Logic (PHIL2080, COMP2620, COMP6262)
Chapter: Introduction to Logic

Yoshihiro Maruyama

on the Legacy of John Slaney and Pascal Bercher

Australian National University

Motivation

Making good Arguments: What is Logic?

- Logic is the science of reasoning, i.e., making arguments.
  - Correct reasoning vs. wrong reasoning
  - Making (and reasoning about) valid arguments
  - ⇒ See (famous) Monty Python sketch “argument clinic” (e.g., `https://www.dailymotion.com/video/x2hwqn9`)
- There are separates slides in keynote (which I will talk about if time permits)

Propositional Logic

Basic Definitions: Generators of Formulae

the simplest elements of the logical language and represent statements that can be either true or false, without any further analysis of their internal structure.

Atoms (generators of logical formulae) are:

- truth (denoted by $\top$, T, or 1)
- falsity (denoted by $\perp$, F, or 0)
- propositional variables (denoted by $p, q, r, \ldots$)
  p, q, r are varibales that can be T/F

Atoms are bases or generators for more complex propositions with various connectives.

Basic Definitions: Syntax of Connectives

Which logical connectives exist in propositional logic?

- ... and ...: $\wedge$ conjunction e.g., $(p \wedge \top)$ or $(p \wedge (q \wedge r))$
- ... or ...: $\vee$ disjunction e.g., $(\bot \vee \top)$ or $(p \vee (q \wedge r))$
- if ..., then ...: $\rightarrow$ implication e.g., $(p \rightarrow q)$ or $((p \wedge q) \rightarrow (p \vee q))$
  also: ... implies ...
- ... if and only if ...: $\leftrightarrow$ biconditional, $\equiv$ e.g., $(p \leftrightarrow q)$ or $((p \wedge q) \leftrightarrow (q \wedge p))$
- not ...: $\neg$ negation e.g., $((\neg p) \rightarrow q)$ or $\neg(p \rightarrow q)$

A formula (other than atoms) is generated using atoms and logical connectives. formula = atoms + connectives

the symbol of equivalence ($\equiv$) denotes logical equivalence, indicating that two statements have the same truth value under certain conditions;
the symbol of equality ($=$) denotes numerical or qualitative equality, indicating that two quantities or objects are identical in value or characteristics.

Basic Definitions: Semantics of Connectives

What do these connectives *mean*?

- The semantics is defined in terms of truth tables.
- A truth table for a formula tells us for each interpretation of the propositional variables whether the formula is true or false.
- Examples:
  - $(\neg p)$ inverts p's truth value: $\top$ is switched to $\bot$, and vice versa.
  - $(p \wedge q)$ is true if and only if both p and q are true.
  - $(p \vee q)$ is true if and only if at least one of p and q is true.

p	*q*	F^0	NOR^1	$\nleftarrow^2$	$\neg p^3$	NIMPLY^4	$\neg q^5$	XOR^6	NAND^7	AND^8	XNOR^9	q^{10}	IMPLY^{11}	p^{12}	$\leftarrow^{13}$	OR^{14}	T^{15}
T	T	F	F	F	F	F	F	F	F	T	T	T	T	T	T	T	T
T	F	F	F	F	F	T	T	T	T	F	F	F	F	T	T	T	T
F	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F	T	T
F	F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T
Com		✓	✓					✓	✓	✓	✓					✓	✓
Assoc		✓						✓		✓	✓	✓		✓		✓	✓
Adj		F^0	NOR^1	$\nrightarrow^4$	$\neg q^5$	NIMPLY^2	$\neg p^3$	XOR^6	NAND^7	AND^8	XNOR^9	p^{12}	IMPLY^{13}	q^{10}	$\rightarrow^{11}$	OR^{14}	T^{15}
Neg		T^{15}	OR^{14}	$\leftarrow^{13}$	p^{12}	IMPLY^{11}	q^{10}	XNOR^9	AND^8	NAND^7	XOR^6	$\neg q^5$	NIMPLY^4	$\neg p^3$	$\nleftarrow^2$	NOR^1	F^0
Dual		T^{15}	NAND^7	$\rightarrow^{11}$	$\neg p^3$	$\leftarrow^{13}$	$\neg q^5$	XNOR^9	NOR^1	OR^{14}	XOR^6	q^{10}	$\nleftarrow^2$	p^{12}	$\nrightarrow^4$	AND^8	F^0
L id				F				F		T	T	T,F	T			F	
R id						F		F		T	T			T,F	T	F	

https://en.wikipedia.org/wiki/Truth_table

Basic Definitions: Semantics of Connectives

What do these connectives *mean*?

- The semantics is defined in terms of truth tables.
- A truth table for a formula tells us for each interpretation of the propositional variables whether the formula is true or false.
- Examples: (expressed as *truth tables*)

p	$\neg$
$\bot$	$\top$
$\top$	$\bot$

p	q	$\wedge$
$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\bot$
$\top$	$\top$	$\top$

p	q	$\vee$
$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$
$\top$	$\top$	$\top$

Basic Definitions: Semantics of Connectives

What do these connectives *mean*?

- The semantics is defined in terms of truth tables.
- A truth table for a formula tells us for each interpretation of the propositional variables whether the formula is true or false.
- Examples: (expressed as *truth tables*)

p	$\neg$
0	1
1	0

p	q	$\wedge$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$\vee$
0	0	0
0	1	1
1	0	1
1	1	1

We will henceforth use 0/1 because its readability is so much improved!

Semantics: Semantics of $\neg$, $\wedge$, $\vee$, $\rightarrow$

- The semantics of propositional logic is given by truth tables. (Which you already saw.)
- Truth tables:

p	$\neg$
0	1
1	0

p	q	$\wedge$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$\vee$
0	0	0
0	1	1
1	0	1
1	1	1

p	q	$\rightarrow$
0	0	1
0	1	1
1	0	0
1	1	1

- Example implication: If the light is red (p), you must stop (q).

Semantics: Semantics of $\neg$, $\wedge$, $\vee$, $\rightarrow$

- The semantics of propositional logic is given by truth tables. (Which you already saw.)
- Truth tables:

p	$\neg$
0	1
1	0

p	q	$\wedge$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$\vee$
0	0	0
0	1	1
1	0	1
1	1	1

p	q	$\rightarrow$
0	0	1
0	1	1
1	0	0
1	1	1

- Example implication: If the light is red (p), you must stop (q).

Semantics: Semantics of $\leftarrow$, $\leftrightarrow$

- Some "additional" truth tables: p ➝ q ↔ ¬p ∨ q

p	q	$\rightarrow$
0	0	1
0	1	1
1	0	0
1	1	1

p	q	$\leftarrow$
0	0	1
0	1	0
1	0	1
1	1	1

p	q	$\leftrightarrow$
0	0	1
0	1	0
1	0	0
1	1	1

- We will not need them since we restrict to the standard connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$.

➝: logical implication if p then q; ⇒: material conditional p implies q

In summary, ⇒ is often preferred in formal contexts, such as formal logic and proof theory, to denote logical entailment or consequence, whereas ➝ is more commonly used in introductory logic and informal discourse to represent conditional implication.

Semantics: Expressing Arbitrary Formulae with Truth Tables

Truth tables can be used to express arbitrary formulae, e.g.,

- $p \wedge \neg q$ $\qquad$ $\neg p \vee q$ that's $p \to q$!

p	q	$\neg q$	$\wedge$
0	0	1	0
0	1	0	0
1	0	1	1
1	1	0	0

p	q	$\neg p$	$\neg p \vee q$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

- $p \to (q \to p)$

p	q	$q \to p$	$p \to (q \to p)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

Such a formula, which always evaluates to true is called a tautology.

Semantics: Interpretations and Properties of Formulae

Definition:
An *Interpretation* of a formula ϕ, defined over a set P of propositional variables is an assignment of truth values to variables in P.

Example:

- Let $p =$ *LogicIsInteresting*
- Let $q =$ *PascalsSlidesAreWellDesigned*
- Let $r =$ *studentsUnderstandContent*
- Now consider $(p \wedge q) \rightarrow r$

p	q	r	$(p \wedge q) \rightarrow r$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Here, an interpretation could be:
$I(p) = 1$, $I(q) = 1$, $I(r) = 0$

This interpretation does not make the formula true!
(Interpretations can be thought of as rows in the table)

Semantics: Interpretations and Properties of Formulae, cont'd

- A formula ϕ is a ***tautology*** *(or true)* if: all true
  ϕ is true under every interpretation.
- A formula ϕ is ***satisfiable*** if: any true
  There exists an interpretation that makes ϕ true.
- A formula ϕ is ***unsatisfiable*** if: all false
  There does not exist an interpretation that makes ϕ true.
  Or equivalently: If ϕ is false under every interpretation.

Summary

Content of this Lecture

- Organizational Matters
- Introduction to *Propositional Logic*
  - Syntax: formulae = atoms bound together by logical connectives
  - Semantics: truth tables and interpretations; specify the notion of truth

→ Logic Notes sections:
  - Complete *1. Introduction* except *Inference in the abstract*
  - *3. More about propositional logic*: *Truth tables*.

- Study the following slides by the next lecture.

Syntax Simplifications: Precedence of Connectives

Our connectives use some precedence, which we exploit to eliminate parentheses! Connectives, ordered by precedence:

- Highest: $\neg$ e.g., $\neg p \to q \equiv (\neg p) \to q$
- Second-highest: $\wedge$ e.g., $p \wedge q \vee r \equiv (p \wedge q) \vee r$
- Mid: $\vee$ e.g., $p \to q \vee r \equiv p \to (q \vee r)$
- Second-Lowest: $\to$ e.g., $p \to \neg q \leftrightarrow r \equiv (p \to (\neg q)) \leftrightarrow r$
- Lowest: $\leftrightarrow$ e.g., $\neg p \vee q \leftrightarrow q \wedge r \equiv ((\neg p) \vee q) \leftrightarrow (q \wedge r)$

We reduce parentheses to simplify and avoid confusion by exploiting:

- *precedence*, e.g., we write: $\neg p \to q$ instead of $((\neg p) \to q)$
- *associativity*, e.g., we write:
  - $p \wedge q \wedge r$ instead of $(p \wedge (q \wedge r))$
  - $(p \wedge \neg q \wedge r) \to (p \vee \neg q \vee r)$ instead of $(((p \wedge (\neg q)) \wedge r) \to (p \vee ((\neg q) \vee r)))$

Connective Scopes and Main Connective: Connective Scopes

- Every connective has a *scope*.
- "[The scope of a connective] is defined to be the shortest formula or subformula in which that occurrence lies." (Logic Notes)
- Examples: In the formula $\neg(p \wedge q) \rightarrow ((p \vee r) \rightarrow \neg s)$
  - ... the scope of its first $\neg$ is $(p \wedge q)$
  - ... the scope of its second $\neg$ is s

Connective Scopes and Main Connective: Main Connective

- Every formula has a ***main connective***:
- "[T]he main connective of any formula [...] is the one which is not inside the scope of any other. [...] The scope of the main connective is the whole formula." (Logic Notes)
- Examples: The main connective of . . .
  - $\neg(p \wedge q) \to ((p \vee r) \to \neg s)$ is: the first $\to$
  - $(p \wedge q) \vee r$ is: $\vee$
  - What's the main connective of $(p \wedge q) \vee r \vee (q \to r)$?
    Recall that "$(p \wedge q) \vee r \vee (q \to r)$" is only syntactic sugar!
    - It was either $((p \wedge q) \vee r) \vee (q \to r)$ [then, it's the right $\vee$],
    - or it was $(p \wedge q) \vee (r \vee (q \to r))$ [then, it's the left $\vee$]

    $\to$ Formally, assiciativity defines *uniquely* what a formula with missing parentheses defines. (But that's not important for this course.)
- Why is it important to identify the main connective?
  Because the main connective defines the "type" of the formula, which defines what we are allowed to do in our proofs.

Connective Scopes and Main Connective: Type of Formula

The main connective dictates the type of a formula:

- if main connective is $\neg$, formula is a *negation*
- ... $\wedge$, ... *conjunction*
- ... $\vee$, ... *disjunction*
- ... $\rightarrow$, ... *implication*
- ... $\leftrightarrow$, ... *double-implication*

Substitution: Substitutions of Formulae

replace each occurrence of the varibale with a corresponding new varibale

What is a substitution?

- "Formula A is a substitution instance of formula B if and only if A results from B by substitution of formulas for sentence letters." (Logic Notes) – (Definition is specific to *propositional* logic.)

Example:

- "Original" formula: $q \lor p$
- One of its substitution instances is $(p \land q) \lor \neg r$, because:
  - q got substituted by $(p \land q)$
  - p got substituted by $\neg r$

Non-Example:

- "Original" formula: $q \lor q$
- The formula $(p \land q) \lor \neg r$ is *not a* substitution instance of it (because the left part had to be the same as the right)