SEQUENT CALCULUS

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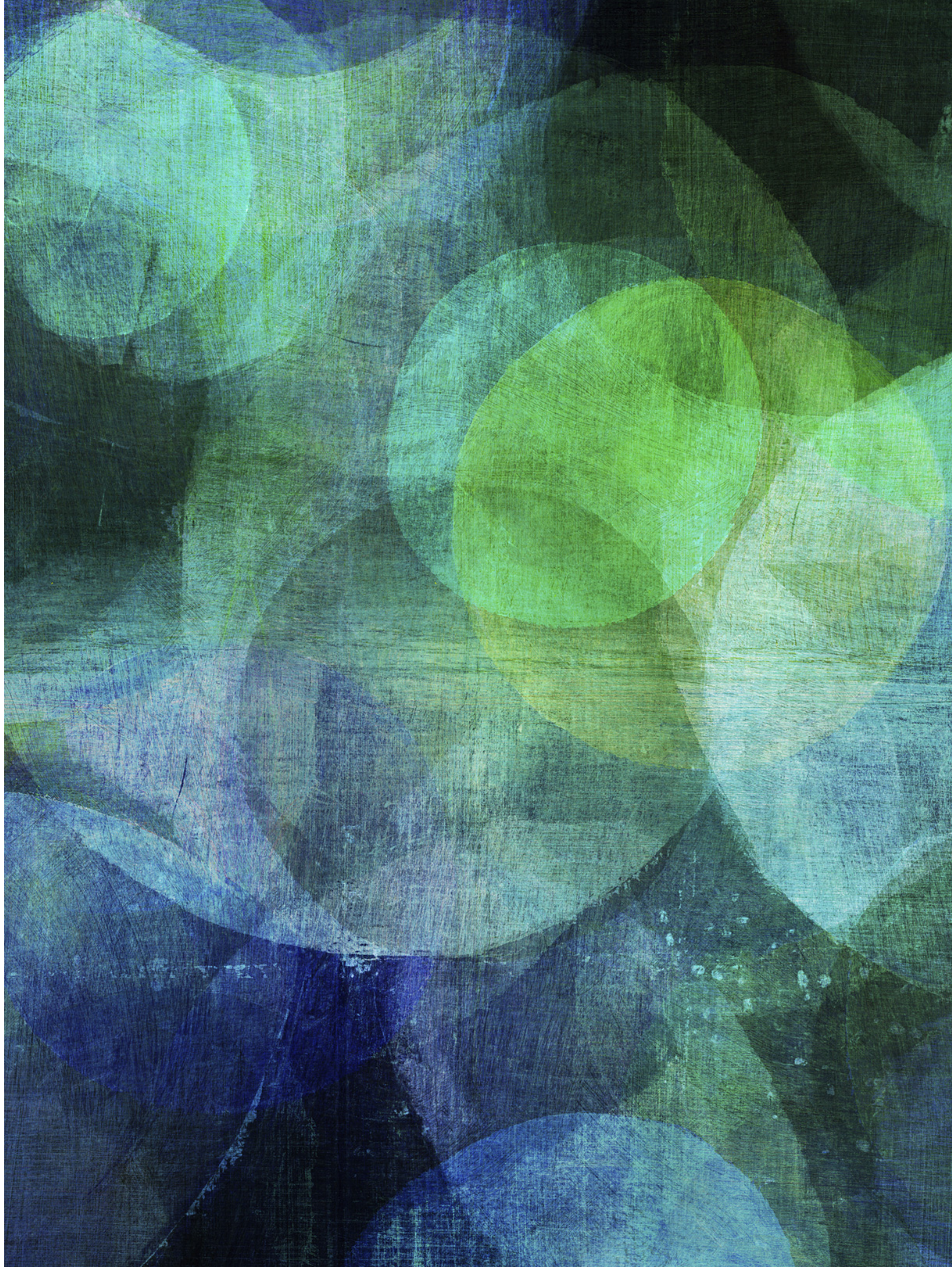
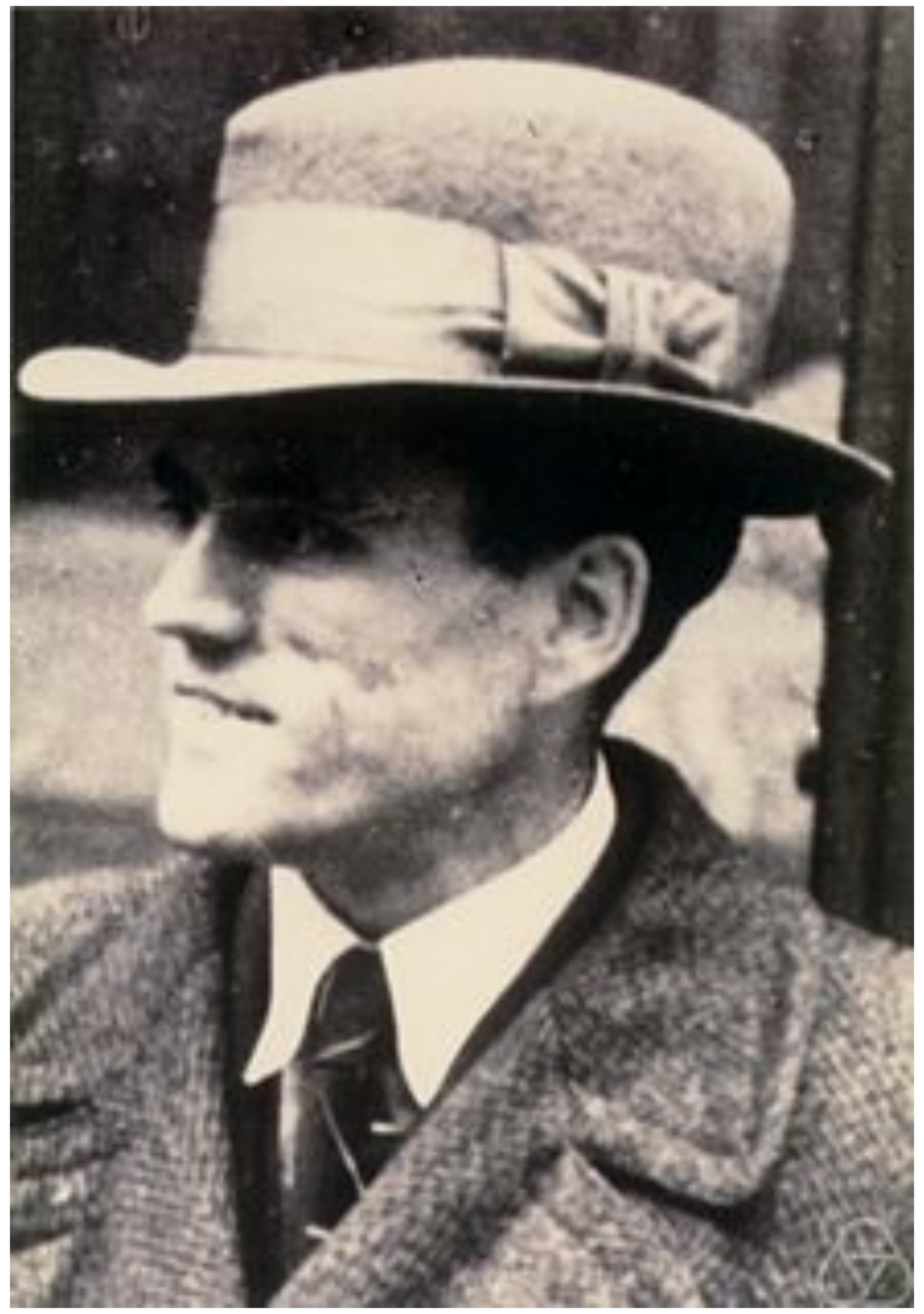


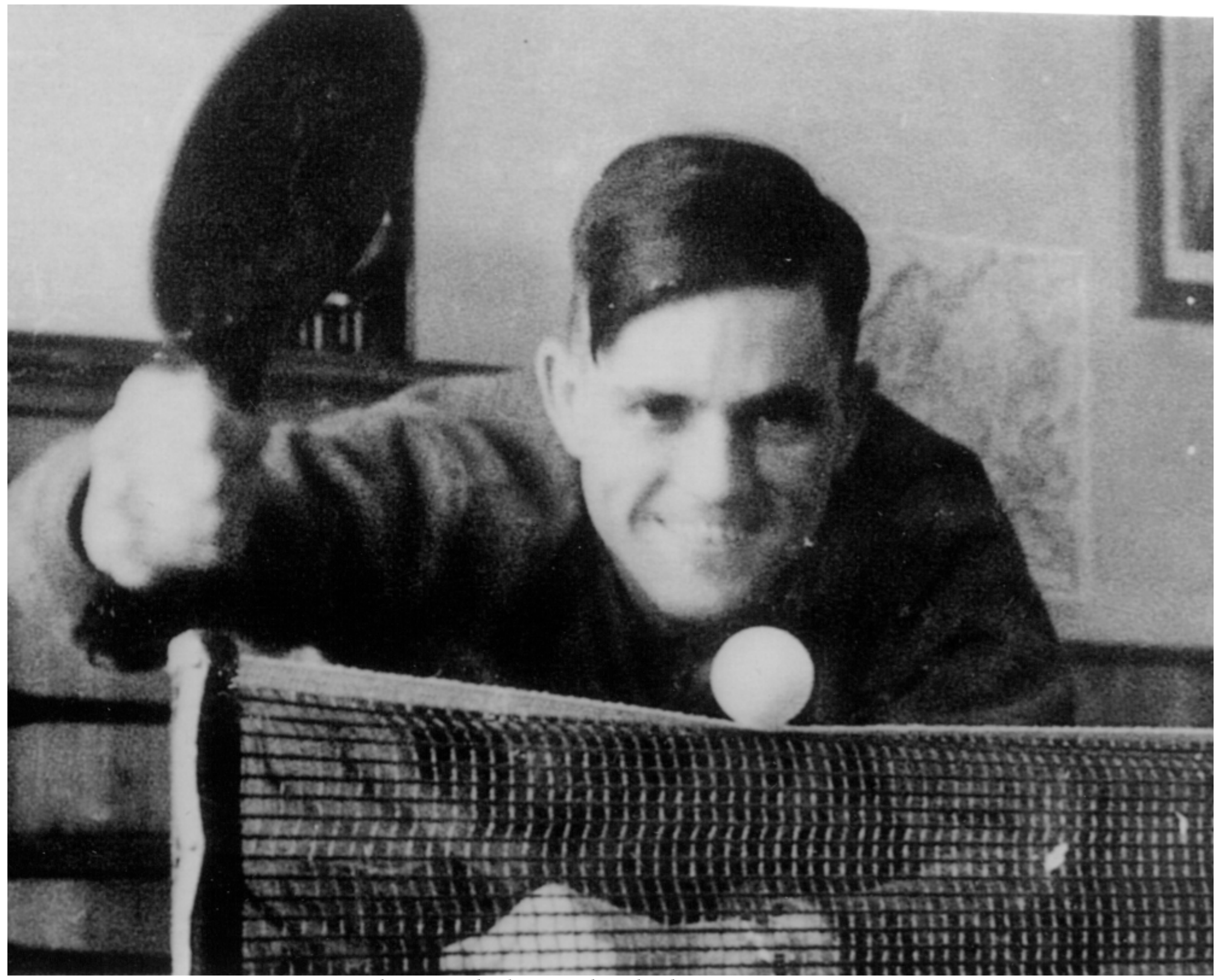
INTRODUCTION: FROM NATURAL DEDUCTION TO SEQUENT CALCULUS

Gentzen is the discoverer of natural deduction (1934; Szabo's translation):

"We wish to set up a formalism that reflects as accurately as possible the actual logical reasoning involved in mathematical proofs."

Is natural deduction so natural? If so, why do we struggle with ND proofs of trivial logical truths such as "A or not A". If it is natural, everything should follow naturally.





https://prabook.com/web/gerhard.gentzen/3757132

Gerhard Gentzen (1909-1945)

Died at 35 for starvation (WWII)

Gödel died for starvation too, yet for a different reason (mental issue)

INTRODUCTION (CONT'D)

Gentzen discovered both natural deduction and sequent calculus. Why did he need sequent calculus in addition to natural deduction?

Five years after Gödel's discovery of incompleteness theorems (one of which says it is impossible to prove the consistency of a system within that system), Gentzen proved the consistency of arithmetic via sequent calculus.

Sequent calculus is now broadly applied in logic and computer science (and even physics) as well as for consistency proofs in foundations of mathematics.

INTRODUCTION (CONT'D)

"Gentzen was motivated by a desire to establish the consistency of number theory. He was unable to prove the main result required for the consistency result, the cut elimination theorem—the Hauptsatz—directly for natural deduction. For this reason he introduced his alternative system, the sequent calculus, for which he proved the Hauptsatz" (Hauptsatz in logic means cut elimination theorems in sequent calculus.)

Quote from: https://en.wikipedia.org/wiki/Natural_deduction

A famous saying by Jean-Yves Girard: A logic without cut-elimination is like a car without an engine. Girard discovered linear logic (we shall learn similar logic).

Cut elimination is used to define the identity of proofs, and represents computational process (so it is important in theoretical computer science). Natural deduction and sequent calculus can be seen as programming languages: the proof-as-program correspondence (Curry-Howard isomorphism); based on this you can even extract programs from proofs (constructive programming).

https://romanripa.typepad.com/.a/6a0148c792648c970c022ad3b6118d200d-popup

https://cacm.acm.org/magazines/2010/10/99493-linear-logic/fulltext

GOAL

$$\frac{X_1 \vdash A, Y_1 \quad X_2, A \vdash Y_2}{X_1, X_2 \vdash Y_1, Y_2}\ \text{Cut}$$

$$\frac{}{X, \bot \vdash Y}\ \bot\vdash \qquad \frac{}{X, A \vdash A, Y}\ \text{Axiom} \qquad \frac{}{X \vdash \top, Y}\ \vdash\top$$

$$\frac{X, A, B \vdash Y}{X, A \land B \vdash Y}\ \land\vdash \qquad \frac{X_1 \vdash A, Y_1 \quad X_2 \vdash B, Y_2}{X_1, X_2 \vdash A \land B, Y_1, Y_2}\ \vdash\land$$

$$\frac{X_1, A \vdash Y_1 \quad X_2, B \vdash Y_2}{X_1, X_2, A \lor B \vdash Y_1, Y_2}\ \lor\vdash \qquad \frac{X \vdash A, B, Y}{X \vdash A \lor B, Y}\ \vdash\lor$$

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\ \neg\vdash \qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\ \vdash\neg$$

$$\frac{X_1 \vdash A, Y_1 \quad X_2, B \vdash Y_2}{X_1, X_2, A \to B \vdash Y_1, Y_2}\ \to\vdash \qquad \frac{X, A \vdash B, Y}{X \vdash A \to B, Y}\ \vdash\to$$

$$\frac{X, A \vdash Y}{X, \forall x A \vdash Y}\ \forall\vdash \qquad \frac{X \vdash A, Y}{X \vdash \forall x A, Y}\ \vdash\forall$$

$$\frac{X, A \vdash Y}{X, \exists x A \vdash Y}\ \exists\vdash \qquad \frac{X \vdash A, Y}{X \vdash \exists x A, Y}\ \vdash\exists$$

It may look complex, but actually **easier** to prove in SC than in ND

The same side conditions as in natural deduction

Assumptions and conclusions are **symmetric**. The rules are **symmetric** too.

The same side conditions as in natural deduction

SEQUENT

Definition

In natural deduction, a **sequent** consisted of a <u>set</u> of formulae called its *assumptions* and a <u>single</u> formula called its *conclusion*: $X \vdash A$.

In **sequent calculus**, sequents are defined to have a <u>set</u> of *conclusions:* $X \vdash Y$. Both X and Y are sets; if we have two C's in X or Y, we can delete one of them; it's called ***contraction***.

*Sequents in some non-classical logics, especially substructural logics such as *relevant logic* or some *fuzzy logics* may have more complex structures in place of sets.

COMMAS IN SEQUENT CALCULUS

We use commas to put multiple premises and conclusions together. In particular, commas represent set unions. For instance, X, A, B is the set:

$$X \cup \{A, B\}$$

Importantly, comma on the left (of $\vdash$) between premises should read as **and** while comma on the right between conclusions should read as **or**. X, Y usually represent sets (potentially singletons or even empty sets) and A, B represent single formulae as in:

$$X, A \vdash B, Y$$

HOW TO READ RULES

Consider, e.g., the right conjunction rule:

$$\frac{X \vdash A, Y \qquad\qquad X \vdash B, Y}{X \vdash A \wedge B, Y} \vdash \wedge$$

The way to prove in sequent calculus is usually upwards. In this case, read it as follows: When you have to prove $X \vdash A \wedge B, Y$, it suffices to prove $X \vdash A, Y$ and $X \vdash B, Y$.

To derive $A \wedge B$ **or** any alternative conclusion in the set Y from the set of premises X, it suffices to derive <u>A **or** anything else in Y from X</u> **AND** <u>B **or** anything else in Y from X</u>.

Simple proof:

$$\frac{\dfrac{}{A, B \vdash A}\,\text{Axiom} \qquad \dfrac{}{A, B \vdash B}\,\text{Axiom}}{A, B \vdash A \wedge B} \vdash \wedge$$

SIMPLE EXAMPLE

The commutativity of conjunction:

$$A \wedge B \vdash B \wedge A$$

Proof:

$$\dfrac{\dfrac{\dfrac{}{A, B \vdash B}\ \text{Ax.}}{A \wedge B \vdash B}\ \wedge\vdash \qquad \dfrac{\dfrac{}{A, B \vdash A}\ \text{Ax.}}{A \wedge B \vdash A}\ \wedge\vdash}{A \wedge B \vdash B \wedge A}\ \vdash\wedge$$

MODUS PONENS

$$\dfrac{\dfrac{}{A \vdash A, B}\,\mathrm{Ax} \qquad \dfrac{}{A, B \vdash B}\,\mathrm{Ax}}{A, A \to B \vdash B} \;\to\vdash$$

$$\dfrac{X_1 \vdash A, Y_1 \qquad X_2, B \vdash Y_2}{X_1, X_2, A \to B \vdash Y_1, Y_2} \;\to\vdash$$

DOUBLE NEGATION ELIMINATION

With $\vdash \neg$ rule, $\neg \vdash$ rule and $\vdash \rightarrow$ rule we can prove the double negation elimination and its converse:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ \text{Ax}}{\vdash A, \neg A}\ \vdash \neg}{\neg\neg A \vdash A}\ \neg \vdash}{\vdash \neg\neg A \rightarrow A}\ \vdash \rightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ \text{Ax}}{A, \neg A \vdash}\ \neg \vdash}{A \vdash \neg\neg A}\ \vdash \neg}{\vdash A \rightarrow \neg\neg A}\ \vdash \rightarrow$$

You don't have to assume the double negation elimination as an additional rule.
Negation works more naturally in sequent calculus than in classical natural deduction.

MORE PROOFS

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{Ax}}{\vdash A, \neg A}\,{\vdash \neg}}{\vdash A \lor \neg A}\,{\vdash \lor} \qquad\qquad \dfrac{\dfrac{\dfrac{}{A, B \vdash A}\,\mathrm{Ax}}{A \vdash B \to A}\,{\vdash \to}}{\vdash A \to (B \to A)}\,{\vdash \to}$$

LEM (the law of excluded middle; $A \lor \neg A$) can be proven much more easily in SC than in ND.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,\mathrm{Ax}}{\vdash \neg B, B}\,{\vdash \neg} \quad \dfrac{\dfrac{}{A \vdash A}\,\mathrm{Ax}}{\neg A, A \vdash}\,{\neg \vdash}}{A, \neg B \to \neg A \vdash B}\,{\to \vdash}}{\neg B \to \neg A \vdash A \to B}\,{\vdash \to}$$

ELIMINATION AND LEFT RULES

The elimination of a connective on the right in natural deduction corresponds to the introduction of it on the left in sequent calculus:

Elimination

$$\frac{X \vdash A \qquad Y \vdash A \to B}{X, Y \vdash B} \to\mathrm{E} \qquad \Leftrightarrow \qquad \frac{X_1 \vdash A, Y_1 \qquad X_2, B \vdash Y_2}{X_1, X_2, A \to B \vdash Y_1, Y_2} \to \vdash$$

Introduction

The cut rule is not really necessary, thanks to the **cut elimination** thm.:

$$\frac{X_1 \vdash A, Y_1 \quad X_2, A \vdash Y_2}{X_1, X_2 \vdash Y_1, Y_2}\ \text{Cut}$$

There is a proof of $X \vdash Y$ iff there is a **cut-free** proof of it

$$\frac{}{X, \bot \vdash Y}\ \bot\vdash \qquad \frac{}{X, A \vdash A, Y}\ \text{Axiom} \qquad \frac{}{X \vdash \top, Y}\ \vdash\top$$

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y}\ \wedge\vdash \qquad \frac{X_1 \vdash A, Y_1 \qquad X_2 \vdash B, Y_2}{X_1, X_2 \vdash A \wedge B, Y_1, Y_2}\ \vdash\wedge$$

$$\frac{X_1, A \vdash Y_1 \qquad X_2, B \vdash Y_2}{X_1, X_2, A \vee B \vdash Y_1, Y_2}\ \vee\vdash \qquad \frac{X \vdash A, B, Y}{X \vdash A \vee B, Y}\ \vdash\vee$$

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\ \neg\vdash \qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\ \vdash\neg$$

$$\frac{X_1 \vdash A, Y_1 \quad X_2, B \vdash Y_2}{X_1, X_2, A \to B \vdash Y_1, Y_2}\ \to\vdash \qquad \frac{X, A \vdash B, Y}{X \vdash A \to B, Y}\ \vdash\to$$

$$\frac{X, A \vdash Y}{X, \forall x A \vdash Y}\ \forall\vdash \qquad \frac{X \vdash A, Y}{X \vdash \forall x A, Y}\ \vdash\forall$$

$$\frac{X, A \vdash Y}{X, \exists x A \vdash Y}\ \exists\vdash \qquad \frac{X \vdash A, Y}{X \vdash \exists x A, Y}\ \vdash\exists$$

THE SUBFORMULA PROPERTY OF SEQUENT CALCULUS WITHOUT CUT

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{Ax}}{\vdash A, \neg A}\,\vdash \neg}{\vdash A \vee \neg A}\,\vdash \vee \qquad\qquad \dfrac{\dfrac{\dfrac{}{A, B \vdash A}\,\mathrm{Ax}}{A \vdash B \to A}\,\vdash \to}{\vdash A \to (B \to A)}\,\vdash \to$$

Notice that formulae only become more complex when you infer from the above to the below, i.e., any formula in the above sequent is a subformula in the below sequent: the **subformula property**.

The cut rule is the only exception in which a formula in the above sequent does not appear in the below sequent.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,\mathrm{Ax}}{\vdash \neg B, B}\,\vdash \neg \qquad \dfrac{\dfrac{}{A \vdash A}\,\mathrm{Ax}}{\neg A, A \vdash}\,\neg \vdash}{A, \neg B \to \neg A \vdash B}\,\to \vdash}{\neg B \to \neg A \vdash A \to B}\,\vdash \to$$

Any proof in sequent calculus has the subformula property if it does not use the cut rule (useful but redundant).

THE EMPTY SEQUENT AS A CONTRADICTION

- Assume that a contradiction $\vdash A \wedge \neg A$ is provable.
- Then, the empty sequent $\vdash$ is provable, since

$$\dfrac{\vdash A \wedge \neg A \qquad \dfrac{\dfrac{A \vdash A}{A, \neg A \vdash}\,\neg\vdash}{A \wedge \neg A \vdash}\,\wedge\vdash}{\vdash}\,\text{Cut}$$

- Thus: if the empty sequent is not provable, the contradiction is not provable.
- So, to give a consistency proof, it suffices to show the empty sequent is not provable.
- We shall come back to this below. For now, it suffices to understand that the empty sequent basically means a contradiction.

FINITARY CONSISTENCY PROOF

- Assume there is a proof of $\vdash$.
  - The cut elimination theorem: any proof can be made cut-free.
  - So there must be a cut-free proof of $\vdash$.
- Any cut-free proof in sequent calculus must have the subformula property.
- But any cut-free proof of $\vdash$ cannot have the subformula property (since any axiom include at least one formula, which cannot be a subformula in $\vdash$ because it includes no formula).
  - Hence: there cannot be a proof of $\vdash$.
  - Thus: the logic is consistent. Cut elim. and subform. prop. give a consistency proof.
- It's a standard method to prove unprovability of sequnets including the empty one.

TWO TRADITIONS IN PROOF THEORY

- **Natural Deduction**: A proof system with introduction and elimination rules which <u>operate on formulae</u> (cf. categorical judgements in philosophy).

- **Sequent Calculus:** A proof system with left and right rules which <u>operate on sequents</u> (cf. hypothetical judgments in philosophy) rather than formulae.
  - Sequent calculus works better for classical logic (the logic you have learned so far).

- If you are interested in the ND vs. SC debate in philosophy of logic, see: P. Schroeder-Heister, The categorical and the hypothetical, Synthese, vol. 187, pp. 925-942, 2012.

- It is based on the inferentialist idea that meaning is inherent within a proof system. Aka. proof-theoretic semantics. Proof theory gives inferentialist semantics.

Peter Schroeder-Heister

coined the term “substructural logic”

http://www.swedishcollegium.se/subfolders/Fellows/Invited_Fellows/2019-20/schroeder-heister.html

Appendix

PROOF-THEORETIC DUALITY

In natural deduction, $\wedge$**E** looks dual to $\vee$**I**:

$$A \wedge B \vdash A \quad \text{dual to} \quad A \vdash A \vee B$$

They are both one-step inferences.

Although $\wedge$**I:**

$$A, B \vdash A \wedge B$$

is a one-step inference, there is no dual one-step version of $\vee$**E**.

PROOF-THEORETIC DUALITY (CONT'D)

When we allow multiple conclusions, we can say that the immediate consequence of $A \vee B$ is that one of the two formulae (A or B) is true, without having to say which one. That is:

$$A, B \vdash A \wedge B \quad \text{dual to} \quad A \vee B \vdash A, B$$

Similar dualities appear for all logical connectives (and quantifiers) in sequent calculus.

THREE PROPERTIES AS LOGICAL CONSEQUENCE RELATION

- **Reflexivity:** $A \vdash A$ is provable.
- **Monotonicity:** If $A \vdash B$ is provable, then $A, X \vdash B, Y$ is provable.
  - To prove this, use the cut rule twice with axioms $X, B \vdash B$ and $A \vdash A, Y$.
- **Cut:** If $X_1 \vdash A, Y_1$ and $X_2, A \vdash Y_2$ are provable, then $X_1, X_2 \vdash Y_1, Y_2$ is provable.
  - Cut is actually redundant due to the cut elimination theorem (see the logic notes).

STRUCTURAL RULES AND SUBSTRUCTURAL LOGICS

- Some logicians argue any logic must satisfy these three properties, but some logics actually don't, esp. resource-sensitive ones (which are concerned, e.g., with how many times you use assumptions, and do matter in computer science and elsewhere).
- Resource-sensitive logics are also called substructural logics, since they lack so-called structural rules, such as ***weakening*** (monotonicity), ***contraction*** (delete one of the same formulae in either side of sequents), ***exchange*** (change the order of formulae).
  - Implicit in regarding both sides of sequents as sets (rather than lists) and in axioms.
- Relevant logic and some fuzzy logics are substructural; we shall learn them later on.

COMPLEX EXAMPLE

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Rxy \to Sxy, Fy, \neg Sxy \vdash Fy}}{Rxy \to Sxy, Fy \land \neg Sxy \vdash Fy}\ \land\vdash \qquad \dfrac{\dfrac{\dfrac{}{Rxy, Fy \land \neg Sxy \vdash Rxy}}{Fy \land \neg Sxy \vdash Rxy, \neg Rxy}\ \vdash\neg \qquad \dfrac{\dfrac{\dfrac{}{Fy, Sxy \vdash \neg Rxy, Sxy}}{Fy, \neg Sxy, Sxy \vdash \neg Rxy}\ \neg\vdash}{Fy \land \neg Sxy, Sxy \vdash \neg Rxy}\ \land\vdash}{Rxy \to Sxy, Fy \land \neg Sxy \vdash \neg Rxy}\ \to\vdash}{Rxy \to Sxy, Fy \land \neg Sxy \vdash Fy \land \neg Rxy}\ \vdash\land}{Rxy \to Sxy, Fy \land \neg Sxy \vdash \exists y(Fy \land \neg Rxy)}\ \vdash\exists}{\forall y(Rxy \to Sxy), Fy \land \neg Sxy \vdash \exists y(Fy \land \neg Rxy)}\ \forall\vdash}{\forall x \forall y(Rxy \to Sxy), Fy \land \neg Sxy \vdash \exists y(Fy \land \neg Rxy)}\ \forall\vdash}{\forall x \forall y(Rxy \to Sxy), \exists y(Fy \land \neg Sxy) \vdash \exists y(Fy \land \neg Rxy)}\ \exists\vdash}{\forall x \forall y(Rxy \to Sxy), \forall x \exists y(Fy \land \neg Sxy) \vdash \exists y(Fy \land \neg Rxy)}\ \forall\vdash}{\forall x \forall y(Rxy \to Sxy), \forall x \exists y(Fy \land \neg Sxy) \vdash \forall x \exists y(Fy \land \neg Rxy)}\ \vdash\forall
$$

SYMMETRY AND DUALITY IN SEQUENT CALCULUS

- Why comma on the left means AND, and comma on the right OR? There are different ways to answer this.
  - Historically, Gentzen, the discoverer of SC, introduced SC to prove the consistency of arithmetic via the cut elimination.
  - Concretely, consider the rules for negation (which have nice symmetry, unlike the ND rules for negation, which are not so elegant): if you interpret comma on the right hand side as conjunction, they are not logically valid rules any more. To enable the symmetry, comma on the left means "and" whilst the comma on the right means "or".

SYMMETRY AND DUALITY IN SEQUENT CALCULUS (CONT'D)

- More philosophically, assertion (on the left) and denial (on the right) are dual to each other, so that conjunction in the left world becomes disjunction in the right world (note that X |- Y can be regarded as meaning it is incoherent to asserting all of X and denying all of Y). A logico-philosophical perspective on this is further developed in bilateralism in philosophy of logic.
- More mathematically, the elegant correspondence between the left world and the right world in sequent calculus, or between assertion and denial in our reasoning practice, fully manifests itself in the concept of star-autonomous categories, although it goes far beyond any logic course taught at an undergrad level. (In mathematics, "right" and "left" can be (dually) equivalent with each other in an elegant way.) What is even more intricate is that the kind of logic we have learned so far can actually collapse the structure of proofs in such categories in a highly intriguing, paradoxical way.)