5.6

5,7,9,11

In order to ease my effort to do homework through latex, I will not pay too much attention toward the format of my output. So there will be no align function in my latex document.

1.

$$\Phi^{-1}(0.1) = -\Phi^{-1}(0.9) = -1.2816$$

$$\Phi^{-1}(0.9) = 1.2816$$

$$\Phi^{-1}(0.5) = 0$$

$$\Phi^{-1}(0.75) = 0.6745$$

$$\Phi^{-1}(0.25) = -0.6745$$

2.

$$F(x) = \Phi(\frac{x-\mu}{\sigma})$$

$$P(X \geqslant 0) = 1 - F(0)$$

$$= 1 - \Phi(-0.5) = 0.6915$$

$$P(-1 < X < 0.5) = F(0.5) - F(-1)$$

$$= \Phi(-0.25) - \Phi(-1) = 0.2426$$

$$P(|X| < 2) = F(2) - F(-2)$$

$$= \Phi(0.5) - \Phi(-1.5) = 0.6247$$

$$P(1 \leqslant -2X + 3 \leqslant 8)$$

$$= P(-2 \leqslant -2x\ leqslant 5)$$

$$= P(-2.5 \leqslant x \leqslant 1)$$

$$= F(1) - F(-2.5) = \Phi(0) - \Phi(-1.75) = 0.4599$$

5.

$$P(X \leqslant 290) = 1 - F(290)$$

$$= 1 - \Phi(-1) = \Phi(1)$$

$$P_{answer} = 1 - (1 - \Phi(1))^3$$

$$= 0.996$$

7.

$$P = P(116 \leqslant X \leqslant 118) = F(118) - F(116)$$

$$= \Phi(-1) - \Phi(-2)$$

$$P_{answer} = P^3 = 0.0025$$

9.

$$X_1 \sim N(20.0.04)$$

$$X_2 \sim N(14, 0.01)$$

$$X_3 \sim N(26, 0.04)$$

$$X_1 + X_2 + X_3 \sim N(60, 0.09)$$

$$P(55.7 + 4 \leqslant X \leqslant 56.3 + 4)$$

$$= F(60.3) - F(59.7) = \Phi(1) - \Phi(-1) = 0.6827$$

11.

$$\bar{X_n} \sim N(\mu, \frac{2}{n})$$

$$Z = (X - \mu) \times \sqrt{\frac{n}{2}}$$

$$P(|Z| < \frac{\sqrt{n}}{20}) \geqslant 0.9$$

$$n = 1083$$

5.7

1.

$$\psi_Y(t) = E(e^{tY}) = E(e^{(tc)X})$$

$$= (\frac{\beta}{\beta - ct})^\alpha = \frac{\frac{\beta}{c}}{\frac{\beta}{c} - t}$$

$$\beta \to \frac{\beta}{c}$$

//The following is a way that I don't know whether it's correct through the derivation of c.d.f and p.d.f, but I decide to reserve it in the homework

$$p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, x > 0$$

$$Y = cX$$

$$F_x(x) = \int_0^x \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx$$

$$F_y(y) = P(Y \leqslant y)$$

$$= P(cX \leqslant y)$$

$$= P(X \leqslant \frac{y}{c})$$

$$= F_x(\frac{y}{c})$$

$$= \int_0^{\frac{y}{c}} \frac{\beta^\alpha}{\Gamma(\alpha)} (\frac{y}{c})^{\alpha-1} e^{-\frac{\beta x}{c}} dx)$$

$$= \int_0^{y} \frac{(\frac{\beta}{c})^\alpha}{\Gamma(\alpha)} (\frac{y}{c})^{\alpha-1} e^{-\frac{\beta x}{c}} dx$$

6.

$$\psi(\bar{X_n}) = E(e^{\frac{t\sum_1^n X_i}{n}})$$

$$= E(e^{\sum_1^n \frac{t}{n} X_i})$$

$$= (\frac{\beta}{\beta - \frac{t}{n}})^{n\alpha}$$

$$= (\frac{n\beta}{n\beta - t})^{n\alpha}$$

For exponential distribution, $\alpha = 1$

$$= (\frac{n\beta}{n\beta - t})^{n}$$

It follows the gamma distribution with parameter $(n, n\beta)$

7.

Let $Y = Max(X_1, X_2 ... X_n)$

$$P(Y < t) = 1 - P(X_1 < t)P(X_2 < t)...P(X_n < t)$$

$$= (\int_0^t \beta e^{-\beta x} dx)^n$$

$$= (1 - e^{\beta t})^n$$

$$P_{answer} = 1 - (1 - e^{\beta t})^3$$

8.

$$Y = Min(X_1, X_2, ... X_n)$$

Follow the similar way in 7:

$$P(Y > y) = e^{-y\sum_{i=1}^n \beta_i}$$

$$F(y) = 1 - e^{-y\sum_{i=1}^n \beta_i}$$

$$p(y) = F^{'}(y) = (\sum_{i=1}^n \beta_i) e^{-t\sum_{i=1}^n \beta_i}$$

So Y follows exponential distribution with parameter $\sum_{i=1}^n \beta_i$

9.

All three components should not fail in order to maintain the system, as three components are independent.

$$P = P(X_1 > 100)P(X_2 > 100)P(X_3 > 100)$$

For exponential distribution

$$P(X > t) = e^{-\beta t}$$

$$P = e^{-100\times(0.001+0.003+0.006)}$$

$$= \frac{1}{e}$$

10.

$$\beta = \frac{1}{\mu}$$

with the conclusion of 7

$$P = e^{-\frac{nt}{\mu}}$$

$$E(Y) = \frac{\mu}{n}$$

$$Var(Y) = \frac{\mu^2}{n^2}$$

6.2

5.

$$P(|\overline{X_n} - E(X)| < 2\sigma) > 0.99$$

$$P(|\overline{X_n} - E(X)| < 2\sigma) > 1 - \frac{\sigma^2}{4n\sigma^2}$$

$$= 1 - \frac{1}{4n} \geqslant 0.99$$

$$n \geqslant 25$$

6.

$$P(|\overline{X_n} - E(X)| \leqslant 0.5) \geqslant 0.8$$

$$P(|\overline{X_n} - E(X)| \leqslant 0.5) > 1 - \frac{\sigma^2}{0.25n}$$

$$1 - \frac{16}{n} \geqslant 0.8$$

$$n \geqslant 80$$

7.

$$n = 16, \mu = 4.5, \sigma^2 = 8.25$$

$$\overline{X_n} \sim N(4.5, 8.25)$$

$$P(4 \leqslant \overline{X_n} \leqslant 6)$$

$$= \Phi(\,)$$

10.

$$P(|\overline{X_n} - E(X)| < \frac{\sigma}{4})$$

$$> 1 - \frac{\sigma^2}{\frac{\sigma^2}{16}n} = \frac{9}{25}$$

$$X_n \sim N(\mu, \frac{\sigma^2}{n})$$

$$P(|\overline{X_n} - E(X)| < \frac{\sigma}{4})$$

$$= P(|Z| < \frac{\sqrt{n}}{4n}) = 2\Phi(\frac{1}{20}) - 1$$

6.3

1.

$$n = 60, \sigma = 5, \mu = 4$$

$$T = \sum_{i=1}^{n} X_i$$

$$T \sim N(240, 1500)$$

$$P(T \geqslant 250) = 1 - \Phi(\frac{\sqrt{15}}{15}) = 0.001$$

3.

$$n = 125, \mu = \sigma^2 = 5$$

$$\overline{X_n} = \frac{\sum_{i=1}^{n} X_i}{n}$$

$$\overline{X_n} \sim N(5, \frac{1}{25})$$

$$P(\overline{X_n} < 5.5) = \Phi(2.5) = 0.9938$$

6.4

4.

$$n = 1200, \mu = \frac{1}{4}, \sigma^2 = \frac{3}{16}$$

$$T = \sum_{i=1}^{n} X_i$$

$$T \sim N(300, 225)$$

$$P(T \leqslant 270) = P(T \leqslant 270.5)$$

$$= \Phi(\frac{-29.5}{15})$$

5.

$$P(\overline{X_n} < 5.5) = P(\overline{X_n} < 6)$$

$$= \Phi()$$

7.5

2.

$$\widehat{p} = \frac{\sum_{i=1}^n x_i}{n} = \frac{29}{35}$$

3.

$$L(p) = p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i}$$

$$\lambda(p) = \sum_{i=1}^n x_i log(p) + (n - \sum_{i=1}^n x_i) log(1-p)$$

$$\frac{d\lambda}{dp} = \frac{\sum_{i=1}^n x_i}{p} + \frac{n - \sum_{i=1}^n x_i}{1-p}$$

$$= \frac{58}{p} + \frac{12}{1-p}$$

$$p \in [\frac{1}{2}, \frac{2}{3}]$$

$$\frac{d\lambda}{dp} > 0$$

$$\widehat{p} = \frac{2}{3}$$

5.

$$p(X=k) = \frac{\theta^k}{k!}e^{-\theta}$$

$$L(\theta) = \frac{e^{-n\theta}\theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n (x_i!)}$$

$$\lambda(\theta) = -n\theta + \sum_{i=1}^n x_i log(\theta) - \sum_{i=1}^n \frac{x_i(x_i+1)}{2}$$

$$\frac{d\lambda}{d\theta} = -n + \frac{\sum_{i=1}^n x_i}{\theta} = 0$$

$$\widehat{\theta} = \overline{x}$$

$$\frac{d^2\lambda}{d\theta^2} = -\frac{\sum_{i=1}^n x_i}{\theta^2} < 0$$

6.
note ...

7.

$$f(x|\theta) = \beta e^{\beta x}$$

$$L(\beta) = \beta^n e^{-\beta \sum_{i=1}^n x_i}$$

$$\lambda(\beta) = nlog(\beta) - \beta \sum_{i=1}^n x_i$$

$$\frac{d\lambda}{d\beta} = \frac{n}{\beta} - \sum_{i=1}^n x_i$$

$$\widehat{\beta} = \frac{n}{\sum_{i=1}^n x_i}$$

$$\frac{d^2\lambda}{d\beta^2} = -\frac{n}{\beta^2} < 0$$

8.

$$L(\theta) = e^{n\theta - \sum_{i=1}^n x_i}$$

$$\lambda(\beta) = n\theta - \sum_{i=1}^n x_i$$

$$\frac{d\lambda}{d\theta} > 0$$

$$\theta \in R$$

9.

$$L(\theta) = \theta^n \prod_{i=1}^n x_i^{\theta-1}$$

$$\lambda(\theta) = nlog(\theta) + (\theta-1)\sum_{i=1}^n x_i$$

$$\frac{d\lambda}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^n log(x_i)$$

$$\widehat{\theta} = -\frac{n}{\sum_{i=1}^n log(x_i)}$$

11.

$$L(\theta_1, \theta_2) = \prod_{i=1}^n \frac{1}{\theta_2 - \theta_1}$$

$$\lambda(\theta_1, \theta_2) = -nlog(\theta_2 - \theta_1)$$

$$\frac{\partial\lambda}{\partial\theta_1} = \frac{n}{\theta_2 - \theta_1}$$

$$\frac{\partial\lambda}{\partial\theta_2} = -\frac{n}{\theta_2 - \theta_1}$$

$$\theta_2 = max, \theta_1 = min$$

7.6

1.

$$\theta = -\frac{n}{\sum_{i=1}^{n} log(x_i)}$$

$$e^{-\frac{1}{\theta}} = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$$

2.

$$\widehat{\theta} = \overline{x}$$

$$D(X) = \sqrt{\theta}$$

3.

$$\widehat{\beta} = \frac{1}{\overline{x}}$$

$$\int_0^s \beta e^{-\beta x} dx = 1 - e^{-\beta s} = \frac{1}{2}$$

$$= \frac{log2}{\beta} = log2\overline{x}$$

5.

$$\frac{a+b}{2}$$

$$a = min, b = max$$

6.

$$\Phi(\frac{s-\mu}{\theta}) = 0.95$$

$$s = 1.645\theta + \mu$$

7.

$$v = 1 - P(x \leqslant 2)$$

$$= 1 - \Phi(\frac{2-\mu}{\theta})$$

7.7

1.

$$f(T|p) = p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i}$$

$$u(x) = 1, v(a,b) = b^a(1-b)^{n-a}$$

2.

$$f(T|p) = p^n(1-p)^{\sum_{i=1}^{n} x_i}$$

$$u(x) = 1, v(a,b) = b^n(1-b)^a$$

3.

$$f(T|p) = \prod_{i=1}^{n} C_{r+x_i-1}^{x_i} p^r (1-p)^{x_i - 1}$$

$$u(x) = \prod_{i=1}^{n} C_{r+x_i-1}^{x_i}$$

$$v(x) = \prod_{i=1}^{n} p^r (1-p)^{x_i - 1}$$

4.

$$f(T|\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\theta} e^{-\frac{(x_i-\mu)^2}{2\theta^2}}$$

$$u(x) = 1$$

$$v(a,b) = (\frac{1}{\sqrt{2\pi b}})^n e^{-\frac{a}{2b}}$$

5.

$$f(T|\beta) = \prod_{i=1}^{n} \frac{\beta^\alpha}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\beta x}$$

$$u(x) = \frac{(\prod_{i=1}^{n} x_i)^{\alpha-1}}{\Gamma^n(\alpha)}$$

$$V(a,b) = b^{n\alpha} e^{-nab}$$

6.

$$u(x) = e^{-\beta \sum_{i=1}^{n} x_i}$$

$$V(a,b)=\frac{\beta^{nb}}{\Gamma^n(b)}a^{b-1}$$

7.

$$f(T|\alpha)=\prod_{i=1}^{n}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x_i^{\alpha-1}(1-x_i)^{\beta-1}$$

$$u(x)=\frac{\prod_{i=1}^{n}(1-x_i)^{\beta-1}}{\Gamma^n(\beta)}$$

$$V(a,b)=(\frac{\Gamma(b+\beta)}{\Gamma(b)})^n a^{b-1}$$

8.

$$f(T|\theta)=(\frac{1}{\theta})^n I_A(x_1,x_2...x_n)I_B(x_1,x_2...x_n)$$

$$u(x)=I_B(x_1,x_2...x_n)$$

$$v(a,b)=(\frac{1}{b})^n I(a)$$

9.

$$f(T|b)=(\frac{1}{b-a})^n I_A(x_1,x_2...x_n)I_B(x_1,x_2...x_n)$$

$$u(x)=I_B(x_1,x_2...x_n)$$

$$v(a,b)=(\frac{1}{b})^n I(a)$$

10.

$$f(T|a)=(\frac{1}{b-a})^n I_A(x_1,x_2...x_n)I_B(x_1,x_2...x_n)$$

$$u(x)=I_A(x_1,x_2...x_n)$$

$$v(a,b)=(\frac{1}{b})^n I(b)$$

7.8

1.

$$f(x_1,x_2...x_n)=\prod_{i=1}^{n}\frac{\beta^\alpha}{\Gamma(\alpha)}x_i^{\alpha-1}e^{-\beta x}$$

$$u(x)=1$$

$$V((T_1,T_2),(\alpha,\theta))=\frac{\beta^{n\alpha}}{\Gamma^n(\alpha)}T_1^{\alpha-1}e^{-\beta T_2}$$

2.

$$f(x_1,x_2...x_n|\alpha,\beta)=\prod_{i=1}^{n}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x_i^{\alpha-1}(1-x_i)^{\beta-1}$$

$$u(x)=1$$

$$V((T_1,T_2),(\alpha,\beta))=(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)})^n T_1^{\alpha-1}T_2^{\beta-1}$$

3.

$$f(x_1,x_2...x_n|\theta)=(\frac{1}{3})^n I_A(x_1,x_2...x_n)I_B(x_1,x_2...x_n)$$

4.

7.9

4*.

5.

$$R(\delta_2,\theta)=Var(\delta)-(E(\delta_2)-\theta)^2$$

$$E(cY)=cE(Y)$$

$$E(\delta_2)-\theta=\frac{nc\theta}{n+1}$$

$$c^*=c=\frac{n+1}{n}$$

6.

$$\delta(x)=\overline{X_n}$$

$$T = \prod_{i=1}^{n} x_i$$

8.

12.

$$E(X_1|T) = 1 \times P(X_1 = 1|T)$$

$$= \frac{P(X_1 = 1)P(X_1 = 1, \sum_{i=1}^{n} x_i = t)}{P(\sum_{i=1}^{n} x_i = t)}$$

$$P(X_1 = 1, \sum_{i=1}^{n} x_i = t) = C_{n-1}^{t-1} p^{t-1} (1-p)^{n-t}$$

$$P(\sum_{i=1}^{n} x_i = t) = C_n^t p^t (1-p)^{n-t}$$

$$origin = \frac{C_{n-1}^{t-1}}{C_n^t}$$

13.

$$E(Y_i|T) = 1 \times P(Y_i = 1|T)$$

$$= \frac{P(x_i = 0, \sum_{i=1}^{n} x_i = t)}{P(\sum_{i=1}^{n} x_i)}$$

$$= \frac{e^{-\theta} \frac{((n-1)\theta)^t e^{-(n-1)\theta)}}{t!}}{\frac{(n\theta)^t e^{-n\theta}}{t!}}$$

$$= (\frac{n-1}{n})^t = (1 - \frac{1}{n})^T$$

$$= ((1 - \frac{1}{n})^n)^{\overline{x}}$$

$$= e^{\overline{x}}$$

14.

8.1

5.

$$P(|\overline{X_n} - p| \leqslant 0.1) \geqslant 1 - \frac{Var(X_n)}{0.01}$$

$$Var(X_n) = \frac{p - p^2}{n} = \frac{0.16}{n}$$
$$n \geqslant 22$$

6.

$$P(|\overline{X_n} - p| \leqslant 0.1) = 2 * \Phi(\frac{\sqrt{n}}{4}) - 1 \geqslant 0.95$$

7.

$$Var(X_n) \leqslant 0.01$$
$$\frac{0.16}{n} \leqslant 0.01$$
$$n \geqslant 16$$

8.

$$\frac{p - p^2}{n} \leqslant 0.01$$
$$n \geqslant 100(p - p^2) \geqslant 25$$

8.2

5.

$$X^2 + Y^2 + Z^2 < 1$$
$$X^2 + Y^2 + Z^2 \sim \chi^2(3) : \frac{1}{2^{\frac{3}{2}}\Gamma(\frac{3}{2})} x^{\frac{1}{2}} e^{-\frac{1}{2}x}$$
$$\int_0^1 \frac{1}{2^{\frac{3}{2}}\Gamma(\frac{3}{2})} x^{\frac{1}{2}} e^{-\frac{1}{2}x} dx$$

6.

$$t = 2, Var = 2\sigma^2$$
$$X^2 + Y^2 + Z^2 \sim \chi^2(3)$$
$$P(X^2 + Y^2 + Z^2) \leqslant 16\sigma^2$$

7.?

$$\psi(Y) = E(e^{ty})$$
$$= E(e^{-2\sum_{i=1}^{n} log(F_i(x_i))t})$$
$$= E(\prod_{i=1}^{n} F_i(x_i)e^{-2t})$$
$$= \prod_{i=1}^{n} E(F_i(x_i)e^{-2t})$$
$$= \prod_{i=1}^{n} \int_{-\infty}^{\infty} F_i(x_i)e^{-2t}x_i dx$$

10.

11.

$$\psi(\sqrt{x}) = E(e^{t\sqrt{x}})$$

13.

$$\hat{\delta_0^2} = \frac{\sigma^2}{n}\sum_{i=1}^{n} Z_i^2$$
$$\psi(\hat{\delta_0^2}) = E(e^{t\frac{\sigma^2}{n}\sum_{i=1}^{n} Z_i^2})$$
$$= \prod_{i=1}^{n}(\frac{1}{1-\frac{2\sigma^2 t}{n}})^{\frac{n}{2}}$$
$$\psi_{target} = (\frac{\frac{n}{2\sigma^2}}{\frac{n}{2\sigma^2}-t})^{n/2}$$