

10.6 Alternating Series, Conditional Convergence

Main Ideas

- **Alternating Series Test**

the series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$ converges if and only if

1. The u_n terms are all positive
2. The u_n terms are eventually non-increasing ($u_{n+1} \leq u_n$ for all $n \geq N$ for some N)
3. The u_n terms approach 0 ($\lim u_n = 0$ as $n \rightarrow \infty$)

- **Conditional Convergence**

If a series is convergent, but not absolutely convergent, then it is conditionally convergent

if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

Summary of Convergence Tests

1. **N -th term test**
2. **Geometric series**
3. **P -series**
4. **Integral test**
5. **Direct comparison**
6. **Limit comparison**
7. **Root test**
8. **Ratio test**
9. **Absolute convergence**
10. **Alternating series test**

Homework Problems

1.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$$

1. terms are all positive since $n > 0 \implies \sqrt{n} > 0 \implies \frac{1}{\sqrt{n}} > 0$ always

2. terms are decreasing since $d \left[\frac{1}{\sqrt{x}} \right] = \frac{-1}{2x^{3/2}} < 0$ always

$$3. \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

\implies conv by alternating series test

3.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n3^n}$$

1. terms are all positive since $n > 0 \implies n3^n > 0 \implies \frac{1}{n3^n} > 0$ always

2. terms are decreasing since $d \left[\frac{1}{x3^x} \right] = \frac{-1}{x3^x} \cdot (3^x + x3^x \ln 3) < 0$ always

$$3. \lim_{n \rightarrow \infty} \frac{1}{n3^n} = 0$$

\implies conv by alternating series test

5.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + 1}$$

1. u_n terms are all positive since top and bottom always positive when $n \geq 1$

2. terms are decreasing since $d \left[\frac{x}{x^2 + 1} \right] = \frac{(x^2 + 1) - x \cdot 2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} < 0$ when $x \geq 1$

$$3. \lim_{n \rightarrow \infty} \frac{n}{n^2 + 1} = 0$$

\implies conv by alternating series test

7.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n^2}$$

3. does not meet condition because

$$\lim_{n \rightarrow \infty} \frac{2^n}{n^2} \xrightarrow{\text{L'Hopital's}} \lim_{n \rightarrow \infty} \frac{2^n \cdot \ln 2}{2n} \xrightarrow{\text{L'Hopital's}} \lim_{n \rightarrow \infty} \frac{2^n \cdot \ln 2 \cdot \ln 2}{2} = \infty \neq 0$$

\implies div by n -th term test

11.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$$

1. u_n terms are all positive since top and bottom always positive when $n > 1$

2. terms are decreasing since $d \left[\frac{\ln x}{x^2} \right] = \frac{x^{-1} \cdot x^2 - \ln x \cdot 2x}{(x^2)^2} = \frac{x(1 - 2 \ln x)}{(x^2)^2} > 0$ when $x \geq 1$

$$3. \lim_{n \rightarrow \infty} \frac{\ln n}{n^2} \xrightarrow{\text{L'Hopital's}} \frac{1/n}{2n} = \frac{1}{2n^2} = 0$$

\implies conv by alternating series test

15.

$$\sum_{n=1}^{\infty} (-1)^{n+1} (0.1)^n$$

$$\sum_{n=1}^{\infty} |(-1)^{n+1} (0.1)^n| = \sum_{n=1}^{\infty} (0.1)^n$$

conv absolutely since geometric series with $r = 0.1 \implies |r| < 1$

17.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

not abs conv since p -series div with $p = 1/2 < 1$

alternating series test:

1. terms are all positive since $\sqrt{n} > 0$ always
2. terms are decreasing since $d \left[\frac{1}{\sqrt{x}} \right] = \frac{-1}{2x^{3/2}} < 0$ always when $x \geq 1$
3. $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$

\Rightarrow converges by alternating series test \Rightarrow conditionally conv

19.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3 + 1}$$

compare with $b_n = \frac{1}{n^2}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{b_n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n^3 + 1} \div \frac{1}{n^2} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3 + 1} = 1$$

\Rightarrow both converge or both diverge by limit comparison
 \Rightarrow conv since $1/n^2$ is p -series with $p = 2 > 1 \Rightarrow$ abs conv

21.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+3}$$

not abs conv since p -series with $p = 1$

1. all terms are positive since $n+3 > 0$ always when $n \geq 1$
2. terms are decreasing since $d \left[\frac{1}{x+3} \right] = \frac{-1}{(x+3)^2} < 0$ always since $(x+3)^2 > 0$ always when $x \geq 1$
3. $\lim_{n \rightarrow \infty} \frac{1}{n+3} = 0$

\Rightarrow conv by alternating series test \Rightarrow conditionally convergent

23.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3+n}{5+n}$$

$$\lim_{n \rightarrow \infty} \frac{3+n}{5+n} = 1 \neq 0 \implies \text{div by } n\text{-th term test}$$

31.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0 \implies \text{div by } n\text{-th term test}$$