10.5 Absolute Convergence, Ratio, Root Test

Main Ideas

- **Absolute Convergence**

$$\text{If} \quad \sum_{n=1}^{\infty} |a_n| \quad \text{converges, then} \quad \sum_{n=1}^{\infty} a_n \quad \text{converges}$$

- **Ratio Test**

$$\text{If} \quad \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho \quad \text{then the series} \quad \sum_{n=1}^{\infty} a_n$$

1. **Converges** if $\rho < 1$
2. **Diverges** if $\rho > 1$
3. **Inconclusive** if $\rho = 1$

- **Root Test**

$$\text{If} \quad \lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho \quad \text{then the series} \quad \sum_{n=1}^{\infty} a_n$$

1. **Converges** if $\rho < 1$
2. **Diverges** if $\rho > 1$
3. **Inconclusive** if $\rho = 1$

Homework Problems

1.

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}/(n+1)!}{2^n/n!}\right| = \frac{2 \cdot 2^n/(n+1)\cdot n!}{2^n/n!} = \frac{2}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1 \implies \text{conv}$$

3.

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n!/(n+2)^2}{(n-1)!/(n+1)^2}\right| = \frac{n\cdot(n-1)!/(n+2)^2}{(n-1)!/(n+1)^2} = \frac{n(n+2)^2}{(n+1)^2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n(n+2)^2}{(n+1)^2} = \infty > 1 \implies \text{div}$$

5.

$$\sum_{n=1}^{\infty} \frac{n^4}{(-4)^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^4}{(-4)^{n+1}}\Big/\frac{n^4}{(-4)^n}\right| = \frac{(n+1)^4}{4^{n+1}}\Big/\frac{n^4}{4^n} = \frac{(n+1)^4}{4n^4}$$

$$\lim_{n\to\infty}\frac{(n+1)^4}{4n^4} = \frac{1}{4} < 1 \implies \text{conv}$$

7.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2(n+2)!}{n!\ 3^{2n}} \quad = \quad \sum_{n=1}^{\infty}(-1)^n\frac{n^2(n+2)(n+1)}{3^{2n}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2(n+3)(n+2)}{3^{2(n+1)}} \ \Big/\ \frac{n^2(n+2)(n+1)}{3^{2n}}$$

$$= \frac{(n+1)^2(n+3)(n+2)}{9\cdot 3^{2n}}\cdot\frac{3^{2n}}{n^2(n+2)(n+1)} = \frac{(n+1)(n+3)}{9n^2}$$

$$\lim_{n\to\infty}\frac{(n+1)(n+3)}{9n^2} = \frac{1}{9} < 1 \implies \text{conv}$$

9.

$$\sum_{n=1}^{\infty} \frac{7}{(2n+5)^n} \qquad \lim_{n\to\infty} \sqrt[n]{\frac{7}{(2n+5)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{7}}{2n+5} = 0 < 1 \implies \text{conv}$$

11.

$$\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n \qquad \lim_{n\to\infty} \sqrt[n]{\left(\frac{4n+3}{3n-5)}\right)^n} = \lim_{n\to\infty} \frac{4n+3}{3n-5} = \frac{4}{3} > 1 \implies \text{div}$$

17.

$$\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{\sqrt{2}}}{2^{(n+1)}} / \frac{n^{\sqrt{2}}}{2^n} = \frac{(n+1)^{\sqrt{2}}}{2n^{\sqrt{2}}}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^{\sqrt{2}}}{2n^{\sqrt{2}}} = \frac{1}{2} < 1 \implies \text{conv by ratio test}$$

21.

$$\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^{10}}{10^n}} = \lim_{n\to\infty} \frac{n^{\frac{10}{n}}}{10} = \frac{1}{10} < 1 \implies \text{conv by root test}$$

27.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^3} \qquad \text{compare with } \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{\ln n}{n^3} / \frac{1}{n^2} = \lim_{n\to\infty} \frac{\ln n}{n} \xrightarrow{\text{L'Hopital's}} \lim_{n\to\infty} \frac{1/n}{1} = 0 \implies \text{conv by limit comparison}$$

33.

$$\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+2)(n+3)}{(n+1)!} / \frac{(n+1)(n+2)}{n!}\right| = \frac{(n+2)(n+3)}{(n+1)n!} \cdot \frac{n!}{(n+1)(n+2)} = \frac{(n+3)}{(n+1)^2}$$

$$\lim_{n\to\infty} \frac{(n+3)}{(n+1)^2} = 0 < 1 \implies \text{conv by ratio test}$$