Lecture 2

Formal schemes:

Setup:

A topological ring complete for I-adic topology, $I \subseteq A$ is f.g.

$$\begin{array}{ccccc} \mathrm{Spf}(A) & \hookrightarrow & \mathrm{Spa}(A,A) & \hookrightarrow & \mathrm{Spd}(A,A) \\ \| & & & & \| \\ \mathfrak{X} & & & & \mathfrak{X}^{\diamond} \end{array}$$

when $x \in \mathrm{Spa}(A,A)$

$$\mathrm{sp}(x) : \mathfrak{p}_x \subseteq A \quad \text{prime ideal}$$

$$\mathfrak{p}_x := \{ a \in A \mid |a|_x < 1 \}$$

Since $|i| < 1 \quad \forall i \in I \qquad \mathfrak{p}_x \in \mathrm{Spec}(A/I)$

$$\mathrm{sp} : \mathrm{Spa}(A,A) \longrightarrow \mathrm{Spec}(A/I)$$

Fact: $sp: \mathrm{Spa}(A,A)^{an} \longrightarrow \mathrm{Spec}(A/I)$

is spectral map of spectral spaces.

Variant: (A, A^+) uniform Tate Huber pair

$$\mathrm{Spa}(A, A^+) = \mathrm{Spa}(A^+, A^+)^{an} \longrightarrow \mathrm{Spec}(A^+/\varpi)$$

Example: $A = \mathbb{Z}_p\langle T \rangle$ $\quad \mathbb{B}^1_{\mathbb{Z}_p} = \mathrm{Spa}(A)$

$\mathrm{Spec}(A/p) = \mathbb{A}^1_{\mathbb{F}_p}$ $\quad sp: \mathbb{B}^1_{\mathbb{Q}_p} \longrightarrow \mathbb{A}^1_{\mathbb{F}_p}$

$$sp^{-1}(\eta) = \{ \eta \}, \quad \left| \sum a_i p^i \right|_\eta = \sup |a_i|$$

$$sp^{-1}(0) = \mathbb{B}^{<1}_{\mathbb{Q}_p} \cup \{ \epsilon \}, \quad \left| \sum a_i p^i \right|_\epsilon = \sup |a_i \varepsilon^i|$$

doesn't have structure of adic space! $\quad \forall n \quad 1 - \frac{1}{n} < |\varepsilon| < 1$

"tends infinitesimally towards 0."

Question: More generally for X a v-sheaf $x \in |X|$ what is $sp(x)$?

Reduction Functor:

$\left\{ \begin{matrix} \text{Perf} \\ \text{sch} \end{matrix} \right\}$ = perfect schemes in characteristic p endowed with scheme-theoretic v-topology

$$\left\{ \begin{matrix} \text{Perf} \\ \text{sch} \end{matrix} \right\} \xrightarrow{\diamond} \{ v\text{-sheaves} \}$$

$\mathrm{spec}(A)^\diamond = \mathrm{spd}(A, A)$, continuous admits right adjoint $(-)^{\mathrm{red}}$

$$\mathrm{Hom}(\mathrm{Spec}(A), F^{\mathrm{red}}) = \mathrm{Hom}(\mathrm{Spec}(A)^\diamond, F)$$

Ex: • $(X^\diamond)^{\mathrm{red}} = X$

• B I-adic ring over $\mathbb{Z}_p$

$$\mathrm{Spd}(B, B)^{\mathrm{red}} = \underbrace{\mathrm{Spec}(B/I)^{\mathrm{perf}}}_{\text{target of specialization.}}$$

• (R, R^+) perfectoid, then

$$\mathrm{Spd}(R^+)^{\mathrm{red}} = \mathrm{Spec}(R^+/R^{\circ\circ})$$

• If X is qs diamond $X^{red} = \emptyset$.

We get an adjunction map

$$(X^{red})^{\diamond} \longrightarrow X$$

Definition

a) $X \to Y$ is formally adic if:

$$\begin{array}{ccc} (X^{red})^{\diamond} & \to & (Y^{red})^{\diamond} \\ \downarrow & & \downarrow \\ X & \to & Y \end{array}$$

is Cartesian.

Ex:

$\mathbb{B}^{1\,\diamond}_{\mathbb{Z}_p} \to \mathrm{Spd}\,\mathbb{Z}_p$, formally adic

Non-Ex:

$\mathrm{Spd}(\mathbb{F}_p[[t]], \mathbb{F}_p) \to \mathrm{Spd}(\mathbb{F}_p)$ NOT formally adic

b) X is formally separated if

$\Delta : X \longrightarrow X \cdot X$ is closed immersion and formally adic.

Key point: $\mathrm{Spa}(C, C^+) \subseteq \mathrm{Spd}(C^+, C^+)$ is not dense, but it is "formally dense".

We want to define $sp : |X| \longrightarrow |X^{red}|$ in functorial way:

$\forall x \in X$ we have

$$\begin{tikzcd} x : \mathrm{Spa}(C, C^+) \arrow[r] & X \\ \tilde{x} : \mathrm{Spa}(C^+, C^+) \arrow[u, phantom, "\cup"] \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

$$sp(x) = \tilde{x}^{red} : \mathrm{Spec}(C^+/\varpi)^{perf} \longrightarrow X^{red}$$

<u>Definition:</u> X is a <u>specializing</u> v-sheaf if

a) $\exists$ v-cover $\coprod_{j \in J} \mathrm{Spd}(A_j, A_j) \longrightarrow X$

b) X is formally separated.

<u>Fact:</u> $\exists!$ continuous specialization map $sp : |X| \longrightarrow |X^{red}|$

$$\begin{tikzcd} \coprod_{j \in J} |\mathrm{Spd}(A, A)| \arrow[r] \arrow[d, "sp"] & |X| \arrow[d, "sp"] \\ \coprod_{j \in J} |\mathrm{Spec}(A/I)| \arrow[r] & |X^{red}| \end{tikzcd}$$

Imitation game:

{specializing v-sheaves}

- General setup where specialization map is defined.

$\cup$

{Pre-Kimberlites}

- We ask for reduced special fiber to be well-behaved.
- Good notion of étale site / nearby cycles

$\cup$

{smelted Kimberlites}

- We ask for generic fiber to be well-behaved

$\cup$

{Kimberlites}

- We ask for analytic locus to be well-behaved

$\cup$

{Spatial Kimberlites}

- We ask for reasonable qcqs properties.

$\cup$

{Formal Schemes}$^{\diamond}$

<u>Imitation game:</u>

Conditions:

$\{\text{Pre-Kimberlites}\}$ — • $(X^{red})^{\diamond} \to X$ closed immersion
• X^{red} perfect scheme

$\supseteq$

$\{\text{smelted Kimberlites}\}$ — * Pre-Kimberlite + generic fiber is a locally spatial diamond.

$\supseteq$

$\{\text{Kimberlites}\}$ — * Pre-Kimberlite + analytic locus $X^{an} := X \setminus (X^{red})^{\diamond}$ is a spatial diamond.

$\supseteq$

$\{\text{spatial Kimberlites}\}$ — Kimberlite + $X \to *$ rep in a locally spatial diamonds + $\sqcup \mathrm{Spd}(A) \to X$ qcqs formally adic cover.

Theorem 1 : (smelted - kimberlite)

If X is a kimberlite, the specialization map is continuous for the constructible topology of X^{red} and X^{an}.

Moreover, it is a closed map.
(quotient)

strategy:

- To prove X^{an} is connected we can prove X^{red} is connected and $sp^{-1}(x)$ is connected.

- $sp^{-1}(x)^{int}$ has geometric structure

- Is $sp^{-1}(x) \supseteq sp^{-1}(x)^{int}$ dense?

constructible topology is key to settle this.

Specialization triples:

Given a Kimberlite $\mathfrak{X}$ we can attach a triple $(\mathfrak{X}^{an}, \mathfrak{X}^{red}, sp)$

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{Nice}^{*} \\ \text{formal} \\ \text{schemes}\end{array}\right\} \arrow[rr, leftrightarrow] \arrow[d, phantom, sloped, "\subset"] \arrow[ddr, bend right=60, "\text{(Lorenzo) Full-faithful}"'] \arrow[drr] & \Diamond & \left\{\begin{array}{c}\text{Nice}^{@} \\ \text{kimberlites}\end{array}\right\} \arrow[d, phantom, sloped, "\subset"] \arrow[ddl, bend left=60, "\text{(AGLR) Full-faithful} \leadsto \underline{\text{Theorem 2}}"] \\
\left\{\begin{array}{c}\text{Formal} \\ \text{Schemes over} \\ \mathbb{Z}_p\end{array}\right\} \arrow[rr] \arrow[dr] & \Diamond & \left\{\begin{array}{c}\text{Kimberlites} \\ \text{over} \\ \mathbb{Z}_p\end{array}\right\} \arrow[dl] \\
& \left\{\begin{array}{c}\text{specialization} \\ \text{triples}\end{array}\right\} &
\end{tikzcd}$$

$*$ weakly normal, topologically of finite type, flat formal schemes.

@ • $\mathbb{C}_p$-point are dense and formalizable,
• X^{red} is perfectly finite type
• X is flat (weaker but similar to spatial).

Tubular and formal neighborhoods:

Recall that on $\mathbb{B}^1_{\mathbb{Z}_p}$ $sp^{-1}(0) = \mathbb{B}^{<1}_{\mathbb{Q}_p} \cup \{\epsilon\}$

$sp^{-1}(0)^{int}$ has diamond structure.

this is the Berthelot tube.

Let X be a pre-Kimberlite and $S \subseteq |X^{red}|$ a locally closed subset we define:

$$\begin{tikzcd} \hat{X}_{/S} \arrow[r, phantom, "\subseteq"] & X \\ & \mathrm{Spa}(R,R^+) \arrow[ul, dashed, "\exists!"] \arrow[u, "f"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X^{\odot}_{/S} \arrow[r, phantom, "\subseteq"] & X^{an} \\ & \mathrm{Spa}(R,R^+) \arrow[ul, dashed, "\exists!"] \arrow[u, "f"'] \end{tikzcd}$$

whenever $sp(f(spa(R))) \subseteq S$.

formal neighborhood

tubular neighborhood

Ex $X = Spd(B)$ $S \subseteq Spf(B)$, $S = V(J)$, $I \subseteq J$

analytic localization

$\hat{X}_S = Spd(\hat{B}_J) = \bigcap N_{j<<1}$

Heuer's specialization map:

Recall $\mathrm{spec}(A)^{\diamond/\circ\circ}(R, R^+)$

$= \{ A \to R^+/R^{\circ\circ} \}$

Fact (Heuer) $\mathrm{spec}(A)^{\diamond/\circ\circ}$ is a v-sheaf.

Construction: If X is a pre-kimberlite there is a map of v-sheaves

$SP: X \rightsquigarrow (X^{red})^{\diamond/\circ\circ}$, given

by formula $SP \circ f \sim f^{red}$

$$\begin{tikzcd}
\mathrm{Spa}(R, R^+) \arrow[r, "f"] & X \arrow[r, "SP"] & (X^{red})^{\diamond/\circ\circ} \\
& \arrow[d, leftrightsquigarrow] & \\
\mathrm{spec}(R^+/R^{\circ\circ}) \arrow[r, "f^{red}"] & X^{red} &
\end{tikzcd}$$

$\mathrm{spec}(R^+/R^{\circ\circ}) = \mathrm{Spd}(R^+)^{red}$

Reinterpretation of formal neighborhoods:

$$\begin{tikzcd} \hat{X}_{/S} \arrow[r, squiggly] \arrow[d] & X \arrow[d, "sp"] \\ S^{\diamond/\circ\circ} \arrow[r, squiggly] & (X^{red})^{\diamond/\circ\circ} \end{tikzcd}$$

<u>Étale and Zariski sites:</u>

Fix X pre-kimberlite

we let

$$(X)_{for,\acute{e}t} = \left\{ f : T \to X \;\middle|\; \begin{array}{l} T \text{ is prekimberlite,} \\ f \text{ is formally adic,} \\ \text{étale and} \\ \text{quasi-compact} \end{array} \right\}$$

$$(X^{red})_{qc,\acute{e}t,sep}$$

<u>Fact</u> : The functor

$$\mathrm{red} : (X)_{\mathrm{for}, \acute{e}t} \longrightarrow (X^{\mathrm{red}})_{\mathrm{qc},\acute{e}t,\mathrm{sep}}$$

is an equivalence.

The inverse has formula:

$$S \longmapsto \begin{tikzcd} \hat{X}_S \arrow[r] \arrow[d] & X \arrow[d, "sp"] \\ S^{\mathrm{q/oo}} \arrow[r] & (X^{\mathrm{red}})^{\mathrm{q/oo}} \end{tikzcd}$$

<u>Definition:</u> We get a morphism of sites $\psi_{\mathrm{for}} : (X^{an})_{\acute{e}t} \longrightarrow (X^{\mathrm{red}})_{\acute{e}t}$

$$\psi^{-1}(S) = X_S^{\circ} = \hat{X}_S \cap X^{an}.$$

This gives the formal nearby cycles functor

$$\Psi_{\mathrm{for}} : D_{\acute{e}t}(X^{an}, \Lambda) \to D_{\acute{e}t}(X^{\mathrm{red}}, \Lambda)$$

We write $j : X^{an} \hookrightarrow X \hookleftarrow (X^{red})^{\diamond}$

and we have analytic nearby cycles

$$\Psi_{an} := j_* i^* : D_{ét}(X^{an}, \Lambda) \longrightarrow D_{ét}(X^{red,\diamond}, \Lambda)$$

$$\begin{tikzcd}
D_{ét}(X^{an}) \arrow[r, "\Psi_{for}"] \arrow[d, "\Psi_{an}"'] & D_{ét}(X^{red}) \arrow[dl, shift left, "c^*"] \\
D_{ét}(X^{red,\diamond}) \arrow[ur, shift left, "c_*"] &
\end{tikzcd}$$

$$c^{-1}(U) = X^{red,\diamond} \cap U^{\diamond}$$

<u>Theorem 3</u>

1) $c_* \Psi_{an} = \Psi_{for}$

2) If $\Psi_{an}(A) = c^*(B)$ then $B \simeq \Psi_{for}(A)$.

Warning: $R\Psi_{an}(A)$ is not always of this form!!!