Numerical Methods for PDEs (Spring 2017)

Solutions 2

Consider the heat equation

$$u_t - K u_{xx} = 0 \quad \text{for} \quad 0 < x < 1, \quad t > 0, \tag{1}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \tag{2}$$

and the initial condition

$$u(x,0) = u_0(x). \tag{3}$$

Problem 4. Show that the Du Fort - Frankel method for Eq. (1), given by

$$\frac{w_{k,j+1} - w_{k,j-1}}{2\tau} - K\frac{w_{k+1,j} - w_{k,j-1} - w_{k,j+1} + w_{k-1,j}}{h^2} = 0,$$

has the local truncation error $O\left(\tau^2 + h^2 + \tau^2/h^2\right)$.

Solution. The local truncation error is given by

$$\tau_{kj} = \frac{u_{k,j+1} - u_{k,j-1}}{2\tau} - K\frac{u_{k+1,j} - u_{k,j-1} - u_{k,j+1} + u_{k-1,j}}{h^2} \tag{4}$$

where $u_{kj} = u(x_k, t_j)$. Assuming that the solution $u(x,t)$ is smooth enough, we expand $u_{k\pm1,j}$ and $u_{k,j\pm1}$ in Taylor's series at point (x_k, t_j):

$$\begin{aligned}
u_{k\pm1,j} = u(x_{k\pm1}, t_j) = u(x_k,t_j) \pm h\frac{\partial u}{\partial x}(x_k,t_j) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_k,t_j) \pm \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x_k,t_j) + O(h^4),\\
u_{k,j\pm1} = u(x_k, t_{j\pm1}) = u(x_k,t_j) \pm \tau\frac{\partial u}{\partial t}(x_k,t_j) + \frac{\tau^2}{2}\frac{\partial^2 u}{\partial t^2}(x_k,t_j) \pm \frac{\tau^3}{6}\frac{\partial^3 u}{\partial t^3}(x_k,t_j) + O(\tau^4).
\end{aligned}$$

It follows that

$$\begin{aligned}
u_{k+1,j} + u_{k-1,j} &= 2u(x_k,t_j) + h^2\frac{\partial^2 u}{\partial x^2}(x_k,t_j) + O(h^4),\\
u_{k,j+1} - u_{k,j-1} &= 2\tau\frac{\partial u}{\partial t}(x_k,t_j) + \frac{\tau^3}{3}\frac{\partial^3 u}{\partial t^3}(x_k,t_j) + O(\tau^5),\\
u_{k,j+1} + u_{k,j-1} &= 2u(x_k,t_j) + \tau^2\frac{\partial^2 u}{\partial t^2}(x_k,t_j) + O(\tau^4).
\end{aligned}$$

Substituting these in (1), we find that

$$\begin{aligned}
\tau_{kj} &= \frac{\partial u}{\partial t}(x_k,t_j) + \frac{\tau^2}{6}\frac{\partial^3 u}{\partial t^3}(x_k,t_j) + O(\tau^4)\\
&\quad -\frac{K}{h^2}\left[2u(x_k,t_j) + h^2\frac{\partial^2 u}{\partial x^2}(x_k,t_j) + O(h^4) - 2u(x_k,t_j) - \tau^2\frac{\partial^2 u}{\partial t^2}(x_k,t_j) + O(\tau^4)\right],\\
&= \frac{\partial u}{\partial t}(x_k,t_j) - K\frac{\partial^2 u}{\partial x^2}(x_k,t_j) + \frac{\tau^2}{6}\frac{\partial^3 u}{\partial t^3}(x_k,t_j) + O(\tau^4) + K\frac{\tau^2}{h^2}\frac{\partial^2 u}{\partial t^2}(x_k,t_j) + O(h^2) + O(\tau^4) + O(\tau^4/h^2),\\
&= O(\tau^2 + h^2 + \tau^2/h^2).
\end{aligned}$$

Problem 5. The initial boundary value problem (1)–(3) is solved numerically using the finite-difference method:

$$\begin{aligned}
&w_{k0} = u_0(x_k), \quad w_{0j} = 0, \quad w_{Nj} = 0,\\
&\frac{w_{k,j+1} - w_{k,j}}{\tau} - K(1-\sigma)\frac{w_{k+1,j} - 2w_{kj} + w_{k-1,j}}{h^2} - K\sigma\frac{w_{k+1,j+1} - 2w_{k,j+1} + w_{k-1,j+1}}{h^2} = 0,
\end{aligned} \tag{5}$$

for $k = 1, 2, \ldots, N-1$ and $j = 0, 1, \ldots$. Here w_{kj} is an approximation to $u(x_k, y_j)$ and $x_k = kh$ $(k = 0, 1, \ldots, N)$, $t_j = j\tau$ $(j = 0, 1, \ldots)$, $h = \frac{1}{N}$. In Eqs. (5), σ is a real parameter such that $0 \le \sigma \le 1$. Show that the method is stable if

$$\sigma \ge \frac{1}{2}\left(1 - \frac{1}{2\gamma}\right)$$

where $\gamma = K\tau/h^2$.

Solution. The perturbation z_{kj} satisfies the difference equation

$$\frac{z_{k,j+1} - z_{k,j}}{\tau} - K(1-\sigma)\frac{z_{k+1,j} - 2z_{kj} + z_{k-1,j}}{h^2} - K\sigma\frac{z_{k+1,j+1} - 2z_{k,j+1} + z_{k-1,j+1}}{h^2} = 0 \tag{6}$$

for $k = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots$. We will seek a particular solution of (6) in the form

$$z_{k,j} = \rho_q^j e^{iqx_k}, \quad q \in \mathbb{R}. \tag{7}$$

The finite-difference method (5) is stable with respect to initial condition, if $|\rho_q| \le 1$ for all $q \in \mathbb{R}$. Substitution of (7) into (5) yields

$$\frac{e^{iqx_k}}{\tau}\left(\rho_q^{j+1} - \rho^j\right) - K(1-\sigma)\frac{\rho_q^j}{h^2}\left(e^{iqx_{k+1}} - 2e^{iqx_k} + e^{iqx_{k-1}}\right) - K\sigma\frac{\rho_q^{j+1}}{h^2}\left(e^{iqx_{k+1}} - 2e^{iqx_k} + e^{iqx_{k-1}}\right) = 0$$

or

$$\rho_q - 1 - \gamma(1-\sigma)\left(e^{iqh} - 2 + e^{-iqh}\right) - \rho_q\gamma\sigma\left(e^{iqh} - 2 + e^{-iqh}\right) = 0.$$

Since

$$e^{iqh} - 2 + e^{-iqh} = \left(e^{iqh/2} - e^{-iqh/2}\right)^2 = -4\sin^2\frac{qh}{2},$$

we obtain

$$\rho_q = \frac{1 - 4\gamma(1-\sigma)\sin^2\frac{qh}{2}}{1 + 4\gamma\sigma\sin^2\frac{qh}{2}}.$$

Further, we have

$$|\rho_q| \le 1 \quad \Rightarrow \quad -1 \le \rho_q \quad \Rightarrow \quad 4\gamma(1-2\sigma)\sin^2\frac{qh}{2} \le 2.$$

The last inequality holds for all q, provided that

$$4\gamma(1-2\sigma) \le 2 \quad \text{or} \quad \sigma \ge \frac{1}{2}\left(1 - \frac{1}{2\gamma}\right),$$

which is the required stability condition.

Problem 6. Show that if

$$\gamma \equiv \frac{K\tau}{h^2} = \frac{1}{6}$$

in the explicit forward-difference method for Eq. (1):

$$\frac{w_{k,j+1} - w_{kj}}{\tau} - K\frac{w_{k+1,j} - 2w_{kj} + w_{k-1,j}}{h^2} = 0,$$

then the local truncation error is $O(\tau^2)$ or, equivalently $O(h^4)$.

Solution. The local truncation error is given by

$$\tau_{kj} = \frac{u_{k,j+1} - u_{kj}}{\tau} - K\frac{u_{k+1,j} - 2u_{kj} + u_{k-1,j}}{h^2} \tag{8}$$

where $u_{kj} = u(x_k, t_j)$. Assuming that the solution $u(x,t)$ is smooth enough, we expand $u_{k\pm1,j}$ and $u_{k,j+1}$ in Taylor's series at point (x_k, t_j):

$$u_{k\pm1,j} = u(x_{k\pm1}, t_j) = u(x_k, t_j) \pm h\frac{\partial u}{\partial x}(x_k, t_j) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_k, t_j) \pm \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x_k, t_j) + \frac{h^4}{24}\frac{\partial^2 u}{\partial x^4}(x_k, t_j) + O(h^5),$$

$$u_{k,j+1} = u(x_k, t_{j+1}) = u(x_k, t_j) + \tau\frac{\partial u}{\partial t}(x_k, t_j) + \frac{\tau^2}{2}\frac{\partial^2 u}{\partial t^2}(x_k, t_j) + \frac{\tau^3}{6}\frac{\partial^3 u}{\partial t^3}(x_k, t_j) + O(\tau^4).$$

It follows that

$$\frac{u_{k,j+1}-u_{kj}}{\tau} = \frac{\partial u}{\partial t}(x_k,t_j) + \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x_k,t_j) + O(\tau^2),$$
$$\frac{u_{k+1,j}-2u_{kj}+u_{k-1,j}}{h^2} = \frac{\partial^2 u}{\partial x^2}(x_k,t_j) + \frac{h^2}{12}\frac{\partial^2 u}{\partial x^4}(x_k,t_j) + O(h^4).$$

Hence,

$$\begin{aligned}\tau_{kj} &= \frac{\partial u}{\partial t}(x_k,t_j) - K\frac{\partial^2 u}{\partial x^2}(x_k,t_j) + \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x_k,t_j) - K\frac{h^2}{12}\frac{\partial^2 u}{\partial x^4}(x_k,t_j) + O(\tau^2) + O(h^4)\\ &= \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x_k,t_j) - K\frac{h^2}{12}\frac{\partial^2 u}{\partial x^4}(x_k,t_j) + O(\tau^2) + O(h^4). \qquad (9)\end{aligned}$$

From Eq. (1), we have

$$\frac{\partial^2 u}{\partial t^2} = K\frac{\partial^2}{\partial x^2}\frac{\partial u}{\partial t} = K^2\frac{\partial^4 u}{\partial x^4}.$$

Therefore, we can rewrite the above formula for τ_{kj} in the form

$$\tau_{kj} = K\frac{h^2}{2}\left(\frac{\tau K}{h^2} - \frac{1}{6}\right)\frac{\partial^2 u}{\partial x^4}(x_k,t_j) + O(\tau^2) + O(h^4).$$

It follows that if

$$\frac{K\tau}{h^2} = \frac{1}{6},$$

then $\tau_{kj} = O(\tau^2) + O(h^4)$ or, in view of the last formula,

$$\tau_{kj} = O(\tau^2) = O(h^4).$$

Problem 7. Devise a backward difference scheme of $O(\tau+h^2)$ for the non-homogeneous heat equation eq.(2.73) in the notes with boundary conditions as given in eq.(2.80). This means you need to derive an expression for $w_{0,j+1}$ similar to eq.(2.78) and also a similar expression for $w_{N,j+1}$.

Solution. The backward difference formula at the left endpoint $x_0 = 0$ is

$$\frac{w_{0j}-w_{0,j-1}}{\tau} - K\frac{w_{1j}-2w_{0j}+w_{-1,j}}{h^2} = f(0,t_j).$$

Using the central difference formula to approximate the derivative term in the boundary condition

$$\frac{\partial u}{\partial x}(0,t) + c_1(t)u(0,t) = \mu_1(t)$$

gives

$$w_{-1,j} = w_{1j} + 2hc_1(t_j)w_{0j} - 2h\mu_1(t_j).$$

Substituting this into the backward difference formula and multiplying by τ gives

$$w_{0j} - w_{0,j-1} - \gamma\left(w_{1j} - 2w_{0j} + w_{1j} + 2hc_1(t_j)w_{0j} - 2h\mu_1(t_j)\right) = \tau f(0,t_j).$$

Collecting terms gives

$$w_{0j}\left(1 + 2\gamma(1 - hc_1(t_j))\right) = 2\gamma w_{1j} + w_{0,j-1} + \tau f(0,t_j) - 2h\gamma\mu_1(t_j).$$

Thus

$$w_{0j} = \frac{2\gamma}{1+2\gamma(1-hc_1(t_j))}w_{1j} + \frac{w_{0,j-1}+\tau f(0,t_j) - 2h\gamma\mu_1(t_j)}{1+2\gamma(1-hc_1(t_j))}.$$

Similarly for the right endpoint $x_N = L$ we have the backward difference formula

$$\frac{w_{N,j}-w_{N,j-1}}{\tau} - K\frac{w_{N+1,j}-2w_{N,j}+w_{N-1,j}}{h^2} = f(L,t_j).$$

Using the central difference formula to approximate the derivative term in the boundary condition

$$\frac{\partial u}{\partial x}(L,t) + c_2(t)u(L,t) = \mu_2(t)$$

gives

$$w_{N+1,j} = w_{N-1,j} - 2hc_2(t_j)w_{N,j} + 2h\mu_2(t_j).$$

Substituting this into the backward difference formula and multiplying by τ gives

$$w_{N,j} - w_{N,j-1} - \gamma\left(w_{N-1,j} - 2hc_2(t_j)w_{N,j} + 2h\mu_2(t_j) - 2w_{N,j} + w_{N-1,j}\right) = \tau f(L,t_j).$$

Collecting terms gives

$$w_{N,j}\left(1 + 2\gamma(1 + hc_2(t_j))\right) = 2\gamma w_{N-1,j} + w_{N,j-1} + \tau f(L,t_j) + 2h\gamma\mu_2(t_j).$$

Thus

$$w_{N,j} = \frac{2\gamma}{1 + 2\gamma(1 + hc_2(t_j))}w_{N-1,j} + \frac{w_{N,j-1} + \tau f(L,t_j) + 2h\gamma\mu_2(t_j)}{1 + 2\gamma(1 + hc_2(t_j))}.$$

Problem 8. Consider the equation

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} - 2a(x,t)\frac{\partial u}{\partial x} + b(x,t) \quad \text{for} \quad 0 < x < 1, \quad t > 0,$$

subject to the initial and boundary conditions

$$u(0,t) = \mu_1(t), \quad u(1,t) = \mu_2(t), \quad u(x,0) = u_0(x).$$

Obtain a finite-difference approximation to this boundary-value problem and show that your finite-difference method is consistent with the equation, i.e. that the local truncation errors tend to zero as step sizes in x and in t go to zero.

Solution. Let (x_k, t_j) be the grid points, where $x_k = hk$ $(k = 0,1,\ldots,N)$, $t_j = \tau j$ $(k = 0,1,\ldots)$, $h = 1/N$. Employing the central difference formulae for the first and second order derivatives with respect to x and the two-point forward difference formula for $\partial u/\partial t$ at point (x_k,t_j), we obtain the following finite-difference equation

$$\frac{w_{k,j+1} - w_{k,j}}{\tau} = 3\frac{w_{k+1,j} - 2w_{k,j} + w_{k-1,j}}{h^2} - 2a(x_k,t_j)\frac{w_{k+1,j} - w_{k-1,j}}{2h} + b(x_k,t_j),$$

for $k = 1,2,\ldots,N-1$, $j = 0,1,\ldots$. From initial and boundary conditions, we obtain

$$w_{0,j} = \mu_1(t_j), \quad w_{N1,j} = \mu_2(t_j), \quad w_{k,0} = u_0(x_k).$$

The local truncation error in given by

$$\tau_{kj} = \frac{u_{k,j+1} - u_{k,j}}{\tau} - 3\frac{u_{k+1,j} - 2u_{k,j} + u_{k+1,j}}{h^2} + 2a(x_k,t_j)\frac{u_{k+1,j} - u_{k-1,j}}{2h} - b(x_k,t_j),$$

where $u_{kj} = u(x_k,t_j)$. Since (see Solution to Problem 6 above)

$$\frac{u_{k,j+1} - u_{kj}}{\tau} = \frac{\partial u}{\partial t}(x_k,t_j) + O(\tau), \qquad \frac{u_{k+1,j} - 2u_{kj} + u_{k-1,j}}{h^2} = \frac{\partial^2 u}{\partial x^2}(x_k,t_j) + O(h^2)$$

and since

$$\frac{u_{k+1,j} - u_{k-1,j}}{2h} = \frac{\partial u}{\partial x}(x_k,t_j) + O(h^2),$$

We obtain

$$\tau_{kj} = \frac{\partial u}{\partial t}(x_k,t_j) - 3\frac{\partial^2 u}{\partial x^2}(x_k,t_j) + 2a(x_k,t_j)\frac{\partial u}{\partial x}(x_k,t_j) - b(x_k,t_j) + O(\tau) + O(h^2) = O(\tau + h^2).$$

Hence,

$$\tau_{kj} \to 0 \quad \text{as} \quad \tau \to 0, \quad h \to 0,$$

so that the method is consistent.

Remark. The method described above is not the only possible finite-difference method to solve the given problem.