Logic: Tutorial Week 6

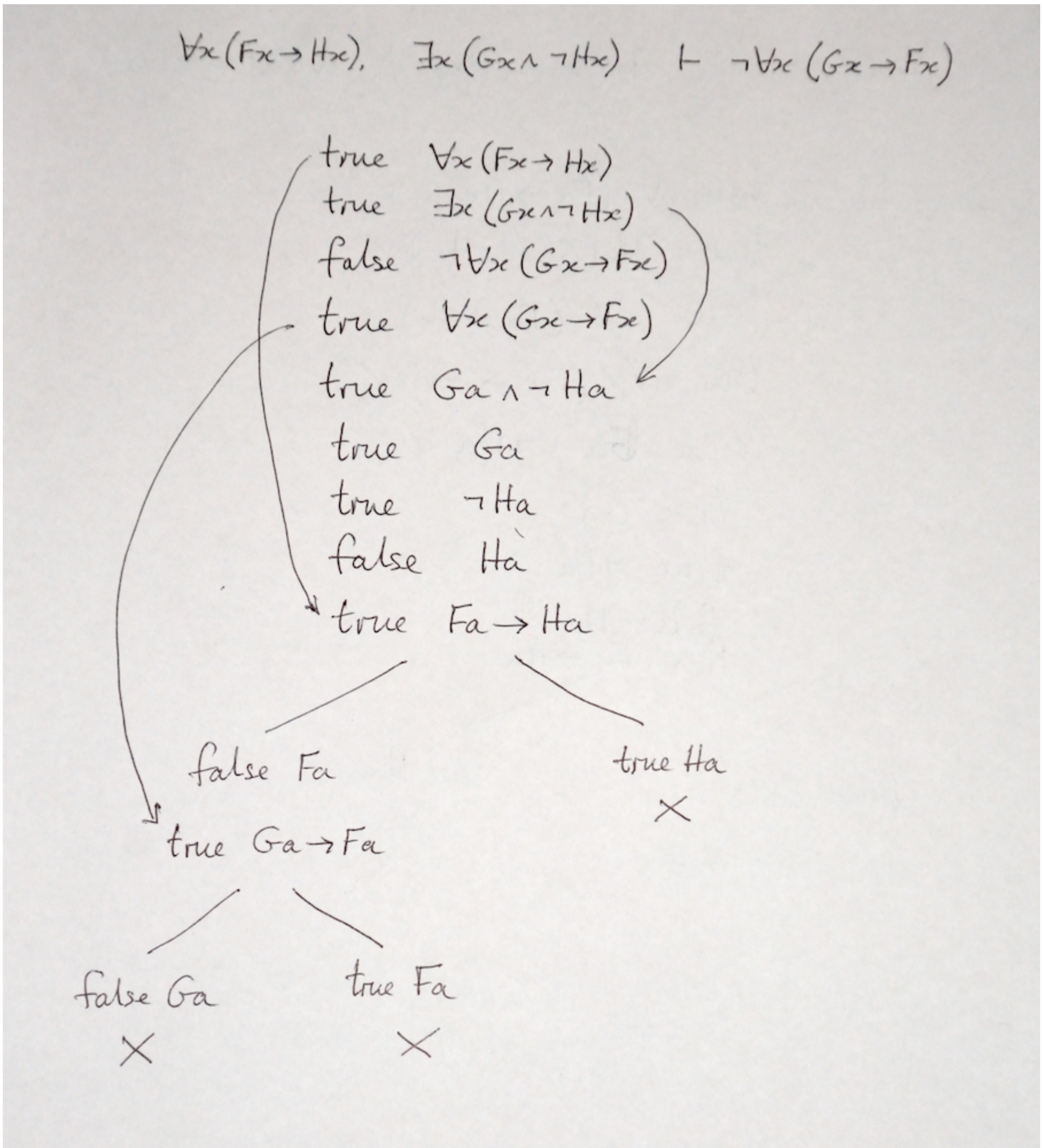
All (or at least most) of the solutions provided here are not in plain LaTeX, but instead hand-drawn graphics (still created by our predecessor), we hope that's okay to read.

You will see that there are some minor differences to the notation we use in our course:

- The lables are simetimes t/f, sometimes $true/false$. Although it's clearly obvious what they mean, please be consistent and use only T/F as used in the lecture.
- Sometimes lines are not even numbered: You should always number them.
- For multiple branches the same line number is used. We prefer using different line numbers in such cases.
- Sometimes no line numbers are used at all, but instead arrows are used to show where some line comes from. Semantically that is perfectly fine (it serves exactly the same purpose), but we still ask you to use line numbers for the sake of consistency.
- Contradictory branches are not indicated as such using the lightning bolt (that's usually used in math for contradictions), but instead with a big X. We prefer consistency, so please use the lightning bolt.
- Some exercises us a format notation for the interpretation that is only going to be introduced during week 7. You can re-visit this notation after the respective content was taught. Until then you can simply use the notation that was used until week 6, i.e., an interpretation (in case of an open branch!) simply "collects" all the truth assignments to atoms, which thereby shows yuo which predicates are true for which objects (constants) thereby showing that it is possible to make all the assumptions true while making the formula on the right false (thus contradicting validity).

Proof 1

$$\forall x(Fx \to Hx), \exists x(Gx \land \neg Hx) \vdash \neg\forall x(Gx \to Fx)$$



Proof 2

$$\forall x(Fx \to Hx), \exists x(Gx \land \neg Hx) \vdash \neg \forall x(Fx \to Gx)$$

$\forall x(Fx \to Hx),\ \exists x(Gx \land \neg Hx) \vdash \neg\forall x(Fx \to Gx)$

true $\forall x(Fx \to Hx)$ ✓ ^{a}

true $\exists x(Gx \land \neg Hx)$ ✓

false $\neg\forall x(Fx \to Gx)$ ✓

true $\forall x(Fx \to Gx)$ ✓ ^{a}

true $Ga \land \neg Ha$ ✓

true Ga

true $\neg Ha$ ✓

false Ha

true $Fa \to Ha$ ✓

false Fa	true Ha
true $Fa \to Ga$ ✓	×

false Fa	true Ga

already present in the branch

Domain = $\{\alpha\}$. $I(a) = \alpha$.

$I(F) = \emptyset$

$I(G) = \{\alpha\}$

$I(H) = \emptyset$

Proof 3

$$\forall x(Fx \to Hx), \forall x(Gx \to \neg Hx) \vdash \forall x(Fx \to (Gx \to Rxx))$$

(1) t: ∀x(Fx → Hx)

(2) t: ∀x(Gx → ¬Hx)

(3) f: ∀x(Fx → (Gx → Rxx)

(4) f: Fa → (Ga → Raa) instance of 3

(5) t: Fa

(6) f: Ga → Raa

(7) t: Ga

(8) f: Raa

(9) t: Fa → Ha from 1

(10) f: Fa | t: Ha

(11) × | t: Ga → ¬Ha from 2

(12) f: Ga | t: ¬Ha

(13) × | f: Ha

×

All branches close: sequent is valid

Proof 4

$$\forall x\,\forall y\forall z((Rxy \wedge Ryz) \to Rxz), \forall x\forall y(Rxy \to Ryx) \vdash \forall x\forall y(Rxy \to Rxx)$$

1. $t: \forall x\forall y\forall z((Rxy \wedge Ryz) \to Rxz)$
2. $t: \forall x\forall y(Rxy \to Ryx)$
3. $f: \forall x\forall y(Rxy \to Rxx)$
4. $f: \forall y(Ray \to Raa)$ from 3
5. $f: Rab \to Raa$ from 4
6. $t: Rab$ from 5
7. $f: Raa$ from 5
8. $t: \forall y(Ray \to Rya)$ from 2
9. $t: Rab \to Rba$ from 8
10. $f: Rab$ – from 9 – $t: Rba$

    × (6)

11. $t: \forall y\forall z((Ray \wedge Ryz) \to Raz)$ from 1
12. $t: \forall z((Rab \wedge Rbz) \to Raz)$ from 11
13. $t: (Rab \wedge Rba) \to Raa$ from 12
14. $f: Rab \wedge Rba$ — from 13 — $t: Raa$

    × (7)

15. $f: Rab$ – from 14 – $f: Rba$

    × (6) × (10)

Proof 5

$$\exists x \exists y Rxy \vdash \exists x \exists y Ryx$$

t: $\exists x \exists y\, Rxy$

f: $\exists x \exists y\, Ryx$

t: $\exists y\, Ray$

t: Rab

f: $\exists y\, Ryb$

f: Rab

×

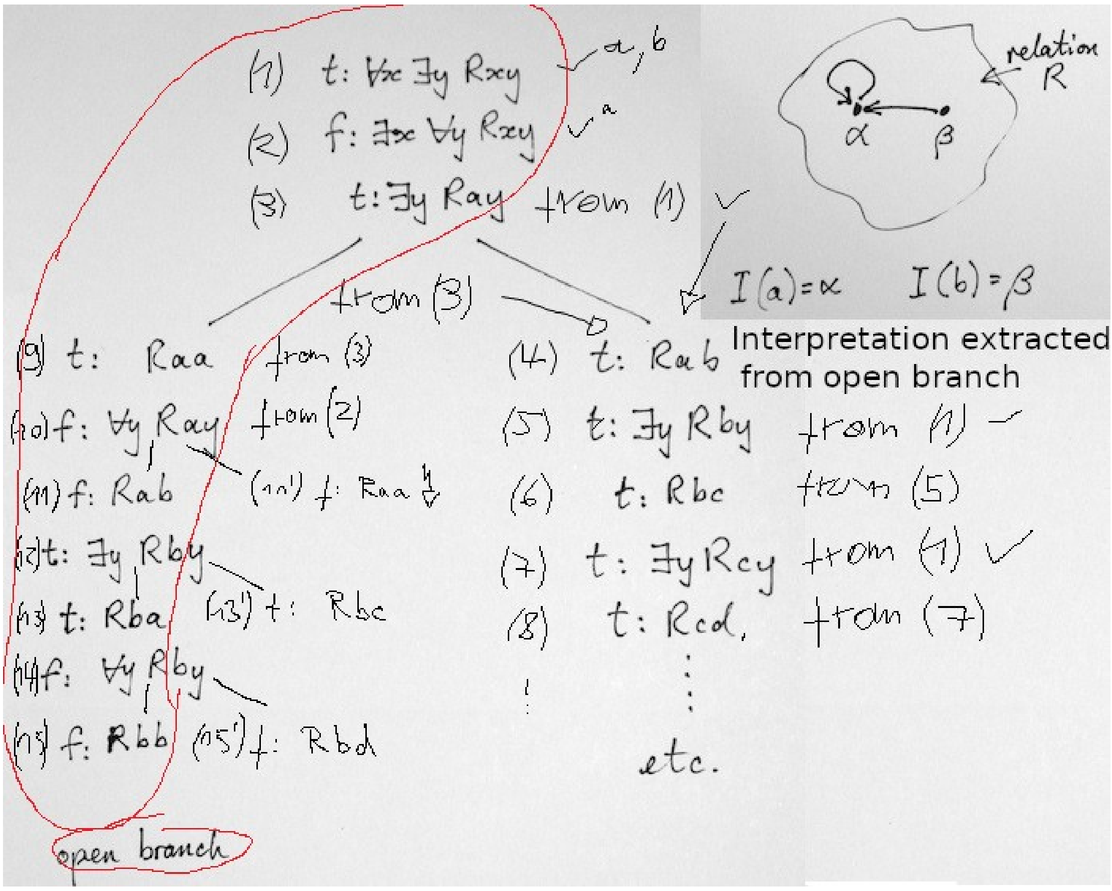
premise true

conclusion false

Sequent is valid.

Proof 6

$$\forall x \exists y Rxy \vdash \exists x \forall y Rxy$$



This is another example where we need to use the "updated" true existential elimination rule for invalid sequents – as well as the similar (dual) rule for false universal quantifiers (note that the latter was not included in the main lecture, but it works in *exactly* the same way as the (updated) true existential rule, similar to how the (standard) true existential behaves like (standard) false universal and how the (standard) false existential behaves like the (standard) true universal, cf. page 10 of the cheat sheet – and note that page 11 now also contains the (updated) false universal rule).

Semantic Tableaux can always prove the validity of a valid sequent, and sometimes the invalidity of invalid sequents. However, not knowing whether a sequent is valid or not makes the proof much harder, because then you might not know the "proof strategy" that you are supposed to attempt. Keep in mind that the "updated" rules (cf. page 11 of the cheat sheet) are only for invalid sequents, so it would be great if you knew in advance whether a sequent is valid or not so that you know whether these rules are available to you. So if we don't tell you whether a sequent is valid or not, it's worth thinking a minute or two about whether it's valid or not, so you can be more efficient doing the proof. (Also note that technically it's not correct that the "updated rules" are only "available" to you in case a sequent is invalid, because these rules technically substitute their "standard" counter parts. However, the left branches of these rules (which use already existent constants) can only be used in case a sequent is invalid, otherwise they can't.)

Let's start analyzing the proof!

The right branch is the one that you probably produce first (potentially without even knowing that you will produce another branch next to it since you might not have seen yet that the sequent is invalid), since that one relies on the standard rule for true existential elimination. This is why the line number starts with (4).

But (at the latest) after noticing that this results into an infinite branch (thus preventing you from either producing a contradiction or providing an open branch and thus an interpretation) you recall that we can branch over existing variables for true existential quantifiers (in case the sequent is invalid) and create an alternative branch on the left by instantiating line (3) with an already existing constant. We wrote down line number (9) here, but in principle you could even write down (4) and cross out the right branch as that branch did not help anyway – so if you would have seen that earlier you could also have started with the left branch right away.

Even starting to write down the left branch means that you've committed to try proving the invalidity of the sequent, as its pure existence bases on using a rule (or one of its branches) that's only available for invalid sequents. That means that from now on you can always choose any branch of the updated rules that helps you to show that it is open (i.e., you may pick any branch of the "updated" rules so that no contradiction occurs and no more rules are applicable).

In line (10) we have a false universally quantified formula. So we know that we can either pick an existing constant, or create a new one (according to the updated rule for invalid sequents and false universally quantified formulae). Let's assume we choose to pick an already existing constant (to get a shorter proof), then we end up in line (11'), which results in a contradiction. But that doesn't help us since we try to prove the invalidity! But for this we need an *open* branch! So picking this does not help. Thus, instead we use the "standard" version of the rule for false universals (or, equivalently: the right-most branch of the updated version of that rule), leading to line (11) with a new constant b. It's noteworthy that this b is new although it was already used in the right-most (infinite) branch, simply because it's in another branch! In fact, that's what branches mean: they are "different worlds", i.e., different possibilities. So, "new" always refers to the current branch (or "path").

The next line where we have a *choice* on what to do is line (12), where again we can choose to either re-use an existent constant or create a new one. Here, we do *not* want to create a new one (which would create a new constant c as indicated in a new branch, shown in line (13')), because that was our problem in the first place! Creating this new constant lead to the infinite branch on the very right, so instead we re-use the existent constant a leading to line (13).

Finally, we again have the choice what to do in line (14). We could again re-use an existing constant leading to line (15) or to create a new one leading to line (15'). If you want to prove the invalidity (which you do in this branch) it's a good idea to try re-using constants first since you want the rule applications to terminate. This is what the left branch, i.e. line (15) shows. Line (15') is just shown for the sake of completeness. This is the branch where a new constant d gets introduced. However, we do not need or use that branch, since the left one finally produced an open branch. (We only showed that right option to indicate that the rule we had available allows for branching.)

At the end, (due to applying the new/updated rules for invalid sequents) we were able to derive an interpretation based on two constants a and b (which represent the objects α and β in the graphical illustration). Going through all true atoms in the left branch, we obtain $T : Raa$, $T : Rba$, all the rest is false. So we can construct a model in which exactly these relations are true, which do not contradict all our premises. A graphical illustration of that interpretation is shown at the top-right (where the notation introduced in week 7 is used; here, α and β are objects from the domain).

Line (1) told us that for all x there exists some y, such that Rxy is true. This is true for that interpretation, since if x is a we get Raa, and for $x = b$ we get Rba, so that is satisfied. For the second premise of the proof, i.e., line (2), we can also see that it's satisfied: It must be false that there exists an x, such that for all y Rxy holds. Well, we can just check that: For $x = a$ it's not true that Raa and Rab holds (since only Raa holds) and for $x = b$ it's also not true that Rba and Rbb hold (since only Rba holds). You do not need to do this reasoning, providing an interpretation is enough. We just provided this so you understand better what finding an open branch and extracting an interpretation means.

Some final general remark:

Dealing with invalid sequents based on predicate logic is certainly far from trivial. The reason is that for invalid sequents we have more choices: We can either produce a new constant or re-use an existent one, so the more constants there are, the more choices we have. But the ultimate goal (for showing invalidity) is still to obtain an open branch, which means that all lines in that open branch are "completed" in the sense that no further rule can be applied. It's a good idea to indicate all branches for all lines/formulae that do allow branching (like below lines (10), (12), and (14), so you can easily choose a different branch if one doesn't work out (like line (11'), or even (4)).