Logic: Tutorial Week 5

Proof 1

$$\forall x(Fx \to Gx),\ \forall x(Gx \to Hx)\ \vdash\ \forall x(Fx \to Hx)$$

α_1	(1)	$\forall x(Fx \to Gx)$	A (given)
α_2	(2)	$\forall x(Gx \to Hx)$	A (given)
α_3	(3)	Fa	A (antecedent of $Fa \to Ha$)
α_1	(4)	$Fa \to Ga$	1 $\forall$E
α_1, α_3	(5)	Ga	3, 4 $\to$E
α_2	(6)	$Ga \to Ha$	2 $\forall$E
$\alpha_1, \alpha_2, \alpha_3$	(7)	Ha	5, 6 $\to$E
α_1, α_2	(8)	$Fa \to Ha$	7 $[\alpha_3]$ $\to$I
α_1, α_2	(9)	$\forall x(Fx \to Hx)$	8 $\forall$I

The conclusion $\forall x(Fx \to Hx)$ is universal in form, so we aim to get it from a typical instance $Fa \to Ha$. This in turn is a conditional, so the usual $\to$I strategy applies: we assume its antecedent (line 3) and derive its consequent (line 7). The overall shape of the proof is quite common in first order natural deduction: the quantifiers in the premises are stripped off so that we can do some propositional logic in the middle of the proof, and then the quantifiers are put back at the end.

Proof 2

$$\forall x(\neg Fx \to Gx)\ \vdash\ \forall x(\neg Gx \to Fx)$$

α_1	(1)	$\forall x(\neg Fx \to Gx)$	A
α_2	(2)	$\neg Ga$	A (for $\to$I)
α_3	(3)	$\neg Fa$	A (for RAA)
α_1	(4)	$\neg Fa \to Ga$	1 $\forall$E
α_1, α_3	(5)	Ga	4, 5 $\to$E
α_1, α_2	(6)	$\neg\neg Fa$	2, 5 $[\alpha_3]$ RAA
α_1, α_2	(7)	Fa	6 $\neg\neg$E
α_1	(8)	$\neg Ga \to Fa$	7 $[\alpha_2]$ $\to$I
α_1	(9)	$\forall x(\neg Gx \to Fx)$	8 $\forall$I

Here the propositional logic in the middle of the proof involves the negation rules, including $\neg\neg$E. The strategy for handling the universal quantifier is essentially the same as in proof 1 above.

Proof 3

$$\forall x(Fx \to Gx),\ \exists x(Fx \wedge Hx) \ \vdash\ \exists x(Gx \wedge Hx)$$

α_1	(1)	$\forall x(Fx \to Gx)$	A
α_2	(2)	$\exists x(Fx \wedge Hx)$	A
α_3	(3)	$Fa \wedge Ha$	A (instance of 2 for $\exists$E)
α_3	(4)	Fa	3 $\wedge$E
α_1	(5)	$Fa \to Ga$	1 $\forall$E
α_1, α_3	(6)	Ga	4, 5 $\to$E
α_3	(7)	Ha	3 $\wedge$E
α_1, α_3	(8)	$Ga \wedge Ha$	6, 7 $\wedge$I
α_1, α_3	(9)	$\exists x(Gx \wedge Hx)$	8 $\exists$I
α_1, α_2	(10)	$\exists x(Gx \wedge Hx)$	2, 9 [α_3] $\exists$E

This is a very straightforward $\exists$E proof. Note that the $\exists$I move has to be performed *before* the $\exists$E, so that the name a does not occur in the conclusion.

Proof 4

$$\forall x(Fx \to \neg Gx) \ \vdash\ \neg\exists x(Fx \wedge Gx)$$

α_1	(1)	$\forall x(Fx \to \neg Gx)$	A
α_2	(2)	$\exists x(Fx \wedge Gx)$	A
α_3	(3)	$Fa \wedge Ga$	A
α_3	(4)	Fa	3 $\wedge$E
α_1	(5)	$Fa \to \neg Ga$	1 $\forall$E
α_1, α_3	(6)	$\neg Ga$	4, 5 $\to$E
α_3	(7)	Ga	3 $\wedge$E
α_1, α_3	(8)	$\perp$	6, 7 $\neg$E
α_1, α_2	(9)	$\perp$	2, 8 [α_3] $\exists$E
α_1	(10)	$\neg\exists x(Fx \wedge Gx)$	9 [α_2] $\neg$I

The primitive negation rules are better than RAA for the purposes of this proof, since using RAA at line 8 would have made the formula at line 9 $\neg(Fa \wedge Ga)$ which contains the name a, so the application of $\exists$E at line 9 would have been blocked.

Proof 5

$$\forall x(Fx \to Hx),\ \forall x(Gx \to \neg Hx)\ \vdash\ \neg\exists x(Fx \wedge Gx)$$

α_1	(1)	$\forall x(Fx \to Hx)$	A
α_2	(2)	$\forall x(Gx \to \neg Hx)$	A
α_3	(3)	$\exists x(Fx \wedge Gx)$	A
α_4	(4)	$Fa \wedge Ga$	A
α_4	(5)	Fa	4 $\wedge$E
α_1	(6)	$Fa \to Ha$	1 $\forall$E
α_1, α_4	(7)	Ha	5, 6 $\to$E
α_2	(8)	$Ga \to \neg Ha$	2 $\forall$E
α_4	(9)	Ga	4 $\wedge$E
α_2, α_4	(10)	$\neg Ha$	8, 9 $\to$E
$\alpha_1, \alpha_2, \alpha_4$	(11)	$\bot$	7, 10 $\neg$E
$\alpha_1, \alpha_2, \alpha_3$	(12)	$\bot$	3, 11 [α_4] $\exists$E
α_1, α_2	(13)	$\neg\exists x(Fx \wedge Gx)$	12 [α_3] $\neg$I

This is very similar to the last proof, and similar remarks apply.

Proof 6

$$\neg\forall x(Fx \to Gx)\ \vdash\ \exists x(Fx \wedge \neg Gx)$$

α_1	(1)	$\neg\forall x(Fx \to Gx)$	A
α_2	(2)	$\neg\exists x(Fx \wedge \neg Gx)$	A
α_3	(3)	Fa	A
α_4	(4)	$\neg Ga$	A
α_3, α_4	(5)	$Fa \wedge \neg Ga$	3, 4 $\wedge$I
α_3, α_4	(6)	$\exists x(Fx \wedge \neg Gx)$	5 $\exists$I
α_2, α_3	(7)	$\neg\neg Ga$	2,6 [α_4] RAA
α_2, α_3	(8)	Ga	7 $\neg\neg$E
α_2	(9)	$Fa \to Ga$	8 [α_3] $\to$I
α_2	(10)	$\forall x(Fx \to Gx)$	9 $\forall$I
α_1	(11)	$\neg\neg\exists x(Fx \wedge \neg Gx)$	1, 10 [α_2] RAA
α_1	(12)	$\exists x(Fx \wedge \neg Gx)$	11 $\neg\neg$E

This requires indirect proof, as $\neg\neg$E must be used in every proof of this sequent. Lines 2 – 10 constitute a proof of the contraposed version

$\neg\exists x(Fx \wedge \neg Gx)\ \vdash\ \forall x(Fx \to Gx)$

which is easier to approach as it falls to the usual $\to$I strategy.

Proof 7

$$\vdash \exists x(Fx \to \forall y Fy)$$

α_1	(1)	$\neg\exists x(Fx \to \forall yFy)$	A
α_2	(2)	$\neg Fa$	A
α_3	(3)	Fa	A
α_2, α_3	(4)	$\neg\neg\forall yFy$	2, 3 [ ] RAA
α_2, α_3	(5)	$\forall yFy$	4 $\neg\neg$E
α_2	(6)	$Fa \to \forall yFy$	5 [α_3] $\to$I
α_2	(7)	$\exists x(Fx \to \forall yFy)$	6 $\exists$I
α_1	(8)	$\neg\neg Fa$	1, 7 [α_2] RAA
α_1	(9)	Fa	8 $\neg\neg$E
α_1	(10)	$\forall yFy$	9 $\forall$I
α_1	(11)	$Fa \to \forall yFy$	10 [ ] $\to$I
α_1	(12)	$\exists x(Fx \to \forall yFy)$	11 $\exists$I
	(13)	$\neg\neg\exists x(Fx \to \forall yFy)$	1, 12 [α_1] RAA
	(14)	$\exists x(Fx \to \forall yFy)$	13 $\neg\neg$E

This is perhaps the most obscure proof in the entire course. The sequent is sometimes called the "key drinker theorem", as an instance of it states: "There is someone such that if he drinks, then everybody drinks!" Its proof requires all the resources of classical logic, including indirect proof and vacuous discharge (twice).