

Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Propositional Logic — Semantic Tableaux

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Introduction

Recap: This course so far!

- What's a formula:
 - **Syntactically** (i.e., how do they look like) **atoms + connectives**
 - **Semantically** (i.e., what do they *mean*, which properties do they have; keywords: *Interpretations*, *Satisfiability*, *Unsatisfiability*, *Tautology*) **truth table**
- What's a sequent:
 - Syntactically (i.e., $X \vdash A$ and $X \models A$)
 - Semantically (i.e., $X \vdash A$ is called *valid*, $X \models A$, if each interpretation that makes all formulae in X true also makes A true.)
- How to prove validity?
 - With **truth tables** (that's the definition; but takes too long)
 - With **Natural deduction** (often much quicker, but 'harder')
tree-like / list-like format

Today: Motivation

- When you were asked to prove $X \vdash A$ with **Natural Deduction (ND)**, then... you were able to do so! The proof existed!
- Why? Because you were only proving *valid* sequents! *already know it's valid*
- Why is that problematic?
 - Because you cannot *decide* validity.
 - Suppose somebody asks: Is $X \vdash A$ valid, what do you do?
 - You can attempt ND, but if you fail: then why? Did you just not try hard enough? Or isn't it possible?
- Today: We learn a second proof system, which cannot only **prove validity** (if it's valid), but it can also **disprove validity** (if it's invalid)!
- We call this: *deciding validity*.
- This is the **Semantic Tableaux proof system**!
- (To be precise, it is not impossible to decide validity with ND.)

*We already know it's valide, only using ND to derive it
But is it valide in the first place?*

Recap on Definitions

- Hopefully everyone recalls the meaning of $X \models A$:
It means that A logically follows from the formulae in X , i.e., that sequent is valid , which is defined in terms of truth tables:
 - Each interpretation that makes all formulae in X true also also makes A true.
 - Or: There is no interpretation that makes X true, but not A .
- So what did $X \vdash A$ mean again?
 - It was actually $X \vdash_{ND} A$. doesn't prove the validity
 - It meant: A can be derived from X using Natural Deduction.
- Today, we learn how to decide validity using **Semantic Tableaux**.
 - Depending on context, $X \vdash A$ might stand for either Natural Deduction ($X \vdash_{ND} A$) or Semantic Tableaux ($X \vdash_{ST} A$).

Semantic Tableaux Proof Idea

General Idea behind Semantic Tableaux

can prove the validity of a sequent

- Semantic tableaux has its name because its proof technique mirrors/directly exploits the definition of validity of a sequent.
- So recall what $X \models A$ means:
 - Each interpretation that makes all formulae in X true also makes A true.
 - Or: There is no interpretation that makes X true, but not A .
- We pursue *proof by contradiction* to exploit this definition!
- General idea: Assume the sequent is invalid and detect a contradiction. From this contradiction we can infer that our assumption of invalidity must be wrong, and we can conclude validity.
- An additional advantage: If we don't get a contradiction we can even prove invalidity! (Which Natural Deduction can't!)

Semantic Tableaux as “Proof by Contradiction”

- For the fourth time today (sorry...), what's **validity**?
 - Each interpretation that makes all formulae in X true also makes A true.
 - Or: There is no interpretation that makes X true, but not A .
- So what's that “inverted” property, i.e., *not* valid or **invalid**?
 - There exists an interpretation such that:
 - ▶ it makes all formulae in X true
 - ▶ but it does *not* make A true!
 - If that leads to a **contradiction**, the sequent was **valid**!
 - If we *find* such an **interpretation**, we have a **counterexample**! :)

Suppose the sequence is invalid

- If it derives a contradiction of invalidity, i.e., all possible interpretations that make X true also make A true, then conclude the sequence is valid
- If it derives an interpretation of invalidity, i.e., there exists at least one interpretation that makes X true but A false, then conclude the sequence is invalid

Example for the Proof Idea

- Suppose we want to decide the sequent $\overbrace{p \wedge q}^X \vdash \overbrace{q \vee \neg p}^A$
- So we **assume invalidity**, meaning:
 - We assume there's an interpretation that makes $p \wedge q$ true
 - and the same interpretation does not make $q \vee \neg p$ true.

p	q	$\overbrace{p \wedge q}^X$	$\overbrace{q \vee \neg p}^A$
0	0	0	1
0	1	0	1
1	0	0	0
1	1	1	1

There's only one interpretation that makes $p \wedge q$ true, namely $I(p) = 1$ and $I(q) = 1$, but this one *also* makes $q \vee \neg p$ true!

- So our assumption that such an interpretation exists was wrong!
- So such an interpretation does not exist.
- But that's the definition of validity! :) So it's valid!

of invalidity
didn't find any counterexample

The Proof Technique in more Detail

- Maintain a list of formulae, label each either **T** (*true*) or **F** (*false*).
Suppose it's invalid
- Initially, **label** all formulae in X with **T**, and A with **F**. *invalid interpretation*
- Then **simplify** each formula and flip truth values as required. E.g.,
 - If some line holds **F** : $q \vee \neg p$, we get two more: **F** : q and **F** : $\neg p$
 - If some line holds **F** : $\neg p$, we get another line: **T** : p
- All lines below each other “belong together” and define **one shared interpretation**. Some rules will **branch**, i.e., create another set of lines existing “**in parallel**”. So different branches describe different interpretations.
- Once **no more formulae can be simplified** (and hence all propositional symbols have a truth value assigned), we either:
Find at least one invalid interpretation, claim it's invalid
 - 1 Have obtained **a consistent (contradiction-free) interpretation** that proves **invalidity**. (Such a branch is called **open branch**.)
 - 2 Or if **each branch leads to a contradiction** (e.g., **T** : p and **F** : p – or even with some formula), we proved **validity**.

The assumption (invalidity) derives contradiction, hence validity.

How to Support Climate Change – or: How to Prove (In)Validity

$$p \vee q, \neg p \vdash q$$

(1) derives p & q are both T

(1)	T:	$p \vee q$	✓	} premises X
(2)	T:	$\neg p$	✓	
(3)	F:	q		} conclusion A
(4)	F:	p		

Assume
invalidity

from (2) Simplify

(5) **T:** p ⚡ contradiction 1
from (1)

(6) **T:** q ⚡ contradiction 2
from (1)

Two contradictions proves the assumption doesn't hold, hence validity.

- Within each path, all lines “accumulate”.
- Branches branch, i.e., they split different possibilities.
- If **all leafs** die, the tree dies: Success! *Sequent is valid*.
- If **some leaf** survives, the tree lives: Failure!
 - The sequent is **invalid**.
 - We can extract an interpretation invalidating the sequent.

All Simplification Rules

Rule Set (And, Or, Not)

And Elimination:

$T: A \wedge B$	
<hr/>	
$T: A, T: B$	
$F: A \wedge B$	
<hr/>	
$F: A$	$F: B$

A	B	\wedge
0	0	0
0	1	0
1	0	0
1	1	1

The comma means that both lines hold in the **same branch**, i.e., we can write them **below each other**!

The bar (“|”) means that we branch over **different possibilities**, so the lines end up in **different branches**!

Rule Set (And, Or, Not)

And Elimination:

$$\frac{T: A \wedge B}{T: A, T: B}$$

$$\frac{F: A \wedge B}{F: A \mid F: B}$$

Or Elimination:

$$\frac{T: A \vee B}{T: A \mid T: B}$$

$$\frac{F: A \vee B}{F: A, F: B}$$

Negation Elimination:

$$\frac{T: \neg A}{F: A}$$

$$\frac{F: \neg A}{T: A}$$

A	B	\wedge
0	0	0
0	1	0
1	0	0
1	1	1

A	B	\vee
0	0	0
0	1	1
1	0	1
1	1	1

A	\neg
0	1
1	0

Rule Set (Implication)

Implication Elimination:

$$A \rightarrow B \leftrightarrow \neg A \wedge B$$

$$\neg(A \rightarrow B) \leftrightarrow \neg(\neg A \wedge B) \leftrightarrow A \vee \neg B$$

$\text{T: } A \rightarrow B$
$\text{F: } A \mid \text{T: } B$
$\text{F: } A \rightarrow B$
$\text{T: } A, \text{ F: } B$

A	B	\rightarrow
0	0	1
0	1	1
1	0	0
1	1	1

- Note that $X \vdash A$ iff $X \models A$ intuitively holds, because these rules mimic the truth tables *exactly*.
- Also keep in mind that we only write $X \vdash A$ instead of $X \vdash_{ND} A$ or $X \vdash_{ST} A$ since the applied proof system is clear from context.

Summary

Content of this Lecture

- We covered *Semantic Tableaux*, which “mimics” the definition of validity.
 - All rules required to simplify formulae as required.
 - You learned (or realized) that:
 - Natural Deduction is difficult to *decide* validity.
 - Semantic Tableaux is easy to decide validity.
- We covered the entire Logic Notes sections:
- More about propositional logic: Semantic tableaux

Examples!

Example (for a valid Sequent)

$$\vdash p \rightarrow (q \rightarrow p)$$

(1) **F:** $p \rightarrow (q \rightarrow p)$ ✓ Initialise A as F

(2) **T:** p from (1)

(3) **F:** $q \rightarrow p$ ✓ from (1)

(4) **T:** q from (3)

(5) **F:** p ⚡ from (3)

$\frac{\mathbf{F}: A \rightarrow B}{\mathbf{T}: A, \mathbf{F}: B}$
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- This sequent is valid, because all branches show a contradiction!
- Here, there was no branching. Normally, we have ≥ 2 branches, and *all* have to show a contradiction.
- We also did not use any assumptions here (that would have been labeled *true* (**T**)), because there weren't any.

Example (for an Invalid Sequent)

Attempting to show validity of $p \rightarrow (q \vee r) \vdash \neg(s \rightarrow \neg q) \rightarrow ((p \wedge s) \rightarrow r)$

- (1) **T:** $p \rightarrow (q \vee r)$ ✓
 (2) **F:** $\neg(s \rightarrow \neg q) \rightarrow ((p \wedge s) \rightarrow r)$ ✓
 (3) **T:** $\neg(s \rightarrow \neg q)$ ✓ from (2)
 (4) **F:** $(p \wedge s) \rightarrow r$ ✓ from (2)
 (5) **F:** $s \rightarrow \neg q$ ✓ from (3)
 (6) **T:** $p \wedge s$ ✓ from (4)
 (7) **F:** r from (4)
 (8) **T:** s from (5)
 (9) **F:** $\neg q$ ✓ from (5)
 (10) **T:** p from (6)
 (11) **T:** q from (9)

$$\frac{\mathbf{T}: A \rightarrow B}{\mathbf{F}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \rightarrow B}{\mathbf{T}: A, \mathbf{F}: B}$$

$$\frac{\mathbf{T}: \neg A}{\mathbf{F}: A}$$

Interpretation: $I(r) = 0, I(s) = I(q) = I(p) = 1$

- (12) **F:** p ⚡ from (1)

- (13) **T:** $q \vee r$ ✓ from (1)

- (14) **T:** q *open!* from (13)

- (15) **T:** r ⚡ from (13)

Comments About Previous Proof(s)

- In the beginning (when only the first lines were shown) we had the choice of which implication to simplify.
- We chose line (2), because its rule does not branch, and it's always good to postpone branching as long as possible so we don't "duplicate" work.
- We detected an *open branch*, i.e., a complete path where no further reductions were possible.
 - Via "collecting" the truth assignments to atoms along the open branch we can construct an interpretation.
 - That interpretation is a witness that:
 - ▶ that there exists an interpretation that makes X true but not A ,
 - ▶ and, thus (by definition), that A does not follow logically from X .
 - ▶ This means that the sequent is *invalid*.
- Also note that we have a contradiction whenever some *formula* appears true and false within the same branch. We do not need to wait until it is atomic.