THE LOGICAL LANDSCAPE

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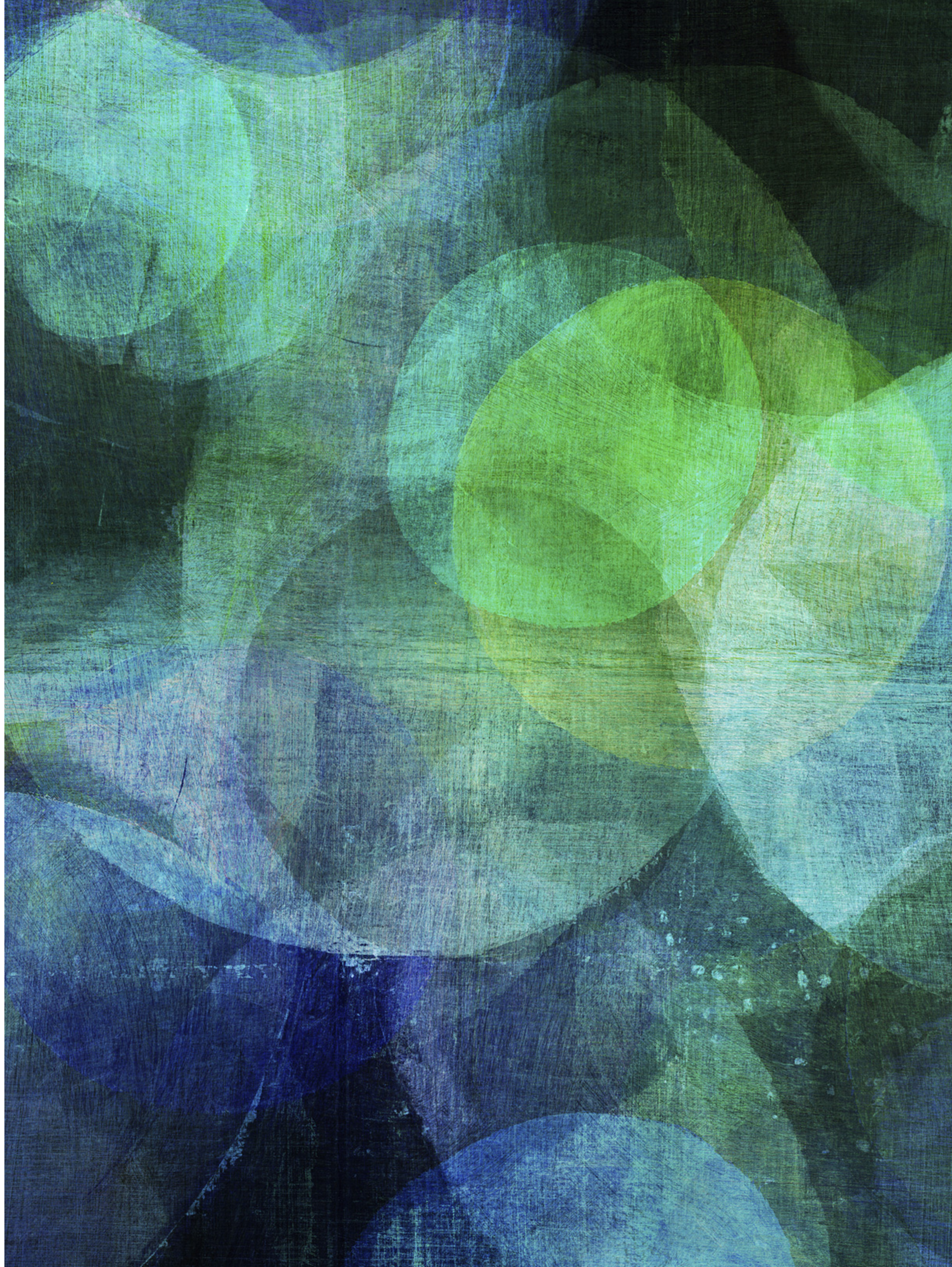


SUMMARY – WHAT DID YOU LEARN?

- 1. Natural deduction (propositional; first-order; equality; restricted quantifiers)
- 2. Semantic tableaux (propositional; first-order)
- 3. Semantics (propositional; first-order; and non-classical three-valued models)
- 4. Sequent calculus (multiple-conclusion CL; and single-conclusion IL)
- 5. Relevant logic and the paradoxes of implications
- 6. Fuzzy logic and the sorites paradox
- 7. Intuitionistic logic and the constructive view of truth

THE LOGICAL LANDSCAPE

(Int. Rel. Log.) → Relevant Logic

Full Lambek Calculus

(single-conclusion SC

without contr./weak.)

Intuitionistic Logic → Classical Logic

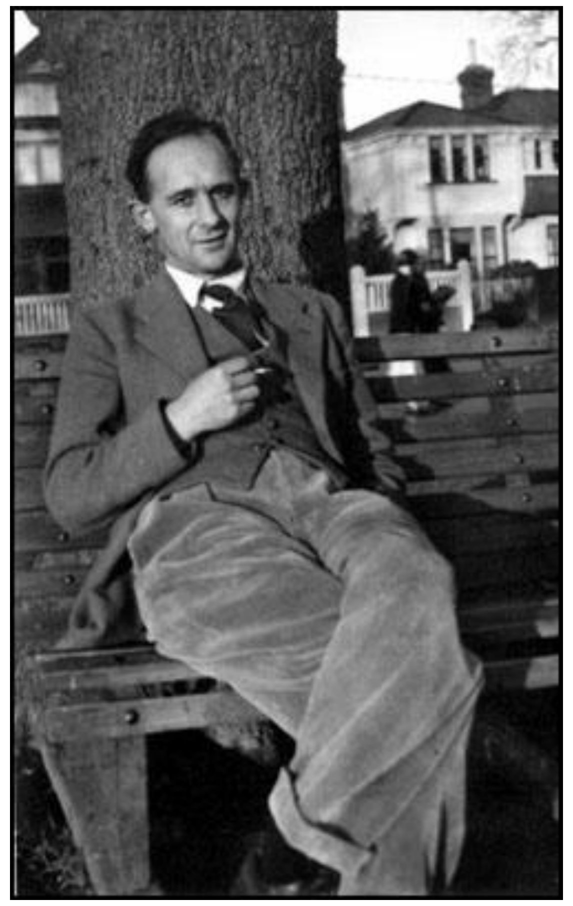
(Int. Fuz. Log.) → Fuzzy Logic

REFLECTION – YOUR FAVOURITE LEADS YOU TO A DEEPER WORLD

- What did you like?
- Proof system or semantics?
- Classical or non-classical (relevant, fuzzy, intuitionistic)?
- Logical pluralism or monism?
- Paradox of implication or Sorites paradox (or Liar paradox)?
- Multiple- or single-conclusion? Realism or antirealism?
- Semantic denotationalism or inferentialism?

PRIOR'S TONK PARADOX

- You can do whatever you want to do in logic; there is no predetermined law in logic.
- Arthur Prior, born in New Zealand, defined a logical connective "tonk" in ND.



https://plato.stanford.edu/entries/prior/

$$\frac{A}{A tonk B}\ tonkI \qquad\qquad \frac{A tonk B}{B}\ tonkE$$

- We can then prove anything implies anything, which basically makes logic inconsistent.
- What's wrong with Prior's tonk? It has both intro. and elim. rules like any other connectives. Why can't we define tonk in natural deduction? Or can we actually?
- Prior's tonk paradox is still debated in philosophy of logic.

THE LOGICAL LANDSCAPE IN ARITHMETIC AND SET THEORY

All this is systems over classical logic.
If we change the underlying logic into FL calculus,
naive set theory is consistent (no Russell paradox).

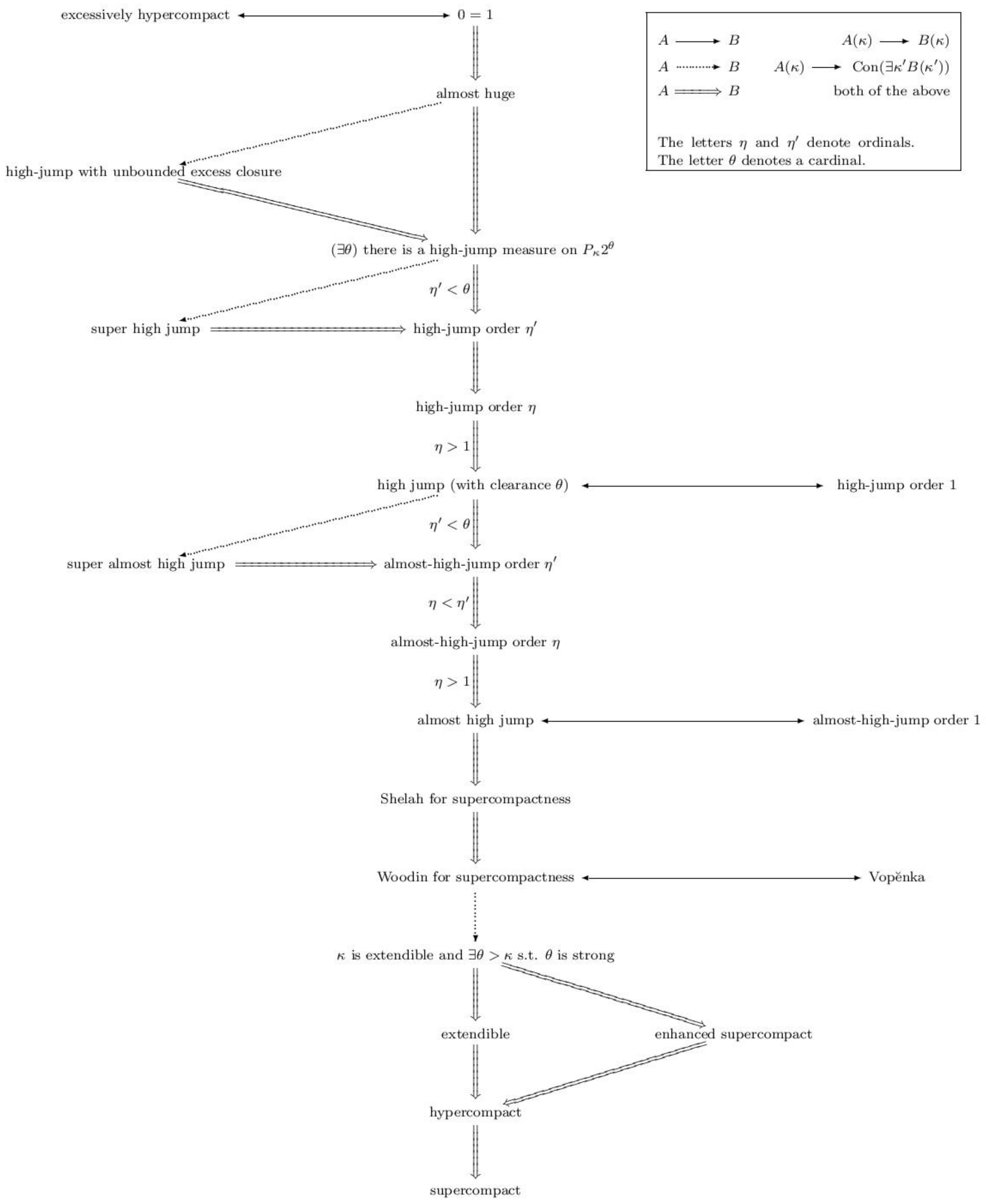
Primitive Recursive Arithmetic → Robinson Arithmetic → Peano Arithmetic → Second-Order Arithmetic → Finite-Order Arithmetic → Zermelo Set Theory → ZF Set Theory

ZF Set Theory → Inconsistent Naive Set Theory

ZF Set Theory → Higher-Order Set Theory → Inconsistent Infinitary Set Theory

Be careful of large infinities,
which sometimes solve your problems
but often lead you to inconsistencies.

THE 0=1 LANDSCAPE IN SET THEORY



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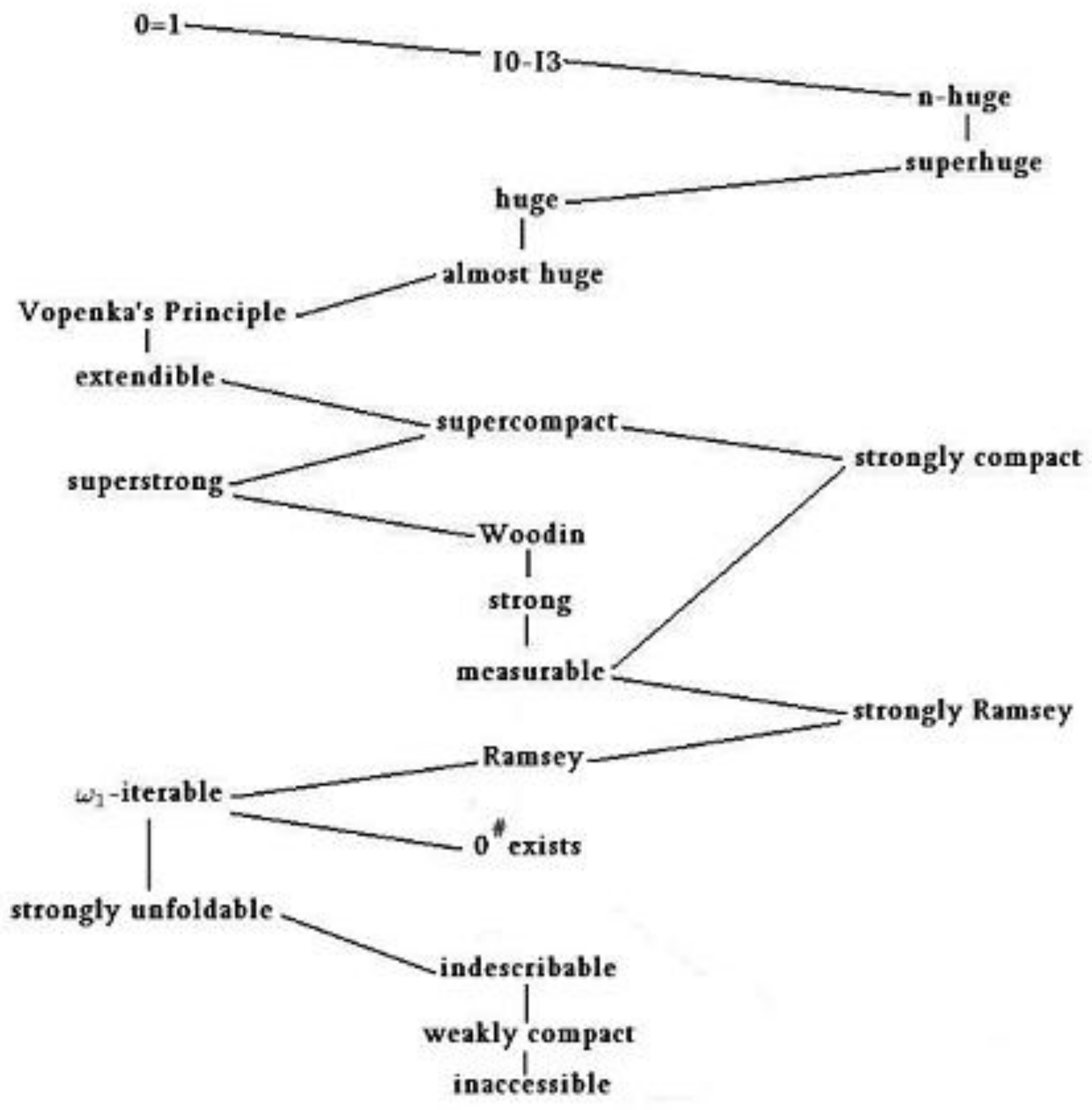
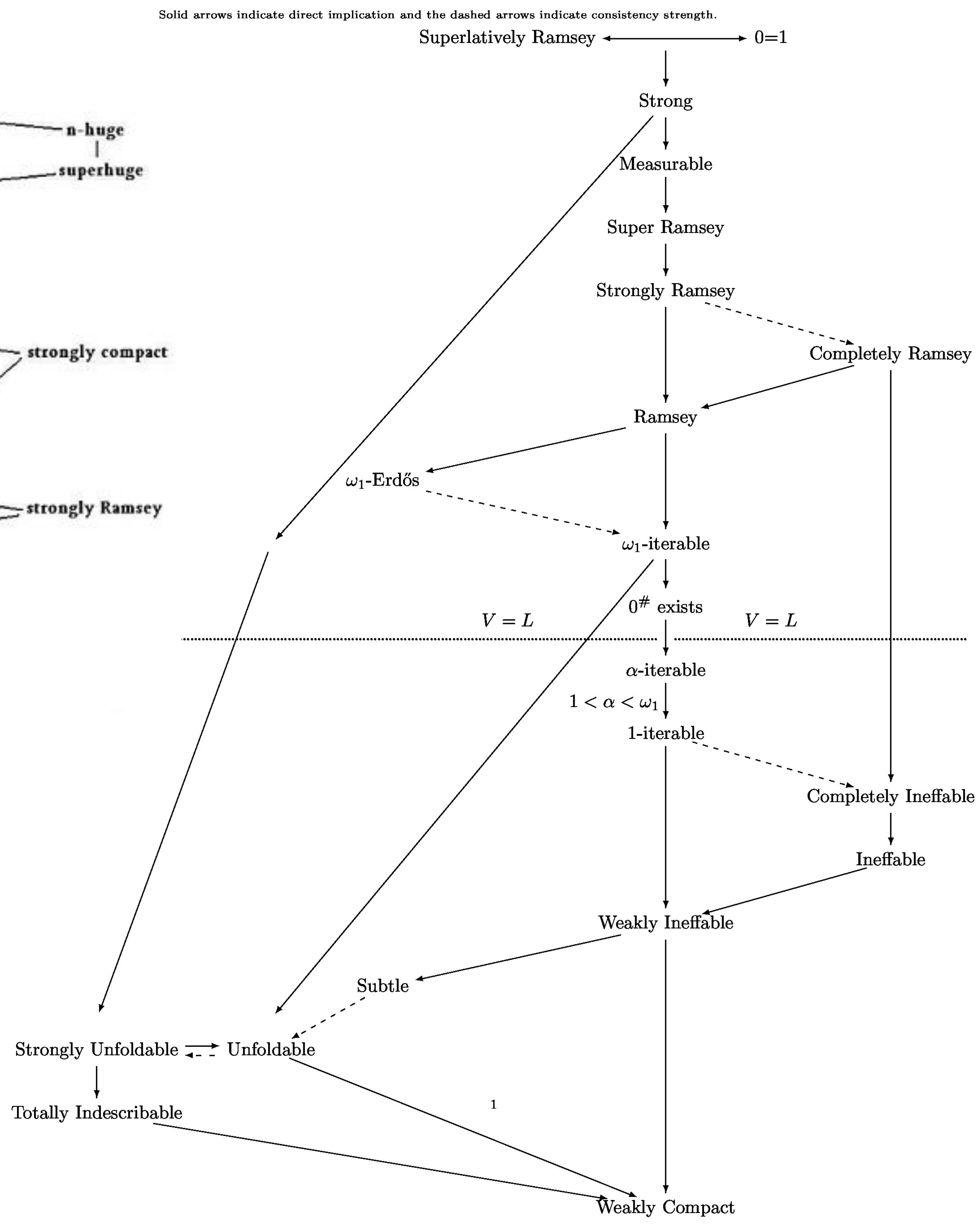


https://ncatlab.org/nlab/show/large+cardinal



https://mathoverflow.net/questions/194486/is-there-a-compendium-of-the-consistency-strength-between-the-most-important-for

PEANO ARITHMETIC

- How to prove 1+1=2? Note: 1=s0 and 2=ss0.
- Then: 1+1=s0+s0=s(s0+0)=ss0=2. Also: 1*1=s0*s0=s0*0+s0=s0=1.

Peano Axioms for natural numbers

PA1 $\forall x(\neg(s(x)=0))$

PA2 $\forall x \forall y(s(x)=s(y) \rightarrow x=y)$

PA3 $\forall x(x+0=x)$

PA4 $\forall x \forall y(x+s(y)=s(x+y))$

PA5 $\forall x(x \cdot 0=0)$

PA6 $\forall x \forall y(x \cdot s(y)=x \cdot y+x)$

PA7 $[A(0) \wedge \forall x(A(x) \rightarrow A(s(x)))] \rightarrow \forall x A(x)$

=: binary relation symbol

0: constant symbol

s: "+1" function symbol

+: "add" function symb.

$\cdot$:"multiply" fun. sym.

PA7: induction axiom

https://www.chegg.com/homework-help/questions-and-answers/relation-u-greater-equal-v-natural-numbers-n-defined-formula-ez-u-z-v--give-formal-proof-u-q64888301

ZF SET THEORY WITH THE AXIOM OF CHOICE

The axioms of Zermelo-Fraenkel set theory with choice ZFC

In principle all of mathematics can be derived from these axioms

Extensionality	$\forall X\, \forall Y\, [\, X = Y \quad \Leftrightarrow \quad \forall z (z \in X \;\Leftrightarrow\; z \in Y)\,]$
Pairing	$\forall x\, \forall y\, \exists Z\, \forall z\, [\, z \in Z \quad \Leftrightarrow \quad z = x$ or $z = y\,]$
Union	$\forall X\, \exists Y\, \forall y\, [\, y \in Y \quad \Leftrightarrow \quad \exists Z (Z \in X$ and $y \in Z)\,]$
Empty set	$\exists X\, \forall y\, [\, y \notin X\,]$ (this set X is denoted by $\emptyset$)
Infinity	$\exists X\, [\, \emptyset \in X$ and $\forall x (x \in X \Rightarrow x \cup \{x\} \in X)\,]$
Power set	$\forall X\, \exists Y\, \forall Z\, [\, Z \in Y \quad \Leftrightarrow \quad \forall z (z \in Z \;\Rightarrow\; z \in X)\,]$
Replacement	$\forall x \in X\, \exists! y\, P(x,y) \quad \Rightarrow \quad [\, \exists Y\, \forall y\, (y \in Y \;\Leftrightarrow\; \exists x \in X\, (P(x,y)))\,]$
Regularity	$\forall X\, [\, X \neq \emptyset \quad \Rightarrow \quad \exists Y \in X\, (X \cap Y = \emptyset)\,]$
Axiom of choice	$\forall X\, [\, \emptyset \notin X$ and $\forall Y, Z \in X (Y \neq Z \;\Rightarrow Y \cap Z = \emptyset)$ $\quad \Rightarrow \quad \exists Y\, \forall Z \in X\, \exists! z \in Z\, (z \in Y)\,]$

$\forall =$ for all $\qquad \exists! =$ there exists a unique $\qquad P$ is any formula that does not contain Y

$z \in X \cup Y \;\Leftrightarrow\; z \in X$ or $z \in Y \qquad z \in X \cap Y \;\Leftrightarrow\; z \in X$ and $z \in Y$

Set theory axioms — Math Poster 2007 — math.chapman.edu

https://cute766.info/the-axioms-of-zermelo-fraenkel-set-theory-with-choice-zfc/

GÖDEL'S FIRST INCOMPLETENESS THEOREM

- Gödel's first incompleteness theorem: if a logical system is finitary and rich enough to express basic arithmetic, there is always an undecidable proposition A within the system (i.e., A is not provable, and its negation is not provable, either).
  - It can happen that the human mind still recognises the truth of that formally undecidable proposition A, and thus Lucas and Penrose argued AI is impossible.
  - Lucas-Penrose have been criticised so much, but Gödel himself argued: "the human mind (even within the realm of pure mathematics) infinitely surpasses the power of any finite machine" or "there exist absolutely unsolvable diophantine problems"
    - Gödel was a Platonist about math, and did not believe in the second disjunct, so he believed the human mind "infinitely surpasses the power of any finite machine."

GÖDEL'S SECOND INCOMPLETENESS THEOREM

- Gödel's second incompleteness theorem: under the same assumptions, it is impossible to prove the consistency of the system within the same system.
  - You cannot prove your own consistency, which can only be proven with a stronger system. This means there are different strengths of consistency of logical systems.
    - If a logical system S1 can prove the consistency of another system S2, then S1 is stronger in terms of consistency strength than S2.
  - The second incompleteness is called intensional whereas the first incompleteness is extensional, due to certain subtleties in the second incompleteness (with a subtle trick, we can actually prove the consistency of a system with the system).

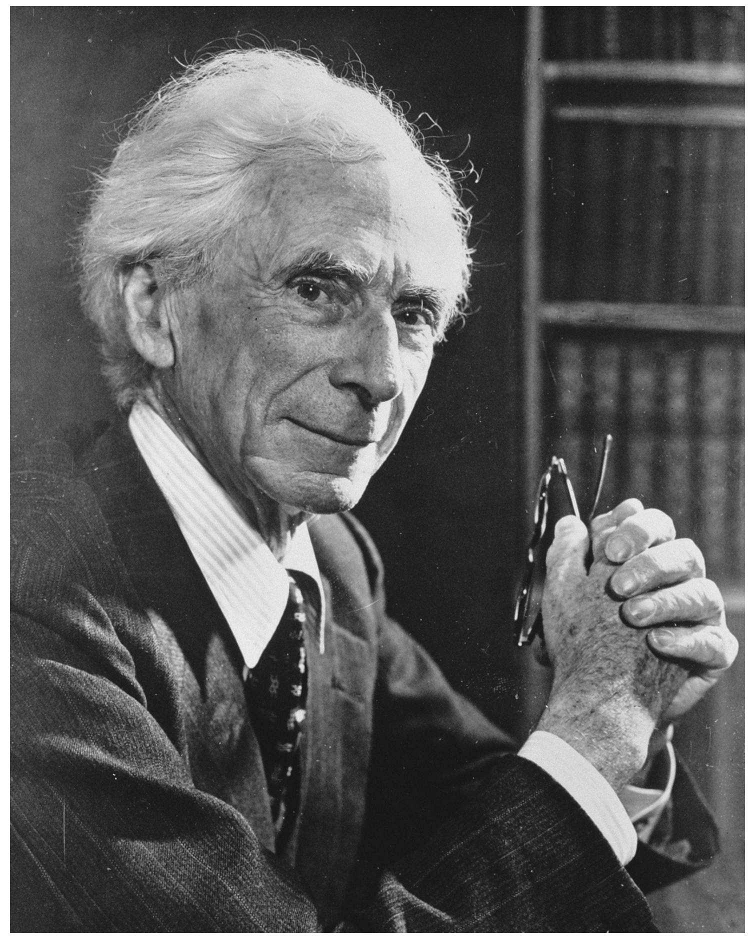
THE PHILOSOPHICAL LOGICAL LANDSCAPE

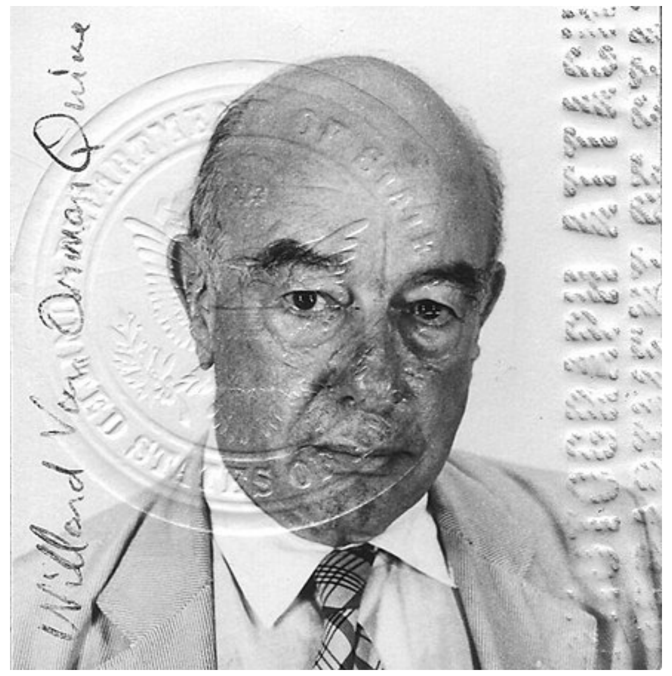
Martin-Löf

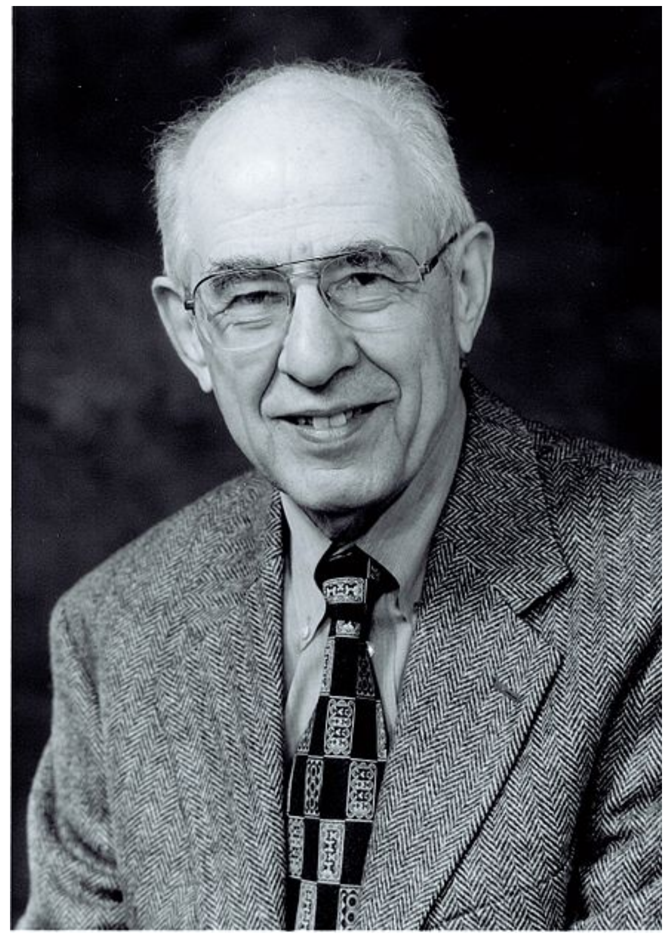
Hintikka

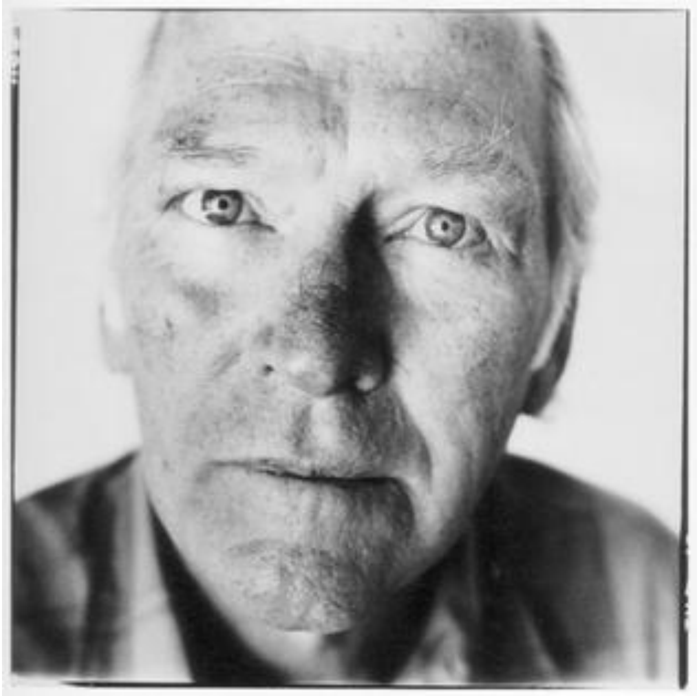
Frege / Russell / Pierce → Carnap / Vienna Circle / Wittgenstein → Quine / Putnam → Dummett / Davidson → Kripke / D. Lewis → … (and many others) → T. Williamson / and many others

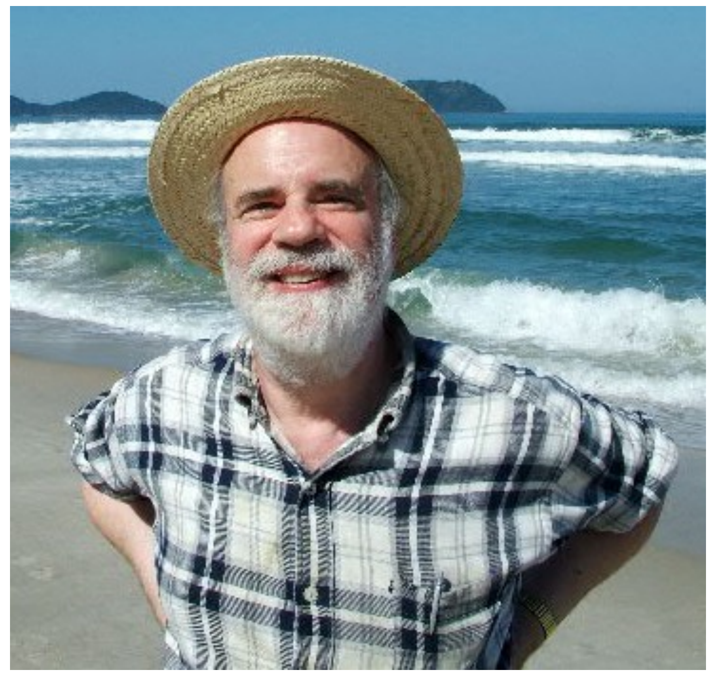
https://undsoc.org/2021/01/07/vienna-circle-in-emerson-hall/

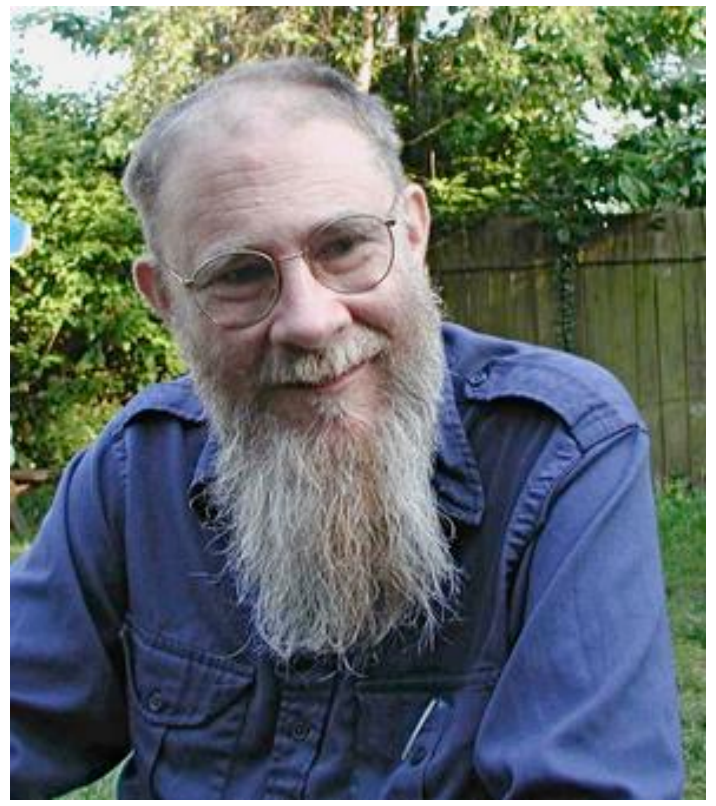












His famous books (further reading):

Vagueness (1994)

Knowledge and its Limits (2000)

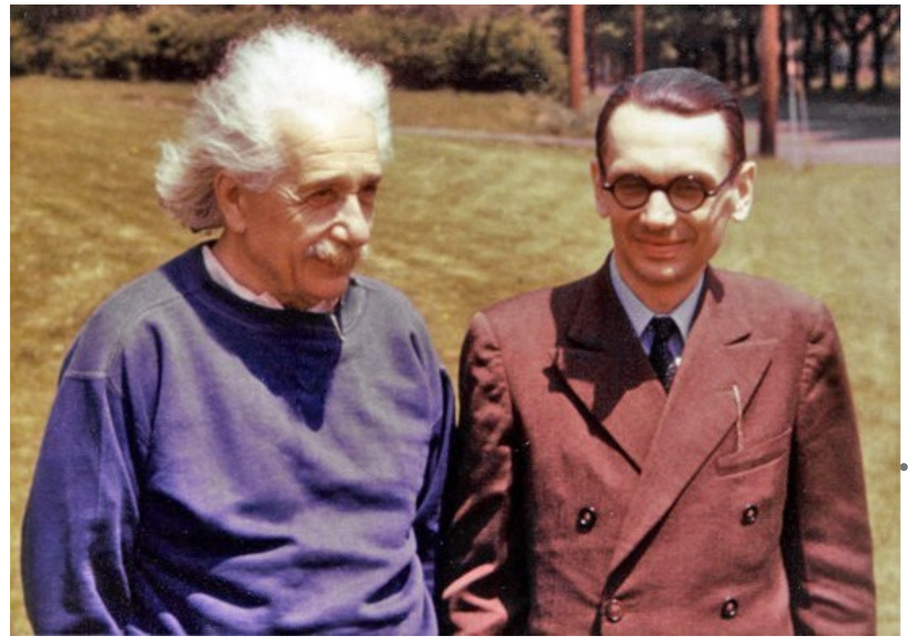
The Philosophy of Philosophy (2007)

Modal Logic as Metaphysics (2013)

HOW TO USE LOGIC IN ORDINARY LIFE

- When someone says “It is logically correct”, you can ask “Which logic do you mean?”
- When someone says “It is obvious like 1+1=2”, you can ask “Can you prove it logically?”
- When someone says “Everything is unambiguously defined in mathematics”, you can ask “How do you define conjunction without using any conjunction at the meta-level?”
- When someone says “There are always answers to mathematical questions”, you can ask “What do you think about the first incompleteness theorem?”
  - When someone says “Machines replace humans”, you can ask the same question.
- When someone says “Mathematics is always consistent and coherent”, you can ask “What do you think about the second incompleteness theorem?”

IT'S FINALLY DONE



https://www.brainpickings.org/2015/05/07/rebecca-goldstein-incompleteness-godel-einstein-time/

https://www.mentalfloss.com/article/501995/12-fascinating-facts-about-rick-and-morty

Thank you!

Hope you enjoyed logic and continue to do so.

Try to use logic in your life so that you don't forget.

Chat with your friends about logic like guys above.

Good luck on exam, but grade is grade, logic is logic.