Notes on SPH

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1 SPH Concept and Essential Formulation

The continuous SPH integral representation for $f(\mathbf{x})$ can be written in the following form of discretized particle approximation

$$\begin{aligned}
f(\mathbf{x}) &= \int_{\Omega} f(\mathbf{x}')W(\mathbf{x}-\mathbf{x}', h)\mathrm{d}\mathbf{x}' \\
&\approx \sum_{j=1}^{N} f(\mathbf{x}_j)W(\mathbf{x}-\mathbf{x}_j, h)\Delta V_j \\
&= \sum_{j=1}^{N} f(\mathbf{x}_j)W(\mathbf{x}-\mathbf{x}_j, h)\frac{1}{\rho_j}(\rho_j \Delta V_j) \\
&= \sum_{j=1}^{N} f(\mathbf{x}_j)W(\mathbf{x}-\mathbf{x}_j, h)\frac{1}{\rho_j}m_j \\
&= \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(\mathbf{x}_j)W(\mathbf{x}-\mathbf{x}_j, h)
\end{aligned} \tag{1.1}$$

For a given particle i, according to the particle approximation, the value of a function and its derivative for particle i are approximated as

$$\langle f(\mathbf{x}_i)\rangle = \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(\mathbf{x_j})W_{ij} \tag{1.2}$$

$$\langle \nabla \cdot f(\mathbf{x}_i)\rangle = \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(\mathbf{x}_j) \cdot \nabla_i W_{ij} \tag{1.3}$$

$$W_{ij} = W(\mathbf{x}_i - \mathbf{x}_j, h) = W(|\mathbf{x}_i - \mathbf{x}_j|, h) \tag{1.4}$$

$$\nabla_i W_{ij} = \frac{\mathbf{x}_i - \mathbf{x}_j}{r_{ij}} \frac{\partial W_{ij}}{\partial r_{ij}} = \frac{x_{ij}}{r_{ij}} \frac{\partial W_{ij}}{\partial r_{ij}} \tag{1.5}$$

Summationa density approach: or the SPH approximation for the density is obtained

$$\rho_i = \sum_{j=1}^{N} m_j W_{ij} \tag{1.6}$$

Here are some techniques in deriving SPH formulations

$$\begin{aligned}
\nabla \cdot f(\mathbf{x}_i) =& \frac{1}{\rho_i}[\nabla \cdot (\rho_i f(\mathbf{x}_i)) - f(\mathbf{x}_i) \cdot \nabla \rho_i] \\
=& \frac{1}{\rho_i}\left[\sum_{j=1}^{N} \frac{m_j}{\rho_j} \rho_j f(\mathbf{x}_j) \cdot \nabla_i W_{ij} - f(\mathbf{x}_i) \sum_{j=1}^{N} m_j \cdot \nabla_i W_{ij}\right] \\
=& \frac{1}{\rho_i}\left[\sum_{j=1}^{N} m_j \left[f(\mathbf{x}_j) - f(\mathbf{x}_i)\right] \cdot \nabla_i W_{ij}\right]
\end{aligned} \tag{1.7}$$

$$\begin{aligned}
\nabla \cdot f(\mathbf{x}_i) =& \rho_i \left[\nabla \cdot \left(\frac{f(\mathbf{x}_i)}{\rho_i}\right) + \frac{f(\mathbf{x}_i)}{\rho_i^2} \cdot \nabla \rho_i\right] \\
=& \rho_i \left[\sum_{j=1}^{N} \frac{m_j}{\rho_j} \frac{f(\mathbf{x}_j)}{\rho_j} \cdot \nabla_i W_{ij} + \frac{f(\mathbf{x}_i)}{\rho_i^2} \sum_{j=1}^{N} m_j \cdot \nabla_i W_{ij}\right] \\
=& \rho_i \left[\sum_{j=1}^{N} m_j \left[\frac{f(\mathbf{x}_j)}{\rho_j^2} + \frac{f(\mathbf{x}_i)}{\rho_i^2}\right] \cdot \nabla_i W_{ij}\right]
\end{aligned} \tag{1.8}$$

There are some properties of SPH formulations as

$$\langle f_1 + f_2 \rangle = \langle f_1 \rangle + \langle f_2 \rangle = \langle f_2 + f_1 \rangle \tag{1.9}$$

$$\langle f_1 f_2 \rangle = \langle f_1 \rangle \langle f_2 \rangle = \langle f_2 f_1 \rangle \tag{1.10}$$

By definition, the support domain for a field point at $\mathbf{x} = (x, y, z)$ is the domain where the information for all the points inside this domain is used to determine the information at the points at $\mathbf{x}$. The influence domain is defined as a domain where a node exerts its influences. Hence the influence domain is associated with a node in the meshfree methods, and the support domain goes with any field point $\mathbf{x}$, which can be, but does not necessarily have to be a node.

2 Constructing Smoothing Functions

In the original SPH paper, the following bell-shaped function was used as the smoothing function

$$W(\mathbf{x} - \mathbf{x}', h) = W(R, h) = \alpha_d \begin{cases} (1 + 3R)(1 - R)^3 & R \leq 1 \\ 0 & R > 1 \end{cases} \tag{2.1}$$

where α_d is $5/4h$, $5/\pi h^2$ and $105/16\pi h^3$ in one-, two- and three-dimensional space, respectively.

Apply the Taylor series expansion of $f(\mathbf{x}')$ in the vicinty of $\mathbf{x}$ yields

$$
\begin{aligned}
f(\mathbf{x}') =& f(\mathbf{x}) + f'(\mathbf{x})(\mathbf{x}' - \mathbf{x}) + \frac{1}{2} f''(\mathbf{X})(\mathbf{x}' - \mathbf{x})^2 + \dots \\
=& \sum_{k=0}^{n} \frac{(-1)^k h^k f^{(k)}(\mathbf{x})}{k!} \left(\frac{\mathbf{x} - \mathbf{x}'}{h} \right)^k + r_n \left(\frac{\mathbf{x} - \mathbf{x}'}{h} \right)
\end{aligned}
\tag{2.2}
$$