Logic: Tutorial Week 4

This tutorial exercise requires you to use the semantic tableau method to analyse sequents featuring the biconditional $\leftrightarrow$. Part of the message to take away from this is that biconditionals do not "fit" so well in tableaux—tableaux are actually rather inefficient and cumbersome when these connectives are involved. (You will see from your proofs that you have to do!)

1 Exercise 1

Give rules for handling the biconditional $\leftrightarrow$ in tableaux. Your rules should not involve any other connectives, so for instance rewriting $A \leftrightarrow B$ as $(A \to B) \wedge (B \to A)$ would not do.

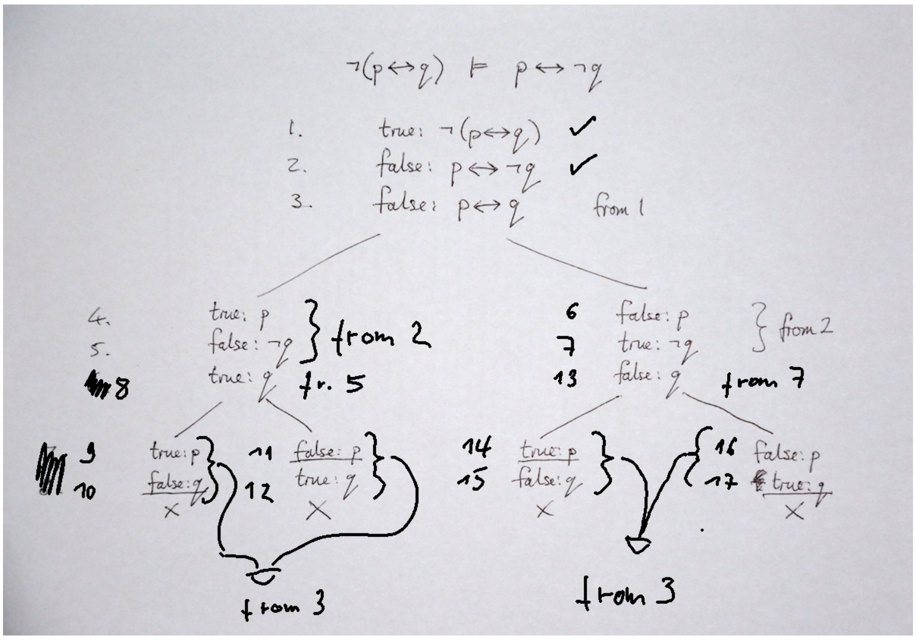
$$\frac{\textbf{t: } A \leftrightarrow B}{\textbf{t: } A,\ \textbf{t: } B \;\Big|\; \textbf{f: } A,\ \textbf{f: } B} \qquad \frac{\textbf{f: } A \leftrightarrow B}{\textbf{t: } A,\ \textbf{f: } B \;\Big|\; \textbf{f: } A,\ \textbf{t: } B}$$

That is, the branch splits and its two principal subformulae appear on each side, with the same labels if the biconditional is true, or opposite ones if it is false.

2 Exercise 2

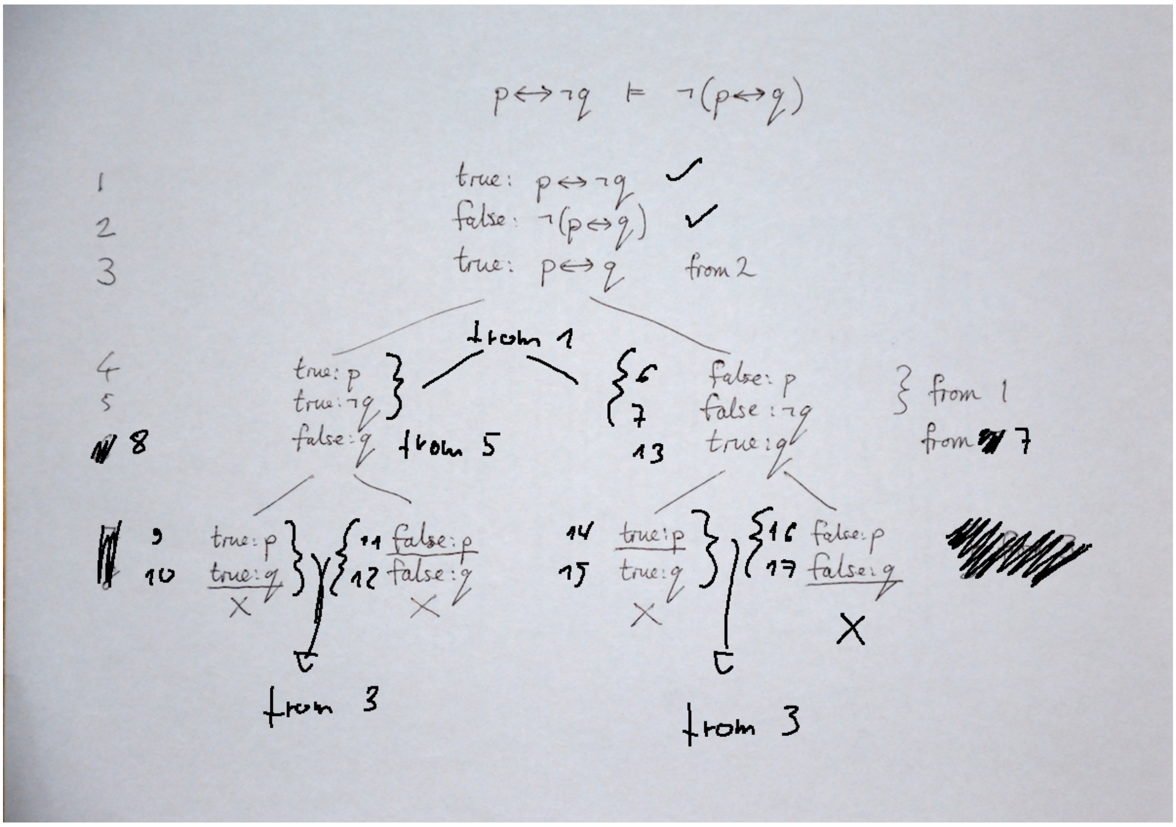
Use your tableau rules to show that $\neg(p \leftrightarrow q)$ is equivalent to $p \leftrightarrow \neg q$.

A being eqivalent to B means that each entails the other, i.e., we need to show both $A \models B$ as well as $B \models A$. We start by showing (via semantic tableau) that $\neg(p \leftrightarrow q)$ semantically entails $p \leftrightarrow \neg q$:



The first split, resulting in the formulae at levels 4 and 5, comes from the analysis of the formula at level 2. Since this is a biconditional laballed as false, its two subformulae p and $\neg q$ appear on the left with values true and false respectively, and on the right with values false and true.

Now here is the very similar tableau showing that the converse entailment is also valid:

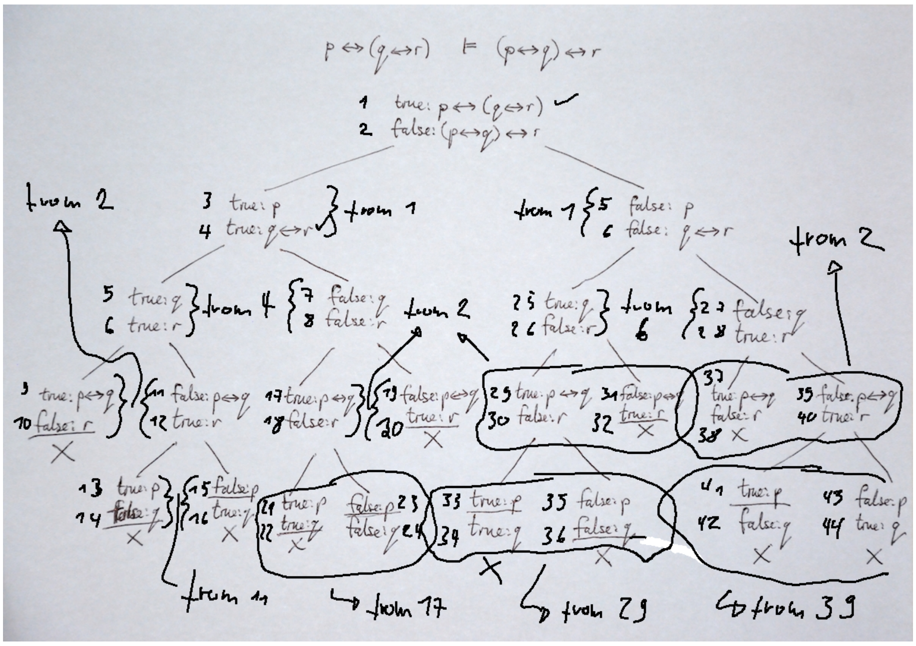


Notice that there are four branches in the above tableaux, just as there are four lines in the truth table with only the two atoms. Hence in this case, the tableau gives no saving in efficiency compared with the crude truth table test: writing out all four lines of the truth table is just as fast as writing out the tableau, and probably easier to read.

3 Exercise 3

Show that $p \leftrightarrow (q \leftrightarrow r)$ **is equivalent to** $(p \leftrightarrow q) \leftrightarrow r$**.**

The fact that biconditionals often cause tableaux to blow up exponentially is seen clearly in the next example:

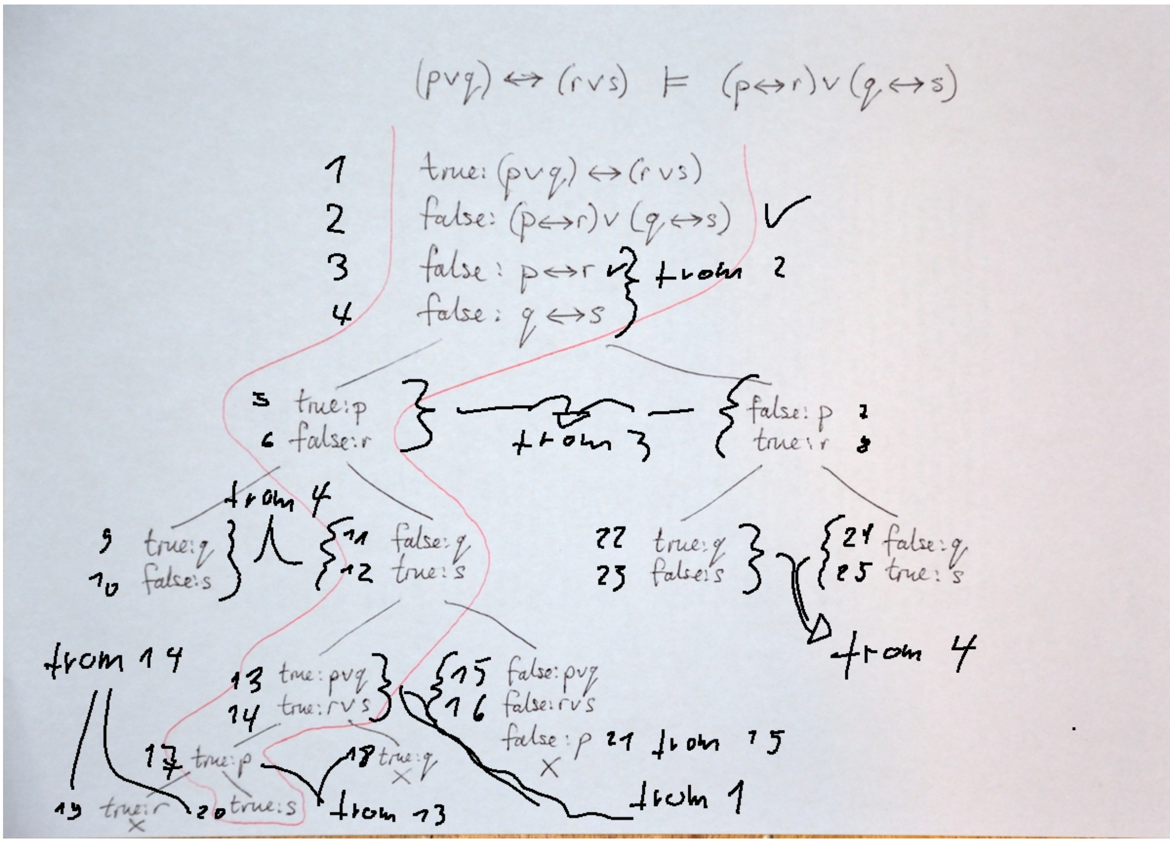


I spare you the converse sequent, which is more of the same. Twelve branches have to be explored, with three or four analysis steps in each one, to obtain the result which could have been observed in eight simple lines of the truth table test.

Tableaux, then, are not the ideal answer to every logical problem, but they certainly have their uses. Unlike natural deduction proofs, they can show sequents to be *invalid* as seen in the next example.

4 Exercise 4

Show that the sequent $((p \vee q) \leftrightarrow (r \vee s) \vdash (p \leftrightarrow r) \vee (q \leftrightarrow s)$ is invalid. From your tableau, extract an assignment of truth values to atoms which invalidates the sequent.



Once we have found an open branch, in which every formula is fully analysed, but in which there is no contradiction, we have shown the sequent to be invalid. There is no need to go on to develop the rest of the branches.

The interpretation which refutes the sequent can easily be read off the open branch: simply trace it back to the top of the tree, collecting the labelled atoms along the way. In this case p is true, q is false, r is also false and s is true. (By inserting these truth values into the the involved formulae of the sequent you can see that they are indeed a witness that show invalidity.)