Numerical Methods for PDEs (Spring 2017)

Problems 3

Hand in your written solution to problem 11 at the start of the lecture on 7 March. Show me your code for that problem in the practical on 10 March.

Problem 9. By expanding $g(x \pm h)$, $Q(x \pm h)$ in Taylor's series at x, show that

$$\frac{d}{dx}\left(Q(x)\frac{dg}{\partial x}\right) = \frac{1}{h^2}\left(Q_+\left[g(x+h) - g(x)\right] - Q_-\left[g(x) - g(x-h)\right]\right) + O(h^2),$$

where

$$Q_\pm = \frac{1}{2}\left[Q(x) + Q(x \pm h)\right].$$

Problem 10. Consider the two-dimensional heat equation

$$\frac{\partial u}{\partial t} - K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \quad \text{for} \quad 0 < x < L_1, \quad y < x < L_2, \quad t > 0, \tag{1}$$

subject to the boundary conditions

$$u(0, y, t) = 0, \quad u(L_1, y, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, L_2, t) = 0,$$

and the initial condition

$$u(x, y, 0) = u_0(x, y).$$

At interior grid points (x_x, y_j, t_n), equation (1) is approximated by the finite-difference scheme

$$\frac{w_{kj}^n - w_{kj}^{n-1}}{\tau} - K\left(\frac{\delta_x^2}{h_1^2} + \frac{\delta_y^2}{h_2^2}\right) w_{kj}^n = 0,$$

where w_{kj}^n are approximations to $u(x_x, y_j, t_n)$; $x_k = kh_1$ for $k = 0, 1, \ldots N_1$, $h_1 = L_1/N_1$; $y_j = jh_2$ for $j = 0, 1, \ldots N_2$, $h_2 = L_2/N_2$; $t_n = n\tau$ for $n = 0, 1, \ldots$ and τ in the length of the time step.

Investigate the stability of this scheme by the Fourier method.

Problem 11. The nonlinear heat equation

$$u_t - Ku_{xx} = f(u) \quad \text{for} \quad 0 < x < L, \quad 0 < t < T$$

(where K is a constant and $f(u)$ is a given function), subject to the initial and boundary conditions

$$u(x, 0) = u_0(x) \ \text{ for } 0 < x < L, \quad u(0, t) = u(L, t) = 0 \ \text{ for } 0 < t < T,$$

is solved using the finite-difference method:

$$\frac{w_{k,j} - w_{k,j-1}}{\tau} - K\frac{w_{k+1,j} - 2w_{k,j} + w_{k-1,j}}{h^2} = f(w_{kj}) \ \text{ for } \ k = 1, 2, \ldots, N-1 \text{ and } j = 1, 2, \ldots, M; \tag{2}$$

$$w_{k,0} = u_0(x_k) \ \text{ for } k = 0, \ldots, N \ \text{ and } \ w_{0,j} = w_{N,j} = 0 \ \text{ for } j = 1, \ldots, M. \tag{3}$$

Obtain the computation formulae for solving the nonlinear equations (2) by the Newton method and implement the solution method in R.