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Exercise 1

(a)

$$[A|b] = \left(\begin{array}{ccc|c} 2 & 7 & 1 & 1 \\ 1 & 4 & 3 & 1 \\ 0 & 2 & 5 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 2 & 7 & 1 & 1 \\ 0 & 2 & 5 & 2 \end{array}\right) \xrightarrow{-2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & -1 & -5 & -1 \\ 0 & 2 & 5 & 2 \end{array}\right)$$

$$\xrightarrow{2R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & -1 & -5 & -1 \\ 0 & 0 & -5 & 0 \end{array}\right)$$

$-5x_3 = 0 \Rightarrow x_3 = 0$

$-x_2 - 5x_3 = -1 \Rightarrow x_2 = 1$

$x_1 + 4x_2 + 3x_3 = 1 \Rightarrow x_1 = -3$

$\therefore S = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \right\}$

(b)

$$[A|b] = \left(\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 3 & 4 & 3 & 5 \end{array}\right) \xrightarrow{-3R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & -2 & -3 & -25 \end{array}\right)$$

$-2x_2 - 3x_3 = -25 \Rightarrow x_2 = \frac{25}{2} - \frac{3}{2}x_3$

$x_1 + 2x_2 + 2x_3 = 10 \Rightarrow x_1 = 10 - 2x_2 - 2x_3 = 10 - (25 - 3x_3) - 2x_3 = -15 + x_3$

$\therefore S = \left\{ \begin{pmatrix} -15 \\ \frac{25}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{pmatrix} x_3 \,\middle|\, x_3 \in \mathbb{R} \right\}$

Exercise 2

Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{pmatrix}$, then $[A|I_3] = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{array}\right)$.

$$[A|I_3] \xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 1 & -4 & -3 & 0 & 1 \end{array}\right) \xrightarrow{-R_2+R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right) \xrightarrow[-2R_3+R_1 \to R_1]{3R_3+R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & -2 & 2 \\ 0 & 1 & 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right) \xrightarrow{-R_2+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -6 & 5 \\ 0 & 1 & 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$= [I_3 | A^{-1}]$

$\therefore A^{-1} = \begin{pmatrix} -2 & -6 & 5 \\ 1 & 4 & -3 \\ 1 & 1 & -1 \end{pmatrix}$

Exercise 3.

(a) No. Let $\lambda = -1$, $\vec{a} = (a_1, a_2, a_3) \in A$, where $a_i \geq 0$

Then $\lambda\vec{a} = -\vec{a} = (-a_1, -a_2, -a_3) \notin A$. because $-a_i \leq 0$

(b) Yes. 1) $B \in \mathbb{R}^3$ 2) $B \neq \phi$, and $\vec{0}_3 = (0,0,0) \in B$

3) Suppose $\vec{b^1} = (b_1^1, b_2^1, b_3^1) \in B$, where $\sum_{i=1}^{3} b_i^1 = 0$

$\vec{b^2} = (b_1^2, b_2^2, b_3^2) \in B$, where $\sum_{i=1}^{3} b_i^2 = 0$

then $\vec{b_1} + \vec{b_2} = (b_1^1 + b_1^2, b_2^1 + b_2^2, b_3^1 + b_3^2) \in B$

because $(b_1^1 + b_1^2) + (b_2^1 + b_2^2) + (b_3^1 + b_3^2) = (b_1^1 + b_2^1 + b_3^1) + (b_1^2 + b_2^2 + b_3^2) = 0 + 0 = 0$

4) Suppose $\lambda \in \mathbb{R}$, $\vec{b} = (b_1, b_2, b_3) \in B$, where $\sum_{i=1}^{3} b_i = 0$

then $\lambda\vec{b} = (\lambda b_1, \lambda b_2, \lambda b_3) \in B$

because $\lambda b_1 + \lambda b_2 + \lambda b_3 = \lambda(b_1 + b_2 + b_3) = \lambda \cdot 0 = 0$

(c) No. Let $\vec{c^1} = (1, 1, 0) \in C$, $\vec{c^2} = (0, 1, 1) \in C$

then $\vec{c^1} + \vec{c^2} = (1, 2, 1) \notin C$

because it has no zero entry

(d) If the solution set is ϕ, then the solution set is not a vector space.

If the solution set is not ϕ,

When $\vec{b} = \vec{0}_3$: • If the solution set only contains $\vec{0}_3$, i.e. $S = \{\vec{0}_3\}$, then S is a vector space.

Prove: ① $S \in \mathbb{R}^3$ ② $S \neq \phi$, and $\vec{0}_3 \in S$

③ Suppose $\vec{x} = \vec{0}_3 \in S$, $\vec{y} = \vec{0}_3 \in S$, then $\vec{x} + \vec{y} = \vec{0}_3 + \vec{0}_3 = \vec{0}_3 \in S$

④ Suppose $\vec{x} = \vec{0}_3 \in S$, $\lambda \in \mathbb{R}$, then $\lambda\vec{x} = \lambda \cdot \vec{0}_3 = \vec{0}_3 \in S$

• If the solution set has infinitely many elements, ($A\vec{x} = \vec{0}$ in REF has free variable(s))

then the solution set S is a vector space.

Prove: ① $S \in \mathbb{R}^3$ ② $S \neq \phi$, and $\vec{0}_3 \in S$ $(A \cdot \vec{0}_3 = \vec{0}_3)$

③ Suppose $\vec{x} \in S$, $\vec{y} \in S$, we have $A\vec{x} = \vec{0}_3$, $A\vec{y} = \vec{0}_3$

then $\vec{x} + \vec{y} \in S$, since $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}_3 + \vec{0}_3 = \vec{0}_3 \in S$

④ Suppose $\vec{x} \in S$, $\lambda \in \mathbb{R}$, we have $\lambda A(\vec{x}) = \vec{0}_3$

then $\lambda \cdot \vec{x} \in S$, since $A(\lambda \cdot \vec{x}) = \lambda A(\vec{x}) = \lambda \cdot \vec{0}_3 = \vec{0}_3 \in S$

When $\vec{b} \neq \vec{0}_3$ • If the solution set has only one element, $S = \{\vec{s}\}$, then S is not a vector space

Because $\vec{0}_3 \notin S$.

• If the solution set has infinitely many elements ($A\vec{x} = \vec{b}$ in REF has free variable(s))

then the solution set S is not a vector space, since $\vec{0}_3 \notin S$.

Exercise 4.

(a) Let $\vec{0}_V \in V$, $\vec{0}_W \in W$. then $T(\vec{0}_V) = T(\vec{0}_V + \vec{0}_V) = T(\vec{0}_V) + T(\vec{0}_V)$

Subtract $T(\vec{0}_V)$ from both sides, we have $T(\vec{0}_V) = \vec{0}_W$.

(b) According to the definition of linear transformation, $T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b})$, $T(c\vec{a}) = cT(\vec{a})$

we have $T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = T(c_1\vec{v}_1) + \cdots + T(c_n\vec{v}_n) = c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n)$

(c) Suppose $c_i \in \mathbb{R}$ $(i=1,\cdots,n)$, $\vec{0}_V \in V$, $\vec{0}_W \in W$.

According to the definition, $T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n) = c_1\vec{w}_1 + \cdots + c_n\vec{w}_1$

$\vec{v}_1 \cdots \vec{v}_n$ are not independent. i.e. if $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_V$, $c_1 \cdots c_n$ can not be all zeros.

Then, we have $T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = T(\vec{0}_V) = \vec{0}_W = c_1\vec{w}_1 + \cdots + c_n\vec{w}_n$.

Thus, $\sum_{i=1}^{n} c\vec{w}_i = \vec{0}_W$ has at least one non zero solution, then $\vec{w}_1 \cdots \vec{w}_n$ are dependent either.

(d) We already knew that $T: V \to W$, $S: W \to X$, then $L: V \to X$ can be represented as

$L: V \to W \to X$. i.e. $L(\vec{v}) = S(T(\vec{v})) = (S \circ T)(\vec{v})$, $(\vec{v} \in T)$

For proving L is a linear transformation, we start from $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ $(c_i \in \mathbb{R}, \vec{v}_i \in T, i=1,\cdots,n)$

then
$$\begin{aligned} L(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) &= S(T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)) \\ &= S(c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n)) \\ &= c_1S(T(\vec{v}_1)) + \cdots + c_nS(T(\vec{v}_n)) \\ &= c_1L(\vec{v}_1) + \cdots + c_nL(\vec{v}_n) \end{aligned}$$

Thus, according to the definition, $L: V \to X$ is a linear transformation.

Exercise 5.

(a) Because the inner product $\langle \cdot, \cdot \rangle$ is linear in its first argument,

then we have $\langle \lambda\vec{a} + \varphi\vec{b}, \vec{c} \rangle = \lambda\langle \vec{a}, \vec{c} \rangle + \varphi\langle \vec{b}, \vec{c} \rangle$. $(\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n, \lambda, \varphi \in \mathbb{R})$

Because $\langle \cdot, \cdot \rangle$ is symmetric, we have

$\langle \lambda\vec{a} + \varphi\vec{b}, \vec{c} \rangle = \langle c, \lambda\vec{a} + \varphi\vec{b} \rangle$, $\lambda\langle \vec{a}, \vec{c} \rangle = \lambda\langle \vec{c}, \vec{a} \rangle$, $\varphi\langle \vec{b}, \vec{c} \rangle = \varphi\langle \vec{c}, \vec{b} \rangle$.

then $\langle \vec{c}, \lambda\vec{a} + \varphi\vec{b} \rangle = \lambda\langle \vec{c}, \vec{a} \rangle + \varphi\langle \vec{c}, \vec{b} \rangle$

Then $\langle \cdot, \cdot \rangle$ is linear in its second argument.

Thus, $\langle \cdot, \cdot \rangle$ is bilinear.

(b) Symmetric: $\langle \vec{x}, \vec{y} \rangle = x_1y_1 + x_2y_2 - (x_1+x_2+y_1+y_2)$

$= y_1x_1 + y_2x_2 - (y_1+y_2+x_1+x_2) = \langle \vec{y}, \vec{x} \rangle$, where $\vec{x}, \vec{y} \in \mathbb{R}^2$

Thus, $\langle \cdot, \cdot \rangle$ is symmetric.

positive define: For $\forall \vec{x} \in \mathbb{R}^2 \setminus \{\vec{0}_2\}$. $\langle \vec{x}, \vec{x} \rangle = x_1^2 + x_2^2 - 2(x_1+x_2)$

Suppose $x_1 = x_2 = 1$, then $\langle \vec{x}, \vec{x} \rangle = 1+1-2\times 2 = -2 < 0$

Thus, $\langle \cdot, \cdot \rangle$ is not positive define.

For $\vec{0}_2 \in \mathbb{R}^2$. $\langle \vec{0}_2, \vec{0}_2 \rangle = \vec{0}_2$

Bilinear: Suppose $\lambda, \varphi \in \mathbb{R}$, $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}_2$

We have $\langle \lambda\vec{x} + \varphi\vec{y}, \vec{z} \rangle = (\lambda x_1 + \varphi y_1)z_1 + (\lambda x_2 + \varphi y_2)z_2$

$- (\lambda x_1 + \varphi y_1 + \lambda x_2 + \varphi y_2 + z_1 + z_2)$

$= \lambda(x_1z_1 + x_2z_2) + \varphi(y_1z_1 + y_2z_2)$

$- \lambda(x_1+x_2) - \varphi(y_1+y_2) - (z_1+z_2)$

$= \lambda(x_1z_1 + x_2z_2 - (x_1+x_2)) + \varphi(y_1z_1 + y_2z_2 - (y_1+y_2))$

$-(z_1+z_2)$

$\lambda \langle \vec{x}, \vec{z} \rangle = \lambda(x_1z_1 + x_2z_2 - (x_1+x_2)) - \lambda(z_1+z_2)$

$\varphi \langle \vec{y}, \vec{z} \rangle = \varphi(y_1z_1 + y_2z_2 - (y_1+y_2)) - \varphi(z_1+z_2)$

Because $\langle \lambda\vec{x} + \varphi\vec{y}, \vec{z} \rangle - (\lambda\langle \vec{x}, \vec{z} \rangle + \varphi\langle \vec{y}, \vec{z} \rangle) \neq 0$

then $\langle \lambda\vec{x} + \varphi\vec{y}, \vec{z} \rangle \neq \lambda\langle \vec{x}, \vec{z} \rangle + \varphi\langle \vec{y}, \vec{z} \rangle$

then $\langle \cdot, \cdot \rangle$ is not linear in its first argument

Thus, $\langle \cdot, \cdot \rangle$ is not bilinear.

Exercise 6.

(a) Prove: Given that $\vec{x}, \vec{y} \in V \setminus \{\vec{0}\}$ $(\vec{x} \neq \vec{y})$, if $\vec{x} \perp \vec{y}$, then we have $\langle \vec{x}, \vec{y} \rangle = 0$

Suppose $\vec{x}$ and $\vec{y}$ are not independent, i.e. $\vec{y} = \lambda\vec{x}$, $\lambda \in \mathbb{R}$.

Then, $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \lambda\vec{x} \rangle = \lambda\langle \vec{x}, \vec{x} \rangle$

According to the definition of $\langle \cdot, \cdot \rangle$, $\langle \vec{x}, \vec{x} \rangle = 0$.

Then, $\langle \vec{x}, \vec{y} \rangle = \lambda\langle \vec{x}, \vec{x} \rangle = \lambda \cdot 0 = 0$ $(\vec{x} \neq \vec{y})$,

which conflicts with the definition $\langle \vec{x}, \vec{y} \rangle > 0$ for $\forall \vec{x}, \vec{y} \in V \setminus \{\vec{0}\}$.

Thus, $\vec{x}$ and $\vec{y}$ are independent.

(b) Disprove: Suppose $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$, $\vec{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$. $\vec{x}$ and $\vec{y}$ are independent.

If we use dot product as $\langle \cdot, \cdot \rangle$, then $\langle \vec{x}, \vec{y} \rangle = 1 \neq 0$

Thus, given $\vec{x}$ and $\vec{y}$ are independent, they are not orthogonal.

(c) $\vec{0}$ is dependent with other vectors in the same vector space.

Prove: Suppose $\vec{x} \in V$, then $\lambda\vec{0} + \varphi\vec{x} = \vec{0}$ has non zero solutions. (e.g. $\lambda = 1, \varphi = 0$)

$\vec{0}$ is orthogonal with other vectors in the same vector space.

Prove: Suppose $\vec{x} \in V$, then $\langle\vec{0}, \vec{x}\rangle = \langle\vec{0}+\vec{0}, \vec{x}\rangle = \langle\vec{0}, \vec{x}\rangle + \langle\vec{0}, \vec{x}\rangle$, then $\langle\vec{0}, \vec{x}\rangle = 0$

For (a) If $\vec{x}$ and $\vec{y}$ are orthogonal, given $\vec{x} = \vec{0}$ and $\vec{y} \neq \vec{0}$ (or $\vec{x} \neq \vec{0}$ and $\vec{y} = \vec{0}$)
then $\vec{x}$ and $\vec{y}$ are not independent

If $\vec{x} = \vec{y} = \vec{0}$, then $\vec{x}$ and $\vec{y}$ are orthogonal and not independent.

For (b) If $\vec{x} = \vec{0}$ and $\vec{y} \neq \vec{0}$ (or $\vec{x} \neq \vec{0}$ and $\vec{y} = \vec{0}$), $\vec{x}$ and $\vec{y}$ are not independent.
the precondition is not satisfied

If $\vec{x} = \vec{y} = \vec{0}$, $\vec{x}$ and $\vec{y}$ are not independent, the precondition is not satisfied