Logic: Tutorial Week 11

3-valued logic

In this class you are supposed to design your own logic and investigate its properties. I.e., you will design your own "mathematical rules" and then take a deep look whether they have meaningful or strange/counter-intuitive consequences.

Introducing the idea

So far, every formula (no matter how complex, or simple "p" or "$\neg q$") was either true of false, so we had exactly two truth values:

- $\top$ (1, or true)
- $\bot$ (0, or false)

This was both true for propositional logic, as well as for predicate logic. For the sake of simplicity, we may want to restrict to propositional logic today.

In this tutorial, you extend this standard 2-valued (propositional) logic to a three-valued one, so we assume that there is a third value that every statement could potentially evaluate to:

- $\top$ (1, or true)
- $\Box$ (you fill define what that's supposed to mean)
- $\bot$ (0, or false)

For the sake of simplicity, we may call that third value "box" ($\Box$). Why? Because there are MANY possible interpretations of what that third value could "mean", and it seems like the box symbol is a great representation for that. :)

Exercise 1: Find possible meanings for the third value

For the first part, you should discuss (in groups) what that third value could possibly represent. We won't give any advice here, because that's the fun part! There are so many possible answers to this!

You may also want to start the discussion by playing Devil's Advocate: Question why such a third value could be required (and argue that it doesn't and try to convince you or your group otherwise).

Exercise 2: Commit to one interpretation and Formalize it! (Use truth tables)

First, it's important that your group commits to one of the many possibilities. Note that there's no right or wrong answer. It's just about investigating one in more detail, so maybe vote for one or just throw a dice.

If you can't decide, an interesting case (for the next exercise) is "unknown" in the sense of "we still live in a world where each statement *should* be true of false, we just don't know (for some) which one it is".

This is even more fun than the last part! Now you have to create all truth tables for all connectives and decide/discuss in the group how the table must look to correctly implement the meaning that you agreed to.

You should start with the easy cases, i.e., with conjunction, disjunction, and negation. Once they are done, do the implication, which is the most challenging one. Once THAT is done investigate how much sense your logic does by looking at some theorems from 2-valued logic and check out whether they still hold in the one you designed:

- $\models p \vee \neg p$ (law of the excluded middle)
- $\models (p \rightarrow q) \leftrightarrow (\neg p \vee q)$, i.e., $(p \rightarrow q) \equiv (\neg p \vee q)$
- Can you think of other theorems in 2-values logics? (E.g., de'Morgan?) Check whether they are still true and if not, whether that's fine!

Early enough (!!) before the workshop is over, come together again and discuss your findings.