Mini-Mapper X:Photoencoder signals

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Introduction

The idea here is to calculate the form of signal that the phototransistor in the photointerrupter on the motor encoder disk will generate as the encoder disk rotates. This isn't strictly necessary, but it's mildly interesting, and a fun geometrical exercise. Figure 1 shows the basic setup, with the ring of holes in the encoder disk rotating past the beam of the photointerrupter.

The important factor for the purposes of determining the current signal generated by the phototransistor in the photointerrupter is the area of overlap between the hole in the encoder disk and the beam of infrared radiation emitted by the photodiode in the photointerrupter.

The following parameters are used in the analysis below:

- d: the diameter of a single hole in the encoder disk;
- D: the diameter of the circle of centres of the holes in the encoder disk;
- w and h: the width and height of the beam from the photodiode in the photointerrupter — for the photointerrupters I've been looking at, the beam aperture is rectangular, with the narrower dimension horizontal.
- ϕ: the angle (anticlockwise positive) between the line between the centre of rotation of the encoder disk and the center of the photointerrupter beam, and the line betwen the centre of rotation of the encoder disk and the centre of the hole under consideration.

We make a few assumptions:

1. $d \leq h$ and $d \geq w$: these assumptions give the geometrical relations between the optical beam and the hole as shown in Figure 1, i.e., the outline of the hole has either zero, two or four intersections with the outline of the optical beam, with all intersections occurring on the vertical sides of the beam.
2. The gaps between the holes in the encoder disk are large enough that only one hole can intersect with the optical beam at a time.

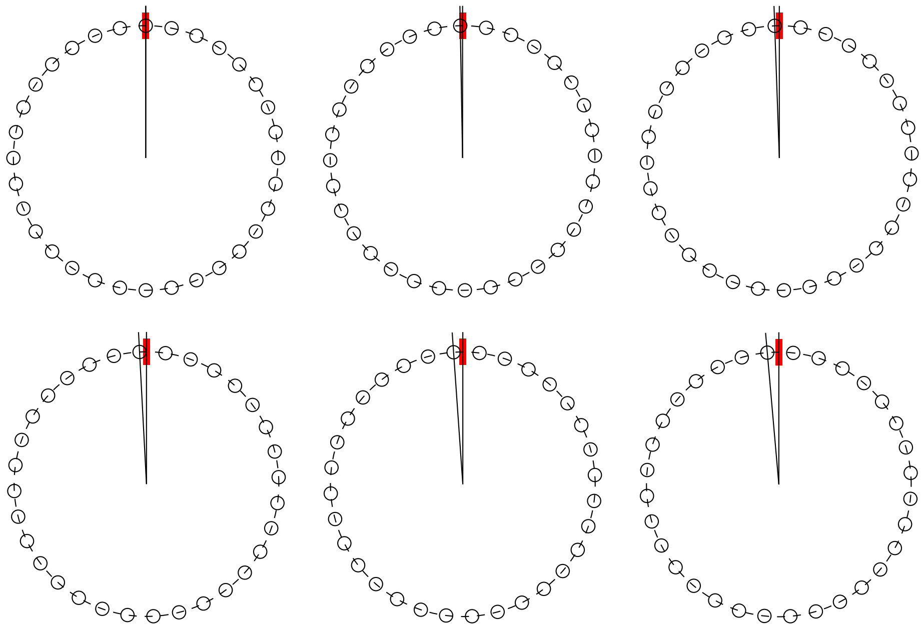


Figure 1: Photointerrupter beam (red, $w = 0.5\,\text{mm} \times h = 2\,\text{mm}$) interfacing with holes in motor encoder disk. Dashed circle shows line of centres of encoder disk holes (diameter $D = 20\,\text{mm}$), with each hole being of diameter $d = 1\,\text{mm}$. The images here show the positions of the encoder disk 1° apart, starting with a hole centrally located in front of the photointerrupter beam ($\phi = 0$) and showing positions of $\phi = 1°, 2°, 3°, 4°, 5°$.

Intersections of holes with optical aperture

Let's think about the overlap of a single hole with the aperture of the photointerrupter. We'll consider only $\phi \geq 0$ (the situation for negative ϕ is symmetrical). Writing $R = D/2$ for the radius of the circle of centres, and $r = d/2$ for the radius of the holes, the centre of the hole we're looking at is at (x_c, y_c):

$$x_c = -R\sin\phi,\ y_c = R\cos\phi$$

and its equation is

$$(x - x_c)^2 + (y - y_c)^2 = r^2$$

or

$$(x + R\sin\phi)^2 + (y - R\cos\phi)^2 = r^2$$

Points of intersection of this circle with the outline of the optical beam are given by solving this equation for y when $x = \pm w/2$:

$$R^2\sin^2\phi \pm wR\sin\phi + w^2/4 + y^2 - 2Ry\cos\phi + R^2\cos^2\phi - r^2 = 0$$

or, simplifying

$$y^2 - 2Ry\cos\phi + R^2 - r^2 + w^2/4 \pm wR\sin\phi = 0. \tag{1}$$

This discriminant of this equation is

$$\Delta_\pm = 4R^2\cos^2\phi \mp 4wR\sin\phi - 4(R^2 - r^2) - w^2. \tag{2}$$

Figure 2 shows values of $\Delta_\pm$ from (2) as a function of ϕ. When the hole extends right across the width of the optical beam aperture, there are four intersections between the circle defining the outline of the hole and the outline of the aperture. In this case, $\Delta_+ \geq 0, \Delta_- \geq 0$. When the outline of the hole intersects with only one vertical side of the optical beam aperture, then there are two intersections between the circle and the outline of the aperture and $\Delta_+ < 0, \Delta_- \geq 0$. When there are no intersections between the circle and the outline of the aperture then both discriminants are negative.

The limits in terms of the angular displacement ϕ of the ranges with solutions are given by $\Delta_\pm = 0$. From (2), this gives

$$4R^2\sin^2\phi \pm 4wR\sin\phi - 4r^2 + w^2 = 0$$

or

$$\begin{aligned}
\sin\phi &= \frac{\mp 4wR \pm \sqrt{16w^2R^2 - 16R^2(w^2 - 4r^2)}}{8R^2} \\
\sin\phi &= \frac{\mp w \pm \sqrt{w^2 - (w^2 - 4r^2)}}{2R} \\
\sin\phi &= \frac{\mp w \pm 2r}{2R}
\end{aligned}$$

$$\sin\phi = \frac{\mp w/2 \pm r}{R}.$$

The four limiting values for ϕ derived from this equation provide the limits of the two intersection and four intersection cases in the positive and negative ϕ directions. Putting $w = 0.5\,\text{mm}$, $r = 0.5\,\text{mm}$ and $R = 10\,\text{mm}$, we get limits of $|\phi| \leq 4.30°$ for the two intersections case, and $|\phi| \leq 1.43°$ for the four intersections case, which matches with the limits seen in Figure 2.

Solutions to the positive and negative versions of (1) exist when $\Delta_\pm \geq 0$ and are given by:

$$y_\pm{}^{\pm} = R\cos\phi \pm \frac{1}{2}\sqrt{\Delta_\pm} = y_c \pm \frac{1}{2}\sqrt{\Delta_\pm} \tag{3}$$

(The sign of the discriminant $\Delta_\pm$ matches the sign on the left hand side. The other plus-or-minus sign is for the usual pair of solutions to a quadratic equation, and corresponds to the two members of the pairs of intersection points.) Figure 3 shows these solutions as a function of ϕ.

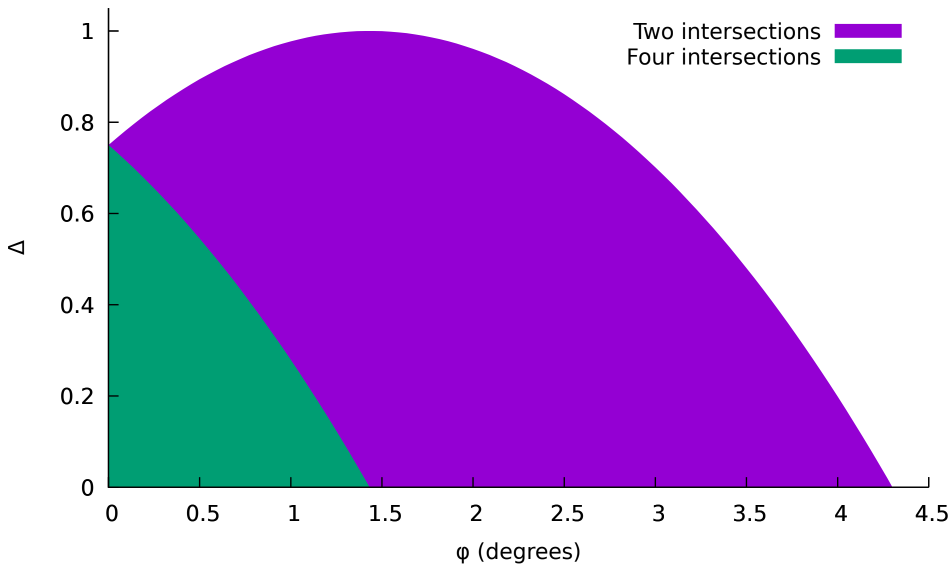


Figure 2: Discriminant for hole/beam intersection equation as a function of angular displacement ϕ. The boundary of the "two intersections" region is $\Delta_-(\phi)$, while that of the "four intersections" region is $\Delta_+(\phi)$.

Hole/aperture overlap area

Given the intersection information from (3), we now need to calculate the area of the overlap between the encoder disk hole and the photointerrupter optical aperture. There are three cases to consider, two requiring calculation and one trivial:

1. There are two intersections between the outline of the hole and the optical aperture. In this case, the overlap between the hole and aperture is formed by a line segment of the vertical side of the aperture and a segment of the circle forming the outline of the hole — this segment lies inside the optical aperture and has endpoints given by the intersection points between the hole and the vertical side of the aperture (Figure 4a).

2. There are four intersections between the outline of the hole and the optical aperture. In this case, the overlap is the encoder disk hole minus two segments of the circle forming the outline of the hole — these segments lie outside the optical aperture and have endpoints given by the intersection points between the hole and the vertical sides of the aperture (Figure 4b).

3. There are no intersections between the outline of the encoder disk hole and the optical aperture. In this trivial case, the overlap area is identically zero.

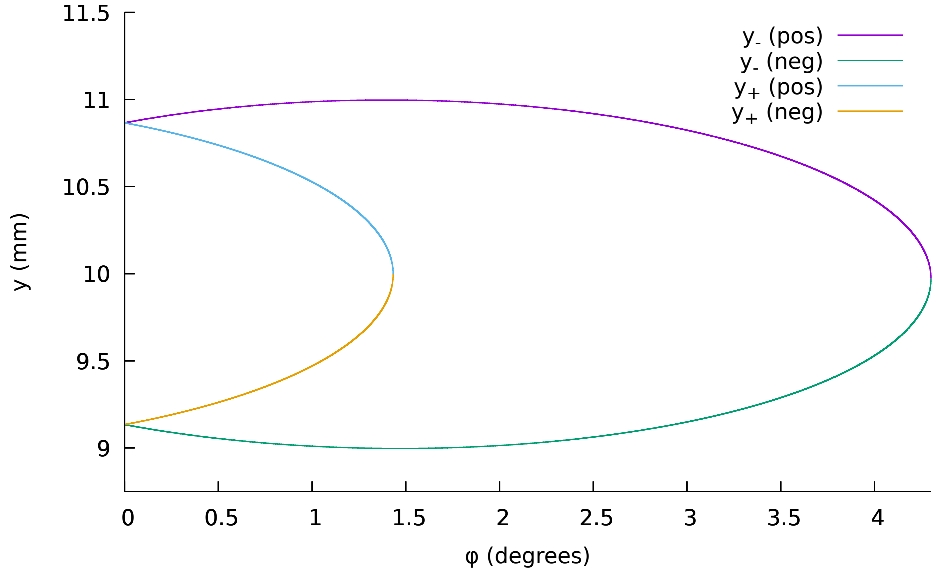


Figure 3: Solutions for hole/beam intersection equation as a function of angular displacement ϕ. The y_- values show intersections with the left-hand edge of the optical aperture (remember positive ϕ means movement of the hole to the left from top dead centre), and y_+ values are for intersection with the right-hand edge of the aperture. The "(pos)" and "(neg)" labels indicate the two signs in (3) for the two solutions to each quadratic equation.

In each of the two cases requiring calculation, the calculation can be performed on the basis of an expression for the area of a segment of a circle. For a circle with radius R, the area A of a segment subtending an angle θ at the centre of the circle is

$$A = \frac{R^2}{2}(\theta - \sin\theta). \tag{4}$$

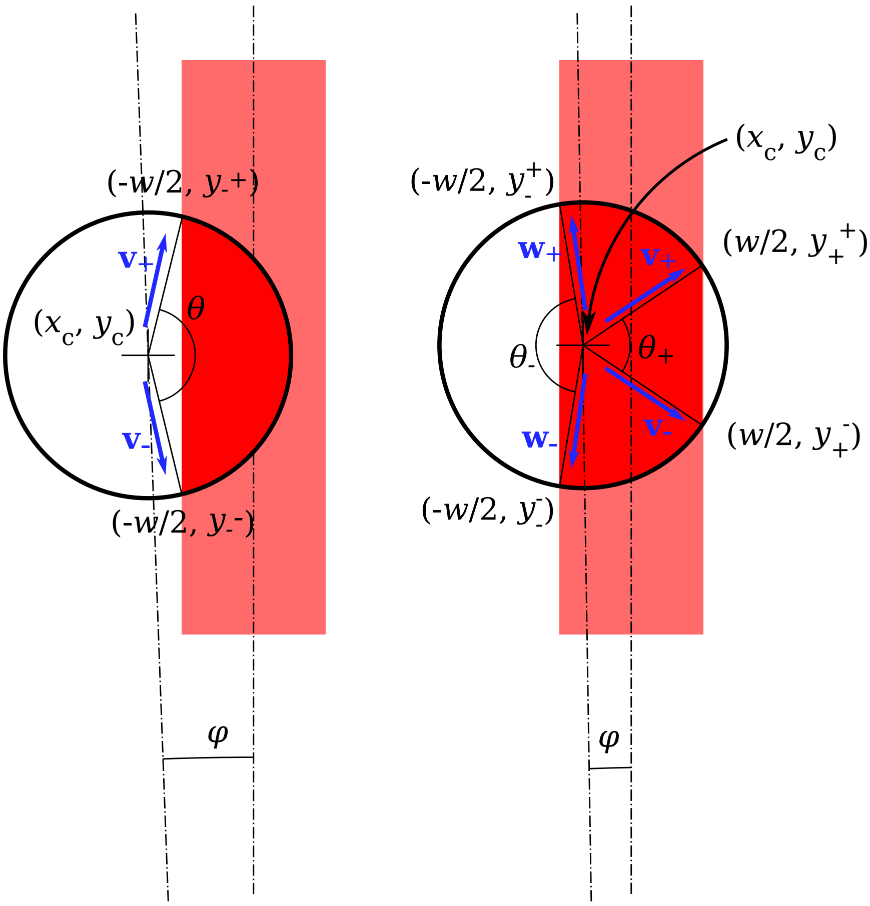
Case 1: two intersections

Figure 4a shows this case. The intersection points of the outline of the encoder disk hole and the optical aperture are $(-w/2, y_-^+)$ and $(-w/2, y_-^-)$. We know the radius of the encoder disk hole, so in order to apply (4) to calculate the area of overlap, we need to find the angle θ. To do this, we construct vectors $\mathbf{v}_+$ and $\mathbf{v}_-$ between the centre of the encoder disk hole and the intersection points:

$$\mathbf{v}_+ = \begin{pmatrix} -w/2 - x_c \\ y_-^+ - y_c \end{pmatrix}, \quad \mathbf{v}_- = \begin{pmatrix} -w/2 - x_c \\ y_-^- - y_c \end{pmatrix}, \tag{5}$$

and then

$$\cos\theta = \hat{\mathbf{v}}_+ \cdot \hat{\mathbf{v}}_-.$$



(a) Two intersections (b) Four intersections

Figure 4: Overlap cases: (a) two intersections, so overlap is a segment of a circle; (b) four intersections, so overlap is a circle minus two segments.

(Here $\hat{\mathbf{v}}$ is the unit vector in the direction of vector $\mathbf{v}$.) First we calculate $\mathbf{v}_+ \cdot \mathbf{v}_-$, making use of the fact that $y_-^{\pm} - y_c = \pm\frac{1}{2}\sqrt{\Delta_-}$ (from (3)):

$$\mathbf{v}_+ \cdot \mathbf{v}_- = \left(\frac{w}{2} + x_c\right)^2 + (y_-^+ - y_c)(y_-^- - y_c) = \left(\frac{w}{2} + x_c\right)^2 - \frac{\Delta_-}{4}. \tag{6}$$

Now, we also find that

$$|\mathbf{v}_\pm|^2 = \left(\frac{w}{2} + x_c\right)^2 + (y_-^\pm - y_c)^2 = \left(\frac{w}{2} + x_c\right)^2 + \frac{\Delta_-}{4}.$$

so that

$$|\mathbf{v}_+||\mathbf{v}_+| = \left(\frac{w}{2} + x_c\right)^2 + \frac{\Delta_-}{4}$$

and we finally arrive at

$$\cos\theta = \frac{\mathbf{v}_+ \cdot \mathbf{v}_-}{|\mathbf{v}_+||\mathbf{v}_+|} = \frac{\left(\frac{w}{2} + x_c\right)^2 - \frac{\Delta_-}{4}}{\left(\frac{w}{2} + x_c\right)^2 + \frac{\Delta_-}{4}}$$

which gives an area for the two intersection case of

$$A_2 = \frac{r^2}{2}(\theta - \sin\theta)$$

Case 2: four intersections

The four intersections case (Figure 4b) is similar, except that we have two angles, θ_- and θ_-, one of each side of the optical aperture, and the area of interest is the total area of the encoder disk hole minus the area of the two segments subtended by θ_- and θ_-.

The left-hand side of Figure 4b can be treated the same way as for the two intersections case:

$$\cos\theta_- = \frac{\mathbf{w}_+ \cdot \mathbf{w}_-}{|\mathbf{w}_+||\mathbf{w}_+|} = \frac{\left(\frac{w}{2} + x_c\right)^2 - \frac{\Delta_-}{4}}{\left(\frac{w}{2} + x_c\right)^2 + \frac{\Delta_-}{4}}. \tag{7}$$

The right-hand side goes the same kind of way, with vectors $\mathbf{v}_+$ and $\mathbf{v}_-$ defined as:

$$\mathbf{v}_+ = \begin{pmatrix} w/2 - x_c \\ y_+^+ - y_c \end{pmatrix}, \quad \mathbf{v}_- = \begin{pmatrix} w/2 - x_c \\ y_+^- - y_c \end{pmatrix}, \tag{8}$$

so that

$$\mathbf{v}_+ \cdot \mathbf{v}_- = \left(\frac{w}{2} - x_c\right)^2 + (y_+^+ - y_c)(y_+^- - y_c) = \left(\frac{w}{2} - x_c\right)^2 - \frac{\Delta_+}{4}. \tag{9}$$

and

$$|\mathbf{v}_\pm|^2 = \left(\frac{w}{2} - x_c\right)^2 + (y_+^\pm - y_c)^2 = \left(\frac{w}{2} - x_c\right)^2 + \frac{\Delta_+}{4}.$$

so that

$$|\mathbf{v}_+||\mathbf{v}_+| = \left(\frac{w}{2} - x_c\right)^2 + \frac{\Delta_+}{4}$$

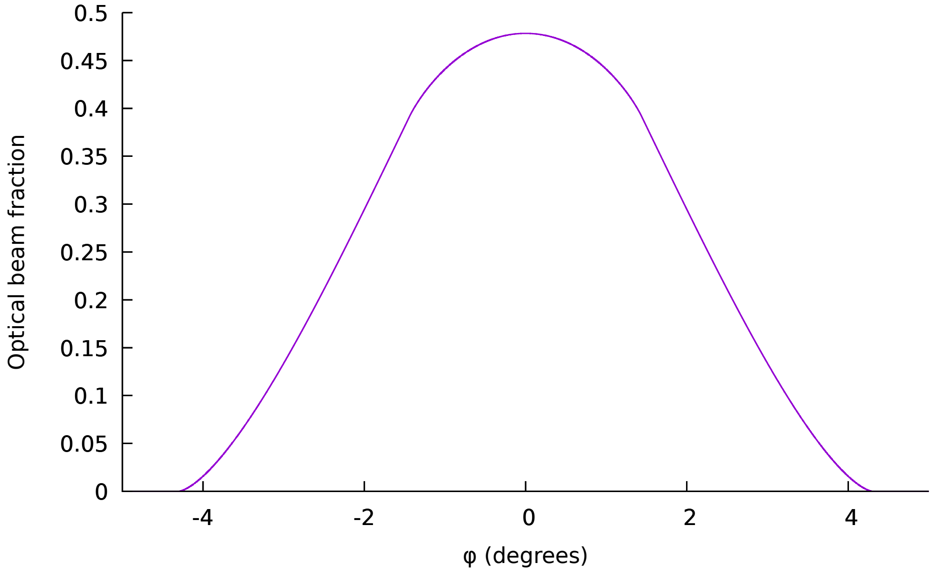


Figure 5: Dependence of fraction of area of photointerrupter optical beam covered by encoder disk hole as a function of angular displacement ϕ, based on $w = 0.5\,\mathrm{mm}$, $r = 0.5\,\mathrm{mm}$ and $R = 10\,\mathrm{mm}$.

and we arrive at

$$\cos\theta_+ = \frac{\mathbf{v}_+ \cdot \mathbf{v}_-}{|\mathbf{v}_+||\mathbf{v}_+|} = \frac{\left(\frac{w}{2} - x_c\right)^2 - \frac{\Delta_+}{4}}{\left(\frac{w}{2} - x_c\right)^2 + \frac{\Delta_+}{4}}.$$

The total area in the four intersections case is then:

$$A_4 = \pi r^2 - \frac{r^2}{2}(\theta_+ - \sin\theta_+) - \frac{r^2}{2}(\theta_- - \sin\theta_-).$$

Final results

The functional dependence of the overlap area between an encoder disk hole and the optical beam of the interrupter is shown in Figure 5, representing the overlap as the fraction of the photointerrupter optical beam covered by the encoder hole disk. The assumption is that the current generated by the phototransistor in the photointerrupter will be proportional to this fractional overlap.

We can now use these results to generate simulated photocurrent signals for use in simulations.