Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: First-Order Logic
— Introduction and Natural Deduction

Yoshihiro Maruyama

Australian National University

Introduction

How to extend Propositional Logic?

$$\overbrace{\underbrace{\text{Socrates}}_{\text{constant}}\ \underbrace{\text{is a goat}}_{\text{predicate}}}^{\text{(natural language) sentence}}$$

Logic is about making statements:

In first-order logic, we:

- can represent individual objects (people, goats, footballs, etc.)
- and express properties and relationships between objects.

In our example,

- the "object" *Socrates* can be represented by a constant,
- the "property" *is a Goat* can be represented by a predicate.

⇒ For example, *isGoat*(*Socrates*)

⇒ In propositional logic, we had to use *SocratesIsGoat*, which is missing some information, since it does not "relate" to another proposition involving Socrates, like *SocratesKicksGoat*.

Predicate Logic

Terminology And Conventions: Terminology

Term: Anything that represents an object, i.e.,

- a constant (representing a fixed object, like the person Socrates)
- a variable (representing a non-specified object)
- a function application (whose general form is $f(t_1, \ldots, t_n)$; e.g., $f(a, b)$, $g(c, x, y)$)

Predicates: Express properties or relations of/between terms:

- Takes as input (or “argument”) a sequence of terms.
  - The sequence length depends on the predicate, e.g., *isGoat* is unary, *kicks* is binary, etc. (some might even be nullary!)
  - This length is called *arity* and can be given as a subscript, e.g., $isGoat_1$, $kicks_2$, but we don’t since it’s clear from context.
- Maps to a truth value, e.g., $isGoat(Socrates)$ might be false, but $isGoat(Jimmy)$ might be true. (Formal semantics is given later.)

Terminology And Conventions: Conventions (cont'd)

- Capital letters are predicate symbols:
  $F, G, H, \ldots, P, Q, R, L, \ldots$
- Lower-case letters represent terms:
  - a, b, c are (usually) used for constants, but we also use them for free variables (as they behave in the same way).
  - f, g, h are used for functions.
  - v and x, y are used for variables.
  - t is used for terms (i.e., any of the above).
- For the sake of simplicity, we do not use parentheses, e.g., $F(a)$, $G(b)$, and $R(a, b)$ become Fa, Gb, and Rab, respectively.
- Now it's clear that the arity is clear from the context! E.g.,
  - Fa represents a predicate F with arity 1 (with term a), and
  - Rab represents a predicate R with arity 2 (with terms a and b).

First-Order Formulae: Possible Quantifiers

We want to "quantify" the objects we talk about.

- For every object x, such that Vx holds, Lx holds.
- *More formally:* $\underbrace{ALL(x : Vx)}_{\text{quantifier!}} Lx$
- *Even more formally:* $\boxed{\underbrace{\forall}_{\text{quantity indicator}} (\underbrace{x}_{\text{variable}} : \underbrace{Vx}_{\text{sort indicator}}) Lx}$

What quantifiers *do* exist? (In our predicate logic!)

- Just two!
- $ALL(x : A)B$, i.e., $\forall(x : A)B$
- $SOME(x : A)B$, i.e., $\exists(x : A)B$

"SOME" means "at least one", so "$\exists$" is also called "exists"
"ALL", i.e., $\forall$, is called the "universal" quantifier

First-Order Formulae: Example (from before)

Propositional logic (not working):

- All logicians are rational p
- Some philosophers are not rational $\neg q$
- Thus, not all philosophers are logicians $\neg r$

Predicate logic (works!):

- All logicians are rational $\forall(x : Lx)Rx$
- Some philosophers are not rational $\exists(x : Px)\neg Rx$
- Thus, not all philosophers are logicians $\neg\forall(x : Px)Lx$

How to prove "$\forall(x : Lx)Rx, \exists(x : Px)\neg Rx \vdash \neg\forall(x : Px)Lx$"?

- Natural Deduction
- Semantic Tableau

From Restricted Quantifiers to Unrestricted Quantifiers: Unrestricted

So far, we were only considering restricted quantifiers:

- $\exists(x : Px)\neg Rx$ (Some philosophers are not rational)
- $\forall(x : Lx)Rx$ (All logicians are rational)

In the following we mainly use unrestricted quantifiers $\forall$ and $\exists$. The relationships between them are as follows:

- Existential quantified formulae become conjunctions:
  E.g., $\exists(x : Gx)Hx$ (some goats are hairy) becomes $\exists x\ Gx \wedge Hx$
- Universally quantified formulae become implications:
  E.g., $\forall(x : Gx)Hx$ (all goats are hairy) becomes $\forall x\ Gx \rightarrow Hx$

Natural Deduction

Introduction

- Instead of re-doing all our previous rules, we will just provide *additional ones*!
  - Two new rules for $\forall$ (introduction and elimination)
  - Two new rules for $\exists$ (introduction and elimination)
- We still perform natural deduction for *propositional logic* in intermediate steps.

Substitutions: Introduction

Our Natural Deduction rules will exploit *substitutions*.

Definition:

- Let A be a formula and t_1 and t_2 be terms.
- $A^{t_1}_{t_2}$ is the result of substituting each *free* (*unbound*) t_2 in A by t_1.
- Any mnemonic? How do I remember what gets substituted by what?
  - *Gravity falls!*
  - $A^{t_1}_{t_2}$ is the result of A after the t_1 “fell down” crushing t_2.

Substitutions: Examples (and Conventions)

- Let $A = \exists x(Px \to Rx)$. Is $A^y_x = \exists y(Py \to Ry)$?
  - No! Recall that x is required to be free/unbound in A!
  - Since there are no free variables, so $A^y_x = A$ here.
- Let $A = Fx \wedge \exists x(Fx \wedge Gx)$. What's A^y_x now?
  - It's $Fy \wedge \exists x(Fx \wedge Gx)$!
  - Because we only substitute free/unbound variables!

Universal Quantifiers

Universal Elimination: Introduction

- Let's assume we want to say that the age of all humans is smaller than 130: $\forall x\ age(x) < 130$
- If we had one constant for each individual (person), we could conclude: $age(a) < 130 \wedge age(b) < 130 \wedge age(c) < 130 \wedge \ldots$ (Though that's clearly not practical! And maybe not even possible if we reason about infinitely many objects like numbers.)
- So we could also conclude $age(x) < 130$ for *any* x! We thus use a (free) *variable* in our rule!
- So, what will the Universal-Elimination rule look like?

$$\boxed{\dfrac{\forall x\ Fx}{Fv}\forall E \qquad \text{more general:} \qquad \dfrac{\forall x\ A}{A_x^t}\forall E}$$

free/unbound x

We do however need a side condition here to make sure our newly introduced term doesn't cause trouble.

Universal Elimination: Side Condition

Assume we had no side condition:

$$\frac{\forall x\, A}{A^t_x}\forall E \qquad \text{in sequent notation:} \qquad \frac{X \vdash \forall x\, A}{X \vdash A^t_x}\forall E$$

Let's consider this sequent: $\forall x\, \exists y(y > x) \vdash \exists y(y > y)$

- Should that be valid? No! No number is larger than itself!
- But we can prove it! (If there's no side condition!)

α_1	(1)	$\forall x\, \exists y(y > x)$	A
α_1	(2)	$\exists y(y > y)$	1 $\forall E$

$$\frac{\forall x\; \overbrace{\exists y\,(y > x)}^{A}}{\underbrace{\exists y\,(y > y)}_{A^t_x \equiv A^y_x}}\forall E$$

So what's missing?

- The "instantiation of x" (the new variable name) must be free! (We don't want it to get captured by another quantifier!)
- This is different from what we demanded for substitutions.

Universal Elimination: The 1-step Rule

So, in conclusion:

Universal Elimination Rule:

$$\frac{X \vdash \forall x\ A}{X \vdash A^t_x}\forall E$$ only if t is not bound in A^t_x!

Universal Introduction: Introduction

- For the *introduction* of the universal quantifier, we would like to have, conceptually, a rule like the following:

$$\frac{Fa \quad Fb \quad Fc \quad \dots}{\forall x\, Fx}\,\forall I$$

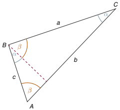
But that's again infeasible.

How about: $\dfrac{Fa}{\forall x\, Fx}\,\forall I$? (as above, a is a constant)

That rule is wrong! Just because Aristotle is a footballer, doesn't mean that everybody is!

But it might work for "typical objects"... (a variable)

Universal Introduction: Typical Objects

- What's a typical object?
- Remember the "undergraduate school" when you have to prove some property of *all* triangles.
  - *Step 1:* Let ABC=  be a triangle.

  - *Step 2:* "some fancy proof"
  - *Step 3:* Thus, ABC has property *P*. Thus *P* holds for all triangles!
- Why is that correct? Since we did not make any assumptions for ABC other than it being a triangle!

  assume the triangle ABC is an universal concept, rather than a specific object

Universal Introduction: The 1-step Rule

- So, we need an "object without any assumption" to generalize its property (formula) to the general case.
- But how to express this "no assumptions"?
- $\dfrac{Fv}{\forall x\ Fx}\forall I$ more general: $\dfrac{A}{\forall x\ A^x_v}\forall I$ with side condition:

  provided the variable v does not occur in any assumption that A depends upon.
- **Universal Introduction Rule**: (in sequent notation)

$$\frac{X \vdash A}{X \vdash \forall x\ A^x_v}\forall I$$

only if v does not occur in X!

More on Syntax

- Note that the Logic Notes refer to constants as “names”.

Example: Example

$$\forall x\ Fx, \forall x\ Gx \vdash \forall x\ (Fx \wedge Gx)$$

α_1	(1)	$\forall x\ Fx$	A
α_2	(2)	$\forall x\ Gx$	A
α_1	(3)	Fa	1 $\forall E$
α_2	(4)	Ga	2 $\forall E$
α_1, α_2	(5)	$Fa \wedge Ga$	3,4 $\wedge I$
α_1, α_2	(6)	$\forall x\ (Fx \wedge Gx)$	5 $\forall I$

$$\boxed{\dfrac{X \vdash A}{X \vdash \forall x\ A^x_v}\forall I}$$

Only if v does not occur in X!

$$\boxed{\dfrac{X \vdash \forall x\ A}{X \vdash A^t_x}\forall E}$$

only if t is not bounded in A_x^t

Example: Example

$$\forall x\ Fx, \forall x\ Gx \vdash \forall x\ (Fx \wedge Gx)$$

α_1	(1)	$\forall x\ Fx$	A
α_2	(2)	$\forall x\ Gx$	A
α_1	(3)	Fa	1 $\forall E$
α_2	(4)	Ga	2 $\forall E$
α_1, α_2	(5)	$Fa \wedge Ga$	3,4 $\wedge I$
α_1, α_2	(6)	$\forall x\ (Fx \wedge Gx)$	5 $\forall I$

$$\frac{X \vdash A}{X \vdash \forall x\ A^x_v}\forall I$$

Only if v does not occur in X!

$$\frac{X \vdash \forall x\ A}{X \vdash A^t_x}\forall E$$

only if t is not bounded in A_x^t

Did we adhere all side conditions? Yes!

- $X \vdash A$ of the $\forall I$ rule corresponds to line 5, which is $\alpha_1, \alpha_2 \vdash Fa \wedge Ga$,
- variable v corresponds to a, and
- although a (of course!) occurs in $Fa \wedge Ga$, it is not in $X = \{\alpha_1, \alpha_2\} = \{\forall x\ Fx, \forall x\ Gx\}$, so all good!

Existential Quantifiers

Existential Introduction: Introduction

- Recall that you can "imagine" the universal quantifier $\forall$ like: $\mathit{age}(a) < 130 \wedge \mathit{age}(b) < 130 \wedge \mathit{age}(c) < 130 \wedge \ldots$
- The existential quantifier $\exists$ can similarly interpreted as: $\mathit{age}(a) > 100 \vee \mathit{age}(b) > 100 \vee \mathit{age}(c) > 100 \vee \ldots$

Existential Introduction: The 1-step Rule

- **Existential Introduction Rule:**

$$\frac{Fv}{\exists x\ Fx}\exists I \quad \text{more general:} \quad \frac{A^t_x}{\exists x\ A}\exists I \quad \text{in sequent notation:} \quad \frac{X \vdash A^t_x}{X \vdash \exists x\ A}\exists I$$

  - Only if x is not bound in A. (I.e., you removed the new quantifier it would be free!)
  - Just make sure the new variable name doesn't get captured by another quantifier.

In simpler terms, if we have evidence that a certain property or predicate holds for a specific object, we can conclude that there exists an object satisfying that property or predicate.

Existential Elimination: Introduction

- We want to *eliminate* the existential quantifier. So can we just use the following rule? $\dfrac{\exists x\ Fx}{Fv}\exists E$? Recall: $\dfrac{\forall x\ Fx}{Fv}\forall E$!
- So, no! Because we don't know *which* object has that property!

Existential Elimination: Introduction, cont'd

- The *idea* behind the rule is the following:

specific case

in general

$$\cfrac{\exists x\ Fx \qquad \begin{array}{c}[Fy]\\ \vdots \\ B\end{array}}{B}\ \exists E$$

for a "typical" y.

- The idea is similar to disjunction elimination: In $A \vee B$, we don't know whether A or B is true, so we assume both and show that either way the derivation can be done.
- Here, we show it for "some instance" that does not pose further restrictions (and then discharge it since we know that such an "instance" exists due to the assumption $\exists x\ Fx$).

Existential Elimination: The 1-step Rule

Existential Elimination Rule:

$$\frac{X \vdash \exists x\, A^x_t \qquad Y, A \vdash B}{X, Y \vdash B}\exists E$$

Provided t does not occur in B or any formula in Y.

- Note what's written here: The assumption formula A in sequent 2 can be regarded an "instantiation" of the derivation in sequent 1 by substituting x by a term.

special case

- We need the side condition so that our choice of the "instance" of x is still "general".
- Otherwise we might be able to derive simply because we chose a specific special case!

In simpler terms, if we know that there exists an object with a certain property or satisfying a certain predicate, we can infer the truth of that property or predicate for a specific instance of such an object.

Example: Example

$\exists x\,(Fx \wedge Gx) \vdash \exists x\, Fx \wedge \exists x\, Gx$

∃_x A_t^x

	α_1	(1)	$\exists x\,(Fx \wedge Gx)$	A
	α_2	(2)	$Fa \wedge Ga$ A	A
	α_2	(3)	Fa	2 $\wedge E$
	α_2	(4)	$\exists x\, Fx$	3 $\exists I$
	α_2	(5)	Ga	2 $\wedge E$
	α_2	(6)	$\exists x\, Gx$	5 $\exists I$
A	α_2	(7)	$\exists x\, Fx \wedge \exists x\, Gx$ B	4,6 $\wedge I$
	α_1	(8)	$\exists x\, Fx \wedge \exists x\, Gx$	1,7[α_2] $\exists E$

$$\frac{X \vdash \exists x\, A^x_t \qquad Y, A \vdash B}{X, Y \vdash B}\,\exists E$$

Provided t does not occur in B or any formula in Y.

$$\frac{X \vdash A^t_x}{X \vdash \exists x\, A}\,\exists I$$

Summary

Content of this Lecture

- We introduced predicate logic:
  - with restricted quantifiers (we re-visit this later)
  - and with unrestricted quantifiers (default!)
- Predicate logic can reason about objects!
- Natural deduction for predicate logics, with additional rules for:
  - Introduction and Elimination rules for $\forall$ and $\exists$
  - For the rest we keep using the rules from propositional logics!
- Many side conditions...
  - Substitutions the substituted varibale must be unbound
  - $\forall E$ and $\exists I$
  - $\forall I$ and $\exists E$

→ The entire Logic Notes sections:
  - 4: Expressing Generality
    - except “Properties of relations”
    - and except “Functions”
    - (You should read them anyway, in particular “Functions”!)