HW2

Deadline: 2016/4/20 (Wed.) 5:20 pm

1. Def. A: the counting process $\{N(t), t \geq 0\}$ is said to be Poisson process having rate λ, $\lambda > 0$ if
   - $N(0) = 0$;
   - The process has independent-increments
   - Number of events in any interval of length t is Poisson dist. with mean λt, that is for all $s, t \geq 0$.

   $$P[N(t+s) - N(s) = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$
   $$n = 0,1,2, \dots$$

   Def. B: the counting process $\{N(t), t \geq 0\}$ is said to be Poisson process having rate λ, $\lambda > 0$ if
   - $N(0) = 0$;
   - The process has stationary and independent increments
   - $P[N(h) = 1] = \lambda h + o(h)$
   - $P[N(h) \geq 2] = o(h)$

   The func. f is said to be $o(h)$ if $\lim_{h \to 0} \frac{f(h)}{h} = 0$

   Prove Def A $\Rightarrow$ Def B.

2. Prove the Poisson Process Property 4:
   If we perform Bernoulli trials to make independent random erasures from a Poisson process, the remaining arrivals also form a Poisson process.

3. If the mean-value function of the renewal process $\{N(t),\ t \geq 0\}$ is given by $m(t) = t/2$, $t \geq 0$, then what is $P\{N(5) = 0\}$?

4. Consider a renewal process $\{N(t),\ t \geq 0\}$ having a gamma (r, λ) interarrival distribution with density $f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{(r-1)!},\ x > 0.$

   (a) Show that

$$P\{N(t) \geq n\} = \sum_{i=nr}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

   (b) Show that

$$m(t) = \sum_{i=r}^{\infty} \left\lfloor \frac{i}{r} \right\rfloor \frac{e^{-\lambda t}(\lambda t)^i}{i!}, \text{where } \lfloor x \rfloor \text{ is the largest integer} \leq x$$

   Hint: use the relationship between the gamma (r, λ) distribution and the sum of r independent exponentials with rate λ to define $N(t)$ in terms of a Poisson process with rate λ.

5. A machine in use is replaced by a new machine either when it fails or when it reaches the age of T years. If the lifetimes of successive machines are independent with a common distribution F with density f, show that

   (1) the long-run rate at which machines are replaced is

$$\left[\int_0^T xf(x) + T\big(1 - F(T)\big)\right]^{-1}$$

   (2) the long-run rate at which machines in use fail equals

$$\frac{F(T)}{\left[\int_0^T xf(x) + T\big(1 - F(T)\big)\right]}$$

   Hint: condition on the lifetime of the first machine

ANSWERING RULES:

1. You can answer the problems in English or Chinese.
2. Please submit a hard copy of your homework in class.
3. Remember to write down your name and student ID, if not you will get 10% penalty.
4. We allow you to hand over the homework after the deadline, but 10% penalty per day.
5. Please write the process of the calculation or some explanations of the answers. Do not just write the answers.
6. Do not cheat, or you will get 0%.