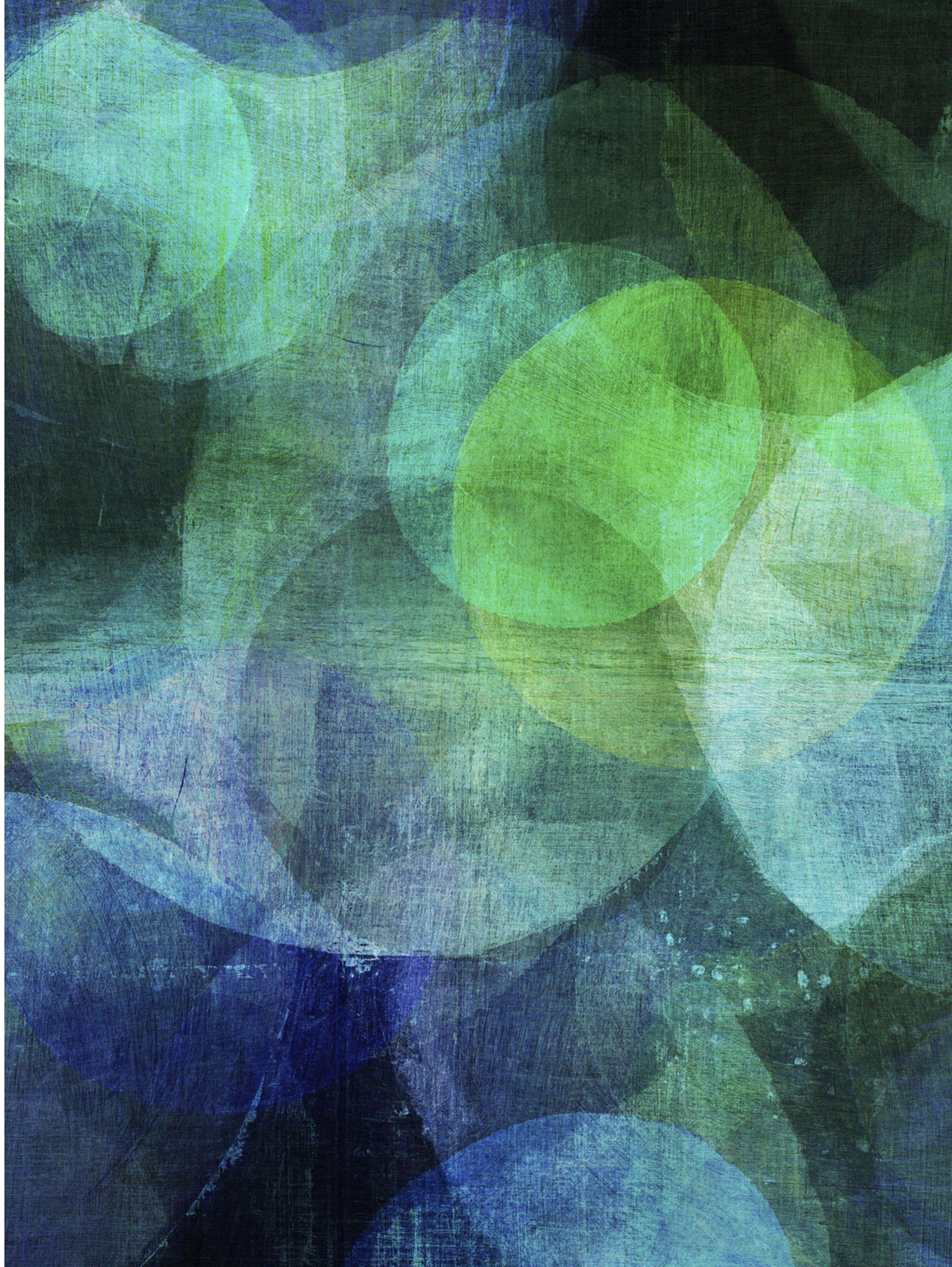
VAGUENESS AND THE SORITES PARADOX



SORITES PARADOX 1

https://www.theifod.com/sorites-paradox/

- $10^{10^{10}}$ grains are a heap (you can choose any sufficiently large number).
- If $10^{10^{10}}$ grains are a heap, then $10^{10^{10}} - 1$ grains are a heap.
- If $10^{10^{10}} - 1$ grains are a heap, $(10^{10^{10}} - 1) - 1$ grains are a heap.
  - In general: if n grains are a heap, $n - 1$ grains are a heap.
  - Removing a single grain does not change the qualitative status of being a heap.
- Thus: zero grains are a heap (by applying the above argument $10^{10^{10}}$ times).
  - The logical form of the argument (apply the $\rightarrow$**E** rule n times): $p_n, (p_n \rightarrow p_{n-1}), \ldots, (p_1 \rightarrow p_0) \vdash p_0$. It's called the Sorites Paradox.

SORITES PARADOX 2

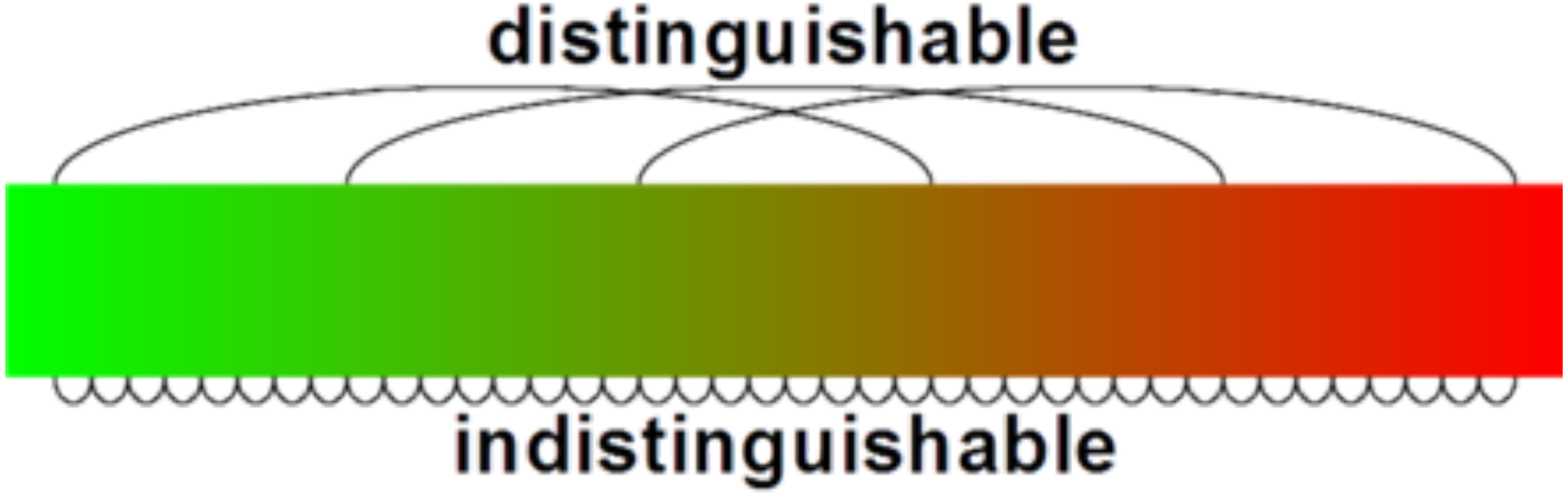
- Suppose this strip is one hundred miles long, and I walk along it from left to right.
  - On the first step is a sign saying “green”.
  - At each step I record the colour of the current step and perform a logical test: If the next step is not directly discriminable from the current step, then I conclude the next step is the same colour as the current one.
- Clearly, by essentially applying the $\to$**E** rule repeatedly together with the initial premises that the first step is green, at the end we will be calling a red step green!

SORITES PARADOX 3

- Now suppose we have a dinosaur.
  - It mates and lays an egg. The egg hatches, releasing an almost identical offspring of the dinosaur.
  - Perform a logical test; if the offspring is almost identical to its parent, then it is of the same species.
  - Return to step one and repeat with the new, almost identical dinosaur.
- 200 million years later you have a chicken, yet according to the logical test with repeatedly appealing to the $\rightarrow$**E** rule, it is a dinosaur. When did the dinosaur become the chicken? (Supposing Darwin's theory of evolution is correct.)

THE PROBLEM

- The problem is: qualitative predicates, such as '… is a heap', '... is green', or '... is a dinosaur', are insensitive to small changes, but is sensitive to sufficiently large changes, and yet there is no obvious borderline between negligible and significant changes.



https://anthropology365.com/2018/06/26/science-race-and-the-sorites-paradox/

A PARTING OF THE WAYS

- Broadly, there are three kinds of responses:

1. Eliminate vague predicates; they are too vague and unimportant for logic, math, or science. Most popular in ordinary science pursuing non-vague truths and clarity.

2. Identify problematic part of the argument, and modify logic so as to make it invalid. Fuzzy logic (and some substructural logics) may arguably work for this purpose. Most popular in formal logic.

3. Keep the truths of classical logic, yet give a logical account of vague predicates via the precisification of vague predicates. Most popular in analytic philosophy.

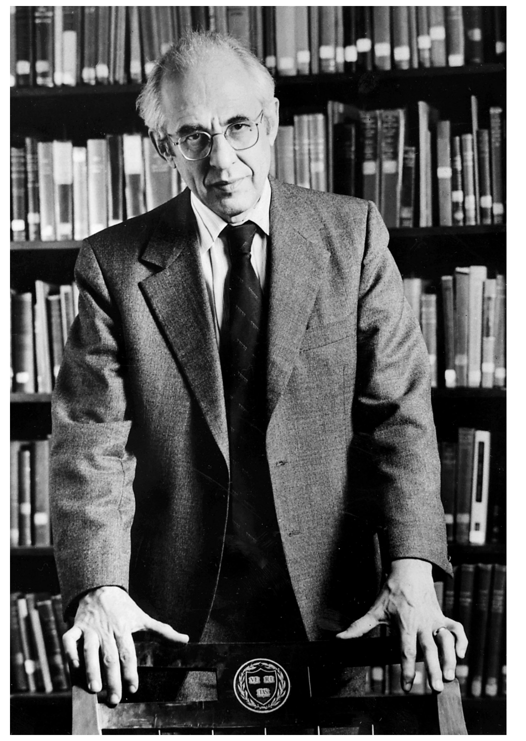
- Option 1: vagueness does not matter for the pursuit of scientific truth; forget about it. Option 2: revise logic to accommodate vagueness. Option 3: accommodate vagueness while keeping classical logic as much as possible. Which one would you choose?

OPTION 1: QUINE'S REGIMENTATION OF LANGUAGE

https://news.harvard.edu/gazette/story/2001/01/quine-92-was-major-philosopher-of-20th-century/

W. V. O. Quine

- Putnam (1983): "Quine has suggested that it really doesn't matter if ordinary language is vague (or even locally inconsistent), as long as we can think of ourselves as approximating a scientific language which is free of these defects."
- Putnam (1983): "In Quine's view, statements in ordinary language aren't true or false; they are only true or false relative to a translation scheme (or 'regimentation') which maps ordinary language onto an ideal language — and there is no fact of the matter as to which is the 'right' translation". Ideal language is fully logical and clear.
- Quine (1977): "Ordinary language is only loosely factual, and needs to be variously regimented when our purpose is scientific understanding. The regimentation is again not a matter of eliciting a latent content. It is again a free creation."
- Simplistically: remove vagueness; it's not part of any serious science or truth.



H. Putnam

https://www.nytimes.com/2016/03/18/arts/hilary-putnam-giant-of-modern-philosophy-dies-at-89.html

OPTION 2: THE FUZZY LOGIC APPROACH

- Fuzzy logic gives a new perspective on the Sorites Paradox.
    - There are more than two truth values in fuzzy logic (possibly infinitely many).
    - We can then argue that each step $p_n \to p_{n-1}$ in the Sorites argument is correct to some degree, but not exactly correct, i.e., being of truth value in between 0 and 1.
- In general we can allow any real number between 0 and 1, but for simplicity, let us think about three-valued fuzzy logic in this course.
- NB: fuzzy logic is also contraction-free, so the same proof-theoretic argument as for contraction-free relevant logic works for fuzzy logic as well. Yet fuzzy logic also gives a many-valued semantic account of invalidity of the Sorites argument.

THE FUZZY LOGIC APPROACH (CONT'D)

- There are three three values, i.e., **F**, **i**, and **T**.
  - We may think of the intermediate value **i** as "half-true", which is the same thing as "half-false".
  - Formally: **F<i<T**.
  - The conditional $A \rightarrow B$, for example, is:

true if the consequent is at least as good as the antecedent;

false if the antecedent is completely true and the consequent completely false;

half-true if the consequent is a half-step down from the antecedent.

THREE-VALUED SEMANTICS

- The three-valued semantics for fuzzy logic:

$\wedge$	T	i	F
T	T	i	F
i	i	i	F
F	F	F	F

$\vee$	T	i	F
T	T	T	T
i	T	i	i
F	T	i	F

$\rightarrow$	T	i	F
T	T	i	F
i	T	T	i
F	T	T	T

A	$\neg A$
T	F
i	i
F	T

  - $A \rightarrow B$ returns **i** if A is **T** and B is **i**, which may happen in the Sorites argument.
  - $\neg A$ is equivalent to $A \rightarrow \bot$ as in ordinary logic.
  - $\neg\neg A$ is equivalent to A, and $\neg(A \vee B)$ to $\neg A \wedge \neg B$.

THREE-VALUED SEMANTICS (CONT'D)

- Comma and semicolon (fuzzy and relevant semicolons have different truth tables):

,	T	i	F
T	T	i	F
i	i	i	F
F	F	F	F

;	T	i	F
T	T	i	F
i	i	F	F
F	F	F	F

- A sequent $X \vdash A$ is valid iff (i) any assignment of values that gives **T** to X also gives **T** to A and (ii) any assignment of values that gives **F** to A gives **F** to X.
  - Tautologies (valid formulae) are those to which any assignment of values gives **T.**

EXAMPLES

- $A \to (A \to B)$ is classically equivalent to $A \to B$, but in fuzzy logic it is not:

  $$\mathbf{i} \to (\mathbf{i} \to \mathbf{F}) \text{ evaluates to } \mathbf{i} \to \mathbf{i} \text{ which is } \mathbf{T}\text{, but } \mathbf{i} \to \mathbf{F} \text{ is } \mathbf{i}.$$

- The following sequent is not valid:

  i T i i F i i i i F

  $$p \to (q \to r) \vdash (p \land q) \to r$$

- The three-valued semantics of semicolon makes contraction invalid. Recall:

$$\frac{X; A; A \vdash B}{X; A \vdash B}\ \textit{contraction}$$

OPTION 3: SUPERVALUATIONISM

- Option 3: define truth by precisifications or sharpenings of vague predicates.
  - Suppose that there is such a step n such that the n-th step is still green but the $(n+1)$-th step is not green: $\exists n \; G(n) \wedge \neg G(n+1)$
  - But where is that borderline? This problem is even more severe if we change the predicate from "… **is** green" to "… **looks** green".
  - How can the two steps immediately before and after the "borderline" be such that no difference of colour between them can be detected just by looking, but one of them looks green and the other does not?

SHARPENINGS OF VAGUE PREDICATES

- Vague predicates can be precisified or sharpened in different ways.
  - For example, in the legal definition of adulthood, there is a precise line drawn for the vague idea of adulthood.
  - In Australia the line is 18 years old, whereas it is 20 in Japan.
  - Each of them is called a precisification or sharpening of the vague predicate.
- Drawing a precise sharp line makes borderline cases disappear, and then vague predicates become ordinary classical predicates with no borderline cases.

SAVING THE CLASSICAL TRUTHS

- How can we determine the truth value of a vague proposition?
  - A vague proposition is **true** if it is true for *every* acceptable sharpening of the vague predicate involved.
  - A vague proposition is **false** if it is false for *every* acceptable sharpening the vague predicate involved.
- This raises the problem of indeterminacy: a vague proposition is typically true for certain sharpenings but false for others; it can then lack a (unique) truth value.
  - Just as we saw previously, a person of 19 years of age would be considered as an adult in Australia but not in Japan. “The person is an adult” may lack a truth value.
  - Yet “the person is an adult or not an adult” is true; the classical truths are saved.

THE BORDERLINE EXISTS, BUT IT'S NOT FIXED

- Even if p and $\neg p$ lack a truth value, $p \vee \neg p$ is true.
  - The sharpening semantics for vagueness is not compositional (truth-functional).
- For any sharpening of the vague predicate $H(n)$ "n grains are a heap", the following is true: $\exists n\ H(n) \wedge \neg H(n+1)$.
  - Because any sharpening gives such an n.
  - There is no fixed borderline, but the existence of some borderline is true. The borderline exists, but it's not fixed.
- Thus: some premise of the Sorites Paradox argument cannot hold, although it depends upon each sharpening which premise it is.

ALL THIS IS JUST A STARTING POINT FOR YOU

- Which solution sounds most plausible to you?
  - Or would you have any other idea?
  - Is there any revenge paradox to each solution?
- There is no canonical or unique solution to any paradox.
  - If there is, it's not a paradox presumably.
- Explore and develop your own idea (rather than supposing something given is absolutely correct).
  - Also: try to find new paradoxes, by which you can leave your name in history.

Appendix

WHAT IS LOGICALLY WRONG WITH THE SORITES ARGUMENT?

- What's logically wrong with the Sorites Paradox argument?
  - Is the $\rightarrow$**E** rule wrong?
  - Abandoning the $\rightarrow$**E** rule entirely is problematic as Michael Dummett puts it:

    "the validity of this rule of inference seems absolutely constitutive of the meaning … of 'if'."
  - How can we naturally/logically justify the invalidity of the rule?

ONE SOLUTION

- How can we resolve the paradox while maintaining something like the $\rightarrow$**E** rule?
  - If $p, p \rightarrow q \vdash q$ is not provable in a logic, then no Sorites paradox in that logic.
    - The assumption is true in some substructural logics.
  - Yet the $\rightarrow$**E** rule with semicolon can be allowed (which yields $p;\ p \rightarrow q \vdash q$), which is the case in some substructural logics.
- As far as the Sorites argument relies upon comma, it does not work in those logics.
- Can we represent the argument using semicolon rather than comma? "And" in the Sorites argument as represented by comma is arguably different from semicolon.

MORE EXAMPLES

- The following sequents are invalid in the three-valued semantics for fuzzy logic.
  - Try to invalidate them by assigning and computing truth values.

$$(p \to q) \land p \vdash q$$

$$p \to \neg p \vdash \neg p$$

$$\vdash \neg(p \land \neg p)$$

$$(p \to q) \land (q \to r) \vdash p \to r$$

$$p \to (q \to r) \vdash (p \land q) \to r$$