On the geometric connected components of moduli spaces of p-adic shtukas and local Shimura varieties.

Ian Gleason

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Abstract

In this article we study the connected components of moduli spaces of p-adic shtukas and local Shimura varieties. Given local shtuka datum (G, b, μ) with G an unramified reductive group over $\mathbb{Q}_p$ and (b, μ) HN-irreducible, we determine the locally profinite set of geometric connected components $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ and describe explicitly the continuous right action of the group $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$ on this space. This confirms and generalizes conjecture 4.26 of [30] in the unramified case. This work builds on and generalizes M. Chen's Theorem B of [4].

Contents

Introduction

In [30] Rapoport and Viehmann propose that there should be a theory of p-adic local Shimura varieties. They conjectured that there should exist towers of rigid-analytic spaces whose cohomology "understands" the local Langlands correspondence for general p-adic reductive groups. In this way, these towers of rigid-analytic varieties would "interact" with the local Langlands correspondence in a similar fashion to how Shimura varieties "interact" with the global Langlands correspondence. Moreover, they conjectured many properties and compatibilities that these towers should satisfy.

In the last decade, the theory of local Shimura varieties went through a drastic transformation with Scholze's introduction of perfectoid spaces and the theory of diamonds. In [34] Scholze and Weinstein construct the sought for towers of rigid analytic spaces and generalized them to what are now known as moduli spaces of p-adic shtukas. Moreover, since then, many of the expected properties and compatibilities for local Shimura varieties have been verified and generalized to moduli spaces of p-adic shtukas. The study of the geometry and cohomology of local Shimura varieties and moduli spaces of p-adic shtukas is still a very active area of research due to their connection to the local Langlands correspondence. The main aim of this article is to study the locally profinite set of connected components, and prove new cases of conjecture 4.26 in [30].

Let us recall the formalism of local Shimura varieties and moduli of p-adic shtukas. Local p-adic shtuka datum over $\mathbb{Q}_p$ is a triple $(G, [b], [\mu])$ where G is a reductive group over $\mathbb{Q}_p$, $[\mu]$ is a conjugacy class of geometric cocharacters $\mu : \mathbb{G}_m \to G$ and $[b]$ is an element of Kottwitz set $B(G, [\mu])$. Whenever $[\mu]$ is minuscule we say that $(G, [b], [\mu])$ is local Shimura datum. We let $E/\mathbb{Q}_p$ denote the reflex field of $[\mu]$. Associated to $(G, [b], [\mu])$ there is a tower of diamonds over $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$, denoted $(\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}})_{\mathcal{K}}$, where $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ ranges over compact subgroup of $G(\mathbb{Q}_p)$. Moreover, whenever $[\mu]$ is minuscule and $\mathcal{K}$ is a compact open subgroup, then $(\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}})_{\mathcal{K}}$ is represented by the diamond associated to a unique smooth rigid-analytic space $\mathbb{M}_{\mathcal{K}}$ over $\breve{E}$. The tower $(\mathbb{M}_{\mathcal{K}})_{\mathcal{K}}$ is the local Shimura variety.

Associated to $[b] \in B(G,\mu)$ one can associate a reductive group J_b over $\mathbb{Q}_p$. After basechange to a completed algebraic closure, each individual space $(\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}} \times \mathbb{C}_p)_{\mathcal{K}}$ comes equipped with continuous and commuting right actions by $J_b(\mathbb{Q}_p)$ and the Weil group W_E. Moreover, the tower receives a right action by the group $G(\mathbb{Q}_p)$ by using correspondances. When we let $\mathcal{K} = \{e\}$ we obtain the space at infinite level, denoted $\mathrm{Sht}_{G,[b],[\mu],\infty} \times \mathbb{C}_p$, which overall comes equipped with a continuous right action by $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$.

This formalism is functorial on the group G in the following way. Whenever we are given a morphism of algebraic groups $f : G \to H$ over $\mathbb{Q}_p$ we obtain a morphism of towers

$$(\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}} \times \mathbb{C}_p)_{\mathcal{K}} \to (\mathrm{Sht}_{H,[f(b)],[f\circ\mu],f(\mathcal{K})} \times \mathbb{C}_p)_{f(\mathcal{K})}$$

and these maps are equivariant with respect to the action induced by the map

$$G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E \to H(\mathbb{Q}_p) \times J_{f(b)}(\mathbb{Q}_p) \times W_E.$$

Since the actions are continuous the groups $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$ act continuously on $\pi_0(\mathrm{Sht}_{G,[b],[\mu],\infty} \times \mathbb{C}_p)$ and our main theorem describes explicitly this action whenever G is an unramified reductive group over $\mathbb{Q}_p$ and $([b],[\mu])$ is HN-irreducible. It is very likely that the methods of this paper could be combined with those of [14] and [12] to remove the HN-irreducible condition. We do not pursue this generality.

Before stating our main theorem we need to set more notation. Let $(G,[b],[\mu])$ be local p-adic shtuka datum with G an unramified reductive group over $\mathbb{Q}_p$. Let G^{der} denote the derived subgroup of G and G^{sc} denote the simply connected cover of G^{der}, let N denote the image of $G^{sc}(\mathbb{Q}_p)$ in $G(\mathbb{Q}_p)$ and let $G^\circ = G(\mathbb{Q}_p)/N$. This is a locally profinite topological group and it is the maximal abelian quotient of $G(\mathbb{Q}_p)$ when this later is considered as an abstract group.

Let $E \subseteq \mathbb{C}_p$ be the field of definition of $[\mu]$, let $\mathrm{Art}_E : W_E \to E^\times$ be Artin's reciprocity character from local class field theory. In §4 we associate to $[\mu]$ a continuous map of topological groups $Nm^\circ_{[\mu]} : E^\times \to G^\circ$ and we associate to $[b]$ a map $det^\circ : J_b(\mathbb{Q}_p) \to G^\circ$.

The general construction of $Nm^\circ_{[\mu]}$ and det° uses z-extensions and we do not review it in this introduction. Nevertheless, whenever $G^{sc} = G^{der}$ we can construct these maps as follows. In this case $G^\circ = G^{ab}(\mathbb{Q}_p)$ where G^{ab} is the co-center of G, which is an algebraic group of multiplicative type (or a torus). If we let $det : G \to G^{ab}$ be the quotient map we can consider the induced data $\mu^{ab} = det \circ [\mu]$ and $[b^{ab}] = [det(b)]$. Then $Nm^\circ_{[\mu]}$ can be defined as the following composition:

$$E^\times \xrightarrow{\mu^{ab}} G^{ab}(E) \xrightarrow{Nm^{G^{ab}}_{E/\mathbb{Q}_p}} G^{ab}(\mathbb{Q}_p) = G^\circ.$$

Here for a torus T over $\mathbb{Q}_p$, like G^{ab}, we are letting $Nm^T_{E/\mathbb{Q}_p} : T^{ab}(E) \to T^{ab}(\mathbb{Q}_p)$ denote the usual norm map

$$t \mapsto \prod_{\gamma \in Gal(E/\mathbb{Q}_p)} \gamma(t).$$

On the other hand, $det^\circ : J_b(\mathbb{Q}_p) \to G^{ab}(\mathbb{Q}_p)$ can be obtained as the composition $det = j_{b^{ab}} \circ det_b$ where the maps $det_b : J_b(\mathbb{Q}_p) \to J_{b^{ab}}(\mathbb{Q}_p)$ and $j_{b^{ab}} : J_{b^{ab}}(\mathbb{Q}_p) \to G^\circ$ can be described as follows. The map det_b is obtained from functoriality of the formation of J_b, and $j_{b^{ab}}$ is the isomorphism $j_{b^{ab}} : J_{b^{ab}}(\mathbb{Q}_p) \cong G^{ab}(\mathbb{Q}_p)$ obtained from regarding $J_{b^{ab}}(\mathbb{Q}_p)$ and $G^{ab}(\mathbb{Q}_p)$ as subgroups of $G^{ab}(K_0)$ and exploiting that G^{ab} is commutative.

Theorem 1. *Let $(G, [b], [\mu])$ be local shtuka datum with G an unramified reductive group over $\mathbb{Q}_p$ and $([b], [\mu])$ HN-irreducible. The following hold:*

1. *The right $G(\mathbb{Q}_p)$ action on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ is trivial on $N = Im(G^{sc}(\mathbb{Q}_p))$ and the induced G°-action is simply-transitive.*

2. *If $s \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ and $j \in J_b(\mathbb{Q}_p)$ then*

$$s \cdot_{J_b(\mathbb{Q}_p)} j = s \cdot_{G^{ab}(\mathbb{Q}_p)} det^\circ(j^{-1}))$$

3. *If $s \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ and $\gamma \in W_E$ then*

$$s \cdot_{W_E} \gamma = s \cdot_{G^{ab}(\mathbb{Q}_p)} [Nm^\circ_{[\mu]} \circ Art_E(\gamma)].$$

Let us comment on previous results in the literature. Before a full theory of local Shimura varieties was available the main example of local Shimura varieties one could work with were the ones obtained as the generic fiber of a Rapoport-Zink space studied in [29]. The most celebrated examples of Rapoport-Zink spaces are of course the Lubin-Tate tower and the tower of covers of Drinfeld's upper half space. In [7] de Jong introduces his version of the fundamental group in rigid-analytic geometry to describe the Grothendieck-Messing period morphism. As an application of his theory of fundamental groups he computes the connected components of the Lubin-Tate tower for $\mathrm{GL}_n(\mathbb{Q}_p)$. In [38] Strauch computes by a very different method the connected components of the Lubin-Tate tower for $\mathrm{GL}_n(F)$ and an arbitrary finite extension F of $\mathbb{Q}_p$ (including ramification).

In [3] M. Chen constructs 0-dimensional local Shimura varieties and studies their geometry. These are the local Shimura varieties associated to tori. In a later paper [4] she constructs her "determinant" map and uses these 0-dimensional local Shimura varieties to describe connected components of Rapoport-Zink spaces of EL and PEL type associated to more general unramified reductive groups. We also use the determinant map, but in our case it is automatically constructed for us from the functoriality (with respect to group morphisms) of moduli spaces of p-adic shtukas. The central strategy of Chen's result builds on and improves the central strategy used by de Jong. Many steps in de Jong's original strategy fail or become technically more challenging when one passes from the Lubin-Tate tower to more general Rapoport-Zink spaces and M. Chen introduces many new ideas to tackle those cases. Two key inputs of Chen's work to the strategy is the use of her "generic" crystalline representations and her collaboration with Kisin and Viehmann on computing the connected components of affine Deligne-Lusztig varieties [5].

Our central strategy builds on the central strategy of de Jong and Chen, but the versatility of Scholze's theory of diamonds and the fully functorial construction of local Shimura varieties allow us to make many simplifications and streamline the proof. This is of course up to the fact that our arguments use Scholze's theory of diamonds rather than rigid analytic spaces. Our new main input to the central strategy is the use of specialization maps. To be able to use specialization maps in a rigorous way we had to develop a formalism that would allow us to use them. The details of this formalism were worked out in detail in a separate paper [13]. Originally, our formalism of specialization maps was developed to address a missing step in our efforts to adapt de Jong and Chen's strategy to the context of diamonds.

Let us sketch the central strategy to prove theorem 1. Once one knows that $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ is a right G°-torsor, computing the actions by W_E and $J_b(\mathbb{Q}_p)$ in terms of the G° action can be reduced to the tori case using functoriality, z-extensions and the determinant map. These uses mainly group theoretic methods and down to earth diagram chases. In the tori case the $J_b(\mathbb{Q}_p)$ action is easy to compute and the W_E action can be bootstrapped to an easier case as follows. For tori T, by the work of Kottwitz, we know that the set $B(T,\mu)$ has a unique element so that the data of b is redundant. We can consider the category of pairs (T,μ) where T is a torus over $\mathbb{Q}_p$ and μ is a geometric cocharacter whose field of definition is E. The construction of moduli spaces of shtukas is functorial with respect to this category. Moreover, this category has an initial object given by $(\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m),\mu_u)$ where

$$\mu_u : \mathbb{G}_m \to \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)_E$$

is the unique map of tori that on E-points is given by the formula

$$f \mapsto f \otimes_{\mathbb{Q}_p} f.$$

After more diagram chasing one can again reduce the tori case to the "universal" case. Finally, this case can be done explicitly using the theory of Lubin-Tate groups and their relation to class field theory. As we have mentioned, the tori case was already handled by M. Chen in [3], but for the convenience of the readers we recall the story in a different language.

Let us sketch how to prove that $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty}\times\mathbb{C}_p)$ is a G° torsor in the simplest case. For this let G be semisimple and simply connected. Our theorem then says that $\mathrm{Sht}_{G,b,[\mu],\infty}\times\mathbb{C}_p$ is connected.

The first step is to prove that $G(\mathbb{Q}_p)$ acts transitively on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty}\times\mathbb{C}_p)$. Using the Grothendieck-Messing period map one realizes that these is equivalent to proving that the b-admissible locus of Scholze's B_{dR}-Grassmanian is connected. This fact is a result of Hansen and Weinstein to which we give an alternative proof.

For the next step, let $x\in\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty}\times\mathbb{C}_p)$ and let $G_x\subseteq G(\mathbb{Q}_p)$ denote the stabilizer of x. Let $\mathcal{K}\subseteq G(\mathbb{Q}_p)$ be a hyperspecial subgroup of G. We claim that it is enough to prove that G_x is open and that $G(\mathbb{Q}_p)=\mathcal{K}\cdot G_x$. Indeed, $\mathcal{K}$ surjects onto $G(\mathbb{Q}_p)/G_x$ so that this space is discrete and compact therefore finite. By a theorem of Margulis [25], since we assumed G to be simply connected, the only open subgroup of finite index is the whole group so that $G_x=G(\mathbb{Q}_p)$.

The proof that G_x is open relies heavily on M. Chen's main technical result on her "generic" crystalline representations. To be able to apply her result in our context one uses that for suitable p-adic fields K, every crystalline representation is realized as a $\mathrm{Spd}(K,O_K)$-valued point in Scholze's B_{dR}-Grassmanian. For the convenience of the reader we include a discussion on how to think of crystalline representations as $\mathrm{Spd}(K,O_K)$-valued points. Finally, proving that $G(\mathbb{Q}_p)=\mathcal{K}\cdot G_x$ is equivalent to proving that $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}}\times\mathbb{C}_p$, the $\mathcal{K}$-level moduli space of shtukas, is connected. This is where our theory of specialization maps gets used. Indeed, in [13] we prove that the specialization map identifies the connected components of moduli spaces of shtukas with the connected components of affine Deligne-Lusztig varieties. To conclude we only need to know that these varieties are connected.

Fortunately for us, the connected components of affine Deligne-Lusztig varieties are now very well understood by the work of many authors [5], [27] [15]. In the HN-irreducible case they can be identified with certain subset of $\pi_1(G)$. Since we assumed G to be simply connected $\pi_1(G)=\{e\}$ which finishes the sketch of the proof for the simply connected case. The central strategy used for general unramified groups G is not very different in spirit and only requires more patience.

Let us comment on the organization of this paper. In section §2 we recall the relation between crystalline representations, Scholze's theory of diamonds, and other geometric constructions that appear in modern p-adic Hodge theory. This part of the paper is purely expository, but we consider it important for the rest or the argument to have these relations in mind. We also decided to include a discussion of Weil descent data and the action of $\underline{J_b(\mathbb{Q}_p)}$ since the author found some of the details in this part of the theory harder to grasp.

In section §3 we reprove M. Chen's results for tori. We do this for several reasons. On one hand, it was a very instructive exercise for the author to do this computation concretely, on the other hand the 0-dimensional local Shimura varieties that appear in Chen's work are constructed in a very different way. It is not clear to the author if proving that Chen's local Shimura varieties agree with Scholze and Weinstein's moduli spaces of shtukas is or not essentially equivalent to doing this computation.

In section §4 the details of the proof of theorem 1 are provided. This is the only part of the paper in which we make claims of originality.

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1 Notation

Let us fix some notation. We let k be a perfect field in characteristic p with algebraic closure $\overline{k}$. For most things the case of interest are when $k = \overline{\mathbb{F}}_p$ or when k is a finite field. In most subsections we will assume that k is algebraically closed, and we will point out when this assumption is taken. We let $W(k)$ (respectively $W(\overline{k})$) denote the ring of p-typical Witt vectors of k, respectively $\overline{k}$, and we let $K_0 = W(k)[\frac{1}{p}]$, respectively $\breve{K}_0 = W(\overline{k})[\frac{1}{p}]$. In the sections in which we assume $k = \overline{k}$ we use the symbols K_0 and $\breve{K}_0$ interchangeably.

We denote by σ the canonical lift of arithmetic Frobenious to $\breve{K}_0$ and abusing notation we will also denote by σ its restriction to K_0. We fix an algebraic closure $\overline{\breve{K}_0}$ of $\breve{K}_0$, and we let C_p denote the p-adic completion of $\overline{\breve{K}_0}$. We use K (respectively $\breve{K}$) to denote subfields of C_p of finite degree over K_0 (respectively $\breve{K}$). We let Γ_K (respectively $\Gamma_{\breve{K}}$) denote the continuous automorphisms of C_p that fix K (respectively $\breve{K}$). If $\overline{K_0}$ is the algebraic closure of K_0 in C_p then Γ_K is canonically isomorphic to $Gal(\overline{K}_0/K)$, since $\overline{K}_0$ is dense in C_p. We will denote by Γ_K^{op} the opposite group which we identify with the group of automorphisms of $\operatorname{Spec}(C_p)$ over $\operatorname{Spec}(K_0)$.

We let $W_{\breve{K}_0}$ denote the subset of continuous automorphisms of $\operatorname{Aut}(C_p)$ that stabilize $\breve{K}_0$ and act as an integral power of σ on $\breve{K}_0$. We topologize $W_{\breve{K}_0}$ so that $\Gamma_{\breve{K}_0}$ is an open subgroup. Suppose $E \subseteq C_p$ is a field of finite degree over $\mathbb{Q}_p$, and let $\mathbb{Q}_{p^s}$ be the maximal unramified extension of $\mathbb{Q}_p$ contained in E. The extension $E/\mathbb{Q}_{p^s}$ is totally ramified and $E \otimes_{\mathbb{Q}_{p^s}} \breve{K}_0$ is canonically isomorphic to the compositum $\breve{E} = E \cdot \breve{K}_0$ inside of C_p, since E and $\breve{K}_0$ are linearly disjoint and have canonical inclusions into C_p. We define an automorphism $\hat{\sigma} \in Aut(\breve{E})$ as the automorphism that maps to $Id \otimes \sigma$ under this identification. We let $W_{\breve{E}/E}$ denote the continuous automorphisms of C_p that stabilize $\breve{E}$, act on $\breve{E}$ as $\hat{\sigma}^{s\cdot n}$ for some $n \in \mathbb{Z}$. Notice that $W_{\breve{E}/E}$ fixes E.

Through out the text G will denote a connected reductive group over $\mathbb{Q}_p$. In certain subsections we will add the additional assumptions that G is quasi-split or even stronger that it is unramified over $\mathbb{Q}_p$. We will point out when one of these two assumptions are taken. Whenever G is quasi-split we will denote by A a maximally split sub-torus of G defined over $\mathbb{Q}_p$, T will denote the centralizer of A which is also a torus and B will denote a $\mathbb{Q}_p$-rational Borel containing T. If G is assumed unramified we will sometimes also assume that G is given as the basechange of a connected reductive group over $\mathbb{Z}_p$ which we will still denote by G.

We will often work in the situation in which we are given an element $b \in G(K_0)$ and/or a cocharacter $\mu : \mathbb{G}_m \to G_K$. In these circumstances $[b]$ always denotes the σ-conjugacy class of b in $G(\breve{K}_0)$ and $[\mu]$ denotes the unique geometric conjugacy class of cocharacters $[\mu] \in Hom(\mathbb{G}_m, G_{\overline{\mathbb{Q}_p}})$ that is conjugate to μ through the action of G_{C_p}. Moreover, we let $E/\mathbb{Q}_p$ denote the field extension contained in C_p over which $[\mu]$ is defined. We let E_0 denote the compositum of E and K_0 in C_p.

2 The geometric perspective of crystalline representations

2.1 Vector bundles, isocrystals and crystalline representations.

Let K_0, K and C_p be as in the notation. With this setup in [10], Fargues and Fontaine construct a remarkable $\mathbb{Q}_p$-scheme, X_{FF,C_p}, which is now known as “the fundamental curve of arithmetic”.

Fargues and Fontaine justify why we can think of X_{FF,C_p} as a “curve” despite the fact that the structure morphism $\mathrm{X}_{FF,C_p} \to \mathrm{Spec}(\mathbb{Q}_p)$ is not of finite type. Moreover, the “curve” is “complete” in an appropriate sense which in particular implies that $H^0(\mathrm{X}_{FF,C_p}, \mathcal{O}_X) = \mathbb{Q}_p$. The curve comes endowed with a section “at infinity” given by a map $\infty : \mathrm{Spec}(C_p) \to \mathrm{X}_{FF,C_p}$ and it also has a $\Gamma_{K_0}^{op}$-action whose unique Γ_K^{op}-fixed point (for all finite extensions K/K_0) is ∞. The completion of the stalk of the structure sheaf at ∞, $\widehat{\mathcal{O}_{X,\infty}}$, is canonically isomorphic to Fontaine's period ring B^+_{dR} and compatibly with the Γ_{K_0}-action. Moreover, $\mathrm{X}_{FF,C_p} \setminus \infty$ is an affine scheme and $\mathrm{H}^0(\mathrm{X}_{FF,C_p} \setminus \infty, \mathcal{O}_X) = B_e = B_{crys}^{\varphi=1}$, which is a principal ideal domain. With this curve at hand Fargues and Fontaine reinterpret geometrically the classical p-adic Hodge theory of Fontaine. We recall this geometric reinterpretation for the case of crystalline representations and the connection with Scholze's theory of diamonds.

Denote by $\varphi-\mathrm{Mod}_{K_0}$ the category of isocrystals over K_0 that has as objects the pairs (D, φ) where D is a finite dimensional K_0 vector space and $\varphi : \sigma^* D \to D$ is an isomorphism. This is a $\mathbb{Q}_p$-linear Tannakian category. Fargues and Fontaine associate to $(D, \varphi) \in \varphi-\mathrm{Mod}_{K_0}$ a vector bundle $\mathcal{E}(D, \varphi)$ that comes equipped with a $\Gamma_{K_0}^{op}$-action that is compatible with the action on X_{FF,C_p} (See [10] 10.2.1, 9.1.1). By this we mean that for any $\gamma^{op} \in \Gamma_{K_0}^{op}$ inducing the associated isomorphism $\theta_{\gamma^{op}} : \mathrm{X}_{FF,C_p} \to \mathrm{X}_{FF,C_p}$ we are given a family of compatible isomorphisms

$$\Theta_{\gamma^{op}} : \theta^*_{\gamma^{op}} \mathcal{E}(D, \varphi) \to \mathcal{E}(D, \varphi).$$

The Beauville-Laszlo theorem (see [35] Lemma 5.2.9), provides us with an equivalence from the category of vector bundles over X_{FF,C_p} to the category of triples (M_e, M^+_{dR}, u) where M_e is a free module over B_e, M^+_{dR} is a free module over B^+_{dR} and $u : M_e \otimes_{B_e} B_{dR} \to M^+_{dR} \otimes_{B^+_{dR}} B_{dR}$

is an isomorphism. This is Berger's category of B-pairs. From this equivalence we get a recipe to construct vector bundles by replacing (or modifying) M_{dR}^+ by some other B_{dR}^+-lattice Λ contained in $M_{dR} := M_{dR}^+ \otimes_{B_{dR}^+} B_{dR}$. If we choose Λ to be stable under the action of Γ_K on M_{dR}, then the new vector bundle produced in this way will have a Γ_K^{op}-action compatible with the one on X_{FF,C_p}. Fortunately, we can understand Γ_K-stable lattices in a concrete way as we recall below.

Given a finite dimensional K vector space V we can let $\mathrm{Fil}^\bullet V$ denote a decreasing filtration of K vector spaces. If $\mathrm{Fil}^\bullet V$ satisfies $\mathrm{Fil}^i V = V$ for $i \ll 0$ and $\mathrm{Fil}^i = 0$ for $i \gg 0$, we say that $\mathrm{Fil}^\bullet V$ is a bounded filtration. To such a filtration we can associate a B_{dR}^+-lattice in $V \otimes_K B_{dR}$ denoted $\mathrm{Fil}^0(V \otimes_K B_{dR})$ and given by the formula:

$$\mathrm{Fil}^0(V \otimes_K B_{dR}) = \sum_{i+j=0} \mathrm{Fil}^i V \otimes_K \mathrm{Fil}^j B_{dR}.$$

Proposition 2.1. *(See [10] 10.4.3) Let V be a finite dimensional vector space over K. The map that assigns to a bounded filtration $\mathrm{Fil}^\bullet V$ the B_{dR}^+-lattice $\mathrm{Fil}^0(V \otimes_K B_{dR})$ in $V \otimes_K B_{dR}$ gives a bijection between the set of bounded filtrations of V and Γ_K-stable B_{dR}^+-lattices Λ in $V \otimes_K B_{dR}$. If we let ξ denote a uniformizer of B_{dR}^+ then the inverse map is given by:*

$$\mathrm{Fil}^i_\Lambda(V) = \left((\xi^i \cdot \Lambda \cap V \otimes_K B_{dR}^+)/(\xi^i \cdot \Lambda \cap V \otimes_K \xi \cdot B_{dR}^+)\right)^{\Gamma_K}.$$

Remark 2.2. *The careful reader may notice that the reference constructs $\mathrm{Fil}^i_\Lambda(V)$ in a slightly different but equivalent way. We also point out the following. Let $(a_1, \dots a_n)$ denote a decreasing sequence of integers and let $\mu : \mathbb{G}_m \to \mathrm{GL}_n$ the character defined by $\mu(t) \cdot e_i = t^{a_i} e_i$. We let $\mathrm{Fil}^\bullet_\mu(K^n)$ denote the decreasing filtration associated μ with $e_j \in \mathrm{Fil}^i_\mu$ if $a_j \geq i$. Then the B_{dR} lattice associated to Fil^i_μ is generated by $\xi^{-a_i} e_i$. Notice the change of signs! Later on we will need to keep track of this.*

Denote by $\varphi{-}\mathrm{ModFil}_{K/K_0}$ the category of filtered φ-modules that has as objects triples $(D, \varphi, \mathrm{Fil}^\bullet D_K)$ where (D, φ) is in $\varphi{-}\mathrm{Mod}_{K_0}$ and $\mathrm{Fil}^\bullet D_K$ is a bounded filtration on $D \otimes_{K_0} K$. To any triple as above Fargues and Fontaine associate a vector bundle $\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K)$ equipped with a Γ_K^{op}-action compatible with the action on X_{FF,C_p}. It is constructed as a modification of $\mathcal{E}(D, \varphi)$ as follows. There is a canonical Γ_K-equivariant identification u between $D \otimes_{K_0} B_{dR}$ and the global sections of the restriction of $\mathcal{E}(D, \varphi)$ to $\mathrm{Spec}(B_{dR})$. Letting $M_e = \mathrm{H}^0(\mathrm{X}_{FF,C_p} \setminus \infty, \mathcal{E}(D, \varphi))$, $M_{dR} = D \otimes_{K_0} B_{dR}$ and $M_{dR}^+ = \mathrm{Fil}^0(D_K \otimes_K B_{dR}^+)$ then $\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K)$ is given by (M_e, M_{dR}^+, u) under the Beauville-Laszlo equivalence.

This induces an exact and fully-faithful functor

$$\varphi{-}\mathrm{ModFil}_{K/K_0} \hookrightarrow \mathrm{Vec}^{\Gamma_K^{op}}_{\mathrm{X}_{FF,C_p}}$$

from the category of filtered isocrystals to the category of Γ_K^{op}-equivariant vector bundles (See [10] 10.5.3). Any object of $\mathrm{Vec}^{\Gamma_K^{op}}_{\mathrm{X}_{FF,C_p}}$ in the essential image of this functor is called a crystalline vector bundle. Moreover, when the filtered isocrystal $(D, \varphi, \mathrm{Fil}^\bullet D_K)$ is "weakly admissible" Fargues and Fontaine prove that $\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K)$ is semi-stable of slope 0 (See [10] 10.5.2, 10.5.6). This in particular implies that $\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K)$ without the Γ_K^{op}-action is non-canonically isomorphic to $\mathcal{O}_X^d$ for $d = dim_K(D)$ so that $\mathrm{H}^0(\mathrm{X}_{FF,C_p}, \mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K))$ is a d-dimensional $\mathbb{Q}_p$-vector space endowed with a continuous Γ_K-action. This construction recovers the classical functor of Fontaine $V_{cris} : \varphi{-}\mathrm{ModFil}^{w.a.}_{K/K_0} \to \mathrm{Rep}_{\Gamma_K}(\mathbb{Q}_p)$ that associates to a weakly admissible filtered isocrystals a crystalline representation.

Remark 2.3. *Since we will need this later, let us be more specific about how Γ_K acts on*

$$V := \mathrm{H}^0(\mathrm{X}_{FF,C_p}, \mathcal{E}(D,\varphi,\mathrm{Fil}^\bullet D_K)).$$

Given an element $\gamma^{op} \in \Gamma_K^{op}$ we have by definition of a Γ_K^{op}-equivariant vector bundle and by adjunction a sequence of maps

$$\mathcal{E}(D,\varphi,\mathrm{Fil}^\bullet D_K) \to \theta_{\gamma^{op},*}\theta^*_{\gamma^{op}}\mathcal{E}(D,\varphi,\mathrm{Fil}^\bullet D_K) \xrightarrow{\theta_{\gamma^{op},*}\Theta_{\gamma^{op}}} \theta_{\gamma^{op},*}\mathcal{E}(D,\varphi,\mathrm{Fil}^\bullet D_K).$$

We can pass to global sections and let $H^0(\gamma^{op}) : H^0(\mathcal{E}) = V \to V = H^0(\theta_{\gamma^{op},*}\mathcal{E})$ denote the operator obtained in this way. Notice that $\gamma^{op} \mapsto H^0(\gamma^{op})$ is contravariant and does not give a group homomorphism. But the composition of maps of sets $\Gamma_K \to \Gamma_K^{op} \to Aut(V)$ given by

$$\gamma \mapsto \mathrm{Spec}(\gamma) \mapsto H^0(\mathrm{Spec}(\gamma))$$

is a group homomorphism.

2.2 Families of B_{dR}-lattices

One can upgrade geometrically the situation using Scholze's theory of diamonds, since this theory allows us to consider "families" of B_{dR}^+-lattices as a geometric object. Recall that the Fargues-Fontaine curve X_{FF,C_p} has a counterpart $\mathcal{X}_{FF,C_p^\flat}$ in the category of adic spaces. Moreover it also has relative analogues. If S be an affinoid perfectoid space in characteristic p, Kedlaya and Liu (See [18] §8.7) associate to S an adic space $\mathcal{X}_{FF,S}$ that they call the relative Fargues-Fontaine curve. This construction is functorial in $\mathrm{Perf}_{\mathbb{F}_p}$, the category of affinoid perfectoid spaces in characteristic p. Moreover, if (D,φ) is an isocrystal over K_0 and S is an affinoid perfectoid space over $\mathrm{Spa}(k,k)$ one can construct a vector bundle $\mathcal{E}_S(D,\varphi)$ over $\mathcal{X}_{FF,S}$. This construction is also functorial in Perf_k and recovers $\mathcal{E}(D,\varphi)$ when $S = \mathrm{Spa}(C_p^\flat, O_{C_p^\flat})$. Strictly speaking this also requires Kedlaya-Liu's GAGA equivalence [18] 8.7.5, 8.7.7.

In the world of diamonds we have a co-equalizer diagram

$$\mathrm{Spd}(K,O_K) = Coeq(\mathrm{Spa}(C_p^\flat, O_{C_p^\flat}) \times_{\mathrm{Spd}(K,O_K)} \mathrm{Spa}(C_p^\flat, O_{C_p^\flat}) \rightrightarrows \mathrm{Spa}(C_p^\flat, O_{C_p^\flat}))$$

and we also have an identification of affinoid perfectoid spaces

$$\mathrm{Spa}(C_p^\flat, O_{C_p^\flat}) \times_{\mathrm{Spd}(K,O_K)} \mathrm{Spa}(C_p^\flat, O_{C_p^\flat}) = \underline{\Gamma_K^{op}} \times \mathrm{Spa}(C_p^\flat, C_p^{\flat,+}).$$

If we let $S_1 = \mathrm{Spa}(C_p, O_{C_p})$ and $S_2 = \underline{\Gamma_{K_0}^{op}} \times \mathrm{Spa}(C_p^\flat, O_{C_p^\flat})$ then the Galois action of $\Gamma_{K_0}^{op}$ on X_{FF,C_p} and $\mathcal{E}(D,\varphi)$ constructed by Fargues and Fontaine can be reinterpreted as glueing datum

$$\mathcal{X}_{FF,S_2} \rightrightarrows \mathcal{X}_{FF,S_1}$$

over the pair of morphisms $S_2 \rightrightarrows S_1$. Neither the Fargues-Fontaine curve as an adic spaces nor the vector bundle $\mathcal{E}(D,\varphi)$ descend to an adic space or a vector bundle over K. But as we will see one can perform some geometric constructions in this context that will make sense as geometric objects over $\mathrm{Spd}(K,O_K)$.

Now, given a perfectoid space $S \in \mathrm{Perf}_{\mathbb{F}_p}$ the data of a map $S \to \mathrm{Spd}(K_0, O_{K_0})$ induces a "section" at infinity $\infty : S^\sharp \to \mathcal{X}_{FF,S}$. This is a closed Cartier divisor as in [34] 5.3.7 and as such it has a good notion of meromorphic functions. We consider the moduli space of meromorphic modifications of $\mathcal{E}_S(D,\varphi)$ along ∞.

Definition 2.4. 1. *We let $Gr(\mathcal{E}(D,\varphi))$ denote the functor from* $\mathrm{Perf}_{\mathrm{Spd}(K_0,O_{K_0})} \to \mathrm{Sets}$ *that assigns:*

$$(S^\sharp, f) \mapsto \{((S^\sharp, f), \mathcal{V}, \alpha)\}/\cong$$

Where $(S^\sharp, f)$ is an untilt of S over $\mathrm{Spa}(K_0, O_{K_0})$*, $\mathcal{V}$ is a vector bundle over $\mathcal{X}_{FF,S}$ and $\alpha : \mathcal{V} \dashrightarrow \mathcal{E}_S(D,\varphi)$ is an isomorphism defined over $\mathcal{X}_{FF,S} \setminus \infty$ and meromorphic along ∞.*

2. *Let Gr_{GL_n} denote the functor from* $\mathrm{Perf}_{\mathbb{Q}_p} \to \mathrm{Sets}$ *that assigns:*

$$(S^\sharp, f) \mapsto \{((S^\sharp, f), \mathcal{V}, \alpha)\}/\cong$$

Where $(S^\sharp, f)$ is an untilt of S over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$*, $\mathcal{V}$ is a vector bundle over* $\mathrm{Spec}(B^+_{dR}(S^\sharp))$ *and $\alpha : \mathcal{V} \dashrightarrow \mathcal{O}^{\oplus n}$ is an isomorphism defined over* $\mathrm{Spec}(B_{dR}(S^\sharp))$.

These moduli spaces are ind-proper ind-diamonds over $\mathrm{Spd}(K_0, O_{K_0})$ (and $\mathrm{Spd}(\mathbb{Q}_p, \mathbb{Z}_p)$ respectively) and after fixing a basis of D we get an identification

$$Gr_{GL_n} \times_{\mathbb{Q}_p} \mathrm{Spd}(K_0, O_{K_0}) \cong Gr(\mathcal{E}(D,\varphi))$$

(See [14] 2.12). The second space is the Beilinson-Drinfeld Grassmanian that appears in the Berkeley notes (See [35] 20.2.1).

We can re-interpret the canonical map $\mathrm{Spa}(C_p, O_{C_p}) \to \mathrm{Spa}(K_0, O_{K_0})$ that comes from thinking of K_0 as a subfield of C_p as a map $m : \mathrm{Spd}(C_p^\flat, O_{C_p^\flat}) \to \mathrm{Spd}(K_0, O_{K_0})$. The basechange

$$Gr(\mathcal{E}_S(D,\varphi)) \times_{\mathrm{Spd}(K_0,O_{K_0}),m} \mathrm{Spd}(C_p^\flat, O_{C_p^\flat})$$

gets identified through Beauville-Laszlo glueing with the moduli space that parametrizes B^+_{dR}-lattices contained in $D \otimes_{K_0} B_{dR}$. This basechange comes equipped with $\Gamma^{op}_{K_0}$-action and the set of Γ_K-invariant B^+_{dR}-lattices in $D \otimes_{K_0} B_{dR}$ are in bijection with natural transformations $\mathrm{Spd}(K, O_K) \to Gr(\mathcal{E}_S(D,\varphi))$.

Indeed, if we parametrize Γ_K-invariant lattices using filtrations as in proposition 2.1, then the B^+_{dR}-lattice induced by a K-filtration $\mathrm{Fil}^\bullet D_K$ allows us to construct a tuple

$$((C_p, m), \mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K), \alpha)$$

where α is the canonical meromorphic isomorphism

$$\alpha : \mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K) \dashrightarrow \mathcal{E}(D, \varphi)$$

over $\mathcal{X}_{FF,C} \setminus \infty$ coming from the construction of $\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K)$ as a modification of $\mathcal{E}(D,\varphi)$. A priori this tuple only defines a map $\mathrm{Spa}(C_p^\flat, O_{C_p^\flat}) \to Gr(\mathcal{E}(D,\varphi))$ but since α is Γ^{op}_K-equivariant this descends to the desired map $\mathrm{Spd}(K, O_K) \to Gr(\mathcal{E}(D,\varphi))$.

Going on with the story one defines $Gr^{adm}(\mathcal{E}(D,\varphi)) \subseteq Gr(\mathcal{E}(D,\varphi))$ to be the subsheaf of tuples for which $\mathcal{V}$ is fiberwise semi-stable of slope 0. From Kedlaya-Liu's semi-continuity theorem (see [35] 22.2.1) we know that this defines an open subfunctor which is called the admissible locus. Additionally, a map $\mathrm{Spd}(K, O_K) \to Gr(\mathcal{E}(D,\varphi))$ factors through $Gr^{adm}(\mathcal{E}(D,\varphi))$ if and only if it is coming from a weakly admissible filtration. A very remarkable aspect of the situation is that if $n = dim_{K_0}(D)$ then $Gr^{adm}(\mathcal{E}(D,\varphi))$ admits a pro-étale $\underline{\mathrm{GL}_n(\mathbb{Q}_p)}$-local system $\mathbb{L}$ that "interpolates" between the n-dimensional crystalline representations associated to (D,φ) (See [14] 2.14). Also See [24] for background on quasi-pro-étale local systems.

Remark 2.5. *To be more specific, a pro-étale local system $\mathbb{L}'$ on $\mathrm{Spd}(K, O_K)$ corresponds to a local system $\mathbb{L}'_{C_p^\flat}$ over $\mathrm{Spa}(C_p^\flat, O_{C_p^\flat})$ together with descent data along $\underline{\Gamma_K^{op}} \times \mathrm{Spa}(C_p^\flat, O_{C_p^\flat}) \rightrightarrows \mathrm{Spa}(C_p^\flat, O_{C_p^\flat})$. But pro-étale local systems over $\mathrm{Spa}(C_p^\flat, O_{C_p^\flat})$ are trivial and of the form $\mathbb{L}'_{C_p^\flat} = \underline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V$ for a $\mathbb{Q}_p$-vector space V. Descent datum will correspond to giving for any $\gamma^{op} \in \Gamma_K^{op}$ an isomorphism $\Theta_{\gamma^{op}} : \gamma^{op,*}\mathbb{L}'_{C_p^\flat} \to \mathbb{L}'_{C_p^\flat}$ in a continuous way. By adjunction and passing to global sections as in remark 2.3 we get a Γ_K-representation with values on $\mathrm{GL}(V)$.*

The precise claim that we will use is the following.

Proposition 2.6. *If $\mathrm{Fil}^\bullet D_K$ is a weakly admissible filtration of (D, φ) and*

$$\iota : \mathrm{Spd}(K, O_K) \to Gr^{adm}(\mathcal{E}(D, \varphi))$$

is the map associated to $\mathrm{Fil}^\bullet D_K$, then $\iota^*\mathbb{L}$ is isomorphic to $V_{cris}(D, \varphi, \mathrm{Fil}^\bullet)$ when we regard $\iota^*\mathbb{L}$ as a continuous Γ_K-representation.

Proof. This follows from the definition of the local system $\mathbb{L}$ through Kedlaya-Liu's equivalence [35] 22.3.1, from the definition of the representation associated to a pro-étale local system discussed in remark 2.5 and from the compatibility discussed in remark 2.3 together with the paragraph preceding it. □

Remark 2.7. *In a computation done below a change of sign will appear. In this remark we discuss why this change of sign appears in a simple case. Let the notation be as in proposition 2.6, let $n = dim(D)$ and let $V = V_{cris}(D, \varphi, \mathrm{Fil}^\bullet)$. If we fix a trivialization $\alpha : \mathbb{Q}_p^n \to V$ we may conjugate the action of Γ_K on V by α to obtain a continuous map that we denote*

$$\rho_{H^0,\alpha} : \Gamma_K \to \mathrm{GL}_n(\mathbb{Q}_p).$$

Now, let $Triv(\iota^*\mathbb{L})$ denote the moduli space of trivializations of $\iota^*\mathbb{L}$. It is a $\underline{\mathrm{GL}_n(\mathbb{Q}_p)}$ right torsor over $\mathrm{Spd}(K, O_K)$. The basechange $Triv(\iota^*\mathbb{L})_{C_p}$ receives a semi-linear action by Γ_K^{op} that we can express as:

$$\gamma^{op} : Triv(\iota^*\mathbb{L}) \times_{\mathrm{Spd}(K,O_K)} \mathrm{Spd}(C_p, O_{C_p}) \xrightarrow{(id,\gamma^{op})} Triv(\iota^*\mathbb{L}) \times_{\mathrm{Spd}(K,O_K)} \mathrm{Spd}(C_p, O_{C_p}).$$

The topological space $|Triv_{C_p}(\iota^*\mathbb{L})|$ becomes a free $\mathrm{GL}_n(\mathbb{Q}_p)$ right torsor. An element $\alpha \in Triv(\iota^*\mathbb{L})(C_p)$ defines a unique point $|\alpha| \in |Triv(\iota^*\mathbb{L})_{C_p}|$. By functoriality of $|\cdot|$ we obtain an element $\gamma^{op}(|\alpha|) \in |Triv(\iota^*\mathbb{L})_{C_p}|$. Since $\mathrm{GL}_n(\mathbb{Q}_p)$ acts simply transitively there is a unique element $g^\alpha_{\gamma^{op}} \in \mathrm{GL}_n(\mathbb{Q}_p)$ with $\gamma^{op}(|\alpha|) = |\alpha| \cdot g^\alpha_{\gamma^{op}}$ this defines a group homomorphism

$$\rho_{|\cdot|,\alpha} : \Gamma_K^{op} \to \mathrm{GL}_n(\mathbb{Q}_p).$$

The careful readers should convince themselves that

$$\rho_{H^0,\alpha} = \rho_{|\cdot|,\alpha} \circ (-)^{Spd,-1}$$

where $(-)^{Spd,-1} : \Gamma_K \to \Gamma_K^{op}$ is the group isomorphism $\gamma \mapsto \mathrm{Spd}(\gamma^{-1})$.

2.3 Isocrystals with G-structure.

We keep the notation as above, we let G denote a connected reductive group over $\mathbb{Q}_p$ and $\mathrm{Rep}_G(\mathbb{Q}_p)$ denote the Tannakian category of $\mathbb{Q}_p$-linear algebraic representations of G. Recall the following definition:

Definition 2.8. *(See [22] §3) An isocrystal with G-structure $\mathcal{F}$, is a $\otimes$-exact functor $\mathcal{F} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \varphi\mathrm{-Mod}_{K_0}$.*

To an element $b \in G(K_0)$ and a representation $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$ we associate the isocrystal

$$(D_{b,\rho}, \varphi_{b,\rho}) := (V \otimes K_0, \rho(b) \cdot (Id \otimes \sigma)),$$

ranging this construction over (V, ρ) defines an isocrystal with G-structure

$$\mathcal{F}_b : \mathrm{Rep}_G(\mathbb{Q}_p) \to \varphi\mathrm{-Mod}_{K_0}.$$

We say that two elements $b_1, b_2 \in G(K_0)$ are σ-conjugate to each other if $b_1 = g^{-1} \cdot b_2 \cdot \sigma(g)$ for some element $g \in G(K_0)$. This defines an equivalence relation and b_1 is σ-conjugate to b_2 if and only if $\mathcal{F}_{b_1}$ is isomorphic to $\mathcal{F}_{b_2}$.

Now, when $k = \overline{k}$ the set of equivalence classes of σ-conjugacy is the set $B(G)$ defined and studied by Kottwitz (See [22] §1.4). In this case, every isocrystal with G-structure is isomorphic $\mathcal{F}_b$ for some $b \in G(\breve{K}_0)$ and consequently $B(G)$ parametrizes isomorphism classes of isocrystals with G-structure. The key input in this case is Steinberg's theorem which shows the vanishing of the Galois cohomology set $H^1(\Gamma_{\breve{K}_0}, G(\breve{K}_0))$ (See [37]). The set $B(G)$ has a very rich theory, we recall some of it below. For the rest of this subsection, we will carry the assumption that $k = \overline{k}$, so that $K_0 = \breve{K}_0$.

Recall that the category of isocrystals over K_0 is semisimple and the simple objects can be parametrized by rational numbers $\lambda \in \mathbb{Q}$. In particular, every object $(D, \varphi) \in \varphi\mathrm{-Mod}_{K_0}$ admits a canonical "slope" decomposition

$$(D, \varphi) = \bigoplus_{\lambda \in \mathbb{Q}} (D_\lambda, \varphi_\lambda).$$

If we let ω_b denote the composition $\mathrm{Forg} \circ \mathcal{F}_b$ where

$$\mathrm{Forg} : \varphi\mathrm{-Mod}_{K_0} \to \mathrm{Vec}(K_0)$$

denotes the forgetful functor to the category of vector spaces over K_0, then the slope decomposition defines $\otimes$-exact $\mathbb{Q}$-grading of ω_b. In turn, this grading can be interpreted as a slope morphism $\nu_b : \mathbb{D} \to G_{K_0}$ of pro-algebraic groups, where $\mathbb{D}$ is the pro-torus with character set $X^*(\mathbb{D}) = \mathbb{Q}$.

Consider the abstract group defined as a semi-direct product $G(K_0) \rtimes \sigma \cdot \mathbb{Z}$ where σ has its natural action on $G(K_0)$.

Definition 2.9. *(See [29] 1.8) For an element $b \in G(K_0) = G(\breve{K}_0)$ with conjugacy class $[b] \in B(G)$ we say that:*

1. *b is decent if there exists an integer s such that $(b\sigma)^s = (s \cdot \nu_b)(p)\sigma^s$ as elements of $G(K_0) \rtimes \sigma \cdot \mathbb{Z}$.*
2. *We say that b is basic if the map $\nu_b : \mathbb{D} \to G_{K_0}$ factors through the center of G.*
3. *We say that $[b] \in B(G)$ is basic if all (equivalently some) element of $[b]$ is basic.*

Since we are assuming $k = \overline{k}$ and that G is connected reductive, every σ-conjugacy class $[b] \in B(G)$ contains a decent element (See [29] 1.11).

Assume for the rest of the subsection that G is quasi-split over $\mathbb{Q}_p$, and fix subgroups $A \subseteq T \subseteq B \subseteq G$ as in the notation section.

For $b \in G(K_0)$ we can let ν_b^{dom} denote the unique map $\nu_b^{dom} : \mathbb{D} \to T_{K_0}$ in the conjugacy class of ν_b that is dominant with respect to B. The map ν_b^{dom} factors through A and is defined over $\mathbb{Q}_p$, so we can write $\nu_b^{dom} \in X_*^+(A)_{\mathbb{Q}} = (X_*^+(T_{\overline{\mathbb{Q}_p}}) \otimes_{\mathbb{Z}} \mathbb{Q})^{\Gamma_{\mathbb{Q}_p}}$ (See [36] 3.2, Introduction of [5]). This gives a well defined map $\mathcal{N} : B(G) \to X_*^+(A)_{\mathbb{Q}}$ usually referred to as the Newton map.

Recall Borovoi's algebraic fundamental group $\pi_1(G)$ which can be defined as the quotient of $X_*(T_{\overline{\mathbb{Q}_p}})$ by the co-root lattice. This group comes equipped with $\Gamma_{\mathbb{Q}_p}$ action and Kottwitz constructs a map $\kappa_G : B(G) \to (\pi_1(G))_{\Gamma_{\mathbb{Q}_p}}$ that is usually referred to as the Kottwitz map.

An important result of Kottwitz [22] states that the map of sets

$$(\nu_b^{dom}, \kappa_G) : B(G) \to \mathcal{N} \times \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$$

is injective. This says that these invariants completely determine the isomorphism classes of isocrystals with G-structure. Now, if we are given an element $\mu \in X_*(T_{\overline{\mathbb{Q}_p}})$ with reflex field E we may define an element

$$\overline{\mu} \in X_*^+(A)_{\mathbb{Q}} = X_*^+(T_{\overline{\mathbb{Q}_p}})_{\mathbb{Q}}^{\Gamma_{\mathbb{Q}_p}}$$

by averaging over the dominant elements inside a conjugacy class in the Galois orbit of μ:

$$\overline{\mu} = \frac{1}{[E : \mathbb{Q}_p]} \sum_{\gamma \in Gal(E/\mathbb{Q}_p)} \mu^\gamma$$

We can now recall Kottwitz' definition of the set $B(G, \mu) \subseteq B(G)$.

Definition 2.10. *The set $B(G, \mu)$ consists of those conjugacy classes $[b] \in B(G)$ for which $\kappa_G([b]) = [\mu]$ in $\pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$ and for which $\overline{\mu} - \nu_b^{dom} \in X_*^+(A)_{\mathbb{Q}}$ is a non-negative $\mathbb{Q}$-linear combination of positive co-roots.*

2.4 G-bundles and G-valued crystalline representations

In this subsection we assume again that k is perfect but not necessarily algebraically closed. We also assume that G is reductive over $\mathbb{Q}_p$ but not necessarily quasi-split over $\mathbb{Q}_p$. Just as in the case of schemes, one has a theory of G-bundles over the relative Fargues-Fontaine curve that uses a Tannakian approach (See [34] Appendix to lecture 19 for the details). Given $S \in \mathrm{Perf}_k$ and $\mathcal{F} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \varphi{-}\mathrm{Mod}_{K_0}$ an isocrystal with G-structure we can define a $\otimes$-exact functor $\mathcal{E}_{\mathcal{F},S} : \mathrm{Rep}_G(\mathbb{Q}_p) \to Vec(\mathcal{X}_{FF,S})$ by letting

$$\mathcal{E}_{\mathcal{F},S}(V, \rho) = \mathcal{E}_S(\mathcal{F}(V, \rho)),$$

this defines a G-bundle over $\mathcal{X}_{FF,S}$. When we are given $b \in G(K_0)$ we write $\mathcal{E}_{b,S}$ instead of $\mathcal{E}_{\mathcal{F}_b,S}$. This allow us to extend Tannakianly definition 2.4.

Definition 2.11. *1. We let $Gr(\mathcal{F})$ denote the functor from $\mathrm{Perf}_{\mathrm{Spd}(K_0, O_{K_0})} \to \mathrm{Sets}$ that assigns:*

$$(S^\sharp, f) \mapsto \{((S^\sharp, f), \mathcal{G}, \alpha)\}/\cong$$

Where $(S^\sharp, f)$ is an untilt of S over $\mathrm{Spa}(K_0, O_{K_0})$, $\mathcal{G}$ is a G-bundle over $\mathcal{X}_{FF,S}$ and $\alpha : \mathcal{G} \dashrightarrow \mathcal{E}_{\mathcal{F},S}$ is an isomorphism defined over $\mathcal{X}_{FF,S} \setminus \infty$ and meromorphic along ∞. When $b \in G(K_0)$ we write $Gr(\mathcal{E}_b)$ instead of $Gr(\mathcal{F}_b)$.

2. *We let Gr_G denote the functor from* $\mathrm{Perf}_{\mathrm{Spd}(\mathbb{Q}_p,\mathbb{Z}_p)} \to \mathrm{Sets}$ *that assigns:*

$$(S^\sharp, f) \mapsto \{((S^\sharp, f), \mathcal{G}, \alpha)\}/\cong$$

Where $(S^\sharp, f)$ is an untilt of S over $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$, *$\mathcal{G}$ is a G-bundle over* $\mathrm{Spec}(B^+_{dR}(S^\sharp))$ *and $\alpha : \mathcal{G} \dashrightarrow G$ is a trivialization defined over* $\mathrm{Spec}(B_{dR}(S^\sharp))$.

In the previous definition the meromorphicity condition asks that for every $(V,\rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$ the associated map of vector bundles $\rho_*(\alpha) : \rho_*\mathcal{G} \dashrightarrow \mathcal{E}(D_{b,\rho}, \varphi_{b,\rho})$ is meromorphic along ∞.

As with the GL_n case, the two moduli spaces become isomorphic after basechange to $\mathrm{Spd}(K_0, O_{K_0})$. Instead of fixing a basis one has to fix an isomorphism of the fiber functors:

$$(\omega_{can} \otimes K_0) \cong \omega_{\mathcal{F}}$$

Here $\omega_{\mathcal{F}} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \varphi{-}\mathrm{Mod}_{K_0} \to K_0 - \mathrm{Vec}$ denotes $\mathrm{Forg} \circ \mathcal{F}$, and if $b \in G(K_0)$ we write ω_b instead of $\omega_{\mathcal{F}_b}$. A careful inspection of the construction of ω_b shows that (in contrast with $\omega_{\mathcal{F}}$) there is a canonical choice of isomorphism $\omega_b \cong \omega_{can}$. We won't really use this.

As with the GL_n case we can define the admissible locus as the subsheaf $Gr^{adm}(\mathcal{E}_b) \subseteq Gr(\mathcal{E}_b)$ of those tuples $((S^\sharp, f), \mathcal{G}, \alpha)$ such that $x^*\mathcal{G}$ is the trivial G-bundle for every geometric point $x : \mathrm{Spa}(C', C'^+) \to S$. This is again an open subsheaf and it admits a pro-étale $\underline{G(\mathbb{Q}_p)}$-torsor which we will also denote by $\mathbb{L}$ (See [34] 22.5.2).

To make contact with crystalline representations one needs to recall how the Tannakian formalism interacts with filtrations, we refer the reader to [31] for the details. Recall that given a fiber functor $\omega : \mathrm{Rep}_G(\mathbb{Q}_p) \to Vec(S)$ one can consider $\otimes$-exact filtrations $\mathrm{Fil}^\bullet(\omega)$ which are sequences of $\otimes$-exact functors $\mathrm{Fil}^n(\omega) : \mathrm{Rep}_G(\mathbb{Q}_p) \to Vec(S)$ indexed by $n \in \mathbb{N}$ such that $\mathrm{Fil}^n(\omega) \supseteq \mathrm{Fil}^{n+1}(\omega)$ and that are subject to various compatibility conditions (See [31] chapitre IV §2.1.1, [6] 4.2.6). To such a filtration one can associate a $\otimes$-grading $(gr(\mathrm{Fil}^\bullet(\omega)))$ which produces a morphism of algebraic groups over S, $\mu_{\mathrm{Fil}^\bullet(\omega)} : \mathbb{G}_m \to \underline{Aut}^\otimes(\omega')$ (See [31] chapitre IV §1.3 [6] 4.2.3). Here $\omega' = (gr(\mathrm{Fil}^\bullet(\omega)))$, denotes the $\otimes$-exact functor obtained from the grading after we forget the graded structure. If $x = \mathrm{Spec}(C)$ is a geometric point of S, we may find an isomorphism $\omega'_x \cong \omega_x$ and this defines a conjugacy class of cocharacters into $\underline{Aut}^\otimes(\omega_x)$. This conjugacy class is independent of the isomorphism chosen and we can denote it $[\mu_{\mathrm{Fil}^\bullet(\omega)}(x)]$.

Now, fix an isomorphism $\omega_b \cong \omega_{can}$, we get an isomorphism $\underline{Aut}^\otimes(\omega_b) \cong G_{K_0}$. Furthermore, if we are given a conjugacy class $[\mu]$ of morphisms $\mu : \mathbb{G}_{m,\overline{K_0}} \to G_{\overline{K_0}}$ with field of definition E_0/K_0 (See [6] 6.1.2) contained in C_p, then we can consider the moduli functor of filtrations of ω_b of *type* $[\mu]$. We denote this moduli space by

$$\mathscr{F}l^{\omega_b}_{E_0,[\mu]} : \mathrm{Sch}_{/E_0} \to \mathrm{Sets},$$

it is given by the formula

$$\mathscr{F}l^{\omega_b}_{E_0,[\mu]}(R) = \left\{\mathrm{Fil}^\bullet(\omega_{b,R}) \mid [\mu_{\mathrm{Fil}^\bullet(\omega)}(x)] = [\mu]\, for\, all\, x \in \mathrm{Spec}(R)\right\}$$

where $\mathrm{Fil}^\bullet(\omega_{b,R})$ ranges over the set of $\otimes$-exact filtrations of ω_b. This functor does not depend of our choice of isomorphism $\omega_b \cong \omega_{can}$.

Since G is defined over $\mathbb{Q}_p$ the conjugacy class $[\mu]$ will be defined over a finite extension E of $\mathbb{Q}_p$ contained in C_p and $\mathscr{F}l^{\omega_b}_{E_0,[\mu]}$ is isomorphic to the basechange of a similarly defined moduli functor $\mathscr{F}l^{\omega_{can}}_{E,[\mu]}$. If F/E is a finite extension and $\mu \in [\mu]$ is a representative defined over F then μ defines a parabolic subgroup $P_\mu \subseteq G_F$ and $\mathscr{F}l^{\omega_{can}}_{F,[\mu]}$ is isomorphic to the generalized flag variety G/P_μ. In particular, $\mathscr{F}l^{\omega_{can}}_{E,[\mu]}$ and $\mathscr{F}l^{\omega_b}_{E_0,[\mu]}$ are represented by geometrically connected smooth projective schemes over $\mathrm{Spec}(E)$ and $\mathrm{Spec}(E_0)$ respectively (See [6] 6.1.4). The associated adic space $(\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{ad}$ evaluates on a complete sheafy Huber pair (R,R^+) over $\mathrm{Spa}(E_0,O_{E_0})$ to the set:

$$(\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{ad}(R,R^+) = \left\{\mathrm{Fil}^\bullet(\omega_{b,R}) \mid [\mu_{\mathrm{Fil}^\bullet(\omega)}(x)] = [\mu]\, for\, all\, x \in \mathrm{Spa}(R,R^+)\right\}$$

This description relies on theorem 2.7.7 [18] of Kedlaya and Liu, and on the fact that a morphism of adic spaces $\mathrm{Spa}(R,R^+) \to (\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{ad}$ is given by a morphism of locally ringed spaces $\mathrm{Spa}(R,R^+) \to \mathscr{F}l^{\omega_b}_{E_0,[\mu]}$ by the construction of $(\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{ad}$ ([16] 3.8). In particular, if K/K_0 is a complete non-Archimedean field extension then

$$(\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{ad}(K,O_K) = \mathscr{F}l^{\omega_b}_{E_0,[\mu]}(K).$$

Just as $[\mu]$ allows us to define $\mathscr{F}l^{\omega_b}_{E_0,[\mu]}$ it also allows us to discuss boundedness conditions for Scholze's affine B_{dR}-Grassmanians. Given an algebraically closed non-Archimedean field C in characteristic p and $C^\sharp$ an untilt over E we have an identification

$$G(B_{dR}(C^\sharp))/G(B^+_{dR}(C^\sharp) = Gr_G((C,C^+))$$

([35] 19.1.2, 19.1.1). By the Cartan decomposition we have another identification

$$G(B^+_{dR}(C^\sharp)\backslash G(B_{dR}(C^\sharp)/G(B^+_{dR}(C^\sharp) = Hom(\mathbb{G}_{m,\overline{\mathbb{Q}}_p}, G_{\overline{\mathbb{Q}}_p})/G.$$

This identification sends a conjugacy class $[\mu]$ to the double coset defined by $\xi^\mu := \mu(\xi)$ where $\xi \in B^+_{dR}(C^\sharp)$ is a uniformizer. Notice that to define the map it is crucial to have a fixed embedding $E \subseteq C^\sharp$ so that the conjugacy class of $\mu_{C^\sharp}$ is well defined.

The set of conjugacy classes of cocharacters comes equipped with a partial order called the Bruhat order. Given a map $m \in Gr_G \times_{\mathbb{Q}_p} \mathrm{Spd}(E,O_E)(R,R^+)$ and a geometric point $x : \mathrm{Spa}(C,C^+) \to \mathrm{Spa}(R,R^+)$ we say that m has relative position of type $[\mu]$ at x, (of type $\leq [\mu]$ at x respectively), if the pullback x^*m lands in the double coset associated to $[\mu]$ (a coset bounded by $[\mu]$ respectively). This allow us to define subsheaves

$$Gr^{[\mu]}_{G,E} \subseteq Gr^{\leq[\mu]}_{G,E} \subseteq Gr_G \times \mathrm{Spd}(E,O_E),$$

given by the condition that for every geometric point, the pullback x^*m has relative position $[\mu]$ (bounded by $[\mu]$ respectively). The space $Gr^{\leq[\mu]}_{G,E}$ is spatial diamond that is proper over $\mathrm{Spd}(E,O_E)$ and $Gr^{[\mu]}_{G,E} \subseteq Gr^{\leq[\mu]}_{G,E}$ is an open subdiamond.

We can now compare the affine B_{dR}-Grassmanian to the flag variety. Recall that there is a Tannakianly defined Bialynicki-Birula map ([35] 19.4.2),

$$\pi^{[\mu]}_{BB} : Gr^{[\mu]}_{G,E} \to (\mathscr{F}l^{\omega_{can}}_{E,[-\mu]})^\diamondsuit.$$

We emphasize that there is a change of signs which is a consequence of the change of signs that appeared in remark 2.2 and of our convention on filtrations. Let us sketch the construction of this map. Let $m \in Gr^{[\mu]}_{G,E}(R,R^+)$ and let $(V,\rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$ be a representation. Then $\rho_*(m) \in Gr^{[\rho\circ\mu]}_{GL_n,E}(R,R^+)$ is a tuple $((R^\sharp,f),\mathcal{V}_{\rho,m},\alpha_{\rho,m})$ where $\mathcal{V}_{\rho,m}$ is a projective $B^+_{dR}(R^\sharp)$-module and $\alpha_{\rho,m}$ an isomorphism of the form:

$$\alpha_{\rho,m} : \mathcal{V}_{\rho,m} \otimes_{B^+_{dR}} B_{dR}(R^\sharp) \to V \otimes_E B_{dR}(R^\sharp)$$

Let $\Lambda_{\rho,m}$ denote $\alpha_{\rho,m}(\mathcal{V}_{\rho,m}) \subseteq V \otimes_E B_{dR}(R^\sharp)$ and identify $V \otimes_E R^\sharp$ with

$$(V \otimes_E B^+_{dR}(R^\sharp))/\xi \cdot (V \otimes_E B^+_{dR}(R^\sharp)).$$

We let

$$\mathrm{Fil}^i_{\rho,m}(V\otimes_E R^\sharp)=(\xi^i\cdot\Lambda_{\rho,m}\cap V\otimes_E B^+_{\mathrm{dR}}(R^\sharp))/(\xi^i\cdot\Lambda_{\rho,m}\cap\xi(V\otimes_E B^+_{\mathrm{dR}}(R^\sharp))).$$

Using the techniques discussed in ([35] 19.4.2) one can justify that each $\mathrm{Fil}^i_{\rho,m}(V\otimes_E R^\sharp)$ is a $R^\sharp$-vector sub-bundle of $V\otimes_E R^\sharp$ and that the family $\mathrm{Fil}^\bullet_m(\omega_{can})[V,\rho]:=\mathrm{Fil}^\bullet_{\rho,m}(V\otimes_E R^\sharp)$ is a $\otimes$-exact filtration of ω_{can} over $R^\sharp$. Then, $\pi^{[\mu]}_{BB}(m)=\mathrm{Fil}^\bullet_m(\omega_{can})$.

Let E_0 denote the compositum of E and K_0 in C_p. With an analogous construction as the one sketched above one can also construct the following variation of the Bialynicki-Birula map

$$\pi^{[\mu]}_{BB}:Gr^{[\mu]}_{E_0}(\mathcal{E}_b)\to\mathscr{F}l^{\omega_b}_{E_0,[-\mu]}.$$

This allows the following group-theoretically enhanced rephrasing of proposition 2.1.

Proposition 2.12. *Let $b\in G(K_0)$ let $[\mu]\in Hom(\mathbb{G}_m,G_{\overline{\mathbb{Q}_p}})/G$ and let K/E_0 be a finite field extension. Then, the Bialynicki-Birula map induces a bijection*

$$\pi^{[\mu]}_{BB}:Gr^{[\mu]}(\mathcal{E}_b)(K,O_K)\cong(\mathscr{F}l^{\omega_b}_{E_0,[-\mu]})^\diamondsuit(K,O_K),$$

of $\mathrm{Spd}(K,O_K)$*-valued points.*

Proof. One may take a faithful representation $\rho:G\to\mathrm{GL}(V)$, this induces the following commutative diagram.

$$\begin{array}{ccc} Gr^{[\mu]}_{E_0}(\mathcal{E}_b) & \xrightarrow{\pi^{[\mu]}_{BB}} & (\mathscr{F}l^{\omega_b}_{E_0,[-\mu]})^\diamondsuit \\ \downarrow & & \downarrow \\ Gr^{[\rho\circ\mu]}_{E_0}(\rho_*\mathcal{E}_b) & \xrightarrow{\pi^{[\rho\circ\mu]}_{BB}} & (\mathscr{F}l^{\omega_{\rho(b)}}_{E_0,[-\rho\circ\mu]})^\diamondsuit \end{array}$$

In this diagram, the two vertical arrows are closed immersions. From proposition 2.1, and by taking into account the boundedness conditions, one can deduce that the horizontal bottom arrow induces a bijection

$$\pi^{[\rho\circ\mu]}_{BB}:Gr^{[\rho\circ\mu]}_{E_0}(\rho_*\mathcal{E}_b)(K,O_K)\to(\mathscr{F}l^{\omega_{\rho(b)}}_{E_0,[-\rho\circ\mu]})^\diamondsuit(K,O_K).$$

Clearly the top horizontal arrow is injective since the vertical arrows will induce injections on (K,O_K)-points.

To prove surjectivity let $m\in(\mathscr{F}l^{\omega_b}_{E_0,[-\mu]})^\diamondsuit(K,O_K)$. We may use that the construction of proposition 2.1 and the Beauville-Laszlo theorem are functorial to produce from m a Γ_K^{op}-equivariant modification of G-bundles

$$\alpha:\mathcal{G}\dashrightarrow\mathcal{E}_{b,C_p}.$$

This induces an element $n:\mathrm{Spd}(K,O_K)\to Gr^{[\mu]}_{E_0}(\mathcal{E}_b)$ with $\pi^{[\mu]}_{BB}(n)=m$. ☐

Let $\mathrm{Rep}^{cont}_{\Gamma_K}(\mathbb{Q}_p)$ denote the category of continuous Galois representations. It is a neutral Tannakian category with canonical fiber functor $\omega^{\Gamma_K}_{can}(W,\tau)=W$. Recall that by the Tannakian formalism to specify a continuous representation $\rho:\Gamma_K\to G(\mathbb{Q}_p)$ (up to $G(\mathbb{Q}_p)$-conjugation) it is sufficient to specify a $\otimes$-exact functor $\mathcal{F}:\mathrm{Rep}_\mathrm{G}(\mathbb{Q}_p)\to\mathrm{Rep}^{cont}_{\Gamma_K}(\mathbb{Q}_p)$ for which $\omega^{\Gamma_K}_{can}\circ\mathcal{F}$ is isomorphic to ω_{can}. Now, the full subcategory $\mathrm{Rep}^{crys}_{\Gamma_K}(\mathbb{Q}_p)$ of crystalline representations is Tannakian and we can define crystalline representations with G-structure as those $\otimes$-exact functors $\mathcal{F} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \mathrm{Rep}^{cont}_{\Gamma_K}(\mathbb{Q}_p)$ such that $\mathcal{F}(V, \rho)$ is crystalline for all $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$.

Given a pair (b, μ) with $b \in \breve{G}(K_0)$ and $\mu : \mathbb{G}_{m,K} \to G_K$ we can construct a filtered isocrystal with G-structure by defining a functor

$$\mathcal{F}_{b,\mu} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \varphi\mathrm{-ModFil}_{K/K_0}$$

such that

$$\mathcal{F}_{b,\mu}(V, \rho) = (D_{b,\rho}, \varphi_{b,\rho}, \mathrm{Fil}^\bullet_\mu)$$

with

$$\mathrm{Fil}^i_\mu(D_{b,\rho} \otimes K) = \oplus_{n \leq i}(V \otimes K)^{(\rho\circ\mu(t)\cdot v = t^n \cdot v)}.$$

Definition 2.13. *(See [29] 1.18). We say that a pair (b, μ) with $b \in G(K_0)$ and $\mu : \mathbb{G}_m \to G_K$ is admissible if the functor $\mathcal{F}_{b,\mu}$ only takes values on weakly admissible filtered isocrystals.*

In general, even if (b, μ) is admissible the functor $V_{cris} \circ \mathcal{F}_{b,\mu}$ might not define a crystalline representation with G-structure. Indeed, the composition $\omega^{\Gamma_K}_{can} \circ V_{cris} \circ \mathcal{F}_{b,\mu}$ might fail to be isomorphic to ω_{can}. Nevertheless, this issue goes away if we impose that $[b]$, the σ-conjugacy class of b in $G(\breve{K}_0)$, lies on the Kottwitz set $B(G, \mu)$ (See [6] 11.4.3).

Associated to the admissible pair (b, μ) there is a map $y_{b,\mu} : \mathrm{Spd}(K, O_K) \to \mathscr{F}l^{\omega_b}_{E_0,[-\mu]}$ defined by the filtration $\mathrm{Fil}^\bullet_\mu$ on ω_b, and we can let $x_{b,\mu} : \mathrm{Spd}(K, O_K) \to Gr^{[\mu]}_{E_{0\mu}}(\mathcal{E}_b)$ denote the unique lift of $y_{b,\mu}$ of proposition 2.12. The following is a group-theoretic refinement of proposition 2.6 and it is one of the key inputs from modern p-adic Hodge theory that we will need later on.

Proposition 2.14. *Suppose that (b, μ) is an admissible pair with $[b] \in B(G, \mu)$, then the map $x_{b,\mu} : \mathrm{Spd}(K, O_K) \to Gr^{[\mu]}_{E_0}(\mathcal{E}_b)$ factors through the admissible locus $Gr^{[\mu],adm}_{E_0}(\mathcal{E}_b)$. Moreover, if $\mathbb{L}$ denotes the pro-étale $\underline{G(\mathbb{Q}_p)}$-torsor on $Gr^{adm}(\mathcal{E}_b)$ then $x^*_{b,\mu}\mathbb{L}$ agrees with the crystalline representation with G-structure defined by the functor $V_{cris} \circ \mathcal{F}_{b,\mu}$.*

Proof. Let $(V, \rho) \in \mathrm{Rep}(\mathbb{Q}_p)$ and consider the Γ_K-equivariant modification

$$\alpha : \mathcal{V}_{x_{(b,\mu)},\rho} \dashrightarrow \mathcal{E}_{b,C_p}(V, \rho)$$

associated to $\rho \circ x_{b,\mu} \in Gr(\mathcal{E}_b(V, \rho))(K, O_K)$. The admissibility of (b, μ) implies that $\mathcal{V}_{x_{(b,\mu)},\rho}$ is a semi-stable vector bundle of slope 0. Moreover, by proposition 2.6 there is a canonical identification

$$H^0(\mathrm{X}_{FF,C_p}, \mathcal{V}_{x_{(b,\mu)},\rho}) = V_{cris} \circ \mathcal{F}_{b,\mu}(V, \rho).$$

Since $\mathcal{V}_{x_{(b,\mu)},\rho}$ is semi-stable of slope 0 we have the identification

$$\mathcal{V}_{x_{(b,\mu)},\rho} = \mathcal{O}_{\mathrm{X}_{FF,C_p}} \otimes H^0(\mathrm{X}_{FF,C_p}, \mathcal{V}_{x_{(b,\mu)},\rho}).$$

Since $[b] \in B(G, \mu)$ then $\omega^{\Gamma_K}_{can} \circ V_{cris} \circ \mathcal{F}_{b,\mu}(V, \rho) \cong \omega_{can}$, and the functor

$$\mathcal{V}_{x_{(b,\mu)},-} : \mathrm{Rep}_G(\mathbb{Q}_p) \to Vec_{\mathrm{X}_{FF,C_p}}$$

is isomorphic to $\omega_{can}(-) \otimes \mathcal{O}_{\mathrm{X}_{FF,C_p}}$. Which says precisely that the G-torsor induced by a geometric point over $x_{(b,\mu)}$ is the trivial G-torsor so that $x_{(b,\mu)}$ lies in the admissible locus.

For the last part of the statement we may reason as in 2.5 by observing that quasi-pro-étale $\underline{G(\mathbb{Q}_p)}$-local systems over $\mathrm{Spd}(C_p^\flat, O_{C_p^\flat})$ are trivial and that $x^*_{b,\mu}\mathbb{L}$ can be interpreted as descent datum, which in turn can be interpreted as continuous Galois representations. Using the identity

$$H^0(\mathrm{X}_{FF,C_p}, \mathcal{V}_{x_{(b,\mu)},\rho}) = V_{cris} \circ \mathcal{F}_{b,\mu}(V, \rho)$$

one can justify that we get the correct Galois representation. □

2.5 M. Chen's result on p-adic Hodge Theory

In this subsection we assume that $k = \overline{k}$ so that $K_0 = \breve{K}_0$, we also assume that G is an unramified reductive group over $\mathbb{Q}_p$. In this case the group is quasi-split and we may choose groups $A \subseteq T \subseteq B \subseteq G$ as we have done in the notation.

Definition 2.15. *(See [4] 5.0.4, [5] 2.5.6) Recall the notation of definition 2.10. We say that a pair $([b], [\mu])$ with $[b] \in B(G, \mu)$ and $[\mu] \in X_*(T_{\overline{\mathbb{Q}_p}})$ is HN-irreducible if all the coefficients of $\overline{\mu} - \nu_b^{dom}$ as a $\mathbb{Q}$-linear combination of simple coroots are strictly positive.*

In section §4 the following result of M. Chen will be a key ingredient.

Theorem 2.16. *(See [4] 5.0.6)*
Let $\mu : \mathbb{G}_m \to G_K$ be a morphism and let $b \in G(K_0)$ be a decent element such that $[b] \in B(G, \mu)$ and $[\mu]$ has reflex field E. Suppose that the map $\mathrm{Spec}(K) \to \mathscr{F}l^{\omega_b}_{\breve{E},[-\mu]}$ induced by the filtration defined by μ maps to the generic point of $|\mathscr{F}l^{\omega_{can}}_{E,[-\mu]}|$ under the map

$$\mathscr{F}l^{\omega_b}_{\breve{E},[-\mu]} = \mathscr{F}l^{\omega_{can}}_{E,[-\mu]} \times_E \breve{E} \to \mathscr{F}l^{\omega_{can}}_{E,[-\mu]},$$

induced from the canonical isomorphism $\omega_{can} \otimes_{\mathbb{Q}_{p^s}} K_0 \cong \omega_b$. Assume further that the pair $([b], [\mu])$ is HN-irreducible, then the following hold:

1. *The pair (b, μ) is admissible and defines a crystalline representation $\xi_{b,\mu} : \Gamma_K \to G(\mathbb{Q}_p)$, well-defined up to conjugation.*

2. *The Zariski closure of $\xi_{b,\mu}(\Gamma_K) \subseteq G$ contains G^{der} and $\xi_{b,\mu}(\Gamma_K)$ contains an open subgroup of $G^{der}(\mathbb{Q}_p)$.*

Remark 2.17. *M. Chen's result is slightly stronger, but this is the formulation that we will use below. Observe that K has infinite transcendence degree over E, so it makes sense for a K-point to lie topologically over the generic point of $\mathscr{F}l^{\omega_{can}}_{E,[-\mu]}$.*

Combining proposition 2.14 with Chen's theorem 2.16 and using the fact that every element $b \in G(K_0)$ is σ-conjugate to a decent one we can deduce the following statement.

Corollary 2.18. *Let $b \in G(K_0)$ and $\mu \in X_*^+(T_{\overline{\mathbb{Q}_p}})$. Suppose that $[b] \in B(G, \mu)$. For every finite extension K/K_0 there is a map $x : \mathrm{Spd}(K, O_K) \to Gr(\mathcal{E}_b)_E^{[\mu],adm}$ such that if $\rho_x : \Gamma_K \to G(\mathbb{Q}_p)$ denotes the Galois representation associated to $x^*\mathbb{L}$, then $\rho_x(\Gamma_K) \cap G^{der}(\mathbb{Q}_p)$ is open in $G^{der}(\mathbb{Q}_p)$.*

2.6 The geometric realization of $\mathbb{L}$ and p-adic shtukas

In this section we assume $k = \overline{k}$ and we let G be any reductive group over $\mathbb{Q}_p$. We fix $b \in G(K_0)$, $[\mu] \in Hom(\mathbb{G}_m, G_{\overline{\mathbb{Q}_p}})$ and we let $E_0 = K_0 \cdot E$ denote the field of definition of $[\mu]$ over K_0. Let $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ denote an open compact subgroup, recall the moduli space of p-adic shtukas that appears in the Berkeley notes.

Definition 2.19. *(See [34] 23.3.1) We define $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} : \mathrm{Perf}_k \to \mathrm{Sets}$ as the presheaf that assigns to $S \in \mathrm{Perf}_k$ isomorphism classes of tuples*

$$((S^\sharp, f), \mathcal{E}, \alpha, \mathbb{P}_{\mathcal{K}}, \iota)$$

such that:

1. *$(S^\sharp, f)$ is an untilt of S over E_0.*

2. *$\mathcal{E}$ is a G-bundle on the relative Fargues-Fontaine $\mathcal{X}_{FF,S}$ curve whose fibers on geometric points of S are isomorphic to the trivial G-torsor.*

3. *$\alpha : \mathcal{E} \dashrightarrow \mathcal{E}_b$ is a modification of G-bundles defined over $\mathcal{X}_{FF,S} \setminus S^\sharp$ meromorphic along $S^\sharp$ and whose type is bounded by $[\mu]$ on geometric points.*

4. *$\mathbb{P}_\mathcal{K}$ is a pro-étale $\underline{\mathcal{K}}$-torsor and ι is an identification of $\mathbb{P}_\mathcal{K} \times^\mathcal{K} \underline{G(\mathbb{Q}_p)}$ with the pro-étale $\underline{G(\mathbb{Q}_p)}$-torsor that $\mathcal{E}$ defines under the equivalence of [34] theorem 22.5.2.*

It is proven in [34] that the presheaves $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}}$ are locally spatial diamonds over $\mathrm{Spd}(E_0, O_{E_0})$, and that whenever μ is a minuscule conjugacy class of cocharacters then $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}}$ is represented by the diamond associated to a smooth rigid-analytic space over $\mathrm{Spa}(E_0, O_{E_0})$. As Scholze and Weinstein prove ([34] 24.3.5) these moduli spaces are group-theoretic generalization of (the generic fiber of) Rapoport-Zink spaces. Since all of our arguments work for the general case of moduli spaces of p-adic shtukas we will not make distinction with the minuscule case.

Scholze and Weinstein construct a family of "Grothendieck-Messing" period morphisms

$$\pi_{GM,\mathcal{K}} : \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \to Gr^{adm,\leq[\mu]}_{E_0}(\mathcal{E}_b)$$

given by the formula:

$$((S^\sharp, f), \mathcal{E}, \alpha, \mathbb{P}_\mathcal{K}, \iota) \mapsto ((S^\sharp, f), \mathcal{E}, \alpha)$$

For every $\mathcal{K}$ this gives a surjective étale morphism of locally spatial diamonds. Moreover, this family is functorial on $\mathcal{K}$. That is, if $\mathcal{K}_1 \subseteq \mathcal{K}_2$ are two compact and open subsets then we get a commutative diagram of étale maps,

$$\begin{tikzcd}
\mathrm{Sht}_{G,b,[\mu],\mathcal{K}_1} \arrow[rr, "\pi_{\mathcal{K}_1,\mathcal{K}_2}"] \arrow[dr, "\pi_{GM,\mathcal{K}_1}"'] & & \mathrm{Sht}_{G,b,[\mu],\mathcal{K}_2} \arrow[dl, "\pi_{GM,\mathcal{K}_2}"] \\
& Gr^{adm,\leq[\mu]}_{E_0}(\mathcal{E}_b) &
\end{tikzcd}$$

where the transition map $\pi_{\mathcal{K}_1,\mathcal{K}_2}$ is the one deduced from assigning to $\mathbb{P}_{\mathcal{K}_1}$ the corresponding $\underline{\mathcal{K}_2}$-torsor $\mathbb{P}_{\mathcal{K}_1} \times^{\mathcal{K}_1} \mathcal{K}_2$. Also, if $\mathcal{K}_1 \subseteq \mathcal{K}_2$ is normal of finite index then the transition maps $\pi_{\mathcal{K}_1,\mathcal{K}_2}$ are surjective and finite étale.

The flexibility of the category of diamonds allows us to define moduli spaces of p-adic shtukas associated to an arbitrary compact subgroup $\mathcal{K}' \subseteq G(\mathbb{Q}_p)$ including the case $\mathcal{K}' = \{e\}$ (which is usually referred to as the infinite level). Indeed, the set of compact open subgroups $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ containing $\mathcal{K}'$ is co-filtered and has intersection equal to $\mathcal{K}'$. We may define the limit of diamonds $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}'} = \varprojlim_{\mathcal{K}' \subseteq \mathcal{K}} \mathrm{Sht}_{G,b,[\mu],\mathcal{K}}$, together with a period map

$$\pi_{GM,\mathcal{K}'} : \mathrm{Sht}_{G,b,[\mu],\mathcal{K}'} \to Gr^{adm,\leq[\mu]}_{E_0}(\mathcal{E}_b).$$

This sheaf has the structure of a locally spatial diamond. Moreover, although the period map in general might not be étale it is always a quasi-proétale map (See [32] 10.1).

Moduli spaces of shtukas at infinite level ($K' = \{e\}$) have the following pleasant description,

$$\mathrm{Sht}_{G,b,[\mu],\infty}(S) = \{(S^\sharp, f), \alpha : G \dashrightarrow \mathcal{E}_b\}$$

where $(S^\sharp, f)$ denotes an untilt of S over E_0, G denotes the trivial G-bundle over $\mathcal{X}_{FF,S}$ and α is a modification of G-bundles over $\mathcal{X}_{FF,S} \setminus S^\sharp$, meromorphic along $S^\sharp$ and whose type is bounded by $[\mu]$ on geometric points. The natural action of $G(\mathbb{Q}_p)$ on the trivial torsor G induces a right action of $\underline{G(\mathbb{Q}_p)}$ on $\mathrm{Sht}_{G,b,[\mu],\infty}$ (See §2.8 to contrast the $\underline{G(\mathbb{Q}_p)}$-action to more obvious $G(\mathbb{Q}_p)$-action). Scholze and Weinstein prove that the period map $\pi_{GM,\infty}$ together with the action of $\underline{G(\mathbb{Q}_p)}$ is the geometric realization of the pro-étale $\underline{G(\mathbb{Q}_p)}$-torsor $\mathbb{L}$ over $Gr^{adm,\leq[\mu]}_{E_0}(\mathcal{E}_b)$. In other words, they prove that the two definitions, the one given directly and the one given in terms of a limit, agree.

2.7 Weil descent

In this section we discuss Weil descent datum and its induced Weil-group action, for this subsection we assume $k = \overline{k}$ so that $K_0 = \breve{K}_0$. Recall that we defined $W_{\breve{E}/E}$ as the subset of continuous automorphisms of C_p that act as $\hat{\sigma} := Id_E \otimes \sigma^{n\cdot s}$ on $\breve{E} = E \cdot K_0$. It evidently contains $\Gamma_{\breve{E}}$ and we may topologize $W_{\breve{E}/E}$ so that $\Gamma_{\breve{E}} \hookrightarrow W_{\breve{E}/E}$ is a topological immersion and an open map. We get a strict exact sequence of topological groups

$$e \to \Gamma_{\breve{E}} \to W_{\breve{E}/E} \to \hat{\sigma}^{\mathbb{Z}} \to e.$$

Whenever $g \in W_{\breve{E}/E}$ we will write $g^{op} \in W^{op}_{\breve{E}/E}$ for the morphism of spaces $g^{op} : \mathrm{Spd}(C_p, O_{C_p}) \to \mathrm{Spd}(C_p, O_{C_p})$ induced by the map of fields. Note that if $g_1 = g_2 \circ g_3$ in $W_{\breve{E}/E}$ then $g_1^{op} = g_3^{op} \circ g_2^{op}$ in $W^{op}_{\breve{E}/E}$.

Definition 2.20. *1. Let $\mathcal{G}$ be a v-sheaf over $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$, a Weil descent datum for $\mathcal{G}$ is an isomorphism $\tau : \mathcal{G} \to \hat{\sigma}^{op,*}\mathcal{G}$ over $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$.*

2. Given Weil descent datum for $\mathcal{G}$ and $n \in \mathbb{N}$ we define inductively

$$\tau^n = \hat{\sigma}^{op,*}(\tau^{n-1}) \circ \tau : \mathcal{G} \to \hat{\sigma}^{op,*}\mathcal{G} \to \hat{\sigma}^{op,*,n}\mathcal{G}.$$

For $-n$ we define $\tau^{-n} = \hat{\sigma}^{op,*,n}([\tau^n]^{-1}) : \mathcal{G} \to \hat{\sigma}^{op,*,-n}\mathcal{G}$. We also define $\tau^0 = Id_{\mathcal{G}}$.

Weil descent datum will provide us with actions by $W^{op}_{\breve{E}}$ instead of only $\Gamma^{op}_{\breve{E}}$. In the following sections we will need to endow our spaces with continuous actions rather than plain actions by an abstract group. An efficient way to provide a v-sheaf with a continuous action is to endow it with the action of the group sheaf $\underline{W^{op}_{\breve{E}}}$ that parametrizes continuous maps $|\mathrm{Spa}(R, R^+)| \to W^{op}_{\breve{E}}$.

Lemma 2.21. *Suppose we are given a right $\underline{\Gamma_{\breve{E}}}$-action on a v-sheaf,*

$$m : \mathcal{F} \times \underline{\Gamma_{\breve{E}}} \to \mathcal{F},$$

and suppose we are given a group homomorphism $\theta : W^{op}_{\breve{E}} \to \mathrm{Aut}(\mathcal{F})$ such that $\theta(\gamma^{op}) = m(-, \gamma)$ for all constant elements $\gamma \in \Gamma_{\breve{E}} \subseteq \underline{\Gamma_{\breve{E}}}$. Then there is a unique right $\underline{W_{\breve{E}/E}}$-action $m' : \mathcal{F} \times \underline{W_{\breve{E}/E}} \to \mathcal{F}$ with $m'_{|\underline{\Gamma_{\breve{E}}}} = m$ and $\theta(\gamma^{op}) = m'(-, \gamma)$ for all constant elements $\gamma \in W_{\breve{E}/E}$.

Proof. Let $W^{disc}_{\breve{E}/E}$ (respectively $\Gamma^{disc}_{\breve{E}}$) denote the sheaf of locally constant maps $|\mathrm{Spa}(R, R^+)| \to W_{\breve{E}/E}$ (respectively $\Gamma_{\breve{E}}$). We observe that any element $g \in \underline{W_{\breve{E}/E}}$ can be written as $g^{disc} \cdot \gamma$ with $\gamma \in \underline{\Gamma_{\breve{E}}}$ and $g^{disc} \in W^{disc}_{\breve{E}}$. Moreover if $g_1^{disc}\gamma_1 = g_2^{disc}\gamma_2$ then $\gamma_1 \cdot \gamma_2^{-1} \in \Gamma^{disc}_{\breve{E}}$. To define an action of $\underline{W_{\breve{E}/E}}$ it is enough to define actions of $\underline{\Gamma_{\breve{E}}}$ and $W^{disc}_{\breve{E}}$ that agree on $\Gamma^{disc}_{\breve{E}}$ because $\underline{W_{\breve{E}/E}}(R, R^+) = W^{disc}_{\breve{E}/E}(R, R^+) \cdot \underline{\Gamma_{\breve{E}}}(R, R^+)$ and $W^{disc}_{\breve{E}}(R, R^+) \cap \underline{\Gamma_{\breve{E}}}(R, R^+) = \Gamma^{disc}_{\breve{E}}(R, R^+)$. Now, θ defines an action $m_\theta : \mathcal{F} \times W^{disc}_{\breve{E}} \to \mathcal{F}$ and the hypothesis ensure that m_θ agrees with m on $\Gamma^{disc}_{\breve{E}}$.

□

Proposition 2.22. *If $(\mathcal{G}, \tau)$ is a v-sheaf over $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$ equipped with a Weil-descent datum then $\mathcal{G} \times_{\breve{E}} \mathrm{Spd}(C_p, O_{C_p})$ comes equipped with a right action by $\underline{W_{\breve{E}/E}}$.*

Proof. We let $\iota : \mathrm{Spd}(C_p, O_{C_p}) \to \mathrm{Spd}(\breve{E}, O_{\breve{E}})$ denote the map induced from the canonical inclusion. By lemma 2.21 it is enough to specify a right action by $\underline{\Gamma_{\breve{E}}}$ and a homomorphism of abstract groups $f : W_{\breve{E}}^{op} \to \mathrm{Aut}(\mathcal{G}_{C_p})$. Since $\mathcal{G}$ is defined over $\breve{E}$ and $\breve{E} = C_p/\underline{\Gamma_{\breve{E}}}$ we already have a well-defined right $\underline{\Gamma_{\breve{E}}}$-action on $\mathcal{G}_{C_p}$. Let $g \in W_{\breve{E}/E}$ restricting to $\hat{\sigma}^n$ on $\breve{E}$, we define $f(g^{op})$ as the g^{op}-linear map that appears in the top triangle of the following commutative diagram with Cartesian squares.

$$\begin{tikzcd}
\mathcal{G}_{C_p} \arrow[r, "\iota^*\tau^n"] \arrow[dd] \arrow[drr, bend right, "f(g^{op})"'] & \iota^*(\hat{\sigma}^{op,*,n}\mathcal{G}) = g^{op,*}(\mathcal{G}_{C_p}) \arrow[rr] \arrow[dd] \arrow[dr, "g^{op}"] & & \mathrm{Spd}(C_p, O_{C_p}) \arrow[dr, "g^{op}"] \arrow[dd, "\iota"] & \\
& & \mathcal{G}_{C_p} \arrow[rr] \arrow[dd] & & \mathrm{Spd}(C_p, O_{C_p}) \arrow[dd, "\iota"] \\
\mathcal{G} \arrow[r, "\tau^n"] & \hat{\sigma}^{op,*,n}\mathcal{G} \arrow[rr] \arrow[dr] & & \mathrm{Spd}(\breve{E}, O_{\breve{E}}) \arrow[dr, "\hat{\sigma}^{op,n}"] & \\
& & \mathcal{G} \arrow[rr] & & \mathrm{Spd}(\breve{E}, O_{\breve{E}})
\end{tikzcd}$$

Checking that f is a group homomorphism is a tedious diagram chase. To prove that the right actions of $\underline{\Gamma_{\breve{E}}}$ and $W_{\breve{E}}^{disc}$ restricted to $\Gamma_{\breve{E}}^{disc}$ are compatible we recall that the action $\underline{\Gamma_{\breve{E}}}$ on $\mathcal{G}_{C_p}$ is constructed as the limit of actions $\Gamma_{F/\breve{E}}$ on $\mathcal{G}_F$ over subfields $F \subseteq C_p$ that are Galois and of finite degree over $\breve{E}$. Each of these actions by a finite discrete group are constructed through a commutative diagram as the one above, except that for $g \in \Gamma_{F/\breve{E}}$ we have a canonical identification $\mathcal{G}_F \to g^{op,*}(\mathcal{G}_F)$. The compatibility boils down to the fact that we defined $\tau^0 = Id_{\mathcal{G}}$. $\square$

Of course given two diamonds with Weil descent datum $(\mathcal{G}_i, \tau_i)$ over $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$ and a map $f : \mathcal{G}_1 \to \mathcal{G}_2$ satisfying a commutative diagram:

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[r, "f"] \arrow[d, "\tau_1"] & \mathcal{G}_2 \arrow[d, "\tau_2"] \\
\hat{\sigma}^{op,*}\mathcal{G}_1 \arrow[r, "\hat{\sigma}^{op,*}f"] & \hat{\sigma}^{op,*}\mathcal{G}_2
\end{tikzcd}$$

the corresponding map $f : \mathcal{G}_1 \times_{\breve{E}} \mathrm{Spd}(C_p, O_{C_p}) \to \mathcal{G}_2 \times_{\breve{E}} \mathrm{Spd}(C_p, O_{C_p})$ will be $\underline{W_{\breve{E}/E}}$-equivariant.

We can give Weil descent datum to the moduli problems we have been working with.

Proposition 2.23.

- *There are canonical isomorphisms of v-sheaves over $\mathrm{Spd}(K_0, O_{K_0})$.*
  1. $\sigma^{op,*} Gr_{K_0}(\mathcal{E}_b) = Gr_{K_0}(\mathcal{E}_{\sigma(b)})$.
  2. $\sigma^{op,*} Gr_{K_0}^{adm}(\mathcal{E}_b) = Gr_{K_0}^{adm}(\mathcal{E}_{\sigma(b)})$.
- *There are canonical isomorphisms of v-sheaves over $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$ compatible with the inclusion and the period morphism.*

1. $\hat{\sigma}^{op,*}Gr^{\leq[\mu]}_{\breve{E}}(\mathcal{E}_b) = Gr^{\leq[\mu]}_{\breve{E}}(\mathcal{E}_{\sigma^s(b)})$.
2. $\hat{\sigma}^{op,*}Gr^{adm,\leq[\mu]}_{\breve{E}}(\mathcal{E}_b) = Gr^{adm,\leq[\mu]}_{\breve{E}}(\mathcal{E}_{\sigma^s(b)})$.
3. $\hat{\sigma}^{op,*}\mathrm{Sht}_{G,b,[\mu],\infty} = \mathrm{Sht}_{G,\sigma^s(b),[\mu],\infty}$

Proof. Recall that $\mathrm{Spd}(K_0, O_{K_0}) = \mathrm{Spd}(k,k)\times_{\mathbb{F}_p^\diamondsuit}\mathbb{Z}_p{}^\diamondsuit$ and that $\sigma^{op} : \mathrm{Spd}(K_0, O_{K_0}) \to \mathrm{Spd}(K_0, O_{K_0})$ gets identified with $\mathrm{Frob}^{op}\times \mathrm{id}$. Given an object

$$[S \to \mathrm{Spd}(K_0, O_{K_0})] \in \mathrm{Perf}_{K_0^\diamondsuit}$$

defined by an untilt $(S^\sharp, f)$ over $\mathrm{Spa}(K_0, O_{K_0})$ we let $S^\sigma \in \mathrm{Perf}_{K_0^\diamondsuit}$ be given by $(S^\sharp, \sigma^{op}\circ f)$. For any sheaf $\mathcal{G}$ over $\mathrm{Spd}(K_0, O_{K_0})$ the functor $\sigma^{op,*}\mathcal{G} : \mathrm{Perf}_{K_0^\diamondsuit} \to \mathrm{Sets}$ is given by the formula $\sigma^{op,*}\mathcal{G}(S) = \mathcal{G}(S^\sigma)$. We remark that although the construction of the relative Fargues-Fontaine curve $\mathcal{X}_{FF,S}$ does not depend on the structure map $S \to \mathrm{Spd}(k,k)$, the construction of the G-bundle $\mathcal{E}_{b,S}$ does. Actually, if $(D,\varphi) \in \varphi{-}\mathrm{Mod}_{K_0}$ then $\mathcal{E}_{S^\sigma}(D,\phi) = \mathcal{E}_S(\sigma^*D, \sigma^*\phi)$, and for isocrystals of the form $(D_{b,\rho}, \varphi_{b,\rho})$, with $b \in G(K_0)$ and $(V,\rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$, one can compute explicitly that

$$(\sigma^*D_{b,\rho}, \sigma^*\varphi_{b,\rho}) = (D_{\sigma(b),\rho}, \varphi_{\sigma(b),\rho}),$$

so that the equalities $\mathcal{E}_{b,S^\sigma} = \mathcal{E}_{\sigma(b),S}$ and $\mathcal{E}_{b,S^{\hat{\sigma}}} = \mathcal{E}_{\sigma^s(b),S}$ hold.

From here the proof of each item is very similar and follows from applying the formula $\sigma^{op,*}\mathcal{G}(S) = \mathcal{G}(S^\sigma)$ (or the analogous formula $\hat{\sigma}^{op,*}\mathcal{G}(S) = \mathcal{G}(S^{\hat{\sigma}})$) to the different moduli spaces. We only spell the details for $\hat{\sigma}^{op,*}Gr^{adm,\leq[\mu]}_{\breve{E}}(\mathcal{E}_b) = Gr^{adm,\leq[\mu]}_{\breve{E}}(\mathcal{E}_{\sigma^s(b)})$.

Fix $S = \mathrm{Spa}(R,R^+)$ together with a map $S \to \mathrm{Spd}(\breve{E}, O_{\breve{E}})$ and a geometric point $x : \mathrm{Spd}(C,C^+) \to S$. Recall that $\hat{\sigma} = Id\otimes\sigma^s$ so that if $\iota : E \to E\otimes_{\mathbb{Q}_{p^s}} K_0 = \breve{E}$ is the natural inclusion then $\hat{\sigma}\circ\iota = \iota$. Recall that $Gr^{adm,\leq[\mu]}_{\breve{E}}(\mathcal{E}_b)(S^{\hat{\sigma}})$ parametrizes modifications $\alpha : \mathcal{G} \dashrightarrow \mathcal{E}_{\sigma(b),S}$ with $\mathcal{G}$ fiberwise the trivial bundle and α bounded on geometric points by $[\mu]$. Now, in the preceding description we use the map $x^{\hat{\sigma}} : \mathrm{Spa}(C,C^+) \to \mathrm{Spd}(\breve{E}, O_{\breve{E}})$ to define the bijection

$$X_*(G_{\overline{\mathbb{Q}_p}})/G \cong_{x^{\hat{\sigma}}} G(B^+_{dR}(C^\sharp))\backslash G(B_{dR}(C^\sharp))/G(B^+_{dR}(C^\sharp))$$

with which we compare against μ. Notice again that the set $G(B^+_{dR}(C^\sharp))\backslash G(B_{dR}(C^\sharp))/G(B^+_{dR}(C^\sharp))$ does not depend of the structure morphism $S \to \mathrm{Spd}(\breve{E}, O_{\breve{E}})$, and that the bijection only depends on the composition $x : \mathrm{Spd}(C,C^+) \to \mathrm{Spd}(E, O_E)$. Since $\hat{\sigma}\circ\iota = \iota$ we may conclude. □

Now, observe that b and $\sigma(b)$ are σ-conjugate by b. More precisely, the family of linear maps

$$\rho(b) : (D_{\sigma(b),\rho}, \varphi_{\sigma(b),\rho}) \to (D_{b,\rho}, \varphi_{b,\rho})$$

is a functorial isomorphism of isocrystals that defines an isomorphism of $\otimes$-exact functors $\phi_b : \mathcal{F}_{\sigma(b)} \to \mathcal{F}_b$. The morphism of isocrystals ϕ_b extends by functoriality to morphisms of G-bundles $\phi_b : \mathcal{E}_{\sigma(b)} \to \mathcal{E}_b$ and allows us to endow our moduli of interest with Weil descent datum, for example:

$$\tau_b : Gr_{K_0}(\mathcal{E}_b) \to \sigma^{op,*}Gr_{K_0}(\mathcal{E}_b) = Gr_{K_0}(\mathcal{E}_{\sigma(b)})$$

and

$$\tau_b : \mathrm{Sht}_{G,b,[\mu],\infty} \to \hat{\sigma}^{op,*}\mathrm{Sht}_{G,b,[\mu],\infty} = \mathrm{Sht}_{G,\sigma^s(b),[\mu],\infty}$$

by the applications

$$[((S^\sharp,f),\mathcal{G},\alpha) \mapsto ((S^\sharp,f),\mathcal{G},(\phi_b^{-1})\circ\alpha)]\ [((S^\sharp,f),\mathcal{G},\alpha) \mapsto ((S^\sharp,f),\mathcal{G},(\phi_b^{-1})^s\circ\alpha)].$$

Moreover, it is not hard to see that the descent datum is compatible with the period morphism π_{GM}. An important feature of the situation is that the Weil descent datum on our moduli spaces only depends on the isomorphism class of the isocrystal $\mathcal{F}_b$. More precisely, if b_1 and b_2 are σ-conjugate by g, $b_1 = g^{-1} b_2 \sigma(g)$ then g induces a commutative diagram like the one below

$$\begin{tikzcd} Gr_{K_0}(\mathcal{E}_{b_1}) \arrow[r, "g"] \arrow[d, "\tau_{b_1}"] & Gr_{K_0}(\mathcal{E}_{b_2}) \arrow[d, "\tau_{b_2}"] \\ \sigma^{op,*}Gr_{K_0}(\mathcal{E}_{b_1}) \arrow[r, "\sigma^{op,*}(g)"] & \sigma^{op,*}Gr_{K_0}(\mathcal{E}_{b_2}). \end{tikzcd}$$

Indeed, this follows from the identity $\sigma(g)b_1^{-1} = b_2^{-1}g$. The same applies to all the spaces considered in proposition 2.23. Using proposition 2.22 we can endow $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ with a right $\underline{W_{\breve{E}/E}}$-action. Moreover, the space $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ with its right $\underline{W_{\breve{E}/E}}$-action are independent of the choice of $b \in [b]$.

2.8 The action of $J_b(\mathbb{Q}_p)$

In this section we let $k = \overline{k}$. In ([22] A.2) Kottwitz shows how to associate to the $\otimes$-functor $\mathcal{F}_b : \mathrm{Rep}_G(\mathbb{Q}_p) \to \varphi\mathrm{-Mod}_{K_0}$ a connected reductive group J_b over $\mathbb{Q}_p$ whose group of $\mathbb{Q}_p$-valued points is the σ-centralizer of b,

$$J_b(\mathbb{Q}_p) = \left\{ g \in G(K_0) \mid g^{-1} \cdot b \cdot \sigma(g) = b \right\}.$$

Let us recall this construction. For any $\mathbb{Q}_p$-algebra R we let $\varphi\mathrm{-Mod}_{K_0} \otimes_{\mathbb{Q}_p} R$ denote the category whose objects are the same as in $\varphi\mathrm{-Mod}_{K_0}$ and morphisms are

$$Hom_R((D_1, \varphi_1), (D_2, \varphi_2)) := Hom_{\varphi\mathrm{-Mod}_{K_0}}((D_1, \varphi_1), (D_2, \varphi_2)) \otimes_{\mathbb{Q}_p} R$$

There is a natural $\otimes$-functor $\beta_R : \varphi\mathrm{-Mod}_{K_0} \to \varphi\mathrm{-Mod}_{K_0} \otimes_{\mathbb{Q}_p} R$ and $J_b(R)$ is defined as $Aut^{\otimes}(\beta_R \circ \mathcal{F}_b)$. With J_b defined in this way we have

$$J_b(\mathbb{Q}_p) = Aut^{\otimes}(\mathcal{F}_b) \subseteq Aut^{\otimes}(\mathrm{Forg} \circ \mathcal{F}_b) = G(K_0).$$

Moreover, recall that the slope decomposition produces a map $\nu_b : \mathbb{D} \to G_{K_0}$, if we denote M_b the centralizer of ν_b in G_{K_0} then $(J_b)_{K_0}$ is isomorphic to M_b. Since the elements of $J_b(\mathbb{Q}_p)$ act on $\mathcal{F}_b$ then we get a homomorphism of abstract groups $J_b(\mathbb{Q}_p) \to Aut(\mathcal{E}_{b,S})$ this already gives an action of $J_b(\mathbb{Q}_p)$ on $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ and the other spaces we have considered, but from this description it is not clear, for example, if this action is continuous with respect to the p-adic topology on $J_b(\mathbb{Q}_p)$. A slightly better approach is to endow our moduli spaces with an action of $\underline{J_b(\mathbb{Q}_p)}$. Let us sketch how to do this following the ideas that author learned from reading ([11] II.4.7).

We let $\mathcal{J}_b : \mathrm{Perf}_{K_0^{\diamond}} \to \mathrm{Sets}$ denote the group sheaf that assigns to $S \to \mathrm{Spd}(K_0, O_{K_0})$ the group of automorphisms of $\mathcal{E}_{b,S}$. This is a sheaf of groups and a locally spatial diamond over $\mathrm{Spd}(K_0, O_{K_0})$. We can endow all of the moduli problems that appear in proposition 2.23 with an evident left action by $\mathcal{J}_b$. Moreover, it is easy to see that this action commutes with the right action of $\underline{G(\mathbb{Q}_p)}$ on $\mathrm{Sht}_{G,b,[\mu],\infty}$.

Recall that the category of isocrystals $\varphi\mathrm{-Mod}_{K_0}$ is naturally $\mathbb{Q}$-graded. This gives a family of compatible $\mathbb{Q}$-gradings on $\mathcal{E}_{b,S}(V, \rho)$ for all $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$ and all $S \in \mathrm{Perf}_{K_0^{\diamond}}$. We let $J_b' \subseteq \mathcal{J}_b$ denote the subsheaf of automorphisms of $\mathcal{E}_b$ that respect the $\mathbb{Q}$-grading. In what follows we construct an injective map $\iota_b : \underline{(J_b(\mathbb{Q}_p))}_{K_0} \to \mathcal{J}_b$ of group diamonds over $\mathrm{Spd}(K_0, O_{K_0})$ that induces an isomorphism onto J'_b. We begin by explaining the vector bundle case.

Suppose (D, φ) is an isocrystal in $\varphi\mathrm{-Mod}_{K_0}$, and that

$$(D, \varphi) = \bigoplus_{\lambda \in \mathbb{Q}} (D_\lambda, \varphi_\lambda)$$

is its slope decomposition. The endomorphism object internal to the category of isocrystals $\underline{E}nd((D, \varphi))$ has as 0-graded piece

$$\bigoplus_{\lambda \in \mathbb{Q}} \underline{E}nd((D_\lambda, \varphi_\lambda)) \subseteq \underline{E}nd((D, \varphi)).$$

Analogously, if we fix $S \in \mathrm{Perf}_{K_0^\diamond}$ we have identifications of internal objects

$$\underline{E}nd(\mathcal{E}_S(D, \phi)) = \mathcal{E}_S(\underline{E}nd((D, \varphi))).$$

The right hand side is naturally graded and we have an injective map from the 0-graded piece

$$\bigoplus_{\lambda \in \mathbb{Q}} \underline{E}nd(\mathcal{E}_S(D_\lambda, \varphi_\lambda)) \subseteq \underline{E}nd(\mathcal{E}_S(D, \phi)).$$

Global sections of this later vector bundle are precisely the endomorphisms of $\mathcal{E}_S(D, \varphi)$ that respect the $\mathbb{Q}$-grading. Now, each term $\underline{E}nd(\mathcal{E}_S(D_\lambda, \varphi_\lambda))$ is an algebra whose underlying vector bundle is trivial. This last implies

$$H^0(\mathcal{X}_{FF,S}, \bigoplus_{\lambda \in \mathbb{Q}} \underline{E}nd(\mathcal{E}_S(D_\lambda, \varphi_\lambda))) = Hom_{cont}(|S|, \bigoplus_{\lambda \in \mathbb{Q}} End_{\varphi-\mathrm{Mod}_{K_0}}(D_\lambda, \varphi_\lambda)).$$

Here the topology on $End_{\varphi-\mathrm{Mod}_{K_0}}(D_\lambda, \varphi_\lambda)$ is the one obtained from knowing that it is a finite dimensional $\mathbb{Q}_p$-vector space. Passing to units and recalling that

$$\bigoplus_{\lambda \in \mathbb{Q}} End_{\varphi-\mathrm{Mod}_{K_0}}(D_\lambda, \varphi_\lambda) = End_{\varphi-\mathrm{Mod}_{K_0}}(D, \varphi)$$

we get our desired map $\iota_{(D,\varphi)} : \underline{Aut(D, \varphi)} \to Aut(\mathcal{E}_S(D, \varphi))$ which identifies the left-hand group with the automorphisms of $\mathcal{E}_S(D, \varphi)$ that respect the $\mathbb{Q}$-grading.

Let us discuss the general case. Given an object $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$ we get a natural map of algebraic groups $J_b \to Aut(\mathcal{F}_b(V, \rho))$. In particular, we get a continuous morphism $\psi_V : J_b(\mathbb{Q}_p) \to Aut(\mathcal{F}_b(V, \rho))(\mathbb{Q}_p)$. Given a continuous map $f : |S| \to J_b(\mathbb{Q}_p)$ we consider the composition $\psi_V \circ f$. This induces an automorphism of $\mathcal{E}_S(\mathcal{F}_b(V, \rho))$ that respects the $\mathbb{Q}$-grading, namely $\iota_{\mathcal{F}_b(V,\rho)}(\psi_V \circ f)$. If we are given a morphism $\pi : (V, \rho_V) \to (W, \rho_W)$ we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{E}_S(\mathcal{F}_b(V, \rho_V)) & \xrightarrow{\mathcal{E}(\mathcal{F}_b(\pi))} & \mathcal{E}_S(\mathcal{F}_b(W, \rho_W)) \\ \downarrow{\scriptstyle \iota_{\mathcal{F}_b(V,\rho)}(\psi_V \circ f)} & & \downarrow{\scriptstyle \iota_{\mathcal{F}_b(V,\rho)}(\psi_W \circ f)} \\ \mathcal{E}_S(\mathcal{F}_b(V, \rho_V)) & \xrightarrow{\mathcal{E}(\mathcal{F}_b(\pi))} & \mathcal{E}_S(\mathcal{F}_b(W, \rho_W)) \end{array}$$

This gives overall an automorphism of $\mathcal{E}_{b,S}$ that respects $\mathbb{Q}$-grading on each $\mathcal{E}_{b,S}(V, \rho)$. This constructs the map $\iota_b : \underline{J_b(\mathbb{Q}_p)} \to \mathcal{J}_b$ which clearly factors through J'_b. Conversely, assume we are given a map $m \in J'_b(S)$. For all $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$ we obtain a continuous map

$$m_{(V,\rho)} : |S| \to Aut(\mathcal{F}_b(V, \rho))(\mathbb{Q}_p) \subseteq End_{\varphi-\mathrm{Mod}_{K_0}}(\mathcal{F}_b(V, \rho)).$$

Moreover, given an arrow $(V,\rho_V)\xrightarrow{\pi}(W,\rho_W)$ we obtain two maps

$$|S|\to Hom_{\varphi-\mathrm{Mod}_{K_0}}(\mathcal{F}_b(V,\rho_V),\mathcal{F}_b(W,\rho_W)).$$

One is given as the composition of $\mathcal{F}_b(\pi)$ with the family of endomorphisms $m_{(W,\rho_W)}$ and the other as the composition of $m_{(V,\rho_V)}$ with $\mathcal{F}_b(\pi)$ in the appropriate order. From the construction of $m_{(V,\rho)}$ these two endomorphisms coincide. We claim this determines a unique continuous map $|S|\to J_b(\mathbb{Q}_p)$. Indeed, $J_b(\mathbb{Q}_p)$ is the subgroup of $\prod_{(V,\rho)} Aut(\mathcal{F}_b(V,\rho))(\mathbb{Q}_p)$ that satisfies the commutativity constraints imposed by the arrows in $Rep_G(\mathbb{Q}_p)$. This gives a map $|S|\to J_b(\mathbb{Q}_p)$ which a priori is only continuous with respect to the weak topology making the maps $J_b(\mathbb{Q}_p)\to Aut(\mathcal{F}_b(V,\rho))(\mathbb{Q}_p)$ continuous. But if (V,ρ) is a faithful representation of G then the map of algebraic groups $J_b\to Aut(V,\rho)$ is a closed immersion. This gives that the weak topology on $J_b(\mathbb{Q}_p)$ is the p-adic topology.

Let us prove that the left action of $\underline{J_b(\mathbb{Q}_p)}$ on our moduli spaces through ι_b commutes, in an appropriate sense, with the Weil group action. The first thing we observe is that the group $\mathcal{J}_b$ itself comes equipped with Weil descent datum. Indeed, $\sigma^{op,*}\mathcal{J}_b$ is canonically identified with $\mathcal{J}_{\sigma(b)}$ and the isomorphism of bundles $\mathcal{E}_{\sigma(b)}\xrightarrow{\phi_b}\mathcal{E}_b$ induces a Weil descent datum

$$\tau_b:\mathcal{J}_b\to\sigma^{op,*}\mathcal{J}_b=\mathcal{J}_{\sigma(b)},$$

obtained from conjugating by ϕ_b. One readily verifies that the action map commutes with Weil descent datum, as in the diagram below.

$$\begin{tikzcd}
\mathcal{J}_b\times_{K_0} Gr_{K_0}(\mathcal{E}_b) \arrow[r, "m"] \arrow[d, "(\tau_b,\tau_b)"] & Gr_{K_0}(\mathcal{E}_b) \arrow[d, "\tau_b"] \\
\sigma^{op,*}\mathcal{J}_b\times_{K_0}\sigma^{op,*}Gr_{K_0}(\mathcal{E}_b) \arrow[r, "\sigma^{op,*}m"] & \sigma^{op,*}Gr_{K_0}(\mathcal{E}_b)
\end{tikzcd}$$

Indeed, both Weil descent data were defined by conjugating by ϕ_b. Mutatis mutandis the same applies to all the moduli spaces that appear in proposition 2.23 and the variants using $\hat{\sigma}$.

The constant group $\underline{J_b(\mathbb{Q}_p)}$ is defined over $\mathrm{Spd}(\mathbb{F}_p)$, this induces a canonical Weil descent datum on $(\underline{J_b(\mathbb{Q}_p)})_{K_0}$. Let us prove that the morphism

$$\iota_b:\underline{J_b(\mathbb{Q}_p)}\to\mathcal{J}_b$$

is compatible with Weil descent datum. Let $S\in\mathrm{Perf}_{\mathbb{F}_p}$, let $f:|S|\to J_b(\mathbb{Q}_p)$ be a continuous map and let $S^\sharp$ denote an untilt of S over K_0. For all $(V,\rho)\in Rep_G(\mathbb{Q}_p)$ we obtain from ι_b and f an automorphism of $\mathcal{E}_S(\mathcal{F}_b(V,\rho))$, and analogously we obtain from $\sigma^*\iota_b$ and f an automorphism of $\mathcal{E}_{S^\sigma}(\mathcal{F}_b(V,\rho))=\mathcal{E}_S(\mathcal{F}_{\sigma(b)}(V,\rho))$. By abuse of notation we let $\sigma:J_b(\mathbb{Q}_p)\to J_{\sigma(b)}(\mathbb{Q}_p)$ denote the group isomorphism obtained from regarding $J_b(\mathbb{Q}_p)$ and $J_{\sigma(b)}(\mathbb{Q}_p)$ as subgroups of $G(K_0)$ and letting σ act on this later group. Consider the following diagram.

$$\begin{tikzcd}
(\underline{J_b(\mathbb{Q}_p)})_{K_0} \arrow[r, "\sigma"] \arrow[d, "\iota_b"] \arrow[dr, "\sigma^*\iota_b"] & (\underline{J_{\sigma(b)}(\mathbb{Q}_p)})_{K_0} \arrow[d, "\iota_{\sigma(b)}"] \\
\mathcal{J}_b \arrow[r, "\tau_b"] & \sigma^*\mathcal{J}_b=\mathcal{J}_{\sigma(b)}
\end{tikzcd}$$

To prove that ι_b is compatible with Weil descent datum one must verify that the lower triangle commutes. One way to do this is to verify that the upper triangle commutes and that the square commutes. Both commutativities are left to the verification of the careful reader. The commutativity of the upper triangle ultimately follows from the fact that if $h : K_0^n \to K_0^n$ is a K_0-linear automorphism given by a matrix (h_{ij}), then $\sigma^* h$ is given by the matrix $(\sigma(h_{ij}))$. The commutativity of the square ultimately follows from the fact that if $g^{-1} \cdot b \cdot \sigma(g) = b$ then the identity $b^{-1} \cdot g \cdot b = \sigma(g)$ also holds.

From this we can conclude that the moduli spaces of proposition 2.23 come equipped with a K_0-linear (respectively $\breve{E}$-linear) left action by $\underline{J_b(\mathbb{Q}_p)}$ that commutes with the σ-linear (respectively $\hat{\sigma}$-linear) right action of $\underline{W_{K_0}}$ (respectively $\underline{W_{\breve{E}/E}}$). Indeed, the action map is compatible with Weil descent datum and since $\underline{J_b(\mathbb{Q}_p)}$ is a constant group defined over $\mathbb{F}_p$ the Weil group action on it is trivial.

2.9 Group functoriality

We start this subsection discussing a convention. As we have discussed above the space $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ comes equipped naturally with a left action by $\underline{J_b(\mathbb{Q}_p)}$ and right actions by $\underline{G(\mathbb{Q}_p)}$ and $\underline{W_{\breve{E}/E}}$. We have also justified that these three actions commute. We may always replace the left $\underline{J_b(\mathbb{Q}_p)}$-action by a right $\underline{J_b(\mathbb{Q}_p)}$-action by defining $\alpha \cdot j := j^{-1} \cdot \alpha$. In this way we can say more succinctly that $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ comes equipped with a right action by the group $\underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \times \underline{W_{\breve{E}/E}}$. Moreover $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ together with its right action by $\underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \times \underline{W_{\breve{E}/E}}$ only depends on b through its associated element $[b] \in B(G)$.

In this section we briefly describe how this action behaves with respect to a morphism of algebraic groups. Fix such a morphism $\text{中} : G \to H$ of reductive groups over $\mathbb{Q}_p$. Let $b_H = \text{中}(b) \in H(L)$ and let $[\mu_H] = [\text{中} \circ \mu]$. From the Tannakian definition of $\mathcal{E}_b := \mathcal{E} \circ \mathcal{F}_b$ and the identity $\mathcal{F}_{b_H} = \mathcal{F}_b \circ \text{中}^*$ we get a canonical identification of H-torsors $\text{中}_* \mathcal{E}_b = \mathcal{E}_{b_H}$ which defines a morphism

$$\text{中}_{\infty,\infty} : \mathrm{Sht}_{G,b,[\mu],\infty} \to \mathrm{Sht}_{H,b_H,[\mu_H],\infty}$$

sending

$$[\alpha : G \dashrightarrow \mathcal{E}_b] \mapsto [\text{中}_* \alpha : H \dashrightarrow \mathcal{E}_{b_H}].$$

Associated to b_H we can form $J_{b_H} = Aut^{\otimes}(\mathcal{F}_{b_H})$ and we get a morphism of algebraic groups $\text{中} : J_b \to J_{b_H}$. We get commutative diagrams

$$\begin{tikzcd} \underline{J_b(\mathbb{Q}_p)} \arrow[r, "\iota_b"] \arrow[d, "\text{中}"] & \mathcal{J}_b(\mathbb{Q}_p) \arrow[d, "\text{中}"] \\ \underline{J_{b_H}(\mathbb{Q}_p)} \arrow[r, "\iota_{b_H}"] & \mathcal{J}_{b_H}(\mathbb{Q}_p) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sht}_{G,b,[\mu],\infty} \arrow[r, "\tau_b"] \arrow[d, "\text{中}_{\infty,\infty}"] & \hat{\sigma}^* \mathrm{Sht}_{G,b,[\mu],\infty} \arrow[d, "\hat{\sigma}^* \text{中}_{\infty,\infty}"] \\ \mathrm{Sht}_{H,b_H,[\mu_H],\infty} \arrow[r, "\tau_{b_H}"] & \hat{\sigma}^* \mathrm{Sht}_{H,b_H,[\mu_H],\infty}. \end{tikzcd}$$

We conclude that the basechange of $\text{中}_{\infty,\infty}$ to $\mathrm{Spd}(C_p, O_{C_p})$ is equivariant with respect to the $\underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \times \underline{W_{\breve{E}/E}}$-action, where $\underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)}$ acts on $\mathrm{Sht}_{H,b_H,[\mu_H],\infty}$ through the map

$$\text{中} : \underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \to \underline{H(\mathbb{Q}_p)} \times \underline{J_{b_H}(\mathbb{Q}_p)}$$

obtained from the map of algebraic groups $\text{中} : G \times J_b \to H \times J_{b_H}$.

We may also impose a level structure $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ to get a family of morphisms

$$\text{中}_{\mathcal{K},\text{中}(\mathcal{K})} : \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \to \mathrm{Sht}_{H,b_H,[\mu_H],\text{中}(\mathcal{K})}.$$

This family of maps ranges over the compact subgroups of $G(\mathbb{Q}_p)$. Notice that even if $\mathcal{K}$ is open in $G(\mathbb{Q}_p)$, 中$(\mathcal{K})$ might not be open in $H(\mathbb{Q}_p)$. Each morphism in this family is $\underline{J_b(\mathbb{Q}_p)}$-equivariant and its basechange to $\mathrm{Spd}(C_p, O_{C_p})$ is $\underline{W_{\breve{E}/E}}$-equivariant.

3 The case of tori

3.1 Norm morphisms

In this section we study $\mathrm{Sht}_{G,b,[\mu],\infty} \times_{\breve{E}} \mathrm{Spd}(C_p, O_{C_p})$ together with its action by $\underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \times \underline{W_{\breve{E}/E}}$ in the case in which G is a torus. We change our notation slightly and let $G = T$ for this case. We remark that this case was tackled by M. Chen in [3] and it was also thoroughly discussed in [9]. We recall the story in a different language.

By the work of Kottwitz we know that every element of $B(T)$ is basic and that the Kottwitz map $\kappa_T : B(T) \to \pi_1(T)_{\Gamma_{\mathbb{Q}_p}} = X_*(T_{\overline{\mathbb{Q}_p}})_{\Gamma_{\mathbb{Q}_p}}$ is a bijection. The sets $B(T, \mu)$ are singletons and are determined by the image of μ in $\pi_1(T)_{\Gamma_{\mathbb{Q}_p}}$.

Let us show that in the case of tori moduli spaces of p-adic shtukas are 0-dimensional.

Proposition 3.1. *If $b \in B(T, \mu)$ then all the maps in the following diagram are isomorphisms:*

$$\begin{tikzcd}
Gr^{adm,[\mu]}_{\breve{E}}(\mathcal{E}_b) \arrow[r] \arrow[dr] & Gr^{[\mu]}_{\breve{E}}(\mathcal{E}_b) \arrow[r] \arrow[d, "\pi_{BB}"] & Gr^{\leq[\mu]}_{\breve{E}}(\mathcal{E}_b) \arrow[d] \\
& (\mathscr{F}l^{\omega_b}_{\breve{E},[-\mu]})^{\lozenge} \arrow[r] & \mathrm{Spd}(\breve{E}, O_{\breve{E}})
\end{tikzcd}$$

Proof. The top and left arrows in the square are isomorphisms since μ is minuscule. Since T is a torus the only parabolic subgroup of T is itself, this gives $\mathscr{F}l^{\omega_b}_{\breve{E},[-\mu]} \cong T_{\breve{E}}/T_{\breve{E}} = \mathrm{Spec}(\breve{E})$.

Now, when $b \in B(T, \mu)$ the admissible locus $Gr^{adm,\leq[\mu]}_{\breve{E}}(\mathcal{E}_b)$ is non-empty and open within $Gr^{[\mu]}_{\breve{E}}(\mathcal{E}_b)$. Since $|\mathrm{Spd}(\breve{E}, O_{\breve{E}})| = \{*\}$ we must have $Gr^{adm,[\mu]}_{\breve{E}}(\mathcal{E}_b) = \mathrm{Spd}(\breve{E}, O_{\breve{E}})$. $\square$

On geometric points the situation is very simple, we have that the natural structure map $Gr^{adm,\leq\mu}_{C_p}(\mathcal{E}_b) \to \mathrm{Spd}(C_p, O_{C_p})$ is an isomorphism and

$$\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p \cong \underline{T(\mathbb{Q}_p)} \times \mathrm{Spd}(C_p, O_{C_p}),$$

since on geometric points every right $\underline{T(\mathbb{Q}_p)}$-torsor is trivial. It becomes more interesting when we compare the action of $\underline{J_b(\mathbb{Q}_p)}$ and $\underline{W_{\breve{E}/E}}$ to that of $\underline{T(\mathbb{Q}_p)}$. We begin by discussing the action of $\underline{J_b(\mathbb{Q}_p)}$.

Recall that if b is basic then J_b is an inner form of T, and that since T is commutative we must have $T = J_b$. More precisely we have a canonical inclusion $J_b(\mathbb{Q}_p) \subseteq T(K_0)$ that induces an isomorphism onto $T(\mathbb{Q}_p)$, we denote by j_b this identification.

Proposition 3.2. *The action of $\underline{T(\mathbb{Q}_p)}$ and $\underline{J_b(\mathbb{Q}_p)}$ are inverse to each other. In other words, if $S \in \mathrm{Perf}_{C_p}$, $f : |S| \to J_b(\mathbb{Q}_p)$ is a continuous map, and $\alpha \in \mathrm{Sht}_{T,b,[\mu],\infty} \times C_p$ then*

$$\alpha \cdot_{J_b(\mathbb{Q}_p)} f = \alpha \cdot_{T(\mathbb{Q}_p)} j_b(f^{-1}).$$

Before starting the proof of proposition 3.2 we recall the following lemma on Tannakian formalism:

Lemma 3.3. *Let X be a quasi-compact separated scheme over $\mathbb{Q}_p$, G an affine algebraic group over $\mathbb{Q}_p$ with center $Z(G)$ and let $\mathcal{T}_1$, $\mathcal{T}_2$ be two G-torsors over X, let $\mathcal{U}$ be a $\mathbb{Q}_p$-linear Tannakian category and let $\mathcal{F} : \mathrm{Rep}_T(\mathbb{Q}_p) \to \mathcal{U}$ denote an exact $\otimes$-functor.*

1. *There is a canonical injection $\iota_{\mathcal{F}} : Z(G)(\mathbb{Q}_p) \to \mathrm{Aut}^{\otimes}(\mathcal{F})$*
2. *There are canonical injections $\iota_i : Z(G)(\mathbb{Q}_p) \to \mathrm{Aut}_X(\mathcal{T}_i)$ for $i \in \{1, 2\}$.*
3. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic over X then the left action of $Z(G)(\mathbb{Q}_p)$ on $\mathrm{Isom}_U(\mathcal{T}_1, \mathcal{T}_2)$ through $\mathrm{Aut}_X(\mathcal{T}_1)$ coincides with the right action of $Z(G)(\mathbb{Q}_p)$ on $\mathrm{Isom}(\mathcal{T}_1, \mathcal{T}_2)$ through $\mathrm{Aut}_X(\mathcal{T}_2)$. That is, $\alpha \circ \iota_1(g) = \iota_2(g) \circ \alpha$ for every $g \in T(\mathbb{Q}_p)$ and $\alpha \in \mathrm{Isom}_X(\mathcal{T}_1, \mathcal{T}_2)$.*

Proof. The proof of the first claim and the second claim are very similar so we only prove the second. Let $\omega_{\mathcal{T}_1}$ and $\omega_{\mathcal{T}_2}$ denote the fiber functors associated to $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. Consider the identity functor $Id : \mathrm{Rep}_G(\mathbb{Q}_p) \to \mathrm{Rep}_G(\mathbb{Q}_p)$, we have that $Z(G)(\mathbb{Q}_p) = \mathrm{Aut}^{\otimes}(Id) \subseteq \mathrm{Aut}^{\otimes}(\omega_{can,\mathbb{Q}_p}) = G(\mathbb{Q}_p)$. For any $g \in Z(G)(\mathbb{Q}_p)$ we let $\eta_g : Id \to Id$ denote the natural transformation that acts on (V, ρ) by $\rho(g)$. Notice that $\eta_g^{(V,\rho)} \in \mathrm{Hom}_{Rep_G}((V, \rho), (V, \rho))$ since g is central.

This gives the desired maps:

$$\iota_i : \mathrm{Aut}^{\otimes}(Id) \to \mathrm{Aut}^{\otimes}(\omega_{\mathcal{T}_i} \circ Id)$$

$$g \mapsto \omega_{\mathcal{T}_i}(\eta_g)$$

Let us prove the third claim, suppose now that $\alpha : \omega_{\mathcal{T}_1} \to \omega_{\mathcal{T}_2}$ is an isomorphism and let $g \in Z(G)(\mathbb{Q}_p)$. We have by definition $\iota_i(g) = \omega_{\mathcal{T}_i}(\eta_g)$. To prove the formula $\alpha \circ \iota_1(g) = \iota_2(g) \circ \alpha$ we must prove that the following diagram is commutative:

$$\begin{tikzcd}
\omega_{\mathcal{T}_1}(V,\rho) \arrow[r, "\alpha"] \arrow[d, "\omega_{\mathcal{T}_1}(\eta_g)"] & \omega_{\mathcal{T}_2}(V,\rho) \arrow[d, "\omega_{\mathcal{T}_2}(\eta_g)"] \\
\omega_{\mathcal{T}_1}(V,\rho) \arrow[r, "\alpha"] & \omega_{\mathcal{T}_2}(V,\rho)
\end{tikzcd}$$

But $\eta_g : (V, \rho) \to (V, \rho)$ is a morphism in $\mathrm{Rep}_G(\mathbb{Q}_p)$, so by definition of natural transformation the diagram must be commutative.

□

Proof of proposition 3.2. We will justify the claim with the aid of the following commutative diagram which we explain below:

$$\begin{tikzcd}
\mathrm{Aut}_{X_{FF,C_p}}(\mathcal{E}_e) & & T(\mathbb{Q}_p) \arrow[ll, "\iota_{\mathcal{E}_b}"'] \arrow[rr, "\iota_{\mathcal{E}_e}"] \arrow[dl, "\iota_{\mathcal{F}_e}"] \arrow[dr, "\iota_{\mathcal{F}_b}"] & & \mathrm{Aut}_{X_{FF,C_p}}(\mathcal{E}_b) \\
J_e(\mathbb{Q}_p) \arrow[u] \arrow[dr, "\iota_e"] & & & & J_b(\mathbb{Q}_p) \arrow[u] \arrow[dl, "j_b"] \\
& & T(K_0) & &
\end{tikzcd}$$

Recall that $\mathcal{F}_b$ and $\mathcal{F}_e$ denote isocrystals with T-structure, that $J_b(\mathbb{Q}_p) = \mathrm{Aut}^{\otimes}(\mathcal{F}_b)$ and that $\mathcal{E}_b = \mathcal{E} \circ \mathcal{F}_b$. The triangles on the left and right of the diagram correspond to the triangles:

$$\begin{tikzcd}
\mathrm{Aut}^{\otimes}(Id) \arrow[rr] \arrow[dr] & & \mathrm{Aut}^{\otimes}(\mathcal{F}_b \circ Id) \arrow[dl] \\
& \mathrm{Aut}^{\otimes}(\mathcal{E} \circ \mathcal{F}_b \circ Id) &
\end{tikzcd}$$

In particular, the triangles on the first diagram are commutative. The bottom square corresponds to the concrete computation of $J_b(\mathbb{Q}_p)$ as a σ-centralizer that is

$$J_b(\mathbb{Q}_p) = \{g \in G(K_0) \mid g^{-1}b\sigma(g) = b\},$$

since T is abelian this is $T(K_0)^{\sigma=Id} = T(\mathbb{Q}_p)$. This implies that the maps $\iota_{\mathcal{F}_b}$ and $\iota_{\mathcal{F}_e}$ of lemma 3.3 are isomorphisms and we have that $j_b = \iota_{\mathcal{F}_b}^{-1}$.

By lemma 3.3, for all $\alpha \in \mathrm{Isom}_{\mathrm{X}_{FF,C_p}\setminus\infty}(\mathcal{E}_e, \mathcal{E}_b)$ and all $t \in T(\mathbb{Q}_p)$ we have $\iota_{\mathcal{F}_b}(t) \circ \alpha = \alpha \circ \iota_{\mathcal{F}_e}(t)$. We can compute the right action of $J_b(\mathbb{Q}_p)$ as follows:

$$\begin{aligned}
\alpha \cdot_{J_b(\mathbb{Q}_p)} j &= j^{-1} \circ \alpha \\
&= \iota_{\mathcal{F}_b}(j_b(j^{-1})) \circ \alpha \\
&= \alpha \circ \iota_{\mathcal{F}_e}(j_b(j^{-1})) \\
&= \alpha \cdot_{T(\mathbb{Q}_p)} j_b(j^{-1})
\end{aligned}$$

On the other hand,

$$\mathrm{Sht}_{T,b,[\mu],\infty}(C_p) \subseteq \mathrm{Isom}_{\mathrm{X}_{FF,C_p}\setminus\infty}(\mathcal{E}_e, \mathcal{E}_b),$$

and this inclusion is $T(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$-equivariant. Moreover, the natural map of sets

$$\mathrm{Sht}_{T,b,[\mu],\infty}(C_p) \to |\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p|$$

is bijective and the $\underline{J_b(\mathbb{Q}_p)}$-action is determined by the $J_b(\mathbb{Q}_p)$-action. This finishes the proof. □

Let us study the Weil group action. In contrast to the actions of $J_b(\mathbb{Q}_p)$ and $T(\mathbb{Q}_p)$ the action of $W_{\breve{E}/E}$ on $\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p$ is not C_p-linear. In particular, we can only compare the actions of $W_{\breve{E}/E}$ and $T(\mathbb{Q}_p)$ on those invariants of $\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p$ that do not depend on the structure morphism to $\mathrm{Spd}(C_p, O_{C_p})$. In our case we compare the continuous actions on the topological space of connected components. As we have seen above this topological space is a topological right $T(\mathbb{Q}_p)$-torsor. Let $x \in \pi_0(\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p)$ and $\gamma \in W_{\breve{E}/E}$. We have

$$x \cdot_{W_{\breve{E}/E}} \gamma = x \cdot_{G(\mathbb{Q}_p)} g_{\gamma,x}$$

for a unique element $g_{\gamma,x} \in T(\mathbb{Q}_p)$. Since the actions of $W_{\breve{E}/E}$ and $T(\mathbb{Q}_p)$ commute we get a group homomorphism $g_{-,x} : W_{\breve{E}}^{op} \to T(\mathbb{Q}_p)$. Since $T(\mathbb{Q}_p)$ is commutative this morphism is independent of x. Moreover, the naive map of sets $\gamma \mapsto g_{\gamma,x}$ which would usually not be a group homomorphism is a group homomorphism again by the commutativity of $T(\mathbb{Q}_p)$. We denote this later group homomorphism by

$$m_{T,\mu} : W_{\breve{E}/E} \to T(\mathbb{Q}_p).$$

The following line of reasoning is taken from [29] lemma 1.22, which in turn is an elaboration of an argument in [21] page 413/41. Let $E \subseteq \overline{\mathbb{Q}_p}$ denote a finite field extension let $\{\mathrm{Tori}_{\mathbb{Q}_p}\}$ denote the category of tori defined over $\mathbb{Q}_p$. Recall the functor $X_*(-) : \{\mathrm{Tori}_{\mathbb{Q}_p}\} \to \mathrm{Sets}$ given by the set of maps $\mathbb{G}_m \to T_{\overline{\mathbb{Q}_p}}$. Consider the subfunctor $X_*^E \subseteq X_*$ given by the subset of maps $\mathbb{G}_m \to T_{\overline{\mathbb{Q}_p}}$ whose field of definition is E. This functor is representable by $\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m$ and comes equipped with a universal cocharacter $\mu_u \in X_*^E(\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m)$. In other words, given a torus $T \in \{\mathrm{Tori}_{\mathbb{Q}_p}\}$ and $\mu \in X_*^E(T)$ there is a unique map $Nm_\mu : \mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m \to T$ of algebraic groups over $\mathbb{Q}_p$ such that $Nm_\mu \circ \mu_u = \mu$ in $X_*(T)$. The universal cocharacter can be expressed on E-points as follows:

$$E^\times \xrightarrow{e \mapsto e\otimes e} (E \otimes E)^\times.$$

Associated to μ_u there is a unique element of $[b_u] \in B(\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m, \mu_u)$ since the Kottwitz map $\kappa : B(G) \to \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$ is bijective for tori. We fix a representative $b_u \in \mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m(\breve{\mathbb{Q}}_p)$ and abreviate by m_{E,μ_u} the map $m_{(\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m, \mu_u)}$ previously constructed.

We can compute the $W_{\breve{E}/E}$-action on $|\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p|$ by reducing it to the universal case. Suppose we are given $\mu \in X_*^E(T)$ and $b \in T(K_0)$ with $[b] \in B(T, \mu)$, then automatically (b, μ) is admissible as in definition 2.13 and from the functoriality of the Kottwitz map we have that $[Nm_\mu(b_u)] = [b]$ in $B(T)$. We may replace b by $Nm_\mu(b_u)$ and we get a norm morphism

$$Nm_\mu : \mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m),b_u,[\mu_u],\infty} \times C_p \to \mathrm{Sht}_{T,b,[\mu],\infty} \times C_p.$$

This map is $E^\times \times W_{\breve{E}/E}$-equivariant when the right space is endowed with the action induced from the map $Nm_\mu : \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)(\mathbb{Q}_p) = E^\times \to T(\mathbb{Q}_p)$. We can deduce the following.

Proposition 3.4. *Let the notation be as above, for all $T \in \{\mathrm{Tori}_{\mathbb{Q}_p}\}$ and $\mu \in X_*^E(T)$ we have*

$$m_{T,\mu} = Nm_\mu \circ m_{E,\mu_u}$$

as maps $W_{\breve{E}/E} \to T(\mathbb{Q}_p)$.

Proof. Fix $x \in \pi_0(\mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m),b_u,[\mu_u],\infty} \times C_p)$ with image $y \in \pi_0(\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p)$ and $\gamma \in W_{\breve{E}/E}$. The equivariance of the norm map with respect to $E^\times$ and $W_{\breve{E}/E}$ allow us to compute:

$$\begin{aligned} y \cdot_{T(\mathbb{Q}_p)} m_{T,\mu}(\gamma) &= y \cdot_{W_{\breve{E}/E}} \gamma \\ &= Nm_\mu(x \cdot_{W_{\breve{E}/E}} \gamma) \\ &= Nm_\mu(x \cdot_{E^\times} m_{E,\mu_u}(\gamma)) \\ &= y \cdot_{T(\mathbb{Q}_p)} Nm_\mu(m_{E,\mu_u}(\gamma)) \end{aligned}$$

□

3.2 The Weil group action on the Lubin-Tate case

Our task now is to compute the action of $W_{\breve{E}/E}$ on $|\mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m),b_u,[\mu_u],\infty} \times C_p|$. This is the only section in which it will pay off to let k be a finite field. Let $E \subseteq \overline{\mathbb{Q}_p}$ be a finite field extension of $\mathbb{Q}_p$, and fix a uniformizer $\pi \in E$. We let $F \subseteq E$ denote the maximal unramified extension, we let $h = [E : \mathbb{Q}_p]$ and we let $s = [F : \mathbb{Q}_p]$. Let $H_{LT,\pi}$ denote a Lubin-Tate formal group law with respect to π [23]. We may think of $H_{LT,\pi}$ as a p-divisible group defined over $\mathcal{O}_E$ and endowed with a strict $\mathcal{O}_E$-action ([8]). This means that the induced $\mathcal{O}_E$-action on $Lie(H_{LT})$ is the canonical one. As a p-divisible group H_{LT} has height h and dimension 1.

We let $\mathbb{M}_{LT} = M(H_{LT,\mathbb{F}_{p^s}})$ denote the covariant Dieudonné module over F obtained from Grothendieck-Messing theory [26]. We normalize the action of Frobenius on the covariant Dieudonné theory as in [2], [35], [33]. Let $\mathcal{M}_{LT}$ denote the Lie algebra of the universal vector extension of H_{LT} over $\mathcal{O}_E$. We have a canonical identification $\mathcal{M}_{LT}[\frac{1}{\pi}] = \mathbb{M}_{LT} \otimes_F E$, this allows us to endow $\mathbb{M}_{LT}$ with the usual one step filtration with $Fil^{-1}(\mathbb{M}_{LT} \otimes_F E) = \mathbb{M}_{LT} \otimes_F E$ and

$$Fil^{-1}(\mathbb{M}_{LT} \otimes_F E)/Fil^0(\mathbb{M}_{LT} \otimes_F E) = Lie(H_{LT})[\frac{1}{\pi}].$$

This data gives an object $D_{LT} = (\mathbb{M}_{LT}, \varphi_{LT}, Fil^{\bullet}(\mathbb{M}_{LT} \otimes_F E))$ in the category of weakly admissible filtered isocrystals. Moreover, due to our normalization of Frobenious action, the crystalline representation associated by Fontaine, $V_{cris}(D_{LT})$, gets identified on the nose with the rational Tate module of H. That is, $V_{cris}(D_{LT}) = T_p(H_{LT})[\frac{1}{p}]$ as Γ_E-representations, we let V_{LT} denote this representation.

The action of $\mathcal{O}_E$ on H_{LT} induces an action of $E^{\times}$ on D_{LT} and on V_{LT} respecting all structures, this way we may endow D_{LT} and V_{LT} with $\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)$-structure if we reason as in [28] remark 3.4. Since $\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)$ is a torus there is a unique cocharacter $\mu_{LT} \in X_*^E(\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m))$ defining the filtration on D_{LT}. We compute μ_{LT}.

We may think of $\mathbb{M}_{LT}$ as an $E \otimes_{\mathbb{Q}_p} F$ module endowed with $Id \otimes \sigma$-linear automorphism φ_{LT}. We get a decomposition

$$\mathbb{M}_{LT} = \bigoplus_{\iota: F \to E} (\mathbb{M}_{LT})_{\iota}$$

of E-vector spaces where F-acts on $(\mathbb{M}_{LT})_{\iota}$ through the embedding $\iota : F \to E$. Since φ_{LT} permutes these embeddings we get that each $(\mathbb{M}_{LT})_{\iota}$ has E-dimension 1. This in particular implies that $(\mathbb{M}_{LT})$ is a rank 1 free $E \otimes_{\mathbb{Q}_p} F$-module. We get a decomposition

$$\mathbb{M}_{LT} \otimes_F E = \bigoplus_{e \in Idem} (\mathbb{M}_{LT} \otimes_F E)_e$$

of $E \otimes_{\mathbb{Q}_p} E$-modules where e ranges over the idempotent elements of $E \otimes_{\mathbb{Q}_p} E$. The cocharacter μ_{LT} corresponds to a grading of $\mathbb{M}_{LT} \otimes_F E$ compatible with this decomposition. Moreover, $gr^{-1}(\mathbb{M}_{LT} \otimes_F E)$ maps isomorphically onto $Lie(H_{LT})[\frac{1}{\pi}]$. Let e_{Δ} denote the idempotent associated to the diagonal map $\Delta : E \otimes_{\mathbb{Q}_p} E \to E$. Since the action of $\mathcal{O}_E$ on $H_{LT,\pi}$ is strict the action of $E \otimes_{\mathbb{Q}_p} E$ on $Lie(H_{LT})[\frac{1}{\pi}]$ is through Δ (i.e. $(e_1 \otimes e_2) \cdot m = e_1 \cdot e_2 \cdot m$). We have that $gr^{-1}(\mathbb{M}_{LT} \otimes_F E) = (\mathbb{M}_{LT} \otimes_F E)_{e_{\Delta}}$ and consequently $gr^0(\mathbb{M}_{LT} \otimes_F E) = \bigoplus_{e \neq e_{\Delta}} (\mathbb{M}_{LT} \otimes_F E)_e$. The cocharacter $\mathbb{G}_{m,E} \to \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)_E$ that defines this grading is on E-valued points the following:

$$E^{\times} \xrightarrow{-1} E^{\times} \xrightarrow{e \mapsto e \otimes e} (E \otimes E)^{\times}$$

In other words, $\mu_{LT} = -\mu_u$. This information is already enough to compute the Weil group action on $\mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m), b_u, [\mu_u], \infty} \times C_p$.

Consider the following identity and notice again the change of signs coming from remark 2.2

$$Gr_E^{adm,[\mu_u]}(\mathcal{E}_{\mathbb{M}_{LT}}) = \mathscr{F}l^{\omega_{b_u}}_{E,[\mu_{LT}]} = \mathrm{Spd}(E, O_E).$$

On this space, $\mathbb{L}$ is characterized by the crystalline representation it defines since this space consists of only one point. See remarks 2.3 and 2.5 and proposition 2.14. From the compatibility of Fontaine's functor with the Tate module we deduce that the crystalline representation associated to $\mathbb{L}$ is the left action of Γ_E on $T_p(H_{LT,\pi})[\frac{1}{p}]$.

After choosing a $E \otimes_{\mathbb{Q}_p} F$ basis for $\mathbb{M}_{LT}$ and letting b_u denote the action of φ_{LT} we get an isomorphism

$$Triv(\mathbb{L}) \times C_p \cong \mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m), b_u, [\mu_u], \infty} \times C_p$$

where the space on the left denotes the moduli space of trivializations of $\mathbb{L}$. The space $Triv(\mathbb{L}) \times C_p$, being defined over $\mathrm{Spd}(E, O_E)$, comes equipped with a canonical Γ_E^{op}-action, but we emphasize that this action is not compatible with the Weil group action $W_{\breve{E}}^{op} \subseteq \Gamma_E^{op}$ on $\mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m), b_u, [\mu_u], \infty} \times C_p$ that we defined in section §2.7. Despite this, the canonical action on $Triv(\mathbb{L}) \times C_p$ will allow us to compute the $W_{\breve{E}/E}$-action we are interested in.

Let k denote an algebraically closed field extension of $\mathbb{F}_{p^s}$ and K_0 as on the notation section. The Weil group action on $Triv(\mathbb{L}) \times C_p$ that we are interested in comes from replacing the canonical Weil descent datum by the Weil descent datum τ induced from the automorphism

$$(\varphi_{LT}^s)^{-1} : \mathbb{M}_{LT} \to (Id \otimes \sigma^s)^* \mathbb{M}_{LT} = \mathbb{M}_{LT}.$$

Let $\gamma \in W_{\breve{E}/E}$ with $\gamma_{|K_0} = \sigma^{n \cdot s}$, and let

$$\Theta_{can}, \Theta_{Weil} : W_{\breve{E}/E}^{op} \to Aut(Triv(\mathbb{L}) \times C_p)$$

denote the action morphisms coming from the canonical and from the "φ_{LT}^s-modified" Weil descent data. Then $\Theta_{can}(\gamma)^{-1} \cdot \Theta_{Weil}(\gamma) = \tau^n$ with τ^n as in definition 2.20.

Now, recall that in the standard (or classical) normalization of covariant Dieudonné theory one defines the isocrystal structure $\psi_{LT} : \sigma^* \mathbb{M}_{LT} \to \mathbb{M}_{LT}$ by defining $\psi_{LT} = \mathbb{M}(\mathcal{V})$ where $\mathcal{V} : H_{LT}^{(p)} \to H_{LT}$ is the Verschiebung map. In the normalization we use we have by definition $\varphi_{LT} := \frac{\psi_{LT}}{p}$. Recall that $\psi_{LT} \circ \mathbb{M}(Frob_{H_{LT}}) = p$, in other words $\varphi_{LT} = \mathbb{M}(Frob_{H_{LT}})^{-1}$. This gives φ_{LT}^s coincides with $\mathbb{M}(Frob_{LT}^{-s})$. If we consider the multiplication map $[\pi] : H_{LT} \to H_{LT}$ restricted to $\mathrm{Spec}(\mathbb{F}_{p^s})$ we see from the definition of a Lubin-Tate formal group law that it agrees with the s-Frobenious automorphism of schemes. That is $Frob_{LT}^{-s}$ coincides with $\frac{1}{\pi}$ as quasi-isogenies. Overall this implies that the action of φ_{LT}^s on $\mathbb{M}_{LT}$ is multiplication by $\frac{1}{\pi} \otimes 1$, and consequently τ acts on $Triv(\mathbb{L})$ via multiplication by $\pi \in E^\times = \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)(\mathbb{Q}_p)$.

We claim now that $m_{\mu,E} = Art_E$ where Art_E denotes Artin's reciprocity map. Indeed, since the crystalline representation associated to $\mathbb{L}$ is the Lubin-Tate character, the action of Θ_{can} on $\pi_0(Triv(\mathbb{L}) \times C)$ when restricted to the inertia subgroup I_E is through the inverse of the Lubin-Tate character. Notice again the sign change, this was discussed on remark 2.7. This also gives the action of Θ_{Weil} since Θ_{can} and Θ_{Weil} agree on I_E. If $\hat{\sigma}_\pi$ denotes the unique lift of Frobenious on W_E^{ab} with $\hat{\sigma}_{\pi|E_\pi} = Id$ with E_π the Lubin-Tate extension associated to π, we see that $\Theta_{can}(\hat{\sigma}_\pi)$ acts trivially on $\pi_0(Triv(\mathbb{L}))$. This gives that $\Theta_{Weil}(\hat{\sigma}_\pi)$ acts on $\pi_0(Triv(\mathbb{L}))$ by τ which is multiplication by π. Specifying the action of I_E and of $\hat{\sigma}_\pi$ is one way of characterizing Artin's reciprocity map Art_E.

The following statement summarizes the results discussed on this section, for this statement we let $k = \overline{k}$:

Theorem 3.5. *(Compare with [3] 4.1) Let T be a torus over $\mathbb{Q}_p$, $b \in T(K_0)$, $\mu \in X_*(T)$ with $[b] \in B(T, \mu)$. Let $E \subseteq C_p$ be the field of definition of μ, let $\mathrm{Art}_E : W_E \to (\Gamma_E)^{ab} \to E^\times$ denote Artin's reciprocity character of local class field theory, let $Nm_\mu : \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m) \to T$ be the unique map with $Nm_\mu \circ \mu_u$ as discussed above and let $Art_{\breve{E}/E}$ denote the composition $Art_{\breve{E}/E} : W_{\breve{E}/E} \to W_E \xrightarrow{Art_E} E^\times$, where the map $W_{\breve{E}/E} \to W_E$ is the one induced by the inclusion of fields $E \subseteq \breve{E} \subseteq C_p$. Then the following hold:*

1. *$\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p$ is a trivial right $\underline{T(\mathbb{Q}_p)}$-torsor over $\mathrm{Spd}(C_p, O_{C_p})$.*
2. *If $s \in \pi_0(\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p)$ and $(g, j, \gamma) \in T(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_{\breve{E}/E}$ then*

$$s \cdot (g, j, \gamma) = s \cdot (g \cdot j_b(j^{-1}) \cdot (Nm_\mu \circ Art_{\breve{E}/E}(\gamma)))$$

where $j_b : J_b(\mathbb{Q}_p) \to T(\mathbb{Q}_p)$ is the isomorphism specified by regarding $J_b(\mathbb{Q}_p)$ as a subgroup of $T(K_0)$.

Since we have a full description of the Galois action we can easily compute from theorem 3.5 the connected components of $\mathrm{Sht}_{T,b,[\mu],\infty}$ as a space over $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$. The computation is easier to explain with the following lemma whose proof we leave to the reader:

Lemma 3.6. *Let $\mathcal{K}$ be a locally profinite group, let L a p-adic field with Galois group Γ_L and $\mathbb{L}_{\mathcal{K}}$ a pro-étale $\underline{\mathcal{K}}$-torsor over $\mathrm{Spd}(L, O_L)$. Define $Triv(\mathbb{L}_{\mathcal{K}})$ as the moduli of trivializations of $\mathbb{L}_{\mathcal{K}}$. Then:*

1. *If C is the p-adic completion of an algebraic closure of L, then the choice of a map $\alpha : \mathrm{Spd}(C, O_C) \to Triv(\mathbb{L}_K)$ determines a group homomorphism $\rho_\alpha : \Gamma_L^{op} \to \mathcal{K}$.*
2. *For any $k \in \mathcal{K}$ we have $\rho_{\alpha \cdot k} = k^{-1} \cdot \rho_\alpha \cdot k$.*
3. *The action of $\mathcal{K}$ on $\pi_0(Triv(\mathbb{L}_{\mathcal{K}}))$ is transitive.*
4. *If $\pi_0(\alpha)$ denotes the unique connected component to which $|\alpha|$ maps to, then the stabilizer subgroup is given by the formula $\mathcal{K}_{\pi_0(\alpha)} = \rho_\alpha(\Gamma_L^{op})$.*

Proposition 3.7. *Let $\mathcal{K} \subseteq T(\mathbb{Q}_p)$ denote the largest compact subgroup, the following statements hold.*

1. *$\pi_0(\mathrm{Sht}_{T,b,[\mu],\infty})$ is a free right $T(\mathbb{Q}_p)/Nm_\mu(Art_{\breve{E}/E}(\Gamma_{\breve{E}}))$-torsor.*
2. *$\pi_0(\mathrm{Sht}_{T,b,[\mu],\mathcal{K}}) = \pi_0(\mathrm{Sht}_{T,b,[\mu],\mathcal{K}} \times C_p)$ and it is a free right $T(\mathbb{Q}_p)/\mathcal{K}$-torsor.*

Proof. The first statement follow directly from lemma 3.6 and theorem 3.5. The second statement follows from the fact that the action of $\Gamma_{\breve{E}}$ is continuous so the action of this compact group factors through the maximal compact subgroup. $\square$

4 On the unramified case.

For this section $k = \overline{k}$. The purpose of this section is to compute $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ together with its right action by $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_{\breve{E}/E}$-action under the assumption that G is an unramified reductive group and that (b, μ) is HN-irreducible (See definition 2.15). We recall that in this case the reflex field is of the form $E = \mathbb{Q}_{p^s}$ for some $s \in \mathbb{N}$ and consequently $\breve{E} = K_0$. Nevertheless, with the notation we have chosen, $W_{\breve{E}/E}$ is the subgroup of W_{K_0} of those automorphisms of C_p that lift a power of $\sigma^s : K_0 \to K_0$.

4.1 Connected components of affine Deligne Lusztig Varieties

As it turns out, the connected components of moduli spaces of p-adic shtukas can be computed from knowledge about the connected components of affine Deligne-Lusztig varieties. In this section we recall the relation. Recall that if G is an unramified group then there is a connected reductive group over $\mathbb{Z}_p$ whose generic fiber is isomorphic to G. Let us fix such a model and by abuse of notation denote it by G. We let $\mathcal{K} = G(\mathbb{Z}_p)$ and we let $\breve{\mathcal{K}} = G(O_{K_0})$. Since we are assuming $k = \overline{k}$, the group G_{K_0} is split over K_0 and we have by the Cartan decomposition a bijection

$$\breve{\mathcal{K}} \backslash G(K_0) / \breve{\mathcal{K}} = X_*(T_{\overline{\mathbb{Q}_p}})$$

given by

$$\mu \mapsto p^\mu := \mu(p) \in T(K_0).$$

We may construct a map $\kappa_G : G(K_0) \to \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$. Given an element $b \in G(K_0)$ there is a unique $\mu' \in X_*(T_{\overline{\mathbb{Q}_p}})$ with $b \in \breve{\mathcal{K}} \backslash p^{\mu'} / \breve{\mathcal{K}}$. Then $\kappa_G(b)$ is defined to be $[\mu']$, the induced class of μ'

in $\pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$. This map is a group homomorphism that is well-defined on σ-conjugacy classes. Moreover, the map constructed in this way descends to the Kottwitz map $\kappa_G : B(G) \to \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$ that we discussed on section §2.3.

Recall that associated to a pair (b, μ) one can associate an affine Deligne Lusztig variety $X^{\leq \mu}_G(b)$. This is a perfect scheme (See [1]) over $\mathrm{Spec}(k)$ whose k-valued points can be described as:

$$X^{\leq \mu}_G(b)(k) = \left\{ g \cdot \breve{\mathcal{K}} \in G(K_0)/\breve{\mathcal{K}} \mid g^{-1} \cdot b \cdot \sigma(g) \in \breve{\mathcal{K}} \backslash p^{\mu'} / \breve{\mathcal{K}} \text{ with } \mu' \leq \mu \right\}$$

In [5], [27] [15], the problem of determining connected components of affine Deligne Lusztig varieties is thoroughly discussed. Although the description in full generality is complicated, in our situation (G reductive and $\mathcal{K}$ hyperspecial) the problem is completely settled. In the references provided above, the connected components are described in three steps. The first step is to pass to the case of a simple adjoint group and it is done as follows:

Theorem 4.1. *(See [5] 2.4.2) Let G^{ad} denote the adjoint quotient of G, then there are natural maps w_G and $w_{G^{ad}}$ and elements $c_{b,\mu} \in \pi_1(G)$ ($c_{b_{ad},\mu_{ad}} \in \pi_1(G^{ad})$ respectively) well-defined up to multiplication by $\pi_1(G)^{\Gamma_{\mathbb{Q}_p}}$ (respectively $\pi_1(G^{ad})^{\Gamma_{\mathbb{Q}_p}}$) making the following diagram commutative and Cartesian:*

$$\begin{tikzcd} X^{\leq \mu}_G(b) \arrow[r] \arrow[d, "w_G"] & X^{\leq \mu_{ad}}_{G^{ad}}(b_{ad}) \arrow[d, "w_{G^{ad}}"] \\ \underline{c_{b,\mu}\pi_1(G)^{\Gamma_{\mathbb{Q}_p}}} \times \mathrm{Spec}(k) \arrow[r] & \underline{c_{b_{ad},\mu_{ad}}\pi_1(G^{ad})^{\Gamma_{\mathbb{Q}_p}}} \times \mathrm{Spec}(k) \end{tikzcd}$$

In the statement above the two sets that appear on the lower horizontal arrow should be interpreted as discrete topological groups so that the product is a disjoint union of copies of $\mathrm{Spec}(k)$. Once one reduces the problem to the adjoint case, one can further simplify to the simple adjoint case by observing that if $G = G_1 \times G_2$ then we get a decomposition

$$X^{\leq \mu}_G(b) = X^{\leq \mu_1}_{G_1}(b_1) \times_k X^{\leq \mu_2}_{G_2}(b_2).$$

This is how the first step is completed in the references.

The second step in the strategy is to reduce the general simple adjoint group case to the case in which (b, μ) is HN-indecomposable. In this work we only consider the case in which (b, μ) is already HN-irreducible which is a stronger condition to being indecomposable. For this reason we do not review this step.

The third and final step is the determination of $\pi_0(X^{\leq \mu}_G(b))$ when G is simple adjoint and (b, μ) is HN-irreducible or when it is HN-indecomposable but not HN-irreducible. Again, we only review the HN-irreducible case.

Theorem 4.2. *([27] 1.1, [5] 1.1, [15] 8.1) If (b, μ) is HN-irreducible and $G = G^{ad}$ is simple and adjoint then $w_G : \pi_0(X^{\leq \mu}_G(b)) \to c_{b,\mu}\pi_1(G^{ad})^{\Gamma_{\mathbb{Q}_p}}$ is a bijection.*

In what follows we rephrase these result in a form that will be more useful for our purposes. For this let G^{der} denote the derived subgroup of G, let $G^{ab} := G/G^{der}$ the maximal abelian quotient and denote by $det : G \to G/G^{der}$ the quotient map. We will often refer to the quotient map $G \to G^{ab}$ as the determinant map.

Corollary 4.3. *If G^{der} is simply connected the natural map $det : X^{\leq \mu}_G(b) \to X^{\leq \mu_{ab}}_{G^{ab}}(b_{ab})$ induced from $det : G \to G^{ab}$ gives a bijection of connected components $\pi_0(X^{\leq \mu}_G(b)) \cong \pi_0(X^{\leq \mu_{ab}}_{G^{ab}}(b_{ab}))$ whenever (b, μ) is HN-irreducible.*

Remark 4.4. *Since $X^{\leq \mu_{ab}}_{G^{ab}}(b_{ab})$ is a disjoint union of copies of* Spec(k) *and* Spec(k) *is algebraically closed, we could say instead that the map $X^{\leq \mu}_{G}(b) \to X^{\leq \mu_{ab}}_{G^{ab}}(b_{ab})$ has geometrically connected fibers.*

Proof. For the convenience of the reader we provide an easy argument using theorems 4.2 and 4.1. A pair (b,μ) is HN-irreducible if and only if for every $\mathbb{Q}_p$-simple factor G_i of G^{ad} with projection map $\pi_i : G \to G_i$ the pair $(b_i,\mu_i) := (\pi_i(b), \pi_i \circ \mu)$ is HN-irreducible. Indeed, the coefficient of $\mu^{dom} - \nu_b^{dom}$ associated to a positive root can be computed on the simple factors of the adjoint quotient. From theorem 4.1 we get a Cartesian diagram:

$$\begin{tikzcd}
\pi_0(X^{\leq \mu}_{G}(b)) \arrow[r] \arrow[d, "w_G"] & \pi_0(X^{\leq \mu_1}_{G_1}(b_1)) \times \cdots \times \pi_0(X^{\leq \mu_n}_{G_n}(b_n)) \arrow[d, "w_{G_i}"] \\
c_{b,\mu}\pi_1(G)^{\Gamma_{\mathbb{Q}_p}} \arrow[r] & c_{b_1,\mu_1}\pi_1(G_1)^{\Gamma_{\mathbb{Q}_p}} \times \cdots \times c_{b_n,\mu_n}\pi_1(G_n)^{\Gamma_{\mathbb{Q}_p}}
\end{tikzcd}$$

The vertical right hand map is a bijection by theorem 4.2 which implies the vertical left hand map is also a bijection by theorem 4.1.

The result follows from showing that in the commutative diagram below the bottom horizontal arrow and the vertical right hand arrow are both bijective.

$$\begin{tikzcd}
\pi_0(X^{\leq \mu}_{G}(b)) \arrow[r] \arrow[d, "w_G"] & \pi_0(X^{\leq \mu_{ab}}_{G^{ab}}(b_{ab})) \arrow[d, "w_{G^{ab}}"] \\
c_{b,\mu}\pi_1(G)^{\Gamma_{\mathbb{Q}_p}} \arrow[r] & c_{b_{ab},\mu_{ab}}\pi_1(G^{ab})^{\Gamma_{\mathbb{Q}_p}}
\end{tikzcd}$$

Since G^{der} is simply connected we have a $\Gamma_{\mathbb{Q}_p}$-equivariant identification $\pi_1(G) \to \pi_1(G^{ab})$ so the bottom map is easily seen to be a bijection. Moreover, the adjoint quotient of G^{ab} is $\{e\}$ and theorem 4.1 says that $w_{G^{ab}} : X^{\leq \mu_{ab}}_{G^{ab}}(b_{ab}) \to c_{b_{ab},\mu_{ab}}\pi_1(G^{ab})^{\Gamma_{\mathbb{Q}_p}}$ is an isomorphism in this case. $\square$

The following theorem explains the role that affine Deligne-Lusztig varieties will play in our computation. This theorem is also the reason that motivated the author to study the specialization map in the context of diamonds and v-sheaves [13].

Theorem 4.5. *(See [13]) Let G be an unramified reductive group over $\mathbb{Q}_p$, μ a conjugacy class of geometric cocharacters and $[b] \in B(G,\mu)$.*

a) There is a continuous and $J_b(\mathbb{Q}_p)$-equivariant specialization map

$$\mathrm{Sp} : |\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p| \to |X^{\leq \mu}_{G}(b)|.$$

b) The specialization map induces a bijection of connected components

$$\pi_0(\mathrm{Sp}) : \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p) \xrightarrow{\cong} \pi_0(X^{\leq \mu}_{G}(b)).$$

4.2 The simply connected case

In this subsection we compute $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty})$ under the assumption that G^{der} is simply connected.

Proposition 4.6. *Suppose that G is as above. The determinant map induces a surjective map of locally spatial diamonds*

$$det : \mathrm{Sht}_{G,b,[\mu],\infty} \to \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty}$$

Proof. We may verify surjectivity after basechanging to an algebraic closure. Moreover, we can choose a section $s : \mathrm{Spa}(C, C^+) \to Gr^{adm,\leq[\mu]}_{K_0}(\mathcal{E}_b)$ and consider the following commutative diagram.

$$\begin{tikzcd}
\underline{G(\mathbb{Q}_p)} \times \mathrm{Spa}(C,C^+) \arrow[rr] \arrow[d, "\cong"] & & \underline{G^{ab}(\mathbb{Q}_p)} \times \mathrm{Spa}(C,C^+) \arrow[d, "\cong"] \\
\mathrm{Sht}_{G,b,[\mu],\infty} \times_{Gr^{\leq[\mu]}_{K_0}(\mathcal{E}_b)} \mathrm{Spa}(C,C^+) \arrow[rr] \arrow[d] & & \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times_{K_0} \mathrm{Spa}(C,C^+) \\
\mathrm{Sht}_{G,b,[\mu],\infty} \times_{K_0} \mathrm{Spa}(C,C^+) \arrow[urr] & &
\end{tikzcd}$$

We can consequently reduce to the surjectivity of $\underline{G(\mathbb{Q}_p)} \to \underline{G^{ab}(\mathbb{Q}_p)}$. That is, we must prove that can lift continuous maps $f \in C^0(|\mathrm{Spa}(R,R^+)|, G^{ab}(\mathbb{Q}_p))$ to a continuous map $\tilde{f} \in C^0(|\mathrm{Spa}(R,R^+)|, G(\mathbb{Q}_p))$. The key point is, of course, that since G^{der} is simply connected by Kneser's theorem [19] the map of groups $G(\mathbb{Q}_p) \to G^{ab}(\mathbb{Q}_p)$ is surjective.

Now, let $Z(G)$ denotes the center of G. We get a strict map of topological abelian groups $Z(G)[\mathbb{Q}_p] \to G^{ab}(\mathbb{Q}_p)$ with finite kernel and cokernel. $Im(Z(G)[\mathbb{Q}_p])$ is an open subgroups and there is a finite number of elements $g_1, \dots, g_n \in G(\mathbb{Q}_p)$ with $\cup_{g_i} g_i \cdot Im(Z(G)[\mathbb{Q}_p]) = G^{ab}(\mathbb{Q}_p)$. The map $\cup_{g_i} g_i \cdot \underline{Z(G)[\mathbb{Q}_p]}) \to \underline{G^{ab}(\mathbb{Q}_p)}$ is surjective and factors through $\underline{G(\mathbb{Q}_p)}$ which finishes the proof. $\square$

Lemma 4.7. *Let G be as above (unramified and such that $G^{der} = G^{sc}$). Let $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ be a hyperspecial subgroup. Suppose (b,μ) is HN-irreducible, then*

$$det : \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \to \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],det(\mathcal{K})}$$

has geometrically connected fibers.

Proof. Since G splits over an unramified extension, we can construct an exact sequence

$$e \to \mathcal{G}^{der} \to \mathcal{G} \to \mathcal{G}^{ab} \to e$$

of reductive groups over $\mathbb{Z}_p$. Indeed, this evident for split groups and we may use étale descent from $\mathrm{Spec}(\mathbb{Z}_{p^s})$ to $\mathrm{Spec}(\mathbb{Z}_p)$ in the general case. An application of Lang's theorem proves that $det(\mathcal{K}) = \mathcal{G}^{ab}(\mathbb{Z}_p)$ which is the maximal bounded subgroup of G^{ab}. By [13] we have a commutative diagram of specialization maps:

$$\begin{tikzcd}
\mid \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times C_p \mid \arrow[r, "det"] \arrow[d, "Sp_{\mathcal{G}}"] & \mid \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],det(\mathcal{K})} \times C_p \mid \arrow[d, "Sp_{\mathcal{G}^{ab}}"] \\
\mid X^{\leq\mu}_{G}(b) \mid \arrow[r, "det"] & \mid X^{\leq\mu_{ab}}_{G^{ab}}(b_{ab}) \mid
\end{tikzcd}$$

The vertical maps give bijections of connected components by theorem 4.5 and the lower horizontal map induces a bijection of connected components by corollary 4.3. $\square$

The following proposition is a particular case of an unpublished result of Hansen and Weinstein that follows from the work done in [14]. We provide an alternative proof that follows the steps of the analogous statement in [4] Lemme 6.1.3.

Proposition 4.8. *Let G be as above and let (b,μ) be HN-irreducible. Then $Gr^{adm,\leq[\mu]}_{K_0}(\mathcal{E}_b)$ is geometrically connected over* $\mathrm{Spd}(K_0, O_{K_0})$.

Proof. Let $\mathrm{Spa}(C, O_C) \to \mathrm{Spd}(K_0, O_{K_0})$ be a map with C algebraically closed non-Archimedean field, $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ a hyperspecial subgroup, and let $\mathcal{M}$ denote a connected component of $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C)$. We consider the restriction of the period morphism $\pi_{GM,\mathcal{K},C} : \mathcal{M} \to Gr_C^{adm,\leq[\mu]}(\mathcal{E}_b)$. By lemma 4.7, $\mathcal{M}$ is an open subdiamond of $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C)$ and by étaleness of $\pi_{GM,\mathcal{K},C}$ the set $U := \pi_{GM,\mathcal{K},C}(\mathcal{M})$ is a connected open subset of $Gr_C^{adm,\leq[\mu]}(\mathcal{E}_b)$. We claim, and prove below, that this open subset doesn't depend on the choice of $\mathcal{M}$. This already implies $Gr_C^{adm,\leq[\mu]}(\mathcal{E}_b) = \pi_{GM,\mathcal{K},C}(\mathcal{M})$ and in particular that it is connected.

Let us prove the claim, for this we take a connected component $\mathcal{M}_\infty$ of $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C)$ that maps to $\mathcal{M}$. Notice that $\pi_{\infty,\mathcal{K}}(\mathcal{M}_\infty) = \mathcal{M}$ since the groups $\mathcal{K}' \subseteq \mathcal{K}$ of finite index are cofinal and for those the transition maps

$$\mathrm{Sht}_{G,b,[\mu],\mathcal{K}'} \times \mathrm{Spd}(C, O_C) \to \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C)$$

are finite étale and surjective so that on topological level the transition maps are open and closed. This also implies $U = \pi_{GM}(\mathcal{M}_\infty)$.

By lemma 4.7 $\pi_0(\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C)) \to \pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],det(\mathcal{K})})$ is a bijection. Let $\mathcal{M}'$ denote some other connected component, and let z and z' denote the elements defined by $\mathcal{M}$ and $\mathcal{M}'$ in $\pi_0(\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C))$. Now, $G^{ab}(\mathbb{Q}_p)$ acts transitively on

$$\pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times \mathrm{Spd}(C, O_C))$$

and consequently $G^{ab}(\mathbb{Q}_p)/det(\mathcal{K})$ acts transitively on

$$\pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],det(\mathcal{K})} \times \mathrm{Spd}(C, O_C)).$$

This allow us to find an element $g \in G(\mathbb{Q}_p)$ with $det(z) \cdot det(g) = det(z')$. Let $x : \mathrm{Spd}(C, C^+) \to U$ be a geometric point and let $\overline{x} : \mathrm{Spd}(C, C^+) \to \mathcal{M}_\infty$ be a lift of x. Consider $\overline{x} \cdot g$. On one hand it is a lift of x, and on the other hand its projection to $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C)$ lands on $\mathcal{M}'$. Indeed, we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Sht}_{G,b,[\mu],\infty}(C, C^+) \arrow[r, "det"] \arrow[d, "\pi_{\infty,\mathcal{K}}"] & \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty}(C, C^+) \arrow[d, "\pi_{\infty,det(\mathcal{K})}"] \\
\mathrm{Sht}_{G,b,[\mu],\mathcal{K}}(C, C^+) \arrow[r, "det"] & \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],det(\mathcal{K})}(C, C^+)
\end{tikzcd}$$

We have:

$$\begin{aligned}
det \circ \pi_{\infty,\mathcal{K}}(\overline{x} \cdot g) &= \pi_{\infty,det(\mathcal{K})} \circ det(\overline{x} \cdot g) \\
&= \pi_{\infty,det(\mathcal{K})}[det(\overline{x}) \cdot det(g)] \\
&= \pi_{\infty,det(\mathcal{K})} \circ det(\overline{x}) \cdot det(g)
\end{aligned}$$

This map lands on $det(z) \cdot det(g)$ which is $det(z')$. This implies that $\pi_{\infty,\mathcal{K}}(\overline{x} \cdot g)$ is a geometric point on $\mathcal{M}'$.

This proves that any topological point of U also comes from a point in $\mathcal{M}'$, and that $\pi_{GM}(\mathcal{M}) \subseteq \pi_{GM}(\mathcal{M}')$. Since the roles of $\mathcal{M}$ and $\mathcal{M}'$ in the proof can be reversed the converse also holds. ☐

Lemma 4.9. *Let $\mathcal{K}$ be a hyperspecial subgroup of $G(\mathbb{Q}_p)$ and let $\mathcal{K}^{der} = \mathcal{K} \cap G^{der}(\mathbb{Q}_p)$. Let $m \in \pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times \mathrm{Spd}(C, O_C))$ and let X_m denote the space defined by the following Cartesian diagram:*

$$\begin{tikzcd}
X_m \arrow[r] \arrow[d] & \mathrm{Spd}(C, O_C) \arrow[d, "m"] \\
\mathrm{Sht}_{G,b,[\mu],\infty} \times C \arrow[r, "det"] & \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times C.
\end{tikzcd}$$

Then $\mathcal{K}^{der}$ acts transitively on $\pi_0(X_m)$.

Proof. Since $\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times \mathrm{Spd}(C, O_C)$ is 0-dimensional, the space X_m is the collection of connected components of $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C)$ that map to m. Let $x, y \in \pi_0(X_m)$, using lemma 4.7 we see that $\pi_{\infty,\mathcal{K}}(x) = \pi_{\infty,\mathcal{K}}(y)$, we let $\mathcal{M}$ denote this connected component. Since $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C)$ is a $\underline{\mathcal{K}}$-torsor over $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C)$, $\mathcal{K}$ acts transitively on the set of connected components of $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C)$ over $\mathcal{M}$. In particular, there is an element $g \in \mathcal{K}$ with $x \cdot g = y$. Since $det(x) = det(y)$ we must have that $m \cdot det(g) = m$, but the action of $G^{ab}(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C))$ is simple so $det(g) = e$ and $g \in G^{der}(\mathbb{Q}_p)$ as we wanted to show. □

We can now describe connected components at infinite level.

Theorem 4.10. *Suppose G is an unramified group over $\mathbb{Q}_p$, suppose that G^{der} is simply connected and suppose that (b, μ) is HN-irreducible, then the determinant map*

$$det_{\infty,\infty} : \mathrm{Sht}_{G,b,[\mu],\infty} \to \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty}$$

has connected geometric fibers.

Proof. Since $\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ is isomorphic to $\underline{G^{ab}}(\mathbb{Q}_p) \times \mathrm{Spd}(C_p, O_{C_p})$, we may prove instead that the determinant map induces a bijection

$$\pi_0(det) : \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})) \to \pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times \mathrm{Spd}(C_p, O_{C_p})).$$

Indeed, we may use [32] 16.2 which says that cohomology of a locally spatial diamond is invariant under the change of geometric point. In particular, this applies to the set of connected components since it is a cohomological invariant.

Let $x \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p}))$. Given K a finite extension of K_0 we let x_K denote the image of x on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K))$ and let $f : \mathrm{Spd}(K, O_K) \to Gr^{adm,\leq[\mu]}(\mathcal{E}_b)$ be a point whose associated crystalline representation is as in corollary 2.18. Let $S_f := Triv(f^*(\mathbb{L}))$ the geometric realization of $f^*\mathbb{L}$. This space is also the fiber over f of the infinite level Grothendieck-Messing period map. Let $s \in \pi_0(S_f)$ be an element mapping to x_K. In summary we have taken a commutative diagram as follows:

$$\begin{tikzcd}
* \arrow[r, "x"] \arrow[d, "s"] \arrow[dr, "x_K"] & \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})) \arrow[d] \\
\pi_0(S_f) \arrow[r, "f"] & \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K))
\end{tikzcd}$$

We let G^{der}_x (respectively $G^{der}_{x_K}$ and G^{der}_s) denote the stabilizer in $G^{der}(\mathbb{Q}_p)$ of its action on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p}))$ (respectively $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K))$ and $\pi_0(S_f)$).

By Chen's theorem 2.16 (phrased in terms of lemma 3.6) G_s is an open subgroup of $G^{der}(\mathbb{Q}_p)$ and we have inclusions $G^{der}_x, G^{der}_s \subseteq G^{der}_{x_K}$. By lemma 4.9, $G^{der}_x \cdot \mathcal{K}^{der} = G^{der}(\mathbb{Q}_p)$ which implies that $G^{der}_{x_K} \cdot \mathcal{K}^{der} = G^{der}(\mathbb{Q}_p)$ as well. In particular, the projection map $\mathcal{K}^{der} \to G^{der}(\mathbb{Q}_p)/G^{der}_{x_K}$ is surjective.

Since $G^{der}(\mathbb{Q}_p)/G^{der}_{x_K}$ has the discrete topology and $\mathcal{K}^{der}$ is compact, we get that $G^{der}_{x_K}$ is closed and of finite index within $G^{der}(\mathbb{Q}_p)$. Moreover, since G^{der} is quasi-split (even unramified) all of the simple factors of G^{der} are isotropic. By Margulis theorem [25] II 5.1 we can conclude that $G^{der}_{x_K} = G^{der}(\mathbb{Q}_p)$. Since the argument doesn't depend on the choice of x the action of $G^{der}(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K))$ is trivial.

Now, $\mathrm{Spd}(C_p, O_{C_p}) = \varprojlim \mathrm{Spd}(K, O_K)$ and we may use [32] 11.22 to compute the action map

$$G^{der}(\mathbb{Q}_p) \times |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})| \to |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})|$$

as the limit of the action maps

$$\varprojlim_{K \subseteq C_p} [G^{der}(\mathbb{Q}_p) \times |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K)| \to |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K)|]$$

Since in the transition maps $|\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K_1, O_{K_1})| \to |\mathrm{Sht}_{G,b,[\mu],\infty}\mathrm{Spd}(K_2, O_{K_2})|$ every connected component on the source surjects onto a connected component on the target we have that $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})) = \varprojlim \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K))$. In particular, $G^{der}(\mathbb{Q}_p)$ acts trivially on the set of connected components. This defines a transitive action of $G^{ab}(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p}))$. The map $\pi_0(det)$ is surjective and equivariant for this action. Since $G^{ab}(\mathbb{Q}_p)$ acts freely on $\pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times \mathrm{Spd}(C_p, O_{C_p}))$, $\pi_0(det)$ must be a bijection.

□

Corollary 4.11. *For G, b and μ as in theorem 4.10 and any compact subgroup $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ the map*

$$\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \to \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],det(\mathcal{K})}$$

has non-empty connected geometric fibers.

Proof. One can deduce the claim for arbitrary compact $\mathcal{K}$ from the identity

$$\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} = \mathrm{Sht}_{G,b,[\mu],\infty}/\underline{\mathcal{K}}.$$

Indeed, the formation of π_0 commutes with colimits, so that $\pi_0(\mathrm{Sht}_{G,b,[\mu],\mathcal{K}}) = \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty})/\mathcal{K}$ which is $\pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty})/det(\mathcal{K})$. □

Using functoriality and equivariance for the three actions we can describe the actions by the three groups on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ in the spirit of theorem 3.5.

Theorem 4.12. *(Compare with [3] 4.1) Let G, b and μ as in theorem 4.10. Let $E \subseteq C_p$ be the field of definition of $[\mu]$, let $Art_{\breve{E}/E} : W_{\breve{E}/E} \to E^\times$ be as in theorem 3.5, let $Nm_{\mu^{ab}} : \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m) \to G^{ab}$ be the norm map associated to μ^{ab} then:*

1. *The $G(\mathbb{Q}_p)$ right action on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ makes it a trivial right $G^{ab}(\mathbb{Q}_p)$-torsor.*

2. *If $s \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ and $j \in J_b(\mathbb{Q}_p)$ then*

$$s \cdot_{J_b(\mathbb{Q}_p)} j = s \cdot_{G^{ab}(\mathbb{Q}_p)} det(j^{-1}))$$

where $det = j_{b^{ab}} \circ det_b$ with $det_b : J_b(\mathbb{Q}_p) \to J_{b^{ab}}(\mathbb{Q}_p)$ the map obtained from functoriality of the formation of J_b, respectively $J_{b^{ab}}$, and where the map $j_{b^{ab}}$ is the isomorphism $j_{b^{ab}} : J_{b^{ab}}(\mathbb{Q}_p) \cong G^{ab}(\mathbb{Q}_p)$ obtained from regarding $J_{b^{ab}}(\mathbb{Q}_p)$ as a subgroup of $G^{ab}(K_0)$.

3. *If $s \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ and $\gamma \in W_{\breve{E}/E}$ then*

$$s \cdot_{W_{\breve{E}/E}} \gamma = s \cdot_{G^{ab}(\mathbb{Q}_p)} [Nm_{\mu^{ab}} \circ Art_{\breve{E}/E}(\gamma)].$$

4.3 z-extensions

In this subsection we extend theorem 4.10 to the case in which G is not necessarily simply connected but we still assume that G is unramified and (b, μ) is HN-irreducible. In what follows we will denote by G^{sc} the central simply connected cover of G^{der} and we denote by $G^\circ = G(\mathbb{Q}_p)/Im(G^{sc}(\mathbb{Q}_p))$. Notice that when G^{der} is simply connected $G^\circ = G^{ab}(\mathbb{Q}_p)$. In general, G° surjects onto $G^{ab}(\mathbb{Q}_p)$ and the kernel is a finite group.

Recall the following definition used extensively by Kottwitz:

Definition 4.13. *A map of connected reductive groups $f : G' \to G$ is a z-extension if:*

- *f is surjective.*
- *$Z = ker(f)$ is central in G'.*
- *Z is isomorphic to a product of tori of the form $\mathrm{Res}_{F_i/\mathbb{Q}_p}\mathbb{G}_m$ for some finite extensions $F_i \subseteq \overline{\mathbb{Q}}_p$.*
- *G' has simply connected derived subgroup.*

By [20] lemma 1.1 whenever G is an unramified group over $\mathbb{Q}_p$ that splits over $\mathbb{Q}_{p^s}$, there exists a z-extension $G' \to G$ with Z isomorphic to a product of tori of the form $\mathrm{Res}_{\mathbb{Q}_{p^s}/\mathbb{Q}_p}\mathbb{G}_m$. In particular, it is unramified as well.

In [22] Kottwitz proves that for any reductive group G and cocharacter μ the natural morphism $B(G) \to B(G^{ad})$ induces a bijection $B(G, \mu) \cong B(G^{ad}, \mu^{ad})$. From here we can easily deduce the following:

Lemma 4.14. *Let $A \subseteq T \subseteq B \subseteq G$ as in the notation section. Assume that $\mathbb{Q}_{p^s}$ is a splitting field for G. Let $\mu \in X_*^+(T)$, $[b] \in B(G, \mu)$, and $f : G' \to G$ a z-extension with $Z = ker(f)$ isomorphic to a finite product of copies of $\mathrm{Res}_{\mathbb{Q}_{p^s}/\mathbb{Q}_p}\mathbb{G}_m$. Let $T' = f^{-1}(T)$ denote the maximal torus of G' projecting onto T. Then:*

1. *For any choice of $\mu' \in X_*(T')^+$ lifting μ there is a unique lift $[b'] \in B(G')$ lifting $[b]$ with $[b'] \in B(G', \mu')$.*
2. *For b' and μ' as in the previous claim (b, μ) is HN-irreducible if and only if (b', μ') is HN-irreducible.*
3. *If E is the field of definition of μ with $\mathbb{Q}_p \subseteq E \subseteq \mathbb{Q}_{p^s}$ then there is a lift $\mu' \in X_*(T')^+$ with field of definition E.*

Proof. The first claim follows directly from the identifications

$$B(G, \mu) = B(G^{ad}, \mu^{ad}) = B(G', \mu').$$

The second claim follows from the first claim, from the fact that $Z := ker(f)$ is central and from the fact that HN-irreducibility can be checked on the adjoint quotient once it is known that $b' \in B(G, \mu')$ holds.

For the third claim consider the exact sequence of $\Gamma_{\mathbb{Q}_p}$-modules:

$$e \to X_*(Z) \to X_*(T') \to X_*(T) \to e$$

Since G and G' split over $\mathbb{Q}_{p^s}$ the subgroup $\Gamma_{\mathbb{Q}_{p^s}} \subseteq \Gamma_E \subseteq \Gamma_{\mathbb{Q}_p}$ acts trivially on all of these groups. We treat this as an exact sequence of $Gal(\mathbb{Q}_{p^s}/E)$-modules. Since $Z = \prod_{i=1}^n \mathrm{Res}_{\mathbb{Q}_{p^s}/\mathbb{Q}_p}(\mathbb{G}_m)$

for some n, we can conclude that $X_*(Z)$ is an induced $\mathbb{Z}[Gal(\mathbb{Q}_{p^s}/E)]$-module and by Shapiro's lemma $H^1(Gal(\mathbb{Q}_{p^s}/E), X_*(Z)) = 0$. This implies that

$$X_*(T')^{\Gamma_E} = X_*(T')^{Gal(\mathbb{Q}_{p^s}/E)} \to X_*(T)^{Gal(\mathbb{Q}_{p^s}/E)} = X_*(T)^{\Gamma_E}$$

is surjective as we wanted to prove.

□

Proposition 4.15. *Suppose that G' is an unramified group, (b', μ') a pair with $[b'] \in B(G', \mu')$, suppose that $Z \subseteq G'$ is a central torus, and let $G = G'/Z$ with projection map $f : G' \to G$. Let $b = f(b')$ and $\mu' = f \circ \mu$ the following hold:*

1. *$Gr^{\leq[\mu']}(\mathcal{E}_{b'}) \to Gr^{\leq[\mu]}(\mathcal{E}_b)$ is an isomorphism.*
2. *$Gr^{adm,\leq[\mu']}(\mathcal{E}_{b'}) \to Gr^{adm,\leq[\mu]}(\mathcal{E}_b)$ is an isomorphism.*
3. *If $\mathbb{L}_{G'}$ (respectively $\mathbb{L}_G$) denotes the pro-étale $\underline{G'(\mathbb{Q}_p)}$-torsor (respectively $\underline{G(\mathbb{Q}_p)}$-torsor) then $\mathbb{L}_G = f_*\mathbb{L}_{G'}$.*

Proof. Both $Gr^{\leq[\mu']}(\mathcal{E}_{b'})$ and $Gr^{\leq[\mu]}(\mathcal{E}_b)$ are spatial diamonds that are proper over $\mathrm{Spd}(K_0, O_{K_0})$, any morphism between them is qcqs and by [32] 12.5 it is enough to prove the map is a bijection at the level of geometric points. In this case after fixing an isomorphism $B_{dR}(C) \cong C((t))$ we may reason as in the classical case. That is, $Gr(\mathcal{E}_{b'})(C, C^+) \cong G'(C((t)))/G'(C[[t]])$, $Gr(\mathcal{E}_b)(C, C^+) \cong G(C((t)))/G(C[[t]])$ and the map $G'(C((t)))/G'(C[[t]]) \to G(C((t)))/G(C[[t]])$ is a $Z(C((t)))/Z(C[[t]])$-torsor. On the other hand $Z(C((t)))/Z(C[[t]]) \cong X_*(Z)$ and we have an exact sequence:

$$e \to X_*(Z) \to X_*(T') \to X_*(T) \to e,$$

and the lifts of μ also form a $X_*(Z)$-torsor. Given a point $x \in Gr(\mathcal{E}_b)(C, C^+)$ of type $\mu \in X_*^+(T)$ and a lift $\mu'' \in X_*^+(T')$ there is a unique $y \in Gr(\mathcal{E}_b)(C, C^+)$ of type μ'' this finishes the proof of the first claim.

Let us prove the second claim, by the previous claim $Gr^{adm,\leq[\mu']}(\mathcal{E}_{b'})$ and $Gr^{adm,\leq[\mu]}(\mathcal{E}_b)$ are two open subdiamonds of $Gr^{\leq[\mu]}(\mathcal{E}_b)$. By [32] 11.15 it is enough to understand the underlying topological space of this open subsheaves. We prove that $Gr^{adm,\leq[\mu']}(\mathcal{E}_{b'})(C, C^+) \to Gr^{adm,\leq[\mu]}(\mathcal{E}_b)(C, C^+)$ is a bijection.

If we represent an element $x \in Gr^{\leq[\mu']}(\mathcal{E}_{b'})(C, C^+)$ by a modification $(\alpha_x : \mathcal{E}_x \dashrightarrow \mathcal{E}_{b'})$, then $f(x)$ is represented by $(f_*\alpha_x : f_*\mathcal{E}_x \dashrightarrow \mathcal{E}_b)$. By definition $x \in Gr^{adm,\leq[\mu']}(\mathcal{E}_{b'})(C, C^+)$ when $\mathcal{E}_x$ is a trivial G'-torsor this implies $f_*\mathcal{E}_x$ is trivial so that $f(x) \in Gr^{adm,\leq[\mu]}(\mathcal{E}_b)(C, C^+)$. Assume instead $f(x) \in Gr^{adm,\leq[\mu]}(\mathcal{E}_b)(C, C^+)$, and let $[b'_x] \in B(G')$ be the unique element with $\mathcal{E}_{b'_x} \cong \mathcal{E}_x$. We need to prove $[b'_x] = [e]$. We begin by proving that $\kappa([b'_x]) = \kappa([b']) - [\mu']$. Indeed using ([9] 2.15) we can deduce that $\kappa(\mathcal{E}_x)$ is independent of $x \in Gr^{\leq[\mu]}(\mathcal{E}_{b'})(C, C^+)$ since $Gr^{\leq[\mu]}(\mathcal{E}_{b'})$ is connected. It is then enough to prove $\kappa([b'_x]) = \kappa([b']) - [\mu']$ when $x \in Gr^{\leq[\mu]}(\mathcal{E}_{b'})(C, C^+)$ is the point associated to ξ^μ. This is precisely the content of ([17] 6.4.1).

By the assumption $b' \in B(G', \mu')$ we have $\kappa([b'_x]) = [e] \in \pi_1(G')_{\Gamma_{\mathbb{Q}_p}}$, so that to prove $[b'_x] = [e]$ it is enough to prove that $[b'_x]$ is basic. But $f([b'_x]) = [e]$ so $\nu_{b'_x}$ must factor through $X_*(Z) \otimes \mathbb{Q}$, and since Z is central $[b'_x]$ is basic.

For the last claim, recall that for any $(V, \rho) \in Rep_{G'}(\mathbb{Q}_p)$ and $x \in Gr^{adm,\leq[\mu]}(\mathcal{E}_{b'})(R, R^+)$, $\rho_*\mathbb{L}_{G'}(x)$ evaluates to $\underline{H^0}(\mathcal{X}_{FF,R}, \rho_*\mathcal{E}_x)$. When $\rho = \tau \circ f$ we get $\underline{H^0}(\mathcal{X}_{FF,R}, \tau_*\mathcal{E}_{f(x)})$ which is the evaluation of $\mathbb{L}_G$ at $(V, \tau) \in Rep_G(\mathbb{Q}_p)$. □

Proposition 4.16. *If (b, μ) is HN-irreducible then the following hold:*

1. *$Gr^{adm,\leq[\mu]}(\mathcal{E}_b) \times \mathrm{Spd}(C, O_C)$ is connected*

2. *The right action of* $G(\mathbb{Q}_p)$ *on* $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C))$ *makes this set into a* G°*-torsor.*

Proof. Using lemma 4.14 we may find a z-extension $f : G' \to G$ and lift (b, μ) to a pair (b', μ') over G' which is also HN-irreducible. The first claim now follows from proposition 4.15 and by proposition 4.8 applied to G', since by definition of z-extension $(G')^{der}$ is simply connected.

Let $Z = Ker(f)$, since this is an induced torus Hilbert's 90 theorem together with Shapiro's lemma proves the surjectivity of the map $f : G'(\mathbb{Q}_p) \to G(\mathbb{Q}_p)$. Using this together with proposition 4.15 we see that

$$f : \mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C, O_C) \to \mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C)$$

is $Z(\mathbb{Q}_p)$-torsor. In particular, the map of sets of connected components is also surjective. Since $\overline{Gr^{adm,\leq[\mu]}}(\mathcal{E}_b)$ is connected the action of $G(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C))$ is transitive. Let $x \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C))$ and denote by G_x the stabilizer of x in $G(\mathbb{Q}_p)$. Let $y \in \pi_0(\mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C, O_C))$ a lift of x, we wish to prove that $Im(G^{sc}(\mathbb{Q}_p)) = G_x$.

By theorem 4.10 the stabilizer of y in $G'(\mathbb{Q}_p)$ is $(G')^{der}(\mathbb{Q}_p)$. By equivariance of f with respect the actions of $G'(\mathbb{Q}_p)$ and $G(\mathbb{Q}_p)$, we have that $Im((G')^{der}(\mathbb{Q}_p)) \subseteq G_x$. Since G' is a z-extension $Im((G')^{der}(\mathbb{Q}_p)) = Im(G^{sc}(\mathbb{Q}_p))$. On the other hand, any $g \in G_x$ has a lift $g' \in G'(\mathbb{Q}_p)$ and we may write $f(y \cdot g') = x \cdot g = x$. Since $f(y \cdot g') = f(y)$, there is an element $z \in Z(\mathbb{Q}_p)$ with $y \cdot g' \cdot z = y$. In other words, $z \cdot g' \in (G')^{der}(\mathbb{Q}_p)$ which implies that $g \in Im(G^{sc}(\mathbb{Q}_p))$ finishing the proof. $\square$

As we have done in previous subsections we can describe the action of $J_b(\mathbb{Q}_p)$ and $W_{\breve{E}/E}$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C))$ in terms of the action of G°. We first describe the action of $J_b(\mathbb{Q}_p)$. To do this we need to construct a map $det^\circ : J_b(\mathbb{Q}_p) \to G^\circ$ that generalizes the determinant map $det : J_b(\mathbb{Q}_p) \to G^{ab}(\mathbb{Q}_p)$ that appears in theorem 4.12. A peculiar aspect of the situation is that G° does not necessarily have algebraic structure (its not the $\mathbb{Q}_p$-points of an algebraic group). Consequently det° is does not come directly from a map of algebraic groups. The map is constructed as follows: Given G and $b \in G(K_0)$ we may choose an unramified z-extension $f : G' \to G$ and a lift $b' \in G'(K_0)$ with $f(b') = b$. Let $Z = Ker(f)$. We get a sequence of maps of reductive groups

$$e \to Z \to J_{b'} \to J_b \to e.$$

Since Z is an induced torus, by Hilbert's theorem 90 and Shapiro's lemma $H^1(\mathbb{Q}_p, Z) = \{0\}$ so that we obtain a surjection $J_{b'}(\mathbb{Q}_p) \to J_b(\mathbb{Q}_p)$. We can construct the following commutative diagram of topological groups:

$$\begin{tikzcd}
Z(\mathbb{Q}_p) \arrow[r] \arrow[d] & G'(\mathbb{Q}_p) \arrow[r, "f"] \arrow[d] & G(\mathbb{Q}_p) \arrow[d] \\
J_{b'}(\mathbb{Q}_p) \arrow[r, "det"] \arrow[dr] \arrow[dd, "f"] & (G')^{ab}(\mathbb{Q}_p) \arrow[r, "f^{ab}"] & G^\circ \\
 & J_{b'^{ab}}(\mathbb{Q}_p) \arrow[u, "\cong_j"] & \\
J_b(\mathbb{Q}_p) \arrow[rruu, bend right, "det^\circ"'] & &
\end{tikzcd}$$

Now, det° is defined as the unique morphism that could make this diagram commutative. More explicitly, if $j \in J_b(\mathbb{Q}_p)$ we pick a lift $j' \in J_{b'}(\mathbb{Q}_p)$, and we define $det^\circ(j) := f^{ab}(det(j'))$. This doesn't depend on the choice of j'. Indeed, two lifts of j differ by an element of $Z(\mathbb{Q}_p)$ but the induced map $Z(\mathbb{Q}_p) \to G^\circ$ is the 0 map, since it factors through the map to G. Similarly the construction of det° does not depend of the choice of $b' \in G'(\mathcal{K}_0)$ lifting b since the possible choices differ by an element of $Z(K_0)$. Finally, we justify that the construction of det° doesn't depend on the choice of z-extension $G' \to G$ taken. This will follow from the fact that the category of z-extensions of G is cofiltered. Given two z-extensions $G_1, G_2 \to G$ we may find a third z-extension making the following diagram commutative:

$$\begin{tikzcd} & G_2 \arrow[dr] & \\ G_3 \arrow[ur, "f_2"] \arrow[dr, "f_1"'] & & G \\ & G_1 \arrow[ur] & \end{tikzcd}$$

Choosing a lift of $b_3 \in G_3(K_0)$ and defining $b_i = f_i(b_3)$ we obtain the following diagram:

$$\begin{tikzcd} J_{b_3}(\mathbb{Q}_p) \arrow[r, "f_i"] \arrow[d, "det"] & J_{b_i}(\mathbb{Q}_p) \arrow[r] \arrow[d, "det"] & J_b(\mathbb{Q}_p) \arrow[d, bend right, "det_i^\circ"'] \arrow[d, bend left, "det_3^\circ"] \\ G_3^{ab}(\mathbb{Q}_p) \arrow[r] & G_i^{ab}(\mathbb{Q}_p) \arrow[r] & G^\circ \end{tikzcd}$$

It is easy to verify $det_i^\circ = det_3^\circ$.

Remark 4.17. *Another way one can define det° is as follows. Since G is quasi-split we may define groups $A \subseteq T \subseteq B \subseteq G$ as in the notation section §2.2. The dominant Newton point ν_b^{dom} is a $\mathbb{Q}_p$-rationally defined map $\mathbb{D} \to A$ and we may define M_b as the centralizer of ν_b in G. One may then reconstruct J_b as a twisted inner form of M_b. Using z-extensions one may construct an isomorphism from $J_b(\mathbb{Q}_p)/[J_b(\mathbb{Q}_p), J_b(\mathbb{Q}_p)]$ and $M_b(\mathbb{Q}_p)/[M_b(\mathbb{Q}_p), M_b(\mathbb{Q}_p)]$ (the maximal abelian quotients when regarded as an abstract groups). The inclusion $M_b(\mathbb{Q}_p) \subseteq G(\mathbb{Q}_p)$ induces a map $M_b(\mathbb{Q}_p) \to G^\circ$ which overall gives a map $J_b(\mathbb{Q}_p) \to G^\circ$. Again, one must justify that this morphism didn't depend of the choices made.*

By functoriality, equivariance and theorem 4.12 we can do the following computation. Pick G', b' and μ' as in the proof of proposition 4.16. We obtain a map

$$f : \mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C, O_C) \to \mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C),$$

let $x \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C))$ and let $y \in \pi_0(\mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C, O_C))$ be a lift of x. Let $j \in J_b(\mathbb{Q}_p)$, and let $j' \in J_{b'}(\mathbb{Q}_p)$ be an element lifting j. We have:

$$\begin{aligned} x \cdot_{J_b(\mathbb{Q}_p)} j &= f(y \cdot_{J_{b'}(\mathbb{Q}_p)} j') \\ &= f(y \cdot_{G'(\mathbb{Q}_p)} j_{b'}(det_{b'}(j^{-1}))) \\ &= x \cdot_{G^\circ} det^\circ(j^{-1}) \end{aligned}$$

We now describe the action of $W_{\breve{E}/E}$, we will also need to introduce a variant of the norm map discussed for tori. Given a connected reductive group G and a conjugacy class of cocharacters $[\mu]$ with reflex field E we define a norm map $Nm^\circ_{[\mu]} : E^\times \to G^\circ$ as follows. Since is G is quasi-split we may fix $\mathbb{Q}_p$-rationally defined Borel a maximal torus $T \subseteq B \subseteq G$ and the unique dominant cocharacter $\mu \in X_*^+(T)$ representing $[\mu]$ and defined over E. We get a norm map $Nm_\mu : E^\times \to T(\mathbb{Q}_p)$ and we may define $Nm^\circ_{[\mu]}$ as the composition:

$$Nm^\circ_{[\mu]} : E^\times \xrightarrow{Nm_\mu} T(\mathbb{Q}_p) \to G(\mathbb{Q}_p) \to G^\circ.$$

We claim that this map is independent of the choice of B and T. Indeed, recall that the action of $G(\mathbb{Q}_p)$ on the set of pairs (B,T) with B a rationally defined Borel and T a rationally defined maximal torus contained in B is transitive. If $(B_2,T_2)=g\cdot(B_1,T_1)\cdot g^{-1}$ for some element $g\in G(\mathbb{Q}_p)$ then $Nm^\circ_{g\cdot\mu\cdot g^{-1}}=g\cdot Nm_\mu g^{-1}$, and since G° is abelian we get $Nm^\circ_{[g\cdot\mu\cdot g^{-1}]}=Nm^\circ_{[\mu]}$.

Proposition 4.18. *With notation as in proposition 4.16 the action of $W_{\breve{E}/E}$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty}\times \mathrm{Spd}(C,O_C))$ is given by the map $Nm^\circ_{[\mu]}\circ Art_{\breve{E}/E}:W_{\breve{E}/E}\to G^\circ$. More precisely, if $x\in\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty}\times\mathrm{Spd}(C,O_C))$ and $\gamma\in W_{\breve{E}/E}$ then:*

$$x\cdot_{W_{\breve{E}/E}}\gamma = x\cdot_{G^\circ} Nm^\circ_{[\mu]}(Art_{\breve{E}/E}(\gamma)).$$

Proof. We let $f:G'\to G$ be a z-extension and we let (b',μ') be a pair over G' lifting (b,μ), and let $Z=ker(f)$. By 4.14 we can always choose G' and μ' so that μ' has the same field of definition as μ. We get a morphism

$$\mathrm{Sht}_{G',b',[\mu'],\infty}\to\mathrm{Sht}_{(G')^{ab},b'^{ab},[\mu'],\infty}.$$

Let $A\subseteq T\subseteq B\subseteq G$ as above and let $T'=f^{-1}(T)$. Recall that for tori the set $B(T',\mu')$ has a unique element, we fix a representative $b_{\mu'}$. This allows us to construct a map

$$\mathrm{Sht}_{T',b_{\mu'},[\mu'],\infty}\to\mathrm{Sht}_{(G')^{ab},b'^{ab},[\mu'],\infty}$$

and by functoriality we also get map

$$\mathrm{Sht}_{T',b_{\mu'},[\mu'],\infty}\to\mathrm{Sht}_{T,b_\mu,[\mu],\infty}$$

We can collect all of these maps in the following commutative diagram of spaces.

$$\begin{tikzcd}
\mathrm{Sht}_{G',b',[\mu'],\infty} \arrow[r] \arrow[d] & \mathrm{Sht}_{(G')^{ab},b'^{ab},[\mu'],\infty} & \mathrm{Sht}_{T',b_{\mu'},[\mu'],\infty} \arrow[l] \arrow[d] \\
\mathrm{Sht}_{G,b,[\mu],\infty} & & \mathrm{Sht}_{T,b_\mu,[\mu],\infty}
\end{tikzcd}$$

Since G' is simply connected we get an equivariant bijection of geometric connected components

$$\pi_0(\mathrm{Sht}_{G',b',[\mu'],\infty}\times\mathrm{Spd}(C,O_C))\to\pi_0(\mathrm{Sht}_{(G')^{ab},b'^{ab},[\mu'^{ab}],\infty}\times\mathrm{Spd}(C,O_C)).$$

After forming geometric connected components and choosing a base point

$$x\in\pi_0(\mathrm{Sht}_{T',b_{\mu'},[\mu'],\infty}\times\mathrm{Spd}(C,O_C))$$

the above diagram looks like this:

$$\begin{tikzcd}
x\cdot G'^{ab}(\mathbb{Q}_p) \arrow[r, "\cong"] \arrow[d] & x\cdot G'^{ab}(\mathbb{Q}_p) & x\cdot T'(\mathbb{Q}_p) \arrow[l] \arrow[d] \\
x\cdot G^\circ & & x\cdot T(\mathbb{Q}_p)
\end{tikzcd}$$

All of the maps are equivariant with respect to the groups involved. Since the map $T'(\mathbb{Q}_p)\to G^\circ$ factors through the map $T'(\mathbb{Q}_p)\to T(\mathbb{Q}_p)$, we get a canonical surjective and $W_{\breve{E}/E}$-equivariant map

$$\pi_0(\mathrm{Sht}_{T,b_\mu,[\mu],\infty}\times\mathrm{Spd}(C,O_C))\to\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty}\times\mathrm{Spd}(C,O_C)).$$

By theorem 3.5, the action on $\pi_0(\mathrm{Sht}_{T,b_\mu,[\mu],\infty})$ is through $Nm_\mu\circ Art_{\breve{E}/E}$. Equivariance and the definition of $Nm^\circ_{[\mu]}$ imply that the action of $W_{\breve{E}/E}$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty}\times\mathrm{Spd}(C,O_C))$ is through $Nm^\circ_{[\mu]}\circ Art_{\breve{E}/E}$. $\square$

References

[1] Bhargav Bhatt and Peter Scholze, *Projectivity of the Witt vector affine Grassmannian*, Invent. Math. **209** (2017), no. 2, 329–423. MR 3674218

[2] Ana Caraiani and Peter Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, Ann. of Math. (2) **186** (2017), no. 3, 649–766. MR 3702677

[3] Miaofen Chen, *Le morphisme déterminant pour les espaces de modules de groupes p-divisibles*, Int. Math. Res. Not. IMRN (2013), no. 7, 1482–1577. MR 3044450

[4] ———, *Composantes connexes géométriques de la tour des espaces de modules de groupes p-divisibles*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 4, 723–764. MR 3250062

[5] Miaofen Chen, Mark Kisin, and Eva Viehmann, *Connected components of affine deligne–lusztig varieties in mixed characteristic*, Compositio Mathematica **151** (2015), no. 9, 1697–1762.

[6] Jean-François Dat, Sascha Orlik, and Michael Rapoport, *Period domains over finite and p-adic fields*, Cambridge Tracts in Mathematics, vol. 183, Cambridge University Press, Cambridge, 2010. MR 2676072

[7] A. J. de Jong, *Étale fundamental groups of non-Archimedean analytic spaces*, vol. 97, 1995, Special issue in honour of Frans Oort, pp. 89–118. MR 1355119

[8] Gerd Faltings, *Group schemes with strict O-action*, vol. 2, 2002, Dedicated to Yuri I. Manin on the occasion of his 65th birthday, pp. 249–279. MR 1944507

[9] Laurent Fargues, *Geometrization of the local langlands correspondence: an overview*, 2016.

[10] Laurent Fargues and Jean-Marc Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Astérisque (2018), no. 406, xiii+382, With a preface by Pierre Colmez. MR 3917141

[11] Laurent Fargues and Peter Scholze, *Geometrization of the local langlands correspondence*, 2021.

[12] Ildar Gaisin and Naoki Imai, *Non-semi-stable loci in hecke stacks and fargues' conjecture*, 2016.

[13] Ian Gleason, *Specialization maps for scholzes category of diamonds*, 2020.

[14] David Hansen, *Moduli of local shtukas and harris's conjecture*, 2021.

[15] Xuhua He and Rong Zhou, *On the connected components of affine deligne-lusztig varieties*, 2016.

[16] R. Huber, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513–551. MR 1306024

[17] Tasho Kaletha, David Hansen, and Jared Weinstein, *On the kottwitz conjecture for local shimura varieties*, 2019.

[18] Kiran S. Kedlaya and Ruochuan Liu, *Relative p-adic Hodge theory: foundations*, Astérisque (2015), no. 371, 239. MR 3379653

[19] Martin Kneser, *Galois-Kohomologie halbeinfacher algebraischer Gruppen über p-adischen Körpern. II*, Math. Z. **89** (1965), 250–272. MR 188219

[20] Robert E. Kottwitz, *Rational conjugacy classes in reductive groups*, Duke Math. J. **49** (1982), no. 4, 785–806. MR 683003

[21] ———, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), no. 2, 373–444. MR 1124982

[22] ———, *Isocrystals with additional structure. II*, Compositio Math. **109** (1997), no. 3, 255–339. MR 1485921

[23] Jonathan Lubin and John Tate, *Formal complex multiplication in local fields*, Ann. of Math. (2) **81** (1965), 380–387. MR 172878

[24] Lucas Mann and Annette Werner, *Local systems on diamonds and p-adic vector bundles*, 2020.

[25] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991. MR 1090825

[26] William Messing, *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, Lecture Notes in Mathematics, Vol. 264, Springer-Verlag, Berlin-New York, 1972. MR 0347836

[27] Sian Nie, *Connected components of closed affine Deligne-Lusztig varieties in affine Grassmannians*, Amer. J. Math. **140** (2018), no. 5, 1357–1397. MR 3862068

[28] M. Rapoport and M. Richartz, *On the classification and specialization of f-isocrystals with additional structure*, Compositio Mathematica **103** (1996), no. 2, 153–181 (en). MR 1411570

[29] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996. MR 1393439

[30] Michael Rapoport and Eva Viehmann, *Towards a theory of local Shimura varieties*, Münster J. Math. **7** (2014), no. 1, 273–326. MR 3271247

[31] Neantro Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics, Vol. 265, Springer-Verlag, Berlin-New York, 1972. MR 0338002

[32] Peter Scholze, *Etale cohomology of diamonds*, 2017.

[33] Peter Scholze and Jared Weinstein, *Moduli of p-divisible groups*, Camb. J. Math. **1** (2013), no. 2, 145–237. MR 3272049

[34] Peter Scholze and Jared Weinstein, *Berkeley lectures on p-adic geometry*, http://www.math.uni-bonn.de/people/scholze/Berkeley.pdf, 2019.

[35] ———, *Berkeley lectures on p-adic geometry: (ams-207)*, Princeton University Press, 2020.

[36] Sug Woo Shin, *Counting points on Igusa varieties*, Duke Mathematical Journal **146** (2009), no. 3, 509 – 568.

[37] Robert Steinberg, *Regular elements of semisimple algebraic groups*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 25, 49–80. MR 180554

[38] Matthias Strauch, *Geometrically connected components of Lubin-Tate deformation spaces with level structures*, Pure Appl. Math. Q. **4** (2008), no. 4, Special Issue: In honor of Jean-Pierre Serre. Part 1, 1215–1232. MR 2441699