Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Propositional Natural Deduction — Negation, Disjunction

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Introduction

Recap on Natural Deduction

- What are theorems? (Sequents without assumptions!)
- Relationship between $\vdash$ and $\rightarrow$:
  - They live in completely different worlds!
  - $\rightarrow$ is a connective and thus part of a formula, just like $\neg$, $\wedge$, and $\vee$.
  - $\vdash$ is *not a connective* and can thus not possibly be part of *any* formula! It only states whether we can derive a single formula A from a set of formulae X, expressed by $X \vdash A$.
- How do proofs in natural deduction look?
  - We use a list/table format with 4 columns.
  - All of these columns are *essential*!
- Introduction and elimination rules for:
  - Conjunction (easy!)
  - Implication (not quite that easy!)
- So what's missing?
  - Negation (not as easy as you might think!)
  - Disjunction (*quite* hard... Practice it!)

Negation

Introduction: Intuitive Meaning

- What does the negation connective in logics mean?
  - Socrates is a goat ($= p$)
  - ISocrates is not a goat ($= \neg p$)
- It inverts truth values (semantic).
- If assuming A leads to a contradiction, then we have the negation of A (syntactic).

Introduction: Truth Table

- Since the not connective simply inverts a single truth value we get a simple truth table:

p	$\neg p$
0	1
1	0

-

p	$\neg p$	$\neg\neg p$
0	1	0
1	0	1

$p \leftrightarrow \neg\neg p$

- I.e., in classical logic, we have the double negation elimination.
- It's not true that it's not true that Socrates is a goat (So it *is* true!)

The 1-Step Rules: Double-Negation Elimination and Introduction

- The (second) truth table gives us the following two rules:
- **Double-Negation Elimination and Introduction Rules**:

$$\frac{\neg\neg A}{A}\,\neg\neg E \qquad\qquad \frac{A}{\neg\neg A}\,\neg\neg I$$

- Again based on sequents:

$$\frac{X \vdash \neg\neg A}{X \vdash A}\,\neg\neg E \qquad\qquad \frac{X \vdash A}{X \vdash \neg\neg A}\,\neg\neg I$$

The 1-Step Rules: A Mistake That Will Cost You Marks

Avoid the next common mistake:

- Look carefully what/where the main connective is!
- The rule refers to a *complete formula*!
- So, e.g., we cannot go from $p \wedge \neg\neg q$ to $p \wedge q$ in just one step!

$$\frac{\neg\neg A}{A}\,\neg\neg E \qquad\qquad \frac{A}{\neg\neg A}\,\neg\neg I$$

The 1-Step Rules: A Mistake That Will Cost You Marks

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- Look carefully what/where the main connective is!
- The rule refers to a *complete formula*!
- So, e.g., we cannot go from $p \wedge \neg\neg q$ to $p \wedge q$ in just one step!
- Correctly handling that: (with a slightly more complex example)

$$p \wedge \neg\neg q \vdash \neg\neg p \wedge q$$

α_1	(1)	$p \wedge \neg\neg q$	A
α_1	(2)	p	1 $\wedge E$
α_1	(3)	$\neg\neg p$	2 $\neg\neg I$
α_1	(4)	$\neg\neg q$	1 $\wedge E$
α_1	(5)	q	4 $\neg\neg E$
α_1	(6)	$\neg\neg p \wedge q$	3,5 $\wedge I$

The 1-Step Rules: Negation-Elimination

- With the double-negation rules we can't introduce or eliminate a *single* negation.
- To deal with single negations, we require the symbol $\bot$.
- We introduced it before: it represents "false", an "absurd" constant that can never be satisfied.
- **Negation-Elimination rule:** (without and with sequent-notation)

$$\frac{A \qquad \neg A}{\bot}\neg E \qquad\qquad \frac{X \vdash A \qquad Y \vdash \neg A}{X, Y \vdash \bot}\neg E$$

(False/0)

The 1-Step Rules: Negation-Introduction

- **Negation-Introduction rule:** (without and with sequent-notation)

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ \bot\end{array}}{\neg A}\neg I \qquad\qquad \dfrac{X, A \vdash \bot}{X \vdash \neg A}\neg I$$

if A leads to a contradiction, then A must not stand

- Negation-Introduction discharges assumption A.
- Interesting fact(s):
  - Since we do not pose further restrictions on A, we can blame the contradiction on anything we want! E.g., if $X = \{A_1, \ldots, A_n\}$ and $X \vdash \bot$, we can conclude $X \setminus \{A_i\} \vdash \neg A_i$ for any $A_i \in X$.
  - This rule is the main proof idea behind the proof technique *"Proof by contradiction"*. (There are, e.g., nice illustrations on YouTube proving that $\sqrt{2}$ is not rational by that technique.)

The 1-Step Rules: Negation-Elimination and -Introduction, Example 1

- If you are in Canberra (p), you are not in Sydney ($\neg q$); thus: if you are in Sydney (q), you are not in Canberra ($\neg p$)

$p \to \neg q \vdash q \to \neg p$

α_1	(1)	$p \to \neg q$	A
α_2	(2)	q	A
α_3	(3)	p	A

The 1-Step Rules: Negation-Elimination and -Introduction, Example 1

- If you are in Canberra (p), you are not in Sydney ($\neg q$); thus: if you are in Sydney (q), you are not in Canberra ($\neg p$)

$p \to \neg q \vdash q \to \neg p$

α_1	(1)	$p \to \neg q$	A
α_2	(2)	q	A
α_3	(3)	p	A
α_1, α_3	(4)	$\neg q$	1,3 $\to E$
$\alpha_1, \alpha_2, \alpha_3$	(5)	$\perp$	2,4 $\neg E$
α_1, α_2	(6)	$\neg p$	5[α_3] $\neg I$
α_1	(7)	$q \to \neg p$	6[α_2] $\to I$

The 1-Step Rules: Negation-Elimination and -Introduction, Example 2

- A contradiction entails anything.

$$p, \neg p \vdash q$$

α_1	(1)	p	A
α_2	(2)	$\neg p$	A
α_1, α_2	(3)	$\bot$	1,2 $\neg E$
α_1, α_2	(4)	$\neg\neg q$	3[] $\neg I$
α_1, α_2	(5)	q	4 $\neg\neg E$

$$\boxed{\dfrac{X \vdash A \qquad Y \vdash \neg A}{X, Y \vdash \bot}\,\neg E}$$

$$\boxed{\dfrac{X, A \vdash \bot}{X \vdash \neg A}\,\neg I} \qquad \boxed{\dfrac{X \vdash \neg\neg A}{X \vdash A}\,\neg\neg E}$$

- Here we have another example of vacuous discharge: We blame the contradiction on a non-existing assumption $\neg q$.

X, ¬q ⊢⊥
---------- ¬I
X ⊢¬¬q

A 2-Step Rule: Reductio ad Absurdum (RAA)

- We can combine Negation-Elimination with its Introduction:

  Again, notations without and with sequents:

$$\dfrac{\begin{matrix}[B] \\ \vdots \\ A\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ \neg A\end{matrix}}{\neg B}\,RAA \qquad\qquad \dfrac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B}\,RAA$$

- The rules discharge assumption B.
- Why is it correct?

$$\dfrac{X, A \vdash \bot}{X \vdash \neg A}\neg I \qquad \dfrac{X \vdash A \qquad Y \vdash \neg A}{X, Y \vdash \bot}\neg E \qquad \dfrac{\dfrac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y, B \vdash \bot}\neg E}{X, Y \vdash \neg B}\neg I$$

A 2-Step Rule: Reductio ad Absurdum (RAA), Example

- $\neg p \to p \vdash p$: if p is even implied by its own negation, then it *must* be true!

Again! Since p and $\neg p$ can't be true at the same time, the implication $\neg p \to p$ cannot be "activated", so its precondition must be false.

$$\neg p \to p \vdash p$$

$$\boxed{\dfrac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B}\ RAA}$$

α_1	(1)	$\neg p \to p$	A
α_2	(2)	$\neg p$	A
α_1, α_2	(3)	p	1,2 $\to E$
α_1	(4)	$\neg\neg p$	2,3[α_2] RAA
α_1	(5)	p	4 $\neg\neg E$

Disjunctions

Introduction: (Our) Or versus Exclusive Or

- Disjunctions are of the form $A \vee B$
- It rains this afternoon or this evening.
  (But it can also be both!)
- The cat is *either* dead *or* alive.
  (Unless it's a physicist's cat, the choice is *exclusive*!
  The cat *cannot be both dead and alive*!)
- We use the first, non-exclusive, notion of or:
  At least one proposition needs to be true!

The 1-Step Rules: Disjunction-Introduction

Disjunction-Introduction Rules:

- Notation without sequents:

$$\frac{A}{A \vee B}\vee I \qquad\qquad \frac{B}{A \vee B}\vee I$$

- Notation with sequents:

$$\frac{X \vdash A}{X \vdash A \vee B}\vee I \qquad\qquad \frac{X \vdash B}{X \vdash A \vee B}\vee I$$

The 1-Step Rules: Disjunction-Elimination, Introduction

- If x is even, then $x^2 + x$ is even.
- If x is odd, then $x^2 + x$ is even.
- x is either odd or even.
- Thus, $x^2 + x$ is even.
- Formally, this can be expressed as $p \rightarrow r, q \rightarrow r, p \vee q \vdash r$

The 1-Step Rules: Disjunction-Elimination Rule (Based on Sequents)

- Disjunction elimination rule:

$$\dfrac{A \vee B \qquad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}\vee E \qquad \dfrac{X \vdash A \vee B \qquad Y, A \vdash C \qquad Z, B \vdash C}{X, Y, Z \vdash C}\vee E$$

The 1-Step Rules: When to Use that Rule

$$\frac{X \vdash A \vee B \qquad Y, A \vdash C \qquad Z, B \vdash C}{X, Y, Z \vdash C} \vee E$$

- Technically, this rule is used to "eliminate" a disjunction.
- But in practice, we use it to prove one!
- How is that possible? Because we can use *any* formula for C!
- I.e., when we want to derive a disjunction, we can use it as C – but this will also require another disjunction for the first sequent!
- We often obtain that one via assuming it.

The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 1

- Disjunction is commutative: $p \lor q \vdash q \lor p$

α_1	(1)	$p \lor q$	A
α_2	(2)	p	A
α_2	(3)	$q \lor p$	2 $\lor I$
α_3	(4)	q	A
α_3	(5)	$q \lor p$	4 $\lor I$
α_1	(6)	$q \lor p$	1,3[α_2],5[α_3] $\lor E$

$$\boxed{\dfrac{X \vdash B}{X \vdash A \lor B}\lor I}$$

$$\boxed{\dfrac{X \vdash A}{X \vdash A \lor B}\lor I}$$

$$\boxed{\dfrac{X \vdash A \lor B \qquad Y, A \vdash C \qquad Z, B \vdash C}{X, Y, Z \vdash C}\lor E}$$

$$X = \overbrace{\{p \lor q\}}^{\alpha_1} \quad A = \alpha_2 = p$$
$$Y = \emptyset \quad B = \alpha_3 = q$$
$$Z = \emptyset \quad C = q \lor p$$

The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 1

- Disjunction is commutative: $p \vee q \vdash q \vee p$

α_1	(1)	$p \vee q$	A
α_2	(2)	p	A
α_2	(3)	$q \vee p$	2 $\vee I$
α_3	(4)	q	A
α_3	(5)	$q \vee p$	4 $\vee I$
α_1	(6)	$q \vee p$	1,3[α_2],5[α_3] $\vee E$

$$\frac{X \vdash B}{X \vdash A \vee B} \vee I$$

$$\frac{X \vdash A}{X \vdash A \vee B} \vee I$$

- In line 3, the q was just some *arbitrary* truth value that we've added due to Disjunction-Introduction!
- Similarly in line 5 the p was arbitrary. Notably, that's not the p from assumption α_2.

The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 2

- Conjunction and disjunction behave just like multiplication and addition, e.g. $p \cdot (q + r) = p \cdot q + p \cdot r$:

$$p, q \vee r \vdash (p \wedge q) \vee (p \wedge r)$$

The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 2

- Conjunction and disjunction behave just like multiplication and addition, e.g. $p \cdot (q + r) = p \cdot q + p \cdot r$:

  $p, q \vee r \vdash (p \wedge q) \vee (p \wedge r)$

α_1	(1)	p	A
α_2	(2)	$q \vee r$	A
α_3	(3)	q	A
α_4	(4)	r	A
α_1, α_3	(5)	$p \wedge q$	1,3 $\wedge I$
α_1, α_3	(6)	$(p \wedge q) \vee (p \wedge r)$	5 $\vee I$
α_1, α_4	(7)	$p \wedge r$	1,4 $\wedge I$
α_1, α_4	(8)	$(p \wedge q) \vee (p \wedge r)$	7 $\vee I$
α_1, α_2	(9)	$(p \wedge q) \vee (p \wedge r)$	2,6[α_3],8[α_4] $\vee E$

Summary

Content of this Lecture

- The remaining rules for natural deduction: negation and disjunction

$\rightarrow$ The entire Logic Notes sections:

  - Propositional natural deduction: Negation
  - Propositional natural deduction: Disjunction

  $\rightarrow$ We are done now with everything until Section 2!