Lecture 2

Formal schemes:

Setup:

A topological ring complete for I-adic topology, $I \subseteq A$ is f.g.

$$\begin{array}{ccccc} \mathrm{Spf}(A) & \longmapsto & \mathrm{Spa}(A,A) & \longmapsto & \mathrm{Spd}(A,A) \\ \| & & & & \| \\ \mathfrak{X} & & & & \mathfrak{X}^{\diamond} \end{array}$$

when $x \in \mathrm{Spa}(A,A)$

$$\mathrm{sp}(x) : \mathfrak{p}_x \subseteq A \quad \text{prime ideal}$$

$$\mathfrak{p}_x = \{ a \in A \mid |a|_x < 1 \}$$

Since $|i| < 1 \ \forall i \in I$ $\quad \mathfrak{p}_x \in \mathrm{Spec}(A/I)$

$$\mathrm{sp} : \mathrm{Spa}(A,A) \longrightarrow \mathrm{Spec}(A/I)$$

Fact: $sp : \mathrm{Spa}(A, A)^{an} \longrightarrow \mathrm{Spec}(A/I)$

is spectral map of spectral spaces.

Variant: (A, A^+) uniform Tate Huber pair

$$\mathrm{Spa}(A, A^+) \subseteq \mathrm{Spa}(A^+, A^+)^{an} \longrightarrow \mathrm{Spec}(A^+/\varpi)$$

Example: $A = \mathbb{Z}_p\langle T\rangle \qquad \mathbb{B}^1_{\mathbb{Z}_p} = \mathrm{Spa}(A)$

$\mathrm{Spec}(A/p) = \mathbb{A}^1_{\mathbb{F}_p} \qquad sp : \mathbb{B}^1_{\mathbb{Q}_p} \longrightarrow \mathbb{A}^1_{\mathbb{F}_p}$

$$sp^{-1}(\eta) = \{ \eta \} , \quad \left|\sum a_i p^i\right|_\eta = \sup |a_i|$$

$$sp^{-1}(0) = \mathbb{B}^{<1}_{\mathbb{Q}_p} \cup \{ \epsilon \} , \quad \left|\sum a_i p^i\right|_\epsilon = \sup |a_i \epsilon^i|$$

doesn't have structure of adic space!

$\forall n \quad 1 - \frac{1}{n} < |\epsilon| < 1$

Question: More generally for X a v-sheaf $x \in |X|$ what is $sp(x)$?

<u>Reduction Functor</u>:

$\left\{ \begin{matrix} \text{Perf} \\ \text{sch} \end{matrix} \right\} =$ perfect schemes in characteristic p endowed with scheme-theoretic v-topology

$$\overbrace{\left\{ \begin{matrix} \text{Perf} \\ \text{sch} \end{matrix} \right\}} \xrightarrow{\diamond} \left\{ \text{v-sheaves} \right\}$$

$\text{spec}(A)^{\diamond} = \text{Spd}(A,A)$, continuous

admits right adjoint $(-)^{red}$

$$\text{Hom}(\text{Spec}(A), F^{red}) = \text{Hom}(\text{Spec}(A)^{\diamond}, F)$$

Ex: • $(X^{\diamond})^{red} = X$

• B I-adic rings over $\mathbb{Z}_p$

$$\text{Spd}(B,B)^{red} = \underbrace{\text{spec}(B/I)^{perf}}_{\text{target of specialization.}}$$

- If X is qs diamond $X^{red} = \emptyset$.

We get an adjunction map

$$(X^{red})^{\diamond} \longrightarrow X$$

Definition

- $X \to Y$ is formally adic if:

$$\begin{tikzcd} (X^{red})^{\diamond} \arrow[r] \arrow[d] & (Y^{red})^{\diamond} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is Cartesian.

- X is formally separated if

$\Delta : X \longrightarrow X \cdot X$ is closed immersion and formally adic.

Key point: $\mathrm{Spa}(C, C^+) \subseteq \mathrm{Spd}(C^+, C^+)$ is not dense, but it is "formally dense".

We want to define $sp : |X| \longrightarrow |X^{red}|$ in functorial way:

$\forall x \in X$ we have

$$\begin{tikzcd} x : \mathrm{Spa}(C, C^+) \arrow[r] & X \\ \tilde{x} : \mathrm{Spa}(C^+, C^+) \arrow[u, phantom, "\cap"] \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

$$sp(x) = \tilde{x}^{red} : \mathrm{Spec}\left(C^+/\varpi\right)^{perf} \longrightarrow X^{red}$$

<u>Definition</u>: X is a <u>specializing</u> v-sheaf if

a) $\exists$ v-cover $\coprod_{j \in J} \mathrm{Spd}(A_j, A_j) \longrightarrow X$

b) X is formally separated.

<u>Fact</u>: $\exists!$ continuous specialization map $sp : |X| \longrightarrow |X^{red}|$

$$\begin{tikzcd} \coprod_{j \in J} |\mathrm{Spd}(A, A)| \arrow[r] \arrow[d, "sp"] & |X| \arrow[d, "sp"] \\ \coprod_{j \in J} |\mathrm{Spec}(A/I)| \arrow[r] & |X^{red}| \end{tikzcd}$$

Imitation game:

{specializing v-sheaves}
- General setup where specialization map is defined.

$\cup$

{Pre-Kimberlites}
- We ask for reduced special fiber to be well-behaved.
- Good notion of étale site / nearby cycles

$\cup$

{smelted Kimberlites}
- We ask for generic fiber to be well-behaved

$\cup$

{Kimberlites}
- We ask for analytic locus to be well-behaved

$\cup$

{spatial Kimberlites}
- We ask for reasonable qcqs properties.

$\cup$

{Formal Schemes}$^{\Diamond}$

Imitation game:

$\{\text{Pre-Kimberlites}\}$ $\begin{cases} \cdot\ (X^{red})^{\diamond} \to X \quad \text{closed immersion} \\ \cdot\ X^{red} \quad \text{perfect scheme} \end{cases}$

$\cup\!\shortmid$

$\{\text{smelted Kimberlites}\}$ $\begin{cases} * \text{ Pre-Kimberlite } + \\ \text{generic fiber is locally spatial diamond.} \end{cases}$

$\cup\!\shortmid$

$\{\text{Kimberlites}\}$ $\begin{cases} * \text{ Pre-Kimberlite } + \\ \text{analytic locus } X \setminus (X^{red})^{\diamond} \text{ is spatial diamond.} \end{cases}$

$\cup\!\shortmid$

$\{\text{spatial Kimberlites}\}$ $\begin{cases} \text{Kimberlite } + \\ X \to * \text{ rep in locally spatial diamonds } + \\ \sqcup \operatorname{Spd}(A) \to X \text{ qcqs formally adic cover.} \end{cases}$

Theorem: (Smelted - kimberlite)

If X is a kimberlite, the specialization map is continuous for the constructible topology of X^{red} and X^{an}.

Moreover, it is a closed map.
(quotient)

Tubular and formal neighborhoods:

Recall that on $\mathbb{B}^1_{\mathbb{Z}_p}$ $\quad sp^{-1}(0) = \mathbb{B}^{<1}_{\mathbb{Q}_p} \cup \{\epsilon\}$

$sp^{-1}(0)^{int}$ has diamond structure.

this is the Berthelot tube.

Let X be a pre-kimberlite and $S \subseteq |X^{red}|$ a locally closed subset we define:

$$\begin{tikzcd} \hat{X}_{/S} \arrow[r, phantom, "\subseteq"] & X \\ & \mathrm{Spa}(R,R^+) \arrow[u, "f"'] \arrow[ul, dashed, "\exists!"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X^{\circledcirc}_{/S} \arrow[r, phantom, "\subseteq"] & X^{an} \\ & \mathrm{Spa}(R,R^+) \arrow[u, "f"'] \arrow[ul, dashed, "\exists!"] \end{tikzcd}$$

whenever $sp(f(\mathrm{Spa}(R))) \subseteq S$.

Heuer's specialization map:

Recall $\quad \mathrm{Spec}(A)^{\diamond/\infty}(R, R^+)$

$$= \{ A \to (R^+/\varpi)^{\mathrm{perf}} \}$$

Theorem (Heuer) $\quad \mathrm{Spec}(A)^{\diamond/\infty}$ is a v-sheaf.

Proposition/Definition: If X is a pre-kimberlite there is a map of v-sheaves

$$SP : X \to (X^{\mathrm{red}})^{\diamond/\infty},$$

given by formula $SP \circ f \sim f^{\mathrm{red}}$

$$\begin{tikzcd}
\mathrm{Spa}(R, R^+) \arrow[r, "f"] & X \arrow[r, "SP"] \arrow[d, leftrightsquigarrow] & (X^{\mathrm{red}})^{\diamond/\infty} \\
 & \mathrm{Spec}\left((R^+/\varpi)^{\mathrm{perf}}\right) \arrow[r, "f^{\mathrm{red}}"] & X^{\mathrm{red}}
\end{tikzcd}$$

Reinterpretation of formal neighborhoods:

$$\begin{tikzcd} \hat{X}_{/S} \arrow[r, squiggly] \arrow[d] & X \arrow[d, "sp"] \\ S^{\diamond/\circ\circ} \arrow[r, squiggly] & (X^{red})^{\diamond/\circ\circ} \end{tikzcd}$$

Étale and Zariski sites:

Fix X pre-kimberlite

we let

$$(X)_{for,\acute{e}t} = \left\{ f: T \to X \;\middle|\; \begin{array}{l} T \text{ is prekimberlite,} \\ f \text{ is formally adic,} \\ \text{étale and} \\ \text{quasi-compact} \end{array} \right\}$$

$$(X^{red})_{qc,\acute{e}t,sep}$$

Theorem: The functor

$$\mathrm{red} : (X)_{\mathrm{for}, \acute{e}t} \longrightarrow (X^{red})_{qc/\acute{e}t, sep}$$

is an equivalence.

The inverse has formula:

$$
\begin{tikzcd}
S \arrow[r, mapsto] & \hat{X}_S \arrow[r] \arrow[d] & X \arrow[d, "sp"] \\
& S^{a/oo} \arrow[r] & (X^{red})^{a/oo}
\end{tikzcd}
$$

Definition: We set a morphism of sites $\Psi_{\mathrm{for}} : (X^{an})_{\acute{e}t} \longrightarrow (X^{red})_{\acute{e}t}$

$$\Psi^{-1}(S) = X^{\Theta}_S = \hat{X}_S \cap X^{an}.$$

This gives the formal nearby cycles functor

$$R\Psi_{\mathrm{for}} : D_{\acute{e}t}(X^{an}, \Lambda) \to D_{\acute{e}t}(X^{red}, \Lambda)$$

We write $\quad j : X^{an} \hookrightarrow X \hookleftarrow (X^{red})^{\diamond}$

and we have analytic nearby cycles

$$R\Psi_{an} := j_* i^* : D_{\acute{e}t}(X^{an}, \Lambda) \to D_{\acute{e}t}(X^{red,\diamond}, \Lambda)$$

$$\begin{tikzcd}
D_{\acute{e}t}(X^{an}) \arrow[r, "R\Psi_{for}"] \arrow[d, "R\Psi_{an}"'] & D_{\acute{e}t}(X^{red}) \arrow[dl, shift left, "c^*"] \\
D_{\acute{e}t}(X^{red,\diamond}) \arrow[ur, shift left, "c_*"] &
\end{tikzcd}$$

Comparison Thm:

1) $c_* R\Psi_{an} = R\Psi_{for}$

2) If $R\Psi_{an}(A) = c^*(B)$ then $B \simeq R\Psi_{for}(A)$.

Warning: $R\Psi_{an}(A)$ is not always of this form!!!