

10.4 Comparison Tests

Main Ideas

- **Direct Comparison Test** (page 568)

$\sum a_n$ and $\sum b_n$ are two series where $0 \leq a_n \leq b_n$ for all n

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

- **Limit Comparison Test** (page 569)

If $a_n > 0$ and $b_n > 0$ for all $n \geq N$ where N is some integer

1. If

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ and $c > 0$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

3. If

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Homework Problems

1.

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 30}$$

is similar to the p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\Rightarrow \frac{1}{n^2 + 30} < \frac{1}{n^2} \text{ for all } n \geq 1$$

since the p -series with $p = 2$ converges, so does our series by direct comparison

3.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} - 1}$$

is similar to the p -series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

$$\Rightarrow \frac{1}{\sqrt{n} - 1} > \frac{1}{\sqrt{n}} \text{ for all } n$$

since the p -series with $p = 1/2$ diverges, so does our series by direct comparison

5.

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$$

since \cos^2 is always between 0 and 1,

$$0 \leq \frac{\cos^2 n}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

since $\frac{1}{n^{3/2}}$ is a p -series with $p = 3/2 > 1$, meaning it converges,

then by the direct comparison test our series also converges

9.

$$\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$$

$$\text{compare with } 1/n^2 \quad \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2}}{\frac{n-2}{n^3-n^2+3}}$$

$$= \lim_{n \rightarrow \infty} \frac{n^3-n^2+3}{n^2(n-2)}$$

$$= \lim_{n \rightarrow \infty} \frac{n^3-n^2+3}{n^3-2n^2} = 1$$

\Rightarrow converges since $1/n^2$ converges

11.

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)} \quad \text{compare with} \quad \sum_{n=2}^{\infty} \frac{1}{n-1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\frac{n(n+1)}{(n^2+1)(n-1)}}{\frac{1}{n-1}} \right| = \lim_{n \rightarrow \infty} \frac{n(n+1)(n-1)}{(n^2+1)(n-1)} = \lim_{n \rightarrow \infty} \frac{n(n+1)}{(n^2+1)} = 1 \Rightarrow \text{both converge or diverge}$$

$$\sum_{n=2}^{\infty} \frac{1}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n} \quad (p\text{-series with } p=1) \Rightarrow \text{both } \underline{\text{diverge}} \text{ by limit comparison}$$

13.

$$\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n} 4^n}$$

$$\text{compare with series } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\frac{5^n}{\sqrt{n} 4^n}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{5^n}{4^n} = \lim_{n \rightarrow \infty} (5/4)^n = \infty$$

since the series on the bottom diverges (p -series where $p = 1/2 \leq 1$)
the series on top diverges also by limit comparison

19.

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n} \quad \text{is bounded between } 0 \text{ and } \frac{1}{2^n}$$

$\frac{1}{2^n}$ is a geometric series with $a=1, r=1/2 < 1 \Rightarrow \underline{\text{converges}}$

21.

$$\sum_{n=1}^{\infty} \frac{2n}{3n-1}$$

$$\lim_{n \rightarrow \infty} \frac{2n}{3n-1} = 2/3 \neq 0 \implies \underline{\text{diverges}} \text{ by } n\text{-th term test}$$

33.

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{(n+1)^2-1}} = \sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n^2+2n}}$$

after change of bounds (start with $n = 1$) is similar to the p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\implies \frac{1}{(n+1)\sqrt{n^2+2n}} < \frac{1}{n^2} \text{ for all } n \geq 1$$

and since $p = 2 > 1$ means the p -series converges, our series also converges by direct comparison