Main Ideas

- **Direct Comparison Test** (page 568)

$$\sum a_n \text{ and } \sum b_n \text{ are two series where } 0 \le a_n \le b_n \text{ for all } n$$

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

- **Limit Comparison Test** (page 569)

$$\text{If } a_n > 0 \text{ and } b_n > 0 \text{ for all } n \ge N \text{ where } N \text{ is some integer}$$

1. If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \text{ and } c > 0 \text{ then } \sum a_n \text{ and } \sum b_n \text{ both converge or both diverge.}$$
2. If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \text{ and } \sum b_n \text{ converges, then } \sum a_n \text{ converges.}$$
3. If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \text{ and } \sum b_n \text{ diverges, then } \sum a_n \text{ diverges.}$$

Homework Problems

1.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+30}$$

$$\text{is similar to the } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\implies \frac{1}{n^2+30} < \frac{1}{n^2} \text{ for all } n \geq 1$$

$$\text{since the } p\text{-series with } p = 2 \text{ converges, so does our series by direct comparison}$$

3.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$$

$$\text{is similar to the } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

$$\implies \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}} \text{ for all n}$$

$$\text{since the } p\text{-series with } p = 1/2 \text{ diverges, so does our series by direct comparison}$$

5.

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$$

$$\text{since } \cos^2 \text{ is always between 0 and 1,}$$
$$0 \leq \frac{\cos^2 n}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

$$\text{since } \frac{1}{n^{3/2}} \text{ is a } p\text{-series with } p = 3/2 > 1\text{, meaning it converges,}$$
$$\text{then by the direct comparison test our series also converges}$$

13.

$$\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\ 4^n}$$

$$\text{compare with series } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$$\implies \lim_{n\to\infty} \frac{\frac{5^n}{\sqrt{n}\ 4^n}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{5^n}{4^n} = \lim_{n\to\infty} (5/4)^n = \infty$$

since the series on the bottom diverges (p-series where $p = 1/2 \leq 1$)
the series on top diverges also by limit comparison

33.

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{(n+1)^2-1}} = \sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n^2+2n}}$$

after change of bounds (start with $n = 1$) is similar to the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\implies \frac{1}{(n+1)\sqrt{n^2+2n}} < \frac{1}{n^2} \text{ for all } n \geq 1$$

and since $p = 2 > 1$ means the p-series converges, our series also converges by direct comparison