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School of Computing
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Tutorial 9

COMP3670/6670: Introduction to Machine Learning

Question 1 **Laplace Expansion**

Given the following matrix $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 & 3 \\ 2 & -1 & 5 & -3 \\ 0 & 4 & 0 & 10 \\ 1 & 3 & 1 & 4 \end{bmatrix}$$

Verify that $\det \mathbf{A} = -16$ by using Laplace expansion.

Question 2 **Upper Triangular Matrix**

An upper triangular matrix is any square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ where all the elements below the main diagonal are zero (that is, for all $i > j$, $A_{ij} = 0$.)

1. Prove that the set of eigenvalues of any upper triangular matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the same as the set of diagonal elements $\{A_{11}, A_{22}, \ldots, A_{nn}\}$.

   (Hint: Use the fact that the determinant of an upper triangular matrix is the product of it's diagonal elements, which you will prove in the assignment.)

2. Prove that the set of all $n \times n$ upper triangular matrices are closed under matrix multiplication.

Question 3 **Fast Matrix Exponentiation**

1. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a diagonal matrix. Prove that for any integer $k \geq 1$ that

$$(\mathbf{A}^k)_{ij} = (\mathbf{A}_{ij})^k$$

2. Let $\mathbf{B} \in \mathbb{R}^{n \times n}$ be a diagonalizable matrix, with an invertible matrix $\mathbf{Q} \in \mathbb{R}^{n \times n}$ and a diagonal matrix $\mathbf{E} \in \mathbb{R}^{n \times n}$ satisfying $\mathbf{B} = \mathbf{QEQ}^{-1}$. Prove that for any integer $k \geq 1$, that

$$\mathbf{B}^k = \mathbf{QE}^k\mathbf{Q}^{-1}$$

   (Hint: Use induction on k. See appendix if you are unfamiliar with an induction proof.)

   Why does this allow $\mathbf{B}^k$ to be computed quickly?

Question 4 **Computing Eigenvalues and Eigenvectors**

Given the following matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

1. Compute the determinant of $\mathbf{A}$.
2. What is the characteristic equation of this matrix?
3. Find the eigenvalues, and their algebraic multiplicity.

4. For each eigenvalue, compute the corresponding eigenspaces.
5. Show that $\mathbf{A}$ is diagonalizable.
6. Find an invertible matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ and a diagonal matrix $\mathbf{D} \in \mathbb{R}^{n \times n}$ such that

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

7. Using Question 3, give a closed form for $\mathbf{A}^n$ for any $n \geq 0$.

Question 5 **Properties of Eigenvalues**

For a given eigenvalue, the corresponding eigenvector may not be unique. Is it true that for every eigenvalue there is a unique *unit* eigenvector?

Question 6 **Singular Value Decomposition**

Compute the singular value decomposition of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Appendix: Induction

The idea behind a proof by induction is usually employed to prove some statement to be true for all natural numbers. Suppose we have some statement $P(n)$ over the natural numbers, for example,

$$P(n) := \text{The sum of the first } n \text{ natural numbers is } \frac{n(n+1)}{2}.$$

We can verify $P(1), P(2), P(3)$ and find them all to be true, so we conjecture that $P(n)$ is true for all $n \in \mathbb{N}$. We cannot check infinitely many statements, so we can use induction. To do a proof by induction, we have to check two things.

1. The base case, usually $P(0)$ (or sometimes $P(1)$).
2. The step case. We assume $P(k)$ for some unknown k, and prove $P(k+1)$ still holds under this assumption.

Since we have $P(0)$ from the base case, the step case tells me that $P(1)$ must also be true. But if $P(1)$ holds, then $P(2)$ holds, and the next, and the next ...

Thus, we have $P(n)$ is true for all $n \in \mathbb{N}$. The assumption of $P(k)$ is called the *inductive hypothesis.*

Let's see an example by proving the above claim.

Proof by induction.
For the base case, prove that $P(0)$ holds, that is, that $\sum_{i=0}^{0} i = \frac{0(0+1)}{2}$. Both sides of the equation can be verified to be zero.

For the step case, we assume that

$$\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$$

and try to prove that

$$\sum_{i=0}^{k+1} = \frac{(k+1)(k+2)}{2}$$

Proof,

$$\begin{aligned}
&\sum_{i=0}^{k+1} i \\
&= (k+1) + \sum_{i=0}^{k} i \\
&= (k+1) + \frac{k(k+1)}{2} \text{ (by the inductive hypothesis)} \\
&= \frac{2(k+1)}{2} + \frac{k(k+1)}{2} \\
&= \frac{2(k+1) + k(k+1)}{2} \\
&= \frac{(k+2)(k+1)}{2}
\end{aligned}$$

as required.