

Logic (PHIL 2080, COMP 2620, COMP 6262)

Chapter: First-Order Logic

— Introduction and Natural Deduction

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Introduction

How to extend Propositional Logic?

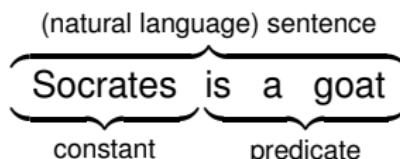
Logic is about making statements:

In first-order logic, we:

- can represent **individual objects** (people, goats, footballs, etc.)
- and express **properties** and **relationships** between objects.

In our example,

- the “**object**” *Socrates* can be represented by a **constant**,
 - the “**property**” *is a Goat* can be represented by a **predicate**.
- ⇒ For example, *isGoat(Socrates)*
- ⇒ In propositional logic, we had to use *SocratesIsGoat*, which is missing some information, since it does not “relate” to another proposition involving Socrates, like *SocratesKicksGoat*.



Predicate Logic

Terminology And Conventions: Terminology

Term: Anything that represents an **object**, i.e.,

- a **constant** (representing a fixed object, like the person Socrates)
- a **variable** (representing a non-specified object)
- a **function** application (whose general form is $f(t_1, \dots, t_n)$; e.g., $f(a, b), g(c, x, y)$)

Predicates: Express **properties** or **relations** of/between **terms**:

- Takes as **input** (or “argument”) a **sequence of terms**.
 - The sequence **length** depends on the predicate, e.g., *isGoat* is unary, *kicks* is binary, etc. (some might even be nullary!)
 - This length is called **arity** and can be given as a subscript, e.g., *isGoat*₁, *kicks*₂, but we don’t since it’s clear from context.
- **Maps** to a **truth value**, e.g., *isGoat(Socrates)* might be false, but *isGoat(Jimmy)* might be true. (Formal semantics is given later.)

Terminology And Conventions: Conventions (cont'd)

- Capital letters are **predicate symbols**:
 $F, G, H, \dots, P, Q, R, L, \dots$
- Lower-case letters represent **terms**:
 - a, b, c are (usually) used for **constants**, but we also use them for **free variables** (as they behave in the same way).
 - f, g, h are used for **functions**.
 - v and x, y are used for **variables**.
 - t is used for **terms** (i.e., any of the above).
- For the sake of simplicity, we **do not use parentheses**, e.g., $F(a), G(b)$, and $R(a, b)$ become Fa, Gb , and Rab , respectively.
- Now it's clear that the **arity** is clear from the context! E.g.,
 - Fa represents a predicate F with arity 1 (with term a), and
 - Rab represents a predicate R with arity 2 (with terms a and b).

First-Order Formulae: Possible Quantifiers

We want to “**quantify**” the objects we talk about.

- For every object x , such that Vx holds, Lx holds.

- More formally:* $\underbrace{ALL(x : Vx)}_{\text{quantifier!}} Lx$

- Even more formally:* $\underbrace{\forall}_{\text{quantity indicator}} \underbrace{(x : \underbrace{Vx}_{\text{sort indicator}})}_{\text{variable}} Lx$

What quantifiers *do* exist? (In our predicate logic!)

- Just two!
- $ALL(x : A)B$, i.e., $\forall(x : A)B$
- $SOME(x : A)B$, i.e., $\exists(x : A)B$

“SOME” means “**at least one**”, so “ \exists ” is also called “**exists**”

“ALL”, i.e., \forall , is called the “**universal**” quantifier

First-Order Formulae: Example (from before)

Propositional logic (not working):

- All logicians are rational p
- Some philosophers are not rational $\neg q$
- Thus, not all philosophers are logicians $\neg r$

Predicate logic (works!):

- All logicians are rational $\forall(x : Lx)Rx$
- Some philosophers are not rational $\exists(x : Px)\neg Rx$
- Thus, not all philosophers are logicians $\neg\forall(x : Px)Lx$

How to prove " $\forall(x : Lx)Rx, \exists(x : Px)\neg Rx \vdash \neg\forall(x : Px)Lx$ "?

- Natural Deduction
- Semantic Tableau

From Restricted Quantifiers to Unrestricted Quantifiers: Unrestricted

So far, we were only considering **restricted quantifiers**:

- $\exists(x : Px) \neg Rx$ (Some philosophers are not rational)
- $\forall(x : Lx) Rx$ (All logicians are rational)

In the following we mainly use **unrestricted quantifiers** \forall and \exists . The relationships between them are as follows:

- Existential quantified formulae become conjunctions:
E.g., $\exists(x : Gx) Hx$ (some goats are hairy) becomes $\exists x Gx \wedge Hx$
- Universally quantified formulae become implications:
E.g., $\forall(x : Gx) Hx$ (all goats are hairy) becomes $\forall x Gx \rightarrow Hx$

Natural Deduction

Introduction

- Instead of re-doing all our previous rules, we will just provide **additional ones!**
 - Two new rules for \forall (introduction and elimination)
 - Two new rules for \exists (introduction and elimination)
- We still perform **natural deduction** for *propositional logic* in intermediate steps.

Substitutions: Introduction

Our Natural Deduction rules will exploit **substitutions**.

Definition:

- Let A be a formula and t_1 and t_2 be terms.
- $A_{t_2}^{t_1}$ is the result of substituting each free (*unbound*) t_2 in A by t_1 .
- Any mnemonic? How do I remember what gets substituted by what?
 - Gravity falls!*
 - $A_{t_2}^{t_1}$ is the result of A after the t_1 “fell down” crushing t_2 .

Substitutions: Examples (and Conventions)

- Let $A = \exists x(Px \rightarrow Rx)$. Is $A_x^y = \exists y(Py \rightarrow Ry)$?
 - No! Recall that **x is required to be free/unbound in A!**
 - Since there are no free variables, so $A_x^y = A$ here.
- Let $A = Fx \wedge \exists x(Fx \wedge Gx)$. What's A_x^y now?
 - It's $Fy \wedge \exists x(Fx \wedge Gx)$!
 - Because we **only substitute free/unbound variables!**

Universal Quantifiers

Universal Elimination: Introduction

- Let's assume we want to say that the age of all humans is smaller than 130: $\forall x \text{age}(x) < 130$
- If we had one constant for each individual (person), we could conclude: $\text{age}(a) < 130 \wedge \text{age}(b) < 130 \wedge \text{age}(c) < 130 \wedge \dots$ (Though that's clearly not practical! And maybe not even possible if we reason about infinitely many objects like numbers.)
- So we could also conclude $\text{age}(x) < 130$ for *any* x !
We thus use a (free) *variable* in our rule!
- So, what will the **Universal-Elimination** rule look like?

$$\frac{\forall x Fx}{Fv} \forall E$$

more general:

$$\frac{\forall x A}{A_x^t} \forall E$$

free/unbound x

We do however need a side condition here to make sure our newly introduced term doesn't cause trouble.

Universal Elimination: Side Condition

Assume we had no side condition:

$$\frac{\forall x A}{A_x^t} \forall E$$

in sequent notation:

$$\frac{X \vdash \forall x A}{X \vdash A_x^t} \forall E$$

Let's consider this sequent: $\forall x \exists y(y > x) \vdash \exists y(y > y)$

- Should that be valid? No! No number is larger than itself!
- But we can prove it! (If there's no side condition!)

$$\begin{array}{lll} \alpha_1 & (1) & \forall x \exists y(y > x) \quad A \\ \alpha_1 & (2) & \exists y(y > y) \quad 1 \forall E \end{array}$$

$$\frac{\forall x \underbrace{\exists y (y > x)}_{A_x^t \equiv A_x^y} \quad A}{\exists y (y > y)} \forall E$$

So what's missing?

- The “instantiation of x ” (the new variable name) must be free!
(We don't want it to get captured by another quantifier!)
- This is different from what we demanded for substitutions.

Universal Elimination: The 1-step Rule

So, in conclusion:

Universal Elimination Rule:

$$\frac{X \vdash \forall x A}{X \vdash A_x^t} \forall E \quad \text{only if } t \text{ is not bound in } A_x^t!$$

Universal Introduction: Introduction

- For the *introduction* of the universal quantifier, we would like to have, conceptually, a rule like the following:

$$\frac{Fa \quad Fb \quad Fc \quad \dots}{\forall x Fx} \forall I$$

But that's again infeasible.

How about: $\frac{Fa}{\forall x Fx} \forall I$? (as above, a is a constant)

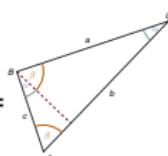
That rule is wrong! Just because Aristotle is a footballer, doesn't mean that everybody is!

But it might work for “typical objects”... (a variable)

Universal Introduction: Typical Objects

- What's a typical object?
- Remember the “undergraduate school” when you have to prove some property of *all* triangles.

- Step 1: Let $ABC = \triangle ABC$ be a triangle.
- Step 2: “some fancy proof”
- Step 3: Thus, $\triangle ABC$ has property P . Thus P holds for all triangles!



- Why is that correct? Since we did not make any assumptions for $\triangle ABC$ other than it being a triangle!
- assume the triangle $\triangle ABC$ is an universal concept, rather than a specific object

Universal Introduction: The 1-step Rule

- So, we need an “object without any assumption” to generalize its property (formula) to the general case.
- But how to express this “no assumptions”?

- $$\frac{Fv}{\forall x \ Fx} \forall I$$
 more general: $\frac{A}{\forall x \ A^x_v} \forall I$ with side condition:

provided the variable v does not occur in any assumption that A depends upon.

- Universal Introduction Rule:** (in sequent notation)

$$\frac{X \vdash A}{X \vdash \forall x \ A^x_v} \forall I \quad \text{only if } v \text{ does not occur in } X!$$

More on Syntax

- Note that the Logic Notes refer to constants as “names”.

Example: Example

$$\forall x Fx, \forall x Gx \vdash \forall x (Fx \wedge Gx)$$

| | | | |
|----------------------|-----|----------------------------|----------------|
| α_1 | (1) | $\forall x Fx$ | A |
| α_2 | (2) | $\forall x Gx$ | A |
| α_1 | (3) | Fa | 1 $\forall E$ |
| α_2 | (4) | Ga | 2 $\forall E$ |
| α_1, α_2 | (5) | $Fa \wedge Ga$ | 3,4 $\wedge I$ |
| α_1, α_2 | (6) | $\forall x (Fx \wedge Gx)$ | 5 $\forall I$ |

$$\frac{X \vdash A}{X \vdash \forall x A^x_v} \forall I$$

Only if v does
not occur in X !

$$\frac{X \vdash \forall x A}{X \vdash A_x^t} \forall E$$

only if t is not
bounded in $A_x \wedge t$

Example: Example

$$\forall x Fx, \forall x Gx \vdash \forall x (Fx \wedge Gx)$$

| | | | |
|----------------------|-----|----------------------------|----------------|
| α_1 | (1) | $\forall x Fx$ | A |
| α_2 | (2) | $\forall x Gx$ | A |
| α_1 | (3) | Fa | 1 $\forall E$ |
| α_2 | (4) | Ga | 2 $\forall E$ |
| α_1, α_2 | (5) | $Fa \wedge Ga$ | 3,4 $\wedge I$ |
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$$\frac{X \vdash A}{X \vdash \forall x A^x_v} \forall I$$

Only if v does
not occur in X !

$$\frac{X \vdash \forall x A}{X \vdash A_x^t} \forall E$$

only if t is not
bounded in $A \wedge x^t$

Did we adhere all side conditions? Yes!

- $X \vdash A$ of the $\forall I$ rule corresponds to line 5, which is $\alpha_1, \alpha_2 \vdash Fa \wedge Ga$,
- variable v corresponds to a , and
- although a (of course!) occurs in $Fa \wedge Ga$, it is not in $X = \{\alpha_1, \alpha_2\} = \{\forall x Fx, \forall x Gx\}$, so all good!

Existential Quantifiers

Existential Introduction: Introduction

- Recall that you can “imagine” the universal quantifier \forall like:
 $age(a) < 130 \wedge age(b) < 130 \wedge age(c) < 130 \wedge \dots$
- The existential quantifier \exists can similarly interpreted as:
 $age(a) > 100 \vee age(b) > 100 \vee age(c) > 100 \vee \dots$

Existential Introduction: The 1-step Rule

- **Existential Introduction Rule:**

$$\frac{Fv}{\exists x \ Fx} \exists I \quad \text{more general: } \frac{A_x^t}{\exists x \ A} \exists I \quad \text{in sequent notation: } \frac{X \vdash A_x^t}{X \vdash \exists x \ A} \exists I$$

- Only if x is not bound in A . (I.e., you removed the new quantifier it would be free!)
- Just make sure the new variable name doesn't get captured by another quantifier.

Existential Elimination: Introduction

- We want to *eliminate* the existential quantifier. So can we just use

the following rule?
$$\frac{\exists x Fx}{Fv} \exists E ? \quad \text{Recall: } \frac{\forall x Fx}{Fv} \forall E !$$

- So, no! Because we don't know *which* object has that property!

Existential Elimination: Introduction, cont'd

- The *idea* behind the rule is the following:
specific case

 $[Fy]$

in general

:

 $\exists x Fx$ B

$$\frac{}{B} \exists E \text{ for a “typical” } y.$$

- The idea is similar to disjunction elimination: In $A \vee B$, we don't know whether A or B is true, so we assume both and show that either way the derivation can be done.
- Here, we show it for “some instance” that does not pose further restrictions (and then discharge it since we know that such an “instance” exists due to the assumption $\exists x Fx$).

Existential Elimination: The 1-step Rule

Existential Elimination Rule:

$$\frac{X \vdash \exists x A_t^x \quad Y, A \vdash B}{X, Y \vdash B} \exists E$$

Provided t does not occur in B or any formula in Y .

- Note what's written here: The assumption formula A in sequent 2 can be regarded an “instantiation” of the derivation in sequent 1 by substituting x by a term.
- We need the side condition so that our choice of the “instance” of x is still “general”.
- Otherwise we might be able to derive simply because we chose a specific special case!

Example: Example

 $\exists x (Fx \wedge Gx) \vdash \exists x Fx \wedge \exists x Gx$
 $\exists_x A_t \wedge x$

- | | | | |
|----------------------|-----|------------------------------------|-------------------------------|
| α_1 | (1) | $\exists x (Fx \wedge Gx)$ | A |
| α_2 | (2) | $Fa \wedge Ga$ | A |
| α_2 | (3) | Fa | 2 $\wedge E$ |
| α_2 | (4) | $\exists x Fx$ | 3 $\exists I$ |
| α_2 | (5) | Ga | 2 $\wedge E$ |
| α_2 | (6) | $\exists x Gx$ | 5 $\exists I$ |
| $\text{A } \alpha_2$ | (7) | $\exists x Fx \wedge \exists x Gx$ | B 4,6 $\wedge I$ |
| α_1 | (8) | $\exists x Fx \wedge \exists x Gx$ | 1,7[α_2] $\exists E$ |

$$\frac{X \vdash \exists x A_t^x \quad Y, A \vdash B}{X, Y \vdash B} \exists E$$

Provided t does not occur in B or any formula in Y.

$$\frac{X \vdash A_x^t}{X \vdash \exists x A} \exists I$$

Introduction
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Predicate Logic
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Natural Deduction
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Universal Quantifiers
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Existential Quantifiers
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Summary
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Summary



Content of this Lecture

- We introduced predicate logic:
 - with restricted quantifiers (we re-visit this later)
 - and with unrestricted quantifiers (default!)
 - Predicate logic can reason about objects!
 - Natural deduction for predicate logics, with additional rules for:
 - Introduction and Elimination rules for \forall and \exists
 - For the rest we keep using the rules from propositional logics!
 - Many side conditions...
 - Substitutions the substituted variable must be unbound
 - $\forall E$ and $\exists I$
 - $\forall I$ and $\exists E$
- The entire Logic Notes sections:
- 4: Expressing Generality
 - ▶ except “Properties of relations”
 - ▶ and except “Functions”
 - ▶ (You should read them anyway, in particular “Functions”!)