Notes: Phase Change Mantle Convection

...

July 22, 2011

1 Introduction

goal: reproduce 2d results for phase changes from Jacobs/van den Berg(2011) and do simulation in 3d

2 setup

- Geometry
  - 2d/3d shell, as we have right now (should be enough for now)
  - nice to have/try: surface features, topology (doesn't make a difference?!)
  - interesting: oblateness (easy to do, likely bigger influence)
- compressibility
- variable viscosity (todo: adapt mass matrix of the stokes preconditioner)
- lookup table for phase changes
  - lookup with temperature and pressure
  - take them from old time step for now
  - quantities to look up: density, viscosity, thermal conductivity...

3 Compressible benchmark

The set of non dimensional equations that have to be solved,

$$\nabla \cdot (\rho_r \mathbf{u}) = \mathbf{0} \tag{1}$$

$$-\nabla P + \nabla \cdot \tau = Ra \mathbf{e_z} \mathbf{T} \tag{2}$$

The model setup to reproduce the compressible benchmark of Tan and Gurnis is as follows

3.1 Model domain

The model domain is a unit box with 16, 32 or 64 elements in each direction (equal spacing) free slip boundary conditions are applied.

3.2 boundary conditions

impermeable free slip boundary conditions are applied to all boundaries.

3.3 viscosity

the viscosity variations are restricted to the vertical direction (1D) to be able to do a FFT decomposition of the equation allowing for a semi analytical solution. The 1D viscosity profile is given as,

$$\eta = e^{az} \tag{3}$$

with a a constant either 0 or 2

3.4 Right hand side

The model is driven by a lateral temperature perturbation of the form

$$T(x,z) = sin(\pi z)cos(\pi kx) \tag{4}$$

with k the wavenumber

3.5 Density

the reference density is expressed as

$$\rho_r(z) = e^{\beta(1-z)} \tag{5}$$

with $\beta = Di/\gamma$
for the incompressible Bousinesq approximation (BA) $Di = 0$ and $\gamma = \inf$ for the truncated anelastic approximation (TALA) $Di = 0.5$ and $\gamma = 1$. The density anomaly is given by,

$$\Delta\rho(x,z) = \rho_r(z)T(x,z)T(x,z) = sin(\pi z)cos(\pi kx) \tag{6}$$

The analytical solution to the flow is given as

$$\partial \begin{bmatrix} U_z \\ U_x \\ \sum_{zz}/2\eta_0 k \\ \sum_{xz}/2\eta_0 k \end{bmatrix} = \begin{bmatrix} \beta & -k & 0 & 0 \\ k & 0 & 0 & 2k/\eta^* \\ 0 & 0 & 0 & -\kappa \\ -\beta\eta^* & 2\kappa\eta^* & k & 0 \end{bmatrix} \cdot \begin{bmatrix} U_z \\ U_x \\ \sum_{zz}/2\eta_0 k \\ \sum_{xz}/2\eta_0 k \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \Omega Ra/2\eta_0 k \\ 0 \end{bmatrix} \tag{7}$$

4 Benchmark results

To show the correctness of our implementation we reproduced the 2D compressible benchmark by [**?**]. Figure 1 gives the relative errors of the solution compared to a pseudo analytical solution as a function of the number of nodal points in the vertical direction.

For the compressible case we present results for $Di/\gamma = 0.5$, the value used in the Tan-Gurnis paper.

4.1 BulkModulus

Bulk Modulus K is the resistance to volume compression, units is Pa
$K = -V\frac{\partial P}{\partial V}$
K_s is de adiabatic bulk modulus (written as $K_s = \gamma P$ (gamma is not Grunneissen parameter in this case but the adiabatic index, just to confuse us)
typical values of K
water 2.2×10^9 solid 5×10^7 Pa
$1/K$ is called compressibility
in the Tan and Gurniss paper they use the ratio Di/γ for the compressibility which is correct since $\gamma = \frac{\alpha K}{c_p \rho}$ and $Di = \frac{\alpha g h}{c_p}$ which results in,
$\frac{Di}{\gamma} = \frac{gh\rho_1}{K}$ with $K = gh\rho_2$
$Pa = kg/ms^2 = \rho gh = (kg/m^3)(m/s^2)(m)$
this means that (if we take g and h constant) we have the change in density over the model domain $\frac{\rho_1}{\rho_2}$.

4.2 Continuity equation

$\nabla \cdot \rho u = 0$ assuming only changes in density in the vertical direction this can be written as, $\nabla \cdot u + \frac{1}{\rho_r}\frac{\partial \rho_r}{\partial z} u_z = 0$ which can be expressed in terms of P, the pressure as, $\nabla \cdot u + \frac{1}{\rho_r}\frac{\partial P}{\partial z} u_z g = 0$
For the benchmark case we have a functional expression for the density ρ_r
$\rho_r(z) = \mathbf{e}\beta(\mathbf{1} - \mathbf{z})$ with $\beta = Di/\gamma = K$ the bulk modulus. (in our case 2.0)

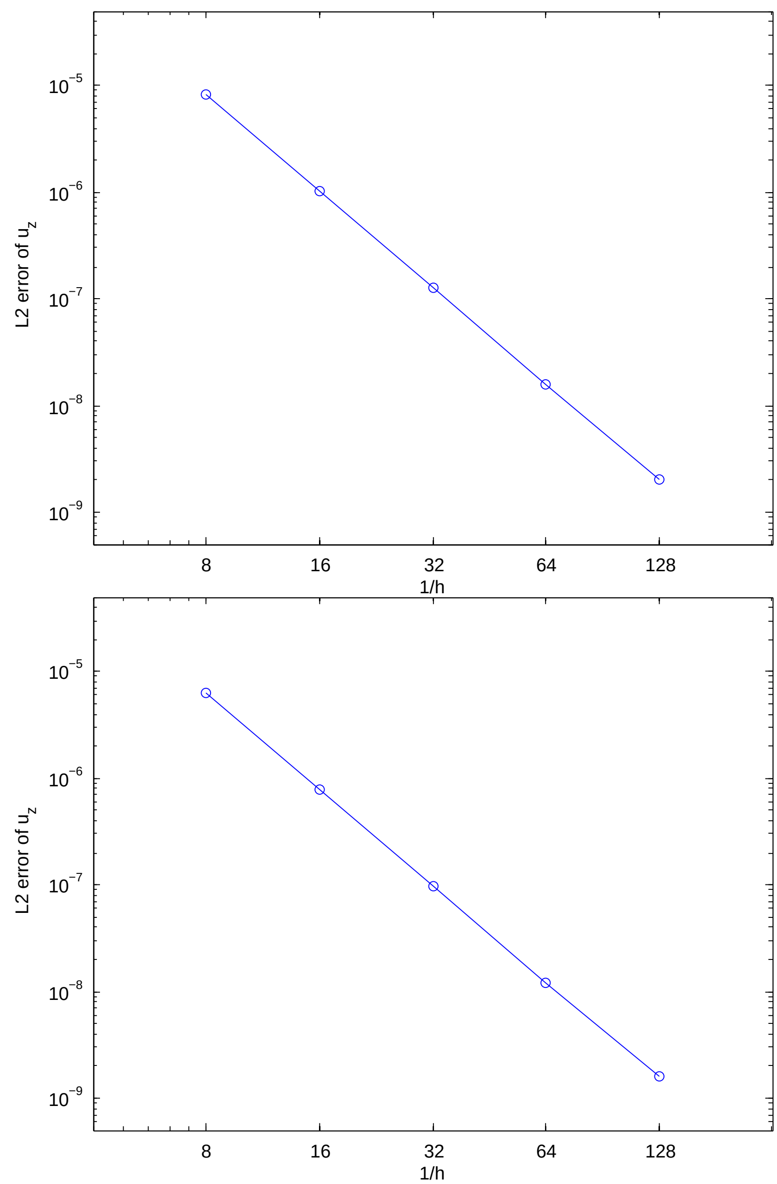

Figure 1: The convergence of the error is order three

that means that we have to put this functional description of $\rho_r(z)$ in our contribution to the lower left of block of the stiffness matrix.

5 Compressible solver

A possible way to solve the compressible Stokes equation is the use of what is called the Schur method in [1].
Here the matrix $\mathbf{A}$ from

$$\mathbf{A}\begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{B^T} \\ \mathbf{B} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \mathbf{b} \tag{8}$$

is decomposed as

$$\begin{bmatrix} \mathbf{F} & 0 \\ \mathbf{B} & -\mathbf{BF^{-1}B^T} \end{bmatrix}\begin{bmatrix} \mathbf{u}^* \\ \delta\mathbf{p} \end{bmatrix} \tag{9}$$

where

$$\begin{bmatrix} \mathbf{u}^* \\ \delta\mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{F^{-1}B^T} \\ 0 & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} \tag{10}$$

the pseudo code for the incompresible case as (figure taken from paper).

Algorithm 8.1 The Schur method

Initialize $u^{(0)}$, $p^{(0)}$ and *maxiter* (maximum iterations)
Compute: $r_u = f - Fu^{(0)} - B^T p^{(0)}$
$r_p = g - Bu^{(0)}$
For $k = 0$ to *maxiter*

1. Solve $Fu_f = r_u$
2. Solve $-BF^{-1}B^T p_\delta = r_p - Bu_f$
3. Update $u_\delta = u_f - u_l$, where u_l is obtained by solving $Fu_l = B^T p_\delta$
4. Update $u^{(k+1)} = u^{(k)} + u_\delta$
5. Update $p^{(k+1)} = p^{(k)} + p_\delta$
6. Update $r_u = f - Fu^{(k+1)} - B^T p^{(k+1)}$
7. Update $r_p = g - Bu^{(k+1)}$
8. **If** converged **Exit**

End For

In this scheme the pressure subsystem in step 2 can be solved by FGMRES (replace $\mathbf{BF^{-1}B^T}$ with $(\mathbf{B}+\mathbf{C})\mathbf{F^{-1}B^T}$ In this case we dont construct $(B+C)F^{-1}B^T$ but approximately solve $(B+C)F^{-1}B^T p_\delta = r_p - Bu_f$. We dont have a good preconditioner for this step but could use something like diagonal scaling.

References

On iterative methods for the incompressible Stokes problem, M. ur Rehman, T. Geenen, C. Vuik, G. Segal, and S.P. MacLachlan, International Journal for Numerical Methods in Fluids, to appear, 2011