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Tutorial 8

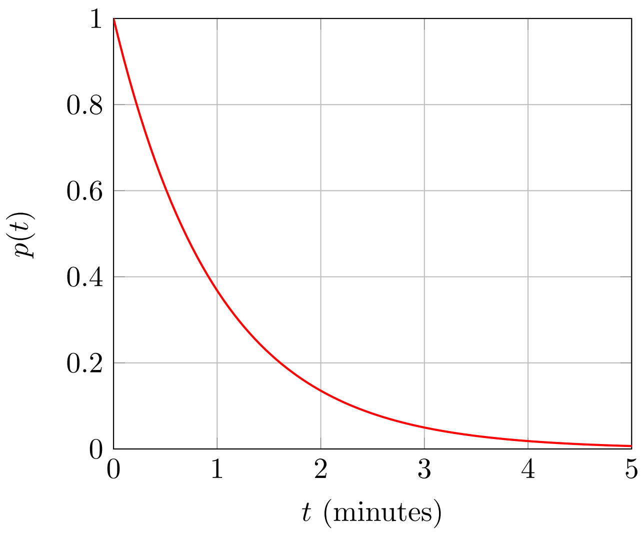
COMP3670/6670: Introduction to Machine Learning

Question 1 **Bomb Defusal**

You are a bomb defusal specialist, and you've come across a bomb that has just armed itself. You know from experience with these kinds of bomb, that the time till it explodes is controlled by a random variable T, sampled from the interval $[0, +\infty)$, according to the prior pdf function

$$p(t) = e^{-t} \text{ minutes}$$

which when plotted, looks like this



a) How likely is it for the bomb to take between 1 and 2 minutes to explode?

b) How long on average until the bomb explodes?

c) For any $\phi \in [0, 1)$, how much time t_ϕ would have to pass such that the probability of the bomb having exploded by then is ϕ?

d) Suppose you waited a minute, and the bomb has not yet exploded. What is the posterior distribution $p(t \mid t \geq 1)$ of the bomb's detonation time? Plot $p(t \mid t \geq 1)$ against the prior $p(t) = e^{-t}$.

e) You're a fast defuser, but an even faster runner. The bomb is placing 5 other people in mortal danger. It would take you 15 seconds to move out of the blast radius of the bomb, and 90 seconds to complete a defusal of the bomb. What action maximizes the expected number of lives saved? Attempting a defuse, or running away?

f) You discover a bomb just as it arms itself. This bomb is equipped with a display, reading out the time left till detonation in minutes in seconds (e.g the display says `1: 30` for 90 seconds). Unfortunately, part of the display is damaged, and you can only read the first digit (the minutes digit), which is a 1. What is the posterior $p(t \mid \text{First digit is } 1)$ based on this evidence? Plot this against $p(t)$.

g) Suppose two bombs[1] (with the same distribution $p(t) = e^{-t}$ as before) are armed simultaneously. Let T_1, T_2 denote random variables for the detonation time of each bomb. We define $T_e = \min(T_1, T_2)$, a random variable for the time taken for either bomb to explode, and $T_b = \max(T_1, T_2)$, a random variable for the time taken for both bombs to explode. Find the pdf $p_e(t)$ corresponding to T_e, and $p_b(t)$ corresponding to T_b, satisfying

$$\int_0^t p_e(x)\,dx = P(T_e \leq t) \qquad \int_0^t p_b(x)\,dx = P(T_b \leq t)$$

Plot $p(t) = e^{-t}$ vs. $p_b(t)$ vs. $p_e(t)$. How long, on average, before any bomb explodes? How long, on average, before both bombs explode?

(Hint: Bayes rule is not useful here.)

Question 2 **Definitions of variance**

Recall that given a continuous random variable X defined over a domain $D \subset \mathbb{R}$ with probability distribution function $p(x) : D \to \mathbb{R}$, and a function $f(x) : \mathbb{R} \to \mathbb{R}$, we define [2]

$$\mathbb{E}_X[f(x)] := \int_D f(x)p(x)\,dx$$

The variance $\mathbb{V}[X]$ of a random variable X is defined as

$$\mathbb{V}_X[x] := \mathbb{E}_X\left[(x - \mathbb{E}_X[x])^2\right]$$

It can also be represented in the alternate form

$$\mathbb{V}_X[x] := \mathbb{E}_X\left[x^2\right] - (\mathbb{E}_X[x])^2$$

Prove this!

Question 3 **Substitution of Random variables**

Assume that we have a random variable X on the interval [0,1] characterized by a pdf $p(x) = \frac{3}{2}\sqrt{x}$. Let Y be a random variable on [0,1] such that $Y = X^3$. Compute the pdf of Y

Question 4 **EM Algorithm**

Suppose that we are fitting a Gaussian mixture model for data items consisting of a single real value, x, using $K = 2$ components. We have $N = 5$ training cases, in which the values of x are as follows:

5, 15, 25, 30, 40

We use the EM algorithm to find the maximum likeihood estimates for the model parameters, which are the mixing proportions for the two components, π_1 and π_2, and the means for the two components, μ_1 and μ_2. The standard deviations for the two components are fixed at 10.

Suppose that at some point in the EM algorithm, the E step found that the responsibilities of the two components for the five data items were as follows:

r_{i1}	r_{i2}
0.2	0.8
0.2	0.8
0.8	0.2
0.9	0.1
0.9	0.1

What values for the parameters π_1, π_2, μ_1, and μ_2 will be found in the next M step of the algorithm?

[1] The bombs are placed sufficiently far apart that if one explodes, the other will be undisturbed. The detonation time of each bomb are independent of each other.

[2] Note that $\int_D$ means to integrate over the entire domain D. For example, if $D = [0,1]$, then $\int_D$ means the same thing as $\int_0^1$.