Section 1.

1. (1) $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $\therefore \det(A) = (-1)^{1+1} a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} + (-1)^{2+1} \cdot d \cdot \begin{vmatrix} b & c \\ h & i \end{vmatrix} + (-1)^{3+1} \cdot g \cdot \begin{vmatrix} b & c \\ e & f \end{vmatrix}$

Let $B = \begin{bmatrix} 3a & b & c \\ 3d & e & f \\ 3g & h & i \end{bmatrix}$ then $\det(B) = (-1)^{1+1} \cdot 3a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} + (-1)^{2+1} \cdot 3d \cdot \begin{vmatrix} b & c \\ h & i \end{vmatrix} + (-1)^{3+1} \cdot 3g \cdot \begin{vmatrix} b & c \\ e & f \end{vmatrix}$

$\det(B) = 3\det(A) = 3 \times 2 = 6$

(2) $\det(A^3) = \det(A \cdot A \cdot A) = \det(A) \cdot \det(A) \cdot \det(A) = 2 \times 2 \times 2 = 8$

(3) $\det(A+A) = \det(2A) = 2^2 \det(A) = 4 \times 2 = 8$

(4) $\because \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix} \xrightarrow{R_1 \Leftrightarrow R_3} \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} \xrightarrow{R_2 \Leftrightarrow R_3} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$\therefore \det(A) = (-1)(-1) \times 2 = 2$

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $Ax = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$

$\|Ax\|_2 = \sqrt{(ax_1+bx_2)^2 + (cx_1+dx_2)^2} = \sqrt{a^2x_1^2 + 2abx_1x_2 + b^2x_2^2 + c^2x_1^2 + 2cdx_1x_2 + d^2x_2^2}$

$= \sqrt{(a^2+c^2)x_1^2 + (b^2+d^2)x^2 + 2(ab+cd)x_1x_2}$

$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$

$\because \|Ax\|_2 = \|x\|_2$

$\therefore (a^2+c^2)x_1^2 + (b^2+d^2)x_2^2 + 2(ab+cd)x_1x_2 = x_1^2 + x_2^2$

$\therefore a^2 + c^2 = 1$, $b^2 + d^2 = 1$, $ab + cd = 0$

$\therefore ab = -cd$.

$\therefore (ab)^2 = (-cd)^2$, $a^2b^2 = c^2d^2$

$\therefore (1-c^2)b^2 = c^2(1-b^2)$, $a^2(1-d^2) = (1-a^2)d^2$

$b^2 - b^2c^2 = c^2 - c^2b^2$, $a^2 - a^2d^2 = d^2 - a^2d^2$

$\therefore b^2 = c^2$, $b = \pm c$

$a^2 = d^2$, $a = \pm d$

$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \begin{bmatrix} \alpha & \beta \\ \pm\beta & \pm\alpha \end{bmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$

If entries along the main diagonal are strictly positive, then

$\alpha = a > 0$, then $a = d = \alpha > 0$, According to $ab + cd > 0$, $b = \pm c$

If $b > 0$, then $ab > 0, d > 0$, thus $c < 0$

If $b < 0$, then $ab < 0, d > 0$, thus $c > 0$

$\therefore b = -c$

$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \middle| \alpha > 0, \beta \in \mathbb{R} \right\}$.

3. Rotation. Let $a = d = \cos\theta$, $b = -\sin\theta$, $c = -b = \sin\theta$.

then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

If $\theta = 45°$, then Ax means $45°$ counterclock wise rotation.