

10.3 Integral Test

Main Ideas

- **Integral Test** If a_n is a sequence of positive terms, and $a_n = f(n)$ for all $n \geq N$, where f is a

1. Continuous
2. Positive
3. Decreasing

function of x , then the series

$$\sum_{n=N}^{\infty} a_n \quad \text{and the integral} \quad \int_N^{\infty} f(x) \, dx$$

either both converge or both diverge.

- **P-Series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if and only if} \quad p > 1$$

Homework Problems

1.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \xrightarrow{\text{convert to integral}} \int_1^{\infty} \frac{dx}{x^2}$$

1. continuous for $x \geq 1$
2. positive since $x^2 \geq 0$ for all x
3. decreasing since $f'(x) = \frac{-2}{x^3}$
which is always negative when $x \geq 1$

$$\begin{aligned} \Rightarrow \int_1^{\infty} \frac{dx}{x^2} &= \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x^2} = \lim_{t \rightarrow \infty} \left. \frac{-1}{x} \right|_1^t \\ &= \lim_{t \rightarrow \infty} \frac{-1}{t} + 1 = 0 + 1 = 1 \neq \infty \Rightarrow \underline{\text{converges}} \end{aligned}$$

3.

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4} \xrightarrow{\text{convert to integral}} \int_1^{\infty} \frac{dx}{x^2 + 4}$$

1. continuous for $x \geq 1$
2. positive since $x^2 \geq 0$ for all x
3. decreasing since $f'(x) = \frac{-1}{(x^2 + 4)^2} \cdot (2x)$
which is always negative when $x \geq 1$

$$\Rightarrow \int_1^{\infty} \frac{dx}{x^2 + 4} = \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x^2 + 4} = \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{\frac{4x^2}{4} + 4} = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{4} \cdot \frac{dx}{\frac{x^2}{4} + 1}$$

$$\text{use } u = x/2, \quad du = dx/2$$

$$\int \frac{1}{2} \cdot \frac{du}{u^2 + 1} = \frac{1}{2} \arctan(u) = \frac{1}{2} \arctan(x/2)$$

$$\Rightarrow \frac{1}{2} \lim_{t \rightarrow \infty} \arctan(x/2)|_1^t = \frac{1}{2} \left(\frac{\pi}{2} - \frac{\pi}{6} \right) \neq \infty \Rightarrow \underline{\text{converges}}$$