Logic: Tutorial Week 8

Proof 1

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$p \to (q \vee r), p, s \;\vdash\; r, s$}
\RightLabel{$\neg \vdash$}
\UnaryInfC{$\neg s, p \to (q \vee r), p, s \;\vdash\; r$}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$p, q, s \;\vdash\; q, r$}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$p, r, s \;\vdash\; q, r$}
\RightLabel{$\vee \vdash$}
\BinaryInfC{$q \vee r, p, s \;\vdash\; q, r$}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$p, s \;\vdash\; p, q, r$}
\RightLabel{$\to \vdash$}
\BinaryInfC{$p \to (q \vee r), p, s \;\vdash\; q, r$}
\RightLabel{$\to \vdash$}
\BinaryInfC{$q \to \neg s, p \to (q \vee r), p, s \;\vdash\; r$}
\RightLabel{$\wedge \vdash$}
\UnaryInfC{$q \to \neg s, p \to (q \vee r), p \wedge s \;\vdash\; r$}
\RightLabel{$\vdash \to$}
\UnaryInfC{$q \to \neg s, p \to (q \vee r) \;\vdash\; (p \wedge s) \to r$}
\RightLabel{$\vdash \to$}
\UnaryInfC{$q \to \neg s \;\vdash\; (p \to (q \vee r)) \to ((p \wedge s) \to r)$}
\end{prooftree}
$$

Proof 2

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p} \qquad \dfrac{}{q \vdash q}}{p, q \vdash p \wedge q}\ \vdash\wedge \qquad \dfrac{\dfrac{}{r \vdash r} \qquad \dfrac{}{s \vdash s}}{r \vee s \vdash r, s}\ \vee\vdash}{(p \wedge q) \to (r \vee s), p, q \vdash r, s}\ \to\vdash}{(p \wedge q) \to (r \vee s), p \vdash r, q \to s}\ \vdash\to}{(p \wedge q) \to (r \vee s) \vdash p \to r, q \to s}\ \vdash\to}{(p \wedge q) \to (r \vee s) \vdash (p \to r) \vee (q \to s)}\ \vdash\vee$$

Notice how smooth and easy this proof is, in comparison with the monster of a natural deduction proof, given as the last propositional logic exercise in the Course Notes.

Proof 3

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$Rba \vdash Rba$}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$Rab \vdash Rab$}
\RightLabel{$\rightarrow\vdash$}
\BinaryInfC{$Rba, Rba \rightarrow Rab \vdash Rab$}
\RightLabel{$\forall\vdash$}
\UnaryInfC{$Rba, \forall y(Rby \rightarrow Ryb) \vdash Rab$}
\RightLabel{$\forall\vdash$}
\UnaryInfC{$Rba, \forall x\forall y(Rxy \rightarrow Ryx) \vdash Rab$}
\RightLabel{$\neg\vdash$}
\UnaryInfC{$Rba, \forall x\forall y(Rxy \rightarrow Ryx), \neg Rab \vdash$}
\RightLabel{$\forall\vdash$}
\UnaryInfC{$Rba, \forall x\forall y(Rxy \rightarrow Ryx), \forall y\neg Ray \vdash$}
\RightLabel{$\exists\vdash$}
\UnaryInfC{$\exists y Rya, \forall x\forall y(Rxy \rightarrow Ryx), \forall y\neg Ray \vdash$}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$Fa \vdash Fa$}
\RightLabel{$\rightarrow\vdash$}
\BinaryInfC{$Fa \rightarrow \exists y Rya, \forall x\forall y(Rxy \rightarrow Ryx), \forall y\neg Ray, Fa \vdash$}
\RightLabel{$\forall\vdash$}
\UnaryInfC{$\forall x(Fx \rightarrow \exists y Ryx), \forall x\forall y(Rxy \rightarrow Ryx), \forall y\neg Ray, Fa \vdash$}
\RightLabel{$\vdash\neg$}
\UnaryInfC{$\forall x(Fx \rightarrow \exists y Ryx), \forall x\forall y(Rxy \rightarrow Ryx), \forall y\neg Ray \vdash \neg Fa$}
\RightLabel{$\vdash\rightarrow$}
\UnaryInfC{$\forall x(Fx \rightarrow \exists y Ryx), \forall x\forall y(Rxy \rightarrow Ryx) \vdash \forall y\neg Ray \rightarrow \neg Fa$}
\RightLabel{$\vdash\forall$}
\UnaryInfC{$\forall x(Fx \rightarrow \exists y Ryx), \forall x\forall y(Rxy \rightarrow Ryx) \vdash \forall x(\forall y\neg Rxy \rightarrow \neg Fx)$}
\end{prooftree}
$$

Note that the rule for $\rightarrow$ on the left has been amended slightly to allow the left and right sequents above the line to share the side formulae rather than both having all of them. This makes no essential difference, but reduces the amount of clutter when the proof is written out.

Proof 4

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Fb,\ \exists yFy\ \vdash\ Fa,\ Fb}\ \mathrm{A}}{Fb\ \vdash\ \exists yFy\to Fb,\ Fa}\ {\vdash}{\to}}{Fb\ \vdash\ \exists x(\exists yFy\to Fx),\ Fa}\ {\vdash}\exists}{\exists yFy\ \vdash\ \exists x(\exists yFy\to Fx),\ Fa}\ \exists{\vdash}}{\vdash\ \exists x(\exists yFy\to Fx),\ \exists yFy\to Fa}\ {\vdash}{\to}}{\vdash\ \exists x(\exists yFy\to Fx)}\ {\vdash}\exists$$

The proof essentially involves contraction (at the last step) and also requires the sequents to be multiple on the right. The contraction move is tricky in the sequent calculus notation, though somehow in the corresponding tableau it happens easily.