Notes on ideal fluids

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I. The equation of continuity

Fluid dynamics concerns itself the study of the motion of fluids (liquids and gases). Fluid is regarded as continuous medium and its phenomena are considered in macro scope. Therefore, any small volume element in the fluid is supposed so large that it still contains a very great number of molecules. Accordingly, infinitely small elements of volume means physically infinitely small.

The mathematical description of the state of a moving fluid is effected by means of functions which give the distribution of the fluid velocity $\mathbf{v} = \mathbf{v}(x, y, z, t)$ and of any two thermodynamic quantities pertaining the fluid, for instance the pressure $p(x, y, z, t)$ and the density $\rho(x, y, z, t)$. All the thermodynamic quantities are determined by the values of any two of them, together with the equation of state; hence, if we are given five quantities, the three components of the velocity $\mathbf{v}$, the pressure p and the density ρ, the state of the moving fluid is completely determined.

All these quantities are, in general, functions of the coordinates x, y, z and of the time t. $\mathbf{v}(x, y, z, t)$ is the velocity of the fluid at a given point (x, y, z) in space and at a given time t. It refers to fixed points in space and not to specific particles of the fluid; in the course of time, the latter move about in space. THe same remarks apply to p and ρ.

Begin with the equation which express the conservation of matter. Consider some volume V_0 of space. The mass of fluid in this volume is $\int \rho \mathrm{d}V$, where ρ is the fluid density, and the integration is taken over the volume V_0. The mass of fluid flowing in unit time through an element $\mathrm{d}\mathbf{f}$ of the surface bounding this volume is $\rho\mathbf{v} \cdot \mathrm{d}\mathbf{f}$; the magnitude of the vector $\mathrm{d}\mathbf{f}$ is equal to the area of the surface element, and its direction is along the normal. By convention, take $\mathrm{d}\mathbf{f}$ along the outward normal. Then $\rho\mathbf{v} \cdot \mathrm{d}\mathbf{f}$ is positive if the fluid is flowing out of the volume, and negative if the flow is into the volume. The total mass of

$$\oint \rho\mathbf{v} \cdot \mathrm{d}\mathbf{f}$$

where the integration is taken over the whole of the closed surface surrounding the volume in question.

Then, the decrease per unit time in the mass of the fluid in the volume V_0 can be written

$$-\frac{\partial}{\partial t} \int \rho \ \mathrm{d}V$$

Equating the two expression, we have

$$\frac{\partial}{\partial t} \int \rho \ \mathrm{d}V = -\oint \rho\mathbf{v} \cdot \mathrm{d}\mathbf{f}$$

The surface integral can be transformed by Green's formula to a volume integral:

$$\oint \rho\mathbf{v} \cdot \mathrm{d}\mathbf{f} = \int \nabla \cdot (\rho\mathbf{v}) \ \mathrm{d}V$$

Thus,

$$\int \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v})\right] \mathrm{d}V = 0$$

Since this equation must hold for any volume, the integration must vanish, i.e.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0 \tag{I.1}$$

This is the equation of continuity. Expanding the expression $\nabla \cdot (\rho\mathbf{v})$,

$$\frac{\partial \rho}{\partial t} + \rho\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla\rho = 0 \tag{I.2}$$

The vector

$$\mathbf{j} = \rho\mathbf{v} \tag{I.3}$$

is called the mass flux density. Its direction is that of the motion of the fluid, while its magnitude equals the mass of fluid flowing in unit time through unit area perpendicular to the velocity.

II. Euler's equation

The total force acting on the volume in the fluid is equal to the integral

$$-\oint p \ \mathrm{d}\mathbf{f}$$

of the pressure, taken over the surface bounding the volume. Transforming it to a volume integral,

$$-\oint p \ \mathrm{d}\mathbf{f} = -\int \nabla p \ \mathrm{d}V$$

Hence, the fluid surrounding any volume element $\mathrm{d}V$ exerts on the element a force $-\mathrm{d}V\nabla p$. In other words, we can say that the force $-\nabla p$ acts on unit volume of the fluid.

Write down the equation of motion of a volume element in the fluid by equating the force $-\nabla p$ to the product of the mass per unit volume (ρ) and the accleration $\mathrm{d}\mathbf{v}/\mathrm{d}t$:

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla p \tag{II.1}$$

The derivative $\mathrm{d}\mathbf{v}/\mathrm{d}t$ which appears here denotes not the rate of change of the fluid velocity at a fixed point in space, but the rate of change of the velocity of a given fluid particle as it moves about in space. This derivative has to be expressed in terms of quantities referring to points fixed in space. The change $\mathrm{d}\mathbf{v}$ in the velocity of the given fluid particle during the time $\mathrm{d}t$ is composed of two parts, the change during $\mathrm{d}t$ in the velocity at a point fixed in space, and the difference between the velocities (at that instant) at two points $\mathrm{d}\mathbf{r}$ apart, where $\mathbf{r}$ is the distance moved by given fluid particle during the time $\mathrm{d}t$. The first part is $(\partial \mathbf{v}/\partial t)\mathrm{d}t$, where the derivative $\partial \mathbf{v}/\partial t$ is taken for constant x, y, z, i.e. at the given point in space. The second part is

$$\mathrm{d}x\frac{\partial \mathbf{v}}{\partial x} + \mathrm{d}y\frac{\partial \mathbf{v}}{\partial y} + \mathrm{d}z\frac{\partial \mathbf{v}}{\partial z} = \mathrm{d}\mathbf{r} \cdot \nabla \mathbf{v}$$

Thus

$$\mathrm{d}\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t}\mathrm{d}t + \mathrm{d}\mathbf{r} \cdot \nabla \mathbf{v}$$

or, dividing both sides by $\mathrm{d}t$,

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$$

Substituting in II.1,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho}\nabla p \tag{II.2}$$

It is called Euler's equation and is one of the fundamental equations of fluid dynamics.

If the fluid is in the gravitational field, an additional force $\rho\mathbf{g}$, where $\mathbf{g}$ is the acceleration due to gravity, acts on any unit volume. This force must be added to the right-hand side of equation II.1, so that equation II.2 takes the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{g} \tag{II.3}$$

Here, it is taken no account of processes of energy dissipation, which may occur in a moving fluid in consequence of internal friction (viscosity) in the fluid and heat exchange between different parts of it. Thermal conductivity and viscosity are unimportant in current setting and such fluids are said to be ideal.

The absence of heat exchange between different parts of the fluid (and also, of course, between the fluid and bodies adjoining it) means that the motion is adiabatic throughout the fluid. Thus the motion of an ideal fluid must necessarily be supposed adiabatic.

In adiabatic motion the entropy of any particle of fluid remains constant as that particle moves about in space. Denoting by s the entropy per unit mass, the condition for adiabatic motion can be expressed as

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0 \tag{II.4}$$

where the total derivative with respect to time denotes the rate of change of entropy for a given fluid particle as it moves about. This condition can be written as

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0 \tag{II.5}$$

This is the general equation describing adiabatic motion of an ideal fluid. Using I.1, it can be written as an equation of continuity for entropy:

$$\begin{aligned}&\rho\left(\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s\right) + s\left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right)\\ =&\frac{\partial(\rho s)}{\partial t} + \nabla \cdot (\rho s \mathbf{v}) = 0\end{aligned} \tag{II.6}$$

The product $\rho s \mathbf{v}$ is the entropy flux density.

The adiabatic equation usually takes a much simpler form. If, as usually happens, the entropy is constant throughout thr volume of the fluid at some initial instant, it retains everywhere the same constant value at all times and for any subsequent motion of the fluid. The adiabatic equation can be simply written as

$$s = \text{constant} \tag{II.7}$$

Such a motion is said to be isentropic.

By employing the thermodynamic relation

$$\mathrm{d}w = T\mathrm{d}s + V\mathrm{d}p$$

where w is the heat function per unit mass of fluid (enthalpy), $V = 1/\rho$ is the specific volume, and T is the temperature. Since $s =$ constant, we have simply

$$\mathrm{d}w = V\mathrm{d}p = \mathrm{d}p/\rho$$

and so

$$\frac{\nabla p}{\rho} = \nabla w$$

Equation II.2 can therefore be written in the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla w \tag{II.8}$$

Using one formula in vector analysis

$$\frac{1}{2}\nabla v^2 = \mathbf{v} \times (\nabla \times \mathbf{v}) + \mathbf{v} \cdot \nabla \mathbf{v}$$

Then rewrite II.8 in the form

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nabla(w + \frac{1}{2}v^2) \tag{II.9}$$

Take the curl of both sides of this equation,

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) = \nabla \times (\mathbf{v} \times (\nabla \times \mathbf{v})) \tag{II.10}$$

which involves only the velocity.

The equations of motion have to be supplemented by the boundary conditions that must be satisfied at the surfaces bounding the fluid. For an ideal fluid, the boundary condition is simply that the fluid cannot penetrate a solid surface. This means that the component of the fluid velocity normal to the bounding surface must vanish if that surface is at rest:

$$v_n = 0 \tag{II.11}$$

IN the general case of a moving surface, v_n must be equal to the corresponding component of the velocity of the surface.

At a boundary between two immiscible fluids, the condition is that the pressure and the velocity component normal to the surface of separation must be the same for the two fluids, and each of these velocity components must be equal to the corresponding component of the velocity of the surface.

As stated previously, the state of a moving fluid is determined by five quantities: the three components of the velocity $\mathbf{v}$ and , for example, the pressure p and the density ρ. Accordingly, a complete system of equations of fluid dynamics should be five in number. For an ideal fluid these are Euler's equations, the equation of continuity, and the adiabatic equation.

III. Hydrostatics

For a fluid at rest in a uniform gravitational field, Euler's equation II.3 takes the form

$$\nabla p = \rho \mathbf{g} \tag{III.1}$$

This equation described the mechanical equilibrium of the fluid. This can be integrated immediately if the density of the fluid may be supposed constant throughout its volume. Taking the z-axis verticaly upward,

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho g$$

Hence

$$p = -\rho g z + \text{contant}$$

If the fluid at rest has a free surface at height h, to which a external pressure p_0, the same at every point, is applied, this surface must be the horizontal plane $z = h$. From the condition, $p = p_0$ for $z = h$, we find that the constant is $p_0 + \rho g h$, so that

$$p = p_0 + \rho g (h - z) \tag{III.2}$$

For large masses of liquid, and for a gas, the density ρ cannot in general be supposed constant; this applies especially to gases (for example, the atmosphere). Let us suppose that the fluid is not only in mechanical equilibrium but also in thermal equilibrium. Then the temperature is the same at every point, and equation II.3 can be integrated as follows. First use the thermodynamic relation

$$\mathrm{d}\Phi = -s\mathrm{d}T + V\mathrm{d}p$$

where Φ is the thermodynamic potential (Gibbs free energy) per unit mass. For constant temperature

$$\mathrm{d}\Phi = V\mathrm{d}p = \mathrm{d}p/\rho$$

Hence, the expression $(\nabla p)/\rho$ can be written in this case as $\nabla\Phi$, so that the equation of equilibrium takes the form

$$\nabla\Phi = \mathbf{g}$$

For a constant vector $\mathbf{g}$ directed along the negative z-axis,

$$\mathbf{g} \equiv -\nabla(gz)$$

Thus

$$\nabla(\Phi + gz) = 0$$

hence we find that throughout the fluid

$$\Phi + gz = \text{constant}$$

gz is the potential energy of unit mass of fluid in the gravitational field. This condition is known from statistical physics to be the condition for thermodynamic equilibrium of a system in an external field.

If a fluid is in mechanical equilibrium in a gravitational field, the pressure in it can be a function only of the altitude z (since, if the pressure were different at different points with the same altitude, motion would result). It then follows that the density

$$\rho = -\frac{1}{g}\frac{\mathrm{d}p}{\mathrm{d}z} \tag{III.3}$$

is also a function of z only. THe pressure and density together determine the temperature, which is therefore again a function of z only. Thus, in mechanical equilibrium in a gravitational field, the pressure, density and temperature distributions depend only on the altitude. If, for example, the temperature is different at different points with the same altitude, then mechanical equilibrium is impossible.

Finally, let us derive the equation of equilibrium for a very large mass of liquid, whose seperate parts are held together by gravitaitonal attraction - a star. Let ϕ be the Newtonian gravitational potential of the field due to the fluid. It statifies the differential equation

$$\Delta\phi = 4\pi G\rho \tag{III.4}$$

where G is the Newtonian constant of gravitation. The gravitational acceleration is $-\nabla\phi$, and the force on a mass ρ is $-\rho\nabla\phi$. The condition of equilibrium is therefore

$$\nabla p = -\rho\nabla\phi$$

Dividing both sides by ρ, taking the divergence of both sides, and using equation III.4,

$$\nabla \cdot \left(\frac{1}{\rho}\nabla p\right) = -4\pi G\rho \tag{III.5}$$

Here only concerns mechanical equilibrium and does not presuppose the existence of complete thermal equilibrium.

If the body is no rotating, it will be spherical when in equilibrium, and the density and pressure distributions will be spherically symmetrical. The equation above in spherical polar coordinates then takes the form

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = -4\pi G\rho \tag{III.6}$$

IV. The condition that convection be absent

A fluid can be in mechanical equilibrium without being in thermal equilibrium. The condition for mechanical equilibrium can be satisfied even if the temperature is not constant throughout the fluid. However, the question then arises of the stability of such an equilibrium. It is found that the equilibrium is stable only when certain condition is fulfilled. Otherwise, the equilibriumis unstable, and this leads to the appearance in the fluid of currents which tend to mix the fluid in such a way as to equalize the temperature. This motion is called convection. Thus the condition for a mechanical equilibrium to be stable is the condition that convection is absent.

Consider a fluid element at height z, having a specific volume $V(p,s)$, where p and s are equilibrium pressure and entropy at height z. Suppose that this fluid element undergoes an adiabatic upward displacement through a small interval ξ; its specific volume then becomes $V(p',s)$, where p' is the pressure at height $z+\xi$. For the equilibrium to be stable, it is necessary that the resulting force on the element should tend to return it to its original position. This means that the element must be heavier than the fluid which is "displaces" in its new position. The specific volume of the latter is $V(p',s')$, where s' is the equilibrium entropy at height $z+\xi$. Thus we have the stability condition

$$V(p',s') - V(p',s) > 0$$

Expanding this difference in powers of $s - s' = \xi \mathrm{d}s/\mathrm{d}z$,

$$\left(\frac{\partial V}{\partial s}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z} > 0 \tag{IV.1}$$

The formula of thermodynamics give

$$\left(\frac{\partial V}{\partial s}\right)_p = \frac{T}{c_p}\left(\frac{\partial V}{\partial T}\right)_p$$

where c_p is the specific heat at constant pressure. Since both c_p and T are positive, so that

$$\left(\frac{\partial V}{\partial T}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z} > 0 \tag{IV.2}$$

THe majority of substances expand on heating, i.e. $\left(\frac{\partial V}{\partial T}\right)_p > 0$. The condition that convection be absent then becomes

$$\frac{\mathrm{d}s}{\mathrm{d}z} > 0 \tag{IV.3}$$

that the entropy must increase with height.

Expanding the dreivative $\frac{\mathrm{d}s}{\mathrm{d}z}$ with Maxwell relations, we have

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}z} &= \left(\frac{\partial s}{\partial T}\right)_p \frac{\mathrm{d}T}{\mathrm{d}z} + \left(\frac{\partial s}{\partial p}\right)_T \frac{\mathrm{d}p}{\mathrm{d}z} \\ &= \frac{c_p}{T}\frac{\mathrm{d}T}{\mathrm{d}z} - \left(\frac{\partial V}{\partial T}\right)_p \frac{\mathrm{d}p}{\mathrm{d}z} > 0 \end{aligned} \tag{IV.4}$$

By substituting $\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{g}{V}$, we obtain

$$-\frac{\mathrm{d}T}{\partial z} < g\beta\frac{T}{c_p} \tag{IV.5}$$

Convection occurs if these conditions are not satisfied.

V. Bernoulli's equation

The equation of fluid dynamics are much simplified in the case of steady flow. By steady flow we mean one in which the velocity is constant in time at any point occupied by fluid. In other words, $\mathbf{v}$ is a function of the coordinates only, so that $\frac{\mathbf{v}}{t} = 0$. Then the Euler's equation reduces to

$$\frac{1}{2}\nabla v^2 - \mathbf{v}\times\nabla\times\mathbf{v} = -\nabla w \tag{V.1}$$

Now introduce the concept of streamlines. There are lines such that the tangent to a streamline at any point gives the direction of the velocity at that point; they are determined by the following system of differential equations:

$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} = \frac{\mathrm{d}z}{v_z} \tag{V.2}$$

In steady flow the streamlines do not vary with time, and coincide with the paths of the fluid particles. In non-steady slow this coincidence no longer occurs: the tangents to the streamlines give the directions of the velocities of fluid particles at various points in space at a given instant, whereas the tangents to the paths give the directions of velocities of given fluid particles at various times.

Form the scalar product of Eq. V.1 with the unit vector tangent to the streamline at each point; this unit vector is denoted by $\mathbf{l}$. The projection of the gradient on any direction is, as we know, the derivative in that direction. Hence the projection of ∇w is $\frac{\partial w}{\partial l}$. The vector $\mathbf{v}\times\nabla\times\mathbf{v}$ is perpendicular to $\mathbf{v}$, and its projection on the direction of $\mathbf{l}$ is therefore zero.

Thus, from Eq. V.1

$$\frac{\partial}{\partial l}\left(\frac{1}{2}v^2 + w\right) = 0$$

It follows from this that $\frac{1}{2}v^2 + w$ is constant along a streamline:

$$\frac{1}{2}v^2 + w = \text{const.} \tag{V.3}$$

In general the constant takes different values for different streamlines.Eq. V.3 is called Bernoulli's equation.

VI. The energy flux

Choose some volume element fixed in space, and find how the energy of the fluid contained in this volume element varies with time. The energy of unit volume of fluid is

$$\frac{1}{2}\rho v^2 + \rho\varepsilon$$

where the first term is the kinetic energy and the second the internal energy, ε being the internal energy per unit mass. The change in this energy is given by the partial derivative

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho\varepsilon\right)$$

By using the equation of continuity I.1 and equation of motion II.2 and thermodynamic relation $\mathrm{d}w = T\mathrm{d}s + \frac{1}{\rho}\mathrm{d}p$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) =& \frac{1}{2}v^2\frac{\partial\rho}{\partial t}+\rho\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t}\\ =& -\frac{1}{2}v^2\nabla\cdot(\rho\mathbf{v})-\mathbf{v}\cdot\nabla p-\rho\mathbf{v}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v}\\ =& -\frac{1}{2}v^2\nabla\cdot(\rho\mathbf{v})-\rho\mathbf{v}\cdot\nabla(\frac{1}{2}v^2+w)+\rho T\mathbf{v}\cdot\nabla s\end{aligned}$$

Then in order to transform the derivative $\frac{\partial\rho\varepsilon}{\partial t}$, ues the thermodynamic relation

$$\mathrm{d}\varepsilon = T\mathrm{d}s - p\mathrm{d}V = T\mathrm{d}s+\frac{p}{\rho^2}\mathrm{d}\rho$$

Since $\varepsilon+\frac{p}{\rho}=\varepsilon+pV$ is simply the heat function w per unit mass

$$\mathrm{d}(\rho\varepsilon)=\varepsilon\mathrm{d}\rho+\rho\mathrm{d}\varepsilon = w\mathrm{d}\rho+\rho T\mathrm{d}s$$

and so sonsider the general adiabatic equation II.5 and the equation of continuity I.1

$$\frac{\rho\varepsilon}{\partial t}=w\frac{\partial\rho}{\partial t}+\rho T\frac{\partial s}{\partial t}=-w\nabla\cdot(\rho\mathbf{v})-\rho T\mathbf{v}\cdot\nabla s$$

Combining the above results, the change in the energy is to be

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho\varepsilon\right)=&-\left(\frac{1}{2}v^2+w\right)\nabla\cdot(\rho\mathbf{v})-\rho\mathbf{v}\cdot\nabla\left(\frac{1}{2}v^2+w\right)\\ =&-\nabla\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2+w\right)\right]\end{aligned}$$

In order to see the meaning of the equation, integrate it over some volume:

$$\frac{\partial}{\partial t}\int\left(\frac{1}{2}\rho v^2+\rho\varepsilon\right)\mathrm{d}V=-\int\nabla\left[\rho\mathbf{v}\left(\frac{1}{2}v^2+w\right)\right]\mathrm{d}V$$

or, cnverting the volume integral on the right into a surface integral,

$$\frac{\partial}{\partial t}\int\left(\frac{1}{2}\rho v^2+\rho\varepsilon\right)\mathrm{d}V=-\oint\rho\mathbf{v}\left(\frac{1}{2}v^2+w\right)\cdot\mathrm{d}\mathbf{f} \tag{VI.1}$$

The left-hand side is the rate of change of the energy of the fluid in some given volume. The right-hand side is therefore the amount of energy flowing out of this volume in unit time. The expression

$$\rho\mathbf{v}\left(\frac{1}{2}v^2+w\right) \tag{VI.2}$$

may be called the energy flux density vector. Its magnitude is the amount of energy passing in unit time through unit area perpendicular to the direction of the velocity.

The energy flux density vector shows that any unit mass of fluid carries with it during its motion an amount of energy $w+\frac{1}{2}v^2$. The fact that the heat function w appears here, and not the internal energy ε, has a simple physical significance. Putting $w=\varepsilon+\frac{p}{\rho}$, the flux of the energy through a closed surface is in the form

$$-\oint\rho\mathbf{v}\left(\frac{1}{2}v^2+\varepsilon\right)\cdot\mathrm{d}\mathbf{f}-\oint\rho\mathbf{v}\cdot\mathrm{d}\mathbf{f}$$

The first term is the energy (kinetic and internal) transported through the surface in unit time by the mass of fluid. The second term is the work done by pressure forces on the fluid within the surface.

VII. The momentum flux

The momentum of unit volume is $\rho\mathbf{v}$. Let us determine its rate of change, $\frac{\partial\rho\mathbf{v}}{\partial t}$. By using the tensor notation

$$\frac{\partial}{\partial t}(\rho v_i)=\rho\frac{\partial v_i}{\partial t}+\frac{\partial\rho}{\partial t}v_i$$

Using the equation of continuity and Euler's equation, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(\rho v_i)=&-\rho v_k\frac{\partial v_i}{\partial x_k}-\frac{\partial p}{\partial x_i}-v_i\frac{\partial(\rho v_k)}{\partial x_k}\\ =&-\frac{\partial p}{\partial x_i}-\frac{\partial}{\partial x_k}(\rho v_i v_k)\\ =&-\delta_{ik}\frac{\partial p}{\partial x_k}-\frac{\partial}{\partial x_k}(\rho v_i v_k)\\ =&-\frac{\partial\Pi_{ik}}{\partial x_k}\end{aligned} \tag{VII.1}$$

where the tensor is defined as

$$\Pi_{ik}=p\delta_{ik}+\rho v_i v_k \tag{VII.2}$$

This tensor is clearly symmetrical.

Integrate the above equation

$$\frac{\partial}{\partial t}\int\rho v_i\mathrm{d}V=-\int\frac{\partial\Pi_{ik}}{\partial x_k}\mathrm{d}V=-\oint\Pi_{ik}\mathrm{d}f_k$$

The left-hand side is the rate of changes of the i-th component of the momentum contained in the volume considered. The surface integral on the right is therefore the amount of momentum flowing out through the bounding surface in unit time. Consequently, $\Pi_{ik}\mathrm{d}f_k$ is the i-th component of the momentum flowing through the surfaces element $\mathrm{d}f$. It we write $\mathrm{d}f_k$ in the form $n_k\mathrm{d}f$, $\Pi_{ik}n_k$ is the flux of the ith component of momentum through unit surface area. Thus Π_{ik} is the i-th component of the amount of momentum flowing in unit time through unit area perpendicular to the x_k-axis. The tensor Π_{ik} is called the momentum flux density tensor. The energy flux is determined by a vector, energy being a scalar; the momentum flux, however, is determined by a tensor of rank two, the momentum itself being a vector.

The vector

$$p\mathbf{n}+\rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n}) \tag{VII.3}$$

gives the momentum flux in the direction of $\mathbf{n}$, through a surface perpendicular to $\mathbf{n}$. In particular, taking the unit vector $\mathbf{n}$ to be directed parallel to the fluid direction, and its flux density is $p + \rho v^2$. In a direction perpendicular to the velocity, only the transverse component (relative to $\mathbf{v}$) of momentum is transported its flux density being just p.

VIII. The conservation of circulation

The integral

$$\Gamma = \oint \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$

taken along some closed contour, is called the velocity circulation round that contour.

Consider a closed contour drawn in the fluid at some instant. Suppose it to be a fluid contour, composed of the fluid particles that lie on it. In the course of time these particles move about, and the contour moves with them. Let us investigate what happens to the velocity circulation. In other words, let us calculate the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint \mathbf{v} \cdot \mathrm{dl}$$

To avoid any diffusion, temporarily denote differentiation with respect to the coordinates by the symbol δ, retaining the symbol d for differentiation with respect to time. Next, an element $\mathrm{d}\mathbf{l}$ of the length of the contour can be written as the difference $\delta\mathbf{r}$ between the position vectors $\mathbf{r}$ of the points at the ends of the element. Thus write the velocity circulation $\oint \mathbf{v} \cdot \delta\mathbf{r}$. In differentiating this integral with respect to time, it must be borne in mind that not only the velocity but also the contour itself changes. Hence, on taking the time differentiation under the integral sign, must differentiate not only $\mathbf{v}$ but also for $\delta\mathbf{r}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint \mathbf{v} \cdot \delta\mathbf{r} = \oint \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \delta\mathbf{r} + \oint \mathbf{v} \cdot \frac{\mathrm{d}\delta\mathbf{r}}{\mathrm{d}t}$$

Since the velocity $\mathbf{v}$ is just the time derivative of the position vector $\mathbf{r}$,

$$\mathbf{v} \cdot \frac{\mathrm{d}\delta\mathbf{r}}{\mathrm{d}t} = \mathbf{v} \cdot \delta\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v} \cdot \delta\mathbf{v} = \delta\left(\frac{1}{2}v^2\right)$$

The integral of a total differential along a closed contour, however, is zero. The second integral therefore vanishes, leaving

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint \mathbf{v} \cdot \delta\mathbf{r} = \oint \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \delta\mathbf{r}$$

It now remains to substitute for the acceleration $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$ from

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla w$$

into

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla w$$

since $\nabla \times \nabla w \equiv 0$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$$

or

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \mathrm{const} \qquad \text{(VIII.1)}$$

Therefore, in an ideal fluid, the velocity circulation round a closed "fluid" contour is constant in time (Kelvin's theorem or the law of conservation of circulation).

It should be emphasized that this result has been obtained by using Euler's equation, and therefore involves the assumption that the flow is isentropic. The theorem does not hold for flows which are not isentropic.

By applying Kevlin's theorem to an infinitesimal closed contour δC and transforming the integral according to Stokes' theorem,

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int \nabla \times \mathbf{v} \cdot \mathrm{d}\mathbf{f} \approx \delta\mathbf{f} \cdot \nabla \times \mathbf{v} = \mathrm{const} \qquad \text{(VIII.2)}$$

where $\mathrm{d}\mathbf{f}$ is a fluid surface elemen spanning the contour δC. The vector $\nabla \times \mathbf{v}$ is often called the vorticity of the fluid flow at a given point. The constancy of the product above can be intuitively interpreted as meaning that the vorticity moves with fluid.

When the flow is not isentropic, the right-hand side of Euler's equation becomes

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) + \frac{1}{\rho^2}\nabla p \times \nabla \rho$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. Since $s = s(p, \rho)$, ∇s is a linear function of ∇p and $\nabla \rho$, and $\nabla s \cdot (\nabla \rho \times \nabla p) = -$. By scalarly multiplying ∇s, the expression on the right-hand side can then be transformed as:

$$\begin{aligned}
\nabla s \cdot \frac{\partial \boldsymbol{\omega}}{\partial t} =& \nabla s \cdot \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) \\
=& - \nabla \cdot [\nabla s \times (\mathbf{v} \times \boldsymbol{\omega})] \\
=& - \nabla \cdot [\mathbf{v}(\boldsymbol{\omega} \cdot \nabla s)] + \nabla \cdot [\boldsymbol{\omega}(\mathbf{v} \cdot \nabla s)] \\
=& - (\boldsymbol{\omega} \cdot \nabla s)\nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla(\boldsymbol{\omega} \cdot \nabla s) \\
& + \boldsymbol{\omega} \cdot \nabla(\mathbf{v} \cdot \nabla s)
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t}(\boldsymbol{\omega} \cdot \nabla s) + \mathbf{v} \cdot \nabla(\boldsymbol{\omega} \cdot \nabla s) + (\boldsymbol{\omega} \cdot \nabla s)\nabla \cdot \mathbf{v} = 0$$

The first two terms can be combined as $\frac{\mathrm{d}(\boldsymbol{\omega} \cdot \nabla s)}{\mathrm{d}t}$, where $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$; in the last term, and from $\rho \nabla \cdot \mathbf{v} = -\frac{\mathrm{d}\rho}{\mathrm{d}t}$. The result is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\omega} \cdot \nabla s}{\rho}\right) = 0$$

which gives a conservation law for isentropic flow.