Bayesian Adaptive Lasso

Please describe all your work in clear terms, before implementing `R` code. Each question should include a description of your approach with clear indication of where I can find the associated source code. Your code should be attached to your assignment or uploaded online to a repository I can freely access.

Consider the implementation of a posterior simulation algorithm for Bayesian adaptive LASSO. More precisely consider the following model:

$$Y \mid \beta, \sigma^2 \sim N(X\boldsymbol{\beta}, \sigma^2 I_n)$$

Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$, the model is completed with priors:

$$\beta_j \mid \tau_j^2 \sim N(0, \tau_j^2); \;\; j = 1, \ldots, p$$

$$p(\tau_j^2 \mid \lambda) \propto \exp\left\{-\frac{\lambda^2}{2}\tau_j^2\right\}; \;\; j = 1, \ldots, p$$

$$h = 1/\sigma^2 \sim Gamma(0.1, rate = 10)$$

a. Consider $p = 1$. Simulate 5,000 Monte Carlo samples from the conditional prior $\boldsymbol{\beta} \mid \tau^2 = 1$ and obtain a plot of the density using the `R` function `density`.

b. Consider $p = 1$. Simulate 5,000 Monte Carlo samples from the marginal prior $\boldsymbol{\beta}$, considering $\lambda^2 = 2$, so that $E(\tau^2 \mid \lambda) = 1$. Obtain a plot of the density as in **a.**

c. Consider $p = 1$. Add a hyper prior on $\gamma = 1/\lambda \sim Gamma(a, rate = b)$. Assess how the marginal prior of $\boldsymbol{\beta}$ changes for $a = 1$ and values of $b \geq 1$.

d. Considering the hyper prior in **c**., describe a Markov Chain Monte Carlo algorithm to sample from the posterior distribution of $\boldsymbol{\beta}$ and σ^2.

e. Implement such algorithm in `R` and compare your results with estimates obtained using `glmnet()`. In particular, you should test your results on the `diabetes` data available from `lars`, (use the matrix of predictors `x`).

f. For the `diabetes` data, fix λ and produce a regularization path for adaptive Bayesian Lasso obtained on a grid of values for the tuning parameter λ. Describe your approach and compare your result with the path obtained using `glmnet()`.

g. Free λ and carry out a sensitivity analysis assessing the behavior of the posterior distribution of $\boldsymbol{\beta}$ and σ^2, as hyper parameters a and b are changed. Explain clearly the rationale you use to assess sensitivity and provide recommendations for the analysis of the diabetes data.

e. [Extra credit] Using `rcpp` attempt and quantify acceleration of your code in (e.).