Image Filtering

Week – 2B

Announcement

- Mathematics Reference Book: Linear algebra, probability
  - https://mml-book.github.io/book/mml-book.pdf
  - eg. CDF (cumulative distribution function, see chapter 6.2 discrete and continuous distribution function)

- Lecture Series by Prof. Shree Nayar, Professor of Computer Science in the School of Engineering at Columbia University, Senior researcher in computer vision with high profile.
  - https://fpcv.cs.columbia.edu/

Outline

- Image Noise
- Linear filter
    - Gaussian filter
    - Edge detection
- Non-linear filter
    - median filter
    - bilateral filter

Outline

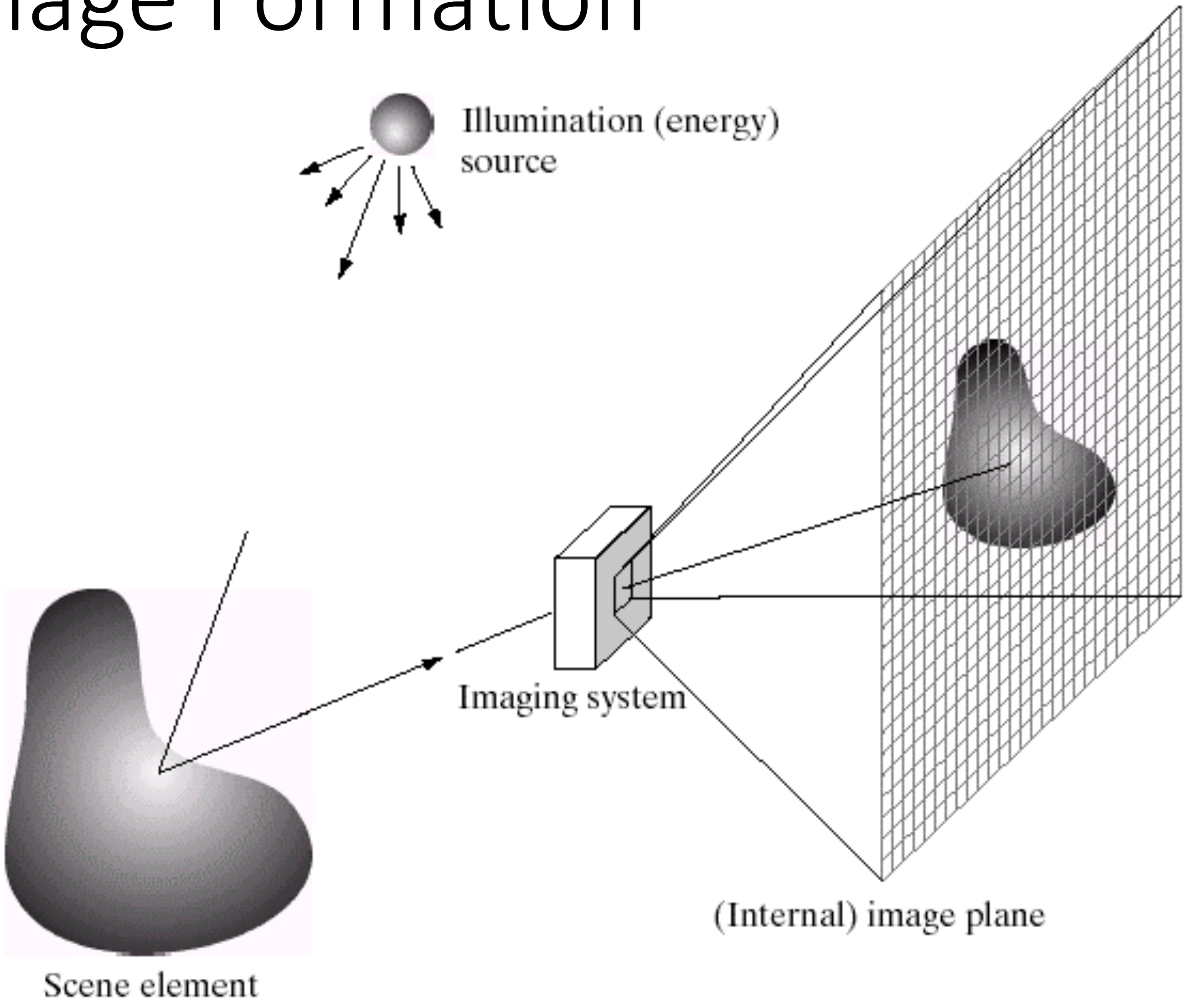
- Image Noise
- Linear filter
  - Gaussian filter
  - Edge detection
- Non-linear filter
  - median filter
  - bilateral filter

Image Filtering

- Filtering (correlation, convolution)
- Gaussian filter
- Application of filters (denoising, edge, contour, corner, texture, template matching and tracking ...,)

Image Noise

Recall: Image Formation



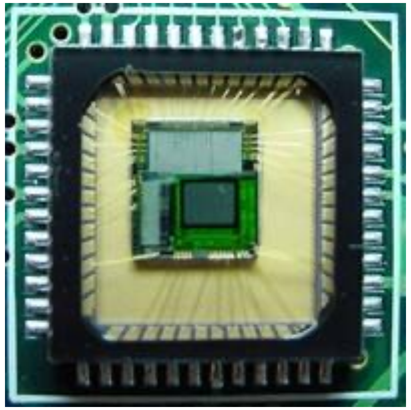
Recall: Digital camera

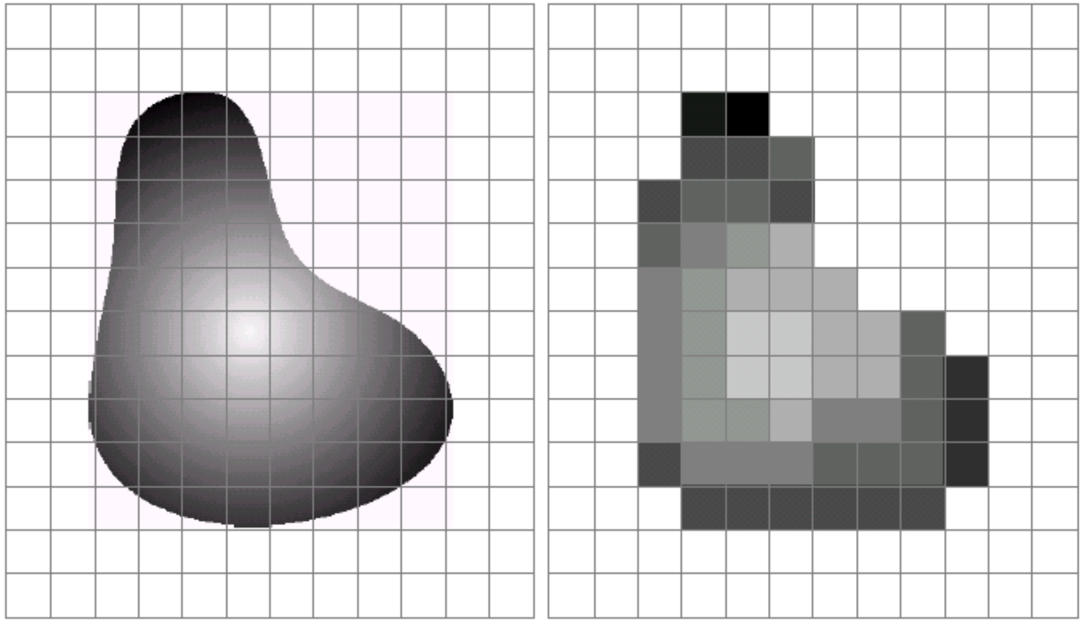
A digital camera replaces film with a sensor array

- Each cell in the array is light-sensitive diode that converts photons to electrons
- http://electronics.howstuffworks.com/digital-camera.htm

Slide by Steve Seitz

Digital images





a b

FIGURE 2.17 (a) Continuos image projected onto a sensor array. (b) Result of image sampling and quantization.

Recall: Images as functions

- We can think of an image as a function, *f*, from R^2 to R: Input: coordinate output: intensity value
  - $f(x, y)$ gives the intensity at position (x, y)
  - Realistically, we expect the image only to be defined over a rectangle, with a finite range:
    - f: $[a,b]$ x $[c,d] \rightarrow [0, 255]$

      limit intensity
- A color image is just three functions pasted together. We can write this as a "vector-valued" function:

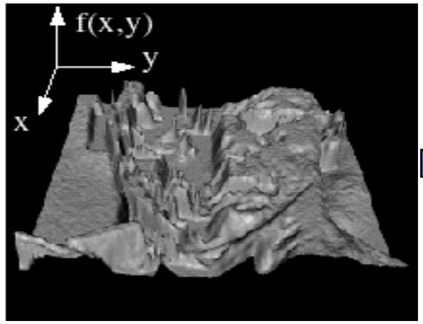
$$f(x,y) = \begin{bmatrix} r(x,y) \\ g(x,y) \\ b(x,y) \end{bmatrix}$$

Source: S. Seitz

Recall: Digital images

- **Sample** the 2D space on a regular grid
- **Quantize** each sample (round to nearest integer)

- Image is thus represented as a matrix of integer values.

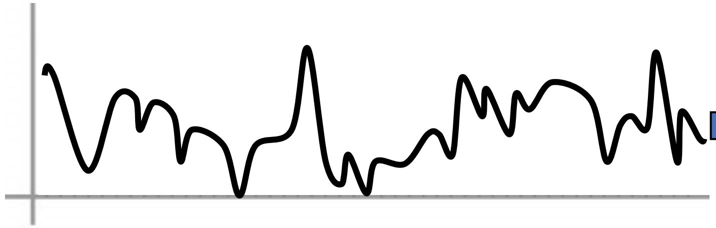


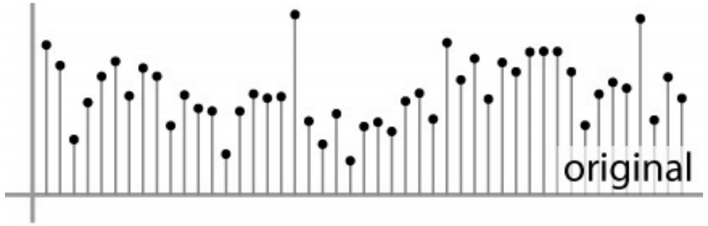
x

y

62	79	23	119	120	105	4	0
10	10	9	62	12	78	34	0
10	58	197	46	46	0	0	48
176	135	5	188	191	68	0	49
2	1	1	29	26	37	0	77
0	89	144	147	187	102	62	208
255	252	0	166	123	62	0	31
166	63	127	17	1	0	99	30

2D

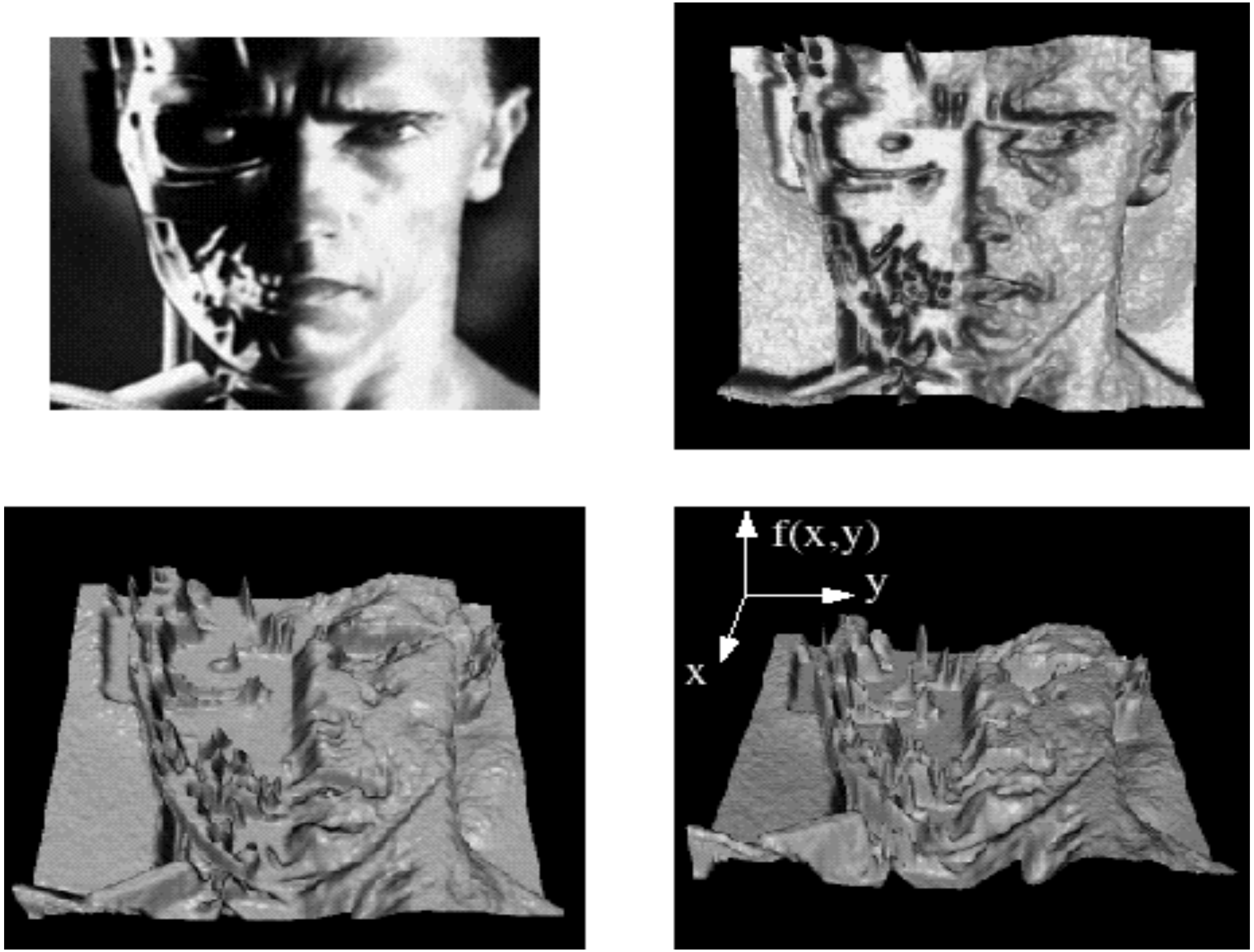




1D

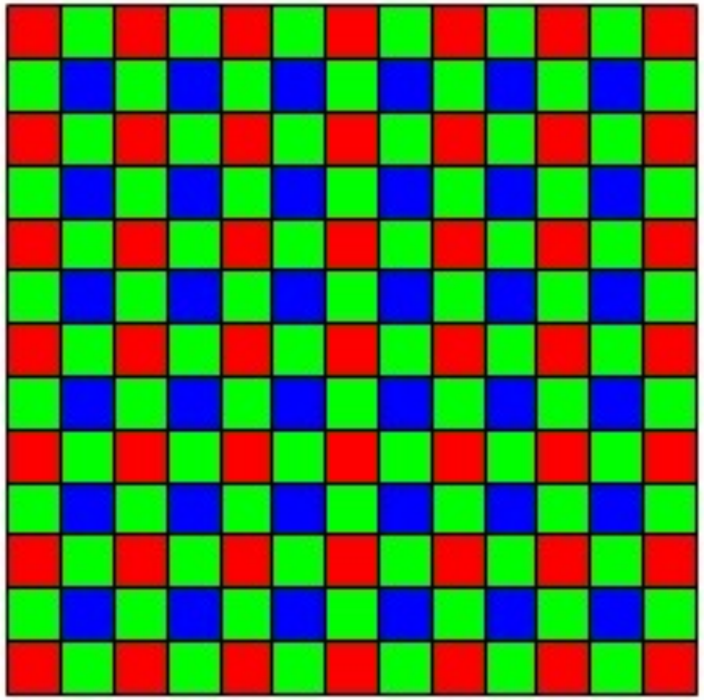
Adapted from S. Seitz

Recall: Images as functions



Source: S. Seitz

Digital color images



Bayer filter

Slide credit: Kristen Grauman

Recall: Digital color images

Color images, RGB color space

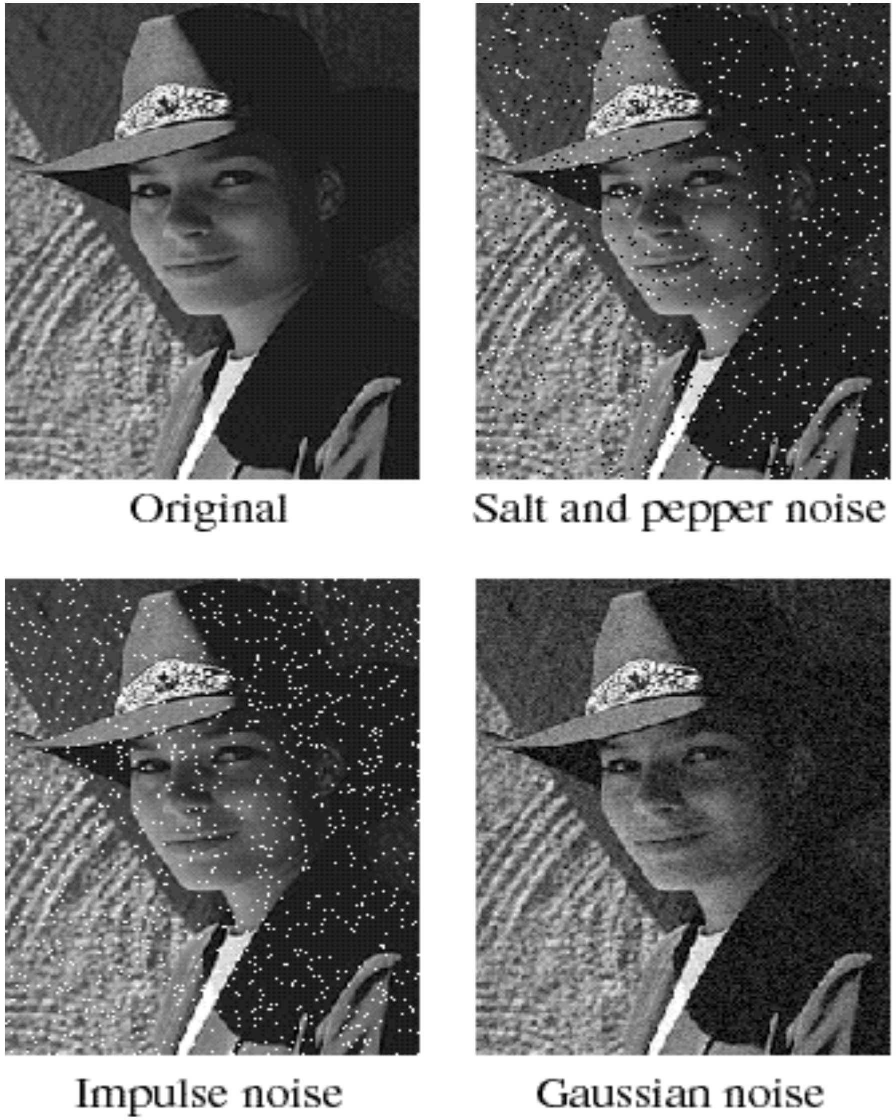
R

G

B

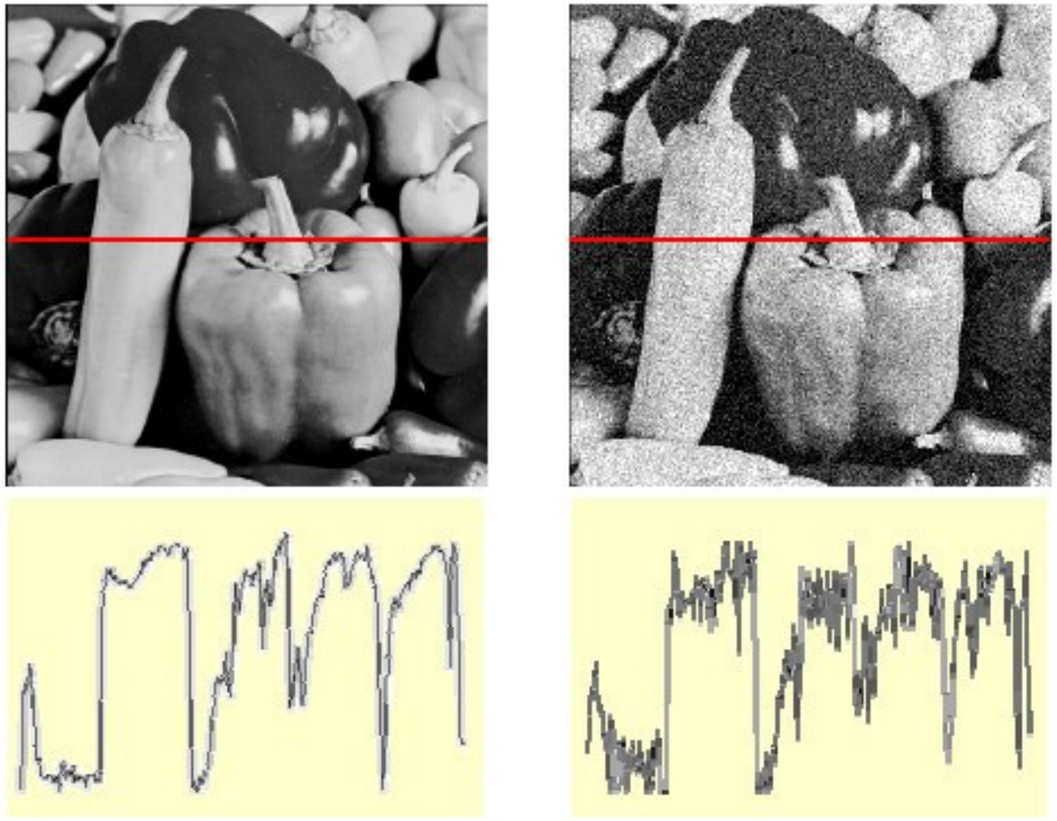
Common types of noise

- **Salt and pepper noise**: random occurrences of black and white pixels
- **Impulse noise:** random occurrences of white pixels, sometimes, it is also referred to as Salt and pepper noise.
- **Gaussian noise**: variations in intensity drawn from a Gaussian normal distribution



Source: S. Seitz

Gaussian noise



$$f(x,y) = \overbrace{\bar{f}(x,y)}^{\text{Ideal Image}} + \overbrace{\eta(x,y)}^{\text{Noise process}}$$

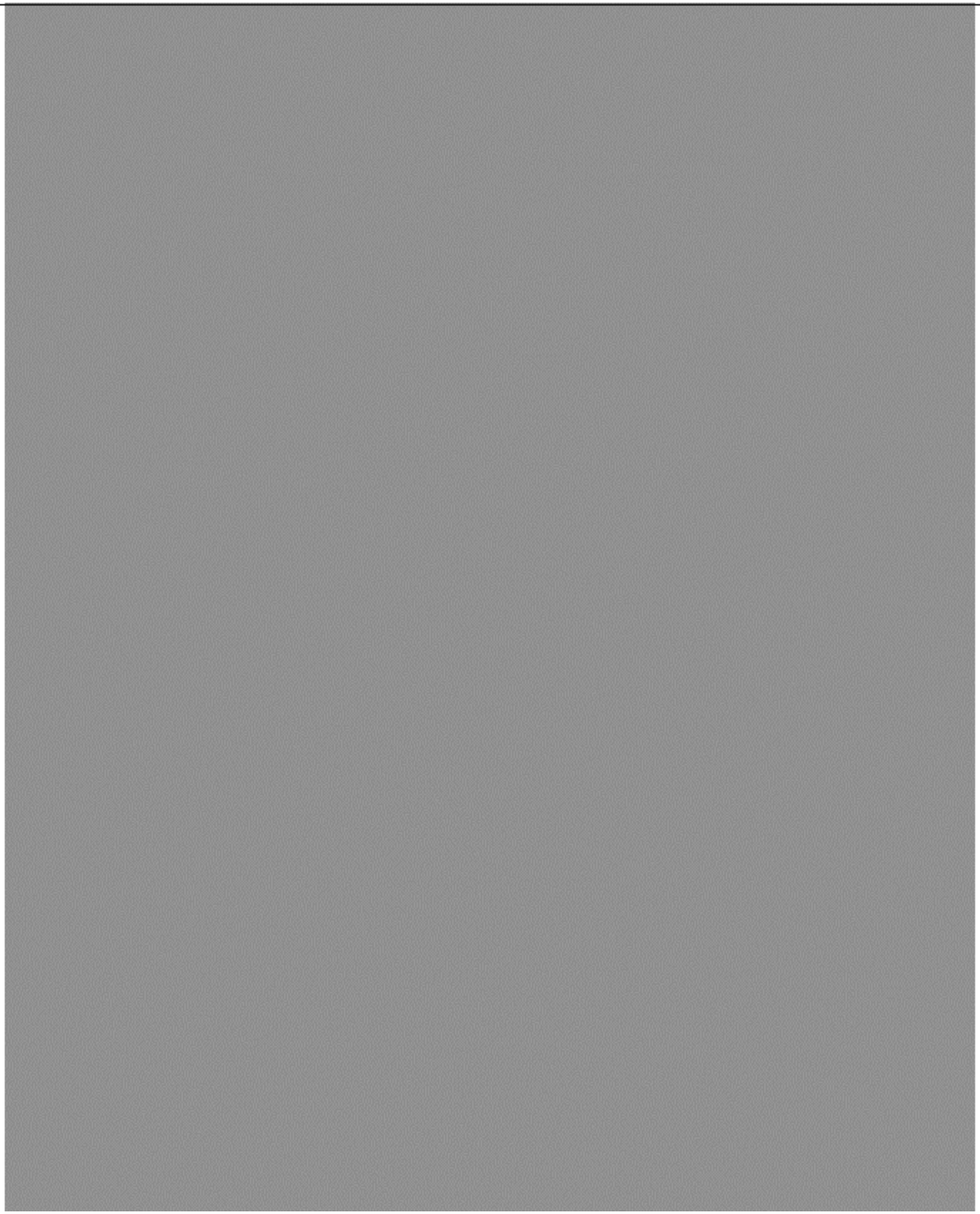
Gaussian i.i.d. ("white") noise:
$\eta(x,y) \sim \mathcal{N}(\mu, \sigma)$

```
>> noise = np.random.normal(mean, std_dev, shape)
>> output = im + noise;
```

What is the impact of the sigma?

Fig: M. Hebert

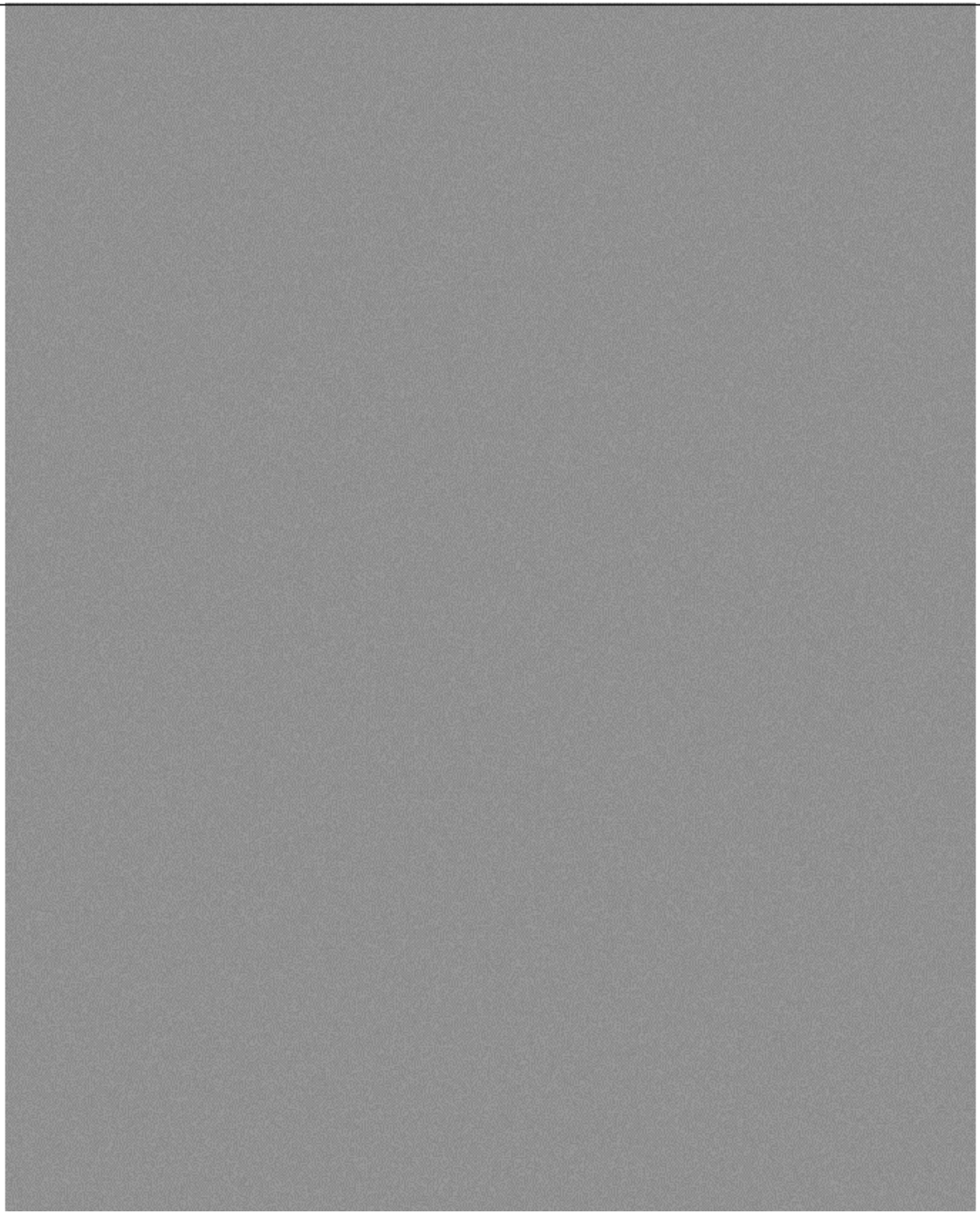
sigma=1



Effect of sigma on Gaussian noise:

Image shows the noise values themselves.

sigma=4



Effect of sigma on Gaussian noise:

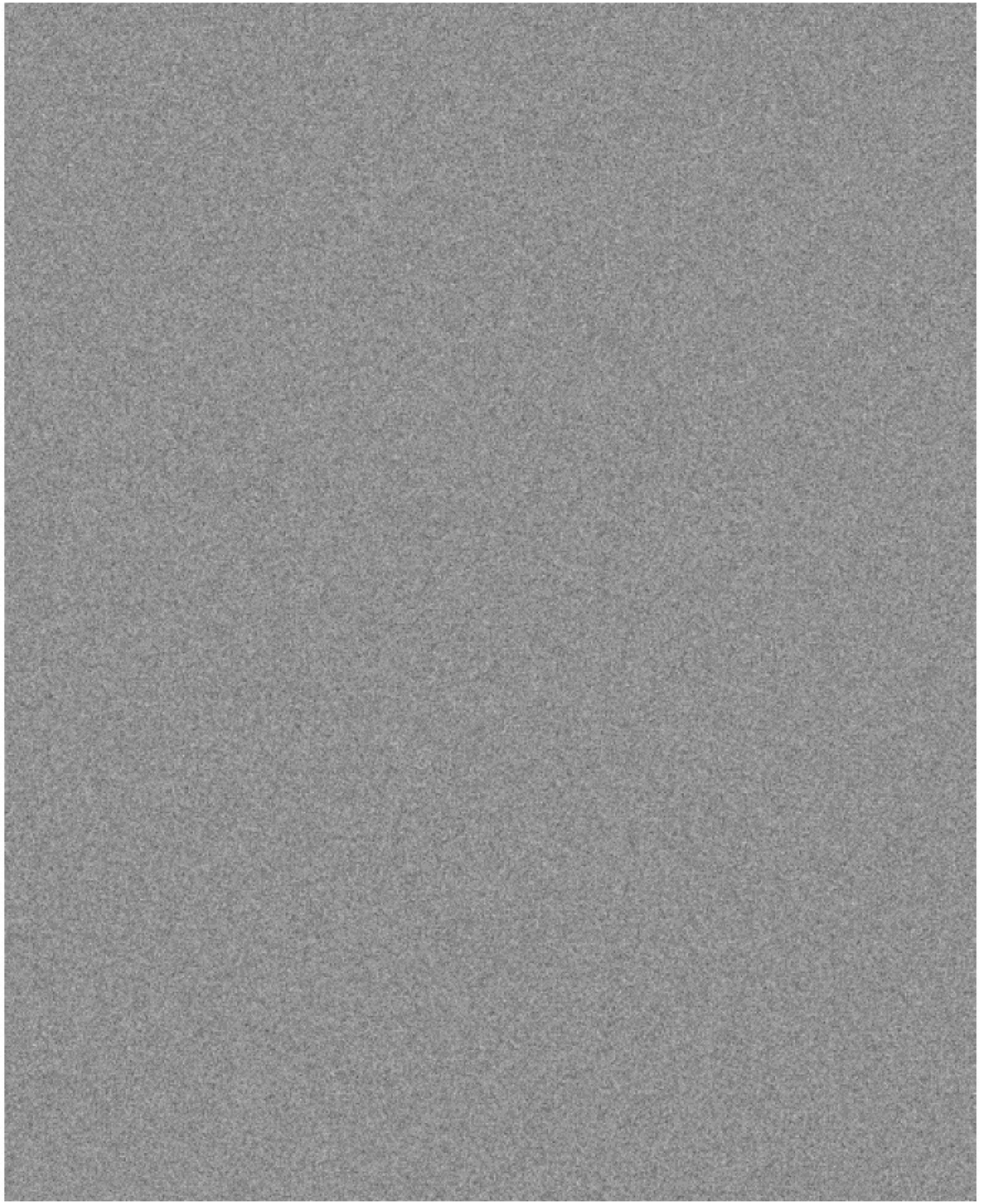
Image shows the noise values themselves.

sigma=1

Effect of sigma on Gaussian noise:

This shows the noise values added to the raw intensities of an image.

sigma=
16



Effect of sigma on Gaussian noise:

Image shows the noise values themselves.

sigma=16

Effect of sigma on Gaussian noise

This shows the noise values added to the raw intensities of an image.

Noise Reduction

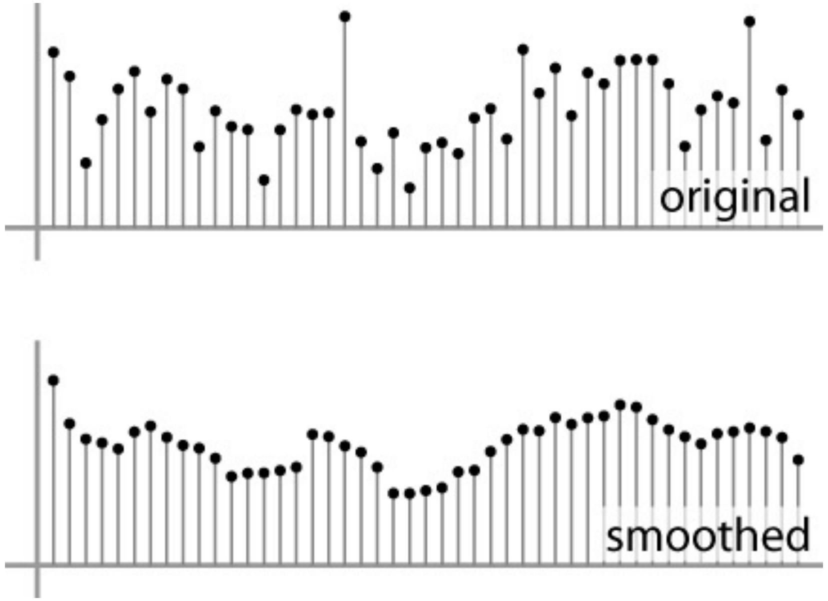
- How should we reduce the noise?

First attempt at a solution

- Let’s replace each pixel with an average of all the values in its neighborhood (neighborhood operation)
- Assumptions:
  - Expect pixel to be like their neighbors
  - Expect noise processes to be independent from pixel to pixel

First attempt at a solution

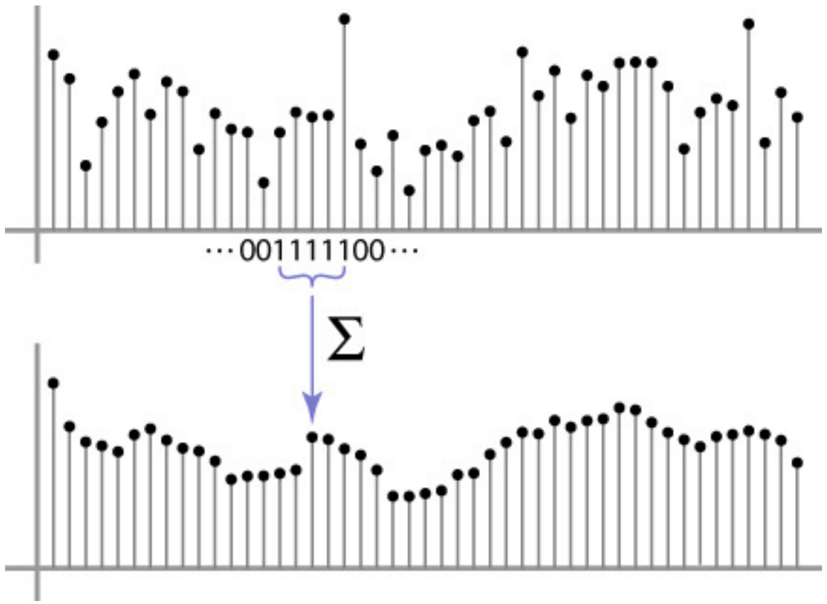
- Let's replace each pixel with an average of all the values in its neighborhood
- Moving average in 1D:



•Source: S. Marschner

Weighted Moving Average

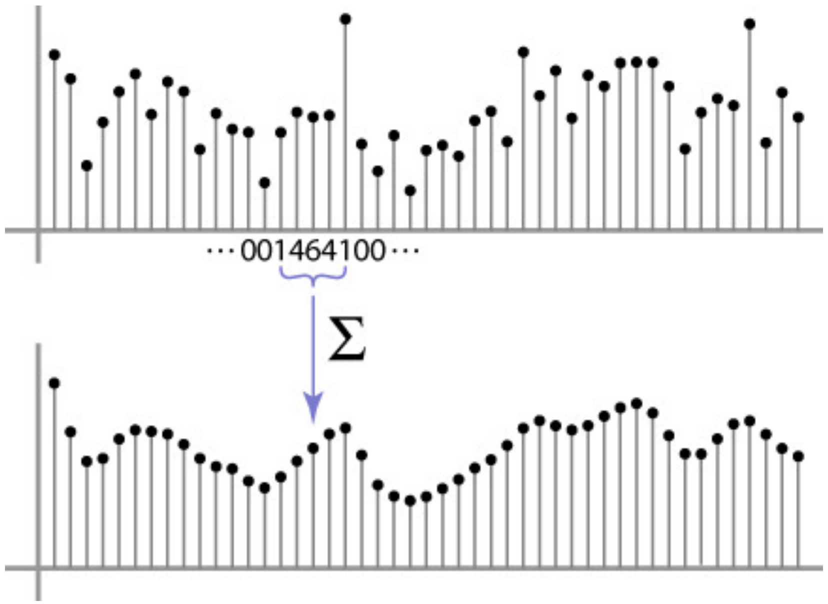
- Can add weights to our moving average
- *Weights* [1, 1, 1, 1, 1] / 5



•Source: S. Marschner

Weighted Moving Average

- Non-uniform weights [1, 4, 6, 4, 1] / 16



•Source: S. Marschner

Moving Average In 2D

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0								

•Source: S. Seitz

Moving Average In 2D

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10							

•Source: S. Seitz

Moving Average In 2D

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20						

•Source: S. Seitz

Moving Average In 2D

$$F[x, y]$$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$$G[x, y]$$

	0	10	20	30					

•Source: S. Seitz

Moving Average In 2D

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30	30				

•Source: S. Seitz

Moving Average In 2D

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

•Source: S. Seitz

Correlation filtering

Say the averaging window size is (2k+1) x (2k+1):

$$\underbrace{G[x,y] = \frac{1}{(2k+1)^2}}_{\text{Attribute uniform weight to each pixel}} \underbrace{\sum_{u=-k}^{k} \sum_{v=-k}^{k} F[x+u, y+v]}_{\text{Loop over all pixels in neighborhood around image pixel F[x,y]}}$$

average over the window

Now we can generalize to allow **different weights** depending on neighboring pixel's relative position:

$$G[x,y] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} \underbrace{H[u,v]}_{\text{Non-uniform weights}} F[x+u, y+v]$$

Slide credit: Kristen Grauman

Correlation filtering

u, v define the size of window

after $$G[x,y] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} H[u,v] F[x+u, y+v]$$ before

moving across the pixels in the window

kernel/mask that depands on the coordinates in the window

This is called cross-correlation, denoted $\boxed{G = H \otimes F}$

- Filtering an image: replace each pixel with a linear combination of its neighbors.
- The filter "kernel" or "mask" *H* [*u*,*v*] defines the weights in the linear combination.

Slide credit: Kristen Grauman

Averaging filter

- What values belong to the kernel *H* for the moving average example?

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$\otimes$

$H[u, v]$

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & ? & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

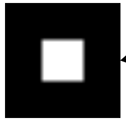
"box filter"

$G[x, y]$

	0	20	40	60	60				

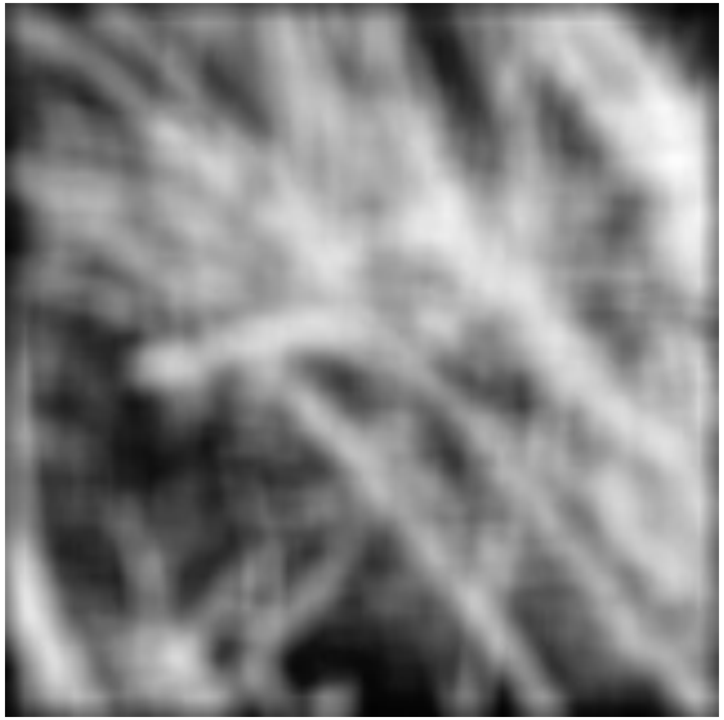
$$G = H \otimes F$$

Smoothing by averaging



depicts box filter:
white = high value, black = low value

original



filtered

We can see the 'Block' effect on the filtered image.

as each neighbour pixel contributes the same

Image Filtering

- Image filtering: compute the function value of local neighbourhood at each pixel.
- Very useful
  - **Enhance images**
    - De-noise, resize, increase contrast, etc.
  - **Extract information from images**
    - Texture, edges, distinctive points, etc.
  - **Detect patterns**
    - Template matching

Gaussian filter

- What if we want nearest neighboring pixels to have the most influence on the output?

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

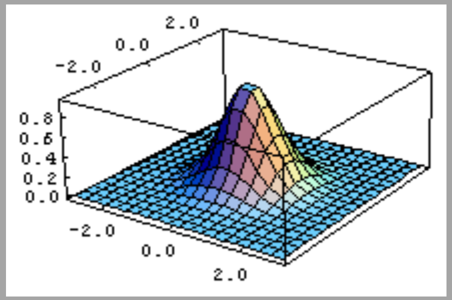
$$F[x,y]$$

$$\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$H[u,v]$$

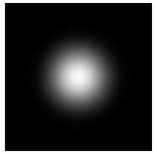
- This kernel is an approximation of a 2d Gaussian function:

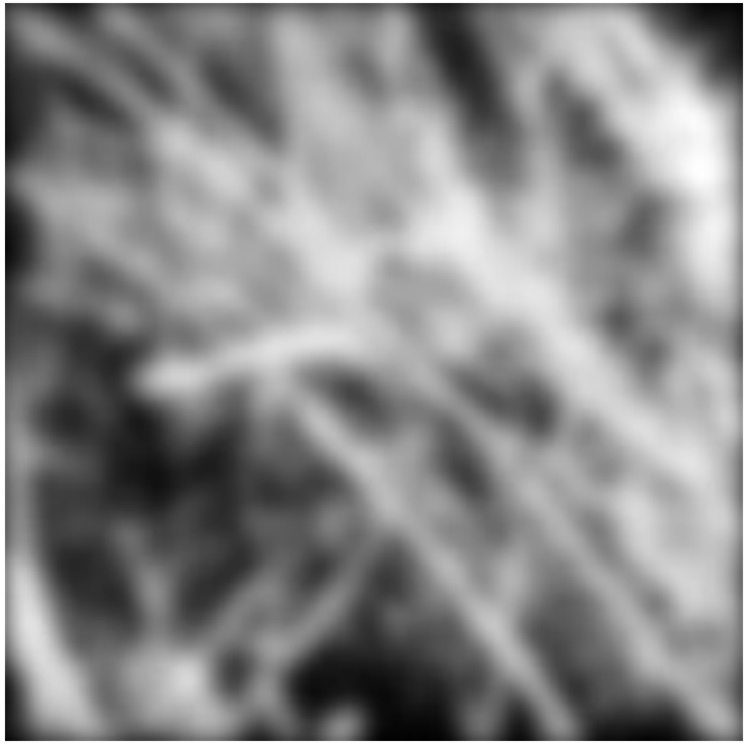
$$h(u,v) = \frac{1}{2\pi\sigma^2} \exp^{-\frac{u^2+v^2}{2\sigma^2}}$$



Source: S. Seitz

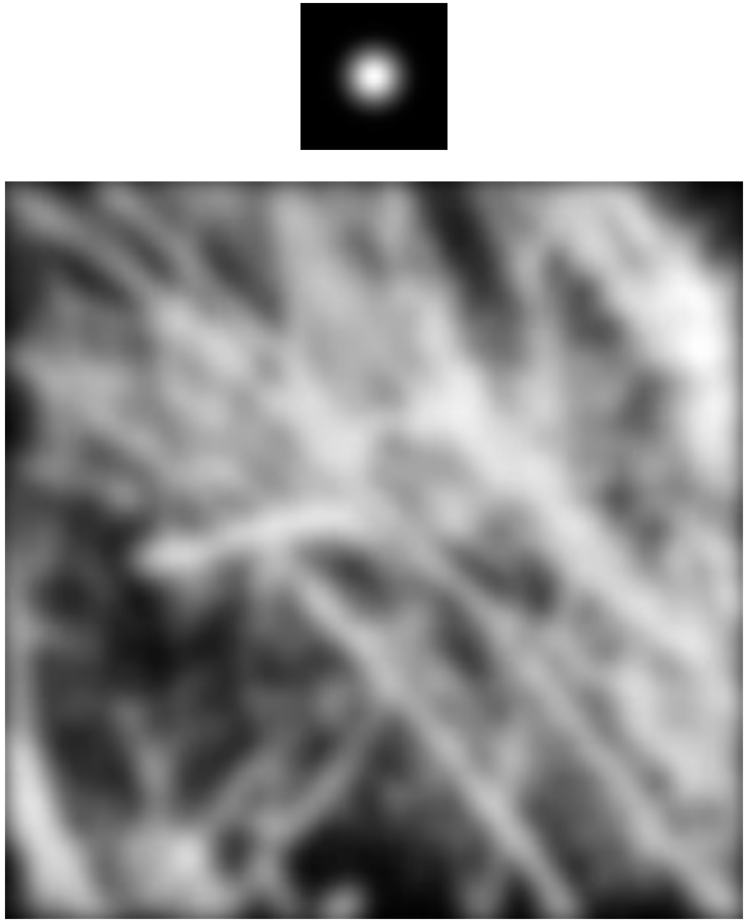
Smoothing with a Gaussian





Compare box filter with Gaussian filter effects

Smoothed image by box filter

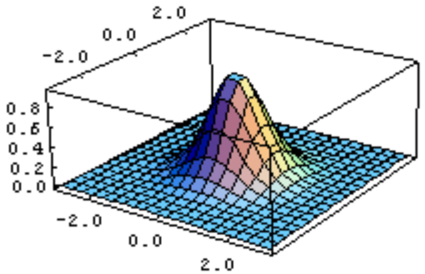


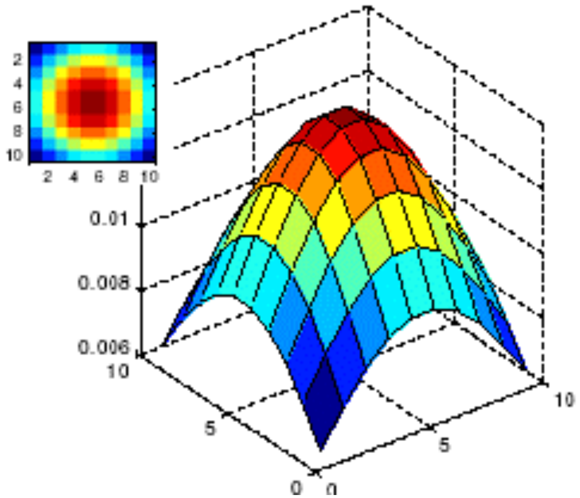
Smoothed image by Gaussian filter

Gaussian filters

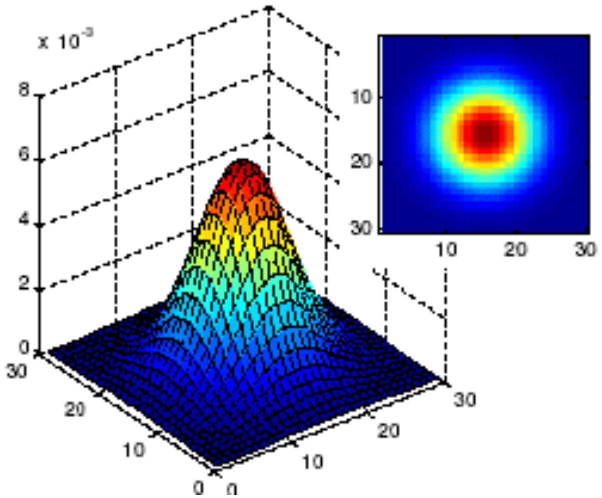
- What parameters matter here?
- **Size** of kernel or mask
  - Note, Gaussian function has infinite support, but discrete filters use finite kernels

$$h(u,v) = \frac{1}{2\pi\sigma^2} \exp^{-\frac{u^2+v^2}{2\sigma^2}}$$





σ = 5 with 10x 10 kernel

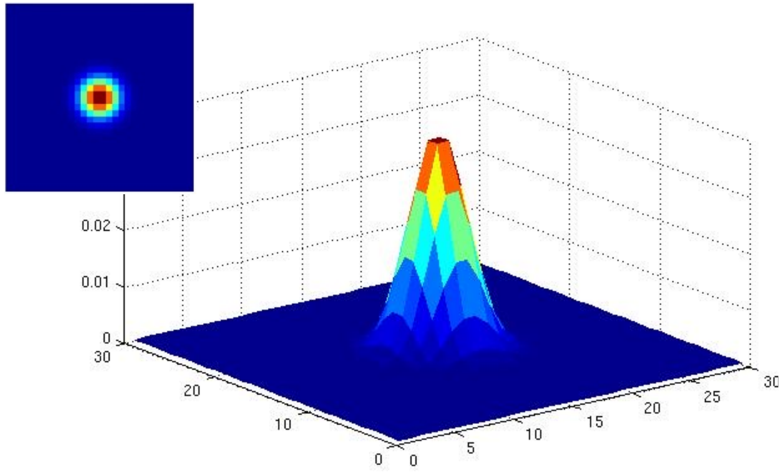


σ = 5 with 30 x 30 kernel

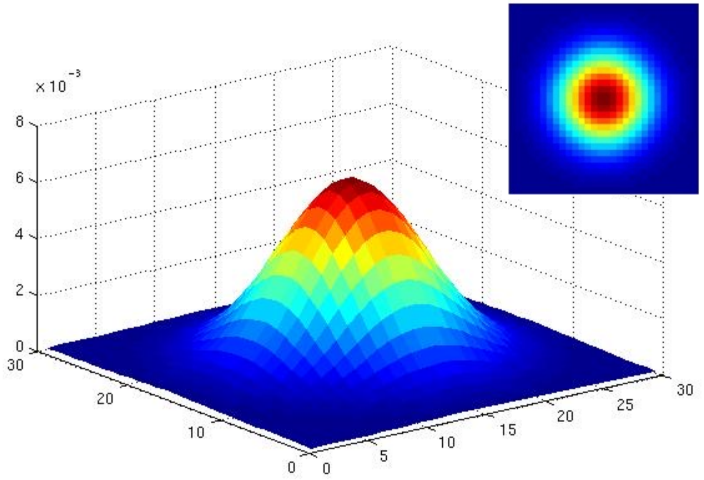
•Slide credit: Kristen Grauman

Gaussian filters

- What parameters matter here?
- **Variance** of the Gaussian function : determines extent of smoothing



σ = 2 with
30 x 30
kernel

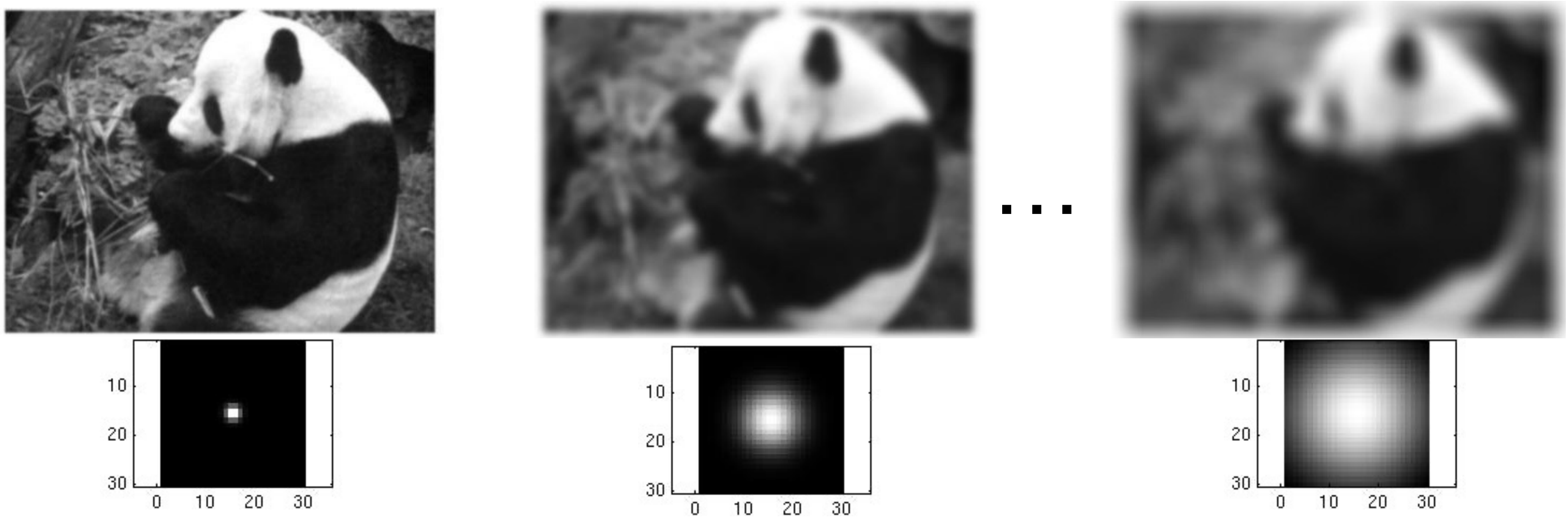


σ = 5 with
30 x 30
kernel

•Slide credit:
Kristen Grauman

Smoothing with a Gaussian

Parameter σ is the "scale" / "width" / "spread" of the Gaussian kernel, and controls the amount of smoothing.



•Slide credit:
Kristen Grauman

Convolution

- Convolution:
  - Flip the filter in both dimensions (bottom to top, right to left)
  - Then apply cross-correlation

$$G[x,y] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} H[u,v]F[x-u,y-v]$$

$$G = H \star F$$

Notation for convolution operator

Convolution

- Convolution:
  - Flip the filter in both dimensions (bottom to top, right to left)
  - Then apply cross-correlation

$$G(x, y) = \sum_{u=0}^{M} \sum_{v=0}^{N} F(u, v) H(x - u, y - v)$$

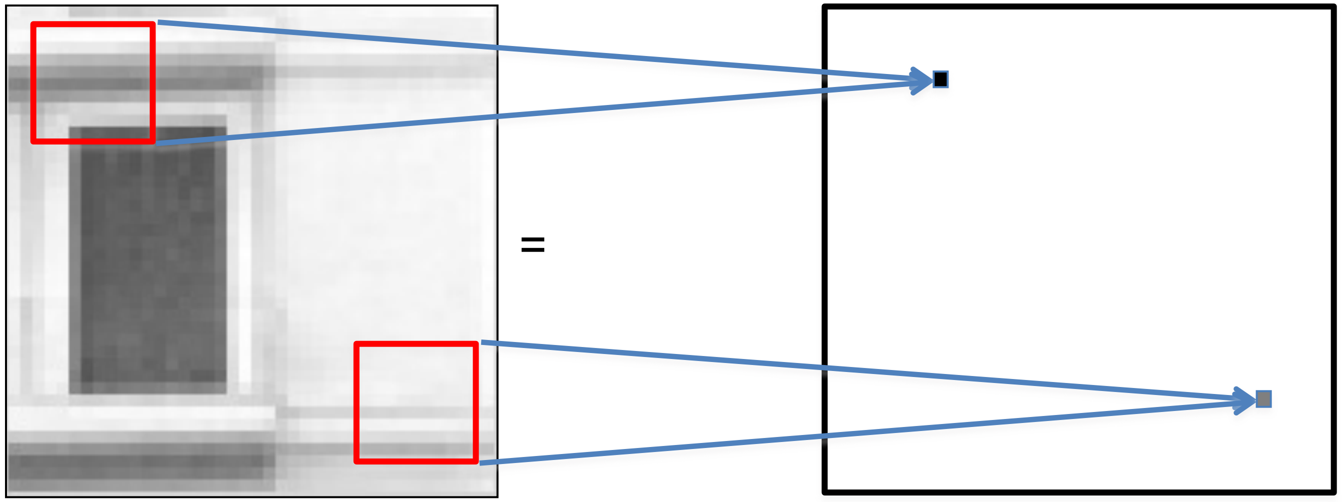
$$G = F \star H$$

Notation for convolution operator

Discrete linear system

In vision, many times, we are interested in operations that are spatially invariant.
For a linear spatially invariant system:

$$g(m,n) = h \star f = \sum_{u=-k}^{u=k} \sum_{v=-k}^{v=k} f(m-u, n-v) h(u,v)$$



Properties of convolution

- **Shift invariant:**
  - Operator behaves the same everywhere, i.e. the value of the output depends on the pattern in the image neighborhood, not the position of the neighborhood.

Properties of convolution

- Commutative:

  $f * g = g * f$

- Associative

  $(f * g) * h = f * (g * h)$

- Distributes over addition

  $f * (g + h) = (f * g) + (f * h)$

- Scalars factor out

  $kf * g = f * kg = k(f * g)$

- Identity:

  unit impulse $e = [\ldots, 0, 0, 1, 0, 0, \ldots]$. $f * e = f$

Separability

- In some cases, the filter is separable, and we can factor into two steps:
  - Convolve all rows with a 1D filter
  - Convolve all columns with a 1D filter

$$\begin{bmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Separability example

- 2D convolution (center location only)(65)

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 5 \\ 4 & 4 & 6 \end{bmatrix}$$

h f

=2 + 6 + 3 = 11

= 6 + 20 + 10 = 36

= 4 + 8 + 6 = 18

65

Source: K. Grauman

Separability example

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 5 \\ 4 & 4 & 6 \end{bmatrix} \quad \begin{aligned} &= 2 + 6 + 3 = 11 \\ &= 6 + 20 + 10 = 36 \\ &= 4 + 8 + 6 = 18 \\ & \overline{\quad 65 \quad} \end{aligned}$$

- The filter is factored into a product of 1D filters.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

Source: K. Grauman

Separability example

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 5 \\ 4 & 4 & 6 \end{bmatrix}$$

$$\begin{aligned} &= 2 + 6 + 3 = 11 \\ &= 6 + 20 + 10 = 36 \\ &= 4 + 8 + 6 = 18 \\ & \overline{ 65} \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

- Perform convolutions along rows.

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 5 \\ 4 & 4 & 6 \end{bmatrix} = \begin{bmatrix} & 11 & \\ & 18 & \\ & 18 & \end{bmatrix}$$

Source: K. Grauman

Separability example

$$\begin{bmatrix}1&2&1\\2&4&2\\1&2&1\end{bmatrix} * \begin{bmatrix}2&3&3\\3&5&5\\4&4&6\end{bmatrix}$$

$= 2 + 6 + 3 = 11$

$= 6 + 20 + 10 = 36$

$= 4 + 8 + 6 = 18$

$\overline{65}$

$$\begin{bmatrix}1&2&1\\2&4&2\\1&2&1\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix} \times \begin{bmatrix}1&2&1\end{bmatrix}$$

$$\begin{bmatrix}1&2&1\end{bmatrix} * \begin{bmatrix}2&3&3\\3&5&5\\4&4&6\end{bmatrix} = \begin{bmatrix}&11&\\&18&\\&18&\end{bmatrix}$$

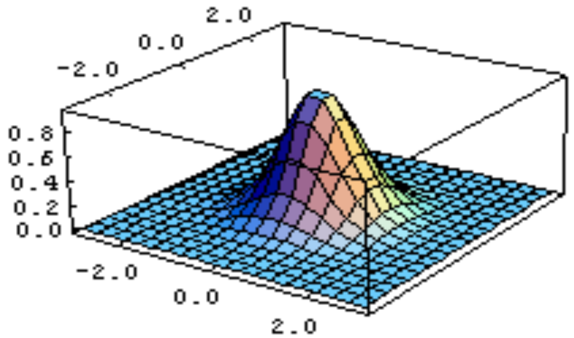
- Perform convolutions along columns.

$$\begin{bmatrix}1\\2\\1\end{bmatrix} * \begin{bmatrix}&11&\\&18&\\&18&\end{bmatrix} = \begin{bmatrix}&&\\&65&\\&&\end{bmatrix}$$

Source: K. Grauman

This kernel is an approximation of a 2d Gaussian function:

$$h(u,v) = \frac{1}{2\pi\sigma^2} \exp^{-\frac{u^2+v^2}{2\sigma^2}}$$



Properties of Gaussian filter

- Rotational symmetry treats features of all orientations equally (isotropy).
- Convolution with self gives another Gaussian
  - So can smooth with small-width kernel, repeat, and get same result as larger-width kernel would have
  - Convolving two times with Gaussian kernel of width σ is same as convolving once with kernel of width $\sigma\sqrt{2}$
- *Separable* kernel
  - Factors into the product of two 1D Gaussians

Source: K. Grauman

Separability of the Gaussian filter

$$G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} \exp^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma} \exp^{-\frac{x^2}{2\sigma^2}}\right) \left(\frac{1}{\sqrt{2\pi}\sigma} \exp^{-\frac{y^2}{2\sigma^2}}\right)$$

The 2D Gaussian can be expressed as the product of two functions, one a function of x and the other a function of y

In this case, the two functions are the (identical) 1D Gaussian

Source: D. Lowe

- What is the complexity of filtering an n×n image with an m×m kernel?
  - $O(n^2 m^2)$
- What if the kernel is separable?
  - $O(n^2 m)$

1. **1D Convolution along Rows:**
   - Convolve each row of the image ($n \times n$) with a 1D kernel of size m.
   - The complexity is $O(n \cdot n \cdot m)$ for the row-wise convolution.

   $$\text{RowConv} = O(n \cdot n \cdot m)$$
2. **1D Convolution along Columns:**
   - Convolve each column of the intermediate result ($n \times n$) with a 1D kernel of size m.
   - The complexity is $O(n \cdot n \cdot m)$ for the column-wise convolution.

   $$\text{ColConv} = O(n \cdot n \cdot m)$$
3. **Total Complexity:**
   - The total complexity is the sum of the row-wise and column-wise convolutions.

   $$\text{Total Complexity} = \text{RowConv} + \text{ColConv} = O(n \cdot n \cdot m)$$

Readings

- Computer Vision: Algorithms and Applications, Chapter 3.1, 3.2 and 3.3