INTRODUCTION TO THE SECOND PART

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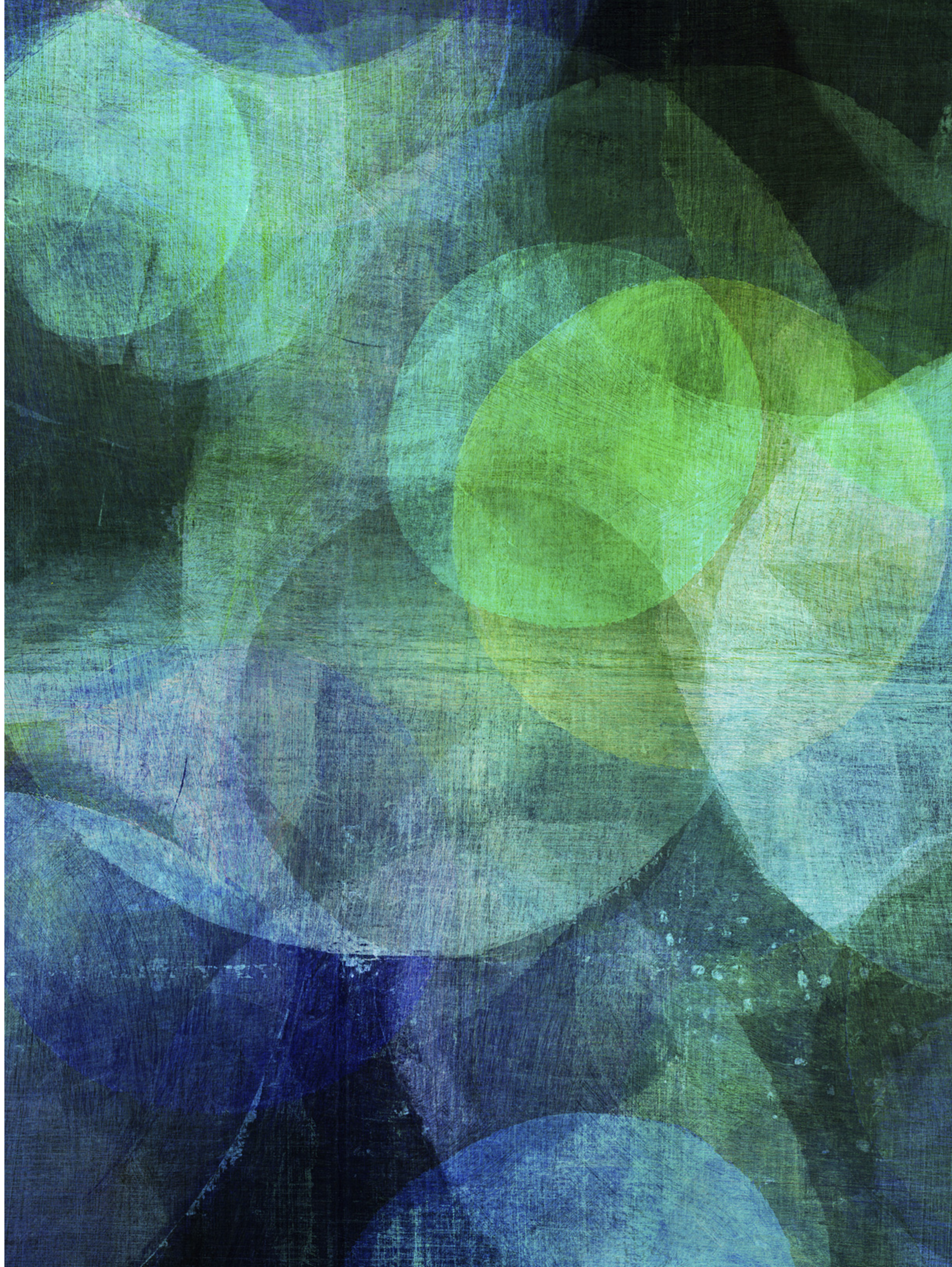

WHAT IS IT ALL ABOUT?

- We have learned the basics of logic (natural deduction calculus, truth table semantics, etc.).
- What is logic all about? What is it doing? And why?
- We applied formal rules so many times, but why? Why do we apply rules and prove theorems? What is the meaning or significance of doing so?
- Why do we need a proof system in the first place? Why do we have to play such a mechanical rule-following game?
- Logic is not a fancy symbol manipulation game. We are doing something more meaningful.

WHAT IS IT ALL ABOUT THEN?

Logic is about characterising the infinite truths in finitary terms; there are infinitely many semantically valid formulae, but they can be finitely generated by a finitary proof system.

Logic is about finitary means to characterise (generate) infinitary truths.

NB. Wittgenstein is one of the discoverers of truth table semantics

Logicomix (2008, 2009)

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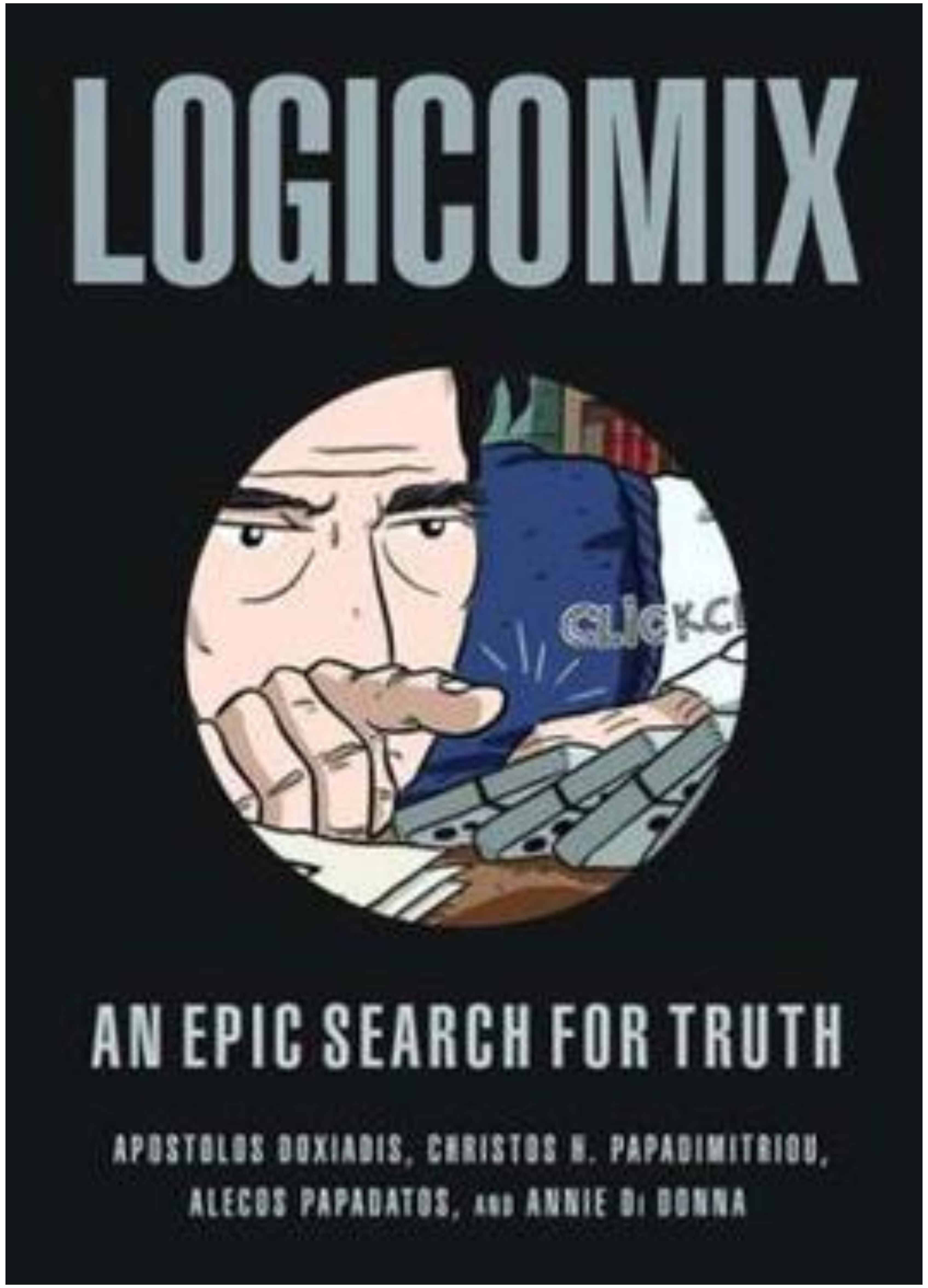
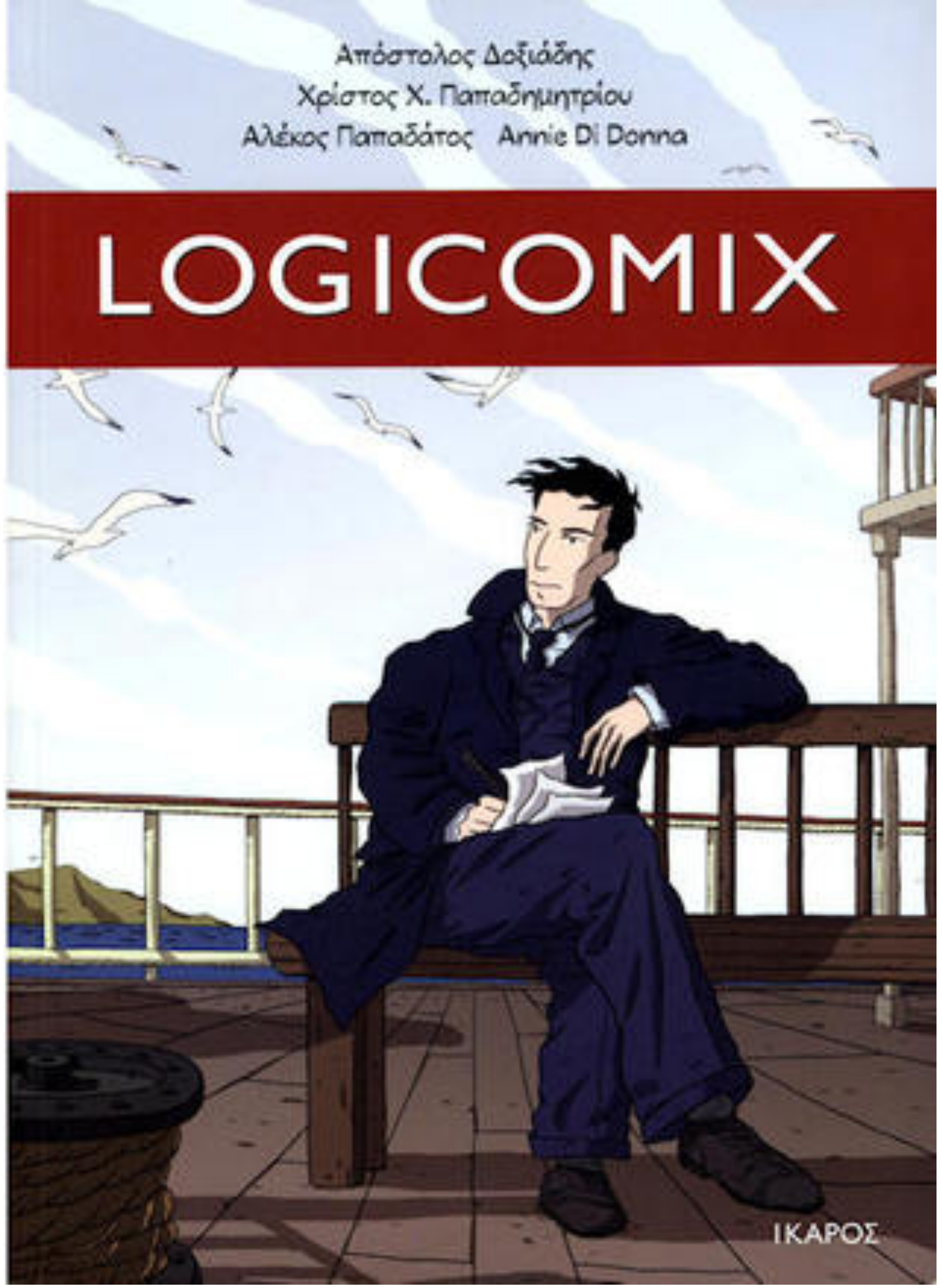




Highly enjoyable,

but not always precise

WHAT IS IT ALL ABOUT THEN?

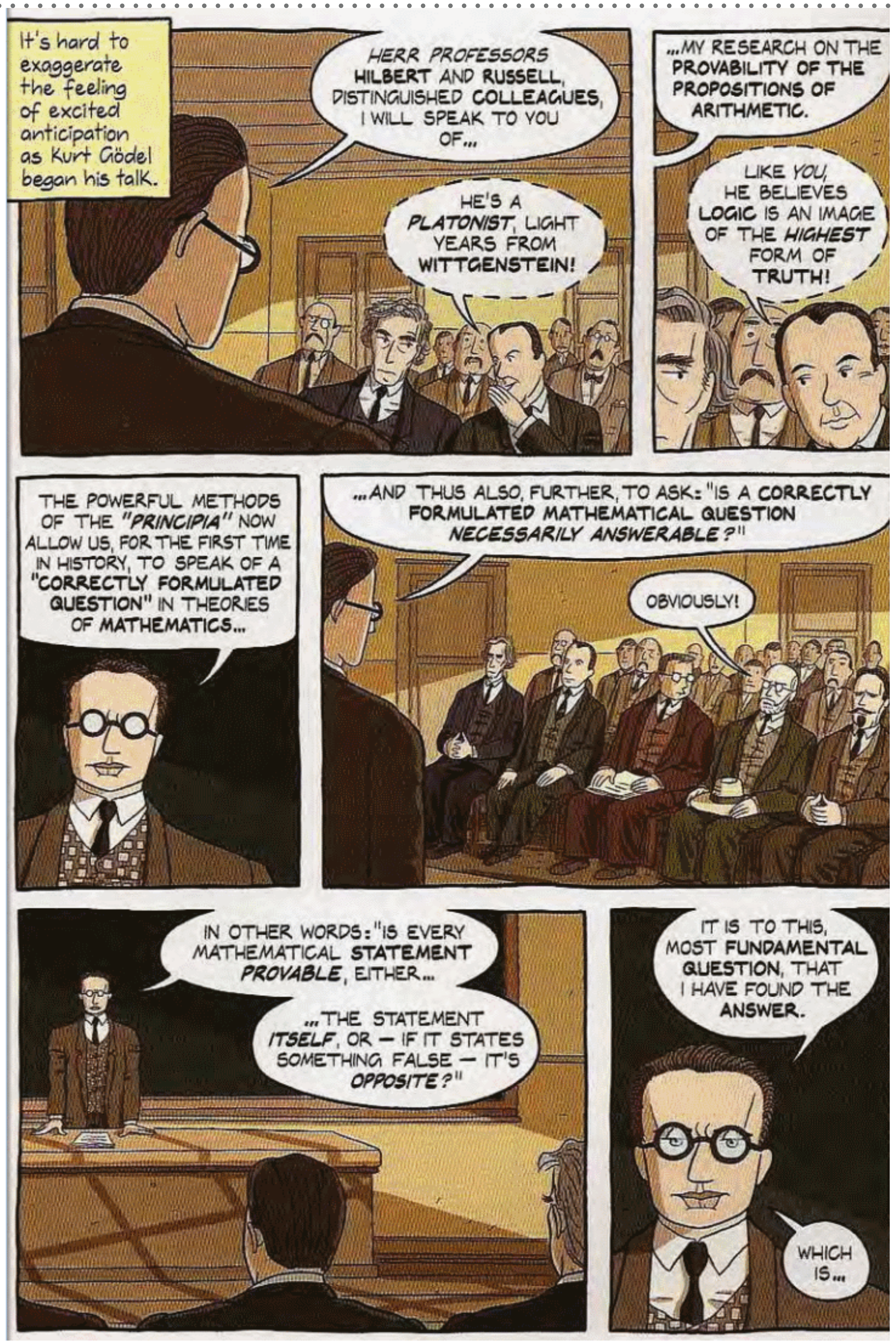
- It is not obvious whether there is any finitary means to characterise (generate) all truths.

- Completeness theorems tell that is possible. It is possible to characterise the infinite set of all logical truths by finitary means, namely by a proof system. Natural deduction allows us to derive all logical truths by finitary proof procedures.

- Incompleteness theorems tell it is impossible; e.g., it is impossible to characterise the infinite set of all arithmetical truths by any finitary means, i.e., by any (computable) proof system.

- There is a mechanical procedure (program) to generate all logical truths, but no such procedure (program) to generate all arithmetical truths, let alone all mathematical truths. A small number of rules allow us to generate all truths, which is a highly non-trivial mathematical phenomenon.

GÖDEL ANNOUNCES INCOMPLETENESS THEOREMS



QUOTES FROM GREAT LOGICIANS

- "Very few logical concepts and axioms" allow us to generate all truths, which is amazing. Note: the human being is a finitary entity, and only comprehends finitary things.

- Gödel (1944): "It was only [Russell's] Principia Mathematica that full use was made […] for actually deriving large parts of mathematics from a very few logical concepts and axioms."

"Large parts" do not mean "all"!

https://en.wikipedia.org/wiki/Logicism

- Kleene (1952): "Leibniz (1666) first conceived of logic as a science containing the ideas and principles underlying all other sciences." Can we generate all scientific truths via logic?

- (It's open, but we can generate physical (e.g., quantum-physical) truths via logic (there is even an automated reasoning system, called Quantomatic, based on category-theoretical logic).)

JUST SEVEN AXIOMS ALLOW YOU TO DERIVE ALMOST EVERYTHING

PA1,…,PA7 $\vdash$ *almost all truths*!

Peano Axioms for natural numbers

PA1 $\forall x(\neg(s(x) = 0))$

PA2 $\forall x \forall y(s(x) = s(y) \to x = y)$

PA3 $\forall x(x + 0 = x)$

PA4 $\forall x \forall y(x + s(y) = s(x + y))$

PA5 $\forall x(x \cdot 0 = 0)$

PA6 $\forall x \forall y(x \cdot s(y) = x \cdot y + x)$

PA7 $[A(0) \wedge \forall x(A(x) \to A(s(x)))] \to \forall x A(x)$

=: binary relation symbol

0: constant symbol

s: “+1” function symbol

+: “add” function symb.

・:“multiply” fun. sym.

PA7: induction axiom

(NB. Just to give a broader perspective on logic; you don’t have to understand this.)

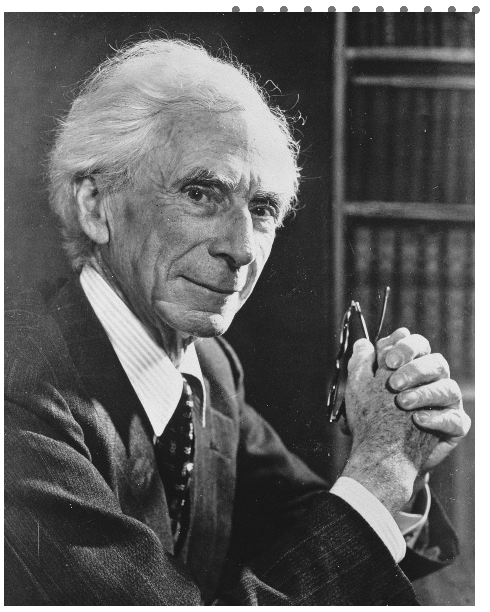
https://www.chegg.com/homework-help/questions-and-answers/relation-u-greater-equal-v-natural-numbers-n-defined-formula-ez-u-z-v--give-formal-proof-u-q64888301

OVERVIEW OF THE SECOND PART

- Semantics of first-order logic; soundness and completeness of first-order logic
- Another (arguably easier) proof system: sequent calculus (a method to prove the consistency of mathematical theories, which is an origin of modern logic. Is the logic we learned consistent? Why are we not able to derive contradictions? We can already prove it semantically.)
- Additional ingredients of first-order logic: identity (equality) and restricted quantifiers
- Introduction to non-classical logic and analyses of paradoxes: modern formal logic allows us to give different mathematical solutions to philosophical paradoxes; simplistically speaking, formal logic allows us to resolve philosophers' long-standing worries since Greek philosophy.

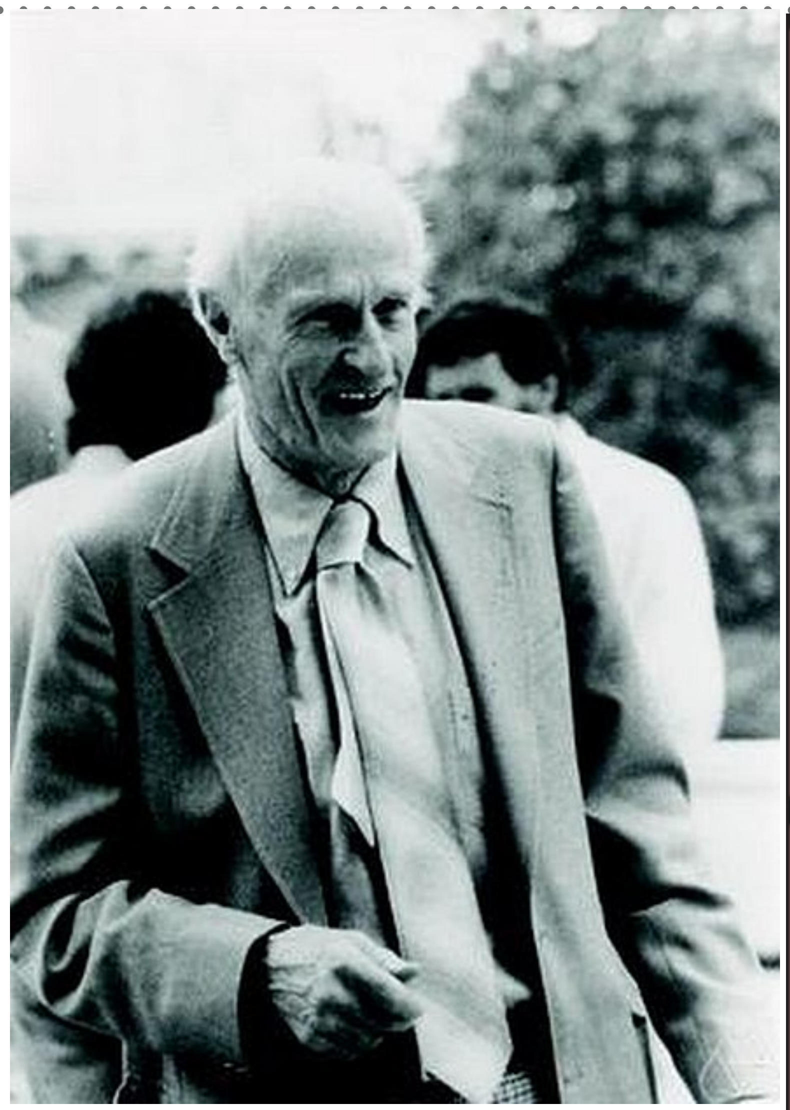
Appendix *(Just for Fun)*

PRINCIPIA MATHEMATICA (1910–1913) AND KLEENE'S ARITHMETICAL HIERARCHY



Just to give a broader perspective on logic.

You do not have to understand this at all!



$*54{\cdot}43.\quad \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota\text{‘}x . \beta = \iota\text{‘}y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51{\cdot}231] \qquad \equiv . \iota\text{‘}x \cap \iota\text{‘}y = \Lambda .$

$[*13{\cdot}12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11{\cdot}11{\cdot}35 . \supset$

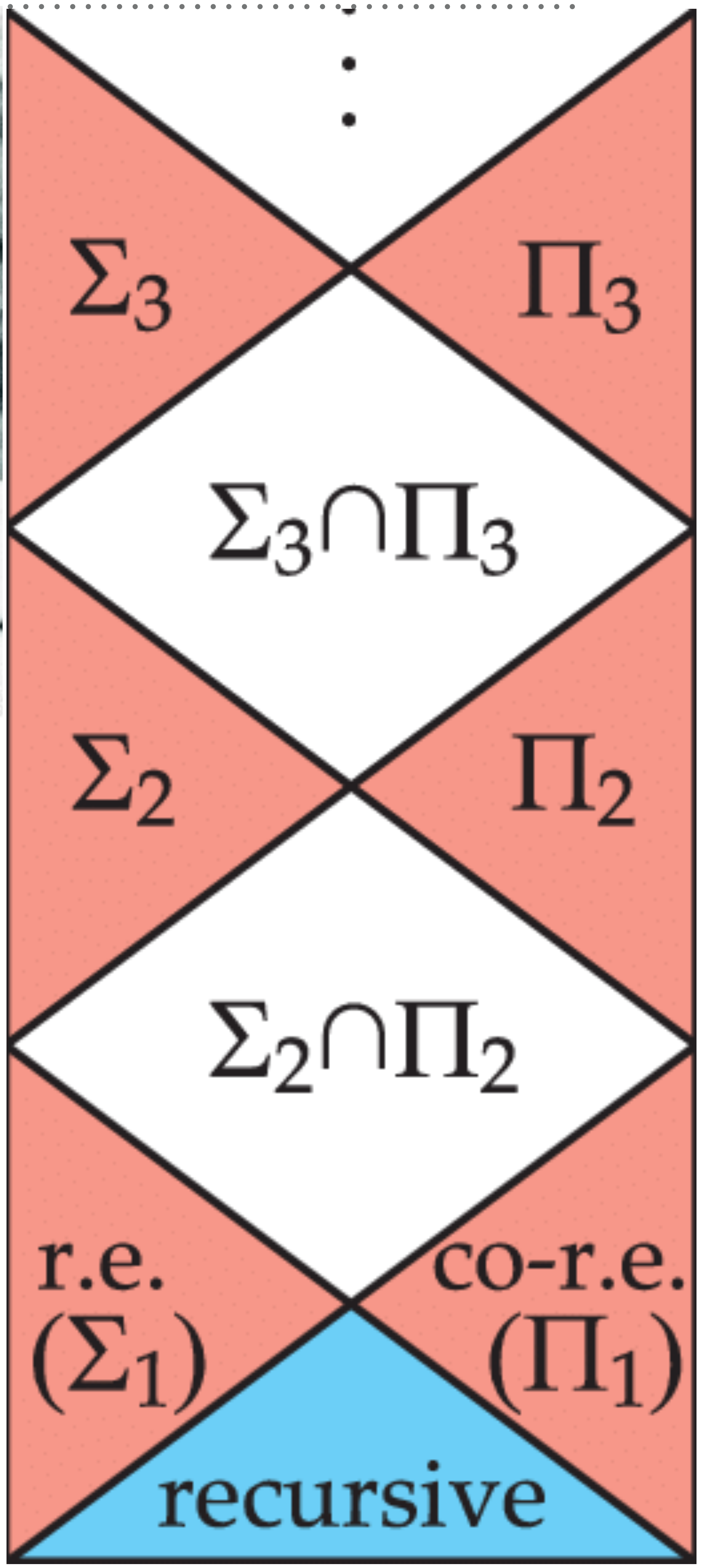
$\vdash :. (\exists x, y) . \alpha = \iota\text{‘}x . \beta = \iota\text{‘}y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash . \text{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

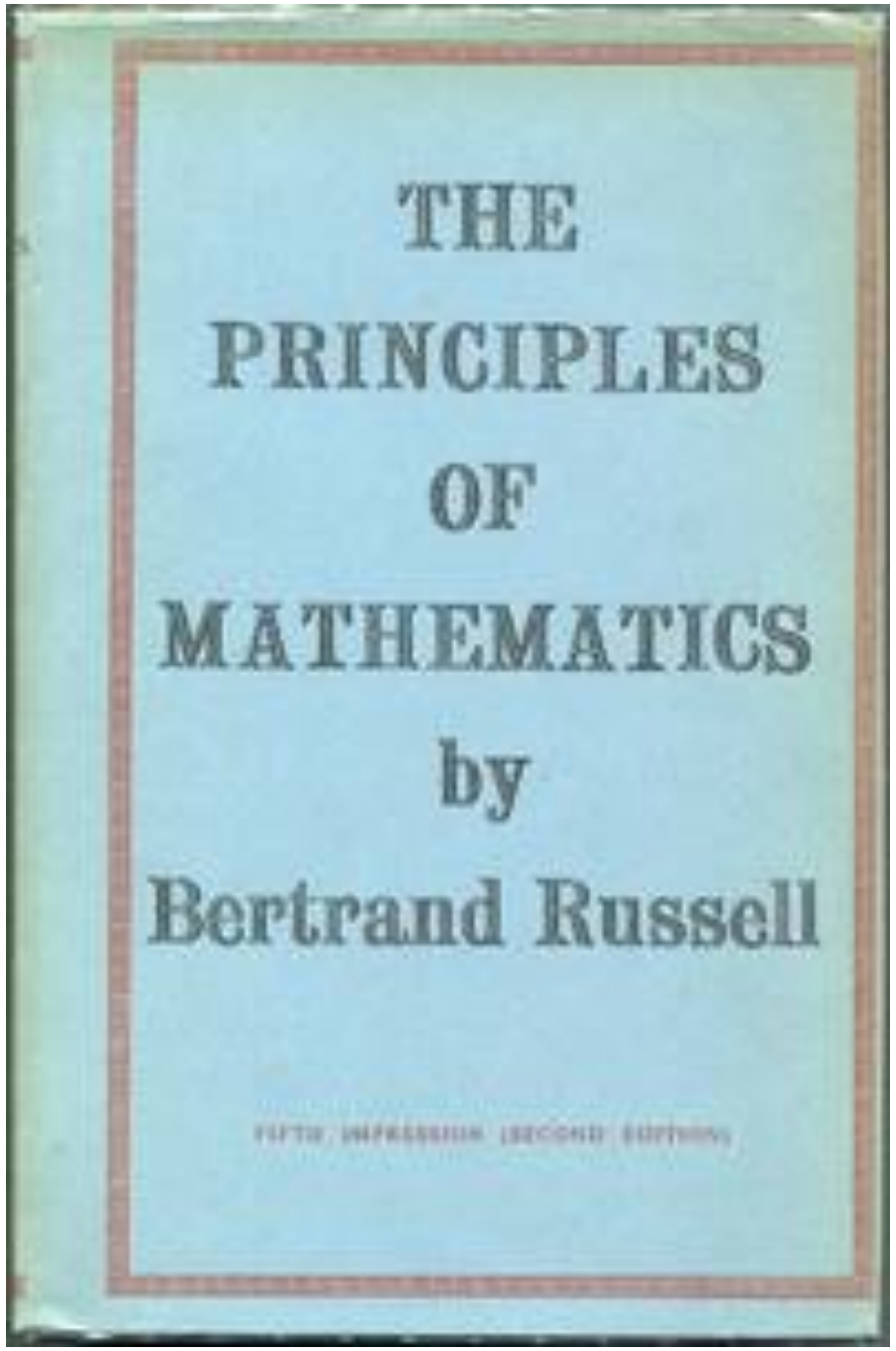
Russell and Whitehead prove that 1+1=2 !

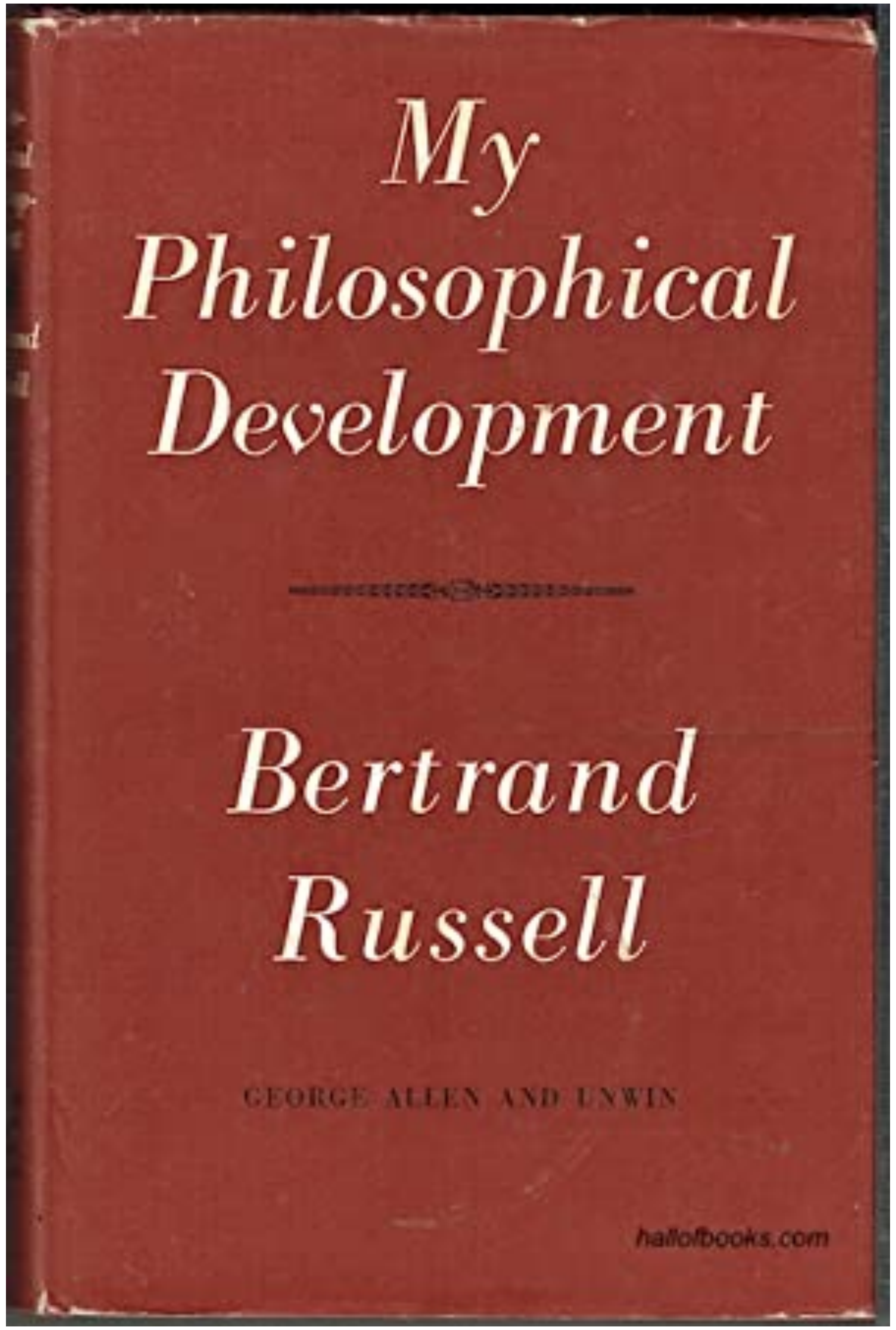
Incompleteness is about complexity of truths:



https://www.researchgate.net/figure/The-arithmetical-hierarchy_fig1_220520596

RUSSELL (1903) AND RUSSELL (1959)





It is said Russell did not understand the differences between ∅ and {∅}, between syntax and semantics. Logic was difficult even for him. Still he understood what logic is (and enjoyed it).

Even if you are currently struggling with formals proofs and/or L4F, you can still become Russell, one of the greatest logicians (and enjoy logic).

https://www.biblio.com/book/principles-mathematics-russell-bertrand/d/1287662772

https://www.abebooks.com/servlet/BookDetailsPL?bi=30812221982&cm_sp=SEARCHREC-_-WIDGET-L-_-BDP-F&searchurl=an%3Dbertrand%2Brussell%26fe%3Don%26sortby%3D17%26tn%3Dmy%2Bphilosophical%2Bdevelopment

RUSSELL AND ELIOT

- An interesting story about Bertrand Russell, one of the founders of (modern) symbolic logic and analytic philosophy (~ philosophy based on logical analysis).

- When Bertrand Russell died, Valerie Eliot contributed the following message to The Times: “My husband, T.S. Eliot, loved to recount how late one evening he stopped a taxi. As he got in, the driver said: ‘You’re T.S. Eliot’. When asked how he knew, he replied: ‘Ah, I’ve got an eye for celebrity. Only the other evening I picked up Bertrand Russell, and I said to him: ‘Well, Lord Russell, what’s it all about’, and do you know, he couldn’t tell me.’”

- Russell still knew what logic is about, as he says (Principles of Mathematics, 1903): “The present work has two main objects. One of these, the *proof* that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles”