10.3 Integral Test

Main Ideas

- **Integral Test** If a_n is a sequence of positive terms, and $a_n = f(n)$ for all $n \geq N$, where f is a
  1. Continuous
  2. Positive
  3. Decreasing

  function of x, then the series

  $$\sum_{n=N}^{\infty} a_n \qquad \text{and the integral} \qquad \int_N^{\infty} f(x)\, dx$$

  either both converge or both diverge.

- **P-Series**

  $$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \text{converges if and only if} \qquad p > 1$$

Homework Problems

1.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \xrightarrow{\text{convert to integral}} \quad \int_1^{\infty} \frac{dx}{x^2}$$

1. continuous for $x \geq 1$
2. positive since $x^2 \geq 0$ for all x
3. decreasing since $f'(x) = \dfrac{-2}{x^3}$

which is always negative when $x \geq 1$

$$\implies \int_1^{\infty} \frac{dx}{x^2} \quad = \quad \lim_{t\to\infty} \int_1^{t} \frac{dx}{x^2} \quad = \quad \lim_{t\to\infty} \left. \frac{-1}{x} \right|_1^t$$

$$= \lim_{t\to\infty} \frac{-1}{t} + 1 = 0 + 1 = 1 \neq \infty \implies \underline{\text{converges}}$$

3.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4} \quad \xrightarrow{\text{convert to integral}} \quad \int_1^{\infty} \frac{dx}{x^2+4}$$

1. continuous for $x \geq 1$
2. positive since $x^2 \geq 0$ for all x
3. decreasing since $f'(x) = \dfrac{-1}{(x^2+4)^2} \cdot (2x)$

which is always negative when $x \geq 1$

$$\implies \int_1^{\infty} \frac{dx}{x^2+4} \quad = \quad \lim_{t\to\infty} \int_1^{t} \frac{dx}{x^2+4} \quad = \quad \lim_{t\to\infty} \int_1^{t} \frac{dx}{\frac{4x^2}{4}+4} \quad = \quad \lim_{t\to\infty} \int_1^{t} \frac{1}{4} \cdot \frac{dx}{\frac{x^2}{4}+1}$$

$$\text{use } u = x/2, \quad du = dx/2$$

$$\int \frac{1}{2} \cdot \frac{du}{u^2+1} = \frac{1}{2} \arctan(u) = \frac{1}{2} \arctan(x/2)$$

$$\implies \frac{1}{2} \lim_{t\to\infty} \arctan(x/2)\big|_1^t = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \neq \infty \implies \underline{\text{converges}}$$

5.

$$\sum_{n=1}^{\infty} e^{-2n} \xrightarrow{\text{convert to integral}} \int_1^{\infty} e^{-2x}$$

1. continuous for $x \geq 1$
2. positive since $e^{-2x} > 0$ for all x
3. decreasing since $f'(x) = -2 \cdot e^{-2x} < 0$ for all x

$$\implies \int_1^{\infty} e^{-2x} = \lim_{t\to\infty} \int_1^{t} e^{-2x} = \lim_{t\to\infty} \frac{-1}{2} e^{-2x}\Big|_1^t = \frac{1}{2e^2} \neq \infty \implies \underline{\text{converges}}$$

7.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+4} \xrightarrow{\text{convert to integral}} \int_1^{\infty} \frac{x\ dx}{x^2+4}$$

1. continuous for $x \geq 1$
2. positive since $\dfrac{x}{x^2+4} > 0$ for all $x > 0$
3. decreasing since $f'(x) < 0$ for all $x \geq 1$

$$\implies \lim_{t\to\infty} \int_1^{t} \frac{x\ dx}{x^2+4} = \lim_{t\to\infty} \int_1^{t} \frac{du}{2u} = \lim_{t\to\infty} \ln|x^2+4|\Big|_1^t = \infty \implies \underline{\text{diverges}}$$

13.

$$\sum_{n=1}^{\infty} \frac{1}{10^n} = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n \quad \text{geometric series with } a = 1, r = 1/10 \implies \text{converges since } |r| < 1$$

15.

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \quad \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0 \implies \text{does not converge by } n\text{-th term test}$$

17.

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} \text{ greater than } p\text{-series } \frac{1}{n^{1/2}} \text{ with } p = 1/2 < 1 \implies \underline{\text{diverges}}$$

21.

23.

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n} \text{ is geometric series with } a = 1, r = 2/3 \implies \underline{\text{converges}}$$

25.

$$\sum_{n=0}^{\infty} \frac{-2}{n+1} \;=\; -2 \sum_{n=1}^{\infty} \frac{1}{n} \quad p\text{-series with } p = 1 \leq 1 \implies \underline{\text{diverges}}$$

27.

$$\sum_{n=1}^{\infty} \frac{2^n}{n+1} \qquad \lim_{n\to\infty} \frac{2^n}{n+1} \xrightarrow{\text{L'Hopital's}} \lim_{n\to\infty} \frac{2^n \cdot \ln 2}{1} = \infty \neq 0 \quad \implies \text{diverges by } n\text{-th term test}$$