10.6 Alternating Series, Conditional Convergence

Main Ideas

- **Alternating Series Test**

$$\text{the series} \qquad \sum_{n=1}^{\infty}(-1)^{n+1}\, u_n = u_1 - u_2 + u_3 - u_4 + \dots \qquad \text{converges if and only if}$$

1. The u_n terms are all positive
2. The u_n terms are eventually non-increasing ($u_{n+1} \leq u_n$ for all $n \geq N$ for some N)
3. The u_n terms approach 0 ($\lim u_n = 0$ as $n \to \infty$)

- **Conditional Convergence**

If a series is convergent, but not absolutely convergent, then it is conditionally convergent

$$\text{if} \quad \sum_{n=1}^{\infty} |a_n| \quad \text{converges, then} \quad \sum_{n=1}^{\infty} a_n \quad \text{is absolutely convergent}$$

Summary of Convergence Tests

1. N**-th term test**
2. **Geometric series**
3. P**-series**
4. **Integral test**
5. **Direct comparison**
6. **Limit comparison**
7. **Root test**
8. **Ratio test**
9. **Absolute convergence**
10. **Alternating series test**

Homework Problems

1.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$$

1. terms are all positive since $n > 0 \implies \sqrt{n} > 0 \implies \dfrac{1}{\sqrt{n}} > 0$ always
2. terms are decreasing since $d\left[\dfrac{1}{\sqrt{x}}\right] = \dfrac{-1}{2x^{3/2}} < 0$ always
3. $\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{n}} = 0$

$\implies$ conv by alternating series test

3.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n3^n}$$

1. terms are all positive since $n > 0 \implies n3^n > 0 \implies \dfrac{1}{n3^n} > 0$ always
2. terms are decreasing since $d\left[\dfrac{1}{x3^x}\right] = \dfrac{-1}{(x3^x)^2}\cdot(3^x + x3^x\ln 3) < 0$ always
3. $\lim\limits_{n\to\infty}\dfrac{1}{n3^n} = 0$

$\implies$ conv by alternating series test

5.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$$

1. u_n terms are all positive since top and bottom always positive when $n \geq 1$
2. terms are decreasing since $d\left[\dfrac{x}{x^2+1}\right] = \dfrac{(x^2+1) - x\cdot 2x}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} < 0$ when $x > 1$
3. $\lim\limits_{n\to\infty}\dfrac{n}{n^2+1} = 0$

$\implies$ conv by alternating series test

7.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$$

3. does not meet condition because

$$\lim_{n\to\infty}\frac{2^n}{n^2} \xrightarrow{\text{L'Hopital's}} \lim_{n\to\infty}\frac{2^n\cdot\ln 2}{2n} \xrightarrow{\text{L'Hopital's}} \lim_{n\to\infty}\frac{2^n\cdot\ln 2\cdot\ln 2}{2} = \infty \neq 0$$

$\Longrightarrow$ div by n-th term test

11.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$$

1. u_n terms are all positive since top and bottom always positive when $n > 1$
2. terms are decreasing since $d\left[\frac{\ln x}{x^2}\right] = \frac{x^{-1}\cdot x^2 - \ln x\cdot 2x}{(x^2)^2} = \frac{x(1-2\ln x)}{(x^2)^2} > 0$ when $x \geq 1$
3. $\lim_{n\to\infty}\frac{\ln n}{n^2} \xrightarrow{\text{L'Hopital's}} \frac{1/n}{2n} = \frac{1}{2n^2} = 0$

$\Longrightarrow$ conv by alternating series test

15.

$$\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$$

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}(0.1)^n\right| = \sum_{n=1}^{\infty}(0.1)^n$$

conv absolutely since geometric series with $r = 0.1 \implies |r| < 1$

17.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$$

$$|a_n| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

not abs conv since p-series div with $p = 1/2 < 1$

alternating series test:

1. terms are all positive since $\sqrt{n} > 0$ always
2. terms are decreasing since $d\left[\frac{1}{\sqrt{x}}\right] = \frac{-1}{2x^{3/2}} < 0$ always when $x \geq 1$
3. $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

$\implies$ converges by alternating series test $\implies$ conditionally conv

19.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$$

$$|a_n| = \frac{n}{n^3+1} \quad \text{compare with } b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{n^3+1} \div \frac{1}{n^2} = \lim_{n\to\infty}\frac{n^3}{n^3+1} = 1$$

$\implies$ both converge or both diverge by limit comparison

$\implies$ conv since $1/n^2$ is p-series with $p = 2 > 1$ $\implies$ abs conv

21.

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n+3}$$

$$|a_n| = \frac{1}{n+3} \quad \text{compare to} \quad \frac{1}{n} = b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{n+3} \div \frac{1}{n} = \lim_{n\to\infty}\frac{n}{n+3} = 1$$

$$\implies \text{ both do the same thing } \implies \text{ not abs conv since } p\text{-series with } p = 1$$

try alternating series test:

1. all terms are positive since $n + 3 > 0$ always when $n \geq 1$

2. terms are decreasing since $d\left[\frac{1}{x+3}\right] = \frac{-1}{(x+3)^2} < 0$ always since $(x+3)^2 > 0$ always when $x \geq 1$

3. $\lim_{n\to\infty}\frac{1}{n+3} = 0$

$$\implies \text{ conv by alternating series test } \implies \text{ conditionally convergent}$$

23.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$$

$$\lim_{n\to\infty}\frac{3+n}{5+n} = 1 \neq 0 \implies \text{ div by } n\text{-th term test}$$

31.

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$$

$$\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0 \implies \text{ div by } n\text{-th term test}$$