Performance Evaluation Math Notes

1. Why z-transform?

   (a) $6g_n - 5g_{n-1} + g_{n-2} = 6(\frac{1}{5})^n, n = 2, 3, 4, ..., g_0 = 0, g_1 = \frac{6}{5}$

$$\begin{aligned}
6\sum_{n=2}^{\infty} g_n z^n - 5\sum_{n=2}^{\infty} g_{n-1} z^n + \sum_{n=2}^{\infty} g_{n-2} z^n &= 6\sum_{n=2}^{\infty} (\frac{1}{5})^n z^n \\
6\sum_{n=2}^{\infty} g_n z^n - 5z\sum_{n=2}^{\infty} g_{n-1} z^{n-1} + z^2\sum_{n=2}^{\infty} g_{n-2} z^{n-2} &= 6\sum_{n=2}^{\infty} (\frac{z}{5})^n \\
6(G(z) - g_0 - g_1 z) - 5z(G(z) - g_0) + z^2 G(z) &= \frac{6z^2}{5(5-z)} \\
G(z)(z-2)(z-3) - \frac{36z}{5} &= \frac{6z^2}{5(5-z)}
\end{aligned}$$

$$\begin{aligned}
G(z) &= \frac{6z(6-z)}{(2-z)(3-z)(5-z)} \\
&= \frac{z(6-z)}{5(1-\frac{z}{2})(1-\frac{z}{3})(1-\frac{z}{5})} \\
&= \frac{8}{1-\frac{z}{2}} + \frac{-9}{1-\frac{z}{3}} + \frac{1}{1-\frac{z}{5}} \\
&= 8\sum_{n=0}^{\infty}(\frac{1}{2})^n z^n - 9\sum_{n=0}^{\infty}(\frac{1}{3})^n z^n + \sum_{n=0}^{\infty}(\frac{1}{5})^n z^n \\
g(z) &= 8(\frac{1}{2})^n - 9(\frac{1}{3})^n + (\frac{1}{5})^n
\end{aligned}$$

(b) $a_{n+1} = 2a_n + 3^n + 5n, n = 1, 2, 3, ..., a_0 = -3$

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+1} z^n &= 2\sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} 3^n z^n + 5\sum_{n=0}^{\infty} n z^n \\
\frac{A(z) - a_0}{z} &= 2A(z) + \frac{1}{1-3z} + \frac{5z}{(1-z)^2} \\
A(z) + 3 &= 2zA(z) + \frac{z}{1-3z} + \frac{5z^2}{(1-z)^2} \\
A(z) &= \frac{-3}{1-2z} + \frac{z}{(1-2z)(1-3z)} + \frac{5z^2}{(1-2z)(1-z)^2} \\
&= \frac{-3}{1-2z} + \frac{1}{1-3z} - \frac{1}{1-2z} - \frac{5}{(1-z)^2} + \frac{5}{1-2z} \\
&= \frac{1}{1-2z} + \frac{1}{1-3z} - \frac{5}{(1-z)^2} \\
&= \frac{1}{1-2z} + \frac{1}{1-3z} - \frac{5z}{(1-z)^2} - \frac{5}{1-z} \\
&= \sum_{n=0}^{\infty} 2^n z^n + \sum_{n=0}^{\infty} 3^n z^n - 5\sum_{n=0}^{\infty} n z^n - 5\sum_{n=0}^{\infty} z^n
\end{aligned}$$

$$a_n = 2^n + 3^n - 5n - 5$$

2. Poisson process Def. B to A:

$$\begin{aligned}
P_0(t+h) &= P(N(t+h) = 0) \\
&= P(N(t) = 0, N(t+h) - N(t) = 0) \\
&= P_0(t)(1 - \lambda h + o(h))
\end{aligned}$$

For $h > 0$, $\frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \frac{o(h)}{h}$.
Let $h \to 0$, $\frac{dP_0(t)}{dt} = -\lambda P_0(t) \Rightarrow P_0(t) = -e^{-\lambda t}$.
Let $n \geq 1$ be an arbitrary integer and assume $P[N(t+s) - N(s) = m] = e^{-\lambda t}\frac{(\lambda t)^m}{m!}$ is true for $1, 2, ..., n-1$.

$$\begin{aligned}
\Rightarrow P_n(t+h) &= \sum_{k=0}^{n} (\text{exactly } k \text{ events in } [0,t], \text{ exactly } n-k \text{ events in } (t, t+h]) \\
&= \sum_{k=0}^{n} P_k(t) P_{n-k}(h) \\
&= P_n(t) P_0(h) + P_{n-1}(t) P_1(h) + o(h) \\
&= P_n(1 - \lambda h) + P_{n-1}(t) \cdot \lambda h + o(h)
\end{aligned}$$

For $h > 0$,
$\frac{P_n(t+h)-P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(h)}{h}$.
Taking $h \to 0$, we get
$\frac{dP_n(t)}{dt} = \lambda P_n(t) + \lambda P_{n-1}(t)$,
$\Rightarrow P_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$