Logic: Tutorial Week 9

Proof 1, Restricted Quantifiers

$$\forall(x\colon \neg Fx)\ Gx \ \vdash\ \forall(x\colon \neg Gx)\ Fx$$

α_1	(1)	$\forall(x\colon \neg Fx)\ Gx$	A
α_2	(2)	$\neg Ga$	A
α_3	(3)	$\neg Fa$	A
α_1, α_3	(4)	Ga	1, 3 $\forall\mathrm{E}_R$
α_1, α_2	(5)	$\neg\neg Fa$	2, 4 [α_3] RAA
α_1, α_2	(6)	Fa	5 $\neg\neg$E
α_1	(7)	$\forall(x\colon \neg Gx)\ Fx$	6 [α_2] $\forall\mathrm{I}_R$

Proof 2, Restricted Quantifiers

$$\exists(x\colon Fx)\ (Gx \wedge Hx) \ \vdash\ \exists(x\colon Gx)\ (Fx \wedge Hx)$$

α_1	(1)	$\exists(x\colon Fx)\ (Gx \wedge Hx)$	A
α_2	(2)	Fa	A (for $\exists\mathrm{E}_R$)
α_3	(3)	$Ga \wedge Ha$	A (for $\exists\mathrm{E}_R$)
α_3	(4)	Ga	3 $\wedge$E
α_3	(5)	Ha	3 $\wedge$E
α_2, α_3	(6)	$Fa \wedge Ha$	2, 5 $\wedge$I
α_2, α_3	(7)	$\exists(x\colon Gx)\ (Fx \wedge Hx)$	4, 6 $\exists\mathrm{I}_R$
α_1	(8)	$\exists(x\colon Gx)\ (Fx \wedge Hx)$	1, 7 [α_2,α_3] $\exists\mathrm{E}_R$

Proof 3, Restricted Quantifiers

$$\exists(x\colon Fx)\ (Gx \vee Hx),\ \forall(x\colon Gx)\ Hx \ \vdash\ \exists(x\colon Hx)\ Fx$$

α_1	(1)	$\exists(x\colon Fx)\ (Gx \vee Hx)$	A
α_2	(2)	$\forall(x\colon Gx)\ Hx$	A
α_3	(3)	Fa	A
α_4	(4)	$Ga \vee Ha$	A
α_5	(5)	Ga	A
α_2, α_5	(6)	Ha	2, 5 $\forall\mathrm{E}_R$
α_7	(7)	Ha	A
α_2, α_4	(8)	Ha	4, 6 [α_5], 7 [α_7] $\vee$E
$\alpha_2, \alpha_3, \alpha_4$	(9)	$\exists(x\colon Hx)\ Fx$	3, 8 $\exists\mathrm{I}_R$
α_1, α_2	(10)	$\exists(x\colon Hx)\ Fx$	1, 9 [α_3,α_4] $\exists\mathrm{E}_R$

Proof 4, Restricted Quantifiers

$$\forall(x\colon Fx)\,\exists(y\colon Gy)\,Rxy,\ \neg\exists(x\colon Gx)\,Hx\ \vdash\ \forall(x\colon Fx)\,\exists(y\colon Rxy)\,\neg Hy$$

α_1	(1)	$\forall(x\colon Fx)\,\exists(y\colon Gy)\,Rxy$	A
α_2	(2)	$\neg\exists(x\colon Gx)\,Hx$	A
α_3	(3)	Fa	A
α_1,α_3	(4)	$\exists(y\colon Gy)\,Ray$	1, 3 $\forall\mathrm{E}_R$
α_5	(5)	Gb	A
α_6	(6)	Rab	A
α_7	(7)	Hb	A
α_5,α_7	(8)	$\exists(x\colon Gx)\,Hx$	5, 7 $\exists\mathrm{I}_R$
α_2,α_5	(9)	$\neg Hb$	2, 8 $[\alpha_7]$ RAA
$\alpha_2,\alpha_5,\alpha_6$	(10)	$\exists(y\colon Ray)\,\neg Hy$	6, 9 $\exists\mathrm{I}_R$
$\alpha_1,\alpha_2,\alpha_3$	(11)	$\exists(y\colon Ray)\,\neg Hy$	4, 10 $[\alpha_5,\alpha_6]$ $\exists\mathrm{E}_R$
α_1,α_2	(12)	$\forall(x\colon Fx)\,\exists(y\colon Rxy)\,\neg Hy$	11 $[\alpha_3]$ $\forall\mathrm{I}_R$

Proof 1, Identity

$$\vdash\ \forall x\forall y(x = y \rightarrow f(x) = f(y))$$

α_1	(1)	$a = b$	A
	(2)	$f(a) = f(a)$	=I
α_1	(3)	$f(a) = f(b)$	1, 2 =E
	(4)	$a = b \rightarrow f(a) = f(b)$	3 $[\alpha_1]$ $\rightarrow$I
	(5)	$\forall y(a = y \rightarrow f(a) = f(y))$	4 $\forall$I
	(6)	$\forall x\forall y(x = y \rightarrow f(x) = f(y))$	5 $\forall$I

Proof 2, Identity

$$\vdash\ \forall x\forall y(\exists z(Rxz \wedge \neg Ryz) \rightarrow x \neq y)$$

α_1	(1)	$\exists z(Raz \wedge \neg Rbz)$	A
α_2	(2)	$a = b$	A
α_3	(3)	$Rac \wedge \neg Rbc$	A
α_3	(4)	Rac	3 $\wedge$E
α_3	(5)	$\neg Rbc$	3 $\wedge$E
α_2,α_3	(6)	Rbc	2, 4 =E
α_3	(7)	$a \neq b$	5, 6 $[\alpha_2]$ RAA
α_1	(8)	$a \neq b$	1, 7 $[\alpha_3]$ $\exists$E
	(9)	$\exists z(Raz \wedge \neg Rbz) \rightarrow a \neq b$	8 $[\alpha_1]$ $\rightarrow$I
	(10)	$\forall y(\exists z(Raz \wedge \neg Ryz) \rightarrow a \neq y)$	9 $\forall$I
	(11)	$\forall x\forall y(\exists z(Rxz \wedge \neg Ryz) \rightarrow x \neq y)$	10 $\forall$I

Proof 3, Identity

$$\forall x \forall y((Fx \wedge Gy) \rightarrow x = y),\ Fa \wedge Ha \ \vdash \ \forall x(Gx \rightarrow Hx)$$

α_1	(1)	$\forall x \forall y((Fx \wedge Gy) \rightarrow x = y)$	A
α_2	(2)	$Fa \wedge Ha$	A
α_3	(3)	Gb	A
α_2	(4)	Fa	2 $\wedge$E
α_2, α_3	(5)	$Fa \wedge Gb$	3, 4 $\wedge$I
α_1	(6)	$\forall y((Fa \wedge Gy) \rightarrow a = y)$	1 $\forall$E
α_1	(7)	$(Fa \wedge Gb) \rightarrow a = b$	6 $\forall$E
$\alpha_1, \alpha_2, \alpha_3$	(8)	$a = b$	5, 7 $\rightarrow$E
α_2	(9)	Ha	2 $\wedge$E
$\alpha_1, \alpha_2, \alpha_3$	(10)	Hb	8, 9 =E
α_1, α_2	(11)	$Gb \rightarrow Hb$	10 [α_3] $\rightarrow$I
α_1, α_2	(12)	$\forall x(Gx \rightarrow Hx)$	11 $\forall$I

Proof 4, Identity

$$\forall x \forall y((Fx \wedge Gy) \rightarrow x \neq y) \ \vdash \ \neg \exists x(Fx \wedge Gx)$$

α_1	(1)	$\forall x \forall y((Fx \wedge Gy) \rightarrow x \neq y)$	A
α_2	(2)	$\exists x(Fx \wedge Gx)$	A
α_3	(3)	$Fa \wedge Ga$	A
α_1	(4)	$\forall y((Fa \wedge Gy) \rightarrow a \neq y)$	1 $\forall$E
α_1	(5)	$(Fa \wedge Ga) \rightarrow a \neq a$	4 $\forall$E
α_1, α_3	(6)	$a \neq a$	3, 5 $\rightarrow$E
	(7)	$a = a$	=I
α_1, α_3	(8)	$\perp$	6, 7 $\neg$E
α_1, α_2	(9)	$\perp$	2, 8 [α_3] $\exists$E
α_1	(10)	$\neg \exists x(Fx \wedge Gx)$	9 [α_2] $\neg$I