10.3 Integral Test

Main Ideas

- **Integral Test** If a_n is a sequence of positive terms, and $a_n = f(n)$ for all $n \geq N$, where f is a
  1. Continuous
  2. Positive
  3. Decreasing

  function of x, then the series

$$\sum_{n=N}^{\infty} a_n \qquad \text{and the integral} \qquad \int_N^{\infty} f(x)\, dx$$

  either both converge or both diverge.

- **P-Series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \text{converges if and only if} \qquad p > 1$$

Homework Problems

1.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \xrightarrow{\text{convert to integral}} \int_1^{\infty} \frac{dx}{x^2}$$

$$\begin{gathered} 1.\ \text{continuous for } x \geq 1 \\ 2.\ \text{positive since } x^2 \geq 0 \text{ for all } x \\ 3.\ \text{decreasing since } f'(x) = \frac{-2}{x^3} \\ \text{which is always negative when } x \geq 1 \end{gathered}$$

$$\implies \int_1^{\infty} \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = \lim_{t\to\infty} \left.\frac{-1}{x}\right|_1^t$$

$$= \lim_{t\to\infty} \frac{-1}{t} + 1 = 0 + 1 = 1 \neq \infty \implies \underline{\text{converges}}$$

3.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4} \xrightarrow{\text{convert to integral}} \int_1^{\infty} \frac{dx}{x^2+4}$$

$$\begin{gathered} 1.\ \text{continuous for } x \geq 1 \\ 2.\ \text{positive since } x^2 \geq 0 \text{ for all } x \\ 3.\ \text{decreasing since } f'(x) = \frac{-1}{(x^2+4)^2} \cdot (2x) \\ \text{which is always negative when } x \geq 1 \end{gathered}$$

$$\implies \int_1^{\infty} \frac{dx}{x^2+4} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2+4} = \lim_{t\to\infty} \int_1^t \frac{dx}{\frac{4x^2}{4}+4} = \lim_{t\to\infty} \int_1^t \frac{1}{4} \cdot \frac{dx}{\frac{x^2}{4}+1}$$

$$\text{use } u = x/2, \quad du = dx/2$$

$$\int \frac{1}{2} \cdot \frac{du}{u^2+1} = \frac{1}{2} \arctan(u) = \frac{1}{2} \arctan(x/2)$$

$$\implies \frac{1}{2} \lim_{t\to\infty} \arctan(x/2)|_1^t = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \neq \infty \implies \underline{\text{converges}}$$