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ASSIGNMENT COVER SHEET

This coversheet must be attached to the front of your assessment

The assessment is due at 5pm unless otherwise specified in the course outline.

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Course Code COMP6261

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Assignment Item Assignment 1

Due Date 2022.8.29 5pm

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Signature Yifan Luo

Q.2

$$(a) H(X) = \sum_x p(x) \log_2 \frac{1}{p(x)} = -q_1 \log_2 q_1 - q_2 \log_2 q_2 \dots - q_m \log_2 q_m = - \sum_{i=1}^m q_i \log_2 q_i$$

$$H(Y) = \sum_y p(y) \log_2 \frac{1}{p(y)} = -\frac{q_1}{5} \log_2 \frac{q_1}{5} - \dots - \frac{q_m}{5} \log_2 \frac{q_m}{5} - \frac{4}{5} \log_2 \frac{4}{5} = -\frac{1}{5} \sum_{i=1}^m q_i (\log_2 q_i - \log_2 5) - \frac{4}{5} \log_2 \frac{4}{5}$$

$$= -\frac{1}{5} \sum_{i=1}^m q_i \log_2 q_i + \frac{1}{5} \sum_{i=1}^m q_i \log_2 5 - \frac{4}{5} \log_2 \frac{4}{5}$$

$$= \frac{1}{5} H(X) - \frac{1}{5} \log_2 \frac{1}{5} - \frac{4}{5} \log_2 \frac{4}{5} = \frac{1}{5} H(X) + \frac{1}{5} \log_2 5 + \frac{4}{5} \log_2 \frac{4}{5}$$

$$(b) I(\text{top side}; \text{bottom side}) = H(\text{top side}) - H(\text{top side} | \text{bottom side}) = \frac{1}{5} H(X) + \log_2 5 - \frac{8}{5}$$

$$= \sum_{t \in \{0,1\}} -p(\text{top side} = t) \log_2 p(\text{top side} = t) - \sum_{b \in \{1,0\}} p(\text{bottom side} = b) H(\text{top side} | \text{bottom side} = b)$$

$$= -2 \times \frac{1}{2} \log_2 \frac{1}{2} - 2 \times \frac{1}{2} \times 0 = 1 - 0 = 1$$

$$(c) I(\text{top side}; \text{front side}) = H(\text{top side}) - H(\text{top side} | \text{front side})$$

$$H(\text{top side}) = - \sum_{t \in \{1, \dots, b\}} p(\text{top side} = t) \log_2 p(\text{top side} = t) = -6 \times \frac{1}{6} \log_2 \frac{1}{6} = \log_2 6$$

$$H(\text{top side} | \text{front side}) = \sum_{f \in \{1, \dots, b\}} p(\text{front side} = f) H(\text{top side} | \text{front side} = f)$$

$$= 6 \times \frac{1}{6} \times 4 \times \frac{1}{4} \log_2 4 = \log_2 4 = 2$$

$$\therefore I(\text{top side}; \text{front side}) = H(\text{top side}) - H(\text{top side}, \text{front side}) = \log_2 6 - 2 = \log_2 3 - 1$$

$$(d) \text{ Let } x \in \{p_1, \dots, p_i, \dots, p_j, \dots, p_m\}, y \in \{p_1, \dots, \frac{p_i+p_j}{2}, \dots, \frac{p_i+p_j}{2}, \dots, p_m\}, H(X) = \sum_{a=1}^m p_a \log_2 \frac{1}{p_a}$$

$$H(Y) = \underbrace{\sum_{a=1}^{i-1} p_a \log_2 \frac{1}{p_a} + \frac{p_i+p_j}{2} \log_2 \frac{1}{\frac{p_i+p_j}{2}}}_{= H(X) - p_i \log_2 \frac{1}{p_i} - p_j \log_2 \frac{1}{p_j}} + \underbrace{\sum_{b=i+1}^{j-1} p_b \log_2 \frac{1}{p_b} + \frac{p_i+p_j}{2} \log_2 \frac{1}{\frac{p_i+p_j}{2}} + \sum_{c=j+1}^m p_c \log_2 \frac{1}{p_c}}_{+ 2 \times \frac{p_i+p_j}{2} \log_2 \frac{1}{\frac{p_i+p_j}{2}}}$$

$$H(Y) - H(X) = (p_i + p_j) \log_2 \frac{2}{p_i + p_j} - p_i \log_2 \frac{1}{p_i} - p_j \log_2 \frac{1}{p_j} = (p_i + p_j) \log_2 \frac{2}{p_i + p_j} - (p_i \log_2 \frac{1}{p_i} + p_j \log_2 \frac{1}{p_j})$$

$$= (p_i \log_2 \frac{2}{p_i + p_j} - p_i \log_2 \frac{1}{p_i}) + (p_j \log_2 \frac{2}{p_i + p_j} - p_j \log_2 \frac{1}{p_j})$$

$$= p_i (\log_2 \frac{2}{p_i + p_j} - \log_2 \frac{1}{p_i}) + p_j (\log_2 \frac{2}{p_i + p_j} - \log_2 \frac{1}{p_j})$$

$$= p_i \log_2 \frac{2p_i}{p_i + p_j} + p_j \log_2 \frac{2p_j}{p_i + p_j}$$

$$\geq p_i \left(1 - \frac{p_i + p_j}{2p_i}\right) + p_j \left(1 - \frac{p_i + p_j}{2p_j}\right)$$

$$= p_i + p_j - \frac{p_i + p_j}{2} - \frac{p_i + p_j}{2} = 0$$

$$\therefore H(Y) \geq H(X)$$

Q3 1. (a) $h(C \text{ a red non-face card}) = h(C = \text{red}, f=0) = h(S \in \{\text{heart, diamond}\}, V \in \{2, 3, \dots, 9\})$

$$= \log_2 \frac{1}{P(SV)} = \log_2 \frac{1}{\frac{1}{48}} = \log_2 \frac{48}{1} = \log_2 48$$

(b) $h(V=k | f=1) = \log_2 \frac{1}{P(V=k | f=1)}$

$$= \log_2 \frac{1}{\frac{1}{4}} = 2$$

(c) $H(S) = \sum_S P(S) \log_2 \frac{1}{P(S)}$

$$= 4 \times \frac{1}{4} \log_2 \frac{1}{\frac{1}{4}} = 2$$

$$H(V, S) = \sum_V \sum_S P(V, S) \log_2 \frac{1}{P(V, S)}$$

$$= 12 \times 4 \times \frac{1}{48} \log_2 \frac{1}{\frac{1}{48}} = \log_2 48$$

(d) $I(V; S) = \sum_{v \in V} \sum_{s \in S} P(V, s) \log_2 \frac{P(V, s)}{P(V)P(s)} = \sum_V \sum_S P(V, S) \log_2 1 = 0$

(e) $I(V; C) = \sum_{v \in V} \sum_{c \in C} P(V, c) \log_2 \frac{P(V, c)}{P(V)P(c)} = \sum_V \sum_C P(V, C) \log_2 1 = 0$

2. (a) $H(S) = \sum_{S \in S} P(S) \log_2 \frac{1}{P(S)} = P(\text{diamond}) \log_2 \frac{1}{P(\text{diamond})} + P(\text{clubs}) \log_2 \frac{1}{P(\text{clubs})}$
 $+ P(\text{hearts}) \log_2 \frac{1}{P(\text{hearts})}$
 $= 2 \times \frac{12}{32} \log_2 \frac{1}{\frac{12}{32}} + \frac{8}{32} \log_2 \frac{1}{\frac{8}{32}}$
 $= \frac{3}{4} \log_2 \frac{8}{3} + \frac{1}{4} \log_2 4 = \frac{1}{2} + \frac{3}{4} \log_2 \frac{8}{3}$

$H(V|S) = \sum_S P(S) H(V|S=S) = \sum_S P(S) \sum_V P(V|S=S) \log_2 \frac{1}{P(V|S=S)}$

$$= 2 \times \frac{12}{32} \times 12 \times \frac{1}{12} \times \log_2 \frac{1}{\frac{1}{12}} + \frac{8}{32} \times 8 \times \frac{1}{8} \times \log_2 \frac{1}{\frac{1}{8}}$$

$$= \frac{3}{4} \times \log_2 12 + \frac{1}{4} \times \log_2 8 = \frac{3}{4} \log_2 12 + \frac{3}{4}$$

$$H(V|S) = H(S) + H(V|S) = \frac{1}{2} + \frac{3}{4} \log_2 \frac{8}{3} + \frac{3}{4} + \frac{3}{4} \log_2 12 = \frac{5}{4} + \frac{3}{4} (\log_2 \frac{8}{3} + \log_2 12)$$

$$= \frac{5}{4} + \frac{3}{4} \log_2 32 = \frac{5}{4} + \frac{15}{4} = 5$$

(b) $I(V; S) = H(V) - H(V|S)$

$$\begin{aligned} &= \sum_V P(V) \log_2 \frac{1}{P(V)} - \left(\frac{3}{4} \log_2 12 + \frac{3}{4} \right) \\ &= 4 \times \frac{12}{32} \log_2 \frac{1}{\frac{12}{32}} + 8 \times \frac{8}{32} \log_2 \frac{1}{\frac{8}{32}} - \left(\frac{3}{4} \log_2 12 + \frac{3}{4} \right) \\ &= \frac{1}{4} \times 4 + \frac{3}{4} \times \log_2 \frac{8}{3} - \frac{3}{4} \log_2 12 - \frac{3}{4} \\ &= \frac{1}{4} + \frac{3}{4} \left(\log_2 \frac{8}{3} - \log_2 12 \right) \\ &= \frac{1}{4} + \frac{3}{4} \log_2 \frac{8}{9} \end{aligned}$$

Because after cards are moved, the information of suit will convey information of value. Diamonds and clubs have chance to observing 2, 3, 4, 5, but hearts not.

$$(c) I(V;S|C) = H(V|C) - H(V|S,C)$$

$$H(V|C) = \sum_c p(c) H(V|C=c)$$

$$\begin{aligned} &= \frac{20}{32} \sum_v p(v|C=r) \log_2 \frac{1}{p(v|C=r)} + \frac{12}{32} \sum_v p(v|C=b) \log_2 \frac{1}{p(v|C=b)} \\ &= \frac{20}{32} (4 \times \frac{1}{20} \log_2 \frac{1}{\frac{1}{20}} + 8 \times \frac{2}{20} \log_2 \frac{1}{\frac{2}{20}}) + \frac{12}{32} (12 \times \frac{1}{12} \log_2 \frac{1}{\frac{1}{12}}) \\ &= \frac{20}{32} (\frac{1}{5} \log_2 20 + \frac{4}{5} \log_2 10) + \frac{12}{32} \log_2 12 \\ &= \frac{1}{8} \log_2 20 + \frac{1}{2} \log_2 10 + \frac{3}{8} \log_2 12 \end{aligned}$$

$$H(V|S,C) = H(V|S) = \frac{3}{4} \log_2 12 + \frac{3}{4}$$

$$\begin{aligned} I(V;S|C) &= \frac{1}{8} \log_2 20 + \frac{1}{2} \log_2 10 - \frac{3}{8} \log_2 12 - \frac{3}{4} \\ &= -\frac{5}{4} + \frac{1}{8} \log_2 25 + \frac{1}{2} \log_2 10 - \frac{3}{8} \log_2 3 \quad \text{X} \end{aligned}$$

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Q4

$$m = p(x) , D = p(x|Y=1)$$

$$q = p(x|Y=0)$$

(a). According to the definition, $p(Y) = \frac{1}{2}$

$$p := p(x|Y=1), q := p(x|Y=0)$$

$$\begin{aligned} m &:= \frac{p+q}{2} = \frac{1}{2}(p(x|Y=1) + p(x|Y=0)) \\ &= p(Y=1)p(x|Y=1) + p(Y=0)p(x|Y=0) \\ &= p(x) \end{aligned}$$

$$\text{Then, } \frac{D_{KL}(p||m) + D_{KL}(q||m)}{2}$$

$$\begin{aligned} &= \frac{1}{2} (D_{KL}(p(x|Y=1)||p(x)) + D_{KL}(p(x|Y=0)||p(x))) \\ &= \frac{1}{2} D_{KL}(p(x|Y=1)||p(x)) + \frac{1}{2} D_{KL}(p(x|Y=0)||p(x)) \\ &= p(Y=1) \cdot \sum_x p(x|Y=1) \log \frac{p(x|Y=1)}{p(x)} + p(Y=0) \cdot \sum_x p(x|Y=0) \log \frac{p(x|Y=0)}{p(x)} \\ &= \sum_x \sum_Y p(Y) p(x|Y) \log \frac{p(x|Y)}{p(x)} \\ &= \sum_x \sum_Y p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \\ &= \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \\ &= I(X;Y) \end{aligned}$$



(b) In (a), we see that $m = \frac{p+q}{2} = p(x)$, $p(x=a) = \frac{2}{5}$, $p(x=b) = \frac{2}{5}$, $p(x=c) = \frac{1}{5}$

$$\begin{aligned} I(X;Y) &= p(Y=1) \sum_x p(x|Y=1) \log \frac{p(x|Y=1)}{p(x)} + p(Y=0) \sum_x p(x|Y=0) \log \frac{p(x|Y=0)}{p(x)} \\ &= \frac{1}{2} \cdot [\frac{3}{5} \cdot \log \frac{\frac{3}{5}}{\frac{2}{5}} + \frac{1}{5} \cdot \log \frac{\frac{1}{5}}{\frac{2}{5}} + \frac{1}{5} \cdot \log \frac{\frac{1}{5}}{\frac{1}{5}}] + \\ &\quad \frac{1}{2} \cdot [\frac{1}{5} \cdot \log \frac{\frac{1}{5}}{\frac{3}{5}} + \frac{3}{5} \cdot \log \frac{\frac{3}{5}}{\frac{2}{5}} + \frac{1}{5} \cdot \log \frac{\frac{1}{5}}{\frac{1}{5}}] \\ &= \frac{1}{2} \cdot [\frac{3}{5} \cdot \log \frac{3}{2} + \frac{1}{5} \cdot \log \frac{1}{2} + 0] + \frac{1}{2} \cdot [\frac{1}{5} \cdot \log \frac{1}{2} + \frac{3}{5} \cdot \log \frac{3}{2} + 0] \\ &= \frac{3}{5} \cdot \log \frac{3}{2} + \frac{1}{5} \cdot \log \frac{1}{2} \\ &= \frac{3}{5} \log 3 - \frac{4}{5} \end{aligned}$$

$$(c) p(Z=x|Y=y) = p(X=x|Y=1-y)$$

$$p = (p(X=a|Y=1) = \frac{3}{5}, p(X=b|Y=1) = \frac{1}{5}, p(X=c|Y=1) = \frac{1}{5})$$

$$q = (p(X=a|Y=0) = \frac{1}{5}, p(X=b|Y=0) = \frac{3}{5}, p(X=c|Y=0) = \frac{1}{5})$$

$$p' = (p(Z=a|Y=1) = p(X=a|Y=0) = \frac{1}{5}, p(Z=b|Y=1) = \frac{3}{5}, p(Z=c|Y=1) = \frac{1}{5})$$

$$q' = (p'(Z=a|Y=0) = p(X=a|Y=1) = \frac{3}{5}, p(Z=b|Y=0) = \frac{1}{5}, p(Z=c|Y=0) = \frac{1}{5})$$

$$\therefore p' = q, q' = p.$$

$$I(Z;Y) = \frac{D_{KL}(p' \parallel m') + D_{KL}(q' \parallel m')}{2}, \text{ where } m' = \frac{p'+q'}{2} = \frac{q+p}{2} = m$$

$$= \frac{D_{KL}(q' \parallel m) + D_{KL}(p' \parallel m)}{2}$$

$$= I(X;Y) = \frac{3}{5} \cdot \log \frac{3}{2} + \frac{1}{5} \cdot \log \frac{1}{2}$$

$\therefore I(Z;Y) = I(X;Y)$ Intuitively, Y brings the same uncertainty reduction for both X and Z . and $p(Z|Y)$ is the same as $p(X|Y)$

$$(d) \text{ Suppose } p(Z=a|Y=0) = p(Z=b|Y=0) = p(Z=c|Y=0) = \frac{1}{3}$$

$$p(Z=a|Y=1) = p(Z=b|Y=1) = p(Z=c|Y=1) = \frac{1}{3}$$

then $p(Z=a) = p(Z=b) = p(Z=c) = \frac{1}{3}$, Z and Y are independent.

$$\text{Thus, } I(X;Y) > I(Z;Y) = 0$$

Intuitively, Y conveys no information of Z , because they are independent.

$$(e) \text{ Suppose } p(Z=a|Y=0) = p(Z=b|Y=0) = p(Z=c|Y=0) = 0$$

$$p(Z=a|Y=1) = p(Z=b|Y=1) = p(Z=c|Y=1) = \frac{1}{3}$$

then $p(Z=a) = p(Z=b) = p(Z=c) = \frac{1}{3}$

$$I(Z;Y) = H(Z) - H(Z|Y)$$

$$= 3 \times \frac{1}{3} \log_2 \frac{1}{\frac{1}{3}} - \frac{1}{2} \times 3 \times \frac{1}{3} \log \frac{1}{\frac{1}{3}}$$

$$= \log_2 3 - \frac{1}{2} \log_2 3$$

$$= \frac{1}{2} \log_2 3$$

$$I(X;Y) < I(Z;Y)$$

Intuitively, given the knowledge of Y , we know that Z will be 0 if Y equals to 0, otherwise Z has equal probability of a, b, c. But X is less dependent on Y , because X is more averagely distributed.

Q.5 (a). Let $a > 0, b > 0$, then represent X as $X = \lambda_1 a + \lambda_2 b$, $\lambda_1 + \lambda_2 = 1$

$\therefore t > 0$ and $a \leq X \leq b$

$$\therefore \lambda_1 = \frac{b-X}{b-a}, \lambda_2 = \frac{X-a}{b-a}$$

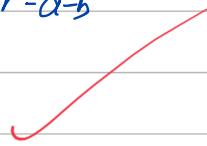
$E(e^{tX})$ is a moment generating function for X .

then $g(t(b-a))$ is the convexion point of e^{tX}

$$\therefore g(u) = \log(1-r+re^u) - ru, r = \frac{a}{b-a}$$

$$\therefore \log E(e^{tX}) \leq g(t(b-a))$$

$$\therefore E(e^{tX}) \leq e^{g(t(b-a))}$$



(b) Taylor theorem: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + h_k(x)(x-a)^k$

$$(\lim_{x \rightarrow a} h_k(x) = 0)$$

$\therefore e^{\lambda X}$ is convex,

$$\therefore e^{\lambda X} \leq \frac{b-X}{b-a} e^{\lambda a} + \frac{X-a}{b-a} e^{\lambda b}$$

$$\therefore E(e^{\lambda X}) \leq \frac{b-EX}{b-a} e^{\lambda a} + \frac{EX-a}{b-a} e^{\lambda b} = \frac{b}{b-a} e^{\lambda a} + \frac{-a}{b-a} e^{\lambda b} = e^{\lambda(a(b-a))}$$

$$\text{Let } L(x) = \frac{x a}{b-a} + \ln\left(1 + \frac{a - e^x a}{b-a}\right)$$

$$\therefore L(0) = L'(0) \text{ and } L''(x) \leq \frac{1}{4}$$

\therefore By Taylor theorem, $L(x) = L(0) + x L'(0) + \frac{1}{2} h^2 L''(x) \leq \frac{1}{8} x^2$

$$\text{Thus } E(e^{\lambda X}) \leq e^{\frac{\lambda^2(b-a)^2}{8}}$$

