3/14/19

EKF and 1st/2nd-order approximations

The error eqn

$$\hat{x} = \bar{x} + \bar{e} \qquad \bar{e} \sim N(\bar{0}, P)$$

EKF reports $\hat{x}, P$

When we propagate, we want to find

$$\mathbb{E}[\dot{\hat{x}}] = \mathbb{E}[\bar{f}(\bar{x},t)] \neq \bar{f}(\underbrace{\mathbb{E}[\bar{x}]}_{\hat{x}},t) \qquad f(\hat{x},t)$$

$\mathbb{E}[\cdot]$ is linear, $\bar{f}(\bar{x},t)$ is NL.

Look @ T.S.E about $\bar{x}$

$$\bar{f}(\bar{x},t) = \bar{f}(\hat{x}-\bar{e},t)$$

$$f(\bar{x}+(-\bar{e}),t) = f(\hat{x},t) + \frac{\partial f}{\partial x}\Big|_{\hat{x}} \cdot (-\bar{e}) + \dots$$

$$\mathbb{E}[f(\bar{x},t)] = \mathbb{E}[\bar{f}(\hat{x},t)] + \mathbb{E}\left[\frac{\partial \bar{f}}{\partial \bar{x}}\Big|_{\hat{x}} \cdot (-\bar{e})\right] + \mathbb{E}[\dots]$$

$$= \bar{f}(\hat{x},t) + \frac{\partial \bar{f}}{\partial \bar{x}}\Big|_{\hat{x}} \cdot \underbrace{\mathbb{E}[(-\bar{e})]}_{\bar{0}} + \mathbb{E}[\dots]$$

$$\hat{f}(\bar{x},t) = \bar{f}(\hat{x},t) + \bar{0} + \underbrace{\mathbb{E}[\dots]}_{\text{2nd \& higher O.T. ignored}}$$

$$\hat{f}(\bar{x},t) \cong \bar{f}(\hat{x},t)$$

Propagation of $\hat{x}_{i-1}^+$ to $\hat{x}_i^-$ is good up to 2nd order

Lets examine covariance

$$\mathbb{E}[\bar{e}_i^- (\bar{e}_i^-)^T] = P_i^-$$

from error eqn,

$$\bar{x}(t) = \hat{x}(t) - \bar{e}(t) = \phi(t; \overbrace{\hat{x}_{i-1} - \bar{e}_{i-1}}^{\bar{x}_{i-1}}, t_{i-1}) + \bar{w}_{t,i-1}$$

taking a T.S.E about $\hat{x}$, using $\bar{e}$

$$\cancel{\hat{x}(t)} - \bar{e}(t) = \cancel{\phi(t; \hat{x}_{i-1}, t_{i-1})} + \frac{\partial \phi}{\partial \bar{x}}\Big|_{\hat{x}} \cdot (-\bar{e}) + [\dots] + \bar{w}_{t,i-1}$$

$$-\bar{e}(t) = \Phi(t,t_{i-1})\Big|_{\hat{x}}(-\bar{e}) + [\dots] + \bar{w}_{t,i-1}$$

$$P_i^- = \mathbb{E}[\bar{e}_i^- (\bar{e}_i^-)^T] = \Phi(t,t_{i-1})\,\mathbb{E}[\bar{e}_{i-1}(\bar{e}_{i-1})^T]\,\Phi(t,t_{i-1}) + [\dots]$$

$\Rightarrow$ covariance propagation only good to 1st-order

covariance prop for EKF is not so great

We have a similar problem w/ meas. prediction $(\bar{r}_i = \bar{y}_i - \hat{h}(\bar{x}_i, t_i))$

$$\hat{h}(\bar{x},t) = \mathbb{E}[\bar{h}(\bar{x},t)] \neq \underbrace{\bar{h}(\mathbb{E}[\bar{x}],t)}_{\bar{h}(\hat{x},t)}$$

Using error eqn,

$$\mathbb{E}[\bar{h}(\bar{x},t)] = \mathbb{E}[\bar{h}(\hat{x}-\bar{e},t)] \qquad H_i$$

$$= \mathbb{E}[\bar{h}(\hat{x},t)] + \mathbb{E}\left[\frac{\partial \bar{h}}{\partial x}\Big|_{\hat{x}} \cdot (-\bar{e})\right] + \mathbb{E}[\dots]$$

$$= \bar{h}(\hat{x},t) + H_i \cdot \bar{0} + \mathbb{E}[\dots]$$

ignore 2nd & higher terms

$\Rightarrow$ meas prediction $\hat{h}(\bar{x},t) \cong \bar{h}(\hat{x},t)$ is good up to 2nd order.

Without proof we can show that the update eqn. (covariance) is only good up to 1st order.

Unscented Transform

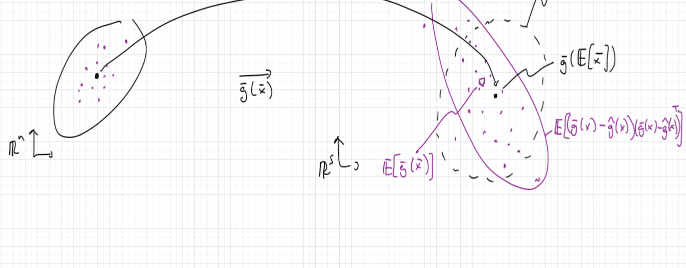
Recall that in general $\mathbb{E}[\bar{g}(\bar{x})] = \hat{g}(\bar{x}) \neq \bar{g}(\mathbb{E}[\bar{x}]) = \bar{g}(\hat{x})$

We want a better way to represent mean/covariance of a post-NL transform PDF

Rather than an analytic T.S.E. about $\hat{x}$ and $\bar{e}$, we observe that

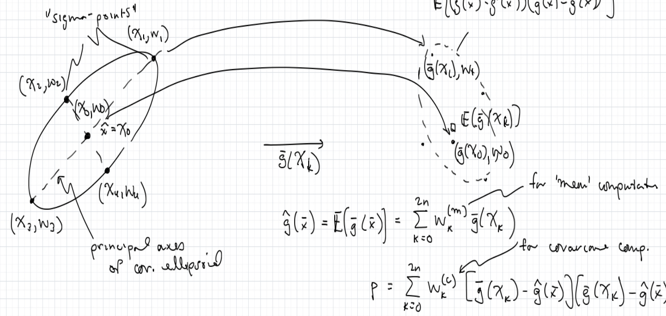
"It is easier to approximate a probability distribution than it is to approximate a NL function."

$$\bar{g}: \mathbb{R}^n \to \mathbb{R}^s \qquad s = \begin{cases} n & \text{prop.} \\ m & \text{meas} \end{cases}$$



The unscented transform helps us judiciously pick points and weights drawn from Gaussian distribution to transform through $\bar{g}(\cdot)$, and exactly compute 1st & 2nd moments of $\bar{g}(x)$ (motivated by monte carlo)

A 2-D representation:



$$\hat{g}(\bar{x}) = \mathbb{E}[\bar{g}(\bar{x})] = \sum_{k=0}^{2n} w_k^{(m)} \bar{g}(\chi_k) \qquad \text{for 'mean' computation}$$

$$P = \sum_{k=0}^{2n} w_k^{(c)} [\bar{g}(\chi_k) - \hat{g}(\bar{x})][\bar{g}(\chi_k) - \hat{g}(\bar{x})]^T \qquad \text{for covariance comp.}$$

How to choose (χ_k, w_k) pairs?

$$\chi_k = \begin{cases} \hat{x} & k=0 \\ \hat{x} + \gamma \bar{\sigma}_k & k = 1, \ldots, n \\ \hat{x} - \gamma \bar{\sigma}_k & k = n+1, \ldots, 2n \end{cases}$$

$2n+1$ sigma points

$S \cdot S \overset{?}{=} P = V D V^{-1}$ ← diagonal

S: principal square root 'sqrtm'

$$\overset{?}{=} \underbrace{(V D^{1/2} V^{-1})}_{S} \cdot \underbrace{(V D^{1/2} V^{-1})}_{S}$$

$$= V D V^{-1}$$

$$\Rightarrow \underset{n \times n}{S} = V D^{1/2} V^{-1} = \begin{bmatrix} \underset{n\times 1}{\sigma_1} & \cdots & \underset{n\times 1}{\sigma_k} & \cdots & \underset{n\times 1}{\sigma_n} \end{bmatrix}_{n \times n}$$

↖ scaled principal axes vector is state spread.

$\gamma = \sqrt{n + \lambda}$

$\lambda = \alpha^2 (\kappa + n) - n$

↖ not k

typically $\kappa = 0$

$0 < \alpha \leq 1$

(smaller α) $\Leftrightarrow$ more nonlinearity

$\alpha = 1 \Leftrightarrow$ linear

without proof

$$w_k^{(m)} = \begin{cases} \dfrac{\lambda}{\lambda + n} & k=0 \\ \dfrac{1}{2(\lambda+n)} & k = 1, \ldots, 2n \end{cases}$$

$$w_k^{(c)} = \begin{cases} \dfrac{\lambda}{\lambda + n} + (1 - \alpha^2 + \beta) & k=0 \\ \dfrac{1}{2(\lambda+n)} & k = 1, \ldots, 2n \end{cases}$$

for Gaussian priors, it can be shown that $\beta = 2$ is the optimal choice.

UKF Algorithm

Given

$\bar{x}_0, P_0, t_0, \alpha, \kappa, \beta$

$t_{i-1} = t_0$

$\hat{x}_{i-1}^+ = \bar{x}_0, \; P_{i-1}^+ = P_0$

↓

Read next Meas $(t_i, \tilde{y}_i, R_i)$

↓

Propagate from t_{i-1} to t_i

compute $S_{i-1}^+ \cdot S_{i-1}^+ = P_{i-1}^+ \Rightarrow \bar{\sigma}_k$

$$\chi_{i-1,k}^+ = \begin{cases} \hat{x}_{i-1}^+ & k=0 \\ \hat{x}_{i-1}^+ + \gamma \bar{\sigma}_k & k = 1, \ldots, n \\ \hat{x}_{i-1}^+ - \gamma \bar{\sigma}_k & k = n+1, \ldots, 2n \end{cases}$$

$\chi_i^- = \phi(t_i, \chi_{i-1,k}^+, t_{i-1})$

$w_k^{(c)}, w_k^{(m)}$ computed $\forall k$

compute $\hat{x}_i^-, P_i^-$ (w/ Proc. noise, add $Q_{i,i-1}$

→

Meas. Update

$$\hat{y}_i^- = \sum_{k=0}^{2n} w_k^{(m)} \bar{h}(\chi_{i,k}, t_i)$$

$$P_{yy,i}^- = \sum_{k=0}^{2n} w_k^{(c)} \left[\bar{h}(\chi_{i,k}) - \hat{y}_i^-\right]\left[\bar{h}(\chi_{i,k}) - \hat{y}_i^-\right]^T$$

$$P_{xy,i}^- = \sum_{k=0}^{2n} w_k^{(c)} \left[\chi_{i,k}^- - \hat{x}_i^-\right]\left[\bar{h}(\chi_{i,k}) - \hat{y}_i^-\right]^T$$

$$K_i = (P_{xy,i}^-)(P_{yy,i}^-)^{-1}$$

$$\hat{x}_i^+ = \hat{x}_i^- + K_i\left[\tilde{y}_i - \hat{y}_i^-\right]$$

$$P_i^+ = P_i^- - K_i (P_{yy,i}^-) K_i^T$$

↓

any meas left? — no → done!

yes? → $i = i+1$ → Read next Meas