Homework Assignment #2

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1. To show definition A implies B, we need to show three things (other conditions already exist):

   (1) $P(N(h)=1)=\lambda h+o(h)$:

$$\begin{aligned} P(N(h)=1) &= e^{-\lambda h}\frac{(\lambda h)^1}{1!} \\ &= (\lambda h)e^{-\lambda h} \\ &= (\lambda h)\left(1+(-\lambda h)+\frac{(-\lambda h)^2}{2!}+\cdots\right) \\ &= \lambda h-\left((\lambda h)^2+\frac{(\lambda h)^3}{2!}-\cdots\right) \\ &= \lambda h+o(h) \end{aligned}$$

   (2) $P(N(h)\geq 2)=o(h)$:

$$\begin{aligned} P(N(h)\geq 2) &= 1-P(N(h)=0)-P(N(h)=1) \\ &= 1-e^{-\lambda h}-(\lambda h)e^{-\lambda h} \\ &= 1-\left(1+(-\lambda h)+\frac{(-\lambda h)^2}{2!}+\cdots\right)-\left(\lambda h-\left((\lambda h)^2+\frac{(\lambda h)^3}{2!}-\cdots\right)\right) \\ &= 1-1+\lambda h-\lambda h-\left(\frac{(-\lambda h)^2}{2!}+\frac{(-\lambda h)^3}{3!}+\cdots\right)+\left((\lambda h)^2+\frac{(\lambda h)^3}{2!}-\cdots\right) \\ &= 0-\left(\frac{(-\lambda h)^2}{2!}+\frac{(-\lambda h)^3}{3!}+\cdots\right)+\left((\lambda h)^2+\frac{(\lambda h)^3}{2!}-\cdots\right)=o(h) \end{aligned}$$

   (3) The process has the property of stationary increments:

$$P(N(t+s)-N(s)=n)=e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

   →The number of events that occur in any time interval $[s,s+t]$ only depends on the length of the time interval →Stationary increments.

   All of the three were proved, therefore, definition A implies B.

2. Let p denotes the probability of an event being preserved, and hence $1-p$ is the probability of an event being erased. We want to show that $P(N_{new}(t)=i)=e^{-p\lambda t}\frac{(p\lambda t)^i}{i!}$:

$$P(N_{new}(t)=i)$$

$$=\sum_{j=i}^{\infty} P(i \text{ out of } j \text{ events were preserved} | j \text{ events occurred}) P(j \text{ events occurred})$$

$$=\sum_{j=i}^{\infty} P(i \text{ out of } j \text{ events were preserved})$$

$$=\sum_{j=i}^{\infty} e^{-\lambda t}\frac{(\lambda t)^j}{j!} C_i^j p^i (1-p)^{j-i}$$

$$=\sum_{j=i}^{\infty} e^{-\lambda t}\frac{(\lambda t)^j}{j!}\frac{j!}{i!\,(j-i)!} p^i (1-p)^{j-i}$$

$$=e^{-\lambda t}p^i\frac{(\lambda t)^i}{i!}\sum_{j=i}^{\infty}\frac{\left(\lambda t(1-p)\right)^{j-i}}{(j-i)!}$$

$$=e^{-\lambda t}p^i\frac{(\lambda t)^i}{i!}e^{\lambda t(1-p)} \text{ (by applying the Maclaurin series)}$$

$$=e^{-p\lambda t}\frac{(p\lambda t)^i}{i!}$$

3. Since the form of $m(t)=\frac{t}{2}$ follows the mean-value function of the Poisson process $m(t)=\lambda t$, $\lambda=0.5$ in this case. Therefore, $P[N(5)=0]=e^{-0.5\times 5}\frac{(0.5\times 5)^0}{0!}=e^{-2.5}$.

4. $P[N(t)\geq n]=P[N(t)=n]+P[N(t)=n+1]+\cdots=\sum_{i=nr}^{\infty} e^{-\lambda t}\frac{(\lambda t)^i}{i!}$; $m(t)=\sum_{n=1}^{\infty} P[N(t)\geq n]=\sum_{n=1}^{\infty}\sum_{i=nr}^{\infty} e^{-\lambda t}\frac{(\lambda t)^i}{i!}=\sum_{i=r}^{\infty}\left\lfloor\frac{i}{r}\right\rfloor e^{-\lambda t}\frac{(\lambda t)^i}{i!}$.

5. The long-run time at which machines are replaced is the sum of the expected lifetime within T years and the expected lifetime after T years, that is, the long-run time equals to the $E[\text{expected lifetime within } T \text{ years}]=\int_0^T xf(x)dx$ plus $E[\text{expected lifetime after } T \text{ years}]=T\big(1-F(T)\big)$. The long-run rate is the reciprocal of the long-run time and is $\frac{1}{\int_0^T xf(x)dx+T\left(1-F(T)\right)}$; the long-run rate at which machines in use fail $=P(\text{use fail}|\text{ replaced})=\frac{P(\text{replaced because of use fail})}{P(\text{replaced})}=\frac{F(T)}{\int_0^T xf(x)dx+T\left(1-F(T)\right)}$.