

COMP3670: Introduction to Machine Learning

Note: For the purposes of this assignment, we let lowercase p denote probability density functions (pdf's), and upper case P denote probabilities. If a random variable Z is characterized by a probability density function p , we have that

$$P(a \leq Z \leq b) = \int_a^b p(z) dz$$

You should show your derivations, but you may use a computer algebra system (CAS) to assist with integration or differentiation.¹

Question 1 Bayesian Inference (40 credits)

Let X be a random variable representing the outcome of a biased coin with possible outcomes $\mathcal{X} = \{0, 1\}$, $x \in \mathcal{X}$. The bias of the coin is itself controlled by a random variable Θ , with outcomes² $\theta \in \theta$, where

$$\theta = \{\theta \in \mathbb{R} : 0 \leq \theta \leq 1\}$$

The two random variables are related by the following conditional probability distribution function of X given Θ .

$$p(X = 1 \mid \Theta = \theta) = \theta$$

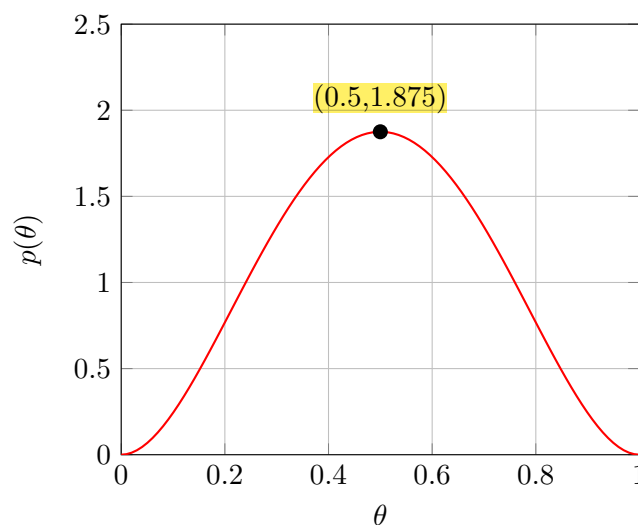
$$p(X = 0 \mid \Theta = \theta) = 1 - \theta$$

We can use $p(X = 1 \mid \theta)$ as a shorthand for $p(X = 1 \mid \Theta = \theta)$.

We wish to learn what θ is, based on experiments by flipping the coin. Before we flip the coin, we choose as our prior distribution

$$p(\theta) = 30\theta^2(1 - \theta)^2$$

which, when plotted, looks like this:



¹For example, asserting that $\int_0^1 x^2 (x^3 + 2x) dx = 2/3$ with no working out is adequate, as you could just plug the integral into Wolfram Alpha using the command `Integrate[x^2(x^3 + 2x), {x, 0, 1}]`

²For example, a value of $\theta = 1$ represents a coin with 1 on both sides. A value of $\theta = 0$ represents a coin with 0 on both sides, and $\theta = 1/2$ represents a fair, unbiased coin.

- a) (3 credits) Verify that $p(\theta) = 30\theta^2(1 - \theta)^2$ is a valid probability distribution on $[0, 1]$ (i.e that it is always non-negative and that it is normalised.)

We flip the coin a number of times.³ After each coin flip, we update the probability distribution for θ to reflect our new belief of the distribution on θ , based on evidence.

Suppose we flip the coin four times, and obtain the sequence of coin flips ⁴ $x_{1:4} = 0101$. For its two subsequences 01 and 0101, denoted by $x_{1:2}, x_{1:4}$ (and for the case before any coins are flipped), complete the following questions.

- b) (15 credits) Compute their probability distribution functions after observing the two subsequences $x_{1:2}$ and $x_{1:4}$, respectively.
- c) (3 credits) Compute their expectation values μ of θ before any evidence as well as after observing the two subsequences $x_{1:2}$ and $x_{1:4}$, respectively.
- d) (3 credits) Compute their variances σ^2 of θ before any evidence as well as after observing the two subsequences $x_{1:2}$ and $x_{1:4}$, respectively.
- e) (5 credits) Compute their maximum a posteriori estimations θ_{MAP} of θ before any evidence as well as after observing the two subsequences $x_{1:2}$ and $x_{1:4}$, respectively.

Present your results in a table like as shown below.

Posterior	PDF	μ	σ^2	θ_{MAP}
$p(\theta)$	$30\theta^2(1 - \theta)^2$?	?	?
$p(\theta x_{1:2} = 01)$?	?	?	?
$p(\theta x_{1:4} = 0101)$?	?	?	?

- f) (5 credits) Plot each of the probability distributions $p(\theta), p(\theta|x_{1:2} = 01), p(\theta|x_{1:4} = 0101)$ over the interval $0 \leq \theta \leq 1$ on the same graph to compare them.
- g) (6 credits) What behaviour would you expect of the posterior distribution $p(\theta|x_{1:n})$ if we updated on a very long sequence of alternating coin flips $x_{1:n} = 01010101 \dots$?
 What would you expect $\mu, \sigma^2, \theta_{MAP}$ to look like for large n ?
 Sketch/draw an estimate of what $p(\theta|x_{1:n})$ would approximately look like against the other distributions.

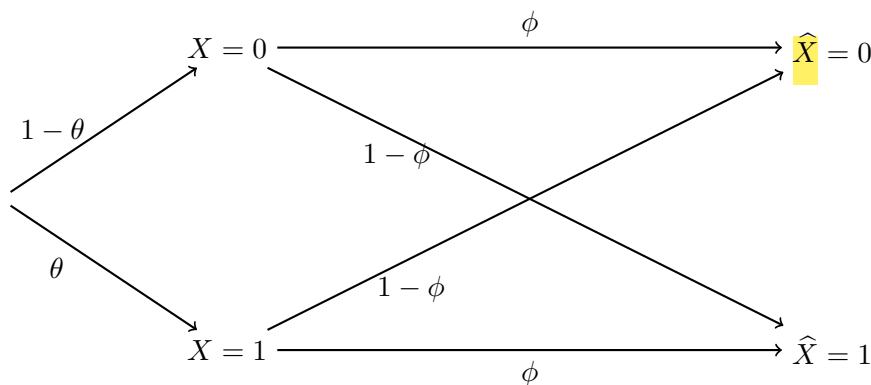
Question 2 Bayesian Inference on Imperfect Information (50 credits)

We have a Bayesian agent running on a computer, trying to learn information about what the parameter θ could be in the coin flip problem, based on observations through a noisy camera. The noisy camera takes a photo of each coin flip and reports back if the result was a 0 or a 1. Unfortunately, the side of the coin with a "1" on it is very shiny, and the reflected light causes the camera to sometimes report back the wrong result.⁵ The probability that the camera returns a correct answer is parameterised by $\phi \in [0, 1]$. Letting X denote the true outcome of the coin, and \hat{X} denoting what the camera reported back, we can draw the relationship between X and \hat{X} as shown.

³The coin flips are independent and identically distributed (i.i.d.).

⁴We write $x_{1:n}$ as shorthand for the sequence $x_1 x_2 \dots x_n$.

⁵The errors made by the camera are i.i.d, in that past camera outputs do not affect future camera outputs.



So, we have

$$\begin{aligned}
 p(\hat{X} = 0 \mid \phi, X = 0) &= \phi \\
 p(\hat{X} = 0 \mid \phi, X = 1) &= 1 - \phi \\
 p(\hat{X} = 1 \mid \phi, X = 1) &= \phi \\
 p(\hat{X} = 1 \mid \phi, X = 0) &= 1 - \phi
 \end{aligned}$$

We would now like to investigate what **posterior distributions** are obtained, as a **function of the parameter ϕ** . Let $\hat{x}_{1:n}$ be a sequence of coin flips as **observed by the camera**.

- (5 credits) **Briefly comment** about how the **camera behaves** for $\phi = 1, \phi = 0.5, \phi = 0$. How you expect **this** would change how the **agent updates its prior to a posterior on θ** , given an observation of \hat{X} . **(No equations required.)**
- (10 credits) Compute $p(\hat{X} = x \mid \theta)$ for all $x \in \{0, 1\}$.
- (15 credits) The coin is flipped, and the camera reports **seeing a zero**. (i.e. that $\hat{X} = 0$.) Given an **arbitrary prior $p(\theta)$** , compute the posterior $p(\theta \mid \hat{X} = 0)$. What does $p(\theta \mid \hat{X} = 0)$ simplify to when $\phi = 1$? When $\phi = 1/2$? When $\phi = 0$? **Explain** your observations.
- (10 credits) Compute $p(\theta \mid \hat{X} = 0)$ for the same choice of prior $p(\theta) = 30\theta^2(1 - \theta)^2$ as before. **Simplify your expression**.
- (10 credits) Plot $p(\theta \mid \hat{X} = 0)$ as a function of θ , for all $\phi \in \{0, 0.25, 0.5, 0.75, 1\}$ on the same graph to compare them. **Comment** on how the **shape** of the **distribution changes with ϕ** . Explain your observations.

Question 3

Relating Random Variables

(10 credits)

Let X be a random variable, on $[0, 1]$, with **probability density function**

$$p(x) = x^2 + \frac{2}{3}x + \frac{1}{3}$$

Let Y be a random variable on $[2, 3]$, such that **$Y = X^2 + 2$** . Find the **probability density function** for Y .