Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: First-Order Logic
— Properties of Proof Systems and Semantic Tableaux

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Introduction

Recap: Predicate Logics

- We now (since week 5) know Predicate Logics as a means to express properties of and relationships between objects.
- For example:
  - If everyone plays football, and everyone is a goat,
    then everyone is a football-playing goat
  - $\forall x\ Fx, \forall x\ Gx \vdash \forall x\ (Fx \land Gx)$
- We know how to prove sequents involving Predicate Logic using Natural Deduction.
  - We "only" needed additional elimination and introduction rules for the exists ($\exists$) and universal ($\forall$) quantifiers.
  - Other than that we just re-used the rules for Propositional Logic.

Recap: Semantic Tableau

- Today, we cover Semantic Tableau for *Predicate Logic*.
- But first a recap on Semantic Tableau for *Propositional Logic*!
- If we want to prove $X \vdash A$ (with $X = \{A_1, \ldots, A_n\}$), then, we:
  - Label each assumption $A_1, \ldots, A_n$ as being *true* (**T**),
  - Label A as being *false* (**F**),
  - Simplify each formula (according to the connectives corresponding to truth tables) thus eventually obtaining:
    1. a contradiction in all the branches, or
    2. ≥ 1 open branch (i.e., none of its formulae can be simplified further and there's no contradiction).

  In case 1 the sequent is *valid*.

  In case 2 the sequent is *invalid*, and we can construct an interpretation that makes all formulae in X true, but A false (which is a witness for invalidity).

Today: Properties of Logics and Proof Systems

Properties of Logics:

- What does it mean to *decide* validity?
- Is that always possible for sequents in Propositional Logic? What about Predicate Logic?

Properties of Proof Systems:

- Are all proofs correct? (Soundness)
- Can we always prove validity? (Completeness)

Today: Semantic Tableau for Predicate Logic

- We still use the same rules as we had in the propositional case.
- But now we introduce four additional rules, namely for:
  - ∃-formulae which are labeled *true*
  - . . . *false*
  - ∀-formulae which are labeled *true*
  - . . . *false*

Properties

Recap on our Symbols and their Meanings

We differentiate between *validity* and *provability*:

- $X \models A$ (A follows logically from X)
  - → Every interpretation that makes X true also makes A true.
- $X \vdash_{ND} A$ ($X \vdash A$ can be proved via *Natural Deduction*)
  - → A can be *derived* from X. (Syntax manipulation.)
- $X \vdash_{ST} A$ ($X \vdash A$ can be proved via *Semantic Tableau*)
  - → We can't find an interpretation that makes X true but not A. (Exploits validity definition.)
- There are many more proof systems!

Syntax vs. Semantics

- So, what's the relation between $X \models A$ and $X \vdash A$?
- A desirable situation would be $X \models A$ iff $X \vdash A$.
- Our proof systems could do *anything*! So what *could* happen?
- Let $\mathcal{X}$ be *some* proof system (like, e.g., ND).
  1. $X \vdash_{\mathcal{X}} A$, but not $X \models A$
     → The proof system is wrong! (I.e., not sound.)
  2. $X \models A$, but not $X \vdash_{\mathcal{X}} A$
     → The proof system is incomplete! (I.e., not complete.)
- What we want:

Soundness	Every provable sequent is valid. (Cf. above's 1)
Completeness	Every valid sequent is provable. (Cf. above's 2)

Decision Procedure

- Let $\mathcal{X}$ be some proof system that's sound and complete.
- So, can we also *decide* validity of each sequent with $\mathcal{X}$?
- I.e., we want to know whether $X \models A$ holds, by using $\mathcal{X}$. Can we find out?
- Again, $\mathcal{X}$ is sound and complete, so we can check validity, right?
- No, not necessarily! Both just mention validity, not invalidity!
- We only know: $X \models A$ iff $X \vdash A$
- But we don't necessily know wheter $X \models A$ holds since a sequent could also be invalid! (In that case maybe the proof system just keeps running... So we don't get $X \vdash A$, but we also don't get an output saying "$X \models A$ is false")

Properties of Logics and proof systems

Decidability of Logics:

- *Decidability of a Logic* means determining for an arbitrary sequent whether it's valid or not.
- Propositional Logic: Yes, decidable.
- Predicate Logic: No, undecidable. No such algorithm can exist.

Soundness and Completeness of Proof Systems:

- Natural Deduction:
  - → Sound and complete for Propositional and Predicate Logic
- Semantic Tableau:
  - → Also Sound and complete for Propositional and Predicate Logic

Semantic Tableau Rules

Simplifying a *true* ∃ Quantifier (Intuition)

$$\frac{\textbf{T: } \exists x\ Fx}{\textbf{T: } Fa} \quad \text{provided } a \text{ is new to the branch}$$

- Why does *a* need to be new?
- Think of the triangle ABC! If *a* would exist already in the branch it would not be general (e.g., we could “accidentally” assume that our triangle is rectangular).

Simplifying a *false* ∀ Quantifier (Intuition)

$$\frac{\textbf{F: } \forall x\ Fx}{\textbf{F: } Fa} \quad \text{provided } a \text{ is new to the branch}$$

- This corresponds to the true existential quantifier!
- Recall $\neg\forall x\ Fx \equiv \exists x\ \neg Fx$

Rules For *true* ∃ and *false* ∀, formally

$$\boxed{\begin{array}{c}\dfrac{\mathbf{T:}\ \exists x\ Fx}{\mathbf{T:}\ Fa}\\[1ex] \textit{if a is new to the branch}\end{array}}$$

$$\equiv$$

$$\boxed{\begin{array}{c}\dfrac{X, \mathbf{T:}\ \exists x\ A}{X, \mathbf{T:}\ A^a_x}\\[1ex] \textit{for a not in X or A}\end{array}}$$

- The *X* represents all other lines we have in that branch.

Rules For *true* ∃ and *false* ∀, formally

$$\frac{\textbf{T: } \exists x\ Fx}{\textbf{T: } Fa}$$

if a is new to the branch

$$\frac{\textbf{F: } \forall x\ Fx}{\textbf{F: } Fa}$$

if a is new to the branch

$$\equiv \qquad \equiv$$

$$\frac{X, \textbf{T: } \exists x\ A}{X, \textbf{T: } A^a_x}$$

for a not in X or A

$$\frac{X, \textbf{F: } \forall x\ A}{X, \textbf{F: } A^a_x}$$

for a not in X or A

Simplifying a *true* ∀ Quantifier (Intuition)

$$\frac{\textbf{T: } \forall x\ Fx}{\textbf{T: } Fa, \textbf{T: } Fb, \ldots} \quad \text{for all } a, b, \ldots \text{ in the branch (present and future!)}$$

- This rule will continue being available for new constants/terms produced later on. (Then we have to apply the rule again!)
- If we already obtained a contradiction, we are clearly done. But if we want to show that a branch is open we need to have applied this rule to *all* constants! (I.e., also those that get created after we already applied the rule to all constants that existed until then.)

Simplifying a *false* $\exists$ Quantifier (Intuition)

$$\frac{\textbf{F: } \exists x\ Fx}{\textbf{F: } Fa, \textbf{F: } Fb, \ldots}$$ for all $a, b, \ldots$ in the branch (present and future!)

- Again, this rule will never be finished! If a new constant/term gets introduced we need to apply the rule again!
- Recall from last week that $\neg\exists x\ Fx \equiv \forall x\ \neg Fx$

Rules for *true* ∀ and *false* ∃, formally

$$\frac{\textbf{T: } \forall x\, Fx}{\textbf{T: } Fa, \textbf{T: } Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

$$\frac{\textbf{F: } \exists x\, Fx}{\textbf{F: } Fa, \textbf{F: } Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

$$\equiv \qquad \equiv$$

$$\frac{X, \textbf{T: } \forall x\, A}{X, \textbf{T: } \forall x\, A, \textbf{T: } A^a_x}$$

for a in X or A

$$\frac{X, \textbf{F: } \exists x\, A}{X, \textbf{F: } \exists x\, A, \textbf{F: } A^a_x}$$

for a in X or A

Special case for false Existential and true Universal

- Recall the rules for false existentials and true universals:

$$\frac{\textbf{F: } \exists x\; Fx}{\textbf{F: } Fa, \textbf{F: } Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

$$\frac{\textbf{T: } \forall x\; Fx}{\textbf{T: } Fa, \textbf{T: } Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

- They state that you only “use” constants which are already there.
- Sometimes, however, there one no such constants! Then, you are also allowed to create a new one.

Examples

Example 1

$\forall x(Fx \lor Gx) \vdash^? \forall x\ Fx \lor \forall x\ Gx$

(1)	**T:**	$\forall x(Fx \lor Gx)$		
(2)	**F:**	$\forall x\ Fx \lor \forall x\ Gx$	✓	
(3)	**F:**	$\forall x\ Fx$		from (2)
(4)	**F:**	$\forall x\ Gx$		from (2)

$$\frac{\textbf{T: } \forall x\ Fx}{\textbf{T: } Fa, \textbf{T: } Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

Note that we did not apply the rule for false universal quantifier here because the formula is actually a false *disjunction*, not a false universally quantified formula.

$$\frac{\textbf{F: } \forall x\ Fx}{\textbf{F: } Fa}$$

if a is new to
the branch

Example 1

$\forall x(Fx \vee Gx) \vdash^{?} \forall x\, Fx \vee \forall x\, Gx$

(1) **T:** $\forall x(Fx \vee Gx)$ $\checkmark^{a,b}$

(2) **F:** $\forall x\, Fx \vee \forall x\, Gx$ ✓

(3) **F:** $\forall x\, Fx$ ✓ from (2)

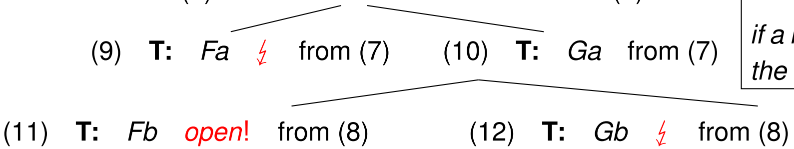
(4) **F:** $\forall x\, Gx$ ✓ from (2)

(5) **F:** Fa from (3)

(6) **F:** Gb from (4)

(7) **T:** $Fa \vee Ga$ ✓ from (1)

(8) **T:** $Fb \vee Gb$ ✓ from (1)



(9) **T:** Fa ↯ from (7)

(10) **T:** Ga from (7)

(11) **T:** Fb *open!* from (8)

(12) **T:** Gb ↯ from (8)

Extracted interpretation: see next slide.

$$\frac{\textbf{T: } \forall x\, Fx}{\textbf{T: } Fa, \textbf{T: } Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

$$\frac{\textbf{F: } \forall x\, Fx}{\textbf{F: } Fa}$$

if a is new to the branch

Example 1 (cont'd)

So? Is $\forall x(Fx \vee Gx) \vdash \forall x\ Fx \vee \forall x\ Gx$ valid?

- Let's see... Not all branches are contradictory.
- Thus, there is an open branch:

(5)	**F:**	Fa	from (3)
(6)	**F:**	Gb	from (4)
(10)	**T:**	Ga	from (7)
(11)	**T:**	Fb	from (8)

- We can, as usual, extract an *interpretation* I that answers for which objects F and G is true:
  - Informally: $I(Fa) = \bot$ and $I(Fb) = \top$
  - Informally: $I(Ga) = \top$ and $I(Gb) = \bot$
  - Thus, showing that there is an interpretation that makes the assumption true, but the formula false!

  → So the sequent is invalid!

The formal definition will be provided in week 7

Example 2

$$\exists x\; Fx, \exists x\; Gx \vdash^{?} \exists x\; (Fx \wedge Gx)$$

(1) **T:** $\exists x\; Fx$ ✓

(2) **T:** $\exists x\; Gx$ ✓

(3) **F:** $\exists x\; (Fx \wedge Gx)$ $\checkmark^{a,b}$

(4) **T:** Fa from (1)

(5) **T:** Gb from (2)

(6) **F:** $Fa \wedge Ga$ ✓ from (3)

(7) **F:** $Fb \wedge Gb$ ✓ from (3)

(8) **F:** Fa ↯ from (6) (9) **F:** Ga from (6)

(10) **F:** Fb *open*! from (7) (11) **F:** Gb ↯ from (7)

T: $\exists x\; Fx$

T: Fa

if a is new to the branch

F: $\exists x\; Fx$

F: Fa, **F:** $Fb, \ldots$

for all a, b, . . . in the branch – present and future!

Extracted interpretation: see next slide.

Example 2 (cont'd)

So? Is $\exists x\ Fx, \exists x\ Gx \vdash^{?} \exists x\ (Fx \wedge Gx)$ valid?

- Let's see... Not all branches are contradictory.
- Thus, there is an open branch:

(4)	**T:**	Fa	from (1)
(5)	**T:**	Gb	from (2)
(9)	**F:**	Ga	from (6)
(10)	**F:**	Fb	from (7)

- Again we can design an interpretation that answers for which objects F and G become true:
  - F is true for exactly a
  - G is true for exactly b
  - Thus, showing that there is an interpretation that makes the assumption true, but the formula false!
  - → So the sequent is invalid!

Invalid Sequents

Advanced Remarks: Sequent is invalid, so?

- There are some invalid sequents for which you can’t find a proof that shows invalidity.
- (We were however still able to find invalidity proofs for *some* invalid sequents as above.)
- In some cases, however, we *could* prove invalidity by modifying rules in a suitable way.
- Even with such rules, though, we still can’t *always* prove invalidity. (Since Predicate Logic is undecidable.)

Advanced Remarks: Motivating Example

Assume we are deep within some branch:

(n)	**T:**	$\forall x\, \exists y\, Rxy$	$\checkmark^{a,b,c}$	from (k<n)
(n+1)	**T:**	$\exists y\, Ray$	✓	from (n)
(n+2)	**T:**	Rab		from (n+1)
(n+3)	**T:**	$\exists y\, Rby$	✓	from (n)
(n+4)	**T:**	Rbc		from (n+3)
(n+5)	**T:**	$\exists y\, Rcy$	✓	from (n)
(n+6)	**T:**	Rcd		from (n+5)

- So we have an infinite branch!
- We will *never* be able to show that it is open.

$$\frac{X, \mathbf{T{:}}\ \forall x\ A}{X, \mathbf{T{:}}\ \forall x\ A, \mathbf{T{:}}\ A^a_x}$$

for a in X or A

$$\frac{X, \mathbf{T{:}}\ \exists x\ A}{X, \mathbf{T{:}}\ A^a_x}$$

for a not in X or A

Summary

Content of this Lecture

- Properties of Logics and Proof Systems (soundness, completeness, decidability)
- Semantic Tableau for Predicate Logics
  - We added several additional rules, but kept using the old ones.
  - We can prove validity and invalidity. (If you are interested, there is another complex example given below.)
  - Invalidity cannot *always* be proved, which shows an essential difference between propositional and predicate logics.
- This week covered the following sections in the Logic Notes:
  - 5: More about first order logic
    - Quantifiers in semantic tableaux

Example by de’Morgan

Consider the following argument:

- All horses are animals.
- Therefore, any horse’s head is an animal head!

We formalize this in terms of Predicate Logic.

- Instead of:
  “any horse’s head is an animal head”
- We formalize that as: “each part of a horse is part of an animal”
  $\forall x(\exists y(Hy \land Pxy) \rightarrow \exists y(Ay \land Pxy))$

Thus we get:

$$\forall x\ Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \land Pxy) \rightarrow \exists y(Ay \land Pxy))$$

Example 3

$$\forall x\, Hx \to Ax \vdash \forall x(\exists y(Hy \land Pxy) \to \exists y(Ay \land Pxy))$$

(1)	**T:**	$\forall x\, Hx \to Ax$		
(2)	**F:**	$\forall x(\exists y(Hy \land Pxy) \to \exists y(Ay \land Pxy))$	✓	
(3)	**F:**	$\exists y(Hy \land Pay) \to \exists y(Ay \land Pay)$	✓	from (2)
(4)	**T:**	$\exists y(Hy \land Pay)$	✓	from (3)
(5)	**F:**	$\exists y(Ay \land Pay)$		from (3)
(6)	**T:**	$Hb \land Pab$	✓	from (4)
(7)	**T:**	Hb		from (6)
(8)	**T:**	Pab		from (6)

$$\frac{\textbf{T: } \forall x\, Fx}{\textbf{T: } Fa, \textbf{T: } Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

$$\frac{\textbf{F: } \forall x\, Fx}{\textbf{F: } Fa}$$

if a is new to
the branch

$$\frac{\textbf{F: } \exists x\, Fx}{\textbf{F: } Fa, \textbf{F: } Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

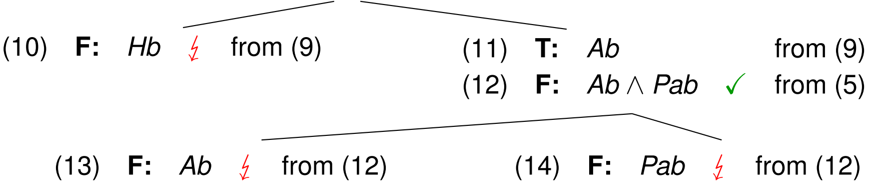
$$\frac{\textbf{T: } \exists x\, Fx}{\textbf{T: } Fa}$$

if a is new to
the branch

Example 3

$$\forall x\, Hx \to Ax \vdash \forall x(\exists y(Hy \land Pxy) \to \exists y(Ay \land Pxy))$$

(1)	**T:**	$\forall x\, Hx \to Ax$	$\checkmark^{b}$	
(2)	**F:**	$\forall x(\exists y(Hy \land Pxy) \to \exists y(Ay \land Pxy))$	$\checkmark$	
(3)	**F:**	$\exists y(Hy \land Pay) \to \exists y(Ay \land Pay)$	$\checkmark$	from (2)
(4)	**T:**	$\exists y(Hy \land Pay)$	$\checkmark$	from (3)
(5)	**F:**	$\exists y(Ay \land Pay)$	$\checkmark^{b}$	from (3)
(6)	**T:**	$Hb \land Pab$	$\checkmark$	from (4)
(7)	**T:**	Hb		from (6)
(8)	**T:**	Pab		from (6)
(9)	**T:**	$Hb \to Ab$	$\checkmark$	from (1)



(10) **F:** Hb ↯ from (9)

(11) **T:** Ab from (9)

(12) **F:** $Ab \land Pab$ $\checkmark$ from (5)

(13) **F:** Ab ↯ from (12)

(14) **F:** Pab ↯ from (12)

All branches are contradictory. Sequent is *valid*!

Example 3 (Again with a different Order)

$$\forall x\, Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

(1) **T:** $\forall x\, Hx \rightarrow Ax$ $\checkmark^{b}$

(2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ $\checkmark$

(3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ $\checkmark$ from (2)

(4) **T:** $\exists y(Hy \wedge Pay)$ $\checkmark$ from (3)

(5) **F:** $\exists y(Ay \wedge Pay)$ $\checkmark^{b}$ from (3)

(6) **T:** $Hb \wedge Pab$ $\checkmark$ from (4)

(7) **T:** Hb from (6)

(8) **T:** Pab from (6)

(9) **F:** $Ab \wedge Pab$ $\checkmark$ from (5)

(10) **F:** Ab from (9) (11) **F:** Pab $\lightning$ from (9)

(12) **T:** $Hb \rightarrow Ab$ $\checkmark$ from (1)

(13) **F:** Hb $\lightning$ from (12) (14) **T:** Ab $\lightning$ from (12)

All branches are contradictory. Sequent is *valid*!