Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Propositional Logic — Semantic Tableaux

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Introduction

Recap: This course so far!

- What's a formula:
  - Syntactically (i.e., how do they look like) atoms + connectives
  - Semantically (i.e., what do they *mean*, which properties do they have; keywords: *Interpretations*, *Satisfiability*, *Unsatisfiability*, *Tautology*) truth table
- What's a sequent:
  - Syntactically (i.e., $X \vdash A$ and $X \models A$)
  - Semantically (i.e., $X \vdash A$ is called *valid*, $X \models A$, if each interpretation that makes all formulae in X true also makes A true.)
- How to prove validity?
  - With truth tables (that's the definition; but takes too long)
  - With Natural deduction (often much quicker, but 'harder')
    tree-like / list-like format

Today: Motivation

validity

- When you were asked to prove $X \vdash A$ with Natural Deduction (ND), then... you were able to do so! The proof existed!
- Why? Because you were only proving *valid* sequents! already know it's valid
- Why is that problematic?
  - Because you cannot *decide* validity.
  - Suppose somebody asks: Is $X \vdash A$ valid, what do you do?
  - You can attempt ND, but if you fail: then why? Did you just not try hard enough? Or isn't it possible?
- Today: We learn a second proof system, which cannot only prove validity (if it's valid), but it can also *disprove* validity (if it's invalid)!
- We call this: *deciding validity*.

→ This is the Semantic Tableaux proof system!

- (To be precise, it is not impossible to decide validity with ND.)

We already know it's valide, only using ND to derive it
But is it valide in the first place?

Recap on Definitions

- Hopefully everyone recalls the meaning of $X \models A$:
  It means that A logically follows from the formulae in X, i.e., that sequent is *valid*, which is defined in terms of truth tables:
  - Each interpretation that makes all formulae in X true also also makes A true.
  - Or: There is no interpretation that makes X true, but not A.
- So what did $X \vdash A$ mean again?
  - It was actually $X \vdash_{ND} A$.
  - It meant: A can be derived from X using Natural Deduction.

doen't prove the validity

- Today, we learn how to decide validity using Semantic Tableaux.
  - Depending on context, $X \vdash A$ might stand for either Natural Deduction ($X \vdash_{ND} A$) or Semantic Tableaux ($X \vdash_{ST} A$).

Semantic Tableaux Proof Idea

General Idea behind Semantic Tableaux

can prove the validity of a sequent

- Semantic tableaux has its name because its proof technique mirrors/directly exploits the definition of validity of a sequent.
- So recall what $X \models A$ means:
  - Each interpretation that makes all formulae in X true also makes A true.
  - Or: There is no interpretation that makes X true, but not A.
- We pursue *proof by contradiction* to exploit this definition!
- General idea: Assume the sequent is invalid and detect a contradiction. From this contradiction we can infer that our assumption of invalidity must be wrong, and we can conclude validity.
- An additional advantage: If we don't get a contradiction we can even prove invalidity! (Which Natural Deduction can't!)

Semantic Tableaux as "Proof by Contradiction"

- For the fourth time today (sorry...), what's validity?
  - Each interpretation that makes all formulae in X true also makes A true.
  - Or: There is no interpretation that makes X true, but not A.
- So what's that "inverted" property, i.e., *not* valid or *invalid*?
  - There exists an interpretation such that:
    - it makes all formulae in X true
    - but it does *not* make A true! counter example
  - If that leads to a contradiction, the sequent was valid!
  - If we *find* such an interpretation, we have a counterexample (!) of validity

Suppose the sequence is invalid
- If it derives a contradiction of invalidity, then conclude it's valid
- If it derives an interpretation of invalidity, then conclude it's invalid

Example for the Proof Idea

- Suppose we want to decide the sequent $\overbrace{p \wedge q}^{X} \vdash \overbrace{q \vee \neg p}^{A}$
- So we assume invalidity, meaning:
  - We assume there's an interpretation that makes $p \wedge q$ true
  - and the same interpretation does not make $q \vee \neg p$ true.

p	q	$\overbrace{p \wedge q}^{X}$	$\overbrace{q \vee \neg p}^{A}$
0	0	0	1
0	1	0	1
1	0	0	0
1	1	1	1

There's only one interpretation that makes $p \wedge q$ true, namely $I(p) = 1$ and $I(q) = 1$, but this one *also* makes $q \vee \neg p$ true!

of invalidity

- So our assumption that such an interpretation exists was wrong!
- So such an interpretation does not exist.
- But that's the definition of validity! :) So it's valid!

didn't find any counterexample

The Proof Technique in more Detail

- Maintain a list of formulae, label each either **T** (*true*) or **F** (*false*).

  Suppose it's invalid
- Initially, label all formulae in X with **T**, and A with **F**. invalid interpretation
- Then simplify each formula and flip truth values as required. E.g.,
  - If some line holds **F** : $q \vee \neg p$, we get two more: **F** : q and **F** : $\neg p$
  - If some line holds **F** : $\neg p$, we get another line: **T** : p
- All lines below each other "belong together" and define one shared interpretation. Some rules will *branch*, i.e., create another set of lines existing "in parallel". So different branches describe different interpretations.
- Once no more formulae can be simplified (and hence all propositional symbols have a truth value assigned), we either:

  Find at least one invalid interpretation, claim it's invalid
  1. Have obtained a consistent (contradiction-free) interpretation that proves invalidity. (Such a branch is called **open branch**.)
  2. Or if each branch leads to a contradiction (e.g., **T** : p and **F** : p – or even with some formula), we proved validity.

  The assumption (invalidity) derives contradiction, hence validity.

How to Support Climate Change – or: How to Prove (In)Validity

$$p \vee q, \neg p \vdash q$$

(1) derives p & q are both T

(1)	**T:**	$p \vee q$ ✓	premises X	Assume invalidity
(2)	**T:**	$\neg p$ ✓		
(3)	**F:**	q	conclusion A	
(4)	**F:**	p	from (2)	Simplify

contradiction 1

(5) **T:** p ↯ from (1)

contradiction 2

(6) **T:** q ↯ from (1)

Two contradictions proves the assumption doesn't hold, hence validity.

- Within each path, all lines "accumulate".
- Branches branch, i.e., they split different possibilities.
- If all leafs die, the tree dies: Success! *Sequent is valid.*
- If some leaf survives, the tree lives: Failure!
  - The sequent is invalid.
  - We can extract an interpretation invalidating the sequent.

All Simplification Rules

Rule Set (And, Or, Not)

And Elimination:

$$\frac{\textbf{T: } A \wedge B}{\textbf{T: } A\ ,\ \ \textbf{T: } B}$$

$$\frac{\textbf{F: } A \wedge B}{\textbf{F: } A \;\big|\; \textbf{F: } B}$$

A	B	$\wedge$
0	0	0
0	1	0
1	0	0
1	1	1

The comma means that both lines hold in the same branch, i.e., we can write them below each other!

The bar ("|") means that we branch over different possibilities, so the lines end up in different branches!

Rule Set (And, Or, Not)

And Elimination:

$$\frac{\textbf{T: } A \wedge B}{\textbf{T: } A \,,\; \textbf{T: } B}$$

$$\frac{\textbf{F: } A \wedge B}{\textbf{F: } A \;\Big|\; \textbf{F: } B}$$

Or Elimination:

$$\frac{\textbf{T: } A \vee B}{\textbf{T: } A \;\Big|\; \textbf{T: } B}$$

$$\frac{\textbf{F: } A \vee B}{\textbf{F: } A \,,\; \textbf{F: } B}$$

Negation Elimination:

$$\frac{\textbf{T: } \neg A}{\textbf{F: } A}$$

$$\frac{\textbf{F: } \neg A}{\textbf{T: } A}$$

A	B	$\wedge$
0	0	0
0	1	0
1	0	0
1	1	1

A	B	$\vee$
0	0	0
0	1	1
1	0	1
1	1	1

A	$\neg$
0	1
1	0

Rule Set (Implication)

Implication Elimination:

$A \to B \leftrightarrow \neg A \wedge B$

$\neg(A \to B) \leftrightarrow \neg(\neg A \wedge B) \leftrightarrow A \vee \neg B$

$$\frac{\textbf{T: } A \to B}{\textbf{F: } A \mid \textbf{T: } B}$$

$$\frac{\textbf{F: } A \to B}{\textbf{T: } A \text{ , } \textbf{F: } B}$$

A	B	$\to$
0	0	1
0	1	1
1	0	0
1	1	1

- Note that $X \vdash A$ iff $X \models A$ intuitively holds, because these rules mimic the truth tables *exactly*.
- Also keep in mind that we only write $X \vdash A$ instead of $X \vdash_{ND} A$ or $X \vdash_{ST} A$ since the applied proof system is clear from context.

Summary

Content of this Lecture

- We covered *Semantic Tableaux*, which “mimics” the definition of validity.
- All rules required to simplify formulae as required.
- You learned (or realized) that:
  - Natural Deduction is difficult to *decide* validity.
  - Semantic Tableaux is easy to decide validity.

→ We covered the entire Logic Notes sections:
  - More about propositional logic: Semantic tableaux

Examples!

Example (for a valid Sequent)

$$\vdash p \to (q \to p)$$

(1)	**F:**	$p \to (q \to p)$ ✓	Initialise A as F	
(2)	**T:**	p		from (1)
(3)	**F:**	$q \to p$ ✓		from (1)
(4)	**T:**	q		from (3)
(5)	**F:**	p ↯		from (3)

$$\frac{\textbf{F: } A \to B}{\textbf{T: } A\ ,\ \textbf{F: } B}$$

- This sequent is valid, because all branches show a contradiction!
- Here, there was no branching. Normally, we have ≥ 2 branches, and *all* have to show a contradiction.
- We also did not use any assumptions here (that would have been labeled *true* (**T**)), because there weren't any.

Example (for an Invalid Sequent)

Attempting to show validity of $p \to (q \vee r) \vdash \neg(s \to \neg q) \to ((p \wedge s) \to r)$

(1)	**T:**	$p \to (q \vee r)$ ✓		
(2)	**F:**	$\neg(s \to \neg q) \to ((p \wedge s) \to r)$ ✓		
(3)	**T:**	$\neg(s \to \neg q)$ ✓	from (2)	
(4)	**F:**	$(p \wedge s) \to r$ ✓	from (2)	
(5)	**F:**	$s \to \neg q$ ✓	from (3)	
(6)	**T:**	$p \wedge s$ ✓	from (4)	
(7)	**F:**	r	from (4)	
(8)	**T:**	s	from (5)	
(9)	**F:**	$\neg q$ ✓	from (5)	
(10)	**T:**	p	from (6)	
(11)	**T:**	q	from (9)	

(12) **F:** p ↯ from (1)

(13) **T:** $q \vee r$ ✓ from (1)

(14) **T:** q *open!* from (13)

(15) **T:** r ↯ from (13)

Interpretation: $I(r) = 0$, $I(s) = I(q) = I(p) = 1$

$$\frac{\mathbf{T}: A \to B}{\mathbf{F}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \to B}{\mathbf{T}: A\,,\ \mathbf{F}: B}$$

$$\frac{\mathbf{T}: \neg A}{\mathbf{F}: A}$$

Comments About Previous Proof(s)

- In the beginning (when only the first lines were shown) we had the choice of which implication to simplify.
- We chose line (2), because its rule does not branch, and it's always good to postpone branching as long as possible so we don't "duplicate" work.
- We detected an *open branch*, i.e., a complete path where no further reductions were possible.
  - Via "collecting" the truth assignments to atoms along the open branch we can construct an interpration.
  - That interpretation is a witness that:
    - that there exists an interpretation that makes X true but not A,
    - and, thus (by definition), that A does not follow logically from X.
    - This means that the sequent is *invalid*.
- Also note that we have a contradiction whenever some *formula* appears true and false within the same branch. We do not need to wait until it is atomic.