Logic: Tutorial Week 3

Proof 1

$$\neg p \to \neg q \ \vdash \ q \to p$$

α_1	(1)	$\neg p \to \neg q$	A (given)
α_2	(2)	q	A (antecedent of $q \to \neg p$)
α_3	(3)	$\neg p$	A (for contradiction)
α_1, α_3	(4)	$\neg q$	1, 3 $\to$E
$\alpha_1, \alpha_2, \alpha_3$	(5)	$\perp$	2, 4 $\neg$E
α_1, α_2	(6)	$\neg\neg p$	5 [3] $\neg$I
α_1, α_2	(7)	p	6 $\neg\neg$E
α_1	(8)	$q \to p$	7 [α_2] $\to$I

Here the desired conclusion $q \to p$ is a conditional, so we start by assuming its antecedent q. Then we are aiming to prove the sequent $\neg p \to \neg q, q \ \vdash \ p$ which will be achieved at line 7. The conclusion of this sub-goal is p, so the only reasonable way to get it is to assume $\neg p$ (line 3) and derive a contradiction (easy: line 4 contradicts line 2). The rule RAA enables us to compress lines 5 and 6 into a single line, giving the slightly shorter proof:

$$\neg p \to \neg q \ \vdash \ q \to p$$

α_1	(1)	$\neg p \to \neg q$	A (given)
α_2	(2)	q	A (antecedent of $q \to \neg p$)
α_3	(3)	$\neg p$	A (for contradiction)
α_1, α_3	(4)	$\neg q$	1, 3 $\to$E
α_1, α_2	(5)	$\neg\neg p$	2, 4 [α_3] RAA
α_1, α_2	(6)	p	5 $\neg\neg$E
α_1	(7)	$q \to p$	6 [α_2] $\to$I

Proof 2

$$p \to r, \ q \to \neg r \ \vdash \ \neg(p \wedge q)$$

α_1	(1)	$p \to r$	A
α_2	(2)	$q \to \neg r$	A
α_3	(3)	$p \wedge q$	A
α_3	(4)	p	3 $\wedge$E
α_1, α_3	(5)	r	1, 4 $\to$E
α_3	(6)	q	3 $\wedge$E
α_2, α_3	(7)	$\neg r$	2, 6 $\to$E
α_1, α_2	(8)	$\neg(p \wedge q)$	5, 7 [α_3] RAA

To derive the negated formula $\neg(p \wedge q)$, we follow the usual strategy of assuming the formula to be negated (i.e. $p \wedge q$ at line 3) and deriving a contradiction (lines 5 and 7) so that the negation rules can apply.

Proof 3

$$\neg p \ \vdash \ p \to q$$

α_1	(1)	$\neg p$	A
α_2	(2)	p	A
α_1, α_2	(3)	$\neg\neg q$	1, 2 [ ] RAA
α_1, α_2	(4)	q	3 $\neg\neg$E
α_1	(5)	$p \to q$	4 [α_2] $\to$I

This illustrates vacuous discharge with RAA. Compare it with the proof in the course notes of the related sequent $p, \neg p \ \vdash \ q$.

Proof 4

$$(p \wedge q) \to r,\ p \to (q \vee r) \ \vdash \ p \to r$$

α_1	(1)	$(p \wedge q) \to r$	A
α_2	(2)	$p \to (q \vee r)$	A
α_3	(3)	p	A
α_2, α_3	(4)	$q \vee r$	2, 3 $\to$E
α_5	(5)	q	A (left disjunct for $\vee$E)
α_3, α_5	(6)	$p \wedge q$	3, 5 $\wedge$I
$\alpha_1, \alpha_3, \alpha_5$	(7)	r	1, 6 $\to$E
α_8	(8)	r	A (right disjunct for $\vee$E)
$\alpha_1, \alpha_2, \alpha_3$	(9)	r	4, 7 [α_5], 8 [α_8] $\vee$E
α_1, α_2	(10)	$p \to r$	9 [α_3] $\to$I

This sequent requires $\vee$E for its proof. Recall that the abstract statement of $\vee$E is:

$$\frac{X \ \vdash \ A \vee B \qquad Y, A \ \vdash \ C \qquad Z, B \ \vdash \ C}{X, Y, Z \ \vdash \ C}$$

Match up the components of the inference to line 9 with the parts of this rule template. The disjuncts A and B are q and r respectively, and the conclusion C is r. The set of assumptions X is $\{p \to (q \vee r), p\}$, represented in the written proof by the list of assumption numbers "2,3". Y is $\{(p \wedge q) \to r, p\}$, and Z is the empty set. That is, we have:

$$\frac{p \to (q \vee r), p \ \vdash \ q \vee r \qquad (p \wedge q) \to r, p, q \ \vdash \ r \qquad r \ \vdash \ r}{p \to (q \vee r), (p \wedge q) \to r, p \ \vdash \ r}$$

or, abbreviating the formulae in X, Y and Z to their assumption numbers:

$$\frac{2, 3 \ \vdash \ q \vee r \qquad 1, 3, q \ \vdash \ r \qquad r \ \vdash \ r}{1, 2, 3 \ \vdash \ r}$$

Proof 5

$$p \to \neg r,\ q \to \neg r \ \vdash \ r \to \neg(p \vee q)$$

α_1	(1)	$p \to \neg r$	A
α_2	(2)	$q \to \neg r$	A
α_3	(3)	r	A (antecedent for $\to$I)
α_4	(4)	$p \vee q$	A (for contradiction)
α_5	(5)	p	A (left disjunct for $\vee$E)
α_1, α_5	(6)	$\neg r$	1, 5 $\to$E
α_7	(7)	q	A (right disjunct for $\vee$E)
α_2, α_7	(8)	$\neg r$	2, 7 $\to$E
$\alpha_1, \alpha_2, \alpha_4$	(9)	$\neg r$	4, 6 [α_5], 8 [α_7] $\vee$E
$\alpha_1, \alpha_2, \alpha_3$	(10)	$\neg(p \vee q)$	3, 9 [α_4] RAA
α_1, α_2	(11)	$r \to \neg(p \vee q)$	10 [α_3] $\to$I

This is quite straightforward. The conclusion is a conditional, so we assume its antrecedent r and derive its consequent $\neg(p \vee q)$. This in turn is a negation, so we assume $p \vee q$ and derive a contradiction. In fact, we derive

$\neg r$ which contradicts r. Assumption 4 is a disjunction, so we derive the goal $\neg r$ from each disjunct in turn before applying $\vee$E at line 9.

Proof 6

$$\vdash\ p \vee \neg p$$

α_1	(1)	$\neg(p \vee \neg p)$	A
α_2	(2)	p	A
α_2	(3)	$p \vee \neg p$	2 $\vee$I
α_1	(4)	$\neg p$	1, 3 $[\alpha_2]$ RAA
α_1	(5)	$p \vee \neg p$	4 $\vee$I
	(6)	$\neg\neg(p \vee \neg p)$	1, 5 $[\alpha_1]$ RAA
	(7)	$p \vee \neg p$	6 $\neg\neg$E

This theorem, called the "law of the excluded middle", corresponds to the principle that every proposition is either true or false. To prove it, we have to assume that it is false and derive a contradiction from that. In fact, we derive the contradiction first from p and then from $\neg p$, but since we get it from p, the $\neg p$ follows from the main assumption, which therefore entails its own negation and thus has to be false. Again, $\neg\neg$E is essential to the proof.

Proof 7

$$\neg p \rightarrow (q \vee r)\ \vdash\ \neg q \rightarrow (p \vee r)$$

α_1	(1)	$\neg p \rightarrow (q \vee r)$	A
α_2	(2)	$\neg q$	A
α_3	(3)	$\neg(p \vee r)$	A
α_4	(4)	p	A
α_4	(5)	$p \vee r$	4 $\vee$I
α_3	(6)	$\neg p$	3, 5 $[\alpha_4]$ RAA
α_1, α_3	(7)	$q \vee r$	1, 6 $\rightarrow$E
α_8	(8)	q	A
α_2, α_8	(9)	$\neg\neg(p \vee r)$	2, 8 [ ] RAA
α_2, α_8	(10)	$p \vee r$	9 $\neg\neg$E
α_5	(11)	r	A
α_5	(12)	$p \vee r$	11 $\vee$I
$\alpha_1, \alpha_2, \alpha_3$	(13)	$p \vee r$	7, 10 $[\alpha_8]$, 12 $[\alpha_5]$ $\vee$E
α_1, α_2	(14)	$\neg\neg(p \vee r)$	3, 13 $[\alpha_3]$ RAA
α_1, α_2	(15)	$p \vee r$	14 $\neg\neg$E
α_1	(16)	$\neg q \rightarrow (p \vee r)$	15 $[\alpha_2]$ $\rightarrow$I

This is a little more complicated. Note that part of the reasoning (to get line 10) uses the fact that a contradiction entails everything. Note also the overall strategy of proving the disjunction (line 15) by showing that its negation (line 3) is self-contradictory (line 13). RAA and $\neg\neg$E then work together to produce the desired $p \vee r$. The final $\rightarrow$I move is routine.