Section 3

1. We know that $X \in \{x_1, x_2\}$. $Y \in \{y_1, y_2\}$.

Assume x_2 is not evidence again y_1.

then $P(Y=y_1 \mid X=x_2) > P(Y=y_1)$

We know that $P(Y=y_1 \mid X=x_1) > P(Y=y_1)$

then $P(Y=y_1) = P(Y=y_1, X=x_1) + P(Y=y_1, X=x_2)$

$= P(X=x_1) P(Y=y_1 \mid X=x_1) + P(X=x_2) P(Y=y_1 \mid X=x_2)$

$= P(X=x_1) P(Y=y_1 \mid X=x_1) + (1-P(X=x_1)) P(Y=y_1 \mid X=x_2)$

If $P(Y=y_1 \mid X=x_2) > P(Y=y_1)$, then

$P(Y=y_1) = P(X=x_1) P(Y=y_1 \mid X=x_1) + (1-P(X=x_1)) P(Y=y_1 \mid X=x_2)$

$> P(X=x_1) P(Y=y_1) + (1-P(X=x_1)) P(Y=y_1)$

$= P(X=x_1) P(Y=y_1) + P(Y=y_1) - P(X=x_1) P(Y=y_1)$

$= P(Y=y_1)$.

i.e. $P(Y=y_1) > P(Y=y_1)$

which is a contradiction.

Thus x_2 must be an evidence against y_1.

2. (1) When $h=1$, i.e. N times flips have 1 head,

then $P(H=0) = \binom{N}{0} q^0 (1-q)^{N-0} = (1-q)^N$

Assume $H=n$ $P(H=n) = \binom{N}{n} q^n (1-q)^{N-n}$

then $P(H=n+1) = P(H=n) q \cdot \binom{n+1}{1} = \binom{N}{n}\binom{n+2}{1} q \cdot q^n \cdot (1-q)^{N-n}$

$= \binom{N}{n+1} q^{n+1} (1-q)^{N-(n+1)}$

(2) $P(\text{Coin}=\text{real} \mid h \text{ heads in } N \text{ flips}) = \dfrac{P(h \text{ heads in } N \text{ flips} \mid \text{Coin}=\text{real})\, P(\text{Coin}=\text{real})}{P(h \text{ heads in } N \text{ flips})}$

RHS = $\dfrac{P(h \text{ heads in } N \text{ flips} \mid \text{Coin}=\text{real})\, P(\text{Coin}=\text{real})}{P(h \text{ heads in } N \text{ flips} \mid \text{Coin}=\text{real})\, P(\text{Coin}=\text{real}) + P(h \text{ heads in } N \text{ flips} \mid \text{Coin}=\text{fake})\, P(\text{Coin}=\text{fake})}$

We know that $P(\text{Coin}=\text{real}) = P(\text{Coin}=\text{fake}) = \frac{1}{2}$

$P(h \text{ heads in } N \text{ flips} \mid \text{Coin}=\text{real}) = \binom{N}{h} q^h (1-q)^{N-h}$

$P(h \text{ heads in } N \text{ flips} \mid \text{Coin}=\text{fake}) = \binom{N}{h} p^h (1-p)^{N-h}$

$$\therefore RHS = \frac{\frac{1}{2} \times \binom{N}{n} q^n (1-q)^{N-n}}{\frac{1}{2} \times \binom{N}{n} q^n (1-q)^{N-n} + \frac{1}{2} \times \binom{N}{n} p^n (1-p)^{N-n}}$$

$$= \frac{q^n (1-q)^{N-n}}{q^n (1-q)^{N-n} + p^n (1-p)^{N-n}}$$