Numerical Methods for PDEs (Spring 2017)

Problems 4

Hand in your written solution to problem 14 at the start of the lecture on 14 March.

Problem 12. Consider the following equation

$$u_{xx} + (x^2 + y^2)u_{yy} + 2u_x = f(x, y) \quad \text{for} \quad 0 < x < 1, \quad 0 < y < 1, \tag{1}$$

subject to the boundary conditions

$$u(0, y) = u(1, y) = 0, \quad u(x, 0) = u(x, 1) = 0. \tag{2}$$

(a) Obtain a finite-difference approximation to this boundary-value problem and show that your finite-difference method is consistent with the equation, i.e. that the local truncation errors tend to zero as step sizes in x and in y go to zero.

(b) Modify your finite-difference method for the case of the following boundary conditions:

$$u_x(0, y) = u_x(1, y) = 0, \quad u(x, 0) = u(x, 1) = 0. \tag{3}$$

Problem 13. Consider the boundary value problem

$$\begin{aligned}
&u_{xx} + u_{yy} = 0 \quad \text{for} \quad 0 < x < 1, \quad 0 < y < 1; \\
&u(0, y) = y, \quad u(1, y) = 1 - y \quad \text{for} \quad 0 < y < 1; \\
&u(x, 0) = x, \quad u(x, 1) = 1 - x \quad \text{for} \quad 0 < x < 1.
\end{aligned} \tag{4}$$

For the grid $(x_k, y_j) = (kh, jh)$ for $k, j = 0, 1, 2, 3$, with $h = 1/3$, write down the system of linear equations for $w_{1,1}$, $w_{1,2}$, $w_{2,1}$, $w_{2,2}$ [where $w_{k,j} \approx u(x_k, t_j)$], solve it and compare the result with the exact solution given by

$$u = -2xy + x + y.$$

Problem 14. Consider the following elliptic equation

$$u_{xx} + u_{xy} + u_{yy} = f(x, y) \quad \text{for} \quad 0 < x < 1, \quad 0 < y < 1, \tag{5}$$

subject to the boundary conditions

$$u(0, y) = u(1, y) = 0, \quad u(x, 0) = u(x, 1) = 0. \tag{6}$$

Consider the square grid with step size $h = 1/N$ in both x and y directions and obtain a finite-difference approximation to this boundary-value problem with truncation error $O(h^2)$ (calculate the truncation error of your scheme explicitly using Taylor expansions).