

Homework Assignment #2

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1. To show definition A implies B, we need to show three things (other conditions already exist):

(1) $P(N(h) = 1) = \lambda h + o(h)$:

$$\begin{aligned}P(N(h) = 1) &= e^{-\lambda h} \frac{(\lambda h)^1}{1!} \\&= (\lambda h) e^{-\lambda h} \\&= (\lambda h) \left(1 + (-\lambda h) + \frac{(-\lambda h)^2}{2!} + \dots \right) \\&= \lambda h + \left(-(\lambda h)^2 + \frac{(\lambda h)^3}{2!} - \dots \right) \\&= \lambda h + o(h)\end{aligned}$$

(2) $P(N(h) \geq 2) = o(h)$:

$$\begin{aligned}P(N(h) \geq 2) &= 1 - P(N(h) = 0) - P(N(h) = 1) \\&= 1 - e^{-\lambda h} - (\lambda h) e^{-\lambda h} \\&= 1 - \left(1 + (-\lambda h) + \frac{(-\lambda h)^2}{2!} + \dots \right) - \left(\lambda h + \left(-(\lambda h)^2 + \frac{(\lambda h)^3}{2!} - \dots \right) \right) \\&= 1 - 1 + \lambda h - \lambda h - \left(\frac{(-\lambda h)^2}{2!} + \frac{(-\lambda h)^3}{3!} + \dots \right) - \left(-(\lambda h)^2 + \frac{(\lambda h)^3}{2!} - \dots \right) \\&= 0 - \left(\frac{(-\lambda h)^2}{2!} + \frac{(-\lambda h)^3}{3!} + \dots \right) - \left(-(\lambda h)^2 + \frac{(\lambda h)^3}{2!} - \dots \right) = o(h)\end{aligned}$$

(3) The process has the property of stationary increments:

$$P(N(t+s) - N(s) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

→ The number of events that occur in any time interval $[s, s+t]$ only depends on the length of the time interval → Stationary increments.

All of the three were proved, therefore, definition A implies B.

2. Let p denotes the probability of an event being preserved, and hence $1 - p$ is the probability of

an event being erased. We want to show that $P(N_{new}(t) = i) = e^{-p\lambda t} \frac{(p\lambda t)^i}{i!}$:

$$\begin{aligned}
P(N_{new}(t) = i) &= \sum_{j=i}^{\infty} P(i \text{ out of } j \text{ events were preserved} | j \text{ events occurred}) P(j \text{ events occurred}) \\
&= \sum_{j=i}^{\infty} P(i \text{ out of } j \text{ events were preserved}) \\
&= \sum_{j=i}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} C_i^j p^i (1-p)^{j-i} \\
&= \sum_{j=i}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \frac{j!}{i! (j-i)!} p^i (1-p)^{j-i} \\
&= e^{-\lambda t} p^i \frac{(\lambda t)^i}{i!} \sum_{j=i}^{\infty} \frac{(\lambda t (1-p))^{j-i}}{(j-i)!} \\
&= e^{-\lambda t} p^i \frac{(\lambda t)^i}{i!} e^{\lambda t (1-p)} \text{ (by applying the Maclaurin series)} \\
&= e^{-p\lambda t} \frac{(p\lambda t)^i}{i!}
\end{aligned}$$

3. Since the form of $m(t) = \frac{t}{2}$ follows the mean-value function of the Poisson process $m(t) = \lambda t$,

$$\lambda = 0.5 \text{ in this case. Therefore, } P[N(5) = 0] = e^{-0.5 \times 5} \frac{(0.5 \times 5)^0}{0!} = e^{-2.5}.$$

4. $P[N(t) \geq n] = P[N(t) = n] + P[N(t) = n+1] + \dots = \sum_{i=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^i}{i!}$; $m(t) =$

$$\sum_{n=1}^{\infty} P[N(t) \geq n] = \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^i}{i!} = \sum_{i=r}^{\infty} \left[\frac{i}{r} \right] e^{-\lambda t} \frac{(\lambda t)^i}{i!}.$$

5. The long-run time at which machines are replaced is the sum of the expected lifetime within T years and the expected lifetime after T years, that is, the long-run time equals to the $E[\text{expected lifetime within } T \text{ years}] = \int_0^T x f(x) dx$ plus $E[\text{expected lifetime after } T \text{ years}] = T(1 - F(T))$. The long-run rate is the reciprocal of the long-run time and is $\frac{1}{\int_0^T x f(x) dx + T(1 - F(T))}$;

the long-run rate at which machines in use fail = $P(\text{use fail} | \text{replaced}) =$

$$\frac{P(\text{replaced because of use fail})}{P(\text{replaced})} = \frac{F(T)}{\int_0^T x f(x) dx + T(1 - F(T))}.$$