Homework 1

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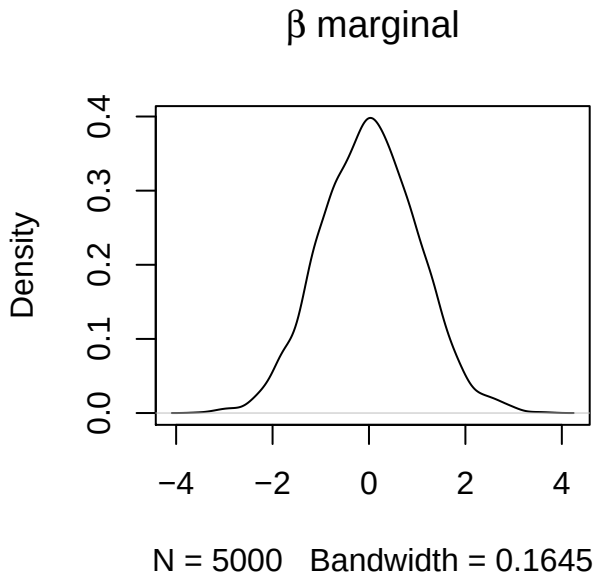
Bayesian Adaptive Lasso

Part a.

Consider $p = 1$. Simulate 5,000 Monte Carlo samples from the conditional prior $\boldsymbol{\beta}|\tau^2 = 1$ and obtain a plot of the density using the R function density.

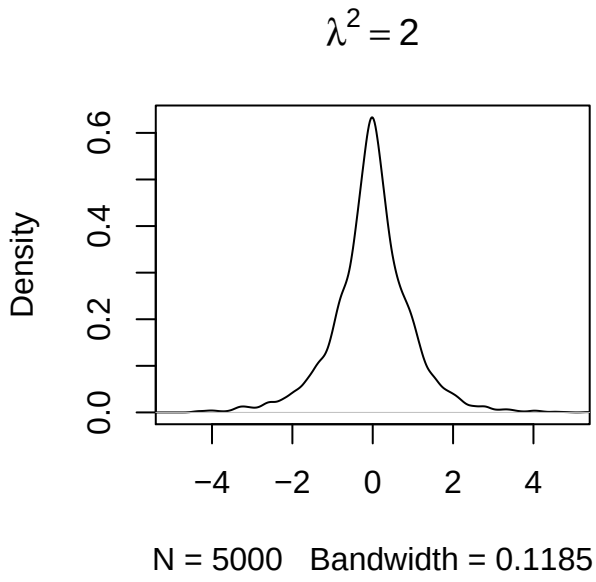
```
n <- 5000
plot(density(rnorm(n,0,1)), main=TeX(paste("$\\beta$", "marginal")))
```



Part b.

Consider $p = 1$. Simulate 5,000 Monte Carlo samples from the marginal prior $\boldsymbol{\beta}$, considering $\lambda^2 = 2$, so that $\mathbb{E}(\tau^2|\lambda) = 1$. Obtain a plot of the density as in **a.**

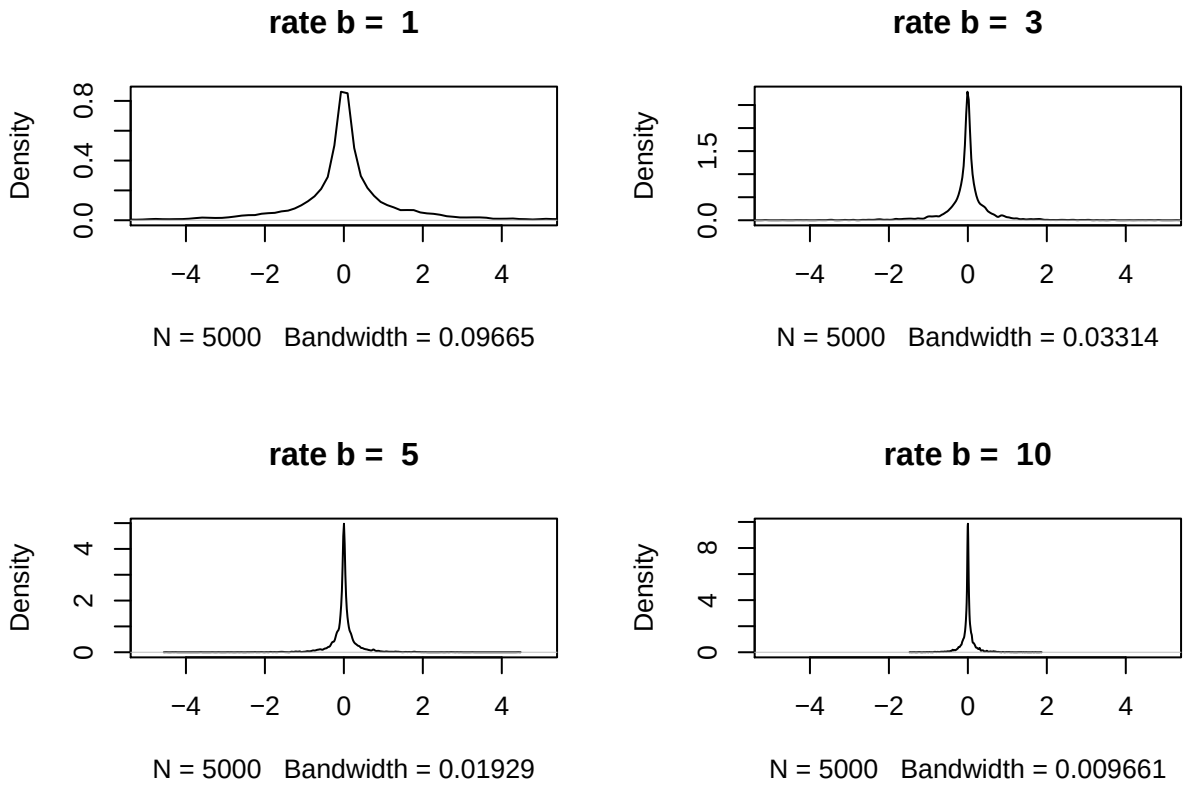
```
lambda <- sqrt(2)
tau.sq <- rgamma(n,shape=1,rate = lambda^2/2)
beta.marginal <- rnorm(n,0,sqrt(tau.sq))
plot(density(beta.marginal), main=TeX(paste("$\\lambda^2 = 2$")), xlim=c(-5,5))
```



Part c.

Consider $p = 1$. Add a hyper prior on $\gamma = 1/\gamma \sim Gamma(a, rate = b)$. Assess how the marginal prior of $\boldsymbol{\beta}$ changes for $a = 1$ and values of $b \geq 1$.

```
set.seed(1)
par(mfrow=c(2,2))
rates <- c(1,3,5,10)
for(b in rates){
  lambda <- 1/rgamma(n,1,b)
  tau.sq <- rgamma(n,shape=1,rate = lambda^2/2)
  beta.marginal <- rnorm(n,0,sqrt(tau.sq))
  plot(density(beta.marginal), main=paste("rate b = ",b),xlim=c(-5,5))
}
```



Part d.

Considering the hyper prior in **c.**, describe a Markov Chain Monte Carlo algorithm to sample from the posterior distribution of $\boldsymbol{\beta}$ and σ^2.

I will implement a joint Gibbs and Metropolis sampler. The model is

$$\begin{aligned}
\boldsymbol{Y}|\boldsymbol{\beta}, \sigma^2 &\sim N(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{I}) \\
\beta_j|\tau_j^2 &\sim N(0, \tau_j^2) \\
\tau_j^2 &\sim \text{Gamma}(1, \frac{\lambda^2}{2}) \\
\lambda^2 &\sim \text{Inverse-Gamma}(a, 1/b) \\
\sigma^2 &\sim \text{Inverse-Gamma}(0.1, 0.1).
\end{aligned}$$

I need the full conditionals

$$\begin{aligned}
&\{\beta_1, \ldots, \beta_p|\boldsymbol{Y}, \sigma^2, \tau_1^2, \ldots, \tau_p^2, \lambda\}, \\
&\{\sigma^2|\boldsymbol{Y}, \beta_1, \ldots, \beta_p, \tau_1^2, \ldots, \tau_p^2, \lambda\}, \\
&\{\tau_1^2, \ldots, \tau_p^2|\boldsymbol{Y}, \beta_1, \ldots, \beta_p, \sigma^2, \lambda\}, \\
&\{\lambda|\boldsymbol{Y}, \beta_1, \ldots, \beta_p, \sigma^2, \tau_1^2, \ldots, \tau_p^2\}
\end{aligned}$$

which are all proportional to

$$p(\boldsymbol{Y}|\beta_1, \ldots, \beta_p, \tau_1^2, \ldots, \tau_p^2, \sigma^2, \lambda) \times p(\beta_1, \ldots, \beta_p|\tau_1^2, \ldots, \tau_p^2) \times p(\tau_1^2, \ldots, \tau_p^2|\lambda)p(\lambda)p(\sigma^2)$$

so I'll start with the posterior

$$\begin{aligned}
p(\beta_1, \ldots, \beta_p, \tau_1^2, \ldots, \tau_p^2, \sigma^2, \lambda|\boldsymbol{Y}) \propto\ & p(\boldsymbol{Y}|\beta_1, \ldots, \beta_p, \tau_1^2, \ldots, \tau_p^2, \sigma^2, \lambda) \\
& \times p(\beta_1, \ldots, \beta_p|\tau_1^2, \ldots, \tau_p^2) \\
& \times p(\tau_1^2, \ldots, \tau_p^2|\lambda)p(\lambda)p(\sigma^2).
\end{aligned}$$

As a function of just σ^2, this is proportional to

$$\begin{aligned}
& p(\boldsymbol{Y}|\beta_1, \ldots, \beta_p, \tau_1^2, \ldots, \tau_p^2, \sigma^2, \lambda)p(\sigma^2) \\
= \quad & N(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{I})IG(a, b).
\end{aligned}$$

Time to show this is inverse-gamma distributed.

$$\begin{aligned}
& N(\boldsymbol{y}; \boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{I})IG(\sigma^2; q, r) \\
\propto\ & (\sigma^2)^{-n/2} \exp\big\{ -\frac{1}{2\sigma^2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})^\top(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})\big\}(\sigma^2)^{q-1} \exp\big\{ -\frac{r}{\sigma^2}\big\} \\
=\ & (\sigma^2)^{-(n/2+q)-1} \exp\big\{ -\frac{1}{\sigma^2}(r + \frac{1}{2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})^\top(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})\big\} \\
=\ & IG(n/2 + q, r + (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})^\top(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})/2)|_{q=0.1, r=0.1}
\end{aligned}$$

As a function of $\boldsymbol{\beta}$, the conditional is proportional to

$$\begin{aligned}
& p(\boldsymbol{Y}|\beta_1, \ldots, \beta_p, \tau_1^2, \ldots, \tau_p^2, \sigma^2, \lambda)p(\beta_1, \ldots, \beta_p|\tau_1^2, \ldots, \tau_p^2) \\
= \quad & N(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{I}) \cdot \prod_{i=1}^{p} N(0, \tau_i^2) \\
= \quad & N(\boldsymbol{X}\beta, \Sigma) \cdot N(0, \Omega), \text{ where } \Omega = \text{diag}(\tau_1^2, \ldots, \tau_p^2) \\
= \quad & N(m, \boldsymbol{M})
\end{aligned}$$

with $m = \boldsymbol{M}\boldsymbol{X}^\top\Sigma^{-1}y$ and $\boldsymbol{M} = (\boldsymbol{X}^\top\Sigma^{-1}\boldsymbol{X} + \Omega^{-1})^{-1}$. As a function of $\tau_1^2, \ldots, \tau_p^2$, the conditional is non-standard, but it's proportional to

$$\begin{aligned} & p(\beta_1, \ldots, \beta_p|\tau_1^2, \ldots, \tau_p^2)p(\tau_1^2, \ldots, \tau_p^2|\lambda) \\ = & \prod_{i=1}^{p} N(0, \tau_i^2) \cdot \prod_{i=1}^{p} IG(1, \frac{\lambda^2}{2}) \end{aligned}$$

Finally, as a function of λ, it's proportional to

$$\begin{aligned} & p(\tau_1^2, \ldots, \tau_p^2|\lambda)p(\lambda) \\ = & \prod_{i=1}^{p} IG(1, \frac{\lambda^2}{2}) \cdot IG(0.1, 0.1) \end{aligned}$$

Now I can build an algorithm to iteratively update through these conditional distributions. I take the starting value of $\boldsymbol{\beta}^{(0)}$ to be the least-squares solution $\hat{\boldsymbol{\beta}}$.

Result: Samples from joint posterior $p(\boldsymbol{\beta}, \sigma^2|\boldsymbol{y})$

for *s in # samples* **do**

$\sigma^{2(s+1)} \sim IG(n/2 + a, 2b + (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}^{(s)})^\top(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}^{(s)})/2)$

$\boldsymbol{\beta}^s \sim N_p(m, \boldsymbol{M})$

$m^\top = \qquad \boldsymbol{M} = \tau_1^2, \ldots, \tau_p^2 \sim N(\boldsymbol{\tau}^{(s)}, \delta\boldsymbol{I})^2$

$\lambda^* \sim N(\lambda.s, \delta)^2$

$\log(r) = \log p(\boldsymbol{y}|\boldsymbol{\beta}^*, \tau_1^2, \ldots, \tau_p^2, \sigma^{2(s)}) + \log p(\boldsymbol{\beta}^*|\tau_1^2, \ldots, \tau_p^2)$
$\qquad - \log p(\boldsymbol{y}|\boldsymbol{\beta}^{(s)}, \tau_1^2, \ldots, \tau_p^2, \sigma^{2(s)}) - \log p(\boldsymbol{\beta}^{(s)}|\tau_1^2, \ldots, \tau_p^2)$

$u \sim Unif(0, 1)$

if $\log(u) < \log(r)$ **then**

$\boldsymbol{\tau}^{(s+1)} = \boldsymbol{\beta}^*$

else

$\boldsymbol{\tau}^{(s+1)} = \boldsymbol{\beta}^{(s)}$

end

end

Algorithm 1: Gibbs and Metropolis

Part e.

Implement such algorithm in R and compare your results with estimates obtained using **glmnet()**. In particular, you should test your results on the diabetes data available from lars, (use the matrix of predictors x).

```
loglambda.target <- function(lambda, a, b, tau2){
  (-a - 1)*log(lambda) - (lambda^2/2 * sum(tau2)) - 1/(b*lambda)
}
logtau2j.target <- function(tau2j, lambda, betaj){
  -log(sqrt(tau2j)) - 1/2*(1/tau2j*betaj^2 + lambda^2*tau2j)
}

set.seed(1)
data("diabetes")
X <- cbind(diabetes$x); y <- diabetes$y; n <- nrow(X)
X <- cbind(rep(1,n),X)
p <- ncol(X)
samples <- 50000;
#lambda <- 1e-10
lambda <- 1
tau2 <- rep(1000,p)
beta <- solve(t(X)%*%X)%*%t(X)%*%y
sigma2.keep <- lambda.keep <-rep(0,samples)
beta.keep <-tau2.keep<- matrix(NA,nrow=p,ncol=samples)
sigma2 <-var(y)
sigma2.keep[1] <- sigma2
beta.keep[,1] <- solve(t(X)%*%X)%*%t(X)%*%y
lambda.keep[1] <- lambda
tau2.keep[,1] <- tau2
s <- 2
post.means <- matrix(NA,nrow=p,ncol=length(lambdas))
index <- 1
b<-1
for(s in 2:samples){
  lambda.p <- exp(log(lambda)+rnorm(1,0,lambda))
  logr <- loglambda.target(lambda.p,1,b,tau2)-loglambda.target(lambda,1,b,tau2)+log(lambda.p)-log(lambda
  if(log(runif(1))<logr){lambda <- lambda.p}
  lambda.keep[s] <- lambda
  for(j in 1:length(tau2)){
    tau2j.p <- exp(log(tau2[j])+rnorm(1,0,5))
    logr <-
      logtau2j.target(tau2j.p, lambda, beta[j]) -
      logtau2j.target(tau2[j], lambda, beta[j]) +
      log(tau2j.p)-log(tau2[j])
    if(log(runif(1))<logr){tau2[j] <- tau2j.p}
  }
  tau2 <- 1/rinvgauss(n = length(beta), mean = lambda/abs(beta),
                      shape = lambda^2)
  sigma2 <- rinvgamma(1,shape=n/2+0.1,rate=(0.1+t(y-X%*%beta)%*%(y-X%*%beta)/2))
  M <- solve(t(X)%*%X*1/sigma2+diag(1/tau2))
  m <- M%*%t(X)%*%y/sigma2
  beta <- t(rmvnorm(n=1,mean=m,sigma=M))
```

```
  sigma2.keep[s] <- sigma2
  beta.keep[,s] <- beta
  tau2.keep[,s] <- tau2
}
post.means[,index] <- matrix(rowMeans(beta.keep[,floor(samples/4):samples]),nrow = p)
index <- index +1
}
fit <- glmnet(X, y)
glmnetCoef <- coef(fit,s=1)
plot(fit, xvar="lambda")

fit <- glmnet(X, y)
glmnetCoef <- coef(fit,s=1)
# df <- cbind(beta.ls,colMeans(beta.keep[floor(samples/4):samples,]),glmnetCoef[-1,])
# colnames(df) <- c("Least Square", "Bayes", "Glmnet")
# df
plot(log(lambda.keep[1:samples]), type="l")
beta.ls <- solve(t(X)%*%X)%*%t(X)%*%y
cbind(beta.ls,betaChain)
beta.keep <- t(beta.keep)
tau2.keep <- t(tau2.keep)
plot(sigma2.keep[1:samples],type="l")
plot(log(tau2.keep[,1]),type="l")
plot((tau2.keep[,2]),type="l")
plot((tau2.keep[,3]),type="l")
plot((tau2.keep[,4]),type="l")
plot((tau2.keep[,5]),type="l")
plot((tau2.keep[,6]),type="l")
plot((tau2.keep[,7]),type="l")
```

Part g.

Free λ and carry out a sensitivity analysis assessing the behavior of the posterior distribution of $\boldsymbol{\beta}$ and σ^2, as hyper parameters a and b are changed. Explain clearly the rationale you use to assess sensitivity and provide recommendations for the analysis of the diabetes data.

```
loglambda.target <- function(lambda, a, b, tau2){
  (-a - 1)*log(lambda) - (lambda^2/2 * sum(tau2)) - 1/(b*lambda)
}
logtau2j.target <- function(tau2j, lambda, betaj){
  -log(sqrt(tau2j)) - 1/2*(1/tau2j*betaj^2 + lambda^2*tau2j)
}

set.seed(1)
data("diabetes")
X <- cbind(diabetes$x); y <- diabetes$y; n <- nrow(X)
X <- cbind(rep(1,n),X)
p <- ncol(X)
samples <- 50000;
#lambda <- 1e-10
lambda <- 1
tau2 <- rep(1000,p)
beta <- solve(t(X)%*%X)%*%t(X)%*%y
sigma2.keep <- lambda.keep <-rep(0,samples)
beta.keep <-tau2.keep<- matrix(NA,nrow=p,ncol=samples)
sigma2 <-var(y)
sigma2.keep[1] <- sigma2
beta.keep[,1] <- solve(t(X)%*%X)%*%t(X)%*%y
lambda.keep[1] <- lambda
tau2.keep[,1] <- tau2
s <- 2
lambdas <- seq(from=-12, to = -2, length.out = 12)
post.means <- matrix(NA,nrow=p,ncol=length(lambdas))
index <- 1
b<-1
for(l in lambdas){
for(s in 2:samples){
  lambda.p <- exp(log(lambda)+rnorm(1,0,lambda))
  logr <- loglambda.target(lambda.p,1,b,tau2)-loglambda.target(lambda,1,b,tau2)+log(lambda.p)-log(lambda
  if(log(runif(1))<logr){lambda <- lambda.p}
  lambda.keep[s] <- lambda
  for(j in 1:length(tau2)){
    tau2j.p <- exp(log(tau2[j])+rnorm(1,0,5))
    logr <-
      logtau2j.target(tau2j.p, lambda, beta[j]) -
      logtau2j.target(tau2[j], lambda, beta[j]) +
      log(tau2j.p)-log(tau2[j])
    if(log(runif(1))<logr){tau2[j] <- tau2j.p}
  }
  tau2 <- 1/rinvgauss(n = length(beta), mean = lambda/abs(beta),
                      shape = lambda^2)
  sigma2 <- rinvgamma(1,shape=n/2+0.1,rate=(0.1+t(y-X%*%beta)%*%(y-X%*%beta)/2))
  M <- solve(t(X)%*%X*1/sigma2+diag(1/tau2))
```

```
  m <- M%*%t(X)%*%y/sigma2
  beta <- t(rmvnorm(n=1,mean=m,sigma=M))
  sigma2.keep[s] <- sigma2
  beta.keep[,s] <- beta
  tau2.keep[,s] <- tau2
}
post.means[,index] <- matrix(rowMeans(beta.keep[,floor(samples/4):samples]),nrow = p)
index <- index +1
}
fit <- glmnet(X, y)
glmnetCoef <- coef(fit,s=1)
plot(fit, xvar="lambda")
betaChain <- colMeans(beta.keep[,1:samples])
plot(lambdas,post.means[1,],ylab="Coefficients",xlab="log lambdas",ylim=c(min(post.means),max(post.means
lines(lambdas,post.means[1,])
for(j in 2:10){
points(lambdas,post.means[j,],ylab="Coefficients",xlab="log lambdas", col=j,type="l")
lines(log(lambdas),post.means[j,], col=j)
}

fit <- glmnet(X, y)
glmnetCoef <- coef(fit,s=1)
# df <- cbind(beta.ls,colMeans(beta.keep[floor(samples/4):samples,]),glmnetCoef[-1,])
# colnames(df) <- c("Least Square", "Bayes", "Glmnet")
# df
plot(log(lambda.keep[1:samples]), type="l")
beta.ls <- solve(t(X)%*%X)%*%t(X)%*%y
cbind(beta.ls,betaChain)
beta.keep <- t(beta.keep)
tau2.keep <- t(tau2.keep)
plot(sigma2.keep[1:samples],type="l")
plot(log(tau2.keep[,1]),type="l")
plot((tau2.keep[,2]),type="l")
plot((tau2.keep[,3]),type="l")
plot((tau2.keep[,4]),type="l")
plot((tau2.keep[,5]),type="l")
plot((tau2.keep[,6]),type="l")
plot((tau2.keep[,7]),type="l")
```

Part h.

Implementation and benchmarking in Julia