Question 1.

1. According to the properties of determinants, the determinant of any square matrix A can be represented as $\det(A) = \frac{\prod \operatorname{diag}(B)}{d}$, where B is the REF of A after Gaussian elimination, and d is the product of the scalars by applying Gaussian elimination:

   ① Swapping: multiplies $\det(A)$ by -1

   ② Multiplying a row by a scalar: multiplies $\det(A)$ by the same scalar

   ③ Adding to one row a scalar multiple of another: doesn't change $\det(A)$

   Then, $\det(A) = \frac{1}{d} \prod \operatorname{diag}(B)$, where $\frac{1}{d} \neq 0$.

   After applying Gaussian elimination on a $n \times n$ permutation matrix P, we have the REF of P and denote it as P_{REF}. From the structure of P, we induce that $P_{REF} = I_n$, then $\operatorname{diag}(P_{REF}) = 1 \times 1 \times \cdots \times 1 = 1$. Then, $\det(P) = \frac{1}{d} \neq 0$, where d is a non-zero scalar produced by Gaussian elimination of P.

   Thus, $\det(P) \neq 0 \Rightarrow P$ is always invertible.

2. Suppose p_{ij} is the entry of P, according to the definition of permutation matrices, we have $p_{1,j_1} = p_{2,j_2} = \cdots = p_{n,j_n} = 1$, where $j_1 \neq j_2 \neq \cdots \neq j_n$, and other entries are all 0s.

   Aftere transpose, P^T has entries that $P^T_{j_1,1} = P^T_{j_2,2} = \cdots = P^T_{j_n,n} = 1$, where $j_1 \neq j_2 \neq \cdots \neq j_n$, and other entries are all 0s.

   Thus, based on the definition of permutation matrices, we have P^T is a permutation matrix.

Question 2.

1. Suppose x_i can be written as $x_i = c_1 x_1 + \cdots + c_{i-1} x_{i-1} + c_{i+1} x_{i+1} + \cdots + c_n x_n$, where $c_1 \cdots c_{i-1}, c_{i+1} \cdots c_n$ can not all be 0s.

   Then
   $$\begin{aligned} Ax_i &= \lambda_i x_i \\ &= \lambda_i (c_1 x_1 + \cdots + c_{i-1} x_{i-1} + c_{i+1} x_{i+1} + \cdots + c_n x_n) \\ &= \lambda_i c_1 x_1 + \cdots + \lambda_i c_{i-1} x_{i-1} + \lambda_i c_{i+1} x_{i+1} + \cdots + \lambda_i c_n x_n \quad ① \end{aligned}$$

   Also,
   $$\begin{aligned} Ax_i &= A(c_1 x_1 + \cdots + c_{i-1} x_{i-1} + c_{i+1} x_{i+1} + \cdots + c_n x_n) \\ &= c_1 A x_1 + \cdots + c_{i-1} A x_{i-1} + c_{i+1} A x_{i+1} + \cdots + c_n A x_n \\ &= c_1 \lambda_1 x_1 + \cdots + c_{i-1} \lambda_{i-1} x_{i-1} + c_{i+1} \lambda_{i+1} x_{i+1} + \cdots + c_n \lambda_n x_n \quad ② \end{aligned}$$

   $① - ② = 0 \Rightarrow c_1(\lambda_i - \lambda_1)x_1 + \cdots + c_{i-1}(\lambda_i - \lambda_{i-1})x_{i-1} + c_{i+1}(\lambda_i - \lambda_{i+1})x_{i+1} + \cdots + c_n(\lambda_i - \lambda_n)x_n = 0$

   From assumptions, we know that $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$, which implies $\lambda_i - \lambda_k \neq 0$, $k = 1, \cdots, i-1, i+1, \cdots, n$

   We also know that x_k, $k = 1, \cdots, i-1, i+1, \cdots, n$, is not a zero vector.

   It implies that $c_1 = c_2 = \cdots = c_{i-1} = c_{i+1} = \cdots = c_n = 0$, which conflicts with assumptions.

   Thus, $\{x_1, \cdots, x_n\}$ are linear independent vectors. Q.E.D.

2. We know that $X \in \mathbb{R}^{n\times n}$, and applying Gaussian elimination only changes the scalar product of $\det(X)$. Following the steps, let $A = B - \lambda I \in \mathbb{R}^{n\times n}$. After Gaussian elimination, the REF of A, denoted as A_{REF}, has as most n non-zero entries on its diagonal i.e. full rank, then $\det(A_{REF}) = 0$ if there is at least one zero entry on the diagonal, otherwise, $\det(A_{REF}) = C \times \prod_{i=1}^{n} a_{REF\,i,i}$, where $C \in \mathbb{R}$, $a_{REF\,i,i} \in \mathbb{R}$ is the diagonal entry.
We know that $\prod \operatorname{diag}(A_{REF})$ can be represented as $(a_{11}-\lambda)\cdot(a_{22}-\lambda)\cdot \dots \cdot (a_{nn}-\lambda)$, ①
When ① $=0$, it has at most n roots. (fundamental theorem of algebra) i.e. at most n distinct eigenvalues $\lambda_1 \cdots \lambda_n$. Q.E.D.

Question 3

1. Suppose we have a lower triangular matrix $A \in \mathbb{R}^{n\times n}$, $n \in \mathbb{Z}^+$. From the definition of lower triangular matrices, we know that for $1 < i < j < n$, $a_{ij} = 0$, where a_{ij} is an entry of A, and for $1 < j \leq i < n$, $a_{ij} \in \mathbb{R}$.
Let $B \in \mathbb{R}^{n\times n}$, $b_{ij} \in \mathbb{R}$. Then, we have $C = AB \in \mathbb{R}^{n\times n}$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.
For $k < j$, $c_{ij} = \sum_{k=1}^{n} 0 \cdot b_{kj} = 0$; For $k \geq j$, $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \in \mathbb{R}$.
Thus, according to the definition of lower triangular matrices, $C = AB$ is a lower triangular matrix.
If $C = BA \in \mathbb{R}^{n\times n}$, then $c_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}$. For $k < j$, $a_{kj} = 0$, then $c_{ij} = 0$. For $k \geq j$, $a_{kj} \in \mathbb{R}$, $c_{ij} \in \mathbb{R}$. Thus, $C = BA$ is also a lower triangular matrix.
In summary, for any lower triangular matrix A, its matrix multiplication is also a lower triangular matrix.
Q.E.D.

2. When $d = 1$ $U^1 = [U_{11}]$, then $\det(U^1) = \prod \operatorname{diag}(U^1) = U_{11}$. The base case is true
Assume for $d = n-1$, $U^{n-1} = \begin{bmatrix} U_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ U_{n1} & \cdots & U_{n-1\,n-1} \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}$, $\det(U^{n-1}) = \prod \operatorname{diag}(U^{n-1}) = U_{11} \times U_{22} \times \cdots \times U_{n-1}$

Then, for $k = n$, $U^n = \begin{bmatrix} U^{n-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ U_{n1} & \cdots & U_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}$, $\det(U^n) = U_{1n} \cdot |\cdot| + U_{2n} \cdot |\cdot| + \cdots + U_{nn} \cdot |U^{n-1}|$
where $U_{1n} = U_{2n} = \cdots = U_{n-1\,n} = 0$, $U_{nn} \in \mathbb{R}$.
Then, $RHS = U_{nn} \cdot \det(U^{n-1}) = U_{nn} \cdot \prod \operatorname{diag}(U^{n-1}) = U_{11} \times U_{22} \times \cdots \times U_{nn} = \prod \operatorname{diag}(U^n)$
Thus, the induction holds for $k = 1, \dots, n$, i.e. for $U \in \mathbb{R}^{n\times n}$, $\det(U) = \prod \operatorname{diag}(U)$
Q.E.D.

Question 4.

1. Suppose $M = Av_1 = [m_1, \dots m_n]^T \in \mathbb{R}^{n\times 1}$, where $m_k = \sum_{i=1}^{n} a_{ki} \cdot v_{1i}$

Then $v_2^T M = v_2^T [m_1, \cdots, m_n]^T \in \mathbb{R}$, $RHS = \sum_{j=1}^{n} v_{2j} m_j = \sum_{j=1}^{n}\sum_{i=1}^{n} v_{2j} v_{1i} a_{ji}$ ①

For $N = Av_2$, following the same step, we have $v_1^T N = \sum_{i=1}^{n}\sum_{j=1}^{n} v_{1i} v_{2j} a_{ij}$ ②

From ① and ②, it can be easily showed that $v_2^T M = v_1^T N$

i.e. $v_2^T A v_1 = v_1^T A v_2$, then $\lambda_1 v_2^T v_1 = \lambda_2 v_1^T v_2$.

Given the identity $a^T b = b^T a$, we have $\lambda_1 v_1^T v_2 = \lambda_2 v_1^T v_2$.

Because $\lambda_1 \neq \lambda_2$, then $v_1^T v_2 = 0$, i.e. v_1 and v_2 are orthogonal.

Q.E.D.

Question 5.

1. Let $Ax = \lambda x$, where x is its eigenvector and λ is the corresponding eigenvalue.

Then, let $Ax = \lambda I_2 x$, $\Rightarrow (A - \lambda I_2)x = 0$, i.e. $\begin{bmatrix} 6-\lambda & 4 \\ 3 & 5-\lambda \end{bmatrix} x = 0$

Suppose $B = A - \lambda I_2$, then we have

$\det(B) = (6-\lambda)(5-\lambda) - 3\times 4$. when $\det(B) = 0$, we have $\lambda_1 = 2$ and $\lambda_2 = 9$.

2. For $\lambda_1 = 2$. $(A - \lambda_1 I_2)x = 0 \Rightarrow \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ①

Solve ①, we have $x_1 = x_2 = 0$

Then $E_{\lambda_1} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

For $\lambda_2 = 9$. $(A - \lambda_2 I_2)x = 0 \Rightarrow \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ②

Solve ②. we have $x_1 = 4n$. $x_2 = 3n$. $n \in \mathbb{R}$.

Then $E_{\lambda_2} = \text{span}\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right)$

3. From the above, we know that $E_{\lambda_1} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$, $E_{\lambda_2} = \text{span}\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$, Let $a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Suppose $A = \begin{bmatrix} a^T \\ b^T \end{bmatrix}$, then $Ax = 0$ has only a trivial solution set, i.e. $x_1 = x_2 = 0$

which means a and b are independent vectors in $\mathbb{R}^2$.

Suppose $ma + nb = \begin{bmatrix} x \\ y \end{bmatrix}$, $x, y \in \mathbb{R}$, then we have $m = \frac{4}{7}x - \frac{3}{7}y$, $n = \frac{x}{7} + \frac{y}{7}$

i.e. a and b span $\mathbb{R}^2$

Thus, all eigenvectors of A span $\mathbb{R}^2$.

4. Following the steps of eigen decomposition. given the eigenvalues and its corresponding eigenvectors, we have $\lambda_1 = 2$, $\lambda_2 = 9$, $E_{\lambda_1} = \{\begin{pmatrix}1\\-1\end{pmatrix}\}$, $E_{\lambda_2} = \text{span}(\begin{pmatrix}4\\3\end{pmatrix})$

Then we construct $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathbb{R}^{2\times 2}$ as $a = \lambda_1 = 2$ $b = \lambda_2 = 9$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix}$

and $P = \begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}$, $\det(P) = 1\times 3 - 4\times(-1) = 7 \neq 0$, P^{-1} exists

5. $A^n = (PDP^{-1})^n = PDP^{-1} \times PDP^{-1} \times \cdots \times PDP^{-1} = PD^nP^{-1}$

Thus, $D^n = \begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix} \cdots \begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 2^n & 0 \\ 0 & 9^n \end{bmatrix}$

For P^{-1}, $[P \mid I] = \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right]$

$$\xrightarrow{r_2 \leftarrow r_1 + r_2} \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 7 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{r_2 \leftarrow \frac{1}{7} r_2} \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & \frac{1}{7} & \frac{1}{7} \end{array}\right]$$

$$\xrightarrow{r_1 \leftarrow -4r_2 + r_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{7} & -\frac{4}{7} \\ 0 & 1 & \frac{1}{7} & \frac{1}{7} \end{array}\right] = [I \mid P^{-1}] \quad \therefore P^{-1} = \begin{bmatrix} \frac{3}{7} & -\frac{4}{7} \\ \frac{1}{7} & \frac{1}{7} \end{bmatrix}$$

From above, we know that $P = \begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}$.

Then $A^n = \begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 9^n \end{bmatrix}\begin{bmatrix} \frac{3}{7} & -\frac{4}{7} \\ \frac{1}{7} & \frac{1}{7} \end{bmatrix} = \begin{bmatrix} \frac{3\times 2^n + 4\times 9^n}{7} & \frac{-4(2^n - 9^n)}{7} \\ \frac{-3(2^n - 3^{2n})}{7} & \frac{2^{n+2} + 3^{2n+1}}{7} \end{bmatrix}$

$$= \frac{1}{7}\begin{bmatrix} 3\times 2^n + 4\times 9^n & -4(2^n - 9^n) \\ -3(2^n - 3^{2n}) & 2^{n+2} + 3^{2n+1} \end{bmatrix}$$