Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Propositional Logic
— Recap on Proof Strategies

Yoshihiro Maruyama

Australian National University

Introduction

Recap: Sequents, Validity, Proof Systems

- You know how to prove $X \vdash A$ via ND
- You know how to prove $X \vdash A$ via ST
- ND is useful to show $X \vdash A$ is valid.
- To show *invalidity*, ST is more convenient.

So, when to use which?

- If we don’t tell whether it’s valid or invalid:
  - If you think the sequent is invalid: ST is most useful
  - If you think the sequent is valid: Choose what you are stronger in!
  - If you don’t know either way: Use ST and *let it tell you*!

Strategies: Overview

Tips for Semantic Tableau and Natural Deduction

Semantic Tableau:

- Always apply non-branching rules first.
- In case of invalid sequents you could follow down branches leading to an open branch more quickly – which requires “seeing” which interpretation proves invalidity.

Natural Deduction:

- See the next slide.
- We also sometimes need the fall-back strategy: Assume negation of final derivation and exploit contradiction.

Natural Deduction: Overview

How to show $X \vdash A$? Depends on A!

A could be . . .

$\underline{B \wedge C}$:	$\underline{B \vee C}$:	$\underline{B \rightarrow C}$:	$\underline{\neg B}$:	p (atom)
derive B	assume B	assume B	assume B	
derive C	assume C	derive C	derive contradiction	
$\wedge I$	$\vee E$	$\rightarrow I$	use negation rules	

Note:

- $X \vdash A$ can also refer to sub steps!
- Usually, you will need $\vee E$ if $B \vee C \in X$, not if $B \vee C = A$.

Examples for Natural Deduction

$X \vdash A$, A is an Implication

$$p \to ((q \wedge r) \to s) \vdash q \to ((p \wedge r) \to s)$$

α_1	(1)	$p \to ((q \wedge r) \to s)$	A
α_2	(2)	q	A
α_3	(3)	$p \wedge r$	A
α_3	(4)	p	3 $\wedge E$
α_1, α_3	(5)	$(q \wedge r) \to s$	1,4 $\to E$
α_3	(6)	r	3 $\wedge E$
α_2, α_3	(7)	$q \wedge r$	2,6 $\wedge I$
$\alpha_1, \alpha_2, \alpha_3$	(8)	s	5,7 $\to E$
α_1, α_2	(9)	$(p \wedge r) \to s$	8[α_3] $\to I$
α_1	(10)	$q \to ((p \wedge r) \to s)$	9[α_2] $\to I$

$X \vdash A$, A is a Negation

$$\boxed{\dfrac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B}\ RAA}$$

$\neg(p \lor q) \vdash \neg p$

α_1	(1)	$\neg(p \lor q)$	A
α_2	(2)	p	A
α_2	(3)	$p \lor q$	2 $\lor I$
α_1	(4)	$\neg p$	1,3[α_2] RAA

$X \vdash A$, A is a Disjunction (here: in one of the Substeps)

$$\neg p \land \neg q \vdash \neg(p \lor q)$$

$$\boxed{\dfrac{X \vdash A \lor B \qquad Y, A \vdash C \qquad Z, B \vdash C}{X, Y, Z \vdash C}\,\vee E}$$

$$\boxed{\dfrac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B}\,RAA}$$

α_1	(1)	$\neg p \land \neg q$	A
α_2	(2)	$p \lor q$	A
α_3	(3)	p	A
α_4	(4)	q	A
α_1	(5)	$\neg p$	1 $\land E$
α_3	(6)	$\neg(\neg p \land \neg q)$	3,5[α_1] RAA
α_1	(7)	$\neg q$	1 $\land E$
α_4	(8)	$\neg(\neg p \land \neg q)$	4,7[α_1] RAA
α_2	(9)	$\neg(\neg p \land \neg q)$	2,6[α_3],8[α_4] $\lor E$
α_1	(10)	$\neg(p \lor q)$	1,9[α_2] RAA

$X \vdash A$, A is an Atom (Or: all Other Strategies Fail)

If everything else fails, assume $\neg A$ and derive A:

$$\frac{X, \neg A \vdash A}{X \vdash A}$$

Closely related is the sequent: $\neg p \rightarrow p \vdash p$
(p is so true that it's even implied by its own negation!)

$$\boxed{\frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B}\ RAA}$$

Example: $(p \rightarrow q) \rightarrow p \vdash p$

α_1	(1)	$(p \rightarrow q) \rightarrow p$	A
α_2	(2)	$\neg p$	A
α_3	(3)	p	A
α_2, α_3	(4)	$\neg\neg q$	2,3[] RAA
α_2, α_3	(5)	q	4 $\neg\neg E$
α_2	(6)	$p \rightarrow q$	5[α_3] $\rightarrow I$
α_1, α_2	(7)	p	1,6 $\rightarrow E$
α_1	(8)	$\neg\neg p$	2,7[α_2] RAA
α_1	(9)	p	8 $\neg\neg E$

Examples for Semantic Tableau

Previous Example, shown with Semantic Tableau

We now show $(p \to q) \to p \vdash p$ via Semantic Tableau.

$$\frac{\mathbf{T}\text{:}\ A \to B}{\mathbf{F}\text{:}\ A \mid \mathbf{T}\text{:}\ B}$$

$$\frac{\mathbf{F}\text{:}\ A \to B}{\mathbf{T}\text{:}\ A\ ,\ \mathbf{F}\text{:}\ B}$$

$(p \to q) \to p \vdash p$

(1) **T:** $(p \to q) \to p$ ✓

(2) **F:** p

(3) **F:** $p \to q$ ✓ from (1)

(5) **T:** p ↯ from (3)

(6) **F:** q from (3)

(4) **T:** p ↯ from (1)

The primary strategy (that often suffices to create small trees) is:

- *Always* apply rules first that don’t branch!
- Don’t forget branches!! And mark lines that are “done”.

Summary

Content of this Lecture

- Today, we did a recap on how to prove various kinds of sequents via Natural Deduction and Semantic Tableau