Solutions to Enderton's Mathematical Introduction to Logic

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Exercise 1.1.1

Problem

Give three sentences in English together with translations into our formal language. The sentences should be chosen so as to have an interesting structure, and the translations should each contain 15 or more symbols.

Solution

We let $A, B, \ldots, Z$ be the first 26 sentence symbols.

English Sentence 1: If someone is a cowboy, then they ride a horse, herd cattle, live in the American southwest, and drink whiskey.

Translation of sentence symbols:
$C =$ a person is a cowboy
$R =$ the person rides a horse
$H =$ the person herds cattle
$A =$ the person lives in the American southwest
$W =$ the person drinks whiskey

Translations of the english sentence: $(C \rightarrow (R \wedge (H \wedge (A \wedge W))))$

English Sentence 2: If the suspect is guilty of an offense and either the suspect is not a previous offender or the offense is not severe, then their punishment is less harsh.

Translation of sentence symbols:
$G =$ the suspect is guilty of an offense

P = the suspect is a previous offender
S = the offense is severe
L = the punishment is less harsh

Translation of the english sentence: $((G \wedge ((\neg P) \vee (\neg S))) \rightarrow L)$

English Sentence 3: If an animal is not both black and white then it is neither a zebra nor a holstein cow.

Translation of sentence symbols:
B = the animal is black
W = the animal is white
Z = the animal is a zebra
H = the animal is a holstein cow

Translation of the english sentence: $((\neg(B \wedge W)) \rightarrow ((\neg Z) \wedge (\neg H)))$

Exercise 1.1.2

Problem

Show that there are no wffs of length 2, 3, or 6, but that any other positive length is possible.

Solution

Suppose α is a wff of length 2, and $\langle \varepsilon_1, \ldots, \varepsilon_n \rangle$ is it's construction sequence. We have that $\alpha = \varepsilon_n$, where ε_n is one of the three possibilities
1) ε_n is a sentence symbol
2) $\varepsilon_n = \varepsilon_\neg(\varepsilon_j)$ where $j < n$
3) $\varepsilon_n = \varepsilon_\square(\varepsilon_j, \varepsilon_k)$ where $j, k < n$ and $\square$ is one of $\wedge, \vee, \rightarrow, \leftrightarrow$

If ε_n is a sentence symbol then α has length 1, a contradiction. Thus, ε_n is not a sentence symbol. If $\varepsilon_n = \varepsilon_\neg(\varepsilon_j)$ then since $\varepsilon_j = \beta$ must have length at least 1, $\alpha = \varepsilon_\neg(\varepsilon_j) = (\neg\beta)$ has length at least 4, a contradiction. Thus $\varepsilon_n = \varepsilon_\square(\varepsilon_j, \varepsilon_k)$ where $j, k < n$ and $\square$ is one of $\wedge, \vee, \rightarrow, \leftrightarrow$ Since $\varepsilon_j = \beta$ and $\varepsilon_k = \gamma$ have length at least 1, $\alpha = \varepsilon_\square(\varepsilon_j, \varepsilon_k) = (\beta \square \gamma)$ will have length at least 5, another contradiction. Therefore, it is impossible for α to have length 2.

The previous argument can be repeated exactly to show that there are no wffs of length 3. Now, suppose that α is a wff of length 6, and $\langle \varepsilon_1, \ldots, \varepsilon_n \rangle$ is its construction sequence. Obviously, α is not a sentence symbol. If $\varepsilon_n = \varepsilon_\neg(\varepsilon_j)$ and $\varepsilon_j = \beta$, then $\alpha = \varepsilon_\neg(\varepsilon_j) = (\neg\beta)$ having length 6 implies that β must have length 3. But this is impossible, since we know there are no wffs with length 3.

Thus, it must be that $\varepsilon_n = \varepsilon_\Box(\varepsilon_j, \varepsilon_k)$. Since $\varepsilon_j = \beta$ and $\varepsilon_k = \gamma$ are wffs, we have that $\alpha = (\beta \Box \gamma)$, so that the concatenation $\beta\gamma$ is of length 3. Since wffs must be of length at least 1, we must have either β or γ of length 2. But this is a contradiction, since there are no wffs of length 2. Therefore, there are no wffs of length 6.

Now, all that remains to be seen is that there are wffs of all positive lengths besides $2, 3$ and 6. Obviously, there exists wffs of length 1 since sentence symbols are wffs. Since negation of a wff of length k yields a new wff with length $k+3$, we know that there exists wffs of lengths $1, 4, 7, 10, \ldots$ given by

$$A_1 \quad (\neg A_1) \quad (\neg(\neg A_1)) \quad (\neg(\neg(\neg A_1)))$$

Now consider the wff $(A_1 \wedge A_2)$ which has length 5. Through repeated negation of this wff as in the above step, we get wffs with lengths $5, 8, 11, 14, \ldots$

The only remaining step is to show that there exists wffs of lengths $9, 12, 15, 18, \ldots$ These can be obtained by considering the wff $(A_1 \vee (A_2 \wedge A_3))$ which has length 9, and performing repeated negations as above. Therefore, there exists wffs with length k for any positive integer k besides $2, 3$ and 6.

Exercise 1.1.3

Problem

Let α be a wff; let c be the number of places at which binary connective symbols $(\wedge, \vee, \rightarrow, \leftrightarrow)$ occur in α; let s be the number of places at which sentence symbols occur in α. (For example, if α is $(A \rightarrow (\neg A))$ then $c = 1$ and $s = 2$.) Show by using the induction principle that $s = c + 1$.

Solution

For an arbitrary formula α, let $s[\alpha]$ denote the number of binary connective symbols which occur in α, and let $c[\alpha]$ denote the numbber of places at which sentence symbols occur in α. Let S denote the set of wffs such that if $\alpha \in S$ then $s[\alpha] = c[\alpha] + 1$.

The goal is to show that S contains *all* wffs. To show this, I show that S contains all the sentence symbols and is closed under all 5 sentence building operations. By the induction principle, this will mean that S is the set of all wffs.

Consider A_i an arbitrary sentence symbol. Since $s[A_i] = 1$ and $c[A_i] = 0$, it is clear that $s[A_i] = c[A_i] + 1$. Thus, $A_i \in S$ for all integers $i \geq 1$. Now suppose that $\alpha, \beta \in S$. Then we have that $s[\alpha] + s[\beta] = c[\alpha] + c[\beta] + 2$. We consider $(\alpha \Box \beta)$ where $\Box$ is one of $\wedge, \vee, \rightarrow, \leftrightarrow$. Then $s[(\alpha \Box \beta)] = s[\alpha] + s[\beta]$ and $c[(\alpha \Box \beta)] = c[\alpha] + c[\beta] + 1$ The equation established earlier then implies that $s[(\alpha \Box \beta)] = c[(\alpha \Box \beta)] + 1$, so that $(\alpha \Box \beta) \in S$. Therefore, by the induction principle, S is the set of all wffs.

Exercise 1.1.4

Problem

Assume we have a construction sequence ending in ε_n, where ε_n does not contain the symbol A_4. Suppose we delete all the expressions in the construction sequence that contain A_4. Show that the result is still a legal construction sequence.

Solution

Let $C = \langle \varepsilon_1, \ldots, \varepsilon_n \rangle$ be the construction sequence. Let $C' = \langle \varepsilon_{k_1}, \ldots, \varepsilon_{k_m} \rangle$ be the same sequence but with all expressions containing A_4 deleted and re-indexed. We have that $m \geq 1$ since ε_n does not contain A_4, and that $\varepsilon_{k_m} = \varepsilon_n$.

Consider ε_{k_i} an arbitrary element in the sequence C'. Since ε_{k_i} is also a member of the original construction sequence C, we have the following three cases to consider.

Case 1: $\varepsilon_{k_i} = \varepsilon_{\neg}(\varepsilon_j)$ for some $j < k_i$ Since ε_{k_i} contains no A_4 symbol and the operation of logical negation does not add an A_4 symbol, it must be that ε_j does not contain A_4 either. Thus, ε_j is present in the sequence C', meaning $\varepsilon_{k_i} = \varepsilon_{\neg}(\varepsilon_{k_r})$ for some $r < i$.

Case 2: $\varepsilon_{k_i} = \varepsilon_{\Box}(\varepsilon_j, \varepsilon_r)$ for some $j, r < k_i$, where $\Box$ is one of $\rightarrow, \leftrightarrow, \wedge, \vee$ Since ε_{k_i} contains no A_4 symbol, and the operation $\varepsilon_{\Box}(\cdot, \cdot)$ does not introduce an A_4 symbol, it must be that ε_j and ε_r do not contain A_4. Thus, ε_j and ε_r are present in the sequence C', meaning $\varepsilon_{k_i} = \varepsilon_{\Box}(\varepsilon_{k_s}, \varepsilon_{k_p})$ where $s, p < i$.

Case 3: ε_{k_i} is a sentence symbol This case requires no further attention, we already know it does not contain A_4

From considering the arbitrary symbol ε_{k_i} from C as a member of its parent sequence C', we see that it follows the rules established for legal construction sequences. Thus, C' is a legal sequence.

Exercise 1.1.5

Suppose that α is a wff not containing the negation symbol $\neg$.

Claim 1. *The length of α is odd.*

Throughout the problem I use $\Box$ to denote an arbitrary choice of $\wedge, \vee, \rightarrow, \leftrightarrow$ Let $\langle \varepsilon_1, \ldots, \varepsilon_n \rangle$ be its corresponding construction sequence. If $n = 1, 2$ then the claim is obvious, since α not containing the $\neg$ symbol implies that ε_1 and ε_2 are sentence symbols. Assume $n > 2$. I prove the claim inductively. If ε_3 is a sentence symbol, then certainly ε_3 has odd length. If $\varepsilon_3 = \varepsilon_\Box(\varepsilon_1, \varepsilon_2)$, then $\varepsilon_3 = (\varepsilon_1 \Box \varepsilon_2)$ has odd length since ε_1 and ε_2 have odd length. Thus, ε_3 has odd length. Now, assume $\varepsilon_1, \ldots, \varepsilon_k$ have odd length. If ε_{k+1} is a sentence symbol then it is obviously of odd length. On the other hand, if $\varepsilon_{k+1} = \varepsilon_\Box(\varepsilon_i, \varepsilon_j)$ where $i, j < k+1$, then $\varepsilon_{k+1} = (\varepsilon_i \Box \varepsilon_j)$ has odd length. Therefore ε_{k+1} has odd length. From this argument it follows that the last construction step of α, that is ε_n, has odd length, and thus α has odd length.

Claim 2. *More than a quarter of the symbols in α are sentence symbols.*

To show this, I will first show that the length of α is of the form $4k+1$ for some integer k, and that α contains $k+1$ sentence symbols.

As before, ε_1 and ε_2 must be sentence symbols, so they have length 1. Since they contain 1 sentence symbols, it's clear that ε_1 and ε_2 have lengths of the form $4k+1$ and contain $k+1$ sentence symbols, where $k = 0$. Now, assume that $\varepsilon_1, \ldots, \varepsilon_j$ have lengths of the form $4k+1$ and contain $k+1$ sentence symbols. If ε_{j+1} is a sentence symbol then we already know it is also of this form. On the other hand, consider the case where $\varepsilon_{j+1} = \varepsilon_\Box(\varepsilon_r, \varepsilon_s)$ for some $r, s < j+1$. Assume that ε_r has length $4q+1$ for some integer q, and thus contains $q+1$ sentence symbols. Assume that ε_s has length $4p+1$ for some integer p, and thus contains $p+1$ sentence symbols. Notice that $\varepsilon_{j+1} = (\varepsilon_r \Box \varepsilon_s)$ then contains $q+p+2$ sentence symbols, and is of length $4q+1+4p+1+3 = 4(q+p+1)+1$. We notice that ε_{j+1} has length of the form $4k+1$ and $k+1$ sentence symbols where $k = q+p+1$. By induction, this argument yields that $\varepsilon_n = \alpha$ has length of the form $4k+1$ and $k+1$ sentence symbols. Since $k+1 > \frac{4k+1}{4}$, it's clear that more than a quarter of the symbols in α are sentence symbols.

Exercise 1.2.1

Problem

Show that neither of the following two formulas tautologically imply each other

$$(A \leftrightarrow (B \leftrightarrow C))$$
$$((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$

Solution

First, I show that the first formula does not tautologically imply the second. Consider the truth assignment v where $v(A) = T$, $v(B) = F$, and $v(C) = F$. Then $\overline{v}$ satisfies the first formula, but does not satisfy the second.

Now, to show that the second formula does not tautologically imply the first, redefine the truth assignment v so that $v(A) = v(B) = v(C) = F$. Then $\overline{v}$ satisfies the second formula, but not the first. Therefore, neither formula tautologically implies the other

Exercise 1.2.2

Part (a)

Is $(((P \to Q) \to P) \to P)$ a tautology?

Yes, it is. To show this, we simply check the four possible truth assignment for P and Q

P	Q	$(P \to Q)$	$((P \to Q) \to P)$	$(((P \to Q) \to P) \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

Since the given formula is true given any truth assignment for P and Q, we conclude it is indeed a tautology.

Part (b)

Define σ_k recursively as follows: $\sigma_0 = (P \to Q)$ and $\sigma_{k+1} = (\sigma_k \to P)$. For which value of k is σ_k a tautology?

I will show that the only k for which σ_k is a tautology are the even $k \geq 2$.

Suppose that σ_k is a tautology for some k, then we have the following truth table:

P	Q	σ_k	$\sigma_{k+1} = (\sigma_k \to P)$	$\sigma_{k+2} = (\sigma_{k+1} \to P)$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	T	F	T

The truth table reveals that σ_{k+2} is also a tautology, and that σ_{k+1} is not. Since part (a) showed that σ_2 is a tautology, we then have that $\sigma_4, \sigma_6, \sigma_8, \ldots$ are tautologies, and that $\sigma_3, \sigma_5, \sigma_7, \ldots$ are not.

All that is left to check are σ_0 and σ_1. $\sigma_0 = (P \to Q)$ and $\sigma_1 = ((P \to Q) \to P$ are clearly not tautologies as showin in the truth table from part (a). Therefore, the only k for which σ_k is a tautology are the even $k \geq 2$.

Exercise 1.2.3

Part (a)

Determine whether or not $((P \to Q) \vee (Q \to P))$ is a tautology.

I will show that, indeed, the formula is a tautology. The only time when the formula would be false is if there exists a truth assignment v for which both $(P \to Q)$ and $(Q \to P)$ are false. Suppose that this is the case. Then $(P \to Q)$ being false implies that $v(P) = T$ and $v(Q) = F$. But then this would implie that $(Q \to P)$ is true, a contradiction. Thus, such a truth assignment does not exist, so that all truth assignments satisfy the formula, i.e. the formula is a tautology.

Part (b)

Determine whether or not $((P \wedge Q) \to R)$ tautologically implies $((P \to R) \vee (Q \to R))$

I will show that there is a tautological implication by using a truth table.

P	Q	R	$(P \wedge Q)$	$(P \to R)$	$(Q \to R)$	$((P \wedge Q) \to R)$	$((P \to R) \vee (Q \to R))$
T	T	T	T	T	T	T	T
T	F	F	F	F	T	T	T
F	T	F	F	T	F	T	T
F	F	T	F	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
F	T	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since all the truth assignments which satisfy $((P \wedge Q) \to R)$ also satisfy $((P \to R) \vee (Q \to R))$, there is a tautological implication.

Exercise 1.2.4

Part (a)

Claim 1. $\Sigma; \alpha \models \beta$ *iff* $\Sigma \models (\alpha \to \beta)$

"$\Rightarrow$"

For the forward direction, assume that $\Sigma; \alpha \models \beta$. Let v be an arbitrary truth assignment which satisfies Σ

Case 1: v also satisfies α.

In this case, since we already know that $\Sigma; \alpha \models \beta$, we know that the truth assignment v will also satisfy β. Thus, $\overline{v}(\alpha) = \overline{v}(\beta) = T$ so that $\overline{v}(\alpha \to \beta) = T$

Case 2: v does not satisfy α.

In this case, $\overline{v}(\alpha) = F$, so no matter the value of $\overline{v}(\beta)$, $\overline{v}(\alpha \to \beta) = T$. Since v is arbitrary in both cases, we see that that $\Sigma \models (\alpha \to \beta)$

"$\Leftarrow$"

For the backwards direction, assume that $\Sigma \models (\alpha \to \beta)$

Let v be a truth assignment satisfying $\Sigma; \alpha$. Then $\overline{v}(\alpha) = T$ and $\overline{v}(\alpha \to \beta) = T$, since $\Sigma \models (\alpha \to \beta)$ Suppose $\overline{v}(\beta) = F$, then since $\overline{v}(\alpha) = T$ and by the axioms which $\overline{v}$ must follow, it must be that $\overline{v}(\alpha \to \beta) = F$, which is a contradiction. Thus, we must have that $\overline{v}(\beta) = T$

Since v was an arbitrary truth assignment satisfying $\Sigma; \alpha$, we have that $\Sigma; \alpha \models \beta$

Part (b)

Claim 1. *α is tautologically equivalent to β if and only if $(\alpha \leftrightarrow \beta)$ is a tautology.*

"$\Rightarrow$"

For the forward direction, assume α and β are tautologically equivalent.

Now, let v be an arbitrary truth assignment.

Case 1: $\overline{v}(\alpha) = F$. In this case, we must have that $\overline{v}(\beta) = F$, since $\beta \vDash \alpha$.

Case 2: $\overline{v}(\alpha) = T$. In this case, we must have that $\overline{v}(\beta) = T$, since $\alpha \vDash \beta$

Both cases show that for an arbitrary truth assignment v, it will be that $\overline{v}(\alpha) = \overline{v}(\beta)$. Thus, $(\alpha \leftrightarrow \beta)$ will be true for any truth assignment, meaning it is a tautology.

"$\Leftarrow$"

Now for the backwards direction, we assume that $(\alpha \leftrightarrow \beta)$ is a tautology.

Let v be a truth assignment such that $\overline{v}(\alpha) = T$, then since $\overline{v}(\alpha \leftrightarrow \beta) = T$, we must have that $\overline{v}(\beta) = T$. Thus, $\alpha \vDash \beta$. Now, let v be a truth assignment such that $\overline{v}(\beta) = T$. By the same logic as the preceding sentences, it must be that $\overline{v}(\alpha) = T$, so that $\beta \vDash \alpha$. Therefore, α and β are tautologically equivalent.

1 Exercise 1.2.5

Part (a)

Prove or disprove: If either $\Sigma \vDash \alpha$ or $\Sigma \vDash \beta$, then $\Sigma \vDash (\alpha \vee \beta)$

I will prove the claim is true. Suppose v is a truth assignment which satisfies Σ, then since $\Sigma \vDash \alpha$ or $\Sigma \vDash \beta$, we will have that $\overline{v}(\alpha) = T$ or $\overline{v}(\beta) = T$. This implies that $\overline{v}(\alpha \vee \beta) = T$. Therefore, $\Sigma \vDash (\alpha \vee \beta)$.

Part (b)

Prove or disprove: If $\Sigma \vDash (\alpha \vee \beta)$, then either $\Sigma \vDash \alpha$ or $\Sigma \vDash \beta$

I will disprove the claim. To do this, I provide a counterexample.

Let $\alpha = (A \leftrightarrow B)$, let $\beta = (B \leftrightarrow C)$, and let $\Sigma = ((A \wedge B) \wedge (\neg C)) \vee ((\neg A) \wedge (B \wedge C))$

I show that $\Sigma \vDash (\alpha \vee \beta)$ with a truth table

A	B	C	$(A \wedge B)$	$(B \wedge C)$	$(\neg A)$	$(\neg C)$	Σ	$\alpha = (A \leftrightarrow B)$	$\beta = (B \leftrightarrow C)$	$(\alpha \vee \beta)$
T	T	T	T	T	F	F	F	T	T	T
T	F	F	F	F	F	T	F	F	T	T
F	T	F	F	F	T	T	F	F	F	F
F	F	T	F	F	T	F	F	T	F	T
T	T	F	T	F	F	T	T	T	F	T
T	F	T	F	F	F	F	F	F	F	F
F	T	T	F	T	T	F	T	F	T	T
F	F	F	F	F	T	T	F	T	T	T

The truth table reveals that $\Sigma \vDash (\alpha \vee \beta)$, however it also shows that $\Sigma \nvDash \alpha$ and $\Sigma \nvDash \beta$. Thus, the counterexample disproves the claim.

Exercise 1.2.6

Part (a)

Show that if v_1 and v_2 are truth assignments which agree on all the sentence symbols in the wff α, then $\overline{v}_1(\alpha) = \overline{v}_2(\alpha)$. Use the induction principle.

Let S be the set of wffs for which $\overline{v}_1$ and $\overline{v}_2$ agree. Since $\overline{v}_i$ is an extension of v_i, and since $v_1(A_k) = v_2(A_k)$ for all sentence symbols A_k in α, all the sentence symbols in α are contained in S. Consider β and γ some wffs in S.

Notice that, by the rules established for the extensions $\overline{v}_i$, we must have that

$$\overline{v}_1((\neg\beta)) = \begin{cases} T & \text{if } \overline{v}_1(\beta) = F \\ F & \text{if } \overline{v}_1(\beta) = T \end{cases}$$

and similarly,

$$\overline{v}_2((\neg\beta)) = \begin{cases} T & \text{if } \overline{v}_2(\beta) = F \\ F & \text{if } \overline{v}_2(\beta) = T \end{cases}$$

But since $\overline{v}_1(\beta) = \overline{v}_2(\beta)$, these functions must coincide so that $\overline{v}_1((\neg\beta)) = \overline{v}_2((\neg\beta))$ Thus, $(\neg\beta) \in S$.

By the same process and logic, we also have that $(\beta \Box \gamma) \in S$ where $\Box$ is one of $\wedge, \vee, \rightarrow, \leftrightarrow$, so that S is closed under the sentence building operations. By the induction principle, S contains all wffs made from the sentence symbols in α. Namely, S contains α, so that $\overline{v}_1(\alpha) = \overline{v}_2(\alpha)$.

Part (b)

Let S be a set of sentence symbols that includes those in Σ and τ (and possibly more). Show that $\Sigma \vDash \tau$ if and only if every truth assignment for S which satisfies every member of Σ also satisfies τ.

First, let S' denote the set of sentence symbols that appear in Σ and τ, and no more. So $S' \subseteq S$.

"$\Rightarrow$" For the forwards direction, we assume that $\Sigma \vDash \tau$. Now, let v be a truth assignment for S which satisfies every member of Σ. The domain of v can be restricted to $S' \subseteq S$, call this new assignment v_0. Since v_0 still satisfies every member of Σ and $\Sigma \vDash \tau$, we have that v_0 satisfies τ. Now since v and v_0 agree on all the sentence symbols in τ, by part (a) we have that $\overline{v}(\tau) = \overline{v}_0(\tau) = T$. Thus, v satisfies τ.

"$\Leftarrow$" Now I prove the backwards direction. Let v be a truth assignment for S' which satisfies Σ. Extend the domain of v, calling the new assignment v', to take on all the symbols of S by the following:

$$v'(A) = \begin{cases} v(A) & \text{if } A \in \Sigma \cup \{\tau\} \\ T & \text{otherwise} \end{cases}$$

Notice that since v' and v agree on all sentence symbols in Σ, $\overline{v}'$ and $\overline{v}$ agree on every member of Σ (by part (a)). Since v satisfies Σ. By our assumption v' must also satisfies τ. Since v and v' agree on all sentence symbols in τ, we must have that $\overline{v}(\tau) = \overline{v}'(\tau) = T$ (by part (a)). Since v was an arbitrary truth assignment, we have that $\Sigma \vDash \tau$.

Exercise 1.2.7

Problem

You are in a land inhabited by people who either always tell the truth or always tell falsehood. You come to a fork in the road and you need to know which fork leads to the capital. There is a local resident there, but he has time only to reply to one yes-or-no question. What one question should you ask so as to learn which fork to take?

Solution

Suppose we label the two roads of the fork as "Left" and "Right", or L and R for short. The important condition we use to solve this riddle is the fact that everyone in the land are either all truth tellers or all liars.

Q: If I ask another man does the left road lead to the capital, would he say yes?

Consider the four possible cases of the man replying:

Case 1: He is a truther, and he responds yes. So if we ask another man if the left road leads to the capital, he will say yes. Since everyone in the land is a truth teller, this hypothetical man would be telling the truth. Thus, the road on the left leads to the capital.

Case 2: He is a truther, and he responds no. So if we ask another man if the left road leads to the capital, he will say no. Since everyone in the land is a truth teller, this hypothetical man would be telling the truth. Thus, it is not the road on the left, but the road on the right which leads to the capital.

Case 3: He is a liar, and he responds yes. Since the man is a liar, the truth is that if we ask another man if the left road leads to the capital, he will say no. Since everyone in the land is a liar, this hypothetical man would be lying. Thus, the road on the left would indeed lead to the capital.

Case 4: He is a liar, and he responds no. Since the man is a liar, the truth is that if we ask another man if the left road leads to the capital, he will say yes. Since everyone in the land is a liar, this hypothetical man would be lying. Thus the road on the left does not lead to the capital, so the road on the right does.

Now, suppose we ask the man the question, and he responds yes. In either case, if he is a truther or a liar, this means that the road on the left leads to the capital. On the contrary, if the man replies no, then it must be that the road on the right leads to the capital.

Exercise 1.2.8

Consider a sequence $\alpha_1, \alpha_2, \ldots$ of wffs. For each wff φ let φ^* be the result of replacing the sentence symbol A_n by α_n for each n.

Part (a)

Let v be a truth assignment for the set of all sentence symbols; define u to be the truth assignment for which $u(A_n) = \overline{v}(\alpha_n)$. Show that $\overline{u}(\varphi) = \overline{v}(\varphi^*)$

Let S be the set of wffs for which $\overline{u}(\varphi) = \overline{v}(\varphi^*)$. If $\varphi = A_n$, then $\varphi^* = \alpha_n$. Thus, since $u(A_n) = \overline{v}(\alpha_n)$ and $\overline{u}$ is an extension of u, we have that $\overline{u}(A_n) = \overline{v}(\alpha_n)$, i.e. $\overline{u}(\varphi) = \overline{v}(\varphi^*)$. Therefore, S contains all sentence symbols.

Now, I just need to show that S is closed under the five sentence building operations. Suppose φ and γ are members of S. Since $\overline{u}(\varphi) = \overline{v}(\varphi^*)$, we will have that $\overline{u}((\neg\varphi)) = \overline{v}((\neg\varphi^*))$ since $\overline{u}$ and $\overline{v}$ both adhere to the preestablished axioms for truth assignment extensions. Now since $\overline{v}((\neg\varphi^*)) = \overline{v}((\neg\varphi)^*)$, we have that $\overline{u}((\neg\varphi)) = \overline{v}((\neg\varphi)^*)$, showing that $(\neg\varphi) \in S$.

Let $\square$ denote one of $\wedge, \vee, \rightarrow, \leftrightarrow$. Then $\overline{u}(\varphi) = \overline{v}(\varphi^*)$ and $\overline{u}(\gamma) = \overline{v}(\gamma^*)$ will imply that $\overline{u}((\varphi\square\gamma)) = \overline{v}((\varphi^*\square\gamma^*))$ by similar logic as above. But since $(\varphi^*\square\gamma^*) = (\varphi\square\gamma)^*$, we have that $\overline{u}((\varphi\square\gamma)) = \overline{v}((\varphi\square\gamma)^*)$, showing that $(\varphi\square\gamma) \in S$.

Since S contains all the sentence symbols and is closed under all sentence building operations, we conclude that S is the set of all wffs by the induction principle. Therefore, for any wff φ, it will be that $\overline{u}(\varphi) = \overline{v}(\varphi^*)$.

Part (b)

Show that if φ is a tautology, then so is φ^*.

Let v be an arbitrary truth assignment, and let u be the truth assignment defined by $u(A_n) = \overline{v}(\alpha_n)$. Then by part (a) we know that $\overline{u}(\varphi) = \overline{v}(\varphi^*)$. Since φ is a tautology, $\overline{u}(\varphi) = T$. Thus, $\overline{v}(\varphi^*) = T$. Since v was arbitrary, it follows that φ^* is a tautology.

Exercise 1.2.9

Problem

Let α be a wff whose only connective symbols are $\wedge, \vee$, and $\neg$. Let α^* be the result of interchanging $\wedge$ and $\vee$ and replacing each sentence symbol by its negation. Show that α^* is tautologically equivalent to $(\neg\alpha)$. Use the induction principle.

Solution

Let S be the set of wffs for which the desired property is true. Consider A an arbitrary sentence symbol. Then $A^* = (\neg A)$ Clearly A^* is tautologically equivalent to $(\neg A)$. Thus, $A \in S$, so S contains all sentence symbols.

Now, suppose α and β are members of S. I want to show that $(\neg\alpha) \in S$. Notice that $(\neg\alpha)^* = (\neg(\neg\alpha))$. Clearly $(\neg\alpha)^*$ is tautologically equivalent to $(\neg(\neg\alpha))$. Thus, $(\neg\alpha) \in S$.

Now I will show that $(\alpha \wedge \beta) \in S$. Suppose that v is a truth assignment which satisfies $(\alpha \wedge \beta)^* = ((\neg\alpha) \vee (\neg\beta))$. By the De Morgan's law tautology, v must also satisfy $(\neg(\alpha \wedge \beta))$, so that $(\alpha \wedge \beta)^*$ tautologically implies $(\neg(\alpha \wedge \beta))$. On the other hand, let v be a truth assignment which satisfies $(\neg(\alpha \wedge \beta))$, then by the De Morgan's law tautology v also satisfies $((\neg\alpha) \vee (\neg\beta)) = (\alpha \wedge \beta)^*$. Thus, $(\neg(\alpha \wedge \beta))$ tautologically implies $(\alpha \wedge \beta)^*$. Therefore, $(\alpha \wedge \beta)^*$ and $(\neg(\alpha \wedge \beta))$ are tautologically equivalent, meaning $(\alpha \wedge \beta) \in S$.

Now I will show that $(\alpha \vee \beta) \in S$. Suppose that v is a truth assignment which satisfies $(\alpha \vee \beta)^* = ((\neg\alpha) \wedge (\neg\beta))$. By the De Morgan's law tautology, v must also satisfy $(\neg(\alpha \vee \beta))$, so that $(\alpha \vee \beta)^*$ tautologically implies $(\neg(\alpha \vee \beta))$. On the other hand, let v be a truth assignment which satisfies $(\neg(\alpha \vee \beta))$, then by the De Morgan's law tautology v also satisfies $((\neg\alpha) \wedge (\neg\beta)) = (\alpha \vee \beta)^*$. Thus, $(\neg(\alpha \vee \beta))$ tautologically implies $(\alpha \vee \beta)^*$. Therefore, $(\alpha \vee \beta)^*$ and $(\neg(\alpha \vee \beta))$ are tautologically equivalent, meaning $(\alpha \vee \beta) \in S$.

Since we are not interested in wffs which contain $\rightarrow$ or $\leftrightarrow$, we do not show that S is closed under their respective sentence building operations.

By the induction principle, it follows that S is the set of all wffs whose only connective symbols are $\neg, \rightarrow$, and $\leftrightarrow$. Thus, if α is a wff whose only connective symbols are $\neg, \rightarrow$, and $\leftrightarrow$, it will be that α^* and $(\neg\alpha)$ are tautologically equivalent.

Exercise 1.2.10

Say that a set Σ_1 of wffs is *equivalent* to a set Σ_2 of wffs iff for any wff α, we have that $\Sigma_1 \vDash \alpha$ iff $\Sigma_2 \vDash \alpha$. A set Σ is *independent* iff no member of Σ is tautologically implied by the remaining members in Σ.

Part (a)

Lemma 1: If $\Sigma = \{\sigma_1, \ldots, \sigma_n\} \nvDash \tau$*, then neither does* $\Sigma' = \{\sigma_1, \ldots, \sigma_{n-1}\}$.

Proof: Let v be a truth assignment which satisfies Σ but for which $\overline{v}(\tau) = F$. Then v also satisfies Σ', but $\overline{v}(\tau) = F$ so that $\Sigma' \not\models \tau$.

Claim 1. *A finite set of wffs has an independent equivalent subset*

Proof: Let there be a finite set of wffs $\Sigma_0 = \{\sigma_1, \ldots, \sigma_n\}$. Now, define $\Sigma_{k+1} \subseteq \Sigma_k$ by

$$\Sigma_{k+1} = \begin{cases} \Sigma_k \setminus \{\sigma_{k+1}\} & \text{if } \Sigma_k \setminus \{\sigma_{k+1}\} \models \sigma_{k+1} \\ \Sigma_k & \text{otherwise} \end{cases}$$

So we have the following chain

$$\Sigma_n \subseteq \cdots \subseteq \Sigma_{k+1} \subseteq \Sigma_k \subseteq \cdots \subseteq \Sigma_1 \subseteq \Sigma_0$$

I first show that Σ_n is independent. Suppose $\sigma_{k+1} \in \Sigma_n \subseteq \Sigma_{k+1}$ for some $k = 0, \ldots, n-1$, then $\Sigma_k \setminus \{\sigma_{k+1}\} \not\models \sigma_k$ by our construction. Since $\Sigma_n \setminus \{\sigma_{k+1}\} \subseteq \Sigma_k \setminus \{\sigma_{k+1}\}$, we have that $\Sigma_n \setminus \{\sigma_{k+1}\} \not\models \sigma_k$ (by lemma1). Therefore, Σ_n is independent.

All that is left to show is that Σ_n is equivalent to Σ_0. To show this, I will need to invoke the following lemma.

Lemma 2: Let $\Sigma_n \subseteq \Sigma_0$ be as constructed above, and suppose that v is a truth assignment which satifies Σ_{k+1} for $k = 0, \ldots, n-1$, then v also satisfies Σ_k.

Proof of Lemma 2: Let v be such a truth assignment satisfying Σ_{k+1} Recall that Σ_{k+1} is defined by

$$\Sigma_{k+1} = \begin{cases} \Sigma_k \setminus \{\sigma_{k+1}\} & \text{if } \Sigma_k \setminus \{\sigma_{k+1}\} \models \sigma_{k+1} \\ \Sigma_k & \text{otherwise} \end{cases}$$

Obviously if $\Sigma_{k+1} = \Sigma_k$ then v satisfies Σ_k. Assume the other case, that is $\Sigma_{k+1} = \Sigma_k \setminus \{\sigma_{k+1}\} \models \sigma_{k+1}$. Then v satisfies σ_{k+1}. Thus, v satisfies $\Sigma_k \setminus \{\sigma_{k+1}\} \cup \{\sigma_{k+1}\} = \Sigma_k$.

Now, we are in good shape to show that Σ_n is equivalent to Σ_0. First, let α be a wff such that $\Sigma_n \models \alpha$. Now, let v be a truth assignment which satisfies Σ_0. Since $\Sigma_n \subseteq \Sigma_0$, v also satisfies Σ_n. Thus, $\overline{v}(\alpha) = T$, so that $\Sigma_0 \models \alpha$.

On the other hand, suppose α is a wff such that $\Sigma_0 \models \alpha$, and let v be a truth assignment which satisfies Σ_n. By applying Lemma 2 repeatedly, we must also have that v satisfies Σ_0, and thus $\overline{v}(\alpha) = T$. Thus, $\Sigma_n \models \alpha$.

Since $\Sigma_n \models \alpha$ iff $\Sigma_0 \models \alpha$, the two sets are equivalent. Therefore, $\Sigma_n \subseteq \Sigma_0$ is the independent equivalent subset desired.

Part (b)

Claim 2. *An infinite set need not have an independent equivalent subset*

To show this, consider the following example. Let $\Sigma = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ where $\alpha_1 = A_1$, $\alpha_2 = (A_1 \wedge A_2)$, and in general $\alpha_{k+1} = (\alpha_k \wedge A_{k+1})$, so that $\Sigma = \{A_1, (A_1 \wedge A_2), ((A_1 \wedge A_2) \wedge A_3), \ldots\}$.

Suppose, for contradiction, that $S \subseteq \Sigma$ is independent and equivalent. It must be that $S \neq \varnothing$ since $\varnothing$ is not equivalent to Σ. This is true since $\Sigma \vDash A_1 \to A_2$, but $\varnothing \nvDash A_1 \to A_2$.

Now, I will show that $|S| = 1$. Suppose otherwise, that $\alpha_i, \alpha_j \in S$ where $i < j$. Then $\alpha_j \vDash \alpha_i$, so that S is not an independent set of wffs. Thus, $|S| = 1$. So $S = \{\alpha_k\}$ for some $k \in \mathbb{N}$. It's obvious that $\Sigma \vDash \alpha_{k+1}$, so we should have that $S \vDash \alpha_{k+1}$. Let v be a truth assignment so that $v(A_1) = v(A_2) = \cdots = v(A_k) = T$ and $v(A_{k+1}) = F$. Then v satisfies $S = \{\alpha_k\}$, but $\overline{v}(\alpha_{k+1}) = F$ showing that $S \nvDash \alpha_{k+1}$, a contradiction. Therefore, Σ does not have an independent equivalent subset.

Part (c)

Let $\Sigma = \{\sigma_0, \sigma_1, \ldots\}$; show that there is an independent equivalent set Σ'.

I have not finished this one yet...

Exercise 1.2.11

Claim 1. *A truth assignment v satisfies the wff*

$$(\cdots(A_1 \leftrightarrow A_2) \leftrightarrow \cdots \leftrightarrow A_n)$$

iff $v(A_i) = F$ for an even number of i's, $1 \leq i \leq n$. (By the associative law for $\leftrightarrow$, the placement of the parentheses is not crucial.)

To prove this claim, we will need to invoke the following statements.

Statement 1: If $v(A_1) = \cdots = v(A_m) = T$, then v satisfies $(\cdots(A_1 \leftrightarrow A_2) \leftrightarrow \cdots \leftrightarrow A_m)$. We can observe this is true by replacing the sentence symbols with their truth values so that the wff becomes $(\cdots(T \leftrightarrow T) \leftrightarrow \cdots \leftrightarrow T)$, which clearly evaluates to a truth value of T.

Statement 2: If $v(A_1) = \cdots = v(A_{2m})$ for some $m \in \mathbb{Z}_{\geq 1}$, then v satisfies $(\cdots(A_1 \leftrightarrow A_2) \leftrightarrow \cdots \leftrightarrow A_{2m})$. We can observe this is true by replacing the sentence symbols with their truth values and applying the associative law so that the wff becomes $((F \leftrightarrow F) \leftrightarrow \cdots \leftrightarrow (F \leftrightarrow F))$, containing m many expressions of $(F \leftrightarrow F)$. Since each $(F \leftrightarrow F)$ evaluates to a truth value of T, we may replace each $(F \leftrightarrow F)$ with a T so that the wff becomes $(T \leftrightarrow \cdots \leftrightarrow T)$ with m many T's. By the first part of this lemma, $(T \leftrightarrow \cdots \leftrightarrow T)$ will evalute to a truth value of T, showing that v indeed satisfies $(\cdots(A_1 \leftrightarrow A_2) \leftrightarrow \cdots \leftrightarrow A_{2m})$.

Statement 3: If $v(A_1) = \cdots = v(A_{2m+1})$ for some $m \in \mathbb{Z}_{\geq 0}$, then v *does not* satisfy $(\cdots(A_1 \leftrightarrow A_2) \leftrightarrow \cdots \leftrightarrow A_{2m+1})$. We can observe this is true by replacing the sentence symbols with their truth values and applying the associative law so that the wff becomes $(F \leftrightarrow (F \leftrightarrow F) \leftrightarrow \cdots \leftrightarrow (F \leftrightarrow F))$, containing m many expressions of $(F \leftrightarrow F)$. Since each $(F \leftrightarrow F)$ evaluates to a truth value of T, we may replace each $(F \leftrightarrow F)$ with a T so that the wff becomes $(F \leftrightarrow T \leftrightarrow \cdots \leftrightarrow T)$ with m many T's. Applying the associatve law again the wff becomes $(F \leftrightarrow (T \leftrightarrow \cdots \leftrightarrow T))$. By the first part of this lemma, $(T \leftrightarrow \cdots \leftrightarrow T)$ will evalute to a truth value of T, so that wff becomes $(F \leftrightarrow T)$ which clearly evalutes to F. Thus, v does not satisfy $(\cdots(A_1 \leftrightarrow A_2) \leftrightarrow \cdots \leftrightarrow A_{2m+1})$.

With the aid of these statements we are ready to prove Claim 1.

Proof of Claim 1:

For the backwards direction, assume that $v(A_i) = F$ for an even number of i's, $1 \leq i \leq n$, say $2m$ of them for some $m \in \mathbb{Z}_{\geq 0}$. Since $\leftrightarrow$ is commutative, we can reindex the A_i yielding the wff

$$(A_{k_1} \leftrightarrow A_{k_2} \leftrightarrow \cdots \leftrightarrow A_{k_n})$$

where $v(A_{k_1}) = \cdots = v(A_{k_{2m}}) = F$ and $v(A_{k_{2m+1}}) = \cdots = v(A_{k_n}) = T$. This reordering yields a wff which is tautologically equivalent to $(\cdots(A_1 \leftrightarrow A_2) \leftrightarrow \cdots \leftrightarrow A_n)$. Notice that by the associative law for $\leftrightarrow$ we can treat this wff as

$$((A_{k_1} \leftrightarrow \cdots \leftrightarrow A_{k_{2m}}) \leftrightarrow (A_{k_{2m+1}} \leftrightarrow \cdots \leftrightarrow A_{k_n}))$$

By the lemma we know that $\overline{v}(A_{k_1} \leftrightarrow \cdots \leftrightarrow A_{k_{2m}}) = T$ and $\overline{v}(A_{k_{2m+1}} \leftrightarrow \cdots \leftrightarrow A_{k_n}) = T$, thus $\overline{v}(((A_{k_1} \leftrightarrow \cdots \leftrightarrow A_{k_{2m}}) \leftrightarrow (A_{k_{2m+1}} \leftrightarrow \cdots \leftrightarrow A_{k_n}))) = T$. Since this new wff is tautologically equivalent to the original, it must be that v satisfies $(\cdots(A_1 \leftrightarrow A_2) \leftrightarrow \cdots \leftrightarrow A_n)$.

To prove the forward direction, we prove the contrapositive, that is if $v(A_i) = F$ for an *odd* number of i's, $1 \leq i \leq n$, then v does *not* satisfy $(\cdots(A_1 \leftrightarrow A_2) \leftrightarrow \cdots \leftrightarrow A_n)$. Assume $v(A_i) = F$ for $2m+1$ i's, where $m \in \mathbb{Z}_{\geq 0}$. Since $\leftrightarrow$ is commutative, we can reindex the A_i yielding the wff

$$(A_{k_1} \leftrightarrow A_{k_2} \leftrightarrow \cdots \leftrightarrow A_{k_n})$$

where $v(A_{k_1}) = \cdots = v(A_{k_{2m+1}}) = F$ and $v(A_{k_{2m+2}}) = \cdots = v(A_{k_n}) = T$. This reordering yields a wff which is tautologically equivalent to $(\cdots(A_1 \leftrightarrow A_2) \leftrightarrow \cdots \leftrightarrow A_n)$. Notice that by the associative law for $\leftrightarrow$ we can treat this wff as

$$((A_{k_1} \leftrightarrow \cdots \leftrightarrow A_{k_{2m+1}}) \leftrightarrow (A_{k_{2m+2}} \leftrightarrow \cdots \leftrightarrow A_{k_n}))$$

By the lemma we know that $\overline{v}(A_{k_1} \leftrightarrow \cdots \leftrightarrow A_{k_{2m+1}}) = F$ and $\overline{v}(A_{k_{2m+2}} \leftrightarrow \cdots \leftrightarrow A_{k_n}) = T$, thus

$$\overline{v}(((A_{k_1} \leftrightarrow \cdots \leftrightarrow A_{k_{2m+1}}) \leftrightarrow (A_{k_{2m+2}} \leftrightarrow \cdots \leftrightarrow A_{k_n}))) = F$$

Since this new wff is tautologically equivalent to the original, it must be that v does not satisfy $(\cdots(A_1 \leftrightarrow A_2) \leftrightarrow \cdots \leftrightarrow A_n)$.

Exercise 1.2.12

Problem

There are three suspects for a murder: Adams, Brown, and Clark. Adams says "I didn't do it. The victim was an old acquaintance of Brown's. But Clark hated him." Brown states "I didn't do it. I didn't even know the guy. Besides I was out of town all that week." Clark says "I didn't do it. I saw both Adams and Brown downtown with the victim that day; one of them must have done it." Assume that the two innocent men are telling the truth, but that the guilty man might not be. Who did it?

Solution

Suppose that Adams and Brown were telling the truth. Then since Adams claims the victim was friends with Brown, but Brown didn't know the victim, their statement contradict each other. Thus, one of Adams or Brown must be lying.

On the other hand, suppose that Brown and Clark were telling the truth. Brown claims he was out of town that week, but Clark claims that he saw Brown with the victim that day. Again, the statements contradict each other, so one of Clark and Brown must be lying.

Since we know that one of Adams or Brown is a liar and one of Brown or Clark is liar, and since there is only one liar, we know that the liar must be Brown. Thus, Brown is guilty of the murder.

Exercise 1.2.13

Problem

An advertisement for a tennis magazine state, "If I'm not playing tennis, I'm watching tennis. And if I'm not watching tennis, I'm reading about tennis." We can assume that the speaker cannot do more than one of these activities at a time. What is the speaker doing?

Solution

To solve this, we will translate the speakers sentences into our formal language and create a truth table. Let A be the statement "I'm playing tennis," let B be the statement "I'm watching tennis," and let C be the statement "I'm reading about tennis." Since the speaker can only be doing one activity at a time, there are only the following four truth assignments to consider

A	B	C	$(\neg A)$	$(\neg B)$	$((\neg A) \to B)$	$((\neg B) \to C)$
T	F	F	F	T	T	F
F	T	F	T	F	T	T
F	F	T	T	T	F	T
F	F	F	T	T	F	F

From the speakers statement, we want a truth assignment where both $((\neg A) \to B)$ and $((\neg B) \to C)$ are true. From our observation of the truth table, we see that there is only one such truth assignment where both statements are true, that is the one where B is true. Thus, the speaker is watching tennis.

2 Exercise 1.2.14

Problem

Let $\mathcal{S}$ be the set of all sentence symbols, and assume that $v : \mathcal{S} \to \{F, T\}$ is a truth assignment. Show there is *at most* one extension $\overline{v}$ meeting conditions 0-5 listed at the beginning of this section.

Solution

Suppose that $\overline{v}_1$ and $\overline{v}_2$ are both such extensions, and let X be the set of wffs for which $x \in X$ iff $\overline{v}_1(x) = \overline{v}_2(x)$. Since $\overline{v}_1$ and $\overline{v}_2$ are extensions of the truth assignment v, we must have that $\overline{v}_1(A) = \overline{v}_2(A)$ for every sentence symbol $A \in \mathcal{S}$. Thus, X contains all of the sentence symbols.

Now, suppose that $\alpha, \beta \in \Sigma$, so that $\overline{v}_1(\alpha) = \overline{v}_2(\alpha)$ and $\overline{v}_1(\beta) = \overline{v}_2(\beta)$. Since both $\overline{v}_1$ and $\overline{v}_2$ meet condition 0-5 listed at the beginning of the section, we must have that $\overline{v}_1((\neg\alpha)) = \overline{v}_2((\neg\alpha))$. Thus, $(\neg\alpha) \in S$.

By conditions 0-5 we also have that

$$\overline{v}_1(\alpha \wedge \beta) = \begin{cases} T & \text{if } \overline{v}_1(\alpha) = T \text{ and } \overline{v}_1(\beta) = T \\ F & \text{otherwise} \end{cases}$$

$$\overline{v}_2(\alpha \wedge \beta) = \begin{cases} T & \text{if } \overline{v}_2(\alpha) = T \text{ and } \overline{v}_2(\beta) = T \\ F & \text{otherwise} \end{cases}$$

Since $\overline{v}_1(\alpha) = \overline{v}_2(\alpha)$ and $\overline{v}_1(\alpha) = \overline{v}_2(\alpha)$. it must also be that $\overline{v}_1(\alpha \wedge \beta) = \overline{v}_2(\alpha \wedge \beta)$. Thus, $(\alpha \wedge \beta) \in S$, meaning S is closed under $\wedge$. By similar reasoning, it is not difficult to see that

$$\begin{aligned} \overline{v}_1(\alpha \vee \beta) &= \overline{v}_2(\alpha \vee \beta) \\ \overline{v}_1(\alpha \to \beta) &= \overline{v}_2(\alpha \to \beta) \\ \overline{v}_1(\alpha \leftrightarrow \beta) &= \overline{v}_2(\alpha \leftrightarrow \beta) \end{aligned}$$

so that $(\alpha \vee \beta), (\alpha \to \beta), (\alpha \leftrightarrow \beta) \in S$. Since S contains all the sentence symbols and is closed under $\neg, \wedge, \vee, \to$, and $\leftrightarrow$, by the induction principle it must be that S is the set of all wffs. Thus, for any wff α, $\overline{v}_1(\alpha) = \overline{v}_2(\alpha)$. Therefore, $\overline{v}_1 = \overline{v}_2$.

Exercise 1.2.15

Problem

Of the following three formulas, which tautologically imply which?

(a) $(A \leftrightarrow B)$

(b) $(\neg((A \to B) \to (\neg(B \to A)))) = \alpha$

(c) $(((\neg A) \vee B) \wedge (A \vee (\neg B))) = \beta$

Solution

I claim that all three statements are tautologically equivalent, consider the following truth table

A	B	$(\neg A)$	$(\neg B)$	$(A \to B)$	$(\neg(B \to A))$	$(A \leftrightarrow B)$	α	β
T	T	F	F	T	F	T	T	T
T	F	F	T	F	F	F	F	F
F	T	T	F	T	T	F	F	F
F	F	T	T	T	F	T	T	T

Notice that an arbitrary truth assignment v satisfies $(A \leftrightarrow B)$ if and only if it also satisfies $\alpha = (\neg((A \to B) \to (\neg(B \to A))))$ and $\beta = (((\neg A) \vee B) \wedge (A \vee (\neg B)))$. Similarly, v satisfies β if and only if it also satisfies $(A \leftrightarrow B)$ and α, and v satisfies α if and only if it also satisfies $(A \leftrightarrow B)$ and β. Thus, all three statements are tautologically equivalent.