10.3 Integral Test

Main Ideas

- **Integral Test** If a_n is a sequence of positive terms, and $a_n = f(n)$ for all $n \geq N$, where f is a

  1. Continuous
  2. Positive
  3. Decreasing

  function of x, then the series

  $$\sum_{n=N}^{\infty} a_n \qquad \text{and the integral} \qquad \int_N^{\infty} f(x)\, dx$$

  either both converge or both diverge.

- **P-Series**

  $$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \text{converges if and only if} \qquad p > 1$$

Homework Problems

1.

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \xrightarrow{\text{convert to integral}} \int_1^{\infty}\frac{dx}{x^2}$$

$$\begin{gathered}\text{1. continuous for } x \geq 1\\ \text{2. positive since } x^2 \geq 0 \text{ for all } x\\ \text{3. decreasing since } f'(x) = \frac{-2}{x^3}\\ \text{which is always negative when } x \geq 1\end{gathered}$$

$$\Longrightarrow \int_1^{\infty}\frac{dx}{x^2} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^2} = \lim_{t\to\infty}\left.\frac{-1}{x}\right|_1^t$$

$$= \lim_{t\to\infty}\frac{-1}{t} + 1 = 0 + 1 = 1 \neq \infty \Longrightarrow \underline{\text{converges}} \text{ by integral test}$$

3.

$$\sum_{n=1}^{\infty}\frac{1}{n^2+4} \xrightarrow{\text{convert to integral}} \int_1^{\infty}\frac{dx}{x^2+4}$$

$$\begin{gathered}\text{1. continuous for } x \geq 1\\ \text{2. positive since } x^2 \geq 0 \text{ for all } x\\ \text{3. decreasing since } f'(x) = \frac{-1}{(x^2+4)^2}\cdot(2x)\\ \text{which is always negative when } x \geq 1\end{gathered}$$

$$\Longrightarrow \int_1^{\infty}\frac{dx}{x^2+4} = \lim_{t\to\infty}\int_1^t\frac{dx}{x^2+4} = \lim_{t\to\infty}\int_1^t\frac{dx}{\frac{4x^2}{4}+4} = \lim_{t\to\infty}\int_1^t\frac{1}{4}\cdot\frac{dx}{\frac{x^2}{4}+1}$$

$$\text{use } u = x/2, \quad du = dx/2$$

$$\int \frac{1}{2}\cdot\frac{du}{u^2+1} = \frac{1}{2}\arctan(u) = \frac{1}{2}\arctan(x/2)$$

$$\Longrightarrow \frac{1}{2}\lim_{t\to\infty}\arctan(x/2)\big|_1^t = \frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) \neq \infty \Longrightarrow \underline{\text{converges}} \text{ by integral test}$$

5.

$$\sum_{n=1}^{\infty} e^{-2n} \xrightarrow{\text{convert to integral}} \int_1^{\infty} e^{-2x}$$

1. continuous for $x \geq 1$
2. positive since $e^{-2x} > 0$ for all x
3. decreasing since $f'(x) = -2 \cdot e^{-2x} < 0$ for all x

$$\implies \int_1^{\infty} e^{-2x} = \lim_{t\to\infty} \int_1^{t} e^{-2x} = \lim_{t\to\infty} \frac{-1}{2} e^{-2x} \Big|_1^t = \frac{1}{2e^2} \neq \infty \implies \underline{\text{converges}} \text{ by integral test}$$

7.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+4} \xrightarrow{\text{convert to integral}} \int_1^{\infty} \frac{x\ dx}{x^2+4}$$

1. continuous for $x \geq 1$
2. positive since $\dfrac{x}{x^2+4} > 0$ for all $x > 0$
3. decreasing since$f'(x) < 0$ for all $x \geq 1$

$$\implies \lim_{t\to\infty} \int_1^{t} \frac{x\ dx}{x^2+4} = \lim_{t\to\infty} \int_1^{t} \frac{du}{2u} = \lim_{t\to\infty} \ln|x^2+4|\Big|_1^t = \infty \implies \underline{\text{diverges}} \text{ by integral test}$$

13.

$$\sum_{n=1}^{\infty} \frac{1}{10^n} = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n \quad \text{geometric series with } a = 1, r = 1/10 \implies \text{converges since } |r| < 1$$

15.

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \quad \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0 \implies \text{does not converge by } n\text{-th term test}$$

17.

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} \text{ greater than } p\text{-series } \frac{1}{n^{1/2}} \text{ with } p = 1/2 < 1 \implies \underline{\text{diverges}} \text{ by direct comparison}$$

21.

$$\sum_{n=2}^{\infty} \frac{\ln n}{n} \xrightarrow{\text{convert to integral}} \int_2^{\infty} \frac{\ln x}{x} dx$$

- continuous and positive since $\ln x$ and x are both > 0 for $x \geq 2$
- decreasing since x grows faster than $\ln x$

$$\int_2^{\infty} \frac{\ln x}{x} dx = \lim_{t \to \infty} \int_2^{t} \frac{\ln x}{x} dx = \lim_{t \to \infty} (\ln x)^2 \Big|_2^t = \infty \implies \underline{\text{diverges}} \text{ by integral test}$$

23.

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n} \text{ is geometric series with } a = 1, r = 2/3 \implies \underline{\text{converges}}$$

25.

$$\sum_{n=0}^{\infty} \frac{-2}{n+1} = -2 \sum_{n=1}^{\infty} \frac{1}{n} \quad p\text{-series with } p = 1 \leq 1 \implies \underline{\text{diverges}}$$

27.

$$\sum_{n=1}^{\infty} \frac{2^n}{n+1} \qquad \lim_{n \to \infty} \frac{2^n}{n+1} \xrightarrow{\text{L'Hopital's}} \lim_{n \to \infty} \frac{2^n \cdot \ln 2}{1} = \infty \neq 0 \implies \text{diverges by } n\text{-th term test}$$

37.

$$\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}} \xrightarrow{\text{convert to integral}} \int_1^{\infty} \frac{e^x}{1+e^{2x}} dx$$

- continuous and positive since $e^x > 1$ for $x \geq 1$
- decreasing since e^{2x} grows faster than e^x

$$= \lim_{t \to \infty} \int_1^t \frac{e^x}{1+e^{2x}} dx$$

$$u = e^x \quad du = e^x \, dx$$

$$\implies \int \frac{e^x}{1+e^{2x}} dx = \int \frac{du}{1+u^2} = \arctan(u) = \arctan(e^x)$$

$$\implies \lim_{t \to \infty} \arctan(e^x)\Big|_1^t = \frac{\pi}{2} - \frac{\pi}{4} \neq \infty \implies \underline{\text{converges}} \text{ by integral test}$$