COMP2610/COMP6261 - Information Theory

Tutorial 11: Noisy Channel Coding

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Week 11 (17th – 21st Oct), Semester 2, 2022

1. Consider a channel with inputs X = {a,b,c}, outputs Y = {a,b,c,d}, and transition matrix

$$Q = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix}.$$

(a) Assuming $p_X = (0.25, 0.25, 0.5)$, what is the mutual information $I(X;Y)$ between the input and output of the channel?

(b) Assuming $p_X = (0.25, 0.25, 0.5)$, what is the average probability of error of the channel?

(c) Calvin claims that he has constructed a block code for Q with rate 0.01 bits per transmission and maximal block error probability 1%. Is his claim possible? Justify your answer.

(d) Hobbes claims that he has constructed a block code for Q with rate 100 bits per transmission and maximal block error probability 1%. Is his claim possible? Justify your answer.

2. *Noisy Coding (Exercise 10.12 in MacKay)*

(a) A binary erasure channel with input $x \in \{0,1\}$ and output $y \in \{0,?,1\}$ has transition matrix

$$Q_E = \begin{bmatrix} 1-q & 0 \\ q & q \\ 0 & 1-q \end{bmatrix}.$$

Find the mutual information $I(X;Y)$ between the input and output for a general distribution $\mathbf{p}_X = (p_0, p_1)$ over inputs. Show that the capacity of this channel is $C_E = 1 - q$ bits.

3. Let X be an ensemble with $\mathcal{A}_X$ = {a,b,c} and probabilities $\mathbf{p}$ = (0.5,0.25,0.25). Write out the alphabet and probabilities for the extended ensemble X^2.

4. Consider a binary erasure channel with input $x \in \{0,1\}$, and $p_X = (0.6, 0.4)$, and transition matrix of

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{3}{4} \end{bmatrix}.$$

An input sequence of length n = 10 is drawn and it is send through the channel. The resulting input and output sequences are 0001111000 and 0011110001, respectively.

(a) Is the drawn input sequence typical in p_X ? Justify your answer.

(b) Is the drawn input and output sequences jointly typical ? Justify your answer.

Tutorial 11 solution

1. Consider a channel with inputs X = {a,b,c}, outputs Y = {a,b,c,d}, and transition matrix

i/p → X

o/p → Y

$$Q = \begin{array}{c} \\ Y=a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} X=a & b & c \end{array} \\ \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix} \end{array}.$$

$P(Y_i/X_i)$

(a) Assuming $p_X = (0.25, 0.25, 0.5)$, what is the mutual information $I(X;Y)$ between the input and output of the channel?

(b) Assuming $p_X = (0.25, 0.25, 0.5)$, what is the average probability of error of the channel?

(c) Calvin claims that he has constructed a block code for Q with rate 0.01 bits per transmission and maximal block error probability 1%. Is his claim possible? Justify your answer.

(d) Hobbes claims that he has constructed a block code for Q with rate 100 bits per transmission and maximal block error probability 1%. Is his claim possible? Justify your answer.

soln.

Given, $X = \{a, b, c\}$ and $Y = \{a, b, c, d\}$

$$Q = \begin{array}{c} \\ y=a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} x=a & x=b & x=c \end{array} \\ \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix} \end{array} \longleftarrow P(Y_i/X_i)$$

i) $P_x = (0.25, 0.25, 0.5)$, To find $I(X;Y)$?

Mutual information $I(X;Y) = \sum_x \sum_y P(x,y) \log_2 \left(\frac{P(x,y)}{P(x)P(y)}\right)$

$= H(X) + H(Y) - H(X,Y)$

$$\boxed{I(X;Y) = H(X) - H(X/Y) = H(Y) - H(Y/X)} \quad \text{---} \; (i)$$

where, $H(Y/X) = \sum p(x)\, H(Y/X=x)$

for $X = a$,

$H(Y/X=a) = -\sum p(y_i/X=a) \log_2 [p(y_i/X=a)]$ $\longrightarrow \log_2 \frac{1}{P(x)} = -\log_2 P(x)$

$= (-0.5 \log_2 0.5) \times 2$

$= 1$

Similararally, for $x=b$,

$$H(Y/x=b) = 2\times\left(-0.5\log_2 0.5\right) = 1$$

and, for $x=c$,

$$H(Y/x=c) = 2\times\left(-0.5\log_2 0.5\right) = 1$$

$$P(Y/X) = \frac{P(X,Y)}{P(X)}$$

$$\Rightarrow P(X,Y) = P(Y/X)\cdot P(X)$$

therefore, $H(Y/X) = P(x=a)\,H(Y/X=a) + P(X=b)\,H(Y/X=b) +$
$P(X=c)\,H(Y/X=c)$

$$\Rightarrow H(Y/X) = \left(\frac{1}{4}\right)\times 1 + \left(\frac{1}{4}\right)\times 1 + \left(\frac{1}{2}\right)\times 1 = 1$$

$$p(y_i,x_i) = \begin{bmatrix} 1/8 & 0 & 0 \\ 1/8 & 1/8 & 0 \\ 0 & 1/8 & 1/4 \\ 0 & 0 & 1/4 \end{bmatrix} \quad \text{--- (ii)}$$

Now, $P(Y)$:

$$P(Y/X) = \begin{array}{c|c} & \begin{array}{ccc} x=a & x=b & x=c \end{array} \\ \begin{array}{c} y=a \\ y=b \\ y=c \\ y=d \end{array} & \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix} \end{array}$$

$$P(Y_i/X_i) = \frac{P(Y_i,X_i)}{P(X_i)}$$

$p(y=a) = p(y=a/x=a)*p(x=a) + p(y=a/x=b)*p(x=b) + p(y=a/x=c)*p(x=c)$
$= 0.5\times 0.25 + 0\times 0.25 + 0\times 0.5$
$= 1/8 + 0 + 0 = 1/8$

Similararally, $p(y=b) = p(y=b/x=a)\,p(x=a) + \text{---}$

$$= 0.5\times 0.25 + 0.5\times 0.25 + 0 = 1/8 + 1/8 = 1/4$$

$p(y=c) = p(y=c/x=a)\,p(x=a) + \text{-----}$

$$= 0 + 0.5\times 0.25 + 0.5\times 0.5 = \frac{1}{8} + 1/4 = 3/8$$

and, $p(y=d) = p(y=d/x=a)\,p(x=a) + \text{----}$

$$= 0 + 0 + 0.5\times 0.5 = 1/4$$

$$p(y) = \left(1/8,\ 1/4,\ 3/8,\ 1/4\right)$$

therefore, $H(Y) = \frac{1}{8}\log_2 8 + \frac{1}{4}\log_2 4 + \frac{3}{8}\log_2(8/3) + 1/4\log_2 4$

$$= 1.9056 \quad \text{--- (iii)}$$

Hence, $I(X;Y) = H(Y) - H(Y/X) = 1.9056 - 1$

$$= 0.9056$$

b> For $P_x = (0.25, 0.25, 0.5)$, the required average probability of error will be,

Prob. of error $(P_{err}) = P(Y \neq a, X = a) +$

$P(Y \neq b, X = b) + P(Y \neq c, X = c)$

Here, $P(Y \neq a, X = a) = P(Y = b, X = a) +$

$P(Y = c, X = a) + P(Y = d, X = a)$

therefore,

$$P_{err} = (0.5 \times 0.25 + 0 + 0) + (0 + 0.5 \times 0.25 + 0)$$
$$+ (0 + 0 + 0.5 \times 0.5)$$
$$= \left(\frac{1}{8}\right) + \left(\frac{1}{8}\right) + \left(\frac{1}{4}\right) = 1/2$$

1.c> According to calvin's claim,

Block rate = 0.01 bits/channel

and, Maxm. error probability = 1 %

from part 1.a, $I(X;Y) = 0.9056$

$\Rightarrow$ channel capacity $(C) = \max_{P_X}(I(X;Y))$

≥ 0.9056

Since, block rate is less than channel capacity $= 0.9056$. Hence, claim is true.

1.d) According to Hobbes,

Block rate = 100 bits/transmission

Maxm. error prob. = 1%

Since, $\max(I(X;Y)) \le \max(\log_2|X|, \log_2|Y|)$

$$\Rightarrow C_{max} \le \max(\log_2 \underbrace{|X|}_{\text{size of } X}, \log_2 \underbrace{|Y|}_{\text{size of } Y})$$

$$= \log_2 4 = 2$$

$I(X;Y) = H(X) - H(X/Y)$

$\le H(X)$ --- (i)

$I(X;Y) = H(Y) - H(Y/X)$

$\le H(Y)$ --- (ii)

$\Rightarrow C = \max(I(X;Y)) \le \max(\log_2|X|, \log_2|Y|)$

For this case,

$$\text{Rate}(P_{error})\Big|_{P_{error}=0.01} = \frac{C_{max}}{1 - H_2(0.01)}$$

$$= \frac{2}{0.9192} = 2.2$$

So, no rate above 2.2 bits/channel can be achieved with Probability of error $(P_{err}) = 0.01$

Hence, False claim.

2. *Noisy Coding (Exercise 10.12 in MacKay)*

(a) A binary erasure channel with input $x \in \{0,1\}$ and output $y \in \{0,?,1\}$ has transition matrix

$$Q_E = \begin{bmatrix} 1-q & 0 \\ q & q \\ 0 & 1-q \end{bmatrix}.$$

Find the mutual information $I(X;Y)$ between the input and output for a general distribution $\mathbf{p}_X = (p_0, p_1)$ over inputs. Show that the capacity of this channel is $C_E = 1 - q$ bits.

soln.: Binary Erasure channel,

i/p: $x \in \{0,1\}$ and o/p: $y \in \{0,?,1\}$

$P_x = \{P_0, P_1\} = \{P_0, 1-P_0\}$

and, $Q_E = \begin{bmatrix} 1-q & 0 \\ q & q \\ 0 & 1-q \end{bmatrix}$ ← $P(Y_i/X_i) = \dfrac{P(X_i, Y_i)}{P(X_i)}$

$$\boxed{\begin{aligned} I(X;Y) &= H(X) - H(X/Y) \\ &= H(Y) - H(Y/X) \end{aligned}}$$

$H(X) = -P_0 \log_2 P_0 - P_1 \log_2 P_1 = -P_0 \log_2 P_0 - (1-P_0)\log_2(1-P_0)$

$= H_2(P_0)$ [You can also write this one as $H_2(P_0, 1-P_0)$]

---- (i)

for $H(X/Y)$:

$$H(X/Y) = -\sum_{\substack{y_i \\ P_0 \\ X_1}} P(y) \sum_{\substack{x_i \\ (1-P_0) \\ X_2}} P(x_i/y=y_i) \log_2 P(x_i/y=y_i) \quad \text{---- (ii)}$$

since, $Q = \begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \end{array} \begin{bmatrix} 1-q & 0 \\ q & q \\ 0 & 1-q \end{bmatrix} = P(Y_i/X_i) \times P(X_i) = P(X_i, Y_i)$

therefore, $$P(x_i, y_i) = \begin{array}{c|cc|c} & 0 & 1 & \\ \hline 0 & P_0(1-q) & 0 & y=0 \\ ? & P_0 q & (1-P_0)q & y=? \\ 1 & 0 & (1-P_0)(1-q_0) & y=1 \end{array}$$

Note: $P(x_i, y_i) = P(y_i/x_i)P(x_i)$

$$P(y) = \begin{cases} P_0(1-q) & y=0 \\ P_0 q + (1-P_0)q = q & y=? \\ (1-P_0)(1-q_0) & y=1 \end{cases}$$

therefore, $P(X|Y) = \dfrac{P(X,Y)}{P(Y)}$

$$\Rightarrow P(X_i/Y_i) = \begin{array}{c|cc} & X=0 & X=1 \\ \hline Y=0 & 1 & 0 \\ Y=? & P_0 & (1-P_0) \\ Y=1 & 0 & 1 \end{array}$$

therefore, $H(X|Y=0) = -P(X|Y=0)\log_2(X/Y=0)$

$$= -1\log_2 1 - 0\log_2 0 = 0$$

similarally, $H(X/Y=?) = H_2(P_0, (1-P_0))$

and $H(X/Y=1) = 0$

$$\Rightarrow H(X/Y) = P(Y=0)H(X/Y=0) + P(Y=?)H(X/Y=?) + P(Y=1)H(X/Y=1)$$

$$= P_0 \cancel{(1-q)} \times 0 + q \times H_2(P_0, 1-P_0) + \cancel{(1-P_0)(1-q)} \times 0$$

$$= q H_2(P_0, 1-P_0) = q H_2(P_0) \quad \text{--- (iii)}$$

Now, $I(X;Y) = H(X) - H(X/Y)$

$$= H_2(P_0) - q H_2(P_0) \quad \text{--- (iv)}$$

Capacity $(C_\epsilon) = \max\limits_{P_x} \left(I(X;Y) \right)$

$$= \max \left(H_2(P_0) [1-q] \right)$$

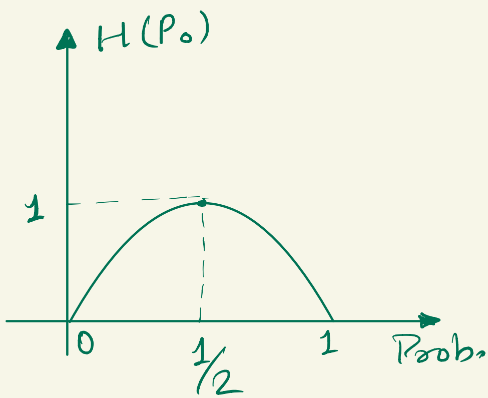
$$= (1-q) \max \left(H_2(P_0) \right)$$

We know that,

$\max(H_2(P_0)) = 1$

therefore,

$C_\epsilon = 1 - q$



Proved

3. Let X be an ensemble with A_X = {a,b,c} and probabilities $\mathbf{p}$ = (0.5,0.25,0.25). Write out the alphabet and probabilities for the extended ensemble X^2.

Soln. Given, $A_X = \{a, b, c\}$ and $p = \{0.5, 0.25, 0.25\}$

Alphabet $X^2 = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$

$$\Rightarrow P(X^2) = (0.5 \times 0.5,\ 0.5 \times 0.25,\ 0.5 \times 0.25, \ldots)$$

$$= \left(\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}\right)$$

4. Consider a binary erasure channel with input $x \in \{0,1\}$, and $p_X = (0.6, 0.4)$, and transition matrix of

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{3}{4} \end{bmatrix}.$$

An input sequence of length n = 10 is drawn and it is send through the channel. The resulting input and output sequences are 0001111000 and 0011110001, respectively.

(a) Is the drawn input sequence typical in p_X ? Justify your answer.
(b) Is the drawn input and output sequences jointly typical ? Justify your answer.

Soln.: Given, $X \in \{0, 1\}$ and $P_X = (0.6, 0.4)$

and transition matrix $\begin{bmatrix} 2/3 & 1/4 \\ 1/3 & 3/4 \end{bmatrix} P(Y/X)$

An input sequence of length $n = 10$ is drawn and is send through the channel. Resultant i/p and o/p

i/p : 0 0 0 1 1 1 1 0 0 0   X

o/p : 0 0 1 1 1 1 0 0 0 1   Y

a) Given, $P_X = (0.6, 0.4)$

Also, $X^{10} = 0001111000$

To check if the drawn sequence is typical in P_X,

Calculate Emperical probabilities:

Out of transmitted data 6 are '0's and 4 are '1's

$\Rightarrow P_{emperical}(x=0) = 6/10 = 0.6$

$\Rightarrow P_{emperical}(x=1) = 4/10 = 0.4$

Here, Emperical probabilities $(P_{emperical}) = (0.6, 0.4)$

Therefore, X^{10} is typical in P_X.

b)

$$Q = \begin{array}{c|cc} & x=0 & x=1 \\ \hline y=0 & 2/3 & 1/4 \\ y=1 & 1/3 & 3/4 \end{array} \longrightarrow P(Y_i/X_i)$$

$$P(y=0) = P(Y=0/x_i)\,P(x_i) = \left(\frac{2}{3}\right) \times 0.6 + \left(\frac{1}{4}\right)(0.4)$$
$$= 0.5$$

$$P(Y=1) = P(Y=1/X_i)\,P(X_i) = \left(\frac{1}{3}\right)(0.6) + \left(\frac{3}{4}\right)(0.4)$$
$$= 0.5$$

Emperical probability : $P_{Y\text{-}emp.}$

Out of received data, 5 are 1's and 5 are 0's

$P_{Y\text{-}emp.}(Y=0) = 5/10 = 1/2$

$P_{Y\text{-}emp}(Y=1) = 5/10 = 1/2$

$\Rightarrow P_{Y\text{-}emp.} = (1/2, 1/2)$

Clearly, Emperical probability of Y $(P_{Y\text{-}emp}) = (\frac{1}{2}, \frac{1}{2})$

$= P(Y)$

Hence, the output sequence is typical in $P(Y)$

Now, check if the drawn input and output sequence is jointly typical.

$$P(Y=0, X=0) = P(Y=0/X=0)\, P(X=0)$$

$$= \left(\frac{2}{3}\right) \times (0.6) = 0.4$$

$$P(Y=1, X=1) = P(Y=1/X=1)\, P(X=1)$$

$$= \left(3/4\right) \times (0.4) = 0.3$$

$$P(Y=1, X=0) = P(Y=1/X=0)\, P(X=0)$$

$$= \left(1/3\right)(0.6) = 0.2$$

$$P(Y, X) = \begin{bmatrix} 4/10 & 1/10 \\ 2/10 & 3/10 \end{bmatrix}$$

Calculate Emperical Prob. for seq.

$$X^{10} = 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0$$

$$Y^{10} = 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1$$

$$P_{emp}(X=0, Y=0) = 4/10$$

$$P_{emp.}(X=1, Y=1) = 3/10$$

$$P_{emp}(Y=1, X=0) = 2/10 = 0.2$$

$$P_{emp.}(Y=0, X=1) = 1/10 = 0.1$$

Clearly, $P_{emp}(X,Y) = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}$

$$P_{X\text{-}emp.}(X) = P(X) = (0.6, 0.4)$$

$$P_{Y\text{-}emp.}(Y) = P(Y) = (1/2, 1/2)$$

$$P_{X,Y\text{-}emp}(Y,X) = P(Y,X)$$

Hence the drawn input and output sequence are jointly typical.