# Homework Assignment #3

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1. The mean duration of a phone conversation is 3 mins $\rightarrow \mu = \frac{1}{3}$ calls per minute ($\lambda$ must be less than $\frac{1}{3}$ calls per minute to keep the system stable).

   The average waiting time $T = \frac{\rho}{\mu(1-\rho)} \leq 3 \rightarrow \rho \leq \frac{1}{2}$ , where $\rho = \frac{\lambda}{\mu}$ is the server utilization.

   Since $\rho = \frac{\lambda}{\mu} \leq \frac{1}{2}, \lambda \leq \frac{1}{2}\mu = \frac{1}{6}$ calls per minute.

2. Denote the arrival rate of customers that choose to eat in the restaurant as $\lambda_1$ and that of those who carry out their orders as $\lambda_2$.

$$\lambda_1 = \lambda_2 = 5 \times 0.5 = 2.5$$

   According to the Little's Law, the average number of customers in the restaurant $N = \lambda_1 \times 3 + \lambda_2 \times (3 + 20) = 65$.

3. Let $\bar{T}_1$ and $\bar{T}_2$ be the expected customer response time in systems 1 and 2, respectively.
   Let $\bar{N}_1$ and denote the mean number of customers in systems 1. According to the stationary queue-length d.f. in an M/M/c queue:

$$\pi(i) = \begin{cases} \pi(0)\dfrac{\rho^i}{i!}, \text{if } i = 0, 1, \ldots, c \\ \pi(0)\dfrac{\rho^i c^{c-i}}{c!}, i \geq c \end{cases}, \text{where}$$

$$\pi(0) = \left[\sum_{i=0}^{c-1} \frac{\rho^i}{i!} + \left(\frac{\rho^c}{c!}\right)\left(\frac{1}{1 - \rho/c}\right)\right]^{-1},$$

   we have $\pi(i) = 2\pi(0)v^i, \forall i \geq 1$, if $v < 1$, from which we deduce that $\pi(0) = \frac{1-v}{1+v}$.

   Thus, for $v < 1$,

$$\bar{N}_1 = \sum_{i=1}^{\infty} i\pi(i)$$

$$= 2\left(\frac{1-v}{1+v}\right)\sum_{i=1}^{\infty} iv^i$$

$$= \frac{2v}{(1-v)(1+v)}$$

by using the well-known identity $\sum_{i=1}^{\infty} iz^{i-1} = {}^{1}\!/_{(1-z)^2}$ for all $0 \le z < 1$. From Little's Law we deduce that

$$\bar{T}_1 = \frac{v}{\lambda(1-v)(1+v)}$$

under the stability condition $v < 1$.

For the M/M/1 queue with arrival rate $2\lambda$ and service rate $2\mu$ we have

$$\bar{T}_2 = \frac{v}{2\lambda(1-v)}$$

under the stability condition $v < 1$. Since $\bar{T}_1 - \bar{T}_2 = \frac{v}{\lambda(1-v)(1+v)} - \frac{v}{2\lambda(1-v)} = \frac{2v-v(1+v)}{2\lambda(1-v)(1+v)} =$

$\frac{1-v}{2\lambda(1-v)(1+v)} > 0$ when $v < 1$, system 2 is better.

4. Let $X(t)$ be the number of machines up at time $t$. It is easy to see that $(X(t), t \ge 0)$ is a birth and death process with rates given by $\lambda_n = \mu$ for $n = 0, 1, \dots, K-1, \lambda_n = 0$ for $n \ge K$ and $\mu_n = n\alpha$ for $n = 1, 2, \dots, K$, respectively. We notice that $(X(t), t \ge 0)$ has the same behavior as the queue-length process of an M/M/K/K queue: $\pi(0)\frac{\rho^i}{i!}$, for $i = 0, 1, \dots, c, \pi(i) = 0$ for $i > c$, where $\pi(0) = \left[\sum_{i=0}^{c} \frac{\rho^i}{i!}\right]^{-1}$. Hence, we have $\pi(i) = \frac{(\mu/\alpha)^i/i!}{C(K, \mu/\alpha)}$ for $i = 0, 1, \dots, K$, where $C(K, a) = \sum_{i=0}^{K} \frac{a^i}{i!}$. The overall failure rate $\lambda_b$ is given by

$$\begin{aligned}
\lambda_b &= \sum_{i=1}^{K} (\alpha i)\pi(i) \\
&= \alpha \frac{\sum_{i=1}^{K} i\,(\mu/\alpha)^i/i!}{C(K,\ \mu/\alpha)} \\
&= \mu \frac{\sum_{i=0}^{K-1} (\mu/\alpha)^i/i!}{C(K,\ \mu/\alpha)} \\
&= \mu \frac{C(K-1,\ \mu/\alpha)}{C(K,\ \mu/\alpha)}.
\end{aligned}$$