1. Question 1

Please note that there are multiple ways of answering these questions. The predicates were stated subsets of D (or $D \times D$), so in the first two solutions we work with that rather than treating each predicate as a function $D \to \{1, 0\}$. You could also use m-variants, as in:

https://users.cecs.anu.edu.au/ jks/LogicNotes/semantics2.html .

As long as your solution constitutes a rigorous proof, it's all good.

Part 1:

$$I(\exists x S(x)) = 1 \text{ iff } a \in I(S) \text{ for some a in the domain}$$
$$\text{iff } a \in \{s1, s2, s3\} \text{ for some a in the domain, which is true.}$$

Part 2:

$I(\forall x C(x, x)) = 1$ iff $(a, a) \in I(C) = 1$ for every a in the domain

iff $(a, a) \in I(C)$ for every a in the domain, which is true because $dis(a, a) = 0 < 2$.

Part 3: In the following solutions, it's notationally much easier to use the alternative interpretations of predicates/relations, as functions to $\{0, 1\}$.

$I(\forall x \forall y \forall z (C(x, y) \land C(y, z) \to C(x, z))) = 1$

iff $I(\forall y \forall z (C(x, y) \land C(y, z) \to C(x, z)))(a) = 1$ for all a in the domain

iff $I(\forall z (C(x, y) \land C(y, z) \to C(x, z)))(a)(b) = 1$ for all a,b in the domain

iff $I((C(x, y) \land C(y, z) \to C(x, z)))(a)(b)(c) = 1$ for all a,b,c in the domain

iff whenever$I((C(x, y) \land C(y, z))(a)(b)(c) = 1$ we have $I(C(x, z)))(a)(b)(c) = 1$ for all a,b,c in the domain.

Now, suppose $I((C(x, y) \land C(y, z))(a)(b)(c) = 1$. This is true iff $I((C(x, y))(a)(b)(c) = 1$ and $C(y, z))(a)(b)(c) = 1$. When a is r, y is s1, and z is s2 (or s3), we get that this is true. In this case however, we do not have $I(C(x, z)))(a)(b)(c) = 1$, because $dis(r, s2) = 2$ (and 2 is not <2). Therefore, the formula is false in the interpretation I.

Note that writing something like $I(C(x, z)))(a)(b)(c)$ seems odd, and we need to implicitly keep track of which element in the domain is being substituted for which free variable. In this case, throughout the proof, we have in mind that $[x \to a], [y \to b], [z \to c]$, or more so, that x takes the first value in the function, y takes the second (which is not there in the formula $C(x, z)$), and z takes the third.

2. Question 3

Again, many answers. Some key points might include:

- Semantics is the world we want to reason about.
- Syntax forms the language and rules we can use to do this reasoning.
- There are many different situations to which we want to apply logic. Separating the syntax allows us to use our logical rules for many different models, so long as we have a sound way to interpret them.
- Separating syntax allows us to question the proposed rules of reasoning itself without any of the messy particulars of the real world to get in the way.