Lecture 4: Moduli spaces of p-adic shtukas.

Definition: • Let (R, R^+) be affinoid perfectoid.

A shtuka with legs at $(R^\sharp, R^+)$ is a pair $(\mathcal{E}, \Phi)$ where $\mathcal{E}$ is a vector bundle over $\mathrm{Spd}\, \mathbb{Z}_p \times \mathrm{Spd}(R, R^+) = \mathcal{Y}^{\diamond}_{[0,\infty)}$ and $\Phi : \varphi^* \mathcal{E} \longrightarrow \mathcal{E}$ is an isomorphism on $\mathcal{Y}_{[0,\infty)} \setminus \mathrm{Spa}(R^\sharp, R^{\sharp,+})$ and meromorphic along this divisor. } modification

• A G-shtuka is a $\otimes$-exact functor

$$\mathrm{Rep}_{G, \mathbb{Z}_p} \longrightarrow \left\{ \begin{array}{c} \text{shtukas with} \\ \text{legs at } R^\sharp \end{array} \right\}.$$

• Let $\mathrm{Sht}_G : \mathrm{Perf} \longrightarrow \{\text{groupoids}\}$

$$\mathrm{Sht}_G(R, R^+) = \left\{ R^\sharp, \mathcal{E}, \Phi \;\middle|\; \begin{array}{c} \text{with } (\mathcal{E}, \Phi) \\ G\text{-shtuka} \\ \text{with legs at } R^\sharp \end{array} \right\}$$

We have a map

$$\mathrm{Sht}_{\mathcal{G}} \longrightarrow \mathrm{HK}_{\mathcal{G}}$$

$$(R^{\sharp}, \mathcal{E}, \Phi) \longmapsto (R^{\sharp}, \varphi^{*}\mathcal{E}, \mathcal{E}, \Phi)$$

Recall: $\mathrm{HK}(R, R^{+}) = \Big\{ (R^{\sharp}, \mathcal{E}_1, \mathcal{E}_2, f) \,\Big|$ $\mathcal{E}_1, \mathcal{E}_2$ $\mathcal{G}$-bundles on $\mathcal{Y}_{[0,\infty)}$, $f : \mathcal{E}_1 \dashrightarrow \mathcal{E}_2$ modification $\mathcal{Y}_{[0,\infty)} \setminus \mathrm{Spa}(R^{\sharp})$. $\Big\}$

We can bound the modification:

$$\begin{tikzcd} \mathrm{Sht}_{\mathcal{G},\mu} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sht}_{\mathcal{G}} \arrow[d] \\ \mathrm{HK}_{\mathcal{G},\mu} \arrow[r] & \mathrm{HK}_{\mathcal{G}} \end{tikzcd}$$

and we obtain a map

$$\mathrm{Sht}_{\mathcal{G},\mu} \longrightarrow \mathrm{Bun}_G \times \mathrm{Spd}\,\mathbb{Z}_p$$

$$(\mathcal{E}, \Phi) \longmapsto \varphi\text{-extension of } \mathcal{E}|_{\mathcal{Y}_{[r,\infty)}}$$

$$|\mathrm{Sht}_{\mathcal{G},\mu}| \longrightarrow B(G,\mu) \subseteq B(G) = |\mathrm{Bun}_G|.$$

On generic fiber:

$$\begin{tikzcd}
\mathrm{Sht}_{\mathcal{G},\mu,\mathcal{O}_p} \arrow[r] \arrow[d, equal] & \mathrm{Bun}_G \\
\underline{\mathcal{G}(\mathbb{Z}_p)}\backslash \mathrm{Gr}_{\mathcal{G},\mu,\mathcal{O}_p} \arrow[ur] & BL_1
\end{tikzcd}$$

We can also look at fiber:

$$\begin{tikzcd}
\text{Moduli space of shtukas of } (\mathcal{G},b,\mu) & \mathrm{Sht}_\mu(b) \arrow[l, squiggly] \arrow[r] \arrow[d] & * \arrow[d, "b"] \\
& \mathrm{Sht}_{\mathcal{G},\mu} \arrow[r] & \mathrm{Bun}_G
\end{tikzcd}$$

<u>Definition</u>: Attached to $(\mathcal{G}, b, \mu)$ we associate $X_\mu(b) \subseteq \mathrm{Fl}_{\mathcal{G}}^{\text{witt}}$ the affine Deligne-Lusztig variety

$$X_\mu(b)(\overline{\mathbb{F}}_p) = \{\, g\mathcal{G} \mid g^{-1} b\, \sigma(g) \in \mathrm{Adm}(\mu) \,\}$$

<u>Theorem</u>: $\mathrm{Sht}_\mu(b)^{\mathrm{red}} = X_\mu(b)$.

- Unravelling $\mathrm{Sht}_\mu(b)(R,R^+) = \{ (R^\#, \mathcal{E}, \Phi, \rho) \mid$

$(\mathcal{E}, \Phi)$ $\mathcal{G}$-shtuka

$\rho: \mathcal{E} \rightsquigarrow \mathcal{E}_b$ φ-equivariant iso

- Key idea: $\mathrm{Sht}_\mu(b)^{\mathrm{red},\diamond}$ is $\mathcal{O}(\varphi)$ where φ is meromorphic

<u>Conjecture</u>: For all affine $U \subseteq X_\mu(b)$, $\widehat{\mathrm{Sht}_\mu(b)_U}$ is a spatial kimberlite.

<u>Theorem</u> $\mathrm{Sht}_\mu(b)$ is a nice unibranch $\underbrace{\text{smelted}}_{\text{generic fiber behaves well}}$ kimberlite. In particular,

$$\mathrm{sp}: \pi_0(\mathrm{Sht}_\mu(b)_{\mathbb{Q}_p}) \xrightarrow{\sim} \pi_0(X_\mu(b)).$$

- Formal separatedness $\underbrace{\mathrm{Sht}_\mu(b)^{\mathrm{red},\diamond} \subseteq \mathrm{Sht}_\mu(b)}_{\text{injectivity implies formal separatedness.}}$

- Locally formal:

$\underbrace{\text{Shtuka on } \mathcal{Y}_{[0,\infty)} \rightsquigarrow}_{\text{glue along } \mathcal{E}_b}$ $\underbrace{\text{Shtuka on } \mathcal{Y}_{[0,\infty]}}_{\text{GAGA}}$ $\rightsquigarrow$ Shtuka on $\mathcal{Y}^{\mathrm{Sch}}_{[0,\infty]}$ $\rightsquigarrow$ Anschütz

$\downarrow$

Shtuka on $\mathrm{Spec}(A_{\inf})$

- Reduced special fiber is nice

  ADLV.

- Generic fiber is nice

  Grothendieck–Messing period morphism

<u>Unibranchness:</u>

v-locally any shtuka is of the form $\mathcal{E} = G$ and $\Phi \in G(A_{inf}[\frac{1}{\xi}])$.

$$WG(R, R^+) = \{ (R^\#, u) \mid u \in G(W(R^+)[\tfrac{1}{\xi}]) \}$$

with vector loop group

$$W^+G(R, R^+) = \{ (R^\#, u) \mid u \in G(W(R^+)) \}$$

$$\widehat{W^+G}(R, R^+) = \left\{ (R^\#, u) \;\middle|\; \begin{array}{l} u \in G(W(R^+)) \text{ and} \\ \exists \text{ uniformizer } \varpi \in R^+ \text{ s.t.} \\ u \in \ker\left(G(W(R^+)) \to G(W(R^+)/[\varpi]) \right) \end{array} \right.$$

This is not a tubular neighborhood.

Theorem:

For all $x \in X_\mu(b)(\bar{\mathbb{F}}_p)$ $\exists$

$y \in \mathcal{M}_{(\mathcal{G},\mu)}(\bar{\mathbb{F}}_p)$ and a correspondence:

$$\widehat{\mathrm{Sht}_\mu(b)}_x \xleftarrow{f} X \xrightarrow{g} \widehat{\mathcal{M}}_{(\mathcal{G},\mu),y}$$

where f and g are $\widehat{W^+\mathcal{G}}$-torsors.

In particular, $\pi_0(\mathrm{Sht}_\mu(b)^\circ_x) = \pi_0(\mathcal{M}^\circ_{(\mathcal{G},\mu),y})$.

$= *$ by Thursday.

Actually given data

$\mathcal{D} = (D, \Phi, \rho)$

D $\mathcal{G}$-bundle over $\mathrm{Spec}(W(\bar{\mathbb{F}}_p))$

$\Phi : \varphi^* D \dashrightarrow D$ map over $\mathrm{Spec}(W(\bar{\mathbb{F}}_p)[\frac{1}{p}])$

$\rho : D \longrightarrow \mathcal{E}_b$ φ-equivariant iso. We get:

$$\begin{array}{ccc} \mathrm{Triv\,Def\,Sht}(\mathcal{D}) & \underset{\text{NOT EQUIVARIANT}}{\overset{\sim}{\longrightarrow}} & \mathrm{Triv\,Def\,Grass} \\ \widehat{W^+\mathcal{G}} \downarrow & & \downarrow \widehat{W^+\mathcal{G}} \\ \widehat{\mathrm{Sht}}_{\mathcal{D}} = \mathrm{Def\,Sht}(\mathcal{D}) & & \mathrm{Def\,Grass}(\mathcal{D}) = \widehat{\mathrm{Gr}}_{\mathcal{G},\mathcal{D}} \end{array}$$

Behind the scenes: Unique liftability of "Isogenies".

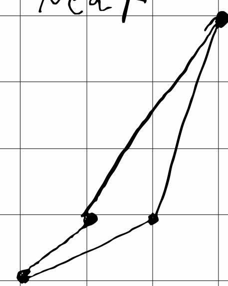
Connected components of ADLV:

Fix (G, b, μ) with (b, μ) HN-irreducible

in terms of Hodge and Newton

polygons this is no

common break point.



We have map

$$\begin{tikzcd} X_\mu(b) \subseteq Fl^{\mathrm{witt}} \arrow[r, "\kappa"] \arrow[dr, "\omega"'] & \pi_1(G)_I \\ & c_{b,\mu} \pi_1(G)_I^{\varphi} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Theorem (G. Lim, Xu)

$$\omega : \pi_0(X_\mu(b)) \longrightarrow c_{b,\mu} \pi_1(G)_I^{\varphi}$$

is bijective if and only if

(b, μ) is HN-irreducible.

Proof sketch: Assume G^{ad} has no

anisotropic factors.

$$\begin{tikzcd} \mathrm{Sht}_{\mu,\infty}(b) \arrow[r, "G(\mathbb{Z}_p)\text{-tors}"] \arrow[rr, bend left, "G(\mathbb{Q}_p)\text{-torsor}"] & \mathrm{Sht}_\mu(b)_{\mathbb{Q}_p} \arrow[r, "\pi_{GM}"] \arrow[d, "sp"] & \mathrm{Gr}_{G,\mu}^{b\text{-adm}} \\ & X_\mu(b) & \end{tikzcd}$$

Compute $\pi_0(\mathrm{Sht}_\mu(b))$ instead.

$G^0 = G(\mathbb{Q}_p)/G^{sc}(\mathbb{Q}_p)$

morally

" $\pi_1(G)_{\Gamma}^{\varphi} = G^0(\mathbb{Q}_p)/G^0(\mathbb{Z}_p)$ "

<u>Thm</u> $\pi_0(\mathrm{Sht}_\mu(b)_{\mathbb{Q}_p})$ is G^0-torsor.

<u>Theorem</u> (G, Lourenço) $\mathrm{Gr}_{G,\mu}^{b\text{-adm}}$ is geometrically connected.

- $G(\mathbb{Q}_p) \circlearrowright \pi_0(\mathrm{Sht}_{\mu,\infty}(b))$ acts transitively

$G_x \subseteq G(\mathbb{Q}_p)$ stabilizer of $x \in \pi_0(\mathrm{Sht}_{\mu,\infty}(b))$

Suppose G^{der} is simply connected

want to show $G_x \supseteq G^{der}(\mathbb{Q}_p)$.

We show:

a) $G_x \cap G^{der}(\mathbb{Q}_p)$ is open in $G^{der}(\mathbb{Q}_p)$

b) $N_x =$ Normalizer of G_x^{der} in G^{der}

then $[G^{der}(\mathbb{Q}_p) : N_x] < \infty$

<u>Fact</u>: If $S \subseteq G^{der}(\mathbb{Q}_p)$ is open and one of TF hold:

a) $[G^{der}(\mathbb{Q}_p) : S] < \infty$ or

b) S is normal

then $S = G^{der}(\mathbb{Q}_p)$.

This gives $N_x = G^{der}(\mathbb{Q}_p)$ and then

$G_x^{der} = G^{der}(\mathbb{Q}_p)$

For openess:

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \mathrm{Spd}(K) \arrow[d, "m"] \\ \mathrm{Sht}_\mu(b) \arrow[r] & \mathrm{Gr}_\mu^b \end{tikzcd}$$

$[K:\breve{\mathbb{Q}}_p] < \infty$

$z \in |T|, \quad G_z \subseteq G_x$

Attached to m we have a conjugacy class of crystalline reps

$$\rho_m : \mathrm{Gal}_K \longrightarrow G(\mathbb{Q}_p)$$

and $G_z = \rho_m(\mathrm{Gal}_K)$

Definition For a Hodge-Tate rep $\rho : \mathrm{Gal}_K \longrightarrow G$ we let MT_ρ be the connected component of Zariski closure of $\rho(\mathrm{Gal}_K) \subseteq G(\mathbb{Q}_p)$.

Theorem (Sen) $\rho(\mathrm{Gal}_K) \cap MT_\rho(\mathbb{Q}_p)$ is open in $MT_\rho(\mathbb{Q}_p)$.

The general crystalline representation that MT group as large as possible contains G^{der}.

b) $[G(\mathbb{Q}_p) : N_x] < \infty$.

Sht_∞ has commuting action $G(\mathbb{Q}_p)$, $J_b(\mathbb{Q}_p)$

$$G_x = G_{x \cdot j} = G_{g_j \cdot x} = g_j G_x g_j^{-1}$$

$\forall j \in J_b(\mathbb{Q}_p)$ $\exists$ $g_j \in G(\mathbb{Q}_p)$ and $g_j \in N_x$.

Moreover $G(\mathbb{Z}_p) \backslash \pi_0(Sht_\infty) / J_b(\mathbb{Q}_p)$

$= \pi_0(Sht_{G(\mathbb{Z}_p)}) / J_b(\mathbb{Q}_p)$

$\supset \pi_0(X_\mu(b)) / J_b(\mathbb{Q}_p)$

$=$ this is finite

This gives $[G(\mathbb{Q}_p) : N_x] < \infty$.