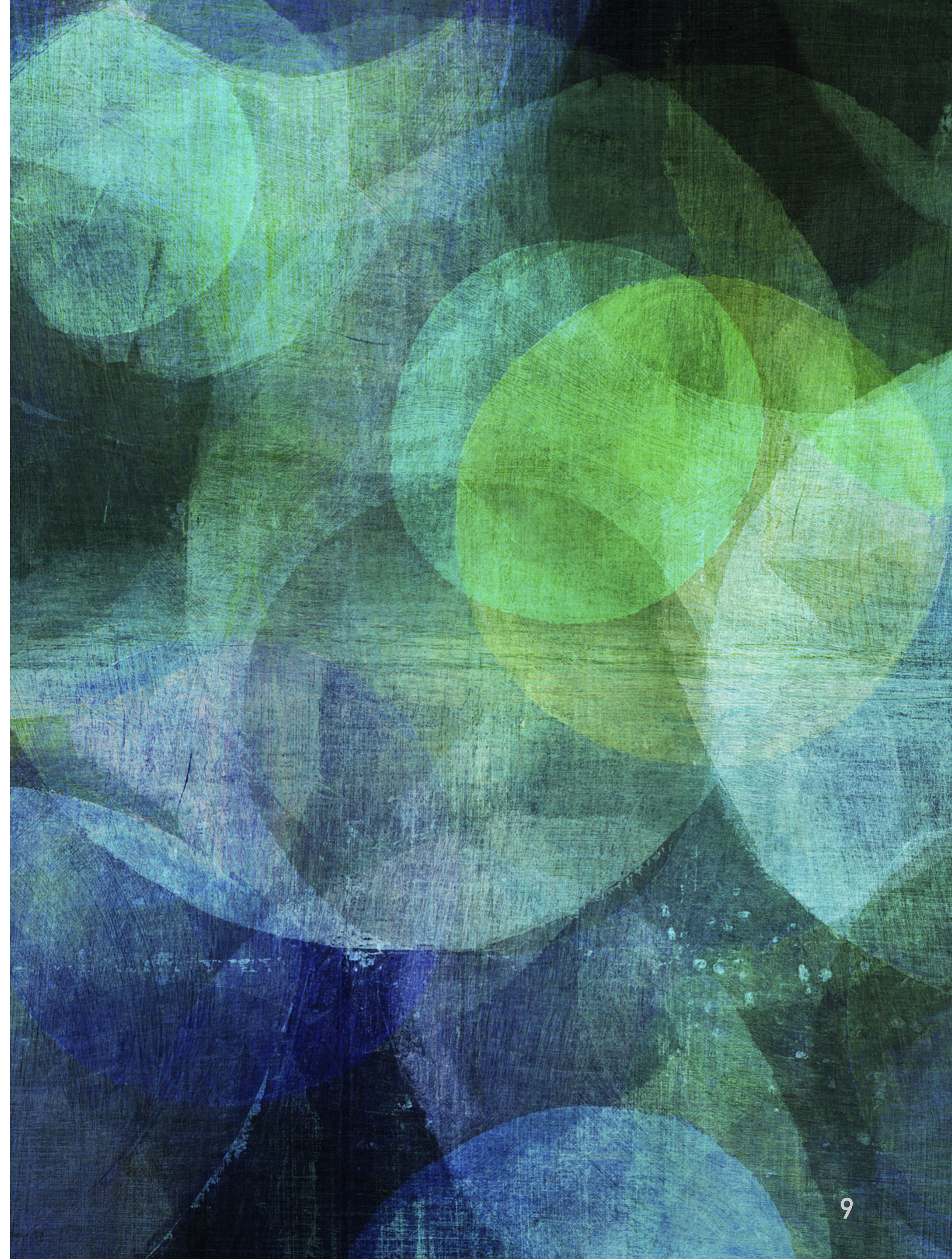


PARADOXES OF IMPLICATION



STRANGE VALID SEQUENTS

In the logic we learned, some strange sequents are valid:

➤ $p \vdash q \rightarrow p$

“I have \$300, therefore if it rains on Tuesday then I have \$300.”

➤ $\neg p \vdash p \rightarrow q$

“I do not have \$300, therefore if I have \$300 it rains on Tuesday.”

➤ $p, \neg p \vdash q$

“The cat is dead and the cat is alive, therefore I am a chicken.”

PARADOXES OF MATERIAL IMPLICATION

- All of the previous sequents are semantically valid and are syntactically provable in the logic we learned so far.
 - However, these arguments do not make much sense.
 - Especially, conclusions are *irrelevant* to premises.
- They are known as **Paradoxes of Material Implication**.
 - Note that $\neg A$ is equivalent to $A \rightarrow \perp$, which can, e.g., be verified via truth tables.
- **Relevant logic** saves us from the paradoxes of implication.

PARADOXES OF MATERIAL IMPLICATION (CONT'D)

- A possible perspective on paradoxes of implication: assumptions are not used in any *relevant* manner for deriving the conclusion.
- Consider a proof of $p, \neg p \vdash q$. RAA in line 4 seemingly suggests the assumption $\neg q$ is what caused the contradiction. But $\neg q$ is *irrelevant* to p and $\neg p$.

$$p, \neg p \vdash q$$

α_1 (1)	p	A
α_2 (2)	$\neg p$	A
α_3 (3)	$\neg q$	A
α_1, α_2 (4)	$\neg \neg q$	1,2[α_3] RAA
α_1, α_2 (5)	q	4 $\neg \neg E$

- Relevant logic allows us to prohibit this sort of *irrelevant reasoning*.

THE LOGIC OF MATH VS. THE LOGIC OF NATURAL LANGUAGE

- Note that these arguments are allowed in the logic of mathematics.
 - Classical logic, the logic you learned, is basically the logic of mathematics. Frege, Russell, and Whitehead (origins of formal logic) all tried to formalise mathematics.
- Is the logic of natural language the same as the logic of mathematics?
 - No obvious answer to this.
- Natural language is seemingly richer and contextual than the logic of mathematics, so they may be different from each other; if so, what is the logic of natural language?
 - Relevant logic sheds light on relevance aspects of natural language reasoning.

SAVING LOGIC FROM PARADOXES

- How to save logic from paradoxes of implications?
- In relevant logic, we reconsider the way assumptions are combined with each other.

THE WAYS ASSUMPTIONS ARE COMBINED

- In the proof of the following sequent, the assumptions α_1 and α_2 :
 - are put together by $\wedge I$;
 - are pulled apart by $\rightarrow I$.
- Any of $\wedge I$ and $\rightarrow I$ is no problem on its own, but the way they combine assumptions are subtly different in the following sense.

$$q \vdash p \rightarrow q$$

α_1	(1)	q	A
α_2	(2)	p	A
α_1, α_2	(3)	$p \wedge q$	1,2 $\wedge I$
α_1, α_2	(4)	q	3 $\wedge E$
α_1	(5)	$p \rightarrow q$	4[α_2] $\rightarrow I$

THE WAYS ASSUMPTIONS ARE COMBINED (CONT'D)

- In the \wedge I rule:

$$\frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \wedge B} \wedge I$$

- The two assumptions X, Y are just collected together and there is no requirement that they interact with each other to yield the conclusion $A \wedge B$.

- Yet in the \rightarrow E rule:

$$\frac{X \vdash A \rightarrow B \quad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

- The assumption Y is “applied” to X so that they yield the conclusion B while A disappears after the application. The same thing happens in \rightarrow I.
- To distinguish the two ways, we introduce a new symbol “;” (semicolon) besides “,”.

RELEVANT LOGIC RULES

- We modify the rules by introducing the **semicolon** to distinguish the different ways assumptions are combined. Here are ND rules for conjunction and implication:

Classical Logic

$$\frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \wedge B} \wedge I$$

$$\frac{X \vdash A \rightarrow B \quad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

Relevant Logic

$$\frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \wedge B} \wedge I$$

$$\frac{X \vdash A \rightarrow B \quad Y \vdash A}{X; Y \vdash B} \rightarrow E$$

$$\frac{X; A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

SAVING LOGIC FROM PARADOX

- With these new rules we can no longer go from line (4) to line (5) as the new $\rightarrow I$ rule requires “;” between the assumptions:

$$\frac{X; A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

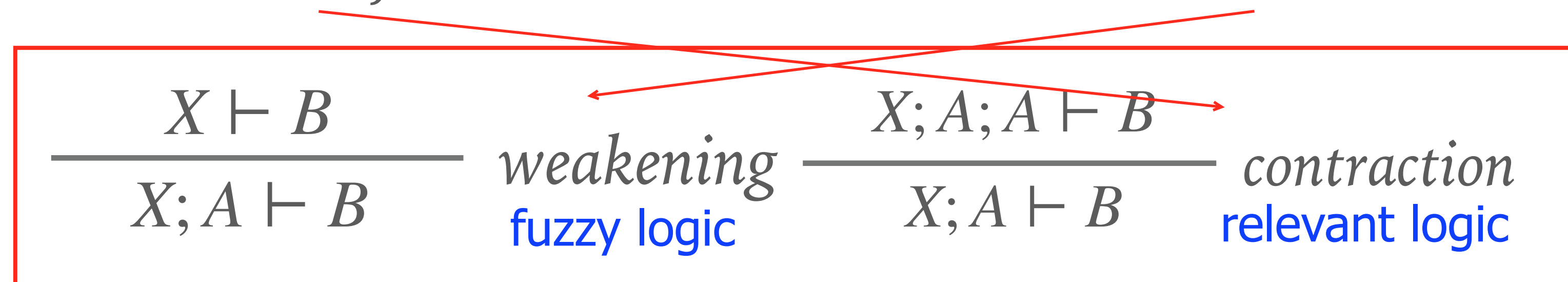
- We can still prove something meaningful via the modified rules: e.g., $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$ (as below)

$$q \vdash p \rightarrow q$$

α_1	(1)	q	A
α_2	(2)	p	A
α_1, α_2	(3)	$p \wedge q$	1,2 $\wedge I$
α_1, α_2	(4)	q	3 $\wedge E$
α_1	(5)	$p \rightarrow q$	4[α_2] $\rightarrow I$

RELEVANT AND FUZZY LOGICS AS SUBSTRUCTURAL LOGICS

- Weakening below is not valid in relevant logic: it does not allow us to add irrelevant assumptions, but it usually allows to reduce duplications, i.e., contraction below.
- (Note: Contraction is not valid in certain fuzzy logic: it does not allow us to reduce two precious resources to just one resource, but it does allow to increase resources.)



- X is not regarded as a set; it matters how many times the same formula appears in X .
- Full proof theory is given in an additional chapter of the logic notes (see 8: Reference), but in this course, you don't have to understand all of them in detail.

CONTRACTION IS ALLOWED IN RELEVANT LOGIC (BUT NOT IN FUZZY LOGIC)

$$p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$$

α_1	(1)	$p \rightarrow (q \rightarrow r)$	A
α_2	(2)	$p \wedge q$	A
α_2	(3)	p	2 $\wedge E$
α_2	(4)	q	2 $\wedge E$
$\alpha_1 ; \alpha_2$	(5)	$q \rightarrow r$	1,3 $\rightarrow E$
$\alpha_1 ; \alpha_2 ; \alpha_2$	(6)	r	4,5 $\rightarrow E$
$\alpha_1 ; \alpha_2$	(7)	r	6 contraction
α_1	(8)	$(p \wedge q) \rightarrow r$	7[α_2] $\rightarrow I$

RESOURCE SENSITIVITY IN SUBSTRUCTURAL LOGIC

- You have three dollars *and* you have three dollars, i.e., you have six dollars!
 - This is a resource-sensitive interpretation of substructural conjunction.
- Having two Tim Tams is different from having one Tim Tam.
 - This world is resource-sensitive (as well as computer science, in which substructural logic has played essential roles, esp. in semantics of computation).



≠



<https://www.arnotts.com/products/tim-tam>

- John told me that there are two possible readings of “A;B”: “A and B” (B. Meyer) or “A is compatible with B” (in the sense that “not (A implies not B)”); S. Read).

SAVING LOGIC FROM ANOTHER PARADOX

- Which step is wrong with the classical proof of the following sequent?

Relevant Logic

$$\frac{X; A \vdash \perp}{X \vdash \neg A} \neg I$$

$$\frac{X \vdash A \quad X \vdash \neg A}{X; Y \vdash \perp} \neg E$$

$$\frac{X \vdash \neg \neg A}{X \vdash A} \neg \neg E$$

$p, \neg p \vdash q$

α_1	(1)	p	A
α_2	(2)	$\neg p$	A
α_3	(3)	$\neg q$	A
α_1, α_2	(4)	$\neg \neg q$	$1, 2[\alpha_3] \text{ RAA}$
α_1, α_2	(5)	q	$4 \neg \neg E$

- We can get $p; \neg p \vdash \perp$, but to derive $\neg \neg q$, we need $p; \neg p; \neg q \vdash \perp$, which is not allowed because we cannot add irrelevant assumptions without the weakening rule.
relevant logic doesn't have the weakening rule
- This means contradictions don't entail explosion (i.e., it is not possible to derive an arbitrary formula from contradictions); it's **paraconsistent** (**contradiction-tolerant**) logic.

MODELLING RELEVANT LOGIC

- We have the following three-valued semantics for relevant logic:
 - We consider **i** to be a “confused” value between true and false. In a way, it can be considered to **be both true and false**; it’s paraconsistent.

\wedge	T	i	F	\vee	T	i	F	\rightarrow	T	i	F	A	$\neg A$
T	T	i	F	T	T	T	T	T	T	F	F	T	F
i	i	i	F	i	T	i	i	i	T	i	F	i	i
F	F	F	F	F	T	i	F	F	T	T	T	F	T

MODELLING RELEVANT LOGIC

- The semicolon and comma are interpreted as follows:

,	T	i	F	;	T	i	F
T	T	i	F	T	T	T	F
i	i	i	F	i	T	i	F
F	F	F	F	F	F	F	F

- In the three-valued relevant logic, $X \vdash A$ is **valid** iff in any assignment of values, the value of A is at least as good as the value of X. Note: T better than i better than F.
 - Concretely: the $T \vdash i$ case is invalid, and the $i \vdash F$ case is invalid; in contrast, the $i \vdash T$ case is valid; the $F \vdash i$ case is valid.
- Tautologies (i.e., valid formulae) are those to which it is impossible to assign F.

INVALID SEQUENTS

- Consider the sequent:

$$(p \wedge q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$$

- Classical logic make $(p \wedge q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ be equivalent with each other.
 - If we assign **T** to p and **i** to both q and r , it is not valid in the three-valued semantics for relevant logic because the premise of the sequent is **i**, while the conclusion is **F**.

$$\begin{array}{ccccccc} \mathbf{T} & \mathbf{i} & \mathbf{i} & & \mathbf{i} & \mathbf{i} & \\ (p \wedge q) \rightarrow r & \vdash & p \rightarrow & (q \rightarrow & r) \end{array}$$

- The converse sequent $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$ is valid.

Appendix

QUESTIONS

- Is relevant logic closer to natural language than classical logic?
 - Vacuous discharge is not allowed in relevant logic since it cannot discharge irrelevant assumptions.
- Can we do ordinary mathematics (e.g., arithmetic) with relevant logic?
- How many logics exist in the world?

INVALID SEQUENTS

➤ The following are not valid (and thus not provable because of soundness):

$$p \vdash q \rightarrow p$$

$$(p \vee q) \wedge \neg p \vdash q$$

$$\neg(p \rightarrow q) \vdash p \wedge \neg q$$

$$p \wedge q \vdash p \leftrightarrow q$$

$$(p \wedge q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$$