EOSC 454 Assignment 2

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1 Problem 1

Derive the analytical solution for v_{int} using the expression for RMS velocity:

$$V_{RMS}^2(t_j) = \frac{1}{t_j} \int_0^{t_{max}} v_{int}^2(u)du \tag{1}$$

First, let $t_{max} = t_j = t$. Rearrange the equation to group the non-integral terms on the left hand side:

$$t \cdot V_{RMS}^2(t) = \int_0^t v_{int}^2(u)du$$

Now, set $t \cdot V_{RMS}^2(t) = F(t)$ and $v_{int}^2(u) = g(u)$, giving us

$$F(t) = \int_0^t g(u)du$$

The fundamental theorem of calculus says that

$$F'(t) = \frac{d}{dt} \int_0^t g(u)du = g(t)$$

Where F' represents the derivative of F with respect to t. Applying this to our equation gives

$$F'(t) = g(t)$$

After rewriting this expression in terms of our original functions we get:

$$V_{RMS}^2(t) + 2t \cdot V_{RMS}(t) \cdot V_{RMS}'(t) = v_{int}(t)^2$$

Finally, solving for $v_{int}(t)$ and rearranging gives:

$$v_{int}(t) = V_{RMS}(t) \cdot \sqrt{1 + \frac{2t \cdot V_{RMS}'(t)}{V_{RMS}(t)}} \tag{2}$$

This matches the analytical solution given in the problem statement.

2 Problem 2

Interval velocity expression:

$$v_{int} = v_0 + \alpha \sin(\omega t) + \beta t \tag{3}$$

2.1 2a

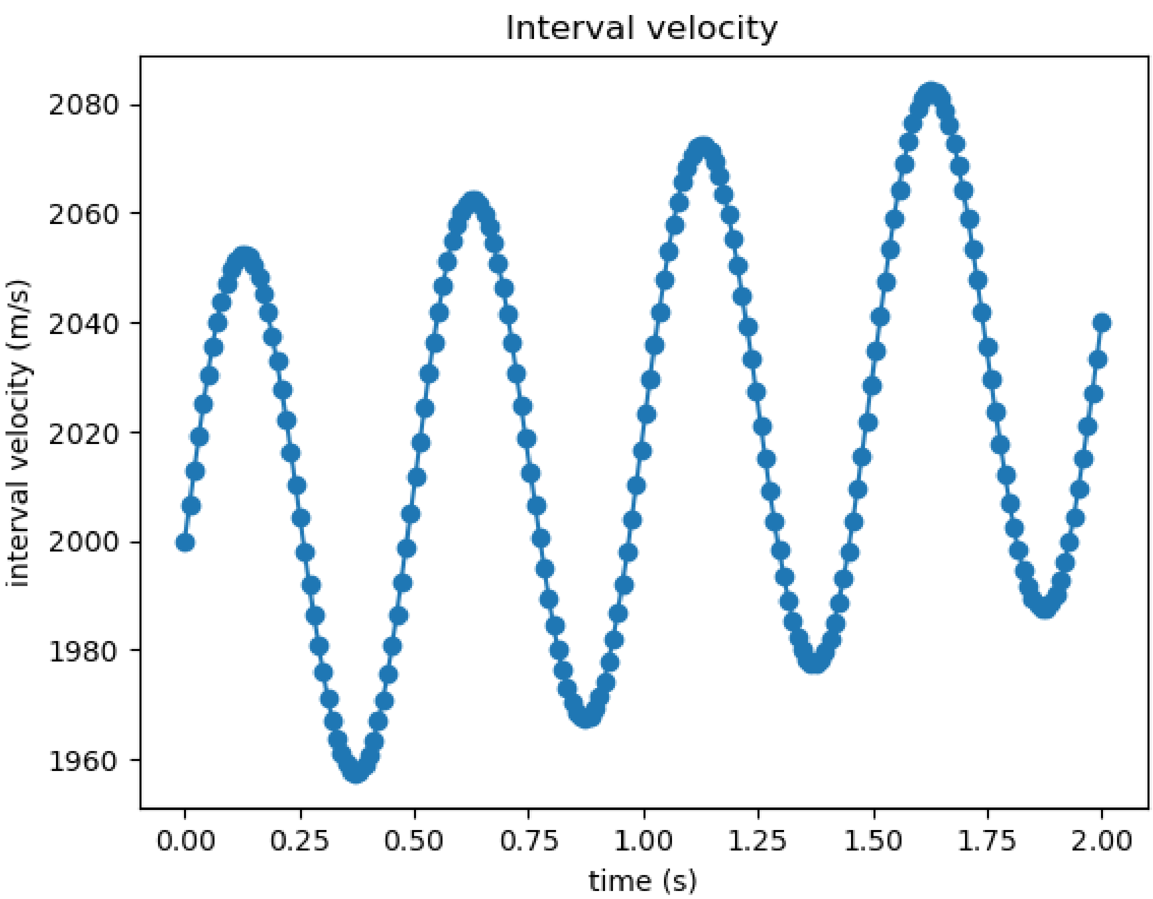

Figure 1: Interval velocity calculated with the following parameters: $v_0 = 2000m/s$, $\alpha = 50m/s$, $\beta = 20m/s^2$, $0 \leq t \leq 2$.

2.2 2b

The influence of parameters v_0, α, ω, and β are discussed in the sections below.

2.2.1 v_0

Changing v_0 sets the initial value of velocity (assuming $t = 0$ initially) and applies a constant offset of that value to the interval velocity. The effects of varying v_0 are shown in the first row of 2.

2.2.2 α

α sets the amplitude of the sine wave, this means that a high alpha results in large oscillations in the interval velocity, while smaller alpha values result in smaller oscillations. The effects of varying α are shown in the second row of figure 2.

2.2.3 ω

ω sets the angular frequency of the sine term. High ω means lots of fast oscillations, while low ω results in slower oscillations. The effects of varying ω are shown in the third row of figure 2.

2.2.4 β

β sets the slope of the change of velocity with time. Positive beta means the velocity increases over time, negative means velocity decreases over time. The effects of varying β are shown in the fourth row of figure 2.

2.3 2c

To find an analytical solution for $V_{RMS}(t)$, we must solve equation 1 given our interval velocity function in equation 3 by solving the integral. Putting the integral into an integral calculator we find that

$$\begin{aligned}\int_0^t v_{\text{int}}^2(u)du = \frac{1}{12\omega^2}\Big[\omega\Big((-24t\alpha\beta - 24v_0\alpha)\cos(t\omega) - 3\alpha^2\sin(2t\omega)\Big) \\ + 24\alpha\beta\sin(t\omega) + (4t^3\beta^2 + 12t^2v_0\beta + 6t\alpha^2 + 12tv_0^2)\omega^2\Big] + \frac{2v_0\alpha}{\omega}\end{aligned} \tag{4}$$

To account for the $\omega = 0$ case, we can also solve the integral with the sine term omitted:

$$\int_0^t (v0 + \beta u)^2 du = \frac{t\,(t^2\beta^2 + 3tv_0\beta + 3v_0^2)}{3} \tag{5}$$

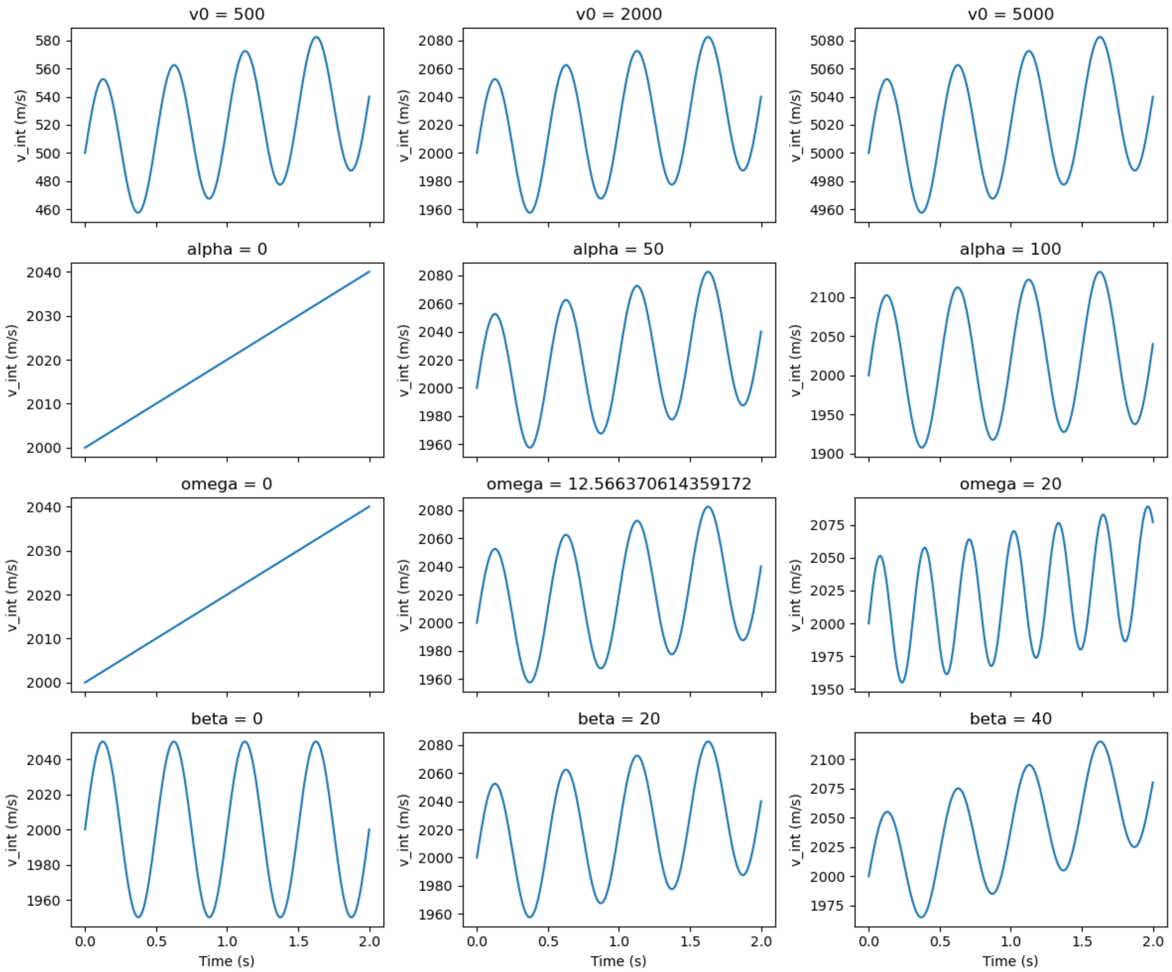

Figure 2: A series of plots where the named value is modified from its default value (see figure 1 caption) where all other parameters are held fixed. One parameter is modified in each row.

2.4 2d

See "approx_v_int" function in Jupyter notebook.

3 Problem 3

3.1 3a

See figure 4.

3.2 3b, 3c

See figure 5.

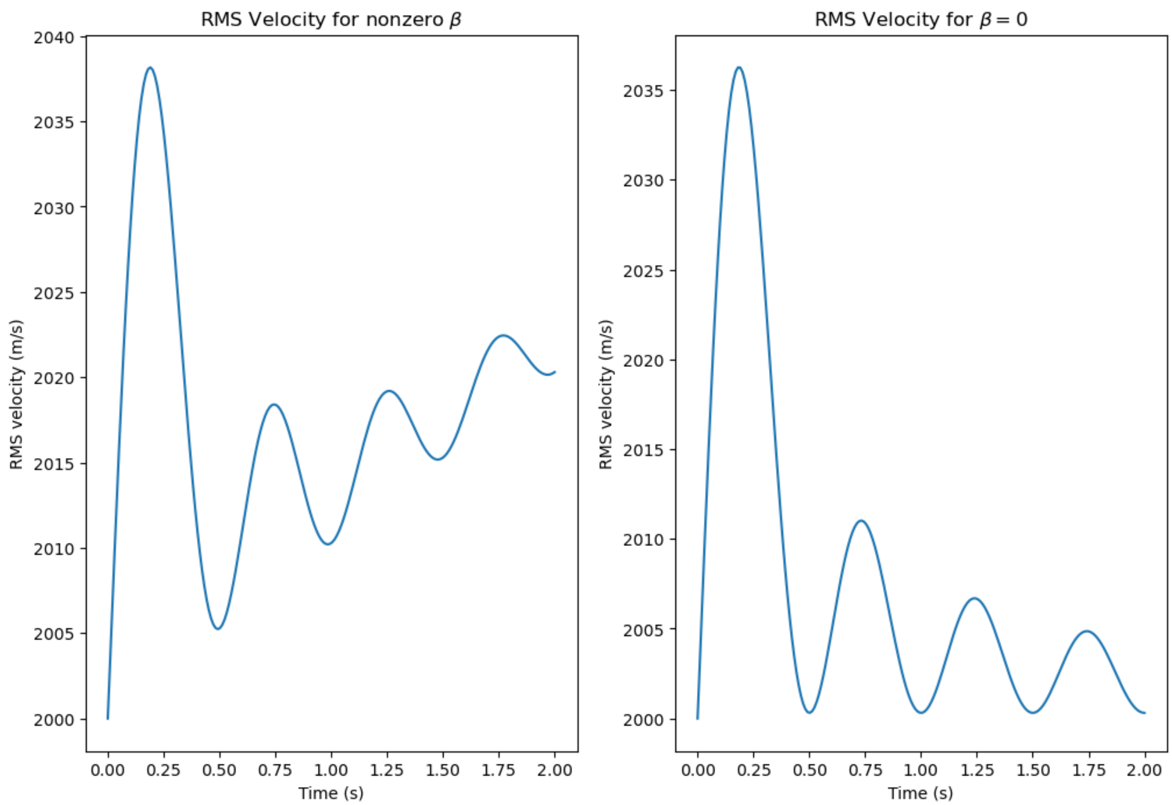

Figure 3: Plots of the RMS velocity computed using the analytical solution to equation 1. The $\beta = 0$ plot matches the RMS velocity computed in class, as expected.

3.3 3d

See figure 6 for plots. As the decimation level increases, the accuracy of the recovered velocity decreases. This is especially evident with the linear interpolation, which becomes increasingly jagged. The cubic spline interpolation interval velocity's accuracy also suffers with higher decimation levels, especially at the endpoints. This is likely because the cubic spline required datapoints on either side of the point being interpolated, which is not possible near the end points.

4 Problem 4

4.1 4a

As seen in figure 7, the recovered interval velocity becomes increasingly noisy as the magnitude of the noise added to the signal increases. The effect of noise on the model is also amplified as time goes on. This is because taking a derivative amplifies high frequency noise and the V'_{RMS} in equation 2 is multiplied by time, meaning later times will have higher amplification factors, while error is not amplified at all for $t = 0$.

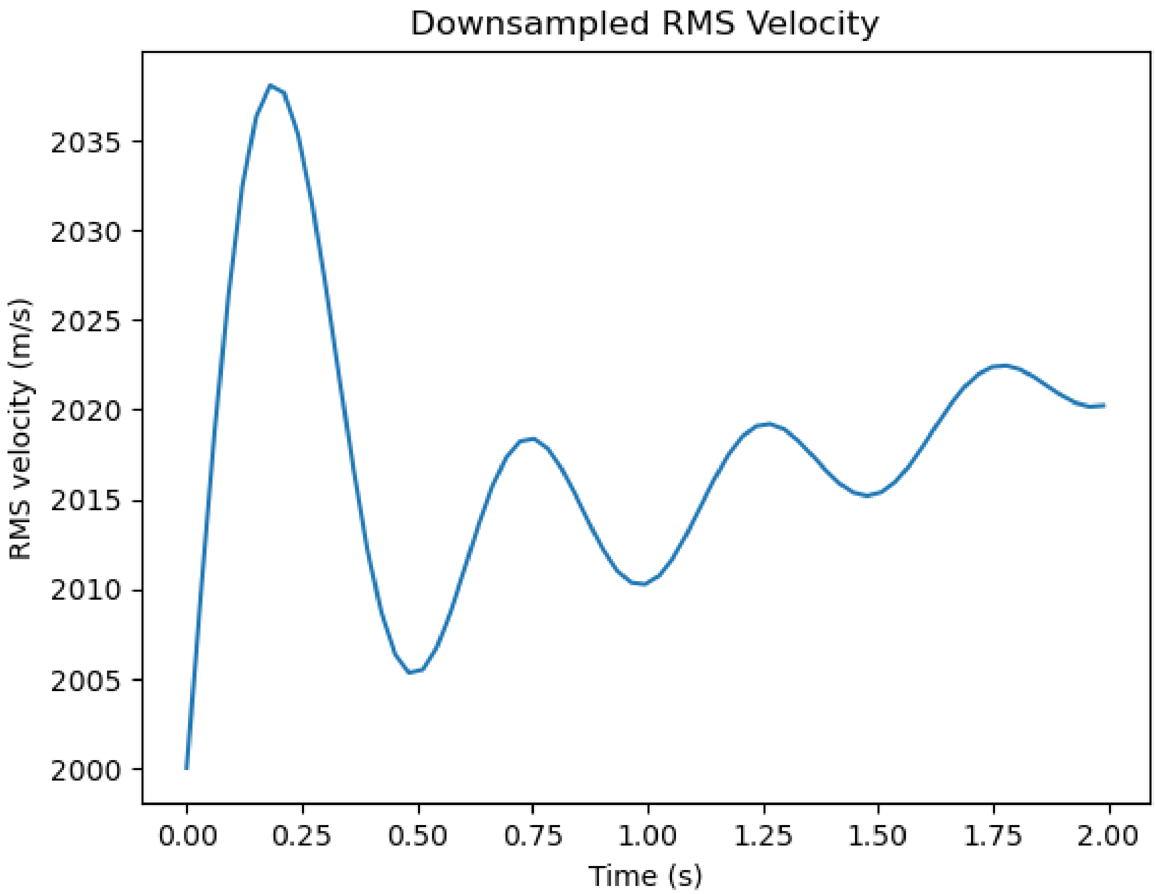

Figure 4: RMS velocity downsampled by taking every third datapoint.

4.2 4b

In the simplified model, when adding gaussian noise, a pattern similar to the previous section appears in the recovered interval velocity, with noise increasing as time goes on. In the correlated noise case (figure 9, we also observe that the magnitude of the error increases with time. The noise in the recovered interval velocity is also slightly out of phase from the actual noise because of the derivative term in equation 2.

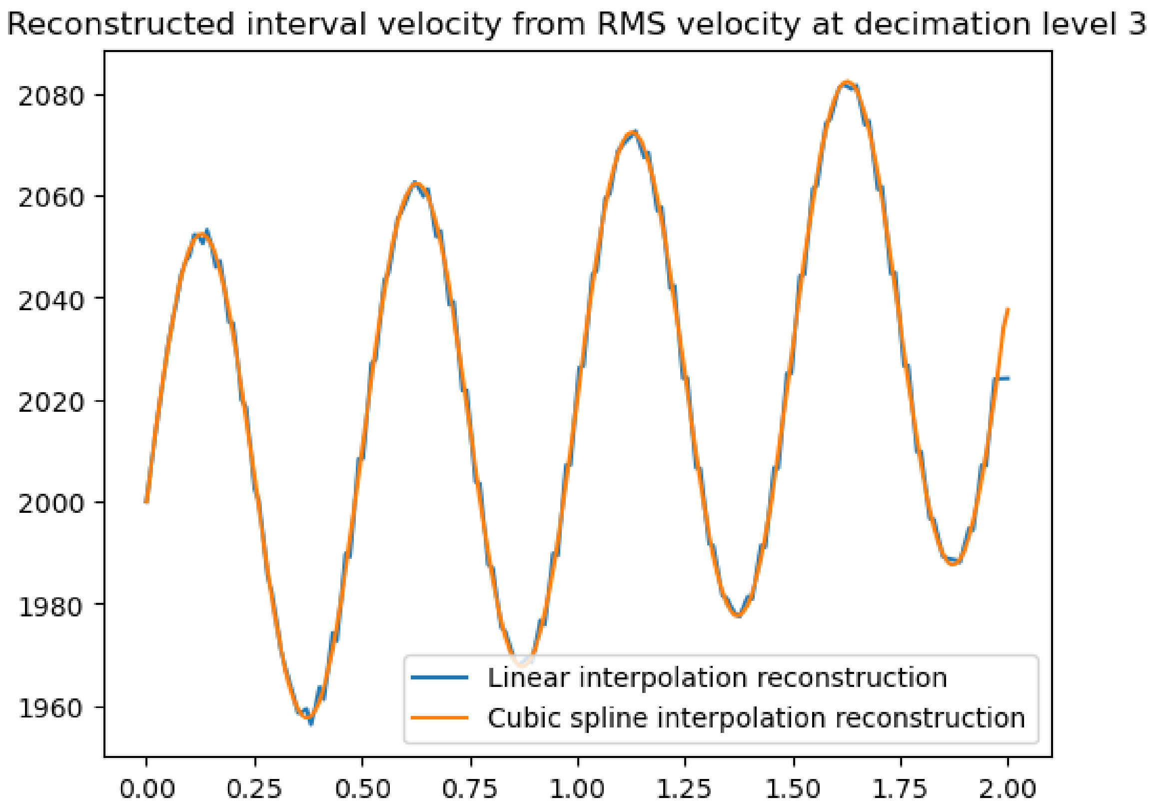

Figure 5: recovered interval velocity using linear and cubic spline interpolation of decimated RMS data.

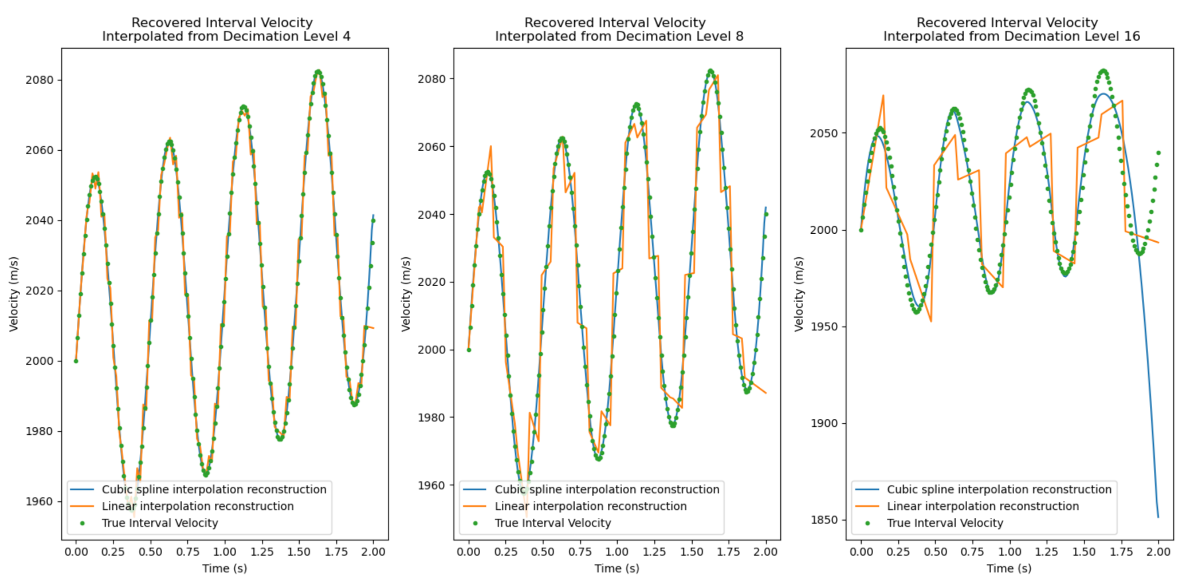

Figure 6: Recovered interval velocity using linear and cubic spline interpolation of decimated RMS data for decimation levels 4, 8, and 16.

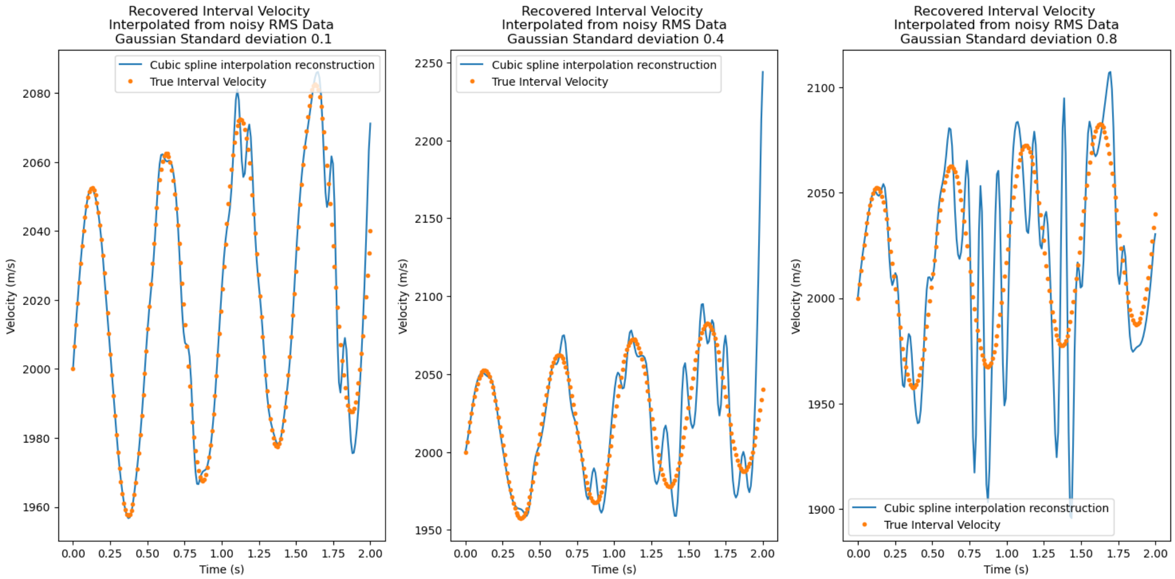

Figure 7: Recovered interval velocity vs true interval velocity. RMS data was first downsampled to every fourth datapoint, then gaussian noise with a standard deviation specified in the plot title was added to each datapoint.

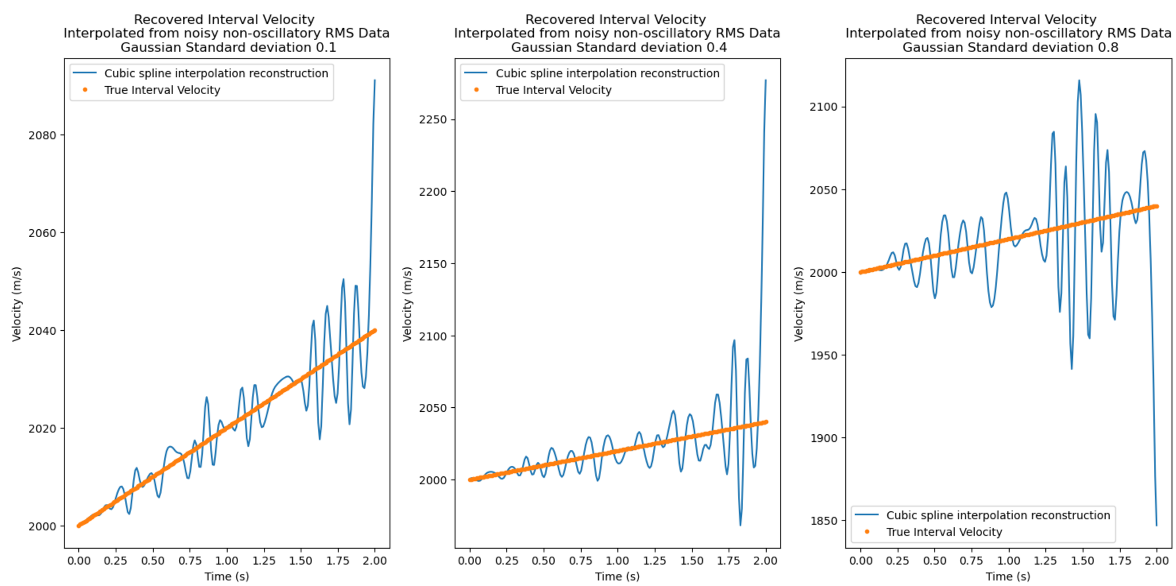

Figure 8: Reconstructed interval velocity data with $\alpha = 0$ and gaussian noise added for different standard deviations.

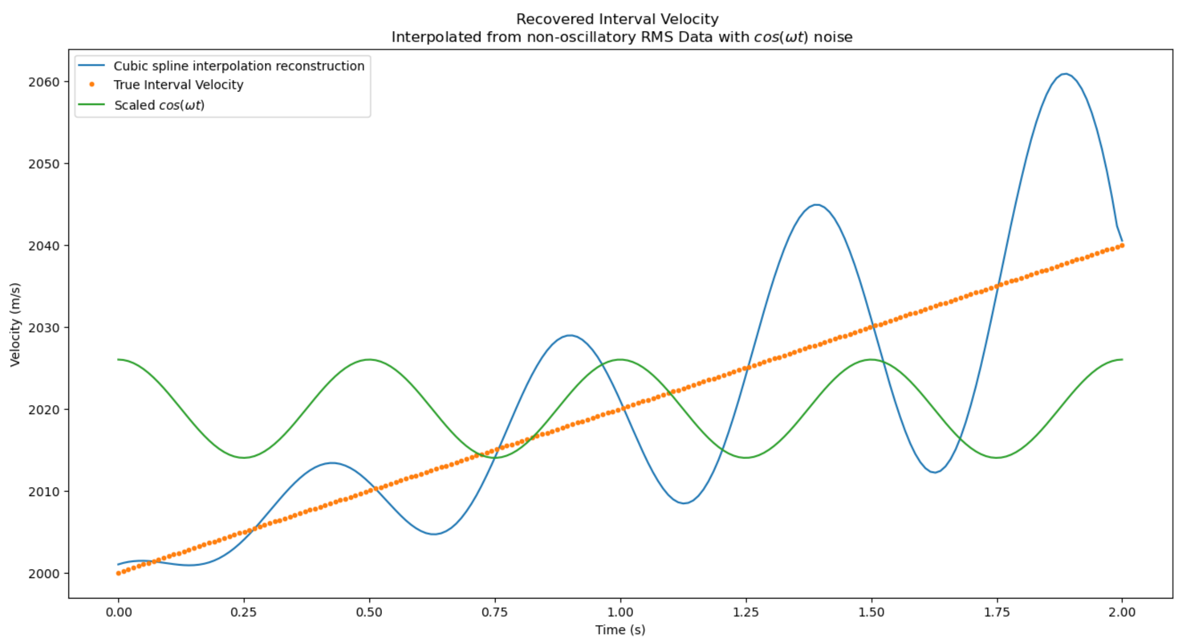

Figure 9: Interval velocity data recovered from rms data with $\alpha = 0$ and $cos(\omega t)$ noise added to it. A scaled and vertically shifted $cos(\omega t)$ function is overlayed for reference.