\phi(3y^2 - z^2) = x^2 - 2y^2.$$

---

Question : $(z^2 - 2yz + y^2)p + (xy + zx)q = xy - zx.$

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}.$$

$$\frac{dy}{xy + zx} = \frac{dz}{xy - zx}.$$

$$\frac{dy}{y + z} = \frac{dz}{y - z}.$$

$$y\,dy - z\,dy = y\,dz + z\,dz.$$

$$y\,dy - d(zy) - z\,dz = 0.$$

$$\frac{y^2}{2} - zy - \frac{z^2}{2} = 0.$$

$$y^2 - 2zy - z^2 = c.$$

$c_2 = x^2+y^2+z^2$ (alwaysa solⁿ).

$\phi = y^2 - 2zy - z^2$

$sol^n = q(y^2 - 2zy - z^2, x^2+y^2+z^2) = 0$

---

Question :-

$(mz - ny)p + (nx - lz)q = ly - mx$ ?

$P(x,y,z) \longrightarrow mz - ny$
$Q(x,y,z) \longrightarrow nx - lz$
$R(x,y,z) \longrightarrow ly - mx$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

$$\frac{l\,dx + m\,dy + n\,dz}{0}$$

$$l\,dx + m\,dy + n\,dz = 0$$

$$\boxed{lx + my + nz = c}$$

and got $x^2+y^2+z^2 = c_2$

$$\phi(x^2+y^2+z^2,\ lx + my + nz) = 0.$$

---

Solve $(z^2+y^2-x^2)p - 2xyq + 2zx = 0$. solve;

$\star$ So non linear PDE $\longrightarrow$

1) Charpit's Method

Let the first order PDE $(x,y,z,p,q)$

auxillary $eq^n$.

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

$$= \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}.$$

find $p$ & $q$

$$dz = p\,dx + q\,dy.$$

---

Question 1: find a complete integral of the $eq^n$ $2z - px^2 - 2qxy + pq = 0$.

$$\frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{dz}{px^2 - pq + 2xyq - pq}$$

$$= \frac{dp}{2z-2qy} = \frac{q}{0}.$$

$$\boxed{q = c}$$

on substitution $q=c$ into given $eq^n$.

$$p = \frac{2x(z-y)}{x^2-c}$$

$$dz = p\,dx + q\,dy = \frac{2x(z-y)\,dx}{x^2-c} + c\,dy.$$

$$\log(z-zy) = \log(x^2-c) + \log b$$

$$\boxed{z-y=(x^2-c)^b}$$

$$z = ay + (x^2-c)b.$$

$$px + qy = pq$$

$$f = px + qy - pq = 0$$

$$\frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} = \frac{dz}{-p(x-q)-y(y-p)} = \frac{dp}{p} = \frac{dq}{q}.$$

taking last 2 terms.

$$\log p = \log q + \log a$$

$$p = aq$$

in the eqn $\Rightarrow p = ax + y.$

$$dz = p\,dx + q\,dy \longrightarrow (ax+y)\,dx + \left(\frac{ax+y}{a}\right)dy$$

$$a\,dz = (ax+y)(a\,dx+dy)$$

$$\boxed{az = a(x+y) + b.}$$

Question $(p^2+q^2)\,y = qz.$

$$\frac{dx}{-2py} = \frac{dy}{-2py+z} = \frac{dz}{-2p^2y \quad -2q^2y + p^2}$$

$$= \frac{dp}{-pq} = \frac{dq}{p^2}$$

$p\,dp + q\,dq = 0$

$p^2 + q^2 = a^2$

$p^2 \Rightarrow a^2 - q^2$

$q = \frac{a^2 y}{z}$

$p = \frac{a}{z}\sqrt{z^2 - a^2 y^2}$

$dz = p\,dx + dy\,q$

$\frac{a}{z}\sqrt{z^2 - a^2 y^2}$

$$\frac{z\,dz - a^2 y\,dy}{\sqrt{z^2 - a^2 y^2}} = a\,dx$$

$\sqrt{z^2 - a^2 y^2} = ax + b.$

$$\boxed{z^2 = a^2 y^2 + (ax+b)^2}$$

special method. 1 type I methode

pde involving only $p$ & $q$ $\longrightarrow$

$z = f(p, q) = 0$

$p = a$

$q = a$

then $q$ or $p$.

$dz = p\,dx + q\,dy$

$p^2 + q^2 = m^2$

$$q+p=a^2$$

$$q=\sqrt{m^2-a^2}$$

$$dz=a\,dx\pm\sqrt{m^2-a^2}\,dy$$

$$z=ax+\sqrt{m^2-a^2}\,y+b.$$

---

Solve $q=3p^2$,

$$p=a$$

$$q=3a^2$$

$$dz=p\,dx+q\,dy$$

$$z=ax+3a^2y+b$$

---

type 2 method.

PDE of the form $\longrightarrow$ $z=px+qy+f(p,q)$.

$\hookrightarrow$ clairaut eqn.

soln $z=ax+by+f(a,b)$.

Solution.

$$z=px+qy+p^2+q^2$$

$$z=ax+by+a^2+b^2.$$

$$z=px+qy+\log pq$$

soln. $z=ax+by+\log ab$

---

type 3: $f(x,p)=g(y,q)=0$

find p & q.

$$p^2+q^2 = x+y$$
$$p^2-x = y-q^2$$

$$p^2 = a+x$$
$$q^2 = y-a$$

$$p = \pm\sqrt{a+x}$$
$$q = \pm\sqrt{y-a}$$

Solⁿ $dz = p\,dx + q\,dy$

$$dz = \sqrt{a+x}\,dx + \sqrt{y-a}\,dy$$
$$z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b.$$

---

$$yp = 2yx + \log q$$
$$p = 2x + \frac{1}{y}\log q$$

let, $p - 2x = \frac{1}{y}\log q = a$

$$p = a+2x$$
$$q = e^{ay}$$

$\rightarrow dz = p\,dx + q\,dy$

$$z = \int (a+2x)dx + \int e^{ay}dy.$$

$$z = ax + x^2 + \frac{e^{ay}}{a} + b.$$

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# Solution of first order but not first degree

Standard form IV ;

given $f(p, q, z) = 0$

Let us assume

$z = f(x + ay)$ as a trial sol$^n$ of (1) or

$z = f(X)$ where $X = x + ay$

$\frac{dX}{dx} = 1.$

$p = \frac{dz}{dx} = \frac{dz}{dX}\left(\frac{dX}{dx}\right) = \frac{dz}{dX}$

$q = \frac{dz}{dy} = \frac{dz}{dX}\left(\frac{dX}{dy}\right) = a\frac{dz}{dX}$

now eq$^n$ 1 reduces to form

$f\left(\frac{dz}{dX}, a\frac{dz}{dX}, z\right) = 0$

which is ODE of order 1.

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$9(p^2 z + q^2) = 4$ :

replacing $p$ by $\frac{dz}{dX}$ and $q$ by $a\frac{dz}{dX}$ we get

$9\left[\left(\frac{dz}{dX}\right)^2 z + \left(a\frac{dz}{dX}\right)^2\right] = 4$

$9\left(\frac{dz}{dX}\right)^2 [z + a^2] = 4.$

$\frac{dz}{dX} = \frac{2}{3\sqrt{z + a^2}}.$

$$\frac{3}{2}\sqrt{z^2+a^2}\,dz = dx$$

$$2X = 2(z+a^2)^{3/2}+c$$

$$X = x+ay$$

$$= 2(x+ay)$$

$$= 2(z+a^2)^{3/2}+c$$

---

Quest 2: $z^2(p^2z^2+q^2)=1$ ?

replacing $p$ by $\frac{dz}{dx}$ and $q$ by $a\frac{dz}{dx}$ when $X=x+ay$.

$$z^2\left[\left(\frac{dz}{dx}\right)^2 z^2 + a^2\left(\frac{dz}{dx}\right)^2\right]=1$$

$$z^2\left[\frac{dz}{dx}\right]^2\left[z^2+a^2\right]=1$$

$$\boxed{\frac{dz}{dx} = \frac{1}{\sqrt{z^2+a^2}}}$$

$$\int z\sqrt{z^2+a^2}\,dz = \int dx$$

Let $z^2+a^2=t$

$$2z\,dz = dt.$$

$$\frac{1}{2}\int \sqrt{t}\,dt = x+c$$

$$\left(\frac{t}{3}\right)^{3/2} \Rightarrow \frac{(z^2+a^2)^{3/2}}{3} = x+c$$

$$\boxed{(z^2+a^2)^{3/2} = 3x+3ay+c'}$$

Ques 3] $pz = 1 + q^2$

replacing $p$ by $\frac{dz}{dx}$ & $q$ by $\frac{dz}{a\,dx}$ where $X = x + ay$

$$z\frac{dz}{dx} = 1 + a^2\frac{dz^2}{dx^2}.$$

---

Linear partial Differential eq^n of higher order with const. coeff.

the general form of linear pde is given as :-

$$a_0\frac{d^nz}{dx^n} + a_1\frac{d^{n-1}z}{dx^{n-1}dy} + a_2\frac{d^nz}{dx^{n-2}dy^2} + \cdots a_n\frac{d^nz}{dy^n} +$$

$$b_0\frac{d^{n-1}z}{dx^{n-1}} + b_1\frac{\partial^{n-1}z}{dx^{n-3}dy},$$

$$b_{n-1}\frac{d^{n-1}z}{dy^{n-1}} + N_0\frac{dz}{dx} + M_1\frac{dz}{dy} + N_0 z = f(x,y).$$

---

PDE

homogeneous PDE

Linear non homogeneous.

when all the derivates are appearing is same order then its known as homogeneous PDE.

when derivative differ in order

$$(D^2 - D'^2 + DD')z = xy$$
$$(D^3 + D'^3 + D^2D' - DD'^2 + D^3)z = \sin(x+y)$$

$(D^2 + D)y = 0$
$(D^3 + D)y = 0$
$(D + DD')y = 0$
$(D^3 + D'^2 + DD')y = 0$

---

Solutions of homogeneous PDE :-

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots a_n D'^n)z = F(x,y)$$

then its sol$^n$ is $z$ = complementary function + P.I.

theorum : if $u$ is the F and $z$ is the P.I. of a linear PDE. then $u + z_1$ general soln.

$u$ is a sol$^n$ of $F(D,D')z = 0$
$z$ is a sol$^n$ $F(D,D')z = f(x,y)$.

---

* Method to find the complemtary function.

$$(a_0 D^n + a_1 D^{n-1} D' \dots + a_n D'^n)z = f(x,y)$$
$$\downarrow$$
complimentary function, $F(D,D')z = 0$.

$$(b_1 D - a_1 D')(b_2 D - a_2 D') \dots (b_n D - a_n D')z = 0$$

Ex $\rightarrow$ $(D^2 - D'^2)z = 0$
$(D + D')(D - D')z = 0$,

case 1) If factors are distinct or non repeative & then corresponding to each non repeative factor. part of cf is taken

$$\phi = (b_i y + a_i x) ; \ \phi \text{ arbitrary fun}^n$$

case II: if factors are repeating upto m time

$(b_i D - a_i x)^m$.

case III:

case $D'^m$ CF: $\phi_1(x) + y\phi_2(x) + y^2\phi_3(x) \cdots + y^{m-1}\phi_{m-1}(x)$

---

Question:- $2r + 5s + 2t = 0$ ?

$$2D^2 + 5DD' + 2D'^2 = 0$$

$$(2D^2 + 4DD' + DD' + 2D'^2)$$

$$(D + 2D')(2D + D')z = 0$$

$$\phi_1(y - 2x) + \phi_2(2y - x) = CF$$

$$PI = RHS = 0$$

$$Sol^n = CF.$$

---

Solve. $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$

$$(D - D')(D - 2D')(D - 3D') = z$$

$$CF = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)$$

$$z = CF + PI$$

$$= CF.$$

## Methods for finding Particular integral

When $f(x,y)$ of $F(D,D')=f(x,y)$ (i) is of the form $F(ax+by)$ and $F(D,D')$ is a homogenous of function of degree 'n'

$$\boxed{PI = \frac{1}{F(D,D')} F(ax+by)}$$

$$PI = \frac{1}{F(D,D')} f(ax+by) = \frac{1}{F(a,b)} \underbrace{\iint \cdots \int}_{n \text{ times}} f(u)\,du\,du\,du$$

$u = ax+by$

only possible provided $F(a,b) = 0$

when $f(a,b) = 0$

$$PI = \frac{1}{F(D,D')} = \frac{x^n}{b^n (n!)} F(ax+by).$$

---

#1. $(D^2+3DD'+2D'^2)\,z = x+y$

CF $\rightarrow$ $(D^2+3DD'+2D'^2)\,z = 0.$

$(D+2D')(D+D')z = 0$

$$CF = \phi_1(y-2x) + \phi_2(y-x)$$

P.I. $\rightarrow$ $\frac{1}{F(D,D')} f(ax+by) = \frac{1}{D^2+3DD'+2D'^2}(x+y)$

$a=1$

$b=1$

$$\therefore \frac{1}{F(a,b)} = \frac{1}{1+3+2} \neq 0.$$

$\frac{1}{6}\int F(u)\, du\, du$.

Let $x+y=u$

2 available

hence integral 2 times

$\frac{1}{6}\int F(u)\, du\, du$

$\int u = \int \frac{u^2}{2} = \frac{u^3}{6}$

$\frac{1}{6}\frac{(x+y)^3}{6}$

$z = CF + PI$

$z = \frac{1}{36}(x+y)^3 + \phi(y-2x) + \phi_2(y-x)$.

---

#2 $[2D^2 - 5DD' + 2D'^2]z = 24(x-y)$

CF $\Rightarrow 2D(D-2D') - D'(D-2D')z = 0$

$(2D-D')(D-2D')z = 0$

$CF = \phi_1(y+2x) + \phi_2(2y+x)$

$PI \rightarrow \frac{1}{2D^2 - 5DD' + 2D'^2} 24[y-x]$

$F(a,b) \neq 0$

$a = -1$

$PI = \frac{4}{9}(y-x)^3$

integraly 2 times dueto 2 variable

---

#3. $(D^2 + 2DD' + D'^2)z = e^{2x+3y}$

$PI = \frac{1}{D^2 + 2DD' + D'^2} e^{2x+3y}$

$a = 2$

$b = 3$

$F(a,b) = \frac{1}{4+2(2)(3)+9} = \frac{1}{25}$.

$PI = \frac{1}{25} e^{2x+3y}$

$e^u$ only 2 times $\longrightarrow e^u$

# $4r - 4s + t = 16\log(x+2y)$

CF $\rightarrow \phi_1(2y+x) + x\phi_2(2y+x)$

$PI : \frac{1}{(2D-D')^2}$

$a = 1$
$b = 2$

$F(a,b) = 0$

using formula :-

$$\frac{x^2}{2^2 \times 2!}[16\log(x+2y)] = 2x^2 \log[x+2y]$$

---

# $[2D^2 - 5DD' + 2D'^2]z = 5\sin(2x+y)$

$CF = \phi_1(2x+y) + \phi_2(2y+x)$

$5\sin(2x+y)$

$a = 2$
$b = 1$

$5\sin(2x+y) \rightarrow$ this part zero

$PI = \frac{1}{2D^2 - 5DD' + 2D'^2}$

wrt to $x$

hence ; $\frac{5}{3(2D')}\ -\cos(2x+y) = \frac{-5}{3}\cos(2x+y)$

---

# $(D^3 - 4D^2D' + 4DD'^2)z = 4\sin(x+y)$

$\phi_1(y+2x) + \phi_2(y) + x\phi_3(y+2x)$

$PI = -x^2\cos(x+y)$

case II : PI of the function $f(x,y)$ in the form of $x^m y^n$.

then $\frac{1}{F(D,D')}$ expand it in an infinite series of ascendency power $D$ & $D'$.

---

# solve $(D^2 - a^2 D'^2) z = x$.

C.F. $\rightarrow \phi_1(y+ax) + \phi_2(y-ax)$

P.I. $\rightarrow$ $\frac{1}{D^2 - a^2 D'^2} x = \frac{1}{D^2 \left[1 - \frac{a^2 D'^2}{D^2}\right]} x - \frac{1}{D^2}\left[1 - \frac{a^2 D'^2}{D^2}\right]^{-1} x$

$\frac{1}{D^2}\left[1 + \frac{a^2 D'^2}{D^2} \cdots\right) x = \frac{1}{D^2} x = \frac{x^3}{6}$

---

# solve $(D^2 - 2DD' - D'^2) z = e^{x+3y} + x^3$

CF : $\phi_1(y+x) + x\,\phi_2(y+x)$

$PI = \frac{1}{(D-D')^2} e^{x+3y} + \frac{1}{(D-D')^2} x^3 y^0$

$PI \longrightarrow \frac{e^{x+3y}}{4} + \frac{1}{D^2}\left[1 - \frac{D'}{D}\right)^{-2} x^3 \quad x^3 y^0.$

$PI \rightarrow \frac{e^{x+3y}}{4} + \frac{1}{D^2}\left[1 + \frac{2D'}{D} \cdots\right) x^3$

$PI \rightarrow \frac{e^{x+3y}}{4} + \frac{x^3}{D^2}$

$PI = \frac{e^{x+3y}}{4} + \frac{x^5}{20}$

Case III : PI of $F(x,y)$ is the form of
$y(g(x))$ $g$ is a function of $x$

then CF →

$\phi(biy + aix) \rightarrow (biD - aix)$

put :- $biy + aix = c$

$$\boxed{y = \frac{c - aix}{bi}}$$

Ans.