Autonomous Vehicle Simulation (AVS) Laboratory, University of Colorado

Basilisk Technical Memorandum

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DERIVATION OF REACTION WHEEL JITTER BACK-SUBSTITUTION

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Scope/Contents
This document derives the back substitution method for reaction wheel jitter to conform with Basilisk's dynamics architecture.

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1 Introduction

The goal is to manipulate the reaction wheel equations of motion to conform to the following form

$$\begin{bmatrix}[A] & [B]\\ [C] & [D]\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{r}}_{B/N}\\ \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{v}_{\text{trans}}\\ \boldsymbol{v}_{\text{rot}}\end{bmatrix} \tag{1}$$

Solving the system-of-equations by

$$\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \Big([D] - [C]][A]^{-1}[B]\Big)^{-1}(\boldsymbol{v}_{\text{rot}} - [C][A]^{-1}\boldsymbol{v}_{\text{trans}}) \tag{2}$$

$$\ddot{\boldsymbol{r}}_{B/N} = [A]^{-1}(\boldsymbol{v}_{\text{trans}} - [B]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}) \tag{3}$$

2 Balanced Reaction Wheel Back-Substitution

2.1 Equations of Motion

The translational equation of motion is not coupled with $\dot{\Omega}$ as seen in the equation below.

$$m_{\text{sc}}[I_{3\times3}]\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{F}_{\text{ext}} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} \tag{4}$$

The rotational equation of motion includes $\dot{\Omega}$ terms, and is thus coupled with wheel motion as seen below.

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N} J_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}\dot{\Omega}_i = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N}(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times J_{\text{s}_i}\Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) + \boldsymbol{L}_B \tag{5}$$

The motor torque equation can be seen below.

$$\dot{\Omega}_i = \frac{u_{\text{s}_i}}{J_{\text{s}_i}} - \hat{\boldsymbol{g}}_{\text{s}_i}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \tag{6}$$

2.2 Back-Substitution Derivation

Plugging Eq. (6) into Eq. (5)

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + ([I_{\text{sc},B}] - \sum_{i=1}^{N} J_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}^T)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N}(\hat{\boldsymbol{g}}_{\text{s}_i}u_{\text{s}_i} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times J_{\text{s}_i}\Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B \quad (7)$$

2.3 Back-Substitution Contribution Matrices

The following can be defined:

$$[A_{\text{contr}}] = [0_{3\times3}] \quad (8)$$

$$[B_{\text{contr}}] = [0_{3\times3}] \quad (9)$$

$$[C_{\text{contr}}] = [0_{3\times3}] \quad (10)$$

$$[D_{\text{contr}}] = -\sum_{i=1}^{N} J_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}^T \quad (11)$$

$$\boldsymbol{v}_{\text{trans,contr}} = \boldsymbol{0} \quad (12)$$

$$\boldsymbol{v}_{\text{rot,contr}} = -\sum_{i=1}^{N}(\hat{\boldsymbol{g}}_{\text{s}_i}u_{\text{s}_i} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times J_{\text{s}_i}\Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) \quad (13)$$

3 Imbalanced Reaction Wheel Back-Substitution

3.1 Equations of Motion

The translational equation of motion is

$$\ddot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N} m_{\text{rw}_i}d_i\hat{\boldsymbol{w}}_{3_i}\dot{\Omega}_i = \ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N} m_{\text{rw}_i}d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} \quad (14)$$

The rotational equation of motion is

$$\begin{aligned} m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N}\Big([I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\Big)\dot{\Omega}_i \\ = \sum_{i=1}^{N}\Big[m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} - [I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\Big([I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\Big)\Big] \\ - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B \end{aligned} \quad (15)$$

The motor torque equation is (note that $J_{12_i} = J_{23_i} = 0$)

$$\begin{aligned} \big[m_{\text{rw}_i}d_i\hat{\boldsymbol{w}}_{3_i}^T\big]\ddot{\boldsymbol{r}}_{B/N} + \big[(J_{11_i} + m_{\text{rw}_i}d_i^2)\hat{\boldsymbol{g}}_{\text{s}_i}^T + J_{13_i}\hat{\boldsymbol{w}}_{3_i}^T - m_{\text{rw}_i}d_i\hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{r}}_{W_i/B}]\big]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \big[J_{11_i} + m_{\text{rw}_i}d_i^2\big]\dot{\Omega}_i \\ = -J_{13_i}\omega_{w_{2_i}}\omega_{s_i} + \omega_{w_{2_i}}\omega_{w_{3_i}}(J_{22_i} - J_{33_i} - m_{\text{rw}_i}d_i^2) - m_{\text{rw}_i}d_i\hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} + u_{s_i} \end{aligned} \quad (16)$$

3.2 Derivation of Back-Substitution

Solve motor torque equation for $\dot{\Omega}_i$ in terms of $\ddot{\boldsymbol{r}}_{B/N}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$

$$\dot{\Omega}_i = \frac{-m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T}{J_{11_i} + m_{\text{rw}_i} d_i^2} \ddot{\boldsymbol{r}}_{B/N} + \frac{-\big[(J_{11_i} + m_{\text{rw}_i} d_i^2)\hat{\boldsymbol{g}}_{\text{s}_i}^T + J_{13_i} \hat{\boldsymbol{w}}_{3_i}^T - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T [\tilde{\boldsymbol{r}}_{W_i/B}]\big]}{J_{11_i} + m_{\text{rw}_i} d_i^2} \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$$
$$+ \frac{1}{J_{11_i} + m_{\text{rw}_i} d_i^2} \Big[\omega_{w_{2_i}} \omega_{w_{3_i}} (J_{22_i} - J_{33_i} - m_{\text{rw}_i} d_i^2) - J_{13_i} \omega_{w_{2_i}} \omega_{s_i} - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] \boldsymbol{r}_{W_i/B} + u_{s_i} \Big] \tag{17}$$

$$\boldsymbol{a}_{\Omega_i} = -\frac{m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}}{J_{11_i} + m_{\text{rw}_i} d_i^2} \tag{18}$$

$$\boldsymbol{b}_{\Omega_i} = -\frac{(J_{11_i} + m_{\text{rw}_i} d_i^2)\hat{\boldsymbol{g}}_{\text{s}_i} + J_{13_i} \hat{\boldsymbol{w}}_{3_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_i/B}] \hat{\boldsymbol{w}}_{3_i}}{J_{11_i} + m_{\text{rw}_i} d_i^2} \tag{19}$$

$$c_{\Omega_i} = \frac{1}{J_{11_i} + m_{\text{rw}_i} d_i^2} \Big[\omega_{w_{2_i}} \omega_{w_{3_i}} (J_{22_i} - J_{33_i} - m_{\text{rw}_i} d_i^2) - J_{13_i} \omega_{w_{2_i}} \omega_{s_i} - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] \boldsymbol{r}_{W_i/B} + u_{s_i} \Big] \tag{20}$$

$$\dot{\Omega}_i = \boldsymbol{a}_{\Omega_i}^T \ddot{\boldsymbol{r}}_{B/N} + \boldsymbol{b}_{\Omega_i}^T \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + c_{\Omega_i} \tag{21}$$

Plugging the equation above into Eq. (14) and multiplying both sides by m_{sc}, (plug $\dot{\Omega}_i$ into translation)

$$\left[[I_{3\times3}] + \frac{1}{m_{\text{sc}}} \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i} \boldsymbol{a}_{\Omega_i}^T \right] \ddot{\boldsymbol{r}}_{B/N} + \left[-[\tilde{\boldsymbol{c}}] + \frac{1}{m_{\text{sc}}} \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i} \boldsymbol{b}_{\Omega_i}^T \right] \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$$
$$= \ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \frac{1}{m_{\text{sc}}} \sum_{i=1}^{N} m_{\text{rw}_i} d_i \left(\Omega_i^2 \hat{\boldsymbol{w}}_{2_i} - c_{\Omega_i} \hat{\boldsymbol{w}}_{3_i} \right) \tag{22}$$

Moving on to rotation, (plug $\dot{\Omega}_i$ into rotation)

$$\left[m_{\text{sc}} [\tilde{\boldsymbol{c}}] + \sum_{i=1}^{N} \Big([I_{\text{rw}_i, W_{c_i}}] \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \hat{\boldsymbol{w}}_{3_i} \Big) \boldsymbol{a}_{\Omega_i}^T \right] \ddot{\boldsymbol{r}}_{B/N}$$
$$+ \left[[I_{\text{sc},B}] + \sum_{i=1}^{N} \Big([I_{\text{rw}_i, W_{c_i}}] \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \hat{\boldsymbol{w}}_{3_i} \Big) \boldsymbol{b}_{\Omega_i}^T \right] \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$$
$$= \sum_{i=1}^{N} \Big[m_{\text{rw}_i} [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] d_i \Omega_i^2 \hat{\boldsymbol{w}}_{2_i} - [I_{\text{rw}_i, W_{c_i}}]' \Omega_i \hat{\boldsymbol{g}}_{s_i} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] \Big([I_{\text{rw}_i, W_{c_i}}] \Omega_i \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \boldsymbol{r}'_{W_{c_i}/B} \Big)$$
$$- \Big([I_{\text{rw}_i, W_{c_i}}] \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \hat{\boldsymbol{w}}_{3_i} \Big) c_{\Omega_i} \Big]$$
$$- [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\text{sc},B}]' \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B \tag{23}$$

Now we have two equations containing $\ddot{\boldsymbol{r}}_{B/N}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$

3.3 Back-Substitution Contribution Matrices

$$[A_{\text{contr}}] = \frac{1}{m_{\text{sc}}} \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i} \boldsymbol{a}_{\Omega_i}^T \tag{24}$$

$$[B_{\text{contr}}] = \frac{1}{m_{\text{sc}}} \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i} \boldsymbol{b}_{\Omega_i}^T \tag{25}$$

$$[C_{\text{contr}}] = \sum_{i=1}^{N} \Big([I_{\text{rw}_i, W_{c_i}}] \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \hat{\boldsymbol{w}}_{3_i} \Big) \boldsymbol{a}_{\Omega_i}^T \tag{26}$$

$$[D_{\text{contr}}] = \sum_{i=1}^{N} \Big([I_{\text{rw}_i, W_{c_i}}] \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \hat{\boldsymbol{w}}_{3_i} \Big) \boldsymbol{b}_{\Omega_i}^T \tag{27}$$

$$\boldsymbol{v}_{\text{trans,contr}} = \frac{1}{m_{\text{sc}}} \sum_{i=1}^{N} m_{\text{rw}_i} d_i \left(\Omega_i^2 \hat{\boldsymbol{w}}_{2_i} - c_{\Omega_i} \hat{\boldsymbol{w}}_{3_i} \right) \tag{28}$$

$$\begin{aligned}\boldsymbol{v}_{\text{rot,contr}} = \sum_{i=1}^{N} \Big[& m_{\text{rw}_i} [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] d_i \Omega_i^2 \hat{\boldsymbol{w}}_{2_i} - [I_{\text{rw}_i, W_{c_i}}]' \Omega_i \hat{\boldsymbol{g}}_{s_i} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] \Big([I_{\text{rw}_i, W_{c_i}}] \Omega_i \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \boldsymbol{r}'_{W_{c_i}/B} \Big) \\ & - \Big([I_{\text{rw}_i, W_{c_i}}] \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \hat{\boldsymbol{w}}_{3_i} \Big) c_{\Omega_i} \Big]\end{aligned} \tag{29}$$