Numerical Methods for PDEs (Spring 2017)

Solutions 3

Problem 9. By expanding $g(x \pm h)$, $Q(x \pm h)$ in Taylor's series at x, show that

$$\frac{d}{dx}\left(Q(x)\frac{dg}{\partial x}\right) = \frac{1}{h^2}\left(Q_+\left[g(x+h)-g(x)\right] - Q_-\left[g(x)-g(x-h)\right]\right) + O(h^2),$$

where

$$Q_\pm = \frac{1}{2}\left[Q(x) + Q(x \pm h)\right].$$

Solution. We have

$$g(x \pm h) = g(x) \pm hg'(x) + \frac{h^2}{2}g''(x) \pm \frac{h^3}{6}g'''(x) + O(h^4).$$

Hence,

$$\begin{aligned} g(x+h) - g(x) &= hg'(x) + \frac{h^2}{2}g''(x) + \frac{h^3}{6}g'''(x) + O(h^4), \\ g(x) - g(x-h) &= hg'(x) - \frac{h^2}{2}g''(x) + \frac{h^3}{6}g'''(x) + O(h^4), \end{aligned} \tag{1}$$

Also, since

$$Q(x \pm h) = Q(x) \pm hQ'(x) + \frac{h^2}{2}Q''(x) \pm \frac{h^3}{6}Q'''(x) + O(h^4),$$

we obtain

$$\begin{aligned} Q_+ &= Q(x) + \frac{h}{2}Q'(x) + \frac{h^2}{4}Q''(x) + \frac{h^3}{12}Q'''(x) + O(h^4), \\ Q_- &= Q(x) - \frac{h}{2}Q'(x) + \frac{h^2}{4}Q''(x) - \frac{h^3}{12}Q'''(x) + O(h^4). \end{aligned} \tag{2}$$

Let

$$X = \frac{d}{dx}\left(Q(x)\frac{dg}{\partial x}\right) - \frac{1}{h^2}\left(Q_+\left[g(x+h)-g(x)\right] - Q_-\left[g(x)-g(x-h)\right]\right). \tag{3}$$

We need to show that $X = O(h^2)$. Substitution of (1) and (2) into (3) yields

$$\begin{aligned} X &= \frac{d}{dx}\left(Q(x)\frac{dg}{\partial x}\right) \\ &\quad -\frac{1}{h^2}\Bigg\{\left(Q(x) + \frac{h}{2}Q'(x) + \frac{h^2}{4}Q''(x) + \frac{h^3}{12}Q'''(x) + O(h^4)\right)\left(hg'(x) + \frac{h^2}{2}g''(x) + \frac{h^3}{6}g'''(x) + O(h^4)\right) \\ &\quad -\left(Q(x) - \frac{h}{2}Q'(x) + \frac{h^2}{4}Q''(x) - \frac{h^3}{12}Q'''(x) + O(h^4)\right)\left(hg'(x) - \frac{h^2}{2}g''(x) + \frac{h^3}{6}g'''(x) + O(h^4)\right)\Bigg\} \\ &= \frac{d}{dx}\left(Q(x)\frac{dg}{\partial x}\right) - \frac{1}{h^2}\Bigg\{hQg' + \frac{h^2}{2}\left(Qg'' + Q'g'\right) + h^3\left(\frac{1}{6}Qg''' + \frac{1}{4}Q'g'' + \frac{1}{4}Q''g'\right) \\ &\quad -\left(hQg' - \frac{h^2}{2}\left(Qg'' + Q'g'\right) + h^3\left(\frac{1}{6}Qg''' + \frac{1}{4}Q'g'' + \frac{1}{4}Q''g'\right)\right) + O(h^4)\Bigg\} \\ &= \frac{d}{dx}\left(Q(x)\frac{dg}{\partial x}\right) - \left(Qg'' + Q'g'\right) + O(h^2) = O(h^2). \end{aligned} \tag{4}$$

Problem 10. Consider the two-dimensional heat equation

$$\frac{\partial u}{\partial t} - K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \quad \text{for} \quad 0 < x < L_1, \quad y < x < L_2, \quad t > 0, \tag{5}$$

subject to the boundary conditions

$$u(0,y,t)=0, \quad u(L_1,y,t)=0, \quad u(x,0,t)=0, \quad u(x,L_2,t)=0,$$

and the initial condition

$$u(x,y,0)=u_0(x,y).$$

At interior grid points (x_x, y_j, t_n), equation (5) is approximated by the finite-difference scheme

$$\frac{w_{kj}^n - w_{kj}^{n-1}}{\tau} - K\left(\frac{\delta_x^2}{h_1^2} + \frac{\delta_y^2}{h_2^2}\right) w_{kj}^n = 0,$$

where w_{kj}^n are approximations to $u(x_x, y_j, t_n)$; $x_k = kh_1$ for $k = 0,1,\dots N_1$, $h_1 = L_1/N_1$; $y_j = jh_2$ for $j = 0,1,\dots N_2$, $h_2 = L_2/N_2$; $t_n = n\tau$ for $n = 0,1,\dots$ and τ in the length of the time step.

Investigate the stability of this scheme by the Fourier method.

Solution. The perturbation z_{kj} satisfies the difference equation

$$\frac{z_{kj}^n - z_{kj}^{n-1}}{\tau} - K\left(\frac{\delta_x^2}{h_1^2} + \frac{\delta_y^2}{h_2^2}\right) z_{kj}^n = 0. \tag{6}$$

We seek a particular solution of (6) in the form

$$z_{k,j}^n = \rho^n e^{iqx_k + ipy_j}. \tag{7}$$

for $q, p \in \mathbb{R}$ and $n = 0,1,\dots$ The finite-difference method is stable if

$$|\rho| \le 1 \quad \text{for all} \quad q,p \in \mathbb{R}.$$

Substituting (7) in (6), we obtain

$$e^{iqx_k+ipy_j}\left(\rho^n - \rho^{n-1}\right) - \gamma_1 \rho^n e^{ipy_j}\left(e^{iqx_{k+1}} - 2e^{iqx_k} + e^{iqx_{k-1}}\right) - \gamma_2 \rho^n e^{iqx_k}\left(e^{ipy_{j+1}} - 2e^{ipy_j} + e^{ipy_{j-1}}\right) = 0$$

or, equivalently,

$$1 - \frac{1}{\rho} - \gamma_1\left(e^{iqh_1} - 2 + e^{-iqh_1}\right) - \gamma_2\left(e^{iph_2} - 2 + e^{-iph_2}\right) = 0,$$

where

$$\gamma_1 = \frac{K\tau}{h_1^2}, \quad \gamma_2 = \frac{K\tau}{h_2^2}.$$

Since

$$e^{iqh_1} - 2 + e^{-iqh_1} = -4\sin^2\frac{qh_1}{2}, \quad e^{iph_2} - 2 + e^{-iph_2} = -4\sin^2\frac{ph_2}{2},$$

we obtain

$$\rho = \frac{1}{1 + 4\gamma_1 \sin^2\frac{qh_1}{2} + 4\gamma_2\sin^2\frac{ph_2}{2}}.$$

Evidently, $0 < \rho \le 1$ for all q and p. Therefore, the scheme is unconditionally stable.

Problem 11. The nonlinear heat equation

$$u_t - Ku_{xx} = f(u) \quad \text{for} \quad 0 < x < L, \ \ 0 < t < T$$

(where K is a constant and $f(u)$ is a given function), subject to the initial and boundary conditions

$$u(x,0) = u_0(x) \ \text{ for } 0 < x < L, \quad u(0,t) = u(L,t) = 0 \ \text{ for } 0 < t < T,$$

is solved using the finite-difference method:

$$\frac{w_{k,j} - w_{k,j-1}}{\tau} - K\frac{w_{k+1,j} - 2w_{k,j} + w_{k-1,j}}{h^2} = f(w_{kj}) \ \text{ for } \ k = 1,2,\dots,N-1 \text{ and } j = 1,2,\dots,M; \tag{8}$$

$$w_{k,0} = u_0(x_k) \ \text{ for } k = 0,\dots,N \ \text{ and } \ w_{0,j} = w_{N,j} = 0 \ \text{ for } j = 1,\dots,M. \tag{9}$$

Obtain the computation formulae for solving the nonlinear equations (8) by the Newton method and implement the solution in R.

Solution. It is convenient to rewrite Eq. (8) in the form

$$-\gamma w_{k-1,j} + (1+2\gamma)w_{k,j} - \gamma w_{k+1,j} - \tau f(w_{kj}) - w_{k,j-1} = 0 \tag{10}$$

where $\gamma = K\tau/h^2$. In vector form, this can be written as

$$\mathbf{\Phi}(\mathbf{w}_j) \equiv A\mathbf{w}_j - \tau\mathbf{F}(\mathbf{w}_j) - \mathbf{w}_{j-1} = \mathbf{0} \tag{11}$$

where

$$\mathbf{w}_j = \begin{bmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ \vdots \\ \vdots \\ w_{N-1,j} \end{bmatrix}, \quad \mathbf{F}(\mathbf{w}_j) = \begin{bmatrix} f(w_{1,j}) \\ f(w_{2,j}) \\ \vdots \\ \vdots \\ \vdots \\ f(w_{N-1,j}) \end{bmatrix}, \quad A = \begin{bmatrix} 1+2\gamma & -\gamma & 0 & \dots & \dots & 0 \\ -\gamma & 1+2\gamma & -\gamma & \ddots & & \vdots \\ 0 & -\gamma & 1+2\gamma & -\gamma & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -\gamma \\ 0 & \dots & \dots & 0 & -\gamma & 1+2\gamma \end{bmatrix}.$$

Equation (10) is equivalent to

$$\Phi_i(\mathbf{w}_j) \equiv \sum_{m=1}^{N-1} a_{im} w_{m,j} - \tau f(w_{i,j}) - w_{i,j-1} = 0, \quad i = 1, \dots, N-1 \tag{12}$$

where a_{im} are the entries of matrix A. Hence,

$$J_{ik}(\mathbf{w}_j) = \frac{\partial \Phi_i}{\partial w_{k,j}} = a_{ik} - \tau f'(w_{k,j})\delta_{ik} \tag{13}$$

where the J_{ik} are the entries of the Jacobian matrix J (see the lecture notes) and δ_{ik} is the Kronecker delta ($\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$ for $i \neq k$). In Newton's method, we compute a sequence $\{\mathbf{w}_j^{(s)}\}$ ($s = 0, 1, \dots$) using the formula

$$\mathbf{w}_j^{(s)} = \mathbf{w}_j^{(s-1)} + \mathbf{r}^{(s)}, \tag{14}$$

where $\mathbf{r}^{(s)}$ is the solution of the linear system

$$J\left(\mathbf{w}_j{}^{(s-1)}\right)\mathbf{r}^{(s)} = -\mathbf{\Phi}\left(\mathbf{w}_j{}^{(s-1)}\right) \tag{15}$$

The initial approximation $\mathbf{w}_j^0$ is chosen as

$$\mathbf{w}_j^0 = \mathbf{w}_{j-1}.$$

Equivalently, equation (15) can be written in the form (cf. Eq. (2.113) in the lecture notes):

$$\begin{aligned} &-\gamma r_{k-1,j}^{(s)} - \gamma r_{k+1,j}^{(s)} + \left(1 + 2\gamma - \tau f'\left(w_{kj}^{(s-1)}\right)\right) r_{k,j}^{(s)} = \\ &= -(1+2\gamma)w_{kj}^{(s-1)} + \gamma\left(w_{k+1,j}^{(s-1)} + w_{k-1,j}^{(s-1)}\right) + \tau f\left(w_{kj}^{(s-1)}\right) + w_{k,j-1} \end{aligned} \tag{16}$$

This can be solved using the double sweep method.