1. What do SAT solvers do? They get a formula (or a set of formulae) and find one (or more) interpretations that make it true. So, since we can't ask for $X \models A$, can we ask for $(A_1 \wedge ... \wedge A_n) \rightarrow A$ instead? (Assuming $X = \{A_1, ..., A_n\}$.)

   No! Consider two examples: $a, a \rightarrow b \models b$ will turn into $(a \wedge (a \rightarrow b)) \rightarrow b$. What will L4F tell? And what does this tell you? Now consider another sequent: $a \vee b \models a \wedge b$. Is it valid? This turns into $(a \vee b) \rightarrow (a \wedge b)$. What will L4F tell? And what does it tell you? In particular when you compare the two sequents? **Discuss in class!**

If you type in $(a \wedge (a \rightarrow b)) \rightarrow b$ into L4F it will output multiple models. If you increase the number of solutions in the settings, you will see that it then outputs four models, one for each of the possible interpretations.

Then, if you see what L4F outputs given the constraint $(a \vee b) \rightarrow (a \wedge b)$, it will output two satisfying constraints. This formula is not a tautology, since as L4F tells us, not every possible interpretation will evaluate to true.

So, does this mean we can use L4F to determine if a sequent is valid by converting it to an implication and seeing if it outputs a model for every possible interpretation? Possibly, but this is impractical - we would have to check that it does so, and this approach will not scale well for sequents with a lot of variables.

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2. Now, consider the deduction equivalence, i.e., $X, A \vdash B$ iff $X \vdash A \rightarrow B$. By using it (and some other intermediate steps), we can turn a sequent $A_1, ..., A_n \vdash A$ into $\vdash (A_1 \wedge ... \wedge A_n) \rightarrow A$. Recall what that means! $(A_1 \wedge ... \wedge A_n) \rightarrow A$ is a tautology. SAT solvers (like L4F) can however not find out that a formula is a tautology, they can only check whether there's a satisfying interpretation. So, how can a SAT-solver be used to find out whether a formula is tautology? **Discuss in class!**

Recall the definitions of *tautology*, *satisfiability* and *unsatisfiability* that you discussed in the week 2 tutorials. A formula is a tautology if it is always true regardless of valuation, satisfiable if there is at least one valuation that makes it true, and unsatisfiable if there is no valuation that makes it true. Also, recall that a formula, given a valuation, will be true if and only if the same valuation makes its negation false. So another way of defining *tautology* is that a formula A is a tautology if and only if $\neg A$ is unsatisfiable.

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3. Finally, putting it all together, how to use a SAT solver to to prove $A_1, ..., A_n \models A$ with a SAT solver? **Discuss in class!**

$(A_1 \wedge ... \wedge A_n) \to A$ will be a tautology if $\neg((A_1 \wedge ... \wedge A_n) \to A)$ is unsatisfiable. This is the same as asking L4F to find a satisfying interpretation to the constraints $(A_1 \wedge ... \wedge A_n)$ and $\neg A$. If L4F says that it cannot find any models, it is a tautology.

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4. Although we now already know how we could check $X \models A$, there's also another way. Consider the following:

   - With the negation introduction rule, we know that we can derive $X \vdash \neg A$ from $X, A \vdash \bot$. (And note that this is equivalent to deriving $X \vdash A$ from $X, \neg A \vdash \bot$.)
   - By realising that A is equivalent to $(\neg A) \to \bot$, we can exploit the deduction equivalence to derive $X, \neg A \vdash \bot$ from $X \vdash A$.

   $\to$ So, in total we just showed that $X \vdash A$ holds iff $X, \neg A \vdash \bot$.

It is possible to define negation in terms of $\to$ and $\bot$. $\neg A$ is equivalent to $A \to \bot$ (try drawing the truth table if you're not convinced). A is equivalent to $\neg\neg A$ (by the $\neg\neg$ introduction and elimination rules), which can also be expressed as $\neg A \to \bot$.

The deduction equivalence means that $X \vdash \neg A \to \bot$ holds iff $X, \neg A \vdash \bot$ holds. This is just an application of the $\to I$ rule in the right-to-left direction and $\to E$ in the left-to-right.

$X \vdash \neg A \to \bot$ is equivalent to $X \vdash \neg\neg A$, which is equivalent to $X \vdash A$ (by applying the $\neg\neg$E rule), we can see that $X \vdash A$ holds if and only if $X, \neg A \vdash \bot$.

$$\dfrac{\dfrac{\dfrac{X, \neg A \vdash \bot}{X \vdash \neg A \to \bot}\to I}{X \vdash \neg\neg A}}{X \vdash A}\neg\neg E \qquad \dfrac{\dfrac{\dfrac{X \vdash A}{X \vdash \neg\neg A}\neg\neg I}{X \vdash \neg A \to \bot} \qquad \neg A \vdash \neg A}{X, \neg A \vdash \bot}\to E$$
