Modal Logic: Completeness, FMP, Translation and Correspondence

Yoshihiro Maruyama

Definition (Syntax of Modal Logic)

- Formulae of propositional modal logic are defined by induction.

1. Propositional variables $p, q, r, \ldots$ are formulae.
2. $\bot$ is a formula.
3. If φ and ψ be formulae then $\varphi \wedge \psi, \varphi \vee \psi, \varphi \to \psi, \neg\varphi, \Box\varphi$ are formulae.
4. $\Diamond\varphi$ abbreviates $\neg\Box\neg\varphi$.

Varieties of Modal Operators

- Necessity/possibility (most standard)
  - Are all logical truths necessary truths?
- Temporality (temporal logic)
  - True in the future, true at present, etc.
- Knowledge/Belief (epistemic logic)
  - A knows P, everyone knows P, it is common knowledge that, etc.
- Norm/Oughtness (deontic logic)

Kripke Semantics

Definition (Kripke Frame)

- A Kripke structure (model) is a structure $\mathscr{M} := \langle W, R, \nu \rangle$ where

  1. W is a nonempty set of possible worlds.
  2. R is an accessibility relation, a binary relation on W.
  3. ν is a valuation, i.e., a function $\nu\colon \mathrm{Prop} \to \mathscr{P}(W)$.

- Without a valuation, we call $\langle W, R \rangle$ a Kripke frame.

Example (Frame Classes)

- A frame class means a class of Kripke frames.
- $\mathbb{F}_{\mathrm{All}}$: the set of all frames.
- $\mathbb{F}_{\mathrm{PreOrd}}$: the set of frames with reflexive and transitive relations.
- $\mathbb{F}_{\mathrm{Equiv}}$: the set of frames with equivalence relations.

Definition (Forcing Relation)

- Let $\mathscr{M} := \langle W, R, \nu \rangle$ be a Kripke structure. We define a forcing relation $\Vdash$ by induction on formulae.

1. $\mathscr{M}, w \Vdash p :\Leftrightarrow w \in \nu(p)$.
2. $\mathscr{M}, w \nVdash \bot$.
3. $\mathscr{M}, w \Vdash \varphi \wedge \psi :\Leftrightarrow \mathscr{M}, w \Vdash \varphi$ and $\mathscr{M}, w \Vdash \psi$.
4. $\mathscr{M}, w \Vdash \varphi \vee \psi :\Leftrightarrow \mathscr{M}, w \Vdash \varphi$ or $\mathscr{M}, w \Vdash \psi$.
5. $\mathscr{M}, w \Vdash \neg\varphi :\Leftrightarrow \mathscr{M}, w \nVdash \varphi$
6. $\mathscr{M}, w \Vdash \varphi \rightarrow \psi :\Leftrightarrow \mathscr{M}, w \nVdash \varphi$ or $\mathscr{M}, w \Vdash \psi$
7. $\mathscr{M}, w \Vdash \Box\varphi :\Leftrightarrow \mathscr{M}, w' \Vdash \varphi$ for all wRw'

Definition (Satisfiability Relation)

- Let $\mathscr{M}$ be a Kripke structure, $F := \langle W, R \rangle$ a Kripke frame, $\mathbb{F}$ a frame class, and T a theory.

  1. We write $\mathscr{M} \models \varphi$ if $\mathscr{M}, w \Vdash \varphi$ for all $w \in W$.
  2. We write $F \models \varphi$ if $\langle W, R, \nu \rangle \models \varphi$ for all valuations ν.
  3. We write $\mathbb{F} \models \varphi$ if $F \models \varphi$ for all frames $F \in \mathbb{F}$
  4. If $\mathscr{M} \models \varphi$ for all $\varphi \in T$, then We say $\mathscr{M}$ is a model of T and we write $\mathscr{M} \models T$.
  5. $F \models T, \mathbb{F} \models T$ are defined in the same way.

Definition (Logical Consequence)

- Let $\mathbb{F}$ be a frame class, T a theory, and φ a formula.
- $\Gamma \vDash_{\mathbb{F}} \varphi$ if and only if $\mathscr{M} \vDash \Gamma$ implies $\mathscr{M} \vDash \varphi$ for all Kripke structures $\mathscr{M} := \langle W, R, \nu \rangle$ such that $\langle W, R \rangle \in \mathbb{F}$.
- In such a case we say φ is an $\mathbb{F}$-logical consequence of Γ.

Definition (Definability of Frame Class)

- Let Γ be a set of formulae and $\mathbb{F}$ be a frame class.
- Γ defines $\mathbb{F}$ if and only if $F \models \Gamma \iff F \in \mathbb{F}$.
- $\mathbb{F}$ is definable if and only if there is some Γ which defines $\mathbb{F}$.

Example (Frame Definability)

If R is ...	*then ... is true in* $\mathfrak{M}$*:*	
serial: $\forall u \exists v Ruv$	$\Box p \rightarrow \Diamond p$	(D)
reflexive: $\forall w Rww$	$\Box p \rightarrow p$	(T)
symmetric: $\forall u \forall v (Ruv \rightarrow Rvu)$	$p \rightarrow \Box\Diamond p$	(B)
transitive: $\forall u \forall v \forall w ((Ruv \wedge Rvw) \rightarrow Ruw)$	$\Box p \rightarrow \Box\Box p$	(4)
euclidean: $\forall w \forall u \forall v ((Rwu \wedge Rwv) \rightarrow Ruv)$	$\Diamond p \rightarrow \Box\Diamond p$	(5)

Table 50.1: Five correspondence facts.

See the textbook and attend tutorials for more details

Proof System

Definition (Uniform Substitution)

- Let σ be a function from propositional variables to (modal) formulae.
- A uniform substitution $\overline{\sigma}$ is defined by induction on formulae.

1. $\overline{\sigma}(p) := \sigma(p)$.
2. $\overline{\sigma}(\perp) := \perp$.
3. $\overline{\sigma}$ commutes with the logical connectives: e.g. $\overline{\sigma}(\varphi \to \psi) := \overline{\sigma}(\varphi) \to \overline{\sigma}(\psi)$.
4. $\overline{\sigma}(\Box\varphi) := \Box\,\overline{\sigma}(\varphi)$.

Definition (Hilbert System for Modal Logic K)

- The axioms of modal logic K are all propositional tautologies and the following: $\Box(p \to q) \to (\Box p \to \Box q)$.
- The rules of inference are:
  - modus ponens (i.e., if you have $p \to q$ and p then you can derive q),
  - necessitation (i.e., if you have p you can derive $\Box p$), and
  - uniform substitution (i.e., if you have φ you can derive any $\overline{\sigma}(\varphi)$, that is, any uniform substitution for φ.

Definition (Normal Modal Logic)

- A set of formulae Λ is a normal modal logic if and only if it is closed under provability in K, i.e., if $\Lambda \vdash_{\mathrm{K}} \varphi$ then $\varphi \in \Lambda$.

- $K(\Lambda)$ denotes the minimum normal modal logic that contains Λ.

Example (Normal Modal Logic)

T: $\Box p \to p$

B: $p \to \Box \Diamond p$

4: $\Box p \to \Box \Box p$

5: $\Diamond p \to \Box \Diamond p$

D: $\Box p \to \Diamond p$

.2: $\Diamond \Box p \to \Box \Diamond p$

$\mathrm{K} := \mathrm{K}(\varnothing)$

$\mathrm{S4} := \mathrm{K}(\{\mathrm{T}, 4\})$

$\mathrm{S5} := \mathrm{K}(\{\mathrm{T}, \mathrm{B}, 4\})$

$\mathrm{S4.2} := \mathrm{K}(\{\mathrm{T}, 4, .2\})$

$\mathrm{KD} := \mathrm{K}(\{\mathrm{K}, \mathrm{D}\})$

Soundness and Completeness

Definition (Notions of Completeness/Soundness)

- Let $\mathbb{F}$ be a frame class and Λ a normal modal logic.
  1. We say Λ is weakly sound for $\mathbb{F}$ if and only if $\vdash_\Lambda \varphi$ implies $\vDash_\mathbb{F} \varphi$ for all formulae φ.
  2. We say Λ is strongly sound for $\mathbb{F}$ if and only if $T \vdash_\Lambda \varphi$ implies $T \vDash_\mathbb{F} \varphi$ for all formulae φ and theories T.
  3. We say Λ is weakly complete for $\mathbb{F}$ if and only if $\vDash_\mathbb{F} \varphi$ implies $\vdash_\Lambda \varphi$ for all formulae φ.
  4. We say Λ is strongly complete for $\mathbb{F}$ if and only if $T \vDash_\mathbb{F} \varphi$ implies $T \vdash_\Lambda \varphi$ for all formulae φ and theories T.

Proposition (Soundness Theorem for K)

Let $\mathscr{M}$ be any Kripke model. If $\vdash_{\mathsf{K}} \varphi$ then $\mathscr{M} \vDash \varphi$, i.e. K is weakly sound for a frame class $\mathbb{F}_{\text{all}}$ where $\mathbb{F}_{\text{all}}$ consists all Kripke frame.

Proof.

We show the proposition by induction on proofs of φ. We check the axiom K $\Box(p \to q) \to \Box p \to \Box q$ is valid in $\mathscr{M}$. Assume $\mathscr{M}, w \Vdash \Box(p \to q)$ and $\mathscr{M}, w \Vdash \Box p$. Let $w' \in W$ be wRw'. By assumption, We have $\mathscr{M}, w' \Vdash p \to q$ and $\mathscr{M}, w' \Vdash p$. We get $\mathscr{M}, w' \Vdash q$ and $\mathscr{M}, w \Vdash \Box q$.

$\Box$

Proposition

Let Λ be a normal modal logic and $\mathbb{F}$ a frame class. The following are equivalent.

1. Λ is weakly sound for $\mathbb{F}$.
2. Λ is strongly sound for $\mathbb{F}$.

Proof.

This follows from the fact that $\mathscr{M} \vDash \varphi \to \psi$ and $\mathscr{M} \vDash \varphi$ imply $\mathscr{M} \vDash \psi$ for every Kripke structure $\mathscr{M}$.

$\square$

Definition (Λ-consistency)

- Let Λ be a normal modal logic and T a theory.

1. T is Λ-consistent if and only if $T \vdash_\Lambda \bot$
2. T is maximally Λ-consistent if and only if it is Λ-consistent and if $S \supseteq T$ is Λ-consistent then $S = T$.

Proposition

Let Λ be a normal modal logic and $\mathbb{F}$ a frame class. The following are equivalent.

1. Λ is strongly complete for $\mathbb{F}$.
2. If T is Λ-consistent then T is satisfiable in $\mathbb{F}$, for any theory T.

Proof.

Follows from: $T \not\models_{\mathbb{F}} \varphi \iff T \cup \{\neg\varphi\}$ is satisfiable and $T \nvdash_{\Lambda} \varphi \iff T \cup \{\neg\varphi\}$ is Λ-consistent.

$\square$

Proposition

Let Λ be a normal modal logic and Γ maximally Λ-consistent. Then the following hold.

1. $\Lambda \subseteq \Gamma$.
2. For all formulae φ, we have $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Proof.

1. is due to maximality of Γ.

2. follows from the fact that either Γ, φ or $\Gamma, \neg\varphi$ is Λ-consistent.

$\square$

Lemma (Lindenbaum's Lemma)

Let Λ be a normal modal logic and T a theory.

If T is Λ-consistent, then there is a maximally Λ-consistent theory T' which extends T.

Proof. The construction is the same as the one for classical propositional logic.

$\square$

Definition (Λ-canonical Model)

- Let Λ be a normal modal logic.
- The Λ-canonical model is the Kripke structure $\mathscr{M}_\Lambda := \langle W_\Lambda, R_\Lambda, \nu_\Lambda \rangle$ such that:

1. W_Λ is the set of maximally Λ-consistent theories.
2. $T\, R_\Lambda\, S$ if and only if $\Box\varphi \in T$ then $\varphi \in S$ for all formulae φ.
3. $T \in \nu_\Lambda(p) :\Leftrightarrow p \in T$.

Lemma (Truth Lemma)

Let Λ be a normal modal logic, T a maximally Λ-consistent theory, $\mathscr{M}_\Lambda$ the Λ-canonical model, and φ a formula.

The following are equivalent.

1. $\mathscr{M}_\Lambda, T \Vdash \varphi$.
2. $T \vdash_\Lambda \varphi$

Proof. We prove the following claim first.

Claim. If $T \nvdash_\Lambda \Box\psi$, then $\{\neg\psi\} \cup \{\chi \mid T \vdash \Box\chi\}$ is Λ-consistent.

If not, then there is a finite set Δ such that $\vdash_\Lambda \bigwedge \Delta \wedge \neg\psi \to \bot$. By propositional logic, we have $\vdash_\Lambda \bigwedge \Delta \to \psi$. Using necessitation and K, we have $\vdash_\Lambda \bigwedge \Box\Delta \to \Box\psi$ where $\Box\Delta := \{\Box\xi \mid \xi \in \Delta\}$. By $T \vdash \Box\xi$ for all $\xi \in \Delta$, we have $T \vdash \Box\psi$.

$\Box$

By the induction on formulae, we prove the truth lemma.

The cases of atomic formulae and logical connectives are done in the same way as in classical propositional logic. We consider the case of modality below.

Assume $T \nvdash_{\Lambda} \Box\varphi$. Using the claim, we have $\{\neg\varphi\} \cup \{\chi \mid T \vdash \Box\chi\}$ is consistent, and then there is a maximal consistent theory S which extends $\{\neg\varphi\} \cup \{\chi \mid T \vdash \Box\chi\}$ by Lindenbaum's Lemma.

We have $T\ R_\Lambda\ S$ and $S \vdash \neg\varphi$. By the induction hypothesis, we have $\mathscr{M}_\Lambda, S \nVdash \varphi$. And thus we get $\mathscr{M}_\Lambda, T \nVdash \Box\varphi$.

Assume $T \vdash_\Lambda \Box\varphi$ and let S be a maximally Λ-consistent theory such that $T\ R_\Lambda\ S$. By the definition of R_Λ, we have $\varphi \in S$. Hence $\mathscr{M}_\Lambda, T \Vdash \Box\varphi$.

$\Box$

Theorem (Completeness Theorem)

Let Λ be a normal modal logic and $\mathbb{F}$ a frame class. If Λ-canonical frame $\mathscr{M}_\Lambda \in \mathbb{F}$, then Λ is strongly complete for $\mathbb{F}$.

Proof.

Assume $T \nvdash_\Lambda \varphi$. Since $T \nvdash_\Lambda \varphi \iff T \cup \{\neg\varphi\}$ is Λ-consistent, the Lindenbaum lemma gives a maximally Λ-consistent theory S extending $T \cup \{\neg\varphi\}$. Since $T \nvDash_\mathbb{F} \varphi \iff T \cup \{\neg\varphi\}$ is satisfiable in $\mathbb{F}$, the truth lemma and $\mathscr{M}_\Lambda \in \mathbb{F}$ give us $T \nvDash_\mathbb{F} \varphi$.

$\square$

Theorem (Completeness Theorem for K)

K is strongly complete for $\mathbb{F}_{\text{all}}$.

Proof. We have $\mathscr{M}_{\mathsf{K}} \in \mathbb{F}_{\text{all}}$ by the definition of $\mathbb{F}_{\text{all}}$.

$\square$

Theorem (Completeness Theorem)

Let Λ be a normal modal logic and $\mathbb{F}$ a frame class. If Λ-canonical frame $\mathscr{M}_\Lambda \in \mathbb{F}$, then Λ is strongly complete for $\mathbb{F}$.

Proof.

Let T be a Λ-consistent theory. By Lindenbaum's Lemma, we have a maximally Λ -consistent theory S extending T. By the truth lemma, $\mathscr{M}_\Lambda, S \models \varphi$ for all $\varphi \in T$. By $\mathscr{M}_\Lambda \in \mathbb{F}$, T is Λ-satisfiable.

$\square$

Finite Model Property, Filtration and Decidability

Definition (Filtration)

- Let $\mathscr{M} := \langle W, R, \nu \rangle$ be a Kripke structure and Γ be a set of propositional variables A filtration of $\mathscr{M}$ via Γ is a model $\mathscr{M}' := \langle W', R', \nu' \rangle$ such that:

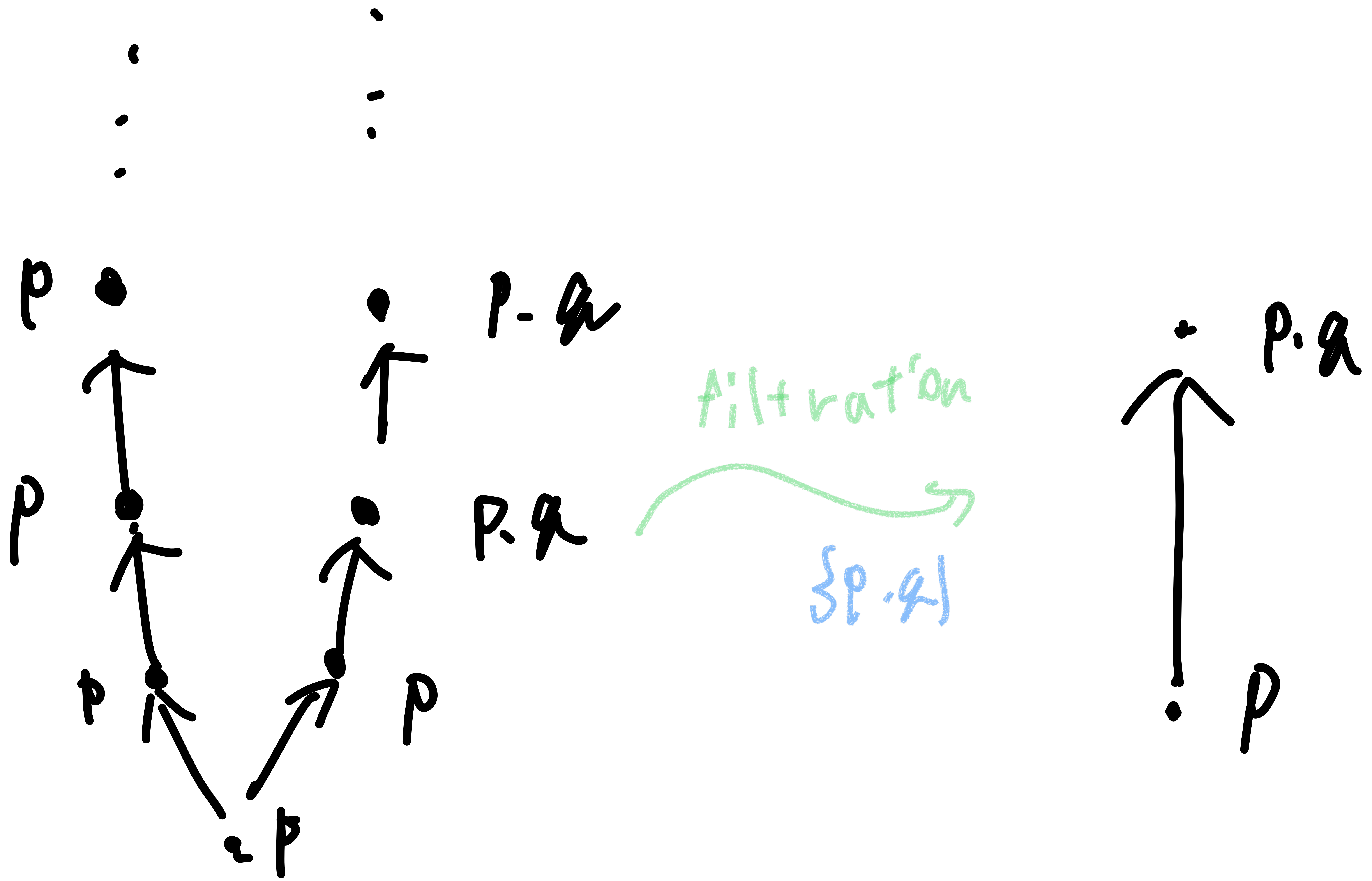
1. $W' = W/\equiv_\Gamma$ where $w \equiv_\Gamma v :\Leftrightarrow \mathscr{M}, w \Vdash p$ if and only if $\mathscr{M}, v \Vdash p$ for all $p \in \Gamma$.

2. wRv implies $[w]_{\equiv_\Gamma} R' [v]_{\equiv_\Gamma}$.

3. If $[w]_{\equiv_\Gamma} R' [v]_{\equiv_\Gamma}$ then $\mathscr{M}, w \vDash \Box \varphi$ implies $\mathscr{M}, v \vDash \varphi$ for all $\Box \varphi \in \Gamma$.

4. $v'(p) = \{[w]_{\equiv_\Gamma} \mid w \in \nu(p)\}$.

Fig: Filtration



Lemma (Filtration Lemma)

Let $\mathscr{M} := \langle W, R, \nu \rangle$ be a Kripke structure and Γ a set of formulae closed under subformulae.

If $\mathscr{M}' := \langle W', R', \nu' \rangle$ is a filtration of $\mathscr{M}$ through Γ, then the following are equivalent for any formula $\varphi \in \Gamma$ and world $w \in W$:

1. $\mathscr{M}, w \models \varphi$.
2. $\mathscr{M}', [w]_\Gamma \models \varphi$

Proof.

We prove the lemma by induction on formulae. We check the case of modality in the following.

Assume $\mathscr{M}, w \models \Box\varphi$ and $[w]_\Gamma R' [v]_\Gamma$. Then $\mathscr{M}, v \models \varphi$. By the induction hypothesis, we have $\mathscr{M}', [v]_\Gamma \models \varphi$.

Conversely, assume $\mathscr{M}', [w]_\Gamma \models \Box\varphi$ and wRv, we get $[w]_\Gamma R' [v]_\Gamma$ and $\mathscr{M}', [v]_\Gamma \models \varphi$. By the induction hypothesis, we have $\mathscr{M}, v \models \varphi$.

$\Box$

Lemma (Finiteness of Filtration)

Let $\mathscr{M} := \langle W, R, \nu \rangle$ be a Kripke structure, Γ a set of formulae closed under subformulae and $\mathscr{M}' := \langle W', R', \nu' \rangle$ a filtration of $\mathscr{M}$ through Γ.

If Γ is finite then so is $\mathscr{M}'$.

Proof.

Let n be the cardinality of Γ. The cardinality of W' is less than or equal to 2^n (by the the definition of $\equiv_\Gamma$, the cardinality of $W' = W/\equiv_\Gamma$ just depends on the truth/falsity of formulae in Γ).

$\square$

Finest / Smallest Filtration

- Let $\mathscr{M} := \langle W, R, \nu \rangle$ be a Kripke structure and Γ a set of propositional variables. The finest filtration of $\mathscr{M}$ via Γ is the following $\mathscr{M}^s := \langle W^s, R^s, \nu^s \rangle$:

  1. $W^s = W/\equiv_\Gamma$.

  2. $[w]_{\equiv_\Gamma} R^s [v]_{\equiv_\Gamma} :\Leftrightarrow w' R v'$ for some $w' \in [w]_{\equiv_\Gamma}, v' \in [v]_{\equiv_\Gamma}$.

  3. $v^s(p) = \{[w]_{\equiv_\Gamma} \mid w \in \nu(p)\}$.

where $w \equiv_\Gamma v :\Leftrightarrow \mathscr{M}, w \Vdash p$ if and only if $\mathscr{M}, v \Vdash p$ for all $p \in \Gamma$.

Coarsest / Largest Filtration

- Let $\mathscr{M} := \langle W, R, \nu \rangle$ be a Kripke structure and Γ a set of propositional variables. The coarsest filtration of $\mathscr{M}$ via Γ is the following $\mathscr{M}^l := \langle W^l, R^l, \nu^l \rangle$:

1. $W^l = W / \equiv_\Gamma$.
2. $[w]_{\equiv_\Gamma} R^l [v]_{\equiv_\Gamma} :\Leftrightarrow \mathscr{M}, w \vDash \Box\varphi$ implies $\mathscr{M}, v \vDash \varphi$ for all $\Box\varphi \in \Gamma$.
3. $v^l(p) = \{[w]_{\equiv_\Gamma} \mid w \in \nu(p)\}$.

where $w \equiv_\Gamma v :\Leftrightarrow \mathscr{M}, w \Vdash p$ if and only if $\mathscr{M}, v \Vdash p$ for all $p \in \Gamma$.

Proposition

Finest and courses filtrations are filtrations.

Proof.

Let $\mathscr{M} := \langle W, R, \nu \rangle$ be Kripke structure, Γ be a set of formulae closed under subformula, and $\mathscr{M}^f := \langle W^f, R^f, \nu^f \rangle$ be the finest or coarsest filtration. We verify the following two claims.

1. wRv implies $[w]_{\equiv_\Gamma} R^f [v]_{\equiv_\Gamma}$,
2. If $[w]_{\equiv_\Gamma} R^f [v]_{\equiv_\Gamma}$, then $\mathscr{M}, w \Vdash \Box\varphi$ implies $\mathscr{M}, v \Vdash \varphi$ for all $\Box\varphi \in \Gamma$.

Let us check the claims for the finest one $\mathscr{M}^s := \langle W^s, R^s, \nu^s \rangle$

The first claim is immediate from the definition of the finest filtration.

To prove the second claim, assume $[w]_{\equiv_\Gamma} R^s [v]_{\equiv_\Gamma}$ and $\mathscr{M}, w \Vdash \Box\varphi$ for $\Box\varphi \in \Gamma$. By the definition of R^s, there is $w' \in [w]_{\equiv_\Gamma}$, $v' \in [v]_{\equiv_\Gamma}$ such that $w' R v'$. By the definition of $\equiv_\Gamma$ and $\Box\varphi \in \Gamma$, we have $\mathscr{M}, w' \Vdash \Box\varphi$. Hence $\mathscr{M}, v' \Vdash \varphi$. By the definition of $\equiv_\Gamma$ and $\varphi \in \Gamma$ (due to closedness of Γ), we have $\mathscr{M}, v \Vdash \varphi$.

We next verify the claim for the coarsest one $\mathscr{M}^l := \langle W^l, R^l, \nu^l \rangle$.

The second claim is immediate from the definition of the coarsest filtration.

To prove the first claim, assume $\Box\varphi \in \Gamma$, wRv and $\mathscr{M}, w \Vdash \Box\varphi$. We have $\mathscr{M}, v \Vdash \varphi$. By definition of R^l we have $[w]_{\equiv_\Gamma} R^l [v]_{\equiv_\Gamma}$.

$\Box$

Theorem (Finite Model Property of K)

K is complete for the frame class $\mathbb{F}_{\text{fin}}$ which consists of all finite frames. That is, it has the FMP (finite model property).

Proof.

Assume φ is K-consistent. By the completeness theorem for K, there is a frame $\mathscr{M}$ which satisfies φ.

Let Γ be the finite set of all subformulae of φ and $\mathscr{M}'$ be a filtration of $\mathscr{M}$ via Γ. By the lemmata above, $\mathscr{M}'$ is finite and satisfies φ

$\square$

Corollary (Decidability of K)

The validity relation of K is decidable, i.e., there is an algorithm which decides whether $\models \varphi$ or not for any formula φ.

Proof.

Let n be the number of subformulae of φ. By FMP and the cardinality consideration, it suffices to check $\mathscr{M} \models \varphi$? for all $\mathscr{M}$ with less than or equal to 2^n elements, which is decidable.

$\square$

Remark

- In general, filtrations do not preserve some frame properties (e.g., symmetry).
- However, the coarsest one preserves symmetry, and the transitive closure of a symmetric relation is symmetric.
- Thus we can prove the FMP and decidability for $\mathsf{S4}, \mathsf{S5}$ in such a way.

The Relationships between Modal and First-Order Logics

Definition (Standard Translation)

- Let $\mathscr{L} := \langle R, P_0, \ldots, P_i, \ldots \rangle$ be a FO language s.t. R is binary and P_i's are unary (corresponding to propositional variables p_i), u a free variable, and φ a modal formula.
- We define the standard translation $\mathrm{ST}_u(\varphi)$ of φ by induction on φ:

1. $\mathrm{ST}_u(\perp) :\equiv \perp$.
2. $\mathrm{ST}_u(p_i) :\equiv P_i(u)$.
3. $\mathrm{ST}_u(\varphi \to \psi) :\equiv \mathrm{ST}_u(\varphi) \to \mathrm{ST}_u(\psi)$
4. $\mathrm{ST}_u(\Box\varphi) :\equiv$ $(\forall x)[uRx \to \mathrm{ST}_u(\varphi)[u := x]]$
5. $\mathrm{ST}_u(\Diamond\varphi) :\equiv$ $(\exists x)[uRx \wedge \mathrm{ST}_u(\varphi)[u := x]]$

Theorem (Translation Theorem)

Let $\mathscr{M} := \langle W, R, \nu \rangle$ be a Kripke model and $\mathscr{M}' := \langle W; R, P_0 \ldots, P_i, \ldots \rangle$ be the $\mathscr{L}$-structure induced by $\mathscr{M}$ where $P_i := \nu(p_i)$.

The following are equivalent with each other:

1. $\mathscr{M}, w \Vdash \varphi$.

2. $\mathscr{M}' \vDash \mathrm{ST}_u(\varphi)[u := \overline{w}]$.

Proof.

We prove it by induction on φ. In the following we consider the cases for propositional variables and modality.

The case for propositional variables is verified by the following equivalences.

$$\begin{aligned}\mathscr{M}, w \Vdash p_i &\iff w \in \nu(p_i) \\ &\iff w \in P_i \\ &\iff \mathscr{M}' \vDash P_i(w).\end{aligned}$$

The case for modality is verified by the following equivalences.

$$\begin{aligned}
\mathscr{M}, w \Vdash \Box\varphi &\iff \mathscr{M}, w' \Vdash \varphi \text{ for all } wRw' \\
&\iff \mathscr{M}' \vDash \mathrm{ST}_u(\varphi)[u := \overline{w'}] \text{ for all } wRw' \text{ (by IH)} \\
&\iff \mathscr{M}' \vDash (\forall w')[R(w, w') \to \mathrm{ST}_u(\varphi)[u := w']] \\
&\iff \mathscr{M}' \vDash \mathrm{ST}_u(\Box\varphi)[u := \overline{w}]
\end{aligned}$$

IH above stands for the induction hypothesis.

Corollary (Trans. b/w K and FOL)

The following are equivalent.

1. $\vdash_{\mathsf{K}} \varphi$.
2. $\vdash (\forall x)\mathrm{ST}_x(\varphi)$

Proof.

Follows from the last theorem and the completeness theorems to translate semantic validities into provabilities.

$\square$

Correspondence Theory

Examples

If R is ...	*then ... is true in* $\mathfrak{M}$:	
serial: $\forall u \exists v Ruv$	$\Box p \to \Diamond p$	(D)
reflexive: $\forall w Rww$	$\Box p \to p$	(T)
symmetric: $\forall u \forall v (Ruv \to Rvu)$	$p \to \Box\Diamond p$	(B)
transitive: $\forall u \forall v \forall w ((Ruv \land Rvw) \to Ruw)$	$\Box p \to \Box\Box p$	(4)
euclidean: $\forall w \forall u \forall v ((Rwu \land Rwv) \to Ruv)$	$\Diamond p \to \Box\Diamond p$	(5)

Table 50.1: Five correspondence facts.

See the textbook and attend tutorials for more details

Definition (Geach Logic)

- Let $s := \langle k, l, m, n \rangle \in \mathbb{N}^4$.
- Geach scheme: $\mathsf{G}^s := \{\Diamond^k \Box^l \varphi \to \Box^m \Diamond^n \varphi \mid \varphi \in \mathrm{Form}\}$.
- Geach Logic $\mathsf{G}(S)$, for S being a subset of $\mathbb{N}^4$, is the minimum normal logic which contains G^s for any $s \in S$.

Definition (Generalized Confluence)

- Let $\langle W, R\rangle$ be a Kripke frame and $i \in \mathbb{N}$ be a natural number. A binary relation R^i on W is defined by induction on i.

  1. $sR^0t :\Leftrightarrow s = t$.
  2. $wR^{i+1}w' :\Leftrightarrow wR^iu$ and uRw' for some $u \in W$.

- Let $s := \langle k, l, m, n\rangle \in \mathbb{N}^4$. $\langle W, R\rangle$ is s-confluent if and only if: For all $w, x, y \in W$ such that wR^kx and wR^my, there is w' such that xR^lw' and yR^nw'.

Proposition (Confluency and Geach Formula)

Let $\langle W, R \rangle$ be a Kripke frame and $s = \langle k, l, m, n \rangle \in \mathbb{N}^4$.

The following are equivalent.

1. $\langle W, R \rangle$ is s-confluent.
2. $\langle W, R \rangle \vDash \mathsf{G}^s$.

Proof.

In a Kripke model, $\Box^i$, $\Diamond^i$ corresponds to R^i and we have the following proposition.

Prop. Let $\mathscr{M} := \langle W, R, \nu \rangle$ be a Kripke model, φ a formula, $w \in W$, and $i \in \mathbb{N}$.

- $w \Vdash \Box^i \varphi$ if and only if $w' \Vdash \varphi$ for all wR^iw'.
- $w \Vdash \Diamond^i \varphi$ if and only if $w' \Vdash \varphi$ for some wR^iw'.

By the prop, we have $1 \implies 2$. We prove $2 \implies 1$ by contraposition. Assume $\langle W, R \rangle$ is not s-confluent. By the definition of s-confluency, we have $w, x, y \in W$ such that wR^kx, wR^my and $x \not R^l w'$ or $y \not R^n w'$ for all $w' \in W$

We define a valuation ν by $u \in \nu(p) :\Leftrightarrow xR^l u$.

By the above prop. and the definition of ν, we have the negation of $\langle W, R, \nu \rangle, w \Vdash \mathsf{G}^s$,
Hence the negation of $w \Vdash \mathsf{G}^s$.

$\square$

Theorem (Completeness Theorem for Geach Logic)

Let $S \subseteq \mathbb{N}^4$ and $\mathbb{F}(S)$ the class of frames that are s-confluent for any $s \in S$. $\mathsf{G}(S)$ is strongly sound and strongly complete for $\mathbb{F}(S)$.

Proof. Soundness follows from the last prop.

To prove completeness, we have to check $\mathscr{M}_{\mathsf{G}(S)}$ is s-confluent for all $s \in S$.

Claim. Let Λ be a normal modal logic, S, T maximally Λ-consistent theories, and $i \in \mathbb{N}$. The following are equivalent.

1. $S \, R^i_\Lambda \, T$.
2. $\{\varphi \mid \Box^i \varphi \in S\} \subseteq T$.
3. $S \supseteq \{\Diamond^i \varphi \mid \varphi \in T\}$.

Shown by induction on i.

Let $s = (k, l, m, n)$ and assume $T_0 \ R_\Lambda^k \ T_1$ and $T_0 \ R_\Lambda^m \ T_2$. We have to show there is a maximally $\mathsf{G}(S)$-consistent theory T_3 such that $T_1 \ R_\Lambda^l \ T_3$ and $T_1 \ R_\Lambda^n \ T_3$. Let T_1', T_2' be a theory such that

- $T_1' := \{\varphi \mid \Box^l \varphi \in T_1\}$.
- $T_2' := \{\varphi \mid \Box^n \varphi \in T_2\}$.

$T_1' \cup T_2'$ is $\mathsf{G}(S)$-consistent. By Lindenbaum's lemma there is a maximally $\mathsf{G}(S)$-consistent theory T_3 which extends $T_1' \cup T_2'$.

By the above claim and the definitions of T_1', T_2', we have $T_1 \ R_\Lambda^l \ T_3$ and $T_1 \ R_\Lambda^n \ T_3$.

$\Box$