

Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Sequents, Semantics, and Propositional Natural
Deduction — Conjunction, Implication, Theorems

Yoshihiro Maruyama



Australian
National
University

Introduction

Recap: Sequents

- We want to know when one logical formula follows logically from another.
- Suppose we know that “ p is true’ (technically: it is *interpreted* as true), and we know that $p \rightarrow q$ holds as well. Then we can logically conclude that q also holds!
- We can express this with *sequents*: $p, p \rightarrow q \models q$
- These conclusions can be arbitrarily complicated, however! I.e., it might not be obvious that the conclusion follows from the premises. truth table & interpretations
- Two ways to **show validity of sequents**: **semantics** (validity by meaning) and **syntactic proof system** (validity by following deductive rules; natural deduction introduced soon).

Sequents

Introduction

In general, a **sequent** is of the following form with **X a set of formulae** and **A a single formula**:

$$X \models A$$

- Read it: A follows from X ; or X entails A .
- For example, “ q follows from p and $p \rightarrow q$ ”
- We write down:
but that just abbreviates:

$$\underbrace{p, p \rightarrow q}_X \models \underbrace{q}_A$$

set of formulae single formula

- Also $X, Y \models A$ abbreviates $X \cup Y \models A$,

Semantically Valid Sequents

Definition:

$X \models A$ means the sequent is *valid*. This is the case if and only if:

- *A is true for every interpretation for which all the formulae in X are true.* Or, equivalently:
- *There is no interpretation that makes X true, but not A.*

How to check/test/prove $X \models A$? Create the *truth tables*! [semantics](#)

- Create a *table t_X* for all formulae in X (all need to be true)
- Create another *table t_A* for A and check validity criterion.

Checking Validity, Example 2

$$\text{Show } \overbrace{(p \vee q) \rightarrow r, p}^X \models \overbrace{(p \rightarrow r) \wedge (q \rightarrow r)}^A$$

Table t_X for premises:

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$	X
0	0	0	0	1	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	0
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	0	0
1	1	1	1	1	1

Table t_A for conclusion:

p	q	r	$p \rightarrow r$	$q \rightarrow r$	A
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Recall the definition: **The sequent is valid if all interpretations that make X true also make A true!**

Checking Validity, Example 2

$$\text{Show } \overbrace{(p \vee q) \rightarrow r, p}^X \models \overbrace{(p \rightarrow r) \wedge (q \rightarrow r)}^A$$

Table t_X for premises:

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$	X
0	0	0	0	1	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	0
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	0	0
1	1	1	1	1	1

Table t_A for conclusion:

p	q	r	$p \rightarrow r$	$q \rightarrow r$	A
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Only two interpretations exist that make all $x \in X$ true: $x_1 \wedge x_2 \wedge \dots \wedge x_n$ is true

- 1 $I_1(p) = I_1(r) = 1, I_1(q) = 0$ 2 $I_2(p) = I_2(q) = I_2(r) = 1$

Both of them make A true! Thus, $X \models A$.

Natural Deduction

Natural Deduction and Derivations

- **Natural deduction** is pure **syntax manipulation** and acts as **proof system** with a formal notion of proof as a mathematical entity (cf. informal proof in ordinary math).
- Natural Deduction exploits **derivations** (or formal proofs).
- A **derivation** is a finite sequence of formulae, which are derived from each other based on elementary formula manipulations (“1-step inference rules”)
[atoms w/ connectives](#) [truth table w/ interpretations](#)
- Syntax (proof system) vs. semantics is arguably the most important distinction in formal logic.

Syntax of Sequents

- From now on, we write $X \vdash A$ rather than $X \models A$.
atoms w/ connectives: natural deduction + derivation
- $X \vdash A$ means A **syntactically** follows from X , i.e., you can formally prove the conclusion A using X as assumptions (within the system of natural deduction).
- $X \models A$ means A **semantically** follows from X .
truth table w/ interpretations
for all interpretations that make the set of formulae in X true, A is also true

Conjunction

The 1-Step Rules: And-Elimination

What are the 1-step rules for dealing with **conjunction**?

Elimination rule:

$$A \wedge B \vdash A$$

$$A \wedge B \vdash B$$

$$\frac{A \wedge B}{A} \wedge E$$

$$\frac{A \wedge B}{B} \wedge E$$

Which reads: If we derived $A \wedge B$, we can derive both A and B .

The 1-Step Rules: And-Introduction

What are the 1-step rules for dealing with conjunction?

Introduction rule:

$$\frac{A \quad B}{A \wedge B} \wedge I$$

Which reads: If we derived A and we derived B , we can derive $A \wedge B$.

Proof Syntax / Notation: Overview

- How to write down proofs?
- There are many different notations that describe the same thing
- We introduce two:
 - **Tree-like representation** of the applied rules (just since it's another standard)
 - **list-like representation** (only use that one!)

Proof Syntax / Notation: Tree- and List-like Representations

- Assume we want to prove $p \wedge q \vdash q \wedge p$
leaf root
- In the **tree-like format**:

$$\begin{array}{c}
 \text{leaf} \qquad \qquad \text{leaf} \\
 p \wedge q \qquad p \wedge q \\
 \hline
 q \quad \wedge E \qquad p \quad \wedge E \\
 \hline
 q \wedge p \quad \wedge I \quad \text{root}
 \end{array}$$

- Leaves** are **assumptions**, **root** is **conclusion**
- Advantages: Makes the proof structure obvious
- In exercises, etc: *Do not use it, unless we ask you to!*

Proof Syntax / Notation: Tree- and List-like Representations

- Assume we want to prove $p \wedge q \vdash q \wedge p$
- In the **list format**: A: assumption; E: elimination; I: introduction

#assum. #line #deriv. how derived

α_1	(1)	$p \wedge q$	A	\equiv	$p \wedge q \vdash p \wedge q$
α_1	(2)	q	1 $\wedge E$	\equiv	$p \wedge q \vdash q$
α_1	(3)	p	1 $\wedge E$	\equiv	$p \wedge q \vdash p$
α_1	(4)	$q \wedge p$	2,3 $\wedge I$	\equiv	$p \wedge q \vdash q \wedge p$

column 1: assumption number *column 2*: line number

column 3: derivation *column 4*: how it was derived

- Note:** Each line represents a sequent! (Sequence of sequents.)

ANU Notes - <https://users.cecs.anu.edu.au/~jks/LogicNotes/sequent-calculus.html>

The 1-Step Rules (Based on Sequents): Derivation Rules

- Derivation Rules as considered so far:

$$\frac{A \wedge B}{A} \wedge E$$

$$\frac{A \wedge B}{B} \wedge E$$

$$\frac{A \quad B}{A \wedge B} \wedge I$$

- Re-written **in terms of sequents**:

$$\frac{X \vdash A \wedge B}{X \vdash A} \wedge E$$

$$\frac{X \vdash A \wedge B}{X \vdash B} \wedge E$$

$$\frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \wedge B} \wedge I$$

→ I.e., now we see how premises accumulate!

The 1-Step Rules (Based on Sequents): Accumulation of Assumptions, Example

$$p, q \vdash p \wedge q$$

	#assum.	#line	#deriv.	how derived		
alpha_1 = p	α_1	(1)	p	A	\equiv	$p \vdash p$ (by assumption)
alpha_2 = q	α_2	(2)	q	A	\equiv	$q \vdash q$ (by assumption)
	α_1, α_2	(3)	$p \wedge q$	1,2 $\wedge I$	\equiv	$p, q \vdash p \wedge q$ ($\wedge I$)
	p, q					

$$\boxed{\frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \wedge B} \wedge I}$$

Implication

The 1-Step Rules: Implication-Elimination and -Introduction

- Elimination rule:** $A, A \rightarrow B \vdash B$

$$\frac{A \rightarrow B \quad A}{B} \rightarrow E$$

- Introduction rule:** $A \vdash B \Rightarrow A \rightarrow B$

if we can derive
 B using A :

then we can derive
 $A \rightarrow B$ and discharge A :

$$\frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \rightarrow B} \rightarrow I$$

\equiv

$$\begin{array}{c} A \\ \vdots \\ B \end{array} +$$

$$\frac{\begin{array}{c} [A] \text{ discharge } A \\ \vdots \\ B \end{array}}{A \rightarrow B} \rightarrow I$$

The 1-Step Rules: Implication-Elimination and -Introduction (Based on Sequents)

- Derivation Rules as considered so far:

$$\frac{A \rightarrow B \quad A}{B} \rightarrow E$$

$$\frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \rightarrow B} \rightarrow I$$

- Re-written in **terms of sequents**:

Introduction rule

$$\frac{X \vdash A \rightarrow B \quad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

Elimination rule

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

Has side effect of
removing the assumption A

- We say that A gets *discharged*, and annotate that in the proof.

The 1-Step Rules: Deduction Equivalence

$$\underbrace{X \vdash A \rightarrow B \quad \text{iff} \quad X, A \vdash B}$$

deduction equivalence
(or deduction theorem)

Why does this hold?

- If $X, A \vdash B$, then $X \vdash A \rightarrow B$:

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

- If $X \vdash A \rightarrow B$, then $X, A \vdash B$:

$$\frac{X \vdash A \rightarrow B \quad A \vdash A}{X, A \vdash B} \rightarrow E$$

(That's the $\rightarrow E$ rule with Y substituted by A)

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1

$p \rightarrow q \vdash (p \wedge r) \rightarrow q$				
#assum.	#line	#deriv.	how derived	
α_1	(1)	$p \rightarrow q$	A	
α_2	(2)	$p \wedge r$	A	
α_2	(3)	p	$2 \wedge E$	
α_1, α_2	(4)	q	$1, 3 \rightarrow E$	
α_1	(5)	$(p \wedge r) \rightarrow q$	$4[\alpha_2] \rightarrow I$	<div> $\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$ </div>
				<div> $\frac{X \vdash A \rightarrow B \quad Y \vdash A}{X, Y \vdash B} \rightarrow E$ </div>
				i.e. line 4 (alpha_1 & alpha_2) - alpha_2
α_1, α_2	(n-1)	q		
α_1	(n)	$(p \wedge r) \rightarrow q$	$(n-1)[\alpha_2] \rightarrow I$	

- Assumption α_2 is a new one, which was not given in the original sequent, so we need to eliminate it later on.
- In the last step, we discharge assumption α_2 .

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1 (cont'd)

The proof of $p \rightarrow q \vdash (p \wedge r) \rightarrow q$ in a tree-like structure:

$$\begin{array}{c}
 \frac{p \rightarrow q \quad \frac{[p \wedge r]^{(1)}}{p} \wedge E}{q} \rightarrow E \\
 \hline
 (p \wedge r) \rightarrow q \rightarrow I(1)
 \end{array}$$

Here, we denote **discharged assumptions** by $[\dots]^{(n)}$, where we number each assumption so that they can be distinguished from each other, i.e., so that we know which rule discharged which assumption(s).

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$$(p \wedge q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$$

$X := \text{alpha}_1 ((p \wedge q) \rightarrow r), \text{alpha}_2 (p)$

$A := \text{alpha}_3 (q)$

$B := r$

α_1	(1)	$(p \wedge q) \rightarrow r$	A
α_2	(2)	p	A
α_3	(3)	q	A
α_2, α_3	(4)	$p \wedge q$	2,3 $\wedge I$
$\alpha_1, \alpha_2, \alpha_3$	(5)	r	1,4 $\rightarrow E$
α_1, α_2	(6)	$q \rightarrow r$	5[α_3] $\rightarrow I$
α_1	(7)	$p \rightarrow (q \rightarrow r)$	6[α_2] $\rightarrow I$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

$$\frac{X \vdash A \rightarrow B \quad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

no need to write this

$\alpha_1, \alpha_2, \alpha_3$	(n-2)	r	
α_1, α_2	(n-1)	$q \rightarrow r$	(n-2)[α_3] $\rightarrow I$
α_1	(n)	$p \rightarrow (q \rightarrow r)$	(n-1)[α_2] $\rightarrow I$

$X := \text{alpha}_1 ((p \wedge q) \rightarrow r)$

$A := \text{alpha}_2 (p)$

$B := q \rightarrow r$

Vacuous Discharge: Discharging Non-existent Assumptions

- We can “discharge” assumptions that are not there; this happens if the conclusion does not depend on its assumption.

$$p \vdash q \rightarrow p$$

$$\begin{array}{llll} \alpha_1 & (1) & p & A \\ \hline \alpha_2 & (2) & q & A \\ \alpha_1 & (2) & q \rightarrow p & 1[] \rightarrow I \end{array}$$

$$\boxed{\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I}$$

$$\begin{array}{llll} \alpha_1, \cancel{\alpha_2} & (n-1) & p & \\ \alpha_1 & (n) & q \rightarrow p & (n-1)[\cancel{\alpha_2}] \rightarrow I \end{array}$$

- We call such a discharge a *vacuous discharge*.
- I.e., whenever we “would remove” some assumption α from a set of assumptions X , but $\alpha \notin X$, we write $i[] \rightarrow I$ instead of $i[\alpha] \rightarrow I$

Excursion: \vdash vs. \rightarrow : An Often Asked Question in Previous Courses

- \vdash and \rightarrow seem to be of a very related nature:
E.g., compare $A, B \vdash C$ with $A \wedge B \rightarrow C$
- So what's the difference?
- \rightarrow is a **logical connective**, whereas \vdash is not.
- \vdash is a **relation** between formulae and cannot be used within a formula.
 $X \vdash A$: whenever any x in X is true, A must also be true
- They are linked by the **deduction theorem**: $X, A \vdash B$ if and only if $X \vdash A \rightarrow B$. In particular: $A \vdash B$ if and only if $\vdash A \rightarrow B$

Theorems

Unconditionally True Formulas

- Sequents that do not depend on anything are called **theorems**!
- Thus, A is a theorem if “ $\vdash A$ ”, e.g., $\vdash p \rightarrow (q \rightarrow p)$.
- Another (slightly more complex) example:

α_1	(1)	p	A
α_2	(2)	q	A
α_1, α_2	(3)	$p \wedge q$	1,2 $\wedge I$
α_1	(4)	$q \rightarrow (p \wedge q)$	3 $[\alpha_2] \rightarrow I$
	(5)	$p \rightarrow (q \rightarrow (p \wedge q))$	4 $[\alpha_1] \rightarrow I$

X: α_1 (p), A: α_2 (q); B: p & q (line 3)

- Thus, we get $\vdash p \rightarrow (q \rightarrow (p \wedge q))$, so its formula is a theorem.
- Note that A in $\vdash A$ is a **tautology**!

Summary

Content of this Lecture

- Sequents and their semantics *truth table w/ interpretations*

- What does $X \models A$ mean? *A semantically follows from X*

- The most important basics of Natural Deduction!

- How can proofs be written? *tree-like format or list-like format*
- What does $X \vdash A$ mean? *A syntaxly follows from X*

- Every logical connective comes with two 1-step rules:**

Introduction and Elimination

- What's a theorem? *$\vdash A$, where A is a theorem that doesn't depend on any other formulae*

→ The Logic Notes sections:

- 3. More about propositional logic: Truth Tables*
- 2. Propositional natural deduction: Conjunction*
- 2. Propositional natural deduction: Implication*
- 2. Propositional natural deduction: Counting assumptions (except Contraction, which you should look up!)*