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Question 1.

(a) True. $H(X) = \sum_x p(x)\log_2\frac{1}{p(x)}$, when $X \sim Uniform(\frac{1}{|X|})$, then $\max H(X) = \sum_x \frac{1}{|X|}\log_2|x| = \log_2|x|$

If $H(X) \le 3$, then $RHS = \log_2|X|^* = 3$, then $|X|^* = 2^3 = 8$. Thus $|X| \le |X|^* = 8$. Q.E.D

(b) True $H(Y) = H(g(x))$: If $g(x)$ is a one-to-one function, then $H(Y) = H(X)$

Else, $g(x)$ is a many-to-one function, then $H(Y) < H(X)$

In this case $3 = H(Y) \le H(X)$ Q.E.D.

(c) True. We know that $H(X) \le L(C, X)$, where $H(X) \le \log_2|X| = \log_2 3 = 1.585$

Then $L(C, X) \geq H(X)$, where $1.585 \geq H(X)$. thus $L(C, X)$ can be 2.6. Q.E.D.

(d) True. $\because I(X;Z|Y) = H(X|Y) - H(X|Z,Y) = 0 \iff H(X|Y) = H(X|Z,Y)$

$\therefore$ Given Y, X and Z are independent. i.e. $p(x|y) = p(x|yz)$

$\therefore p(xyz) = p(z)p(y|z)p(x|yz) = p(z)p(y|z)p(x|y)$

$\therefore Z \to Y \to X$ is a Markov Chain Q.E.D

(e). True $C = \max_{p_X} I(X;Y) = \max_{p_X}(H(X) - H(X|Y)) \le \max_{p_X}(H(X))$

$RHS = \max_{p_X}(H(X)) = \log_2|X| = \log_2|2^X|$

when $\hat{C} = \log_2|2^X| = X$. $\hat{C} = \max C$, then $2^X = 2^{\hat{C}} = 2^C$

Thus $|2^X| \geq 2^C$, input alphabet has at least C elements. Q.E.D

(f) True. $(N, K) = (15, 11)$, i.e. block size is $N = 15$ bits

and the number of total codewords is $2^K = 2^{11}$

Q.E.D

Question 2.

(a) (i) From the table, we know that

$P(X=0) = \frac{1}{6}+\frac{1}{12}+\frac{1}{6}+\frac{1}{12} = \frac{1}{2}$, $P(X=1) = \frac{1}{12}+\frac{1}{6}+\frac{1}{12}+\frac{1}{6} = \frac{1}{2}$

$P(Y=0) = \frac{1}{6}+\frac{1}{12} = \frac{1}{4}$, $P(Y=1) = \frac{1}{12}+\frac{1}{6} = \frac{1}{4}$, $P(Y=2) = \frac{1}{6}+\frac{1}{12} = \frac{1}{4}$, $P(Y=3) = \frac{1}{12}+\frac{1}{6} = \frac{1}{4}$

We also noticed that $P(X=0, Y=0) = \frac{1}{6}$, where $P(X=0)P(Y=0) = \frac{1}{2}\times\frac{1}{4} = \frac{1}{8}$

$\because P(X=0, Y=0) \neq P(X=0)\,P(Y=0)$

$\therefore$ X and Y are not independent

(ii) From (a)(i), we know that

$E[X] = 0\cdot P(X=0) + 1\cdot P(X=1) = 0 + 1\times\frac{1}{2} = \frac{1}{2}$

$E[Y] = 0\cdot P(Y=0) + 1\cdot P(Y=1) + 2\cdot P(Y=2) + 3\cdot P(Y=3) = 0 + \frac{1}{4} + 2\times\frac{1}{4} + 3\times\frac{1}{4} = \frac{3}{2}$

(iii) $E[XY] = 0\times0\times P(X=0,Y=0) + 0\times1\times P(X=0,Y=1) + 0\times2\times P(X=0,Y=2) + 0\times3\times P(X=0,Y=3)$

$+ 1\times0\times P(X=1,Y=0) + 1\times1\times P(X=1,Y=1) + 1\times2\times P(X=1,Y=2) + 1\times3\times P(X=1,Y=3)$

$= 0+0+0+0+0+1\times\frac{1}{6} + 2\times\frac{1}{12} + 3\times\frac{1}{6}$

$= \frac{5}{6}$

(b). (i) $H(X_1, Z) = H(X_1) + H(Z|X_1)$

$H(X_1) = \sum_{x_1} p(x_1)\log_2\frac{1}{p(x_1)} = \sum_{x_1}\frac{1}{|x_1|}\log_2|x_1| = \log_2|x_1| = \log_2 6$

$H(Z|X_1) = \sum_{x_1} p(x_1)\,H(Z|x_1)$ $\qquad Z = |X_1 - X_2| = \{0,1,2,3,4,5\}$.

$H(Z|X_1=1) = \sum_{z\in\{0,1,2,3,4,5\}} P(Z|X_1=1)\log\frac{1}{p(z|X_1=1)} = \frac{1}{3}\times\log 18 + \frac{2}{3}\times\log 36 = 4.84$

(ii) $H(X,Z) = (1+2+2+1)\times\frac{2}{36}\times\log\frac{36}{2} + (6+4+2+2+4+6)\times\frac{1}{36}\log_2 36 = 4.84$

$H(X_1) = 6\times\frac{1}{6}\times\log_2 6 = 2.585$

$H(Z) = 2\times\frac{6}{36}\times\log\frac{36}{6} + \frac{10}{36}\times\log_2\frac{36}{10} + \frac{8}{36}\times\log_2\frac{36}{8} + \frac{4}{36}\times\log_2\frac{36}{4} + \frac{2}{36}\times\log_2\frac{36}{2} = 2.44$

$\therefore I(X, Z) = H(X_1) + H(Z) - H(X_1, Z) = 0.189$

(iii) $I(X_1, X_2; Z) = H(X_1) + H(X_2) + H(Z) - H(X_1 X_2 Z) - I(X_1, X_2)$

$\because X_1 \perp X_2$, then $I(X_1, X_2) = 0$. $H(X_2) = H(X_1)$

$RHS = 2H(X_1) + H(Z) - H(X_1 X_2, Z)$, where $Z = |X_1 - X_2|$ $\qquad \therefore H(X_1, X_2, Z) = H(X_1, X_2) = 2H(X_1)$

$\therefore I(X,Y;Z) = 2H(X_1) + H(Z) - 2H(X_1) = 2.44$

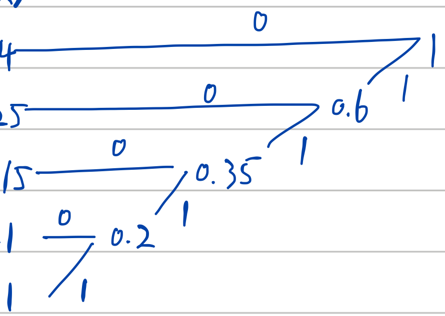
Question 3.

(a) $H(X) = \sum_{i=1}^{5} p(x_i) \log \frac{1}{p(x_i)} = 0.4 \times \log \frac{1}{0.4} + 0.25 \times \log \frac{1}{0.25} + 0.15 \times \log \frac{1}{0.15} + 0.1 \times \log \frac{1}{0.1} + 0.1 \times \log \frac{1}{0.1}$

$= 2.10370$

(b). Huffman code:

X	P(X)	C_{Huff}
x1	0.4	0
x2	0.25	10
x3	0.15	110
x4	0.1	1110
x5	0.1	1111



$L(C_{Huff}, X) = p(x_1) \times l(x_1) + p(x_2) \times l(x_2) + p(x_3)\, l(x_3) + p(x_4)\, l_4 + p(x_5)\, l_5$

$= 0.4 \times 1 + 0.25 \times 2 + 0.15 \times 3 + 0.1 \times 4 + 0.1 \times 4$

$= 2.15$

(c) From the material, we know that X and Z are independent. then

$H(Y) = H(X+Z) = H(X) + H(Z)$

where $H(X) = 2.10370$ from (a)

$H(Z) = p(Z=0) \log_2 \frac{1}{p(Z=0)} + p(Z=1) \log_2 \frac{1}{p(Z=1)}$

$= \frac{1}{2} \log_2 2 + \frac{1}{2} \log_2 2$

$= 1$

Thus. $H(Y) = H(X) + H(Z) = 2.10370 + 1 = 3.10370$

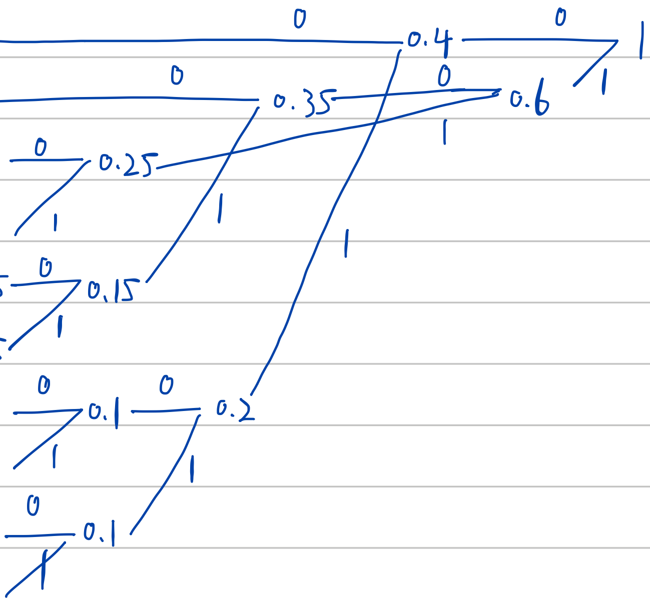
(d) $I(X;Y) = H(Y) - H(Y|X)$, where $H(Y|X) = H(Z) = 1$ from (c).

$H(Y) = 3.10370$ from (c)

Thus, $I(X;Y) = 3.10370 - 1 = 2.10370$

(e) Huffman code (next page)

$Y = X+Z$	$P(Y)$	C_{Huff}
x_1+1	$0.4\times\frac{1}{2}=0.2$	00
x_1+0	$0.4\times\frac{1}{2}=0.2$	100
x_2+1	$0.25\times\frac{1}{2}=0.125$	110
x_2+0	$0.25\times\frac{1}{2}=0.125$	111
x_3+1	$0.15\times\frac{1}{2}=0.075$	1010
x_3+0	$0.15\times\frac{1}{2}=0.075$	1011
x_4+1	$0.1\times\frac{1}{2}=0.05$	0100
x_4+0	$0.1\times\frac{1}{2}=0.05$	0101
x_5+1	$0.1\times\frac{1}{2}=0.05$	0110
x_5+0	$0.1\times\frac{1}{2}=0.05$	0111



$$L(C_{Huff}, Y) = 0.2\times2+0.2\times3+0.125\times3+0.125\times3+0.075\times4+0.075\times4$$
$$+0.05\times4+0.05\times4+0.05\times4+0.05\times4$$
$$=3.15$$

Question 4.

From the material, we know that $X \sim Bern(p_1)$

where p_0 is the probability of 0, and $p_1 = \frac{3}{10}$, $p_0 = \frac{7}{10} = 1 - p_1$

Then, $Y_n = \sum_{i=1}^{n} X_i \sim Binomial(n, p_1)$, $P(Y_k) = \binom{n}{k} p_1^k (1-p_1)^{n-k}$

(a). $E[X_1] = \sum_{x_1 \in \{0,1\}} x_1 p(x_1) = 0 \times p_0 + 1 \times p_1 = 0 + \frac{3}{10} = \frac{3}{10} = 0.3$

$Var[X_1] = E[(X_1 - EX_1)^2] = p_1(1-p_1) = \frac{3}{10} \times \frac{7}{10} = \frac{21}{100} = 0.21$

(b). $H(Y_2) = H(X_1 + X_2) = H(X_1) + H(X_2)$

$H(X_1) = p_1 \log \frac{1}{p_1} + p_0 \log \frac{1}{p_0} = \frac{3}{10} \log \frac{10}{3} + \frac{7}{10} \log \frac{10}{7} = 0.8813$

$\because X_1, \cdots, X_n \overset{i.i.d}{\sim} Bern(p_1)$

$\therefore H(Y_2) = H(X_1) + H(X_2) = 2 \times H(X_1) = 1.7626$

$\therefore H(Y_3) = H(X_1 + X_2 + X_3) = 3 \times H(X_1) = 2.6439$

(c) We know that $Y_n = \sum_{i=1}^{n} X_i$, and $X_1, \cdots X_n$ are i.i.d.

$E[Y_n] = E[X_1 + X_2 + \cdots + X_n] = n \cdot E[X_1] = n \cdot p_1 = n \times \frac{3}{10} = 0.3n$

$Var[Y_n] = Var[X_1 + X_2 + \cdots + X_n] = n^2 \cdot Var[X_1] = n \cdot p_1 (1-p_1) = n \cdot \frac{3}{10} \cdot \frac{7}{10} = 0.21n.$

(d) Markov's inequality: $P(X \geq \lambda \cdot E[X]) \leq \frac{1}{\lambda}$

In this case, $P(Y_n \geq \lambda \cdot E[Y_n]) \leq \frac{1}{\lambda}$

Let $\lambda \cdot E[Y_n] = 0.8n$, then $\lambda \cdot 0.3n = 0.8n \Leftrightarrow \lambda = \frac{0.8n}{0.3n} = \frac{8}{3}$.

then $P(X \geq 0.8n) \leq \frac{1}{\frac{8}{3}} = \frac{3}{8}$

(e) Chebyshev's inequality: $P(|X - E[X]| \geq \lambda) \leq \frac{V[X]}{\lambda^2}$

In this case $P(|Y_n - E[Y_n]| \geq \lambda) \leq \frac{V[Y_n]}{\lambda^2}$

$LHS = P(Y_n \geq \lambda + E[Y_n]$ or $Y_n \leq E[Y_n] - \lambda)$

Let $\lambda + E[Y_n] = 0.8n$, then $\lambda = 0.8n - E[Y_n] = 0.8n - 0.3n = 0.5n$,

$E[Y_n] - \lambda = 0.3n - 0.5n = -0.2n < 0$

Then $P(Y_n \geq 0.8n$ or $Y_n \leq -0.2n) = P(Y_n \geq 0.8n)$

Thus $P(Y_n \geq 0.8n) \leq \frac{V[Y_n]}{\lambda^2} = \frac{0.21n}{(0.5n)^2} = \frac{0.21}{0.25n} = \frac{21}{25n}$

(f). When Chebyshev's inequality has a lower upper bound than Markov's inequality,

$\frac{21}{25n} < \frac{3}{8}$ (from (d) and (e))

we have $n > \frac{56}{25} = 2.24$

Question 5.

(a) $P(Y_2|X) = P(Y_1|X)P(Y_2|Y_1)$

From the diagram, we see that

$P(Y_2=0|X=0) = p_1 \cdot p_2$ $\quad P(Y_2=1|X=0) = p_1(1-p_2)+(1-p_1)(1-p_2)$

$P(Y_2=2|X=0) = (1-p_1)p_2$ $\quad = p_1 - p_1p_2 + 1 - p_2 - (p_1 - p_1p_2)$

$= p_2 - p_1p_2$ $\quad = p_1 - p_1p_2 + 1 - p_2 - p_1 + p_1p_2$

$= 1-p_2$

$P(Y_2=0|X=1) = (1-p_1)p_2 = p_2 - p_1p_2$ $\quad P(Y_2=1|X=1) = (1-p_1)(1-p_2) + p_1(1-p_2)$

$P(Y=2|X=1) = p_1p_2$ $\quad = 1-p_2 - (p_1 - p_1p_2) + p_1 - p_1p_2$

$= 1-p_2 - p_1 + p_1p_2 + p_1 - p_1p_2$

$= 1-p_2$

(b). $P(Y_2) = \sum_{x\in\{0,1\}} P(Y_2, X) = \sum_{x\in\{0,1\}} p(x)\cdot p(Y_2|X)$

i)

then $P(Y_2=0) = p_0p_1p_2 + (1-p_0)(1-p_1)p_2$

$p(Y_2=1) = 1-p_2$

$p(Y_2=2) = p_1p_2(1-p_1) + p_1p_2(1-p_0)$

(c) $H(Y_1|X) = P(X=0)H(Y_1|X=0) + P(X=1)H(Y_1|X=1)$

$= P(X=0)\cdot(p_1\log\frac{1}{p_1} + (1-p_1)\log_2\frac{1}{1-p_1}) + P(X=1)(p_1\cdot\log_2\frac{1}{p_1} + (1-p_1)\log_2\frac{1}{p_1})$

$= H_2(p_1)$

$H(Y_2|X) = P(X=0)H(Y_2|X=0) + P(X=1)H(Y_2|X=1)$

$= P(X=0)\cdot(p_1p_2\cdot\log\frac{1}{p_1p_2} + (1-p_2)\log\frac{1}{1-p_2} + (1-p_1)p_2\log\frac{1}{(1-p_1)p_2}) +$

$p(X=1)\cdot(p_1p_2\cdot\log_2\frac{1}{p_1p_2} + (1-p_2)\cdot\log_2\frac{1}{1-p_2} + (1-p_1)p_2\log\frac{1}{(1-p_1)p_2}$

$= (1-p_1)p_2\cdot\log\frac{1}{1-p_1} + p_1p_2\log\frac{1}{p_1} + H_2(p_2)$

$= p_2((1-p_1)\log\frac{1}{1-p_1} + p_1\log\frac{1}{p_1}) + H_2(p_2)$

$= p_2 H_2(p_1) + H_2(p_2)$

(d) $I(X;Y) = H(Y_2) - H(Y_2|X)$

(f) $C = \max_{p_X} I(X;Y_2)$ when $p_0 = p_1 = p_2 = \frac{1}{2}$. channel capacity $C =$

Question 6.

(a) (i) $H(X|Y) = \sum_{y_i} p(y_i)\, H(X|Y=y_i) = -\sum_{x_i}\sum_{y_i} p(x_i,y_i) \log p(x_i|y_i)$

$$P(x_i,y_i) = p(y_i|x_i)\,p(x_i)$$

$$= \begin{bmatrix} \frac{3}{5}p_0 & 0 \\ \frac{2}{5}p_0 & \frac{2}{5}p_1 \\ 0 & \frac{3}{5}p_1 \end{bmatrix}$$

$P(Y=0) = p_0 \frac{3}{5} p_0 + 0 = \frac{3}{5} p_0^2$

$P(Y=1) = p_0 \frac{2}{5} p_0 + p_1 \frac{2}{5} p_1 = p_0 \frac{2}{5} p_0 + (1-p_0)\frac{2}{5}(1-p_0) =$

$= p_0 \frac{2}{5} p_0 + (\frac{2}{5} - \frac{2}{5}p_0)(1-p_0)$

$= p_0 \frac{2}{5} p_0 + \frac{2}{5} - \frac{2}{5}p_0 - \frac{2}{5}p_0 + \frac{2}{5}p_0^2$

$= \frac{2}{5}$

$P(Y=2) = 0 + p_1 \frac{3}{5} p_1 = p_1^2 \frac{3}{5}$

$$p(X|Y) = \frac{P(X,Y)}{P(Y)} = \begin{bmatrix} \frac{1}{p_0} & 0 \\ p_0 & p_1 \\ 0 & \frac{1}{p_1} \end{bmatrix}$$

$\therefore H(X|Y=0) = -p(x|y=0) \log_2 p(x|y=0)$

$= -\frac{1}{p_0} \log_2 \frac{1}{p_0}$

$H(X|Y=1) = -p(x|y=1) \log_2 p(x|y=1)$

$= -p_0 \log_2 p_0 + (-p_1 \log_2 p_1)$

$H(X|Y=2) = -p(x|y=2) \log_2 p(x|y=2)$

$= -\frac{1}{p_1} \log_2 \frac{1}{p_1}$

$H(X|Y) = p(y=0)\,H(X|Y=0) + p(y=1)\,H(X|Y=1) + p(y=2)\,H(X|Y=2)$

$= \frac{3}{5}p_0^2 \cdot (-\frac{1}{p_0} \log_2 \frac{1}{p_0}) + \frac{2}{5}(-p_0 \log_2 p_0 - p_1 \log_2 p_1) + \frac{3}{5}p_1^2(-\frac{1}{p_1}\log_2 \frac{1}{p_1})$

$= -\frac{3}{5} p_0 \log_2 \frac{1}{p_0} - \frac{2}{5} p_0 \log_2 p_0 - \frac{2}{5} p_1 \log_2 p_1 - \frac{3}{5} p_1 \log_2 \frac{1}{p_1}$

$= \frac{3}{5} p_0 \log_2 p_0 + \frac{2}{5} p_0 \log_2 p_0 + \frac{2}{5} p_1 \log_2 p_1 + \frac{3}{5} p_1 \log_2 p_1$

$= p_0 \log_2 p_0 + p_1 \log_2 p_1 = -H_2(p_0)$

$H(X) = H_2(p_0)$

$\therefore I(X;Y) = H(X) - H(X|Y) = H_2(p_0) - (-H_2(p_1)) = 2H_2(p_0)$

(ii) $C = \max\limits_{p_X} I(X;Y) = 2\max\limits_{p_X} H_2(p_0)$

When $p_0 = p_1 = \frac{1}{2}$, i.e. uniform, then it achieves the capacity.

$C = 2 \times \log_2 2 = 2$ bits

(iii) We know that $P(X=0) = P(X=1) = \frac{1}{2}$,

then in a typical sequence of length N, # 1s = # 0s = Np_0 / Np_1

In this case, $n = 15$. then # 1s = # 0s = $15 \times \frac{1}{2} = 7.5$

But we have 10 1s and 5 0s, #1s ≠ # 0s, then it is not typical.

(iv)

(b) (i)

$$G = \begin{bmatrix} 1 & & & & 1 & 0 & 1 \\ & 1 & & & 1 & 1 & 0 \\ & & 1 & & 1 & 1 & 1 \\ & & & 1 & 0 & 1 & 1 \end{bmatrix} \qquad m_1 = [1001]$$

then $t = G^T m_1 = [1001110]$

(ii) From (i), we have $r = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$

$z_1 = r_1 \oplus r_2 \oplus r_3 \ominus r_5 = 1+1+1-1 = 0$

$z_2 = r_2 \oplus r_3 \oplus r_4 \ominus r_6 = 1+1+1-0 = 1 \qquad \Rightarrow z = [0\,1\,1]$

$z_3 = r_1 \oplus r_3 \oplus r_4 \ominus r_7 = 1+1+1-0 = 1$

From the table, we know the flip bit is r_4.

then unflip r_4, $\hat{s} = [1110]$