COMP3670/6670: Introduction to Machine Learning

Release Date: 3 Aug 2022
Due Date: 23:59pm, 28 Aug 2022
Maximum credit: 100

Exercise 1 **Solving Linear Systems** (4+4 credits)

Find the set $\mathcal{S}$ of all solutions $\mathbf{x}$ of the following inhomogenous linear systems $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{b}$ are defined as follows. Write the solution space $\mathcal{S}$ in parametric form.

(a)

$$\mathbf{A} = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & 3 \\ 0 & 2 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Solution. We form the augmented matrix, and row reduce.

$$\left[\begin{array}{ccc|c} 2 & 7 & 1 & 1 \\ 1 & 4 & 3 & 1 \\ 0 & 2 & 5 & 2 \end{array}\right]$$

$$\Big\downarrow R_1 := R_1 - 2R_2$$

$$\left[\begin{array}{ccc|c} 0 & -1 & 5 & -1 \\ 1 & 4 & 3 & 1 \\ 0 & 2 & 5 & 2 \end{array}\right]$$

$$\Big\downarrow \text{Swap } R_1 \text{ and } R_2$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & -1 & 5 & -1 \\ 0 & 2 & 5 & 2 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_1 := R_1 + 4R_2 \\ R_3 := R_3 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 23 & -3 \\ 0 & -1 & 5 & -1 \\ 0 & 0 & 15 & 0 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_3 := \frac{1}{15}R_3 \\ R_2 := -R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 23 & -3 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_1 := R_1 - 23R_3 \\ R_2 := R_2 + 5R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

We can read off the solution as $x_1 = -3, x_2 = 1, x_3 = 0$. Hence,

$$\mathcal{S} = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

.

(b)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Solution. We form the augmented matrix, and row reduce.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 3 & 4 & 3 & 5 \end{array}\right]$$

$$\Big\downarrow R_2 := R_2 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & -2 & -3 & -25 \end{array}\right]$$

$$\Big\downarrow R_1 := R_1 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -15 \\ 0 & -2 & -3 & -25 \end{array}\right]$$

$$\Big\downarrow R_2 := \frac{-1}{2} R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -15 \\ 0 & 1 & 3/2 & 25/2 \end{array}\right]$$

At this point we can read off the equations $x_1 - x_3 = -15$ and $x_2 + \frac{3}{2}x_3 = \frac{25}{2}$. Rearranging the second gives $x_2 = \frac{25}{2} - \frac{3}{2}x_3$, and rearranging the first gives $x_1 = x_3 - 15$. Here, x_3 is a free variable. So, the solution space is given as

$$\mathcal{S} = \left\{ \begin{bmatrix} x_3 - 15 \\ \frac{25}{2} - \frac{3}{2}x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} -15 \\ 25/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -3/2 \\ 1 \end{bmatrix} x_3 : x_3 \in \mathbb{R} \right\}$$

Exercise 2 **Inverses** (4 credits)

Find the inverse of the following matrix, if an inverse exists.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}$$

Solution. We form an augmented matrix with the identity matrix, and row reduce the original matrix, performing all row operations to the second matrix as well.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\Bigg\downarrow \begin{array}{l} R_3 := R_3 - 3R_1 \\ R_2 := R_2 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 1 & -4 & -3 & 0 & 1 \end{array}\right]$$

$$\Bigg\downarrow \begin{array}{l} R_1 := R_1 - R_2 \\ R_3 := R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 3 & -1 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right]$$

$$\Bigg\downarrow R_3 := -R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 3 & -1 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$\Bigg\downarrow \begin{array}{l} R_2 := R_2 + 3R_3 \\ R_1 := R_1 - 5R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -6 & 5 \\ 0 & 1 & 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

Hence, the inverse is

$$\begin{bmatrix} -2 & -6 & 5 \\ 1 & 4 & -3 \\ 1 & 1 & -1 \end{bmatrix}$$

Exercise 3 **Subspaces** (3+3+3+4 credits)

Which of the following sets are also subspaces of $\mathbb{R}^3$? Prove your answer. (That is, if it is a subspace, you must demonstrate the subspace axioms are satisfied, and if it is not a subspace, you must show which axiom fails.)

(a) $A = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$

Solution. NO, fails closure under scalar multiplication.
$[1, 1, 1]^T \in A$ but $-1 \cdot [1, 1, 1]^T = [-1, -1, -1] \notin A$.

(b) $B = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$.

Solution. YES, we check the requisite properties. Trivially, $B \subseteq \mathbb{R}^3$.

(a) Closure under scalar multiplication.
Let $\mathbf{x} \in B$. Then $\mathbf{x} = [x, y, z]^T$ with $x + y + z = 0$.
Let $c \in \mathbb{R}$ be arbitrary. Then $c\mathbf{x} = [cx, cy, cz]^T$, and $cx + cy + cz = c(x + y + z) = 0c = 0$, so $c\mathbf{x} \in B$.

(b) Closure under vector addition.
Let $\mathbf{x}, \mathbf{y} \in B$. Then $\mathbf{x} = [x_1, x_2, x_3]^T$ with $x_1 + x_2 + x_3 = 0$ and $\mathbf{y} = [y_1, y_2, y_3]^T$ with $y_1 + y_2 + y_3 = 0$. Then $\mathbf{x} + \mathbf{y} = [x_1 + y_1, x_2 + y_2, x_3 + y_3]^T$, and $(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = 0 + 0 = 0$, so $\mathbf{x} + \mathbf{y} \in B$.

(c) Contains the zero vector.
Clearly $\mathbf{0} \in B$, as $\mathbf{0} = [0, 0, 0]^T$ and $0 + 0 + 0 = 0$.

(c) $C = \{(x, y, z) \in \mathbb{R}^3 : x = 0 \text{ or } y = 0 \text{ or } z = 0\}$

Solution. NO, fails closure under addition. We have that $[0, 1, 1]^T \in C$ and $[1, 1, 0]^T \in C$, but $[0, 1, 1]^T + [1, 1, 0]^T = [1, 2, 1]^T \notin C$.

(d) $D =$ The set of all solutions to the matrix equation $\mathbf{Ax} = \mathbf{b}$, for some matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$ and some vector $\mathbf{b} \in \mathbb{R}^3$. (Hint: Your answer may depend on $\mathbf{A}$ and $\mathbf{b}$.)

Solution. Yes if and only if $\mathbf{b} = \mathbf{0}$.
First, note that since $\mathbf{A} \in \mathbb{R}^{3\times 3}$ and $\mathbf{b} \in \mathbb{R}^3$, then all solutions $\mathbf{x}$ to $\mathbf{Ax} = \mathbf{b}$ (if any exist) must be a vector in $\mathbb{R}^3$, so $D \subseteq \mathbb{R}^3$. We check the three axioms.

(a) Closure under scalar multiplication.
Let $\mathbf{x} \in D$. Then $\mathbf{Ax} = \mathbf{b}$. We have that $\mathbf{A}(\lambda\mathbf{x}) = \lambda(\mathbf{Ax}) = \lambda\mathbf{b}$. For $\lambda\mathbf{b} = \mathbf{b}$ for any choice of λ, it must be the case that $\mathbf{b} = \mathbf{0}$. So $\lambda\mathbf{x} \in D$ conditional on $\mathbf{b} = \mathbf{0}$.

(b) Closure under vector addition.
Let $\mathbf{x}, \mathbf{y} \in D$. Then $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ay} = \mathbf{b}$. But then $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{Ax} + \mathbf{Ay} = \mathbf{b} + \mathbf{b} = 2\mathbf{b}$. Now, $\mathbf{b} = 2\mathbf{b}$ is true if and only if $\mathbf{b} = \mathbf{0}$, so we have closure under addition conditional on $\mathbf{b} = \mathbf{0}$.

(c) Contains the zero vector.
$\mathbf{A0} = \mathbf{b}$ is true if and only if $\mathbf{b} = \mathbf{0}$, so this axiom is also conditional on $\mathbf{b} = \mathbf{0}$.

To conclude, the three axioms hold if $\mathbf{b} = \mathbf{0}$, and all of them don't if $\mathbf{b} \neq \mathbf{0}$.

Exercise 4 **Linear Independence** (5+10+15+5 credits)

Let V and W be vector spaces. Let $T : V \to W$ be a linear transformation.

(a) Prove that $T(\mathbf{0}) = \mathbf{0}$.

Solution. $T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$. Since $T(\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$, we subtract $T(\mathbf{0})$ from both sides to obtain $\mathbf{0} = T(\mathbf{0})$, as required.

(b) For any integer $n \geq 1$, prove that given a set of vectors $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ in V and a set of coefficients $\{c_1, \dots, c_n\}$ in $\mathbb{R}$, that

$$T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n)$$

Solution. We proceed by induction. The base case follows immediately from the definition of linearity of T, as $T(c_1\mathbf{v}_1) = c_1T(\mathbf{v}_1)$. Step case, assume that for some integer $n \geq 1$ that

$$T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n) \qquad \text{(Induction Hypothesis)}$$

for any $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ in V, $\{c_1, \dots, c_n\}$ in $\mathbb{R}$. We now prove that

$$T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n + c_{n+1}\mathbf{v}_{n+1}) = c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n) + c_{n+1}T(\mathbf{v}_{n+1})$$

for any $\{\mathbf{v}_1, \dots \mathbf{v}_{n+1}\}$ in V, $\{c_1, \dots, c_{n+1}\}$ in $\mathbb{R}$.

$$\begin{aligned}
&T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n + c_{n+1}\mathbf{v}_{n+1}) & \\
=&T((c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) + c_{n+1}\mathbf{v}_{n+1}) & \text{Vector addition is associative} \\
=&T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) + T(c_{n+1}\mathbf{v}_{n+1}) & T \text{ distributes over vector addition} \\
=&c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n) + T(c_{n+1}\mathbf{v}_{n+1}) & \text{Induction Hypothesis} \\
=&c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n) + c_{n+1}T(\mathbf{v}_{n+1}) & T \text{ distributes over scalar multiplication}
\end{aligned}$$

as required.

(c) Let $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ be a set of linearly **dependent** vectors in V.
Define $\mathbf{w}_1 := T(\mathbf{v}_1), \ldots, \mathbf{w}_n := T(\mathbf{v}_n)$.
Prove that $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a set of linearly **dependent** vectors in W.

Solution. We are given that $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is a set of linearly dependent vectors in V. Then there exists non-trivial[1] solutions to the equation

$$c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

Apply the transformation T to both sides of the equation,

$$\begin{aligned}
T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) &= T(\mathbf{0}) && \\
T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) &= \mathbf{0} && \text{Exercise 4a} \\
c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n) &= \mathbf{0} && \text{Exercise 4b} \\
c_1\mathbf{w}_1 + \ldots + c_n\mathbf{w}_n &= \mathbf{0} && \text{Definition of } \mathbf{w}_1, \ldots, \mathbf{w}_n
\end{aligned}$$

and hence $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a set of linearly dependent vectors in W.

(d) Let X be another vector space, and let $S : W \to X$ be a linear transformation. Define $L : V \to X$ as $L(\mathbf{v}) = S(T(\mathbf{v}))$. Prove that L is also a linear transformation.

Solution. We check the two axioms. Let $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $c \in \mathbb{R}$ be arbitrary. The transformation L distributes over vector addition:

$$\begin{aligned}
&L(\mathbf{v}_1 + \mathbf{v}_2) \\
&= S(T(\mathbf{v}_1 + \mathbf{v}_2)) \\
&= S(T(\mathbf{v}_1) + T(\mathbf{v}_2)) \\
&= S(T(\mathbf{v}_1)) + S(T(\mathbf{v}_2)) \\
&= L(\mathbf{v}_1) + L(\mathbf{v}_2)
\end{aligned}$$

as well as scalar multiplication:

$$\begin{aligned}
&L(c\mathbf{v}_1) \\
&= S(T(c\mathbf{v}_1)) \\
&= S(cT(\mathbf{v}_1)) \\
&= c(S(T(\mathbf{v}_1)) \\
&= cL(\mathbf{v}_1)
\end{aligned}$$

as required.

Exercise 5 **Inner Products** (5+10 credits)

(a) Show that if an inner product $\langle\cdot,\cdot\rangle$ is symmetric and linear in the first argument, then it is bilinear.

Solution. Suppose that $\langle\cdot,\cdot\rangle$ is a symmetric and linear in the first argument inner product. Then,

$$\langle \mathbf{x}, a\mathbf{y} + b\mathbf{z}\rangle = \langle a\mathbf{y} + b\mathbf{z}, \mathbf{x}\rangle = a\,\langle \mathbf{y}, \mathbf{x}\rangle + b\,\langle \mathbf{z}, \mathbf{x}\rangle = a\,\langle \mathbf{x}, \mathbf{y}\rangle + b\,\langle \mathbf{x}, \mathbf{z}\rangle$$

Hence $\langle\cdot,\cdot\rangle$ is linear in the second argument, and hence bilinear.

(b) Define $\langle\cdot,\cdot\rangle$ for all $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^2$ and $\mathbf{y} = [y_1, y_2]^T \in \mathbb{R}^2$ as

$$\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + x_2y_2 - (x_1 + x_2 + y_1 + y_2)$$

Which of the three inner product axioms does $\langle\cdot,\cdot\rangle$ satisfy?

Solution. Symmetry is satisfied, but bilinear and positive definiteness is not.

[1] That is, solutions other than $c_1 = c_2 = \ldots = c_n = 0$.

(a) Symmetry. We verify that $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$.

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y} \rangle =& x_1y_1 + x_2y_2 - (x_1 + x_2 + y_1 + y_2) \\ =& y_1x_1 + y_2x_2 - (y_1 + y_2 + x_1 + x_2) \\ =& \langle \mathbf{y}, \mathbf{x} \rangle\end{aligned}$$

(b) Bilinearity fails. Take $\mathbf{x} = \mathbf{y} = [0, 0]^T$, and $\mathbf{z} = [1, 1]$ and $a = b = 1$. Then

$$\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{0}, \mathbf{z} \rangle = 0z_1 + 0z_2 - (0 + 0 + z_1 + z_2) = -z_1 - z_2 = -2$$

but

$$a\langle \mathbf{x}, \mathbf{z} \rangle + b\langle \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{0}, \mathbf{z} \rangle + \langle \mathbf{0}, \mathbf{z} \rangle = -2(z_1 + z_2) = -4 \neq -2$$

(c) Positive Definiteness fails, choose $\mathbf{x} = [1, 1]^T$. Then

$$\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + x_2^2 - (x_1 + x_2 + x_1 + x_2) = x_1^2 - 2(x_1 + x_2) + x_2^2 = 1^1 - 2(1 + 1) + 1^1 = 1 - 4 + 1 = -2 \not\geq 0$$

Exercise 6 **Orthogonality** (15+6+4 credits)

Let V denote a vector space together with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$.

Let $\mathbf{x}, \mathbf{y}$ be **non-zero** vectors in V.

(a) Prove or disprove that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, then they are linearly independent.

Solution. The statement is true. We are given that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, so $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Assume for a contradiction that $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent, so there exists non-trivial solutions to the equation

$$c_1\mathbf{x} + c_2\mathbf{y} = \mathbf{0}.$$

Since the solution is non trivial, at least one of the c_i is non-zero. Proceed by cases.
Case 1: $c_1 \neq 0$.
Then we inner product both sides with $\mathbf{x}$,

$$\begin{aligned}\langle c_1\mathbf{x} + c_2\mathbf{y}, \mathbf{x} \rangle &= \langle \mathbf{0}, \mathbf{x} \rangle && \\ \langle c_1\mathbf{x} + c_2\mathbf{y}, \mathbf{x} \rangle &= 0 && \text{Tutorial 2} \\ c_1\langle \mathbf{x}, \mathbf{x} \rangle + c_2\langle \mathbf{y}, \mathbf{x} \rangle &= 0 && \text{Bilinearity} \\ c_1\langle \mathbf{x}, \mathbf{x} \rangle &= 0 && \text{Orthogonality of } \mathbf{x} \text{ and } \mathbf{y}\end{aligned}$$

Now, since $c_1 \neq 0$, we have that $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, and then by positive definiteness, $\mathbf{x} = \mathbf{0}$, a contradiction.
Case 2: $c_2 \neq 0$.
Then we inner product both sides with $\mathbf{y}$,

$$\begin{aligned}\langle c_1\mathbf{x} + c_2\mathbf{y}, \mathbf{y} \rangle &= \langle \mathbf{0}, \mathbf{y} \rangle && \\ \langle c_1\mathbf{x} + c_2\mathbf{y}, \mathbf{y} \rangle &= 0 && \text{Tutorial 2} \\ c_1\langle \mathbf{x}, \mathbf{y} \rangle + c_2\langle \mathbf{y}, \mathbf{y} \rangle &= 0 && \text{Bilinearity} \\ c_2\langle \mathbf{y}, \mathbf{y} \rangle &= 0 && \text{Orthogonality of } \mathbf{x} \text{ and } \mathbf{y}\end{aligned}$$

Now, since $c_2 \neq 0$, we have that $\langle \mathbf{y}, \mathbf{y} \rangle = 0$, and then by positive definiteness, $\mathbf{y} = \mathbf{0}$, a contradiction.
So, in either case we get a contradiction, and hence there are no non-trivial solutions to $c_1\mathbf{x} + c_2\mathbf{y} = \mathbf{0}$.
We conclude that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

(b) Prove or disprove that if $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, then they are orthogonal.

Solution. No. For a counter example, choose the vector space $V = \mathbb{R}^2$ equipped with the standard Eucledian dot product. Let $\mathbf{x} = (0, 1)^T$, $\mathbf{y} = (1, 1)^T$. They are linearly independent, as by solving $c_1\mathbf{x} + c_2\mathbf{y} = \mathbf{0}$ for c_1, c_2, we recover the two equations $0c_1 + 1c_2 = 0$ and $1c_1 + 1c_2 = 0$. The first equation gives $c_2 = 0$, substituting into the second gives $c_1 = 0$, so $\mathbf{x}, \mathbf{y}$ are linearly independent. But

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = 0 \cdot 1 + 1 \cdot 1 = 1 \neq 0$$

so they are not orthogonal.

(c) How do the above statements change if we remove the restriction that $\mathbf{x}$ and $\mathbf{y}$ have to be non-zero?

Solution. The disproof for b) still works. a) is now false, by choosing $\mathbf{x} = \mathbf{y} = \mathbf{0}$ they are orthogonal as $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, but still linearly dependant, as $c_1\mathbf{0} + c_2\mathbf{0} = \mathbf{0}$ has non-trivial solutions (say, $c_1 = c_2 = 1$.)