Intuitionistic Logic

Syntax, Semantics and Completeness

Yoshihiro Maruyama

Intuitionsitic Propositional Logic and its Basic Properties

Intuitionistic Propositional Logic (IPL)

1. **Formulae:** $\land$, $\lor$, $\to$, $\bot$ (n.b. $\neg A := A \to \bot$);
2. **Provability:** classical propositional logic (CPL) = *intuitionistic propositional logic (IPL)* + law of excluded middle (LEM);
3. **Formal system:** *natural deduction* in sequent style;
4. **Sound and complete semantics:** *Kripke semantics*.

Provability of IPL (natural deduction)

Notation: Let φ, ψ range over formulae, and Γ over finite sets of formulae.
Recall: Sequents are $\Gamma \Rightarrow \varphi$.

$$\frac{}{\Gamma, \varphi \Rightarrow \varphi}\ \text{Assumption}$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ \wedge\mathrm{I} \qquad \frac{\Gamma \Rightarrow \varphi_0 \wedge \varphi_1}{\Gamma \Rightarrow \varphi_i}\ \wedge\mathrm{E}_i$$

$$\frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_0 \vee \varphi_1}\ \vee\mathrm{I}_i \qquad \frac{\Gamma \Rightarrow \varphi \vee \psi \qquad \Gamma, \varphi \Rightarrow \theta \qquad \Gamma, \psi \Rightarrow \theta}{\Gamma \Rightarrow \theta}\ \vee\mathrm{E}$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ \to\mathrm{I} \qquad \frac{\Gamma \Rightarrow \varphi \to \psi \qquad \Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi}\ \to\mathrm{E}$$

For intuitionistic logic, add *ex falso sequitur quodlibet*:

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}\ \bot\mathrm{E}$$

For classical logic, add *reductio ad absurdum*:

$$\frac{\Gamma, \neg\varphi \Rightarrow \bot}{\Gamma \Rightarrow \varphi}\ \mathrm{RAA}$$

Examples of Sequents Provable in IPL

- We have $\vdash A \Rightarrow \neg\neg A$ in IPL:

$$\dfrac{\dfrac{}{A, \neg A \Rightarrow A} \qquad \dfrac{}{A, \neg A \Rightarrow \neg A}}{\dfrac{A, \neg A \Rightarrow \bot}{A \Rightarrow \neg\neg A}}$$

- We have $\vdash \neg\neg A \Rightarrow A$ in CPL:

$$\dfrac{\dfrac{}{\neg\neg A, \neg A \Rightarrow \neg A} \qquad \dfrac{}{\neg\neg A, \neg A \Rightarrow \neg\neg A}}{\dfrac{\neg\neg A, \neg A \Rightarrow \bot}{\neg\neg A \Rightarrow A}}$$

Remark: The last rule is impossible in IPL; in fact, $\nvdash \neg\neg A \Rightarrow A$ in IPL.

Kripke Semantics

A *Kripke structure* is a triple $\mathscr{M} = (W, \leqslant, \nu)$ of

- A nonempty set W of *possible worlds*;
- A partial order $\leqslant$ on W, called the *accessibility relation*;
- A function $\nu : \mathscr{V} \to \wp(W)$, called the *valuation*, that satisfies *monotonicity*:

$$\text{if } w \in \nu(p) \text{ and } w \leqslant w', \text{ then } w' \in \nu(p),$$

where $\mathscr{V}$ is the set of all propositional variables.

Intuition: $w \leqslant w'$ if w' is a possible future of w.

Forcing Relation

To define *validity* of formulae in IPL w.r.t. a given Kripke structure $\mathscr{M} = (W, \leqslant, \nu)$, we first define a *forcing relation* $\Vdash$ (or $\mathscr{M}, \_ \Vdash \_$) by induction on formulae:

Intuition:

- $\mathscr{M}, w \Vdash \varphi$ <u>if φ holds at w</u>;
- φ is *valid* w.r.t. $\mathscr{M}$ <u>if $\mathscr{M}, w \Vdash \varphi$ for all $w \in W$</u>.

1. $\mathscr{M}, w \Vdash p :\Leftrightarrow w \in \nu(p)$
2. $\mathscr{M}, w \nVdash \bot$
3. $\mathscr{M}, w \Vdash \varphi \wedge \psi :\Leftrightarrow \mathscr{M}, w \Vdash \varphi$ and $\mathscr{M}, w \Vdash \psi$
4. $\mathscr{M}, w \Vdash \varphi \vee \psi :\Leftrightarrow \mathscr{M}, w \Vdash \varphi$ or $\mathscr{M}, w \Vdash \psi$
5. $\mathscr{M}, w \Vdash \varphi \to \psi :\Leftrightarrow \mathscr{M}, w' \nVdash \varphi$ or $\mathscr{M}, w' \Vdash \psi$ for all $w' \geqslant w$.

Notation: We write $w \Vdash \varphi$ for $\mathscr{M}, w \Vdash \varphi$ if $\mathscr{M}$ is clear from the context.

Theories and Models for IPL

Let $\mathscr{M} = (W, \leqslant, \nu)$ be a Kripke structure, and T a theory (i.e., a set of formulae in IPL).

1. φ is *valid* w.r.t. $\mathscr{M}$, written $\mathscr{M} \vDash \varphi$, if $\mathscr{M}, w \Vdash \varphi$ for all $w \in W$;
2. If $\mathscr{M} \vDash \varphi$ for all $\varphi \in T$, then $\mathscr{M}$ is called a *model* of T and written $\mathscr{M} \vDash T$;
3. φ is *valid* w.r.t. T, written $T \vDash \varphi$, if $\mathscr{M} \vDash \varphi$ for all models $\mathscr{M}$ of T.

Notation: We write $T \vdash \Gamma \Rightarrow \varphi$ if $\Gamma \Rightarrow \varphi$ is derivable in IPL with formulae in T potentially used as <u>assumptions</u> (in addition to those in Γ).

An Example of Forcing Relation

In the Kripke structure $\mathscr{M} = (\{0,1,2\}, \{(0,1), (0,2)\}, p \mapsto \{1\})$, or diagramatically

$$\begin{tikzcd} 1 & & 2 \\ & 0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

LEM is *invalid* because at 0 we do not have $p \vee \neg p$ (i.e., $\mathscr{M}, 0 \nVdash p \vee \neg p$).

This motivates us to use Kripke structures as semantics of IPL (not of CPL).

—> As we shall see, Kripke semantics is *sound* and *complete* for IPL.

Monotonicity

Proposition (monotonicity). If $w \Vdash \varphi$ and $w \leqslant w'$ then $w' \Vdash \varphi$.

Proof. We prove it by induction on φ. Atomic case follows from definition, and the cases of $\wedge$, $\vee$ follow from the induction hypothesis.

It remains to consider the case of implication. Assume $w \Vdash \varphi \to \psi$, $w' \geqslant w$ and $w'' \geqslant w'$. It then suffices to assume $w'' \Vdash \varphi$ and show $w'' \Vdash \psi$. But because $w \Vdash \varphi \to \psi$ and $w'' \geqslant w' \geqslant w$, it follows that $w'' \Vdash \varphi$ implies $w'' \Vdash \psi$.

$\square$

Soundness

Theorem (soundness). If $\vdash \Gamma \Rightarrow \varphi$ then $\models \bigwedge \Gamma \to \varphi$.

Proof. By induction on proofs by using monotonicity (on implication $\to$).

$\square$

Completeness Theorem

Overview on the Proof of Completeness

Theorem (completeness of IPL). If $T \vDash \varphi$ then $T \vdash \varphi$.

Plan of the Proof. We shall prove the theorem *by contraposition*, i.e., show

$$T \nvdash \varphi \text{ implies } T \nvDash \varphi.$$

Note: $T \nvDash \varphi \Leftrightarrow \exists \mathcal{M} \vDash T\,.\, \mathcal{M} \nvDash \varphi$

$$\Leftrightarrow \exists \mathcal{M} \vDash T, \exists w \in W\,.\, \mathcal{M}, w \nvDash \varphi,$$

where W is the set of all possible worlds of $\mathcal{M}$. We shall then proceed as:

1. **Lindenbaum's lemma:** construct a *prime* theory $T' \supseteq T$ such that $T' \nvdash \varphi$
2. **Truth lemma:** $\mathcal{M}(T'), \langle\rangle \Vdash \varphi \Leftrightarrow T' \vdash \varphi$ for all prime theories T',

where $\mathcal{M} := \mathcal{M}(T')$ is the *canonical* model of T, and $w := \langle\rangle$ is a possible world of $\mathcal{M}$.

Prime Theory

Definition (prime theories). A theory T of IPL is *prime* if

1. T is *consistent*, i.e., $T \nvdash \bot$
2. T is closed under provability, i.e., $T \vdash \varphi$ implies $\varphi \in T$
3. If $\varphi \vee \psi \in T$, then $\varphi \in T$ or $\psi \in T$.

Lindenbaum's Lemma (1/6)

Lemma (Lindenbaum). If $T \nvdash \varphi$, then there is a prime theory $T' \supseteq T$ such that $T' \nvdash \varphi$.

<u>**Proof idea**</u>. For each $T \vdash \psi \vee \chi$ such that $\psi \notin T$ and $\chi \notin T$, update T by

$$T \leftarrow T \cup \{\psi\} \text{ or } T \leftarrow T \cup \{\chi\}$$

in such a way that T keeps $T \nvdash \varphi$. Then, $T \mapsto T_1 \mapsto T_2 \mapsto \ldots \mapsto T'$ will get the desired T':

- $T' \nvdash \varphi$ by construction
- $T' \nvdash \bot$ because $T' \nvdash \varphi$ (n.b., $T' \vdash \bot$ implies $T' \vdash \varphi$)
- T' is closed under provability: If $T_n \vdash \theta$ and $\theta \notin T_n$, then $T_n \vdash \theta \vee \theta$ so that $\theta \in T_{n+1}$

$$\Rightarrow \text{If } T' \vdash \theta, \text{ then } \theta \in T'$$

- $\psi \vee \chi \in T'$ implies $\psi \in T'$ or $\chi \in T'$ because $\psi \vee \chi \in T_n$ implies $T_n \vdash \psi \vee \chi$ so that

$$\psi \in T_{n+1} \text{ or } \chi \in T_{n+1}.$$

Lindenbaum's Lemma (2/6)

<u>Proof</u>. Let $(\psi_k \vee \chi_k)_{k\in\mathbb{N}}$ be an enumeration of all disjunctions. We define a theory T_n for each $n \in \mathbb{N}$ such that $T = T_0 \subseteq T_1 \subseteq T_2 \ldots$ by induction on n. Our plan is then to define $T' := \bigcup_{n\in\mathbb{N}} T_n$.

1. $T_0 := T$

2. $T_{n+1} := \begin{cases} T_n \cup \{\psi_{i(n)}\} & \text{if } i(n) < \infty \text{ and } T_n \cup \{\psi_{i(n)}\} \nvdash \varphi \\ T_n \cup \{\chi_{i(n)}\} & \text{if } i(n) < \infty \text{ and } T_n \cup \{\psi_{i(n)}\} \vdash \varphi \\ T_n & \text{otherwise} \end{cases}$

where

$$i(n) := \inf\{k \in \mathbb{N} \mid T_n \vdash \psi_k \vee \chi_k, \psi_k \notin T_n, \chi_k \notin T_n\}.$$

Remark: $T_n \vdash \psi_{i(n)} \vee \chi_{i(n)}$, $\psi_{i(n)} \notin T_n$ and $\chi_{i(n)} \notin T_n$, so $T_{n+1} \nvdash \varphi$ (n.b., in the 2nd case of T_{n+1}, $T_n \vdash \chi_{i(n)}$).

Lindenbaum's Lemma (3/6)

Claim. $T' \nvdash \varphi$.

Proof of the claim. If $T' \vdash \varphi$, then there is some $T'_{\text{fin}} \subseteq T'$ such that T'_{fin} is finite and $T'_{\text{fin}} \vdash \varphi$. Taking any $n \in \mathbb{N}$ such that $T'_{\text{fin}} \subseteq T_n$, we have $T_n \vdash \varphi$. Hence, it suffices to prove $T_n \nvdash \varphi$ for all $n \in \mathbb{N}$.

We prove it by induction on $n \in \mathbb{N}$:

Lindenbaum's Lemma (4/6)

Base case: Since $T_0 := T$ and $T \nvdash \varphi$, we have $T_0 \nvdash \varphi$.

Inductive step: By contraposition, i.e., we assume $T_{n+1} \vdash \varphi$ and show $T_n \vdash \varphi$.

- Case 1 - $T_{n+1} = T_n \cup \{\psi_{i(n)}\}$. This case is impossible since $T_{n+1} = T_n \cup \{\psi_{i(n)}\}$ if and only if $T_n \cup \{\psi_{i(n)}\} \nvdash \varphi$.
- Case 2 - $T_{n+1} = T_n \cup \{\chi_{i(n)}\}$. We have $T_n \cup \{\psi_{i(n)}\} \vdash \varphi$ by the definition of T_{n+1}, and $T_n \vdash \psi_{i(n)} \vee \chi_{i(n)}$ by the definition of $i(n)$. Thus, we have $T_n \vdash \varphi$ by the elimination rule on disjunction $\vee$.
- Case 3 - $T_{n+1} = T_n$. This case is immediate.

Lindenbaum's Lemma (5/6)

Claim. T' is consistent.

Proof of the claim. By $T' \nvdash \varphi$, we have $T' \nvdash \bot$ i.e., T' is consistent.

Claim. If $T' \vdash \psi \vee \chi$, then $\psi \in T'$ or $\chi \in T'$.

Proof of the claim. Assume $T' \vdash \psi \vee \chi$ and $\psi \vee \chi = \psi_i \vee \chi_i$.

We have $T_n \vdash \psi_i \vee \chi_i$ for some $n \in \mathbb{N}$.

Then, we have $\psi_i \in T_{k+1}$ or $\chi_i \in T_{k+1}$ for some $k \in \mathbb{N}$.

Hence, $\psi_i \in T'$ or $\chi_i \in T'$.

Lindenbaum's Lemma (6/6)

Claim. T' is closed under provability.

Proof of the claim. Assume $T' \vdash \varphi$. We have $T' \vdash \varphi \vee \varphi$, thus $\varphi \in T'$.

This completes the proof of Lindenbaum's lemma.

$\square$

Canonical Model

Definition (canonical model). Let T be a prime theory, and $(\langle \psi_n, \chi_n \rangle)_{n \in \mathbb{N}}$ an enumeration of all pairs of formulae. We define the canonical model $\mathscr{M}(T) := \langle W, \leqslant, \nu \rangle$ by:

1. $W := \mathbb{N}^{<\omega}$, i.e., the set of all finite sequences of natural numbers
2. $w \leqslant w' :\Leftrightarrow w$ is an initial segment of w'
3. $\nu(p) := \{w \in W \mid p \in T_w\}$, where T_w is defined by

① $T_{\langle\rangle} := T$

② $T_{w^\frown\langle n \rangle} := \begin{cases} (T_w \cup \{\psi_n\})' & \text{if } T_w \cup \{\psi_n\} \nvdash \chi_n \\ T_w & \text{otherwise.} \end{cases}$

Truth Lemma (1/3)

Remark: T_w is prime by construction since so is T.

Lemma (truth lemma). $\mathscr{M}(T), w \Vdash \varphi \Leftrightarrow T_w \vdash \varphi$ for all prime theories T.

Proof. By induction on φ. We focus on the cases of disjunction and implication.

Case of disjunction.

$$\begin{aligned}\mathscr{M}(T), w \Vdash \varphi \vee \psi &\Leftrightarrow \mathscr{M}(T), w \Vdash \varphi \text{ or } \mathscr{M}(T), w \Vdash \psi \\ &\Leftrightarrow T_w \vdash \varphi \text{ or } T_w \vdash \psi \text{ by the induction hypothesis} \\ &\Leftrightarrow T_w \vdash \varphi \vee \psi \text{ by the primeness of } T_w.\end{aligned}$$

Truth Lemma (2/3)

Case of implication.

We first show that $T_w \nvdash \psi \to \chi$ implies $\mathscr{M}(T), w \nVdash \psi \to \chi$.

Assume $T_w \nvdash \psi \to \chi$. Then, $T_w \cup \{\psi\} \nvdash \chi$.

Pick any $n \in \mathbb{N}$ such that $\langle \psi, \chi \rangle = \langle \psi_n, \chi_n \rangle$.

Now, we have $T_{w^\frown\langle n \rangle} = (T_w \cup \{\psi\})'$, $T_{w^\frown\langle n \rangle} \vdash \psi$ and $T_{w^\frown\langle n \rangle} \nvdash \chi$.

By the induction hypothesis, $\mathscr{M}(T), w^\frown\langle n \rangle \Vdash \psi$ and $\mathscr{M}(T), w^\frown\langle n \rangle \nVdash \chi$.

Hence, we have $\mathscr{M}(T), w \nVdash \psi \to \chi$ because $w \leqslant w^\frown\langle n \rangle$.

Truth Lemma (3/3)

We next show the converse: $T_w \vdash \psi \to \chi$ implies $\mathscr{M}(T), w \Vdash \psi \to \chi$. Assume $T_w \vdash \psi \to \chi$ and $w \leqslant w'$; we have to show $\mathscr{M}(T), w' \nVdash \psi$ or $\mathscr{M}(T), w' \Vdash \chi$.

Because $T_w \subseteq T_{w'}$, we have $T_{w'} \nvdash \psi$ or $T_{w'} \vdash \chi$.

By the induction hypothesis, this means that $\mathscr{M}(T), w' \nVdash \psi$ or $\mathscr{M}(T), w' \Vdash \chi$.

$\square$

Completeness Theorem

Theorem (completeness of IPL). If $T \vDash \varphi$ then $T \vdash \varphi$.

__**Proof**__. We prove the theorem by contraposition.

Assume $T \nvdash \varphi$; then, we can take a prime theory $T' \supseteq T$ such that $T' \nvdash \varphi$.

Let $\mathscr{M}(T')$ be the canonical model for T'. Note that $T'_{\langle\rangle} = T'$ and $T'_{\langle\rangle} \nvdash \varphi$.

By the truth lemma, $\mathscr{M}(T'), \langle\rangle \Vdash \psi$ for all $\psi \in T$ (thus, $\mathscr{M}(T')$ is a model of T by monotonicity) and $\mathscr{M}(T'), \langle\rangle \nVdash \varphi$.

Hence, we have shown $T \nvDash \varphi$.

$\square$

Consequences of Completeness Theorem

Double Negation Elimination

Proposition (DNE is not derivable in IPL). $\nvdash \neg\neg p \to p$.

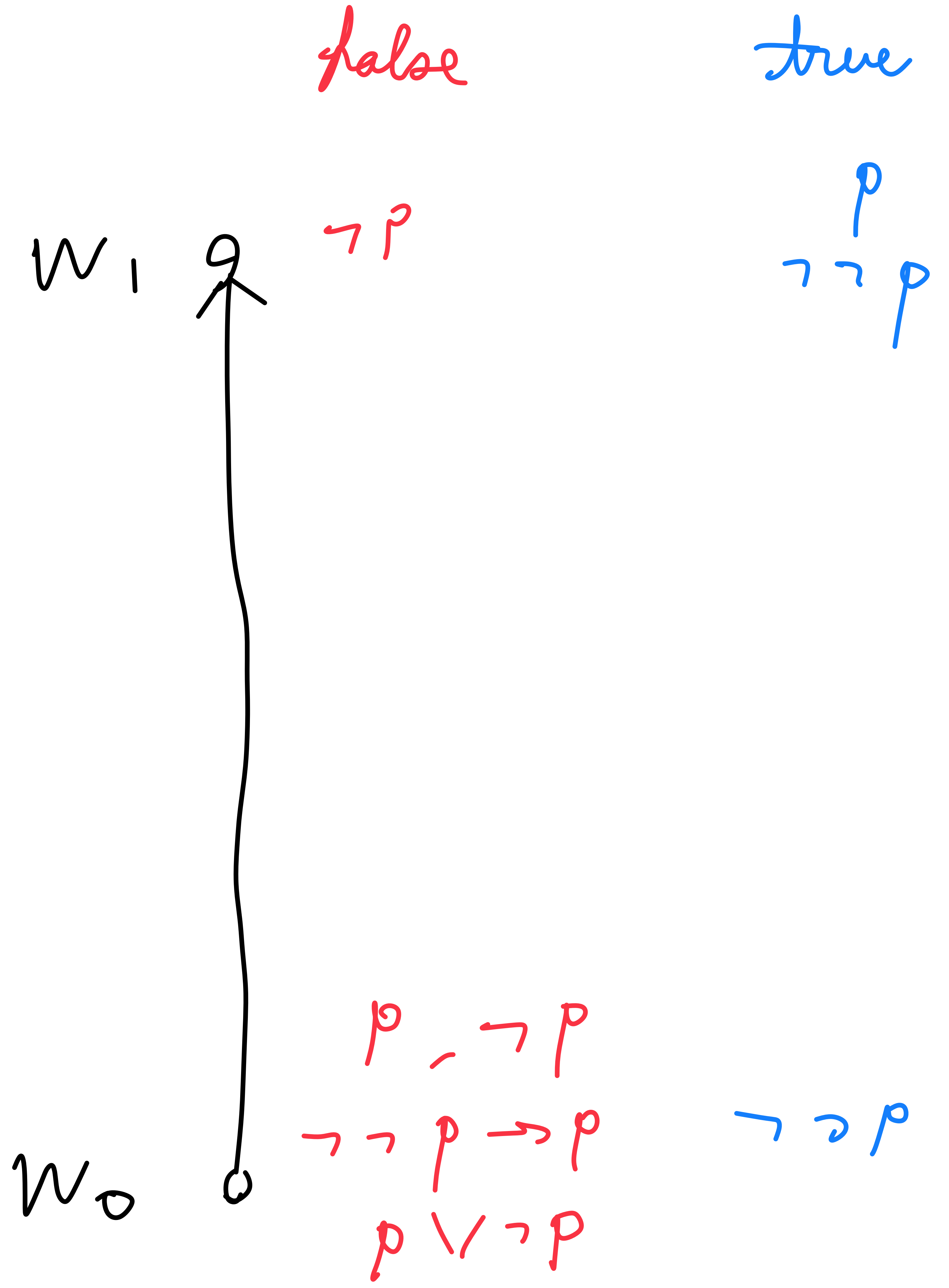
<u>**Proof**</u>. By soundness, it suffices to build a counter model $\mathscr{M} := \langle W, \leqslant, \nu \rangle$ by

- $W := \{w_0, w_1\}$
- $w_0 \leqslant w_1$
- $\nu(p) = \{w_1\}$.

Then, $\neg\neg p \to p$ is not valid at w_0 w.r.t. $\mathscr{M}$.

$\square$

Fig. A countermodel for LEM and DNE



Law of Excluded Middle

Proposition (LEM is not derivable in IPL). $\nvdash \neg p \vee p$.

<u>Proof</u>. Let $\mathscr{M}$ be the counter model for double negation elimination constructed in the last slide. Then, $\neg p \vee p$ is not valid at w_0 w.r.t. $\mathscr{M}$. Hence, again, by soundness, we conclude $\nvdash \neg p \vee p$.

$\square$

De Morgan's Laws

Proposition (De Morgan's laws). We have:

1. $\vdash \neg\varphi \wedge \neg\psi \leftrightarrow \neg(\varphi \vee \psi)$
2. $\vdash \neg\varphi \vee \neg\psi \rightarrow \neg(\varphi \wedge \psi)$
3. $\nvdash \neg(p \wedge q) \rightarrow \neg p \vee \neg q$.

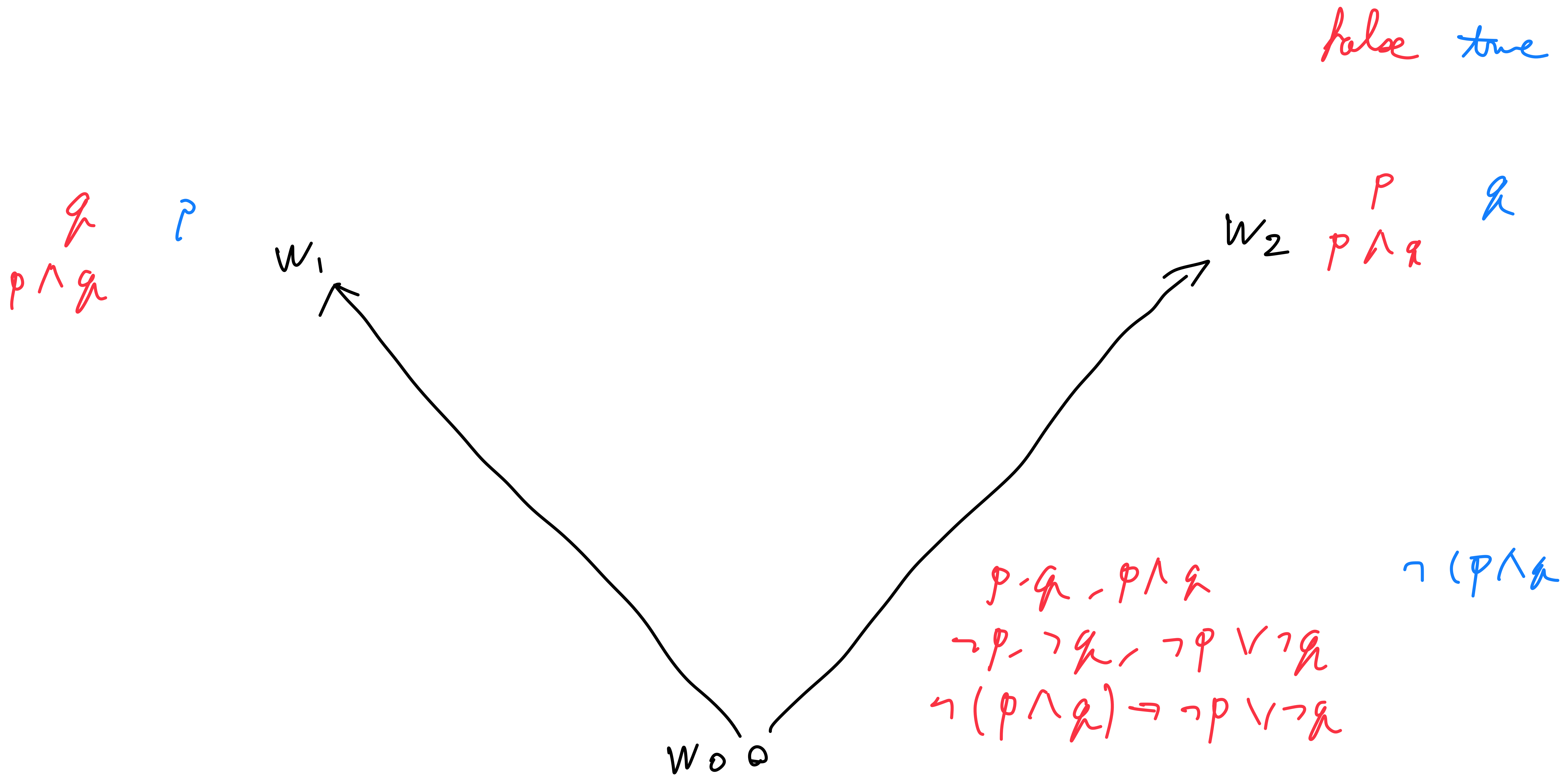
Proof. We focus on the clause 3. We define a counter model $\mathscr{M} := \langle W, \leqslant, \nu \rangle$ by

- $W := \{w_0, w_1, w_2\}$
- $w_0 \leqslant w_1$ and $w_0 \leqslant w_2$
- $\nu(p) := \{w_1\}$ and $\nu(q) := \{w_2\}$.

Then, $(p \wedge q) \rightarrow (\neg p \vee \neg q)$ is not valid at w_0 w.r.t. $\mathscr{M}$.

$\square$

Fig. A countermodel for de Morgan's rule.



Disjunction Property (1/2)

Theorem (disjunction property). If $\vdash \varphi_0 \vee \varphi_1$, then $\vdash \varphi_0$ or $\vdash \varphi_1$.

Proof. We prove the contraposition. Assume $\nvdash \varphi_0$ and $\nvdash \varphi_1$.

By completeness, we get counter models
$\mathscr{M}_0 := \langle W_0, \leqslant_0, \nu_0 \rangle, \mathscr{M}_1 := \langle W_1, \leqslant_1, \nu_1 \rangle$ for φ_0, φ_1.

We then have $\mathscr{M}_0, w_0 \nVdash \varphi_0$ and $\mathscr{M}_1, w_1 \nVdash \varphi_1$ for some $w_0 \in W_0, w_1 \in W_1$.

Disjunction Property (2/2)

We define a Kripke model $\mathscr{M} := \langle W, \leqslant, \nu \rangle$ out of $\mathscr{M}_0$ and $\mathscr{M}_1$ as follows:

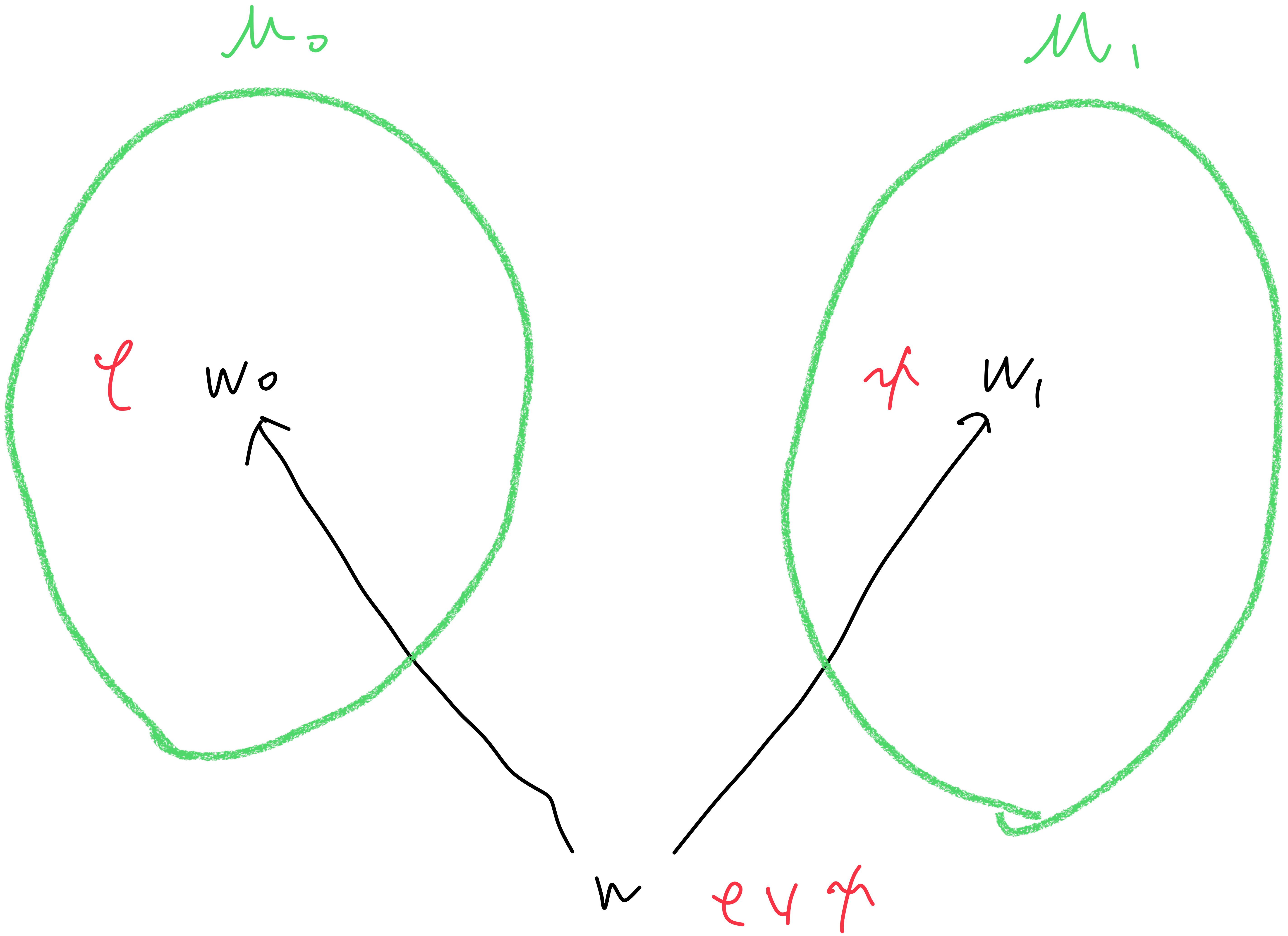
1. $W := \{w\} \uplus W_0 \uplus W_1$ where $\uplus$ stands for disjoint union;
2. $\leqslant$ is the minimal partial order containing both $\leqslant_0$ and $\leqslant_1$ such that $w \leqslant w_0$ and $w \leqslant w_1$ (more formally, $\leqslant$ is the reflexive transitive closure of $\{(w, w_0), (w, w_1)\} \cup \leqslant_0 \cup \leqslant_1$);
3. $\nu(p) := \nu_0(p) \cup \nu_1(p)$.

By construction, we have $\mathscr{M}, w_0 \nVdash \varphi_0$ and $\mathscr{M}, w_1 \nVdash \varphi_1$.

Hence, we get $\mathscr{M}, w \nVdash \varphi_0 \vee \varphi_1$ by monotonicity.

$\square$

Fig. Construction of $\mathcal{M}$.



Rasiowa–Harrop Formulae (appendix)

Definition (Rasiowa–Harrop formulae). *Rasiowa–Harrop formulae* are defined by the following induction:

1. Atomic formulae are Rasiowa–Harrop;
2. $\neg\varphi$ is Rasiowa–Harrop for every formula φ;
3. $\varphi \to \psi$ is Rasiowa–Harrop if and only if so is ψ;
4. $\varphi \wedge \psi$ is Rasiowa–Harrop if and only if so are both φ and ψ;
5. (Intuitionistic predicate logic) $\forall x \,.\, \varphi$ is Rassiowa–Harrop if and only if so is φ.

Rasiowa–Harrop Properties (appendix)

Theorem (Rasiowa-Harrop disjunction property). Let T be a set of Rasiowa–Harrop formulae. If $T \vdash \varphi \vee \psi$, then $T \vdash \varphi$ or $T \vdash \psi$.

Theorem (Rasiowa-Harrop existential property). Let T be a set of Rasiowa–Harrop formulae. If $T \vdash \exists x \,.\, \varphi$, then $T \vdash \varphi[x := t]$ for some term t.