10.4 Comparison Tests

Main Ideas

- **Direct Comparison Test** (page 568)

$$\sum a_n \text{ and } \sum b_n \text{ are two series where } 0 \leq a_n \leq b_n \text{ for all } n$$

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

- **Limit Comparison Test** (page 569)

$$\text{If } a_n > 0 \text{ and } b_n > 0 \text{ for all } n \geq N \text{ where } N \text{ is some integer}$$

1. If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \text{ and } c > 0 \text{ then } \sum a_n \text{ and } \sum b_n \text{ both converge or both diverge.}$$
2. If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \text{ and } \sum b_n \text{ converges, then } \sum a_n \text{ converges.}$$
3. If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \text{ and } \sum b_n \text{ diverges, then } \sum a_n \text{ diverges.}$$

Homework Problems

1.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+30}$$

$$\text{is similar to the } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\implies \frac{1}{n^2+30} < \frac{1}{n^2} \text{ for all } n \geq 1$$

since the p-series with $p = 2$ converges, so does our series by direct comparison

3.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$$

$$\text{is similar to the } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

$$\implies \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}} \text{ for all n}$$

since the p-series with $p = 1/2$ diverges, so does our series by direct comparison

5.

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$$

since $\cos^2$ is always between 0 and 1,

$$0 \leq \frac{\cos^2 n}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

since $\frac{1}{n^{3/2}}$ is a p-series with $p = 3/2 > 1$, meaning it converges,
then by the direct comparison test our series also converges

9.

$$\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$$

$$\text{compare with } 1/n^2 \qquad \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{n-2}{n^3-n^2+3}}$$

$$= \lim_{n\to\infty} \frac{n^3-n^2+3}{n^2(n-2)}$$

$$= \lim_{n\to\infty} \frac{n^3-n^2+3}{n^3-2n^2} = 1$$

$$\implies \text{converges since } 1/n^2 \text{ converges}$$

11.

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)} \quad \text{compare with} \quad \sum_{n=2}^{\infty} \frac{1}{n-1}$$

$$\implies \lim_{n\to\infty} \frac{\frac{n(n+1)}{(n^2+1)(n-1)}}{\frac{1}{n-1}} = \lim_{n\to\infty} \frac{n(n+1)(n-1)}{(n^2+1)(n-1)} = \lim_{n\to\infty} \frac{n(n+1)}{(n^2+1)} = 1 \implies \text{both converge or diverge}$$

$$\sum_{n=2}^{\infty} \frac{1}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n} \quad (p\text{-series with } p=1) \implies \text{both } \underline{\text{diverge}} \text{ by limit comparison}$$

13.

$$\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\,4^n}$$

$$\text{compare with series } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$$\implies \lim_{n\to\infty} \frac{\frac{5^n}{\sqrt{n}\,4^n}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{5^n}{4^n} = \lim_{n\to\infty} (5/4)^n = \infty$$

since the series on the bottom diverges (p-series where $p = 1/2 \leq 1$)
the series on top diverges also by limit comparison

19.

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n} \quad \text{is bounded between 0 and } \frac{1}{2^n}$$

$$\frac{1}{2^n} \text{ is a geometric series with } a=1, r=1/2<1 \implies \underline{\text{converges}}$$

21.

$$\sum_{n=1}^{\infty} \frac{2n}{3n-1}$$

$$\lim_{n\to\infty} \frac{2n}{3n-1} = 2/3 \neq 0 \implies \underline{\text{diverges}} \text{ by } n\text{-th term test}$$

33.

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{(n+1)^2-1}} = \sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n^2+2n}}$$

after change of bounds (start with $n = 1$) is similar to the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\implies \frac{1}{(n+1)\sqrt{n^2+2n}} < \frac{1}{n^2} \text{ for all } n \geq 1$$

and since $p = 2 > 1$ means the p-series converges, our series also converges by direct comparison