Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Sequents, Semantics, and Propositional Natural Deduction — Conjunction, Implication, Theorems

Yoshihiro Maruyama

Australian National University

Introduction

Recap: Sequents

- We want to know when one logical formula follows logically from another. formula = atoms + connectives
- Suppose we know that "p is true' (technically: it is *interpreted* as true), and we know that $p \to q$ holds as well. Then we can logically conclude that q also holds!
- We can express this with *sequents*: $p, p \to q \models q$
- These conclusions can be arbitrarily complicated, however! I.e., it might not be obvious that the conclusion follows from the premises.
- Two ways to show validity of sequents: semantics (validity by meaning) and syntactic proof system (validity by following deductive rules; natural deduction introduced soon).

Sequents

Introduction

In general, a sequent is of the following form with X a set of formulae and A a single formula:

$$X \models A$$

- Read it: A follows from X; or X entails A.
- For example, "q follows from p and $p \to q$"
- We write down: $p, p \to q \models q$
  but that just abbreviates: $\underbrace{\{p, p \to q\}}_{X} \models \underbrace{q}_{A}$

  set of formulae　single formula
- Also $X, Y \models A$ abbreviates $X \cup Y \models A$,

Semantically Valid Sequents

Definition:
$X \models A$ means the sequent is *valid*. This is the case if and only if:

- *A is true for every interpretation for which all the formulae in X are true.* Or, equivalently:
- *There is no interpretation that makes X true, but not A.*

How to check/test/prove $X \models A$? Create the truth tables!

- Create a table t_X for all formulae in X (all need to be true)
- Create another table t_A for A and check validity criterion.

Checking Validity, Example 2

Show $\overbrace{(p \vee q) \to r, p}^{X} \models \overbrace{(p \to r) \wedge (q \to r)}^{A}$

Table t_X for premises:

p	q	r	$p \vee q$	$(p \vee q) \to r$	X
0	0	0	0	1	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	0
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	0	0
1	1	1	1	1	1

Table t_A for conclusion:

p	q	r	$p \to r$	$q \to r$	A
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Recall the definition: The sequent is valid if all interpretations that make X true also make A true!

Checking Validity, Example 2

Show $\overbrace{(p \vee q) \to r, p}^{X} \models \overbrace{(p \to r) \wedge (q \to r)}^{A}$

Table t_X for premises:

p	q	r	$p \vee q$	$(p \vee q) \to r$	X
0	0	0	0	1	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	0
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	0	0
1	1	1	1	1	1

Table t_A for conclusion:

p	q	r	$p \to r$	$q \to r$	A
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Only two interpretations exist that make all $x \in X$ true:

1. $I_1(p) = I_1(r) = 1, I_1(q) = 0$
2. $I_2(p) = I_2(q) = I_2(r) = 1$

Both of them make A true! Thus, $X \models A$.

Natural Deduction

Natural Deduction and Derivations

- Natural deduction is pure syntax manipulation and acts as *proof system* with a formal notion of proof as a mathematical entity (cf. informal proof in ordinary math).
- Natural Deduction exploits *derivations* (or formal proofs).
- A derivation is a finite sequence of formulae, which are derived from each other based on elementary formula manipulations ("1-step inference rules")
- Syntax (proof system) vs. semantics is arguably the most important distinction in formal logic.

Syntax of Sequents

- From now on, we write $X \vdash A$ rather than $X \models A$.
- $X \vdash A$ means A syntactically follows from X, i.e., you can formally prove the conclusion A using X as assumptions (within the system of natural deduction).
- $X \models A$ means A semantically follows from X.

Conjunction

The 1-Step Rules: And-Elimination

What are the 1-step rules for dealing with conjunction?

Elimination rule:

$$\frac{A \wedge B}{A} \wedge E \qquad\qquad \frac{A \wedge B}{B} \wedge E$$

Which reads: If we derived $A \wedge B$, we can derive both A and B.

The 1-Step Rules: And-Introduction

What are the 1-step rules for dealing with conjunction?

Introduction rule:

$$\frac{A \qquad B}{A \wedge B}\wedge I$$

Which reads: If we derived A and we derived B, we can derive $A \wedge B$.

Proof Syntax / Notation: Overview

- How to write down proofs?
- There are many different notations that describe the same thing
- We introduce two:
  - Tree-like representation of the applied rules (just since it's another standard)
  - list-like representation (only use that one!)

Proof Syntax / Notation: Tree- and List-like Representations

- Assume we want to prove $p \wedge q \vdash q \wedge p$
- In the tree-like format:

$$\cfrac{\cfrac{p \wedge q}{q}\,\wedge E \qquad \cfrac{p \wedge q}{p}\,\wedge E}{q \wedge p}\,\wedge I$$

- Leaves are assumptions, root is conclusion
- Advantages: Makes the proof structure obvious
- In exercises, etc: *Do not use it, unless we ask you to!*

Proof Syntax / Notation: Tree- and List-like Representations

- Assume we want to prove $p \wedge q \vdash q \wedge p$
- In the list format: A: assumption; E: elimination; I: introduction

#assum.	#line	#deriv.	how derived		
α_1	(1)	$p \wedge q$	A	$\equiv$	$p \wedge q \vdash p \wedge q$
α_1	(2)	q	1 $\wedge E$	$\equiv$	$p \wedge q \vdash q$
α_1	(3)	p	1 $\wedge E$	$\equiv$	$p \wedge q \vdash p$
α_1	(4)	$q \wedge p$	2,3 $\wedge I$	$\equiv$	$p \wedge q \vdash q \wedge p$

column 1: assumption number *column 2:* line number
column 3: derivation *column 4:* how it was derived

- **Note:** Each line represents a sequent! (Sequence of sequents.)

The 1-Step Rules (Based on Sequents): Derivation Rules

- Derivation Rules as considered so far:

$$\frac{A \wedge B}{A}\wedge E \qquad \frac{A \wedge B}{B}\wedge E \qquad \frac{A \qquad B}{A \wedge B}\wedge I$$

- Re-written in terms of sequents:

$$\frac{X \vdash A \wedge B}{X \vdash A}\wedge E \qquad \frac{X \vdash A \wedge B}{X \vdash B}\wedge E \qquad \frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \wedge B}\wedge I$$

$\rightarrow$ I.e., now we see how premises accumulate!

The 1-Step Rules (Based on Sequents): Accumulation of Assumptions, Example

$p, q \vdash p \wedge q$

#assum.	#line	#deriv.	how derived		
alpha_1 = p $\quad \alpha_1$	(1)	p	A	$\equiv$	$p \vdash p$ (by assumption)
alpha_2 = q $\quad \alpha_2$	(2)	q	A	$\equiv$	$q \vdash q$ (by assumption)
α_1, α_2 p, q	(3)	$p \wedge q$	1,2 $\wedge I$	$\equiv$	$p, q \vdash p \wedge q$ $(\wedge I)$

$$\boxed{\dfrac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \wedge B}\wedge I}$$

Implication

The 1-Step Rules: Implication-Elimination and -Introduction

- **Elimination rule**:

$$\frac{A \to B \qquad A}{B}\ {\to}E$$

- **Introduction rule**:

if we can derive B using A: then we can derive $A \to B$ and discharge A:

$$\frac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \to B}\ {\to}I \quad \equiv \quad \begin{array}{c}A\\ \vdots\\ B\end{array} \quad + \quad \frac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \to B}\ {\to}I$$

The 1-Step Rules: Implication-Elimination and -Introduction (Based on Sequents)

- Derivation Rules as considered so far:

$$\frac{A \to B \qquad A}{B}\to E \qquad\qquad \frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B}\to I$$

- Re-written in terms of sequents:

Introduction rule

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B}\to E \qquad\qquad \frac{X, A \vdash B}{\underbrace{X \vdash A \to B}}\to I$$

Elimination rule

Has side effect of

removing the assumption A

?

- We say that A gets *discharged*, and annotate that in the proof.

The 1-Step Rules: Deduction Equivalence

$$\underbrace{X \vdash A \to B \quad \text{iff} \quad X, A \vdash B}_{\substack{\text{deduction equivalence} \\ \text{(or deduction theorem)}}}$$

Why does this hold?

- If $X, A \vdash B$, then $X \vdash A \to B$:

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

- If $X \vdash A \to B$, then $X, A \vdash B$:

$$\frac{X \vdash A \to B \qquad A \vdash A}{X, A \vdash B} \to E$$

(That's the $\to E$ rule with Y substituted by A)

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1

$p \to q \vdash (p \wedge r) \to q$

#assum.	#line	#deriv.	how derived
α_1	(1)	$p \to q$	A
α_2	(2)	$p \wedge r$	A
α_2	(3)	p	2 $\wedge E$
α_1, α_2	(4)	q	1,3 $\to E$
α_1	(5)	$(p \wedge r) \to q$	4$[\alpha_2]$ $\to I$
α_1, α_2	(n-1)	q	
α_1	(n)	$(p \wedge r) \to q$	(n-1)$[\alpha_2]$ $\to I$

$$\boxed{\frac{X, A \vdash B}{X \vdash A \to B} \to I}$$

$$\boxed{\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B} \to E}$$

$\boxed{?}$

- Assumption α_2 is a new one, which was not given in the original sequent, so we need to eliminate it later on.
- In the last step, we discharge assumption α_2.

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1 (cont'd)

The proof of $p \to q \vdash (p \land r) \to q$ in a tree-like structure:

$$\dfrac{\dfrac{p \to q \qquad \dfrac{[p \land r]^{(1)}}{p}\land E}{q}\to E}{(p \land r) \to q}\to I(1)$$

Here, we denote discharged assumptions by $[\ldots]^{(n)}$, where we number each assumption so that they can be distinguished from each other, i.e., so that we know which rule discharged which assumption(s).

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$$(p \wedge q) \to r \vdash p \to (q \to r)$$

α_1	(1)	$(p \wedge q) \to r$	A
α_2	(2)	p	A
α_3	(3)	q	A
α_2, α_3	(4)	$p \wedge q$	2,3 $\wedge I$
$\alpha_1, \alpha_2, \alpha_3$	(5)	r	1,4 $\to E$
α_1, α_2	(6)	$q \to r$	5[α_3] $\to I$
α_1	(7)	$p \to (q \to r)$	6[α_2] $\to I$

$$\boxed{\dfrac{X, A \vdash B}{X \vdash A \to B} \to I}$$

$$\boxed{\dfrac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B} \to E}$$

$\alpha_1, \alpha_2, \alpha_3$	(n-2)	r	
α_1, α_2	(n-1)	$q \to r$	(n-2)[α_3] $\to I$
α_1	(n)	$p \to (q \to r)$	(n-1)[α_2] $\to I$

Vacous Discharge: Discharging Non-existent Assumptions

- We can "discharge" assumptions that are not there; this happens if the conclusion does not depend on its assumption.

$p \vdash q \rightarrow p$

α_1	(1)	p	A
~~α_2~~	~~(2)~~	~~q~~	~~A~~
α_1	(2)	$q \rightarrow p$	$1[] {\rightarrow} I$
$\alpha_1, \cancel{\alpha_2}$	(n-1)	p	
α_1	(n)	$q \rightarrow p$	$(n\text{-}1)[\cancel{\alpha_2}] \rightarrow I$

$$\boxed{\dfrac{X, A \vdash B}{X \vdash A \rightarrow B} {\rightarrow} I}$$

- We call such a discharge a *vacuous discharge*.
- I.e., whenever we "would remove" some assumption α from a set of assumptions X, but $\alpha \notin X$, we write $i[] {\rightarrow} I$ instead of $i[\alpha] {\rightarrow} I$

Excursion: $\vdash$ vs. $\rightarrow$: An Often Asked Question in Previous Courses

- $\vdash$ and $\rightarrow$ seem to be of a very related nature:
  E.g., compare $A, B \vdash C$ with $A \land B \rightarrow C$
- So what's the difference?
- $\rightarrow$ is a logical connective, whereas $\vdash$ is not.
- $\vdash$ is a relation between formulae and cannot be used within a formula.
- They are linked by the deduction theorem: $X, A \vdash B$ if and only if $X \vdash A \rightarrow B$. In particular: $A \vdash B$ if and only if $\vdash A \rightarrow B$

Theorems

Unconditionally True Formulas

- Sequents that do not depend on anything are called *theorems*!
- Thus, A is a theorem if "$\vdash A$", e.g., $\vdash p \to (q \to p)$.
- Another (slightly more complex) example:

α_1	(1)	p	A
α_2	(2)	q	A
α_1, α_2	(3)	$p \wedge q$	1,2 $\wedge I$
α_1	(4)	$q \to (p \wedge q)$	3[α_2] $\to I$
	(5)	$p \to (q \to (p \wedge q))$	4[α_1] $\to I$

- Thus, we get $\vdash p \to (q \to (p \wedge q))$, so its formula is a theorem.
- Note that A in $\vdash A$ is a tautology!

Summary

Content of this Lecture

- Sequents and their semantics
  - What does $X \models A$ mean?
- The most important basics of Natural Deduction!
  - How can proofs be written?
  - What does $X \vdash A$ mean?
  - Every logical connective comes with two 1-step rules: *Introduction* and *Elimination*
  - What's a theorem?

→ The Logic Notes sections:
  - *3. More about propositional logic*: *Truth Tables*
  - *2. Propositional natural deduction*: *Conjunction*
  - *2. Propositional natural deduction*: *Implication*
  - *2. Propositional natural deduction*: *Counting assumptions* (except *Contraction*, which you should look up!)