The Australian National University
School of Computing
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Semester 2, 2022
Theory Assignment 1 of 4

COMP3670/6670: Introduction to Machine Learning

Release Date: 3 Aug 2022
Due Date: 23:59pm, 28 Aug 2022
Maximum credit: 100

Exercise 1 **Solving Linear Systems** (4+4 credits)

Find the set $\mathcal{S}$ of all solutions $\mathbf{x}$ of the following inhomogenous linear systems $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{b}$ are defined as follows. Write the solution space $\mathcal{S}$ in parametric form.

(a)

$$\mathbf{A} = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & 3 \\ 0 & 2 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Exercise 2 **Inverses** (4 credits)

Find the inverse of the following matrix, if an inverse exists.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}$$

Exercise 3 **Subspaces** (3+3+3+4 credits)

Which of the following sets are also subspaces of $\mathbb{R}^3$? Prove your answer. (That is, if it is a subspace, you must demonstrate the subspace axioms are satisfied, and if it is not a subspace, you must show which axiom fails.)

(a) $A = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$

(b) $B = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$.

(c) $C = \{(x, y, z) \in \mathbb{R}^3 : x = 0 \text{ or } y = 0 \text{ or } z = 0\}$

(d) $D =$ The set of all solutions to the matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$, for some matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$ and some vector $\mathbf{b} \in \mathbb{R}^3$. (Hint: Your answer may depend on $\mathbf{A}$ and $\mathbf{b}$.)

Exercise 4 **Linear Independence** (5+10+15+5 credits)

Let V and W be vector spaces. Let $T : V \to W$ be a linear transformation.

(a) Prove that $T(\mathbf{0}) = \mathbf{0}$.

(b) For any integer $n \geq 1$, prove that given a set of vectors $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ in V and a set of coefficients $\{c_1, \dots, c_n\}$ in $\mathbb{R}$, that

$$T(c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \dots + c_nT(\mathbf{v}_n)$$

(c) Let $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ be a set of linearly **dependent** vectors in V.
Define $\mathbf{w}_1 := T(\mathbf{v}_1), \dots, \mathbf{w}_n := T(\mathbf{v}_n)$.
Prove that $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is a set of linearly **dependent** vectors in W.

(d) Let X be another vector space, and let $S : W \to X$ be a linear transformation. Define $L : V \to X$ as $L(\mathbf{v}) = S(T(\mathbf{v}))$. Prove that L is also a linear transformation.

Exercise 5 **Inner Products** (5+10 credits)

(a) Show that if an inner product $\langle \cdot, \cdot \rangle$ is symmetric and linear in the first argument, then it is bilinear.

(b) Define $\langle \cdot, \cdot \rangle$ for all $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^2$ and $\mathbf{y} = [y_1, y_2]^T \in \mathbb{R}^2$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 - (x_1 + x_2 + y_1 + y_2)$$

Which of the three inner product axioms does $\langle \cdot, \cdot \rangle$ satisfy?

Exercise 6 **Orthogonality** (15+6+4 credits)

Let V denote a vector space together with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$.

Let $\mathbf{x}, \mathbf{y}$ be **non-zero** vectors in V.

(a) Prove or disprove that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, then they are linearly independent.

(b) Prove or disprove that if $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, then they are orthogonal.

(c) How do the above statements change if we remove the restriction that $\mathbf{x}$ and $\mathbf{y}$ have to be non-zero?