Australian National University

ASSIGNMENT COVER SHEET

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This coversheet must be attached to the front of your assessment

The assessment is due at 26/09/2022, 5 pm unless otherwise specified in the course outline.

Student ID	u7351505
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Course Code	COMP6261
Course Name	Information Theory
Assignment Item	Assignment 3
Due Date	24th October 2022
Date Submitted	24th October 2022

I declare that this work:

☑ upholds the principles of academic integrity, as defined in the ANU Policy: [Code of Practice for Student Academic Integrity];

☑ is original, except where collaboration (for example group work) has been authorised in writing by the course convener in the course outline and/or Wattle site;

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Signature Yifan Luo

Comp6261, Yifan Luo, u7351505

Question 1

(a) $H(X_1) = E_x(h(x)) = 4 \times \frac{1}{4} \log 4 = 2$

(b) $H(X_1 X_2 \cdots X_{52}) = \sum_{x_1,\dots,x_{52}} p(x_1,\dots,x_{52}) \log \frac{1}{p(x_1,\dots,x_{52})} = \log_2 \frac{1}{\frac{(13!)^4}{52!}} = 95.4374$

Question 2-I

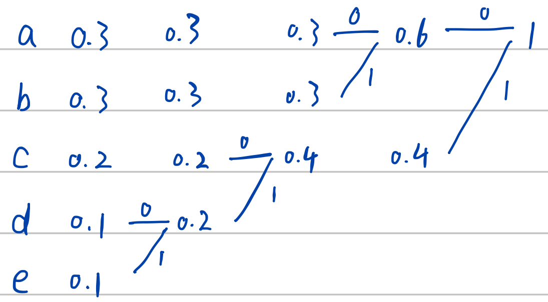
(a) No. 0 is the prefix of 01 and 011.

(b) Yes.

(c) Yes. They are already uniquely decodable.

Question 2-II

Huffman code



$C(a) = 00$

$C(b) = 01$

$C(c) = 10$

$C(d) = 110$

$C(e) = 111$

$L(C, X) = 0.3 \times 2 + 0.3 \times 2 + 0.2 \times 2 + 0.1 \times 3 + 0.1 \times 3 = 2.2$

Shannon code

$l_a = l_b = \lceil \log_2 \frac{1}{0.3} \rceil = 2$

$l_c = \lceil \log_2 \frac{1}{0.2} \rceil = 3$

$l_d = l_e = \lceil \log_2 \frac{1}{0.1} \rceil = 4$

$L(C, X) = 2 \times 0.3 \times 2 + 0.2 \times 3 + 2 \times 0.1 \times 4 = 2.6$

0	a 00 ✓	000	0000
			0001
		001	0010
			0011
	b 01 ✓	010	0100
			0101
		011	0110
			0111
1	10	c 100 ✓	1000
			1001
		101	1010 ✓ d
			1011 ✓ e
	11	110	1100
			1101
		111	1110
			1111

$C(a) = 00$

$C(b) = 01$

$C(c) = 100$

$C(d) = 1010$

$C(e) = 1011$

Question 2 - III - B

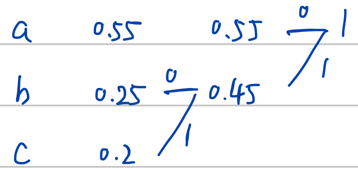
(a) Shannon code

$l_a = \lceil \log_2 \frac{1}{0.55} \rceil = 1$

$l_b = \lceil \log_2 \frac{1}{0.25} \rceil = 2$

$l_c = \lceil \log_2 \frac{1}{0.2} \rceil = 3$

Huffman code



$l_a = 1$, $l_b = l_c = 2$

(b) For Huffman code, when $D \geq 3$, $L(C_{Huff}, X) = 1$

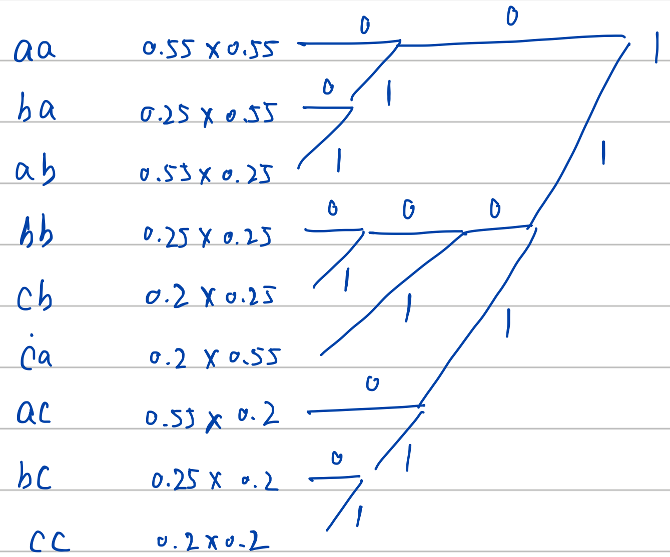
For Shannon code, we what $l_i = \lceil \log_D \frac{1}{p_i} \rceil = 1$.

when $\log_D \frac{1}{0.2} = 1$, $D = 5$

then $D \geq 5$, $L(C_{Shannon}, X) = 1$

Thus, the smallest D is 5.

(c) Huffman code



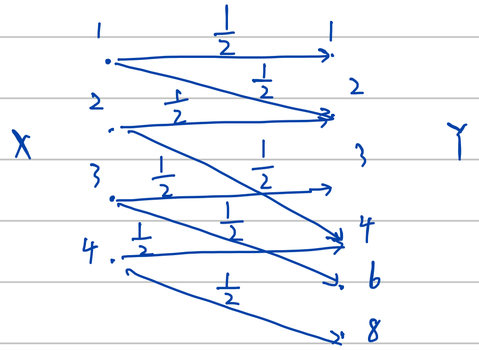
$C(aa) = 00$ $C(ba) = 010$ $C(ab) = 011$ $C(ca) = 101$ $C(ac) = 110$

$C(bb) = 1000$ $C(cb) = 1001$ $C(bc) = 1110$ $C(cc) = 1111$

Question 3-I

(a)

$$Q = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$



(b) $H(Y) = \frac{1}{2}P(X=1)\log\frac{2}{P(X=1)} + \frac{P(X=1)+P(X=2)}{2}\log\frac{2}{P(X=1)+P(X=2)}$
$+ \frac{1}{2}P(X=3)\log\frac{2}{P(X=3)} + \frac{P(X=2)+P(X=4)}{2}\log\frac{2}{P(X=2)+P(X=4)}$
$+ \frac{1}{2}P(X=3)\log\frac{2}{P(X=3)} + \frac{1}{2}P(X=4)\log\frac{2}{P(X=4)}$

$Y=1$	$P(X=1)$
2	$P(X=1)+P(X=2)$
3	$P(X=3)$
4	$P(X=2)+P(X=4)$
6	$P(X=3)$
8	$P(X=4)$

$H(Y|X) = \sum_x p(X=x)H(Y|X=x) = \sum_x p(X=x)\sum_y p(Y=y|X=x)\log\frac{1}{p(Y=y|X=x)}$

$= p(X=1)\cdot(\frac{1}{2}\log 2\times 2) + p(X=2)\cdot(\frac{1}{2}\log 2\times 2) + p(X=3)\cdot(\frac{1}{2}\log 2\times 2)$
$+ p(X=4)(\frac{1}{2}\log 2\times 2)$
$= \log 2\cdot\sum_x p(X=x)$
$= 1$

$I(X;Y) = H(Y) - H(Y|X) = \frac{1}{2}P(X=1)\log\frac{1}{P(X=1)} + \frac{P(X=1)+P(X=2)}{2}\log\frac{1}{P(X=1)+P(X=2)}$
$+ \frac{1}{2}P(X=3)\log\frac{1}{P(X=3)} + \frac{P(X=2)+P(X=4)}{2}\log\frac{1}{P(X=2)+P(X=4)}$
$+ \frac{1}{2}P(X=3)\log\frac{1}{P(X=3)} + \frac{1}{2}P(X=4)\log\frac{1}{P(X=4)}.$

(c) Subset $X' = \{1, 3, 4\}$, then $Y' = \{1, 3, 4, 6, 8\}$

X' and Y' have bijection relationship.

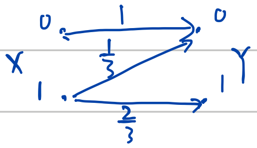
Then, $I(X';Y') = H(X') - H(X'|Y')$
$= p(X=1)\log\frac{1}{p(X=1)} + p(X=3)\log\frac{1}{p(X=3)} + p(X=4)\log\frac{1}{p(X=4)} - 0$
$= H(X')$

$I(X';Y') = H(X') - H(X'|Y') \leq H(X')$

In this case, $I(X';Y')$ has equality when $H(X'|Y')=0$.

i.e. The subset $X' \subseteq X$ satisfies.

Question 3-II



$$I(X;Y) = H(Y) - H(Y|X)$$

$$P(Y=0) = P(Y=0|X=0)\,P(X=0) + P(Y=0|X=1)\,P(X=1)$$

$$= P_0 + (1-P_0)\cdot\tfrac{1}{3} = \tfrac{1}{3} + \tfrac{2}{3}P_0 \quad ①$$

$$P(Y=1) = 1 - P(Y=0) = \tfrac{2}{3} - \tfrac{2}{3}P_0 \quad ②$$

$$H(Y) = P(Y=0)\log\frac{1}{P(Y=0)} + P(Y=1)\log\frac{1}{P(Y=1)} \quad ③$$

$$H(Y|X=0) = P(Y=0|X=0)\log\frac{1}{P(Y=0|X=0)} + P(Y=1|X=0)\log\frac{1}{P(Y=1|X=0)} = 0+0=0$$

$$H(Y|X=1) = P(Y=0|X=1)\log\frac{1}{P(Y=0|X=1)} + P(Y=1|X=1)\log\frac{1}{P(Y=1|X=1)}$$

$$= \tfrac{1}{3}\cdot\log 3 + \tfrac{2}{3}\log\tfrac{3}{2}$$

$$H(Y|X) = P(X=0)\,H(Y|X=0) + P(X=1)\,H(Y|X=1)$$

$$= (1-P_0)\cdot\left(\tfrac{1}{3}\log 3 + \tfrac{2}{3}\log\tfrac{3}{2}\right) \quad ④$$

$$I(X;Y) = H(Y) - H(Y|X) \quad ⑤$$

Combine ① ② ③ ④ ⑤ , we have

$$P_0^* = \arg\max_{P_0} I(X;Y) = 0.58311, \qquad C = \max_{P_0^*} I(X;Y) = 0.46978$$

Question 4-I

Upper bound: $P((\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n) \in A_\varepsilon^n(X,Y,Z)) = \sum_{x,y,z\in A_\varepsilon^n(X,Y,Z)} P(x,y,z) \leq 1$

$$= \sum_{x,y,z\in A_\varepsilon^n(X,Y,Z)} p(x)p(y)p(z)$$

$$\leq \sum_{x,y,z\in A_\varepsilon^n(X,Y,Z)} 2^{-n(H(X)-\varepsilon)}\, 2^{-n(H(Y)-\varepsilon)}\, 2^{-n(H(Z)-\varepsilon)}$$

$$\leq 2^{n(H(X,Y,Z)+\varepsilon)}\, 2^{-n(H(X)-\varepsilon)}\, 2^{-n(H(Y)-\varepsilon)}\, 2^{-n(H(Z)-\varepsilon)}$$

Lower bound: $1 = \sum_{x,y,z} P_{XYZ}(x,y,z) \geq \sum_{x,y,z\in A_\varepsilon^n(X,Y,Z)} P_{XYZ}(x,y,z) \geq \sum_{x,y,z\in A_\varepsilon^n} 2^{-n(H(X,Y,Z)+\varepsilon)} = A_\varepsilon^n(X,Y,Z)\, 2^{-n(H(X,Y,Z)+\varepsilon)}$

$$|A_\varepsilon^n(X,Y,Z)| \leq 2^{n(H(X,Y,Z)+\varepsilon)}$$

$$Pr(A_\varepsilon^n) \geq 1-\varepsilon$$

$$\Rightarrow 1-\varepsilon \leq P((\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n) \in A_\varepsilon^n(X,Y,Z))$$

$$\leq 2^{n(H(X,Y,Z)+\varepsilon)}\, 2^{-n(H(X)-\varepsilon)}\, 2^{-n(H(Y)-\varepsilon)}\, 2^{-n(H(Z)-\varepsilon)}$$

$$\Rightarrow P((\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n) \in A_\varepsilon^n(X,Y,Z)) \geq (1-\varepsilon)\, 2^{-n(H(X,Y,Z)+\varepsilon)}\, 2^{n(H(X)-\varepsilon)} \cdot 2^{n(H(Y)-\varepsilon)} \cdot 2^{-n(H(Z)-\varepsilon)}$$

Question 4-II

(a) $\breve{H}(x^n) = -\frac{1}{n}\log_2(p(x^n))$

where $n = |x| = 15$ $\quad p(x^n) = p(x_1)\cdots p(x_n) = 0.43^7 \times 0.32^5 \times 0.25^3$

Then $\breve{H}(x^{15}) = -\frac{1}{15}\log_2(0.43^7 \times 0.32^5 \times 0.25^3)$

$\approx 1.51616\cdots$

(b) $H(x) = 0.43\log\frac{1}{0.43} + 0.32\log\frac{1}{0.32} + 0.25\log\frac{1}{0.25} = 1.5496$

$|\breve{H}(x^{15}) - H(x)| = 0.03 < \varepsilon = 0.05$

Thus, it is a ε-typical sequence with $\varepsilon = 0.05$

(c) $P(X=a) = 0.2 + 0.08 + 0.15 = 0.43$ $\qquad P(Y=d) = 0.2 + 0.1 + 0.1 = 0.4$

$p(X=b) = 0.1 + 0.15 + 0.07 = 0.32$ $\qquad P(Y=e) = 0.08 + 0.15 + 0.1 = 0.33$

$p(X=c) = 0.1 + 0.1 + 0.05 = 0.25$ $\qquad P(Y=f) = 0.15 + 0.07 + 0.05 = 0.27$

From (a), we know $|\breve{H}(x^{15}) - H(x)| = 0.03$

$\breve{H}(Y^{15}) = -\frac{1}{15}\log_2(0.4^7 \times 0.33^3 \times 0.27^5) = 1.56645$

$H(Y) = 0.4 \times \log\frac{1}{0.4} + 0.33 \times \log\frac{1}{0.33} + 0.27 \times \log\frac{1}{0.27} = 1.56662$

$|\breve{H}(Y^{15}) - H(Y)| = 0.00017$

$\breve{H}(x^{15}, Y^{15}) = -\frac{1}{15}\log_2(p(a,d) \times p(a,f) \times p(b,f) \times p(a,f) \times p(a,d) \times p(b,f) \times p(b,e) \times p(c,d) \times p(a,d) \times p(c,f) \times p(a,d) \times p(b,d) \times p(b,d) \times p(a,e) \times p(c,e))$

$= 3.09507$

$H(X,Y) = \sum_{x,y} p(xy)\log\frac{1}{p(xy)} = 3.05821$

$|\breve{H}(x^{15}, y^{15}) - H(X,Y)| = 0.0368$

If $\varepsilon > |\breve{H}(x^{15}, y^{15}) - H(X,Y)| \approx 0.0368$, then (x,y) are jointly ε-typical; Otherwise, (x,y) are not jointly ε-typical.