ON THE GEOMETRIC CONNECTED COMPONENTS OF MODULI SPACES OF p-ADIC SHTUKAS AND LOCAL SHIMURA VARIETIES.

IAN GLEASON

ABSTRACT. We study connected components of local Shimura varieties. Given local shtuka datum (G, b, μ), with G unramified over $\mathbb{Q}_p$ and (b, μ) HN-irreducible, we determine $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ with its $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$-action. This confirms new cases of a conjecture of Rapoport and Viehmann. We construct and study the specialization map for moduli spaces of p-adic shtukas at parahoric level whose target is an affine Deligne–Lusztig variety.

CONTENTS

Introduction. In [RV14], Rapoport and Viehmann propose that there should be a theory of p-adic local Shimura varieties. They conjectured the existence of towers of rigid-analytic spaces whose cohomology groups "understand" the local Langlands correspondence for general p-adic reductive groups. In this way, these towers of rigid-analytic varieties would "interact" with the local Langlands correspondence in a similar fashion to how Shimura varieties "interact" with the global Langlands correspondence. Moreover, they conjectured many properties and compatibilities that these towers should satisfy.

In the last decade, the theory of local Shimura varieties went through a drastic transformation with Scholze's introduction of perfectoid spaces and the theory of diamonds. In [SW20], Scholze and Weinstein construct the sought for towers of rigid analytic spaces and generalized them to what are now known as moduli spaces of p-adic shtukas. Moreover, since then, many of the expected properties and compatibilities for local Shimura varieties have been verified and generalized to moduli spaces of p-adic shtukas. The study of the geometry and cohomology of local Shimura varieties and moduli spaces of p-adic shtukas is still a very active area of research due to their connection to the local Langlands correspondence. The main aim of this article is to study the locally profinite space of connected components, and describe explicitly the continuous right action of the group $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$ on this space. In particular, we prove and generalize [RV14, Conjecture 4.26] for the case of unramified groups.

Let us recall the formalism of local Shimura varieties and moduli spaces of p-adic shtukas. Local p-adic shtuka datum over $\mathbb{Q}_p$ is a triple $(G, [b], [\mu])$ where G is a reductive group over $\mathbb{Q}_p$, $[\mu]$ is a conjugacy class of geometric cocharacters $\mu : \mathbb{G}_m \to G$ and $[b]$ is an element of Kottwitz set $B(G, [\mu])$. Whenever $[\mu]$ is minuscule we say that $(G, [b], [\mu])$ is local Shimura datum. We let $E/\mathbb{Q}_p$ denote the reflex field of $[\mu]$ and $\breve{E} = E \cdot \breve{\mathbb{Q}}_p$. Associated to $(G, [b], [\mu])$ there is a tower of diamonds over $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$, denoted $(\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}})_\mathcal{K}$, where $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ ranges over compact subgroup of $G(\mathbb{Q}_p)$. Moreover, whenever $[\mu]$

is minuscule and $\mathcal{K}$ is a compact open subgroup, then $(\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}})_\mathcal{K}$ is represented by the diamond associated to a unique smooth rigid-analytic space $\mathbb{M}_\mathcal{K}$ over $\breve{E}$. The tower $(\mathbb{M}_\mathcal{K})_\mathcal{K}$ is the local Shimura variety.

Associated to $[b] \in B(G,\mu)$ there is a reductive group J_b over $\mathbb{Q}_p$. After basechange to a completed algebraic closure, each individual space $(\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}} \times \mathbb{C}_p)_\mathcal{K}$ comes equipped with continuous and commuting right actions by $J_b(\mathbb{Q}_p)$ and the Weil group W_E. Moreover, the tower receives a right action by the group $G(\mathbb{Q}_p)$ by using correspondences. When we let $\mathcal{K} = \{e\}$ we obtain the space at infinite level, denoted $\mathrm{Sht}_{G,[b],[\mu],\infty} \times \mathbb{C}_p$, which overall comes equipped with a continuous right action by $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$.

Since the actions are continuous the groups $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$ act continuously on $\pi_0(\mathrm{Sht}_{G,[b],[\mu],\infty} \times \mathbb{C}_p)$ and our main theorem describes explicitly this action whenever G is an unramified reductive group over $\mathbb{Q}_p$ and $([b],[\mu])$ is HN-irreducible. It is natural to expect that the methods of this paper combined with those of [Han21] and [GI16] could be used to remove the HN-irreducible condition. We do not pursue this generality.

Before stating our main theorem we set some notation. Let $(G,[b],[\mu])$ be local p-adic shtuka datum with G an unramified reductive group over $\mathbb{Q}_p$. Let G^{der} denote the derived subgroup of G and G^{sc} denote the simply connected cover of G^{der}, let N denote the image of $G^{\mathrm{sc}}(\mathbb{Q}_p)$ in $G(\mathbb{Q}_p)$ and let $G^\circ = G(\mathbb{Q}_p)/N$. This is a locally profinite topological group and it is the maximal abelian quotient of $G(\mathbb{Q}_p)$ when this later is considered as an abstract group. Let $E \subseteq \mathbb{C}_p$ be the field of definition of $[\mu]$, let $\mathrm{Art}_E : W_E \to E^\times$ be Artin's reciprocity character from local class field theory. In §4 we associate to $[\mu]$ a continuous map of topological groups $\mathrm{Nm}^\circ_{[\mu]} : E^\times \to G^\circ$ and we associate to $[b]$ a map $\det^\circ : J_b(\mathbb{Q}_p) \to G^\circ$.

The general construction of $\mathrm{Nm}^\circ_{[\mu]}$ and $\det^\circ$ uses z-extensions and we do not review it in this introduction. Nevertheless, whenever $G^{\mathrm{sc}} = G^{\mathrm{der}}$ we can construct them as follows. In this case, $G^\circ = G^{\mathrm{ab}}(\mathbb{Q}_p)$ with G^{ab} is the co-center of G. If we let $\det : G \to G^{\mathrm{ab}}$ be the quotient map we can consider the induced data $\mu^{\mathrm{ab}} = \det \circ [\mu]$ and $[b^{\mathrm{ab}}] = [\det(b)]$. Then $\mathrm{Nm}^\circ_{[\mu]}$ can be defined as:

$$E^\times \xrightarrow{\mu^{\mathrm{ab}}} G^{\mathrm{ab}}(E) \xrightarrow{\mathrm{Nm}^{G^{\mathrm{ab}}}_{E/\mathbb{Q}_p}} G^{\mathrm{ab}}(\mathbb{Q}_p) = G^\circ.$$

Here for a torus T over $\mathbb{Q}_p$, like G^{ab}, we are letting $\mathrm{Nm}^T_{E/\mathbb{Q}_p} : T^{\mathrm{ab}}(E) \to T^{\mathrm{ab}}(\mathbb{Q}_p)$ denote the usual norm map

$$t \mapsto \prod_{\gamma \in \mathrm{Gal}(E/\mathbb{Q}_p)} \gamma(t).$$

On the other hand, $\det^\circ : J_b(\mathbb{Q}_p) \to G^{\mathrm{ab}}(\mathbb{Q}_p)$ is $\det = j_{b^{\mathrm{ab}}} \circ \det_b$ where $\det_b : J_b(\mathbb{Q}_p) \to J_{b^{\mathrm{ab}}}(\mathbb{Q}_p)$ is obtained from functoriality of the formation of J_b, and $j_{b^{\mathrm{ab}}} : J_{b^{\mathrm{ab}}}(\mathbb{Q}_p) \xrightarrow{\cong} G^{\mathrm{ab}}(\mathbb{Q}_p)$ is obtained from regarding $J_{b^{\mathrm{ab}}}(\mathbb{Q}_p)$ and $G^{\mathrm{ab}}(\mathbb{Q}_p)$ as subgroups of $G^{\mathrm{ab}}(K_0)$ and exploiting that G^{ab} is commutative. Our first main theorem is:

Theorem 1. *Let $(G,[b],[\mu])$ be local shtuka datum with G an unramified reductive group over $\mathbb{Q}_p$ and $([b],[\mu])$ HN-irreducible. The following hold:*

(1) *The right $G(\mathbb{Q}_p)$ action on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ is trivial on $N = \mathrm{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$ and the induced G°-action is simply-transitive.*

(2) *If $s \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ and $j \in J_b(\mathbb{Q}_p)$ then*

$$s \cdot_{J_b(\mathbb{Q}_p)} j = s \cdot_{G^\circ} \det^\circ(j^{-1}))$$

(3) *If $s \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ and $\gamma \in W_E$ then*

$$s \cdot_{W_E} \gamma = s \cdot_{G^\circ} [\mathrm{Nm}^\circ_{[\mu]} \circ \mathrm{Art}_E(\gamma)].$$

Let us comment on previous results in the literature. Before a full theory of local Shimura varieties was available the main examples of local Shimura varieties one could work with were the ones obtained as the generic fiber of a Rapoport–Zink space ([RZ96]). The most celebrated examples of Rapoport–Zink spaces are of course the Lubin–Tate tower and the tower of covers of Drinfeld's upper half space. In [dJ95] de Jong, as an application of his theory of fundamental groups, computes the connected components of the Lubin–Tate tower for $\mathrm{GL}_n(\mathbb{Q}_p)$. In [Str08], Strauch computes by a different method the connected components of the Lubin–Tate tower for $\mathrm{GL}_n(F)$ and an arbitrary finite extension F of $\mathbb{Q}_p$ (including ramification). In [Che13], M. Chen constructs 0-dimensional local Shimura varieties and studies their geometry. In a later paper [Che14], she constructs her "determinant" map and uses these 0-dimensional local Shimura varieties to describe connected components of Rapoport–Zink spaces of EL and PEL type

associated to more general unramified reductive groups. Our result goes beyond the previous ones in that the only condition imposed on G is unramifiedness. In this way, our result is the first to cover very general families of local Shimura varieties that can not be constructed from a Rapoport–Zink space. In particular, our result is new for local Shimura varieties associated to reductive groups of exceptional types.

The central strategy of Chen's result builds on and heavily generalizes the central strategy used by de Jong. Two key inputs of Chen's work to the strategy are the use of her "generic" crystalline representations and her collaboration with Kisin and Viehmann on computing the connected components of affine Deligne–Lusztig varieties [CKV15]. Our strategy takes these two inputs as given.

We build on the central strategy employed by de Jong and Chen, but the versatility of Scholze's theory of diamonds and the functorial construction of local Shimura varieties allow us to make simplifications and streamline the proof. Since our arguments take place in Scholze's category of diamonds rather than the category of rigid analytic spaces, our argument works even for moduli spaces of p-adic shtukas that are not a local Shimura variety. In these (non-representable) cases, the result is new even for $G = \mathrm{GL}_n$.

Our new main contribution to the central strategy is the use of specialization maps. To use these specialization maps in a rigorous way, we developed a formalism whose details were worked out in the separate paper [Gle22].

Let us sketch the central strategy to prove Theorem 1. Once one knows that $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ is a right G°-torsor, computing the actions by W_E and $J_b(\mathbb{Q}_p)$ in terms of the G° action can be reduced to the tori case using functoriality, z-extensions and the determinant map. These uses mainly group theoretic methods and down to earth diagram chases. In the tori case, the $J_b(\mathbb{Q}_p)$ action is easy to compute and the W_E action can be bootstrapped to an easier case as follows. For tori T, by the work of Kottwitz, we know that the set $B(T, \mu)$ has a unique element so that the data of b is redundant. We can consider the category of pairs (T, μ) where T is a torus over $\mathbb{Q}_p$ and μ is a geometric cocharacter whose field of definition is E. The construction of moduli spaces of shtukas is functorial with respect to this category. Moreover, this category has an initial object given by $(\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m), \mu_u)$ where

$$\mu_u : \mathbb{G}_m \to \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)_E$$

is the unique map of tori that on E-points is given by the formula

$$f \mapsto f \otimes_{\mathbb{Q}_p} f.$$

After more diagram chasing one can again reduce the tori case to this "universal" case. Finally, this case can be done explicitly using the theory of Lubin–Tate groups and their relation to local class field theory. As we have mentioned, the tori case was already handled by M. Chen in [Che13], but for the convenience of the readers we recall part of the story in a different language.

Let us sketch how to prove that $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ is a G° torsor in the simplest case. For this, let G be semisimple and simply connected. Our theorem then says that $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p$ is connected.

The first step is to prove that $G(\mathbb{Q}_p)$ acts transitively on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$. Using the Grothendieck–Messing period map one realizes that this is equivalent to proving that the b-admissible locus of Scholze's B_{dR}-Grassmannian is connected. This fact is a result of Hansen and Weinstein to which we give an alternative proof.

For the next step, let $x \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ and let $G_x \subseteq G(\mathbb{Q}_p)$ denote the stabilizer of x. Let $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ be a hyperspecial subgroup of G. We claim that it is enough to prove that G_x is open and that $G(\mathbb{Q}_p) = \mathcal{K} \cdot G_x$. Indeed, $\mathcal{K}$ surjects onto $G(\mathbb{Q}_p)/G_x$ so that this space is discrete and compact therefore finite. By a theorem of Margulis [Mar91], since we assumed G to be simply connected, the only open subgroup of finite index is the whole group so that $G_x = G(\mathbb{Q}_p)$. The proof that G_x is open relies heavily on M. Chen's main result of [Che14] on "generic" crystalline representations. To be able to apply her result in our context one uses that for suitable p-adic fields K, every crystalline representation is realized as a $\mathrm{Spd}(K, O_K)$-valued point in Scholze's B_{dR}-Grassmannian. For the convenience of the reader, we include a discussion on how to think of crystalline representations as $\mathrm{Spd}(K, O_K)$-valued points.

Finally, proving that $G(\mathbb{Q}_p) = \mathcal{K} \cdot G_x$ is equivalent to proving that $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathbb{C}_p$, the $\mathcal{K}$-level moduli space of shtukas, is connected. This is where our theory of specialization maps gets used, which leads to our second main theorem. Suppose G general reductive group over $\mathbb{Q}_p$ (no longer assumed to be unramified) and assume that $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ can be realized as the $\mathbb{Z}_p$-points of a parahoric group scheme $\mathscr{G}$

over $\mathbb{Z}_p$. In this circumstance, Scholze and Weinstein, construct a v-sheaf $\mathrm{Sht}_{O_{\breve{E}}}^{\mathscr{G}_b, \leq \mu}$ defined over $\mathrm{Spd}(O_{\breve{E}})$ and whose generic fiber is $\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}}$ ([SW20, §25]).

Theorem 2. *Let $(G, [b], [\mu])$ be local shtuka datum (not necessarily HN-irreducible), let $\mathscr{G}$ be a parahoric model of G and let $\mathcal{K} = \mathscr{G}(\mathbb{Z}_p)$.*

   a) With terminology as in [Gle22, Definition 4.52, Definition 3.12], $(\mathrm{Sht}_{O_{\breve{E}}}^{\mathscr{G}_b, \leq \mu}, \mathrm{Sht}_{G,b,[\mu],\mathcal{K}})$ is a rich smelted kimberlite and the reduced special fiber $(\mathrm{Sht}_{O_{\breve{E}}}^{\mathscr{G}_b, \leq \mu})^{\mathrm{red}}$ is equal to $X_{\mathscr{G}}^{\leq \mu}(b)$, the affine Deligne–Lusztig variety associated to $(\mathscr{G}, [b], \mu)$.[1]

   b) There is a continuous, surjective and $J_b(\mathbb{Q}_p)$-equivariant specialization map

$$\mathrm{Sp} : |\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathbb{C}_p| \to |X_{\mathscr{G}}^{\leq \mu}(b)|.[2]$$

   c) When $\mathscr{G}$ is hyperspecial, $(\mathrm{Sht}_{O_{\breve{E}}}^{\mathscr{G}_b, \leq \mu}, \mathrm{Sht}_{G,b,[\mu],\mathcal{K}})$ is topologically normal and the specialization map induces a bijection of connected components

$$\pi_0(\mathrm{Sp}) : \pi_0(\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathbb{C}_p) \xrightarrow{\cong} \pi_0(X_{\mathscr{G}}^{\leq \mu}(b)).$$

Fortunately for us, the study of connected components of affine Deligne–Lusztig varieties has enough literature [CKV15], [Nie18] [HZ20]. In the HN-irreducible case, and G unramified, they can be identified with certain subsets of $\pi_1(G)$. If we go back to the assumptions of Theorem 1 and assume again that G is semi-simple and simply connected, we get $\pi_1(G) = \{e\}$, which finishes the (sketch of) the proof of Theorem 1 for this case. The central strategy used for general unramified groups G is not very different in spirit and only requires more patience.
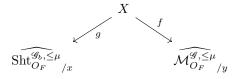
The proof of Theorem 2 uses the machinery from integral p-adic Hodge theory as discussed in [SW20], the formalism developed in [Gle22], and for general parahoric our recent collaboration [AGLR22]. The key inputs to prove that $\mathrm{Sht}_{O_{\breve{E}}}^{\mathscr{G}_b, \leq \mu}$ has a specialization map are Kedlaya's work [Ked20] and Anschütz' work [Ans22, Theorem 1.1] on extending vector bundles and $\mathscr{G}$-torsors over the punctured spectrum of $A_{\inf}$. Recall that $\mathrm{Sht}_{O_{\breve{E}}}^{\mathscr{G}_b, \leq \mu}$ parametrizes triples $(\mathcal{T}, \Phi, \rho)$ where $(\mathcal{T}, \Phi)$ is a shtuka with $\mathscr{G}$ structure and $\rho : \mathcal{T} \to \mathscr{G}_b$ is φ-equivariant trivialization over $\mathcal{Y}_{[r,\infty]}$ for large enough r. A key observation is that $(\mathrm{Sht}_{O_{\breve{E}}}^{\mathscr{G}_b, \leq \mu})^{\mathrm{red}}$ is roughly speaking the locus in which ρ is meromorphic. With this in mind we prove $(\mathrm{Sht}_{O_{\breve{E}}}^{\mathscr{G}_b, \leq \mu})^{\mathrm{red}} = X_{\mathscr{G}}^{\leq \mu}(b)$. The finiteness properties (being rich), are known facts coming from the Grothendieck–Messing period morphism and general results on affine Deligne–Lusztig varieties. Finally, to prove surjectivity of the specialization map and relate the connected components of the generic fiber with the connected components of the reduced special fiber, one is led to study the tubular neighborhoods of $(\mathrm{Sht}_{O_{\breve{E}}}^{\mathscr{G}_b, \leq \mu}, \mathrm{Sht}_{G,b,[\mu],\mathcal{K}})$ (as in [Gle22, Definition 4.18, Definition 4.38]). To do this, we construct a "local model diagram" for tubular neighborhoods. We clarify below.

Before stating our last main theorem we setup some terminology and formulate a conjectural statement that is philosophically aligned with Grothendieck–Messing theory. Let $\mathcal{M}_{O_E}^{\mathscr{G}, \leq \mu}$ denote the local model studied in [AGLR22] and let $\mathcal{A}_{\mathscr{G}, \mu} = (\mathcal{M}_{O_E}^{\mathscr{G}, \leq \mu})^{\mathrm{red}}$ denote its reduced special fiber. This is the μ-admissible locus in the Witt vector affine flag variety. We let $F \supseteq \breve{E}$ be a nonarchimedean field extension with ring of integers O_F and algebraically closed residue field k_F.

Conjecture 1. *For every closed point $x \in |(X_{\mathscr{G}}^{\leq \mu}(b))_{k_F}|$ there exist a closed point $y \in |(\mathcal{A}_{\mathscr{G}, \mu})_{k_F}|$ such that the formal neighborhoods $\widehat{\mathrm{Sht}_{O_F}^{\mathscr{G}_b, \leq \mu}}_{/x}$ and $\widehat{\mathcal{M}_{O_F}^{\mathscr{G}, \leq \mu}}_{/y}$ are isomorphic v-sheaves.*

The weaker version that we are able to prove at the moment is as follows.

Theorem 3. *With the notation as in Conjecture 1 there is a connected v-sheaf in groups $\widehat{L_W^+ G}$ such that for every x there exists y and a diagram*

$$
\begin{array}{ccc}
 & X & \\
 \swarrow{\scriptstyle g} & & \searrow{\scriptstyle f} \\
\widehat{\mathrm{Sht}_{O_F}^{\mathscr{G}_b, \leq \mu}}_{/x} & & \widehat{\mathcal{M}_{O_F}^{\mathscr{G}, \leq \mu}}_{/y}
\end{array}
$$

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[1] We expect these v-sheaves to be rich kimberlites, but we have not proved this yet.

[2] The map is also a spectral map of locally spectral spaces, specializing and a quotient map.

where f and g are both $\widehat{L_W^+ G}$-bundles. In particular, $\widehat{\mathrm{Sht}_{O_F}^{\mathcal{G}_b, \leq \mu}}_{/x}$ is non-empty and $\pi_0(\widehat{\mathrm{Sht}_{O_F}^{\mathcal{G}_b, \leq \mu}}_{/x} \times F) = \pi_0(\widehat{\mathcal{M}_{O_F}^{\mathcal{G}, \leq \mu}}_{/y} \times F)$.

Let us mention that this version of the local model diagram, although not completely satisfactory, has already found some applications in the recent representability results of Pappas and Rapoport [PR21].

Finally, to establish the identity $\pi_0(\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathbb{C}_p) \cong \pi_0(X_{\mathcal{G}}^{\leq \mu}(b))$ one is reduced to proving that all the tubular neighborhoods of the local model $\mathcal{M}_{O_{\mathbb{C}_p}}^{\mathcal{G}, \leq \mu}$ have connected generic fiber. As was observed in [AGLR22], the condition that these tubular neighborhoods are generically connected is a "kimberlite analogue" of normality. When $\mathcal{G}$ is hyperspecial, we prove this normality in [Gle22] using a Demazure resolution. Unfortunately, this part of the argument doesn't generalize directly for general parahoric groups $\mathcal{G}$, and the proof of normality will require more sophisticated tools.

Let us comment on the organization of this paper. The goal of the first chapter is to prove Theorem 1 using mainly generic fiber methods and taking as a black box some "integral method inputs", which we justify in the second chapter. In the first two sections, we recall the relation between crystalline representations, Scholze's theory of diamonds, Chen's "generic" cyrstalline representations, and other geometric constructions that appear in modern rational p-adic Hodge theory. This part of the paper is purely expository, but it is important for the rest of the argument to have these relations in mind. In the third section we discuss local Shimura varieties associated to tori and we review M. Chen's results on this objects. In section four, the details of the proof of Theorem 1 are provided.

The goal of the second chapter is to prove Theorem 2 and Theorem 3. In the first section we collect some facts from integral p-adic Hodge theory required for our argument to go through. In the second section, we recall the kimberlite structure of the local model. In the third section, we establish the main properties we need to construct a specialization map for moduli spaces of shtukas. In the final section, we prove Theorem 3 and finish the proof of Theorem 2.

Notation. When R is a characteristic p ring we let $W(R)$ denote the ring of p-typical Witt vectors of R and we denote by $\varphi : W(R) \to W(R)$ the canonical lift of arithmetic Frobenius. For a Huber pair (R, R^+) we use the abbreviations $\mathrm{Spd}(R)$ and $\mathrm{Spa}(R)$ when the entry R^+ is understood from the context. Whenever $f : R \to R'$ is a ring homomorphism (respectively morphism of Huber pairs), we let $f^{\mathrm{op}} : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$ (respectively $f^{\mathrm{op}} : \mathrm{Spa}(R') \to \mathrm{Spa}(R)$ or $f^{\mathrm{op}} : \mathrm{Spd}(R') \to \mathrm{Spd}(R)$) the morphism of spaces induced by f.

We let k be an algebraically closed field in characteristic p. We let $K_0 = W(k)[\frac{1}{p}]$. We fix an algebraic closure $\overline{K}_0$ of K_0, and we let C_p denote the p-adic completion of $\overline{K}_0$. We use K to denote subfields of C_p of finite degree over K_0. We let Γ_K denote the continuous automorphisms of C_p that fix K. If $\overline{K}_0$ is the algebraic closure of K_0 in C_p then Γ_K is canonically isomorphic to $\mathrm{Gal}(\overline{K}_0/K)$, since $\overline{K}_0$ is dense in C_p. We denote by Γ_K^{op} the opposite group of Γ_K which we identify with the group of automorphisms of $\mathrm{Spec}(C_p)$ over $\mathrm{Spec}(K_0)$. We let W_{K_0} denote the subset of continuous automorphisms of $\mathrm{Aut}(C_p)$ that stabilize K_0 and act as an integral power of φ on K_0. We topologize W_{K_0} so that Γ_{K_0} is an open subgroup. Suppose $E \subseteq C_p$ is a field of finite degree over $\mathbb{Q}_p$, and let $\mathbb{Q}_{p^s}$ be the maximal unramified

extension of $\mathbb{Q}_p$ contained in E. The extension $E/\mathbb{Q}_{p^s}$ is totally ramified and $E \otimes_{\mathbb{Q}_{p^s}} K_0$ is canonically isomorphic to the compositum $E_0 := E \cdot K_0$ inside of C_p. We define an automorphism $\hat{\varphi} \in \mathrm{Aut}(E_0)$ as the automorphism that maps to $\mathrm{Id} \otimes \varphi$ under this identification. We let $W_E^{K_0}$ denote the continuous automorphisms of C_p that stabilize E_0 and act on E_0 as $\hat{\varphi}^{s \cdot n}$ for some $n \in \mathbb{Z}$. Notice that $W_E^{K_0}$ fixes E. The case of interest is when $k = \overline{\mathbb{F}}_p$ but some of arguments require us to pass to larger fields. When $k = \overline{\mathbb{F}}_p$ then $K_0 = \breve{\mathbb{Q}}_p$, $C_p = \mathbb{C}_p$, $E_0 = \breve{E}$ and $W_E^{K_0} = W_E$.

Through out the text, G will denote a connected reductive group over $\mathbb{Q}_p$. In certain subsections we will add the additional assumptions that G is quasi-split or even stronger that it is unramified over $\mathbb{Q}_p$. We will point out when one of these two assumptions are taken. Whenever G is quasi-split we will denote by A a maximally split sub-torus of G defined over $\mathbb{Q}_p$, T will denote the centralizer of A which is also a torus and B will denote a $\mathbb{Q}_p$-rational Borel containing T. We will denote by $\mathscr{G}$ a parahoric model of G over $\mathbb{Z}_p$. Sometimes we will assume $\mathscr{G}$ is hyperspecial in which case we will abuse notation and declare $G = \mathscr{G}$.

We will often work in the situation in which we are given an element $b \in G(K_0)$ and/or a cocharacter $\mu : \mathbb{G}_m \to G_K$. In these circumstances $[b]$ always denotes the φ-conjugacy class of b in $G(K_0)$ and $[\mu]$ denotes the unique geometric conjugacy class of cocharacters $[\mu] \in \mathrm{Hom}(\mathbb{G}_m, G_{\overline{\mathbb{Q}}_p})$ that is conjugate to μ through the action of G_{C_p}. Moreover, we let $E/\mathbb{Q}_p$ denote the field extension contained in C_p over which $[\mu]$ is defined. We let E_0 denote the compositum of E and K_0 in C_p.

1. Geometric connected components.

1.1. The geometric perspective on crystalline representations.

1.1.1. *Vector bundles, isocrystals and crystalline representations.* Let K_0, K and C_p be as above. With this setup, in [FF18], Fargues and Fontaine construct a $\mathbb{Q}_p$-scheme X_{FF,C_p}, known as "the Fargues–Fontaine curve". Denote by $\Phi\text{-Mod}_{K_0}$ the category of isocrystals over K_0, this is a $\mathbb{Q}_p$-linear Tannakian category. Fargues and Fontaine associate to $(D, \Phi) \in \Phi\text{-Mod}_{K_0}$ a vector bundle $\mathcal{E}(D, \Phi)$ that comes equipped with a $\Gamma_{K_0}^{\mathrm{op}}$-action that is compatible with the action on X_{FF,C_p} ([FF18, Définition 10.2.1, Définition 9.1.1]).

The Beauville–Laszlo theorem ([SW20, Lemma 5.2.9]), provides us with an equivalence from the category of vector bundles over X_{FF,C_p} to the category of triples $(M_e, M_{\mathrm{dR}}^+, u)$ where M_e is a free module over B_e, M_{dR}^+ is a free module over B_{dR}^+ and $u : M_e \otimes_{B_e} B_{\mathrm{dR}} \to M_{\mathrm{dR}}^+ \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}$ is an isomorphism. This is Berger's category of B-pairs. From this equivalence we get a recipe to construct vector bundles by replacing (or modifying) M_{dR}^+ by some other B_{dR}^+-lattice Λ contained in $M_{\mathrm{dR}} := M_{\mathrm{dR}}^+ \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}$. If we choose Λ to be stable under the action of Γ_K on M_{dR}, then the new vector bundle produced in this way will have a Γ_K^{op}-action compatible with the one on X_{FF,C_p}. Fortunately, we can understand Γ_K-stable lattices in a concrete way as we recall below.

Given a finite dimensional K vector space V we can let $\mathrm{Fil}^\bullet V$ denote a decreasing filtration of K vector spaces. If $\mathrm{Fil}^\bullet V$ satisfies $\mathrm{Fil}^i V = V$ for $i \ll 0$ and $\mathrm{Fil}^i = 0$ for $i \gg 0$, we say that $\mathrm{Fil}^\bullet V$ is a bounded filtration. To such a filtration we can associate a B_{dR}^+-lattice in $V \otimes_K B_{\mathrm{dR}}$ denoted $\mathrm{Fil}^0(V \otimes_K B_{\mathrm{dR}})$ and given by the formula:

$$\mathrm{Fil}^0(V \otimes_K B_{\mathrm{dR}}) = \sum_{i+j=0} \mathrm{Fil}^i V \otimes_K \mathrm{Fil}^j B_{\mathrm{dR}}.$$

Proposition 1.1. ([FF18, Proposition 10.4.3]) *Let V be a finite dimensional vector space over K. The map that assigns to a bounded filtration $\mathrm{Fil}^\bullet V$ the B_{dR}^+-lattice $\mathrm{Fil}^0(V \otimes_K B_{\mathrm{dR}})$ in $V \otimes_K B_{\mathrm{dR}}$ gives a bijection between the set of bounded filtrations of V and Γ_K-stable B_{dR}^+-lattices Λ in $V \otimes_K B_{\mathrm{dR}}$. If we let ξ denote a uniformizer of B_{dR}^+ then the inverse map is given by:*

$$\mathrm{Fil}_\Lambda^i(V) = \left((\xi^i \cdot \Lambda \cap V \otimes_K B_{\mathrm{dR}}^+)/(\xi^i \cdot \Lambda \cap V \otimes_K \xi \cdot B_{\mathrm{dR}}^+) \right)^{\Gamma_K}.$$

Remark 1.2. The careful reader may notice that the reference constructs $\mathrm{Fil}_\Lambda^i(V)$ in a slightly different but equivalent way. We also point out the following. Let $(a_1, \ldots a_n)$ denote a decreasing sequence of integers and let $\mu : \mathbb{G}_m \to \mathrm{GL}_n$ the character defined by $\mu(t) \cdot e_i = t^{a_i} e_i$. We let $\mathrm{Fil}_\mu^\bullet(K^n)$ denote the decreasing filtration associated μ with $e_j \in \mathrm{Fil}_\mu^i$ if $a_j \geq i$. Then the B_{dR} lattice associated to Fil_μ^i is generated as a B_{dR}^+-module by $\xi^{-a_i} e_i$. Notice the change of signs! It will be important to keep track of this later in a computation.

Denote by $\Phi-\mathrm{ModFil}_{K/K_0}$ the category of filtered Φ-modules that has as objects triples $(D, \Phi, \mathrm{Fil}^\bullet D_K)$ where (D, Φ) is in $\Phi\text{-}\mathrm{Mod}_{K_0}$ and $\mathrm{Fil}^\bullet D_K$ is a bounded filtration on $D \otimes_{K_0} K$. To any triple as above Fargues and Fontaine associate a vector bundle $\mathcal{E}(D, \Phi, \mathrm{Fil}^\bullet D_K)$ equipped with a Γ_K^{op}-action compatible with the action on X_{FF,C_p}.

This induces an exact and fully-faithful functor

$$\Phi-\mathrm{ModFil}_{K/K_0} \hookrightarrow \mathrm{Vec}_{\mathrm{X}_{FF,C_p}}^{\Gamma_K^{\mathrm{op}}}$$

from the category of filtered isocrystals to the category of Γ_K^{op}-equivariant vector bundles ([FF18, Proposition 10.5.3]). Any object of $\mathrm{Vec}_{\mathrm{X}_{FF,C_p}}^{\Gamma_K^{\mathrm{op}}}$ in the essential image of this functor is called a crystalline vector bundle. Moreover, when the filtered isocrystal $(D, \Phi, \mathrm{Fil}^\bullet D_K)$ is "weakly admissible" Fargues and Fontaine prove that $\mathcal{E}(D, \Phi, \mathrm{Fil}^\bullet D_K)$ is semi-stable of slope 0 ([FF18, Définition 10.5.2, Proposition 10.5.6]). This implies that $\mathcal{E}(D, \Phi, \mathrm{Fil}^\bullet D_K)$ without the Γ_K^{op}-action is non-canonically isomorphic to $\mathcal{O}_X^d$ for $d = dim_K(D)$ so that $\mathrm{H}^0(\mathrm{X}_{FF,C_p}, \mathcal{E}(D, \Phi, \mathrm{Fil}^\bullet D_K))$ is a d-dimensional $\mathbb{Q}_p$-vector space endowed with a continuous Γ_K-action. This construction recovers the classical functor of Fontaine $V_{cris} : \Phi-\mathrm{ModFil}_{K/K_0}^{w.a.} \to \mathrm{Rep}_{\Gamma_K}(\mathbb{Q}_p)$ that associates to a weakly admissible filtered isocrystals a crystalline representation.

1.1.2. *Families of B_{dR}-lattices.* One can upgrade geometrically the situation using Scholze's theory of diamonds, since this theory allows us to consider "families" of B_{dR}^+-lattices as a geometric object. Recall that the Fargues-Fontaine curve X_{FF,C_p} has a counterpart $\mathcal{X}_{FF,C_p^\flat}$ in the category of adic spaces. Moreover it also has relative analogues. If S be an affinoid perfectoid space in characteristic p, Kedlaya and Liu ([KL15, §8.7]) associate to S an adic space $\mathcal{X}_{FF,S}$ that they call the relative Fargues-Fontaine curve. This construction is functorial in $\mathrm{Perf}_{\mathbb{F}_p}$, the category of affinoid perfectoid spaces in characteristic p. Moreover, if (D, Φ) is an isocrystal over K_0 and S is an affinoid perfectoid space over $\mathrm{Spa}(k, k)$ one can construct a vector bundle $\mathcal{E}_S(D, \Phi)$ over $\mathcal{X}_{FF,S}$. This construction is also functorial in Perf_k and recovers $\mathcal{E}(D, \Phi)$ when $S = \mathrm{Spa}(C_p^\flat)$. Now, given a perfectoid space $S \in \mathrm{Perf}_{\mathbb{F}_p}$ the data of a map $S \to \mathrm{Spd}(K_0)$ induces a "section" at infinity $\infty : S^\sharp \to \mathcal{X}_{FF,S}$. This is a closed Cartier divisor as in [SW20, Definition 5.3.7] and as such it has a good notion of meromorphic functions. We consider the moduli space of meromorphic modifications of $\mathcal{E}_S(D, \Phi)$ along ∞.

Definition 1.3. (1) We let $\mathrm{Gr}(\mathcal{E}(D, \Phi))$ denote the functor from $\mathrm{Perf}_{\mathrm{Spd}(K_0)} \to \mathrm{Sets}$ that assigns:

$$(S^\sharp, f) \mapsto \{((S^\sharp, f), \mathcal{V}, \alpha)\}/\cong$$

Where $(S^\sharp, f)$ is an untilt of S over $\mathrm{Spa}(K_0)$, $\mathcal{V}$ is a vector bundle over $\mathcal{X}_{FF,S}$ and $\alpha : \mathcal{V} \dashrightarrow \mathcal{E}_S(D, \Phi)$ is an isomorphism defined over $\mathcal{X}_{FF,S} \setminus \infty$ and meromorphic along ∞.

(2) Let Gr_{GL_n} denote the functor from $\mathrm{Perf}_{\mathbb{Q}_p} \to \mathrm{Sets}$ that assigns:

$$(S^\sharp, f) \mapsto \{((S^\sharp, f), \mathcal{V}, \alpha)\}/\cong$$

Where $(S^\sharp, f)$ is an untilt of S over $\mathrm{Spa}(\mathbb{Q}_p)$, $\mathcal{V}$ is a vector bundle over $\mathrm{Spec}(B_{\mathrm{dR}}^+(S^\sharp))$ and $\alpha : \mathcal{V} \dashrightarrow \mathcal{O}^{\oplus n}$ is an isomorphism defined over $\mathrm{Spec}(B_{\mathrm{dR}}(S^\sharp))$.

These moduli spaces are ind-representable by proper spatial diamonds over $\mathrm{Spd}(K_0)$ (and $\mathrm{Spd}(\mathbb{Q}_p, \mathbb{Z}_p)$ respectively) and after fixing a basis of D we get an identification

$$\mathrm{Gr}_{GL_n} \times_{\mathbb{Q}_p} \mathrm{Spd}(K_0) \cong \mathrm{Gr}(\mathcal{E}(D, \Phi))$$

([Han21, Proposition 2.12]). The second space is the B_{dR}-Grassmannian of the Berkeley notes ([SW20, Definition 20.2.1]).

We can re-interpret the canonical map $\mathrm{Spa}(C_p) \to \mathrm{Spa}(K_0)$ that comes from thinking of K_0 as a subfield of C_p as a map $m : \mathrm{Spd}(C_p^\flat) \to \mathrm{Spd}(K_0)$. The basechange

$$\mathrm{Gr}(\mathcal{E}_S(D, \Phi)) \times_{\mathrm{Spd}(K_0), m} \mathrm{Spd}(C_p^\flat)$$

gets identified through Beauville–Laszlo glueing with the moduli space that parametrizes B_{dR}^+-lattices contained in $D \otimes_{K_0} B_{\mathrm{dR}}$. This basechange comes equipped with $\Gamma_{K_0}^{\mathrm{op}}$-action and the set of Γ_K-invariant B_{dR}^+-lattices in $D \otimes_{K_0} B_{\mathrm{dR}}$ are in bijection with natural transformations $\mathrm{Spd}(K) \to \mathrm{Gr}(\mathcal{E}_S(D, \Phi))$.

One defines $\mathrm{Gr}(\mathcal{E}(D, \Phi))^{\mathrm{adm}} \subseteq \mathrm{Gr}(\mathcal{E}(D, \Phi))$ to be the subsheaf of tuples for which $\mathcal{V}$ is fiberwise semi-stable of slope 0. From Kedlaya-Liu's semi-continuity theorem ([SW20, Theorem 22.2.1]) we know that this defines an open subfunctor which is called the admissible locus. Additionally, a map $\mathrm{Spd}(K) \to \mathrm{Gr}(\mathcal{E}(D, \Phi))$ factors through $\mathrm{Gr}(\mathcal{E}(D, \Phi))^{\mathrm{adm}}$ if and only if it is coming from a weakly admissible filtration.

An aspect of the situation is that if $n = dim_{K_0}(D)$ then $\mathrm{Gr}(\mathcal{E}(D, \Phi))^{\mathrm{adm}}$ admits a pro-étale $\underline{\mathrm{GL}_n(\mathbb{Q}_p)}$-local system $\mathbb{L}$ that "interpolates" between the n-dimensional crystalline representations associated to (D, Φ) ([Han21, Proposition 2.14]).

The precise claim that we will use is the following.

Proposition 1.4. *If* $\mathrm{Fil}^\bullet D_K$ *is a weakly admissible filtration of* (D, Φ) *and*

$$\iota : \mathrm{Spd}(K) \to \mathrm{Gr}(\mathcal{E}(D, \Phi))^{\mathrm{adm}}$$

is the map associated to $\mathrm{Fil}^\bullet D_K$, *then* $\iota^* \mathbb{L}$ *is isomorphic to* $V_{cris}(D, \Phi, \mathrm{Fil}^\bullet)$ *when we regard* $\iota^* \mathbb{L}$ *as a continuous* Γ_K-*representation.*

Proof. We omit the details. $\qquad\square$

1.1.3. *Isocrystals with G-structure.* We keep the notation as above, we let G denote a connected reductive group over $\mathbb{Q}_p$ and $\mathrm{Rep}_G(\mathbb{Q}_p)$ denote the Tannakian category of $\mathbb{Q}_p$-linear algebraic representations of G. Recall the following definition:

Definition 1.5. ([Kot97, §3]) An isocrystal with G-structure $\mathcal{F}$, is a $\otimes$-exact functor $\mathcal{F} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \Phi\text{-Mod}_{K_0}$.

To an element $b \in G(K_0)$ and a representation $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$ we associate the isocrystal

$$(D_{b,\rho}, \Phi_{b,\rho}) := (V \otimes K_0, \rho(b) \cdot (\mathrm{Id} \otimes \varphi)),$$

ranging this construction over (V, ρ) defines an isocrystal with G-structure

$$\mathcal{F}_b : \mathrm{Rep}_G(\mathbb{Q}_p) \to \Phi\text{-Mod}_{K_0}.$$

We say that two elements $b_1, b_2 \in G(K_0)$ are φ-conjugate to each other if $b_1 = g^{-1} \cdot b_2 \cdot \varphi(g)$ for some element $g \in G(K_0)$. This defines an equivalence relation and b_1 is φ-conjugate to b_2 if and only if $\mathcal{F}_{b_1}$ is isomorphic to $\mathcal{F}_{b_2}$.

Now, since $k = \bar{k}$ the set of equivalence classes of φ-conjugacy is the set $B(G)$ defined and studied by Kottwitz ([Kot97, §1.4]). Every isocrystal with G-structure is isomorphic $\mathcal{F}_b$ for some $b \in G(K_0)$ and consequently $B(G)$ parametrizes isomorphism classes of isocrystals with G-structure. The set $B(G)$ has a very rich theory, we recall some of it below.

Recall that the category of isocrystals over K_0 is semisimple and the simple objects can be parametrized by rational numbers $\lambda \in \mathbb{Q}$. In particular, every object $(D, \Phi) \in \Phi\text{-Mod}_{K_0}$ admits a canonical "slope" decomposition

$$(D, \Phi) = \bigoplus_{\lambda \in \mathbb{Q}} (D_\lambda, \Phi_\lambda).$$

If we let ω_b denote the composition $\mathrm{Forg} \circ \mathcal{F}_b$ where

$$\mathrm{Forg} : \Phi\text{-Mod}_{K_0} \to \mathrm{Vec}(K_0)$$

denotes the forgetful functor to the category of vector spaces over K_0, then the slope decomposition defines $\otimes$-exact $\mathbb{Q}$-grading of ω_b. In turn, this grading can be interpreted as a slope morphism $\nu_b : \mathbb{D} \to G_{K_0}$ of pro-algebraic groups, where $\mathbb{D}$ is the pro-torus with character set $X^*(\mathbb{D}) = \mathbb{Q}$.

Consider the abstract group defined as a semi-direct product $G(K_0) \rtimes \varphi \cdot \mathbb{Z}$ where φ has its natural action on $G(K_0)$.

Definition 1.6. ([RZ96, Definition 1.8]) For an element $b \in G(K_0) = G(K_0)$ with conjugacy class $[b] \in B(G)$ we say that:

(1) b is decent if there exists an integer s such that $(b\varphi)^s = (s \cdot \nu_b)(p)\varphi^s$ as elements of $G(K_0) \rtimes \varphi \cdot \mathbb{Z}$.
(2) We say that b is basic if the map $\nu_b : \mathbb{D} \to G_{K_0}$ factors through the center of G.
(3) We say that $[b] \in B(G)$ is basic if all (equivalently some) element of $[b]$ is basic.

Since we are assuming $k = \bar{k}$ and that G is connected reductive, every φ-conjugacy class $[b] \in B(G)$ contains a decent element [RZ96, 1.11].

Assume for the rest of the subsection that G is quasi-split. For $b \in G(K_0)$ we can let ν_b^{dom} denote the unique map $\nu_b^{\mathrm{dom}} : \mathbb{D} \to T_{K_0}$ in the conjugacy class of ν_b that is dominant with respect to B. The map ν_b^{dom} factors through A and is defined over $\mathbb{Q}_p$, so we can write $\nu_b^{\mathrm{dom}} \in X_*^+(A)_{\mathbb{Q}} = (X_*^+(T) \otimes_{\mathbb{Z}} \mathbb{Q})^{\Gamma_{\mathbb{Q}_p}}$ ([Shi09, §4], Introduction of [CKV15]). This gives a well defined map $\mathcal{N} : B(G) \to X_*^+(A)_{\mathbb{Q}}$ usually referred to as the Newton map.

Recall Borovoi's algebraic fundamental group $\pi_1(G)$ which can be defined as the quotient of $X_*(T)$ by the co-root lattice. This group comes equipped with $\Gamma_{\mathbb{Q}_p}$ action and Kottwitz constructs a map $\kappa_G : B(G) \to (\pi_1(G))_{\Gamma_{\mathbb{Q}_p}}$ that is usually referred to as the Kottwitz map.

An important result of Kottwitz [Kot97] states that the map of sets

$$(\nu_b^{\mathrm{dom}}, \kappa_G) : B(G) \to \mathcal{N} \times \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$$

is injective. Now, if we are given an element $\mu \in X_*(T)$ with reflex field E we may define an element

$$\overline{\mu} \in X_*^+(A)_{\mathbb{Q}} = X_*^+(T)_{\mathbb{Q}}^{\Gamma_{\mathbb{Q}_p}}$$

by averaging over the dominant elements inside a conjugacy class in the Galois orbit of μ:

$$\overline{\mu} = \frac{1}{[E : \mathbb{Q}_p]} \sum_{\gamma \in \mathrm{Gal}(E/\mathbb{Q}_p)} \mu^{\gamma}$$

We can now recall Kottwitz' definition of the set $B(G, \mu) \subseteq B(G)$.

Definition 1.7. The set $B(G, \mu)$ consists of those conjugacy classes $[b] \in B(G)$ for which $\kappa_G([b]) = [\mu]$ in $\pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$ and for which $\overline{\mu} - \nu_b^{\mathrm{dom}} \in X_*^+(A)_{\mathbb{Q}}$ is a non-negative $\mathbb{Q}$-linear combination of positive co-roots.

1.1.4. *G-bundles and G-valued crystalline representations.* In this section we assume that G is reductive over $\mathbb{Q}_p$, but not necessarily quasi-split. Just as in the case of schemes, one has a theory of G-bundles over the relative Fargues-Fontaine curve that uses a Tannakian approach ([SW20, Appendix to lecture 19]). Given $S \in \mathrm{Perf}_k$ and $\mathcal{F} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \Phi\text{-Mod}_{K_0}$ an isocrystal with G-structure we can define a $\otimes$-exact functor $\mathcal{E}_{\mathcal{F},S} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \mathrm{Vec}(\mathcal{X}_{FF,S})$ by letting

$$\mathcal{E}_{\mathcal{F},S}(V, \rho) = \mathcal{E}_S(\mathcal{F}(V, \rho)),$$

this defines a G-bundle over $\mathcal{X}_{FF,S}$. When we are given $b \in G(K_0)$ we write $\mathcal{E}_{b,S}$ instead of $\mathcal{E}_{\mathcal{F}_b,S}$. This allow us to extend Tannakianly Definition 2.16.

Definition 1.8. (1) Given $\mathcal{F}$ an isocrystal with G-structure, we let $\mathrm{Gr}(\mathcal{F})$ denote the functor from $\mathrm{Perf}_{\mathrm{Spd}(K_0)} \to$ Sets that assigns:

$$(S^{\sharp}, f) \mapsto \{((S^{\sharp}, f), \mathcal{G}, \alpha)\}/\cong$$

Where $(S^{\sharp}, f)$ is an untilt of S over $\mathrm{Spa}(K_0)$, $\mathcal{G}$ is a G-bundle over $\mathcal{X}_{FF,S}$ and $\alpha : \mathcal{G} \dashrightarrow \mathcal{E}_{\mathcal{F},S}$ is an isomorphism defined over $\mathcal{X}_{FF,S} \setminus \infty$ and meromorphic along ∞. When $b \in G(K_0)$ we write $\mathrm{Gr}(\mathcal{E}_b)$ instead of $\mathrm{Gr}(\mathcal{F}_b)$.
(2) We let Gr_G denote the functor from $\mathrm{Perf}_{\mathrm{Spd}(\mathbb{Q}_p)} \to$ Sets that assigns:

$$(S^{\sharp}, f) \mapsto \{((S^{\sharp}, f), \mathcal{G}, \alpha)\}/\cong$$

Where $(S^{\sharp}, f)$ is an untilt of S over $\mathrm{Spa}(\mathbb{Q}_p)$, $\mathcal{G}$ is a G-bundle over $\mathrm{Spec}(B_{\mathrm{dR}}^+(S^{\sharp}))$ and $\alpha : \mathcal{G} \dashrightarrow G$ is a trivialization defined over $\mathrm{Spec}(B_{\mathrm{dR}}(S^{\sharp}))$.

As with the GL_n case, the two moduli spaces become isomorphic after basechange to $\mathrm{Spd}(K_0)$. Instead of fixing a basis one has to fix an isomorphism of the fiber functors:

$$(\omega_{\mathrm{can}} \otimes K_0) \cong \omega_{\mathcal{F}}$$

Here $\omega_{\mathcal{F}} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \Phi\text{-Mod}_{K_0} \to K_0 - \mathrm{Vec}$ denotes $\mathrm{Forg} \circ \mathcal{F}$, and if $b \in G(K_0)$ we write ω_b instead of $\omega_{\mathcal{F}_b}$. A careful inspection of the construction of ω_b shows that (in contrast with $\omega_{\mathcal{F}}$) there is a canonical choice of isomorphism $\omega_b \cong \omega_{\mathrm{can}}$.

As with the GL_n case we can define the admissible locus as the subsheaf $\mathrm{Gr}(\mathcal{E}_b)^{\mathrm{adm}} \subseteq \mathrm{Gr}(\mathcal{E}_b)$ of those tuples $((S^{\sharp}, f), \mathcal{G}, \alpha)$ such that $x^*\mathcal{G}$ is the trivial G-bundle for every geometric point $x : \mathrm{Spa}(C', C'^+) \to S$. This is again an open subsheaf and it admits a pro-étale $G(\mathbb{Q}_p)$-torsor which we will also denote by $\mathbb{L}$ ([SW20, Theorem 22.5.2]).

To make contact with crystalline representations we recall how the Tannakian formalism interacts with filtrations, we refer the reader to [SR72] for the details. Recall that given a fiber functor $\omega : \mathrm{Rep}_G(\mathbb{Q}_p) \to \mathrm{Vec}(S)$ one can consider $\otimes$-exact filtrations $\mathrm{Fil}^{\bullet}(\omega)$ ([SR72, Chapitre IV §2.1.1], [DOR10, Definition 4.2.6]). To such a filtration one can associate a $\otimes$-grading $(gr(\mathrm{Fil}^{\bullet}(\omega)))$ which produces a morphism of algebraic groups over S, $\mu_{\mathrm{Fil}^{\bullet}(\omega)} : \mathbb{G}_m \to \underline{\mathrm{Aut}}^{\otimes}(\omega')$ [SR72, Chapitre IV §1.3] [DOR10, Corollary 4.2.3]. Here $\omega' = (gr(\mathrm{Fil}^{\bullet}(\omega)))$, denotes the $\otimes$-exact functor obtained from the grading after we forget the graded structure. If $x = \mathrm{Spec}(C)$ is a geometric point of S, we may find an isomorphism

$\omega'_x \cong \omega_x$ and this defines a conjugacy class of cocharacters into $\underline{\mathrm{Aut}}^{\otimes}(\omega_x)$. This conjugacy class is independent of the isomorphism chosen and we can denote it $[\mu_{\mathrm{Fil}^{\bullet}(\omega)}(x)]$.

Now, fix an isomorphism $\omega_b \cong \omega_{\mathrm{can}}$, we get an isomorphism $\underline{\mathrm{Aut}}^{\otimes}(\omega_b) \cong G_{K_0}$. Furthermore, if we are given a conjugacy class $[\mu]$ of morphisms $\mu : \mathbb{G}_{m,\overline{K}_0} \to G_{\overline{K}_0}$ with field of definition E_0/K_0 ([DOR10, Definition 6.1.2]) contained in C_p, then we can consider the moduli functor of filtrations of ω_b of *type* $[\mu]$. We denote this moduli space by

$$\mathscr{F}l^{\omega_b}_{E_0,[\mu]} : \mathrm{Sch}_{/E_0} \to \mathrm{Sets},$$

This functor does not depend of our choice of isomorphism $\omega_b \cong \omega_{\mathrm{can}}$.

Since G is defined over $\mathbb{Q}_p$ the conjugacy class $[\mu]$ will be defined over a finite extension E of $\mathbb{Q}_p$ contained in C_p and $\mathscr{F}l^{\omega_b}_{E_0,[\mu]}$ is isomorphic to the basechange of a similarly defined moduli functor $\mathscr{F}l^{\omega_{\mathrm{can}}}_{E,[\mu]}$. If F/E is a finite extension and $\mu \in [\mu]$ is a representative defined over F then μ defines a parabolic subgroup $P_\mu \subseteq G_F$ and $\mathscr{F}l^{\omega_{\mathrm{can}}}_{F,[\mu]}$ is isomorphic to the generalized flag variety G/P_μ. In particular, $\mathscr{F}l^{\omega_{\mathrm{can}}}_{E,[\mu]}$ and $\mathscr{F}l^{\omega_b}_{E_0,[\mu]}$ are represented by geometrically connected smooth projective schemes over $\mathrm{Spec}(E)$ and $\mathrm{Spec}(E_0)$ respectively [DOR10, Theorem 6.1.4]. The associated adic space $(\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{\mathrm{ad}}$ evaluates on a complete sheafy Huber pair (R, R^+) over $\mathrm{Spa}(E_0)$ to the set:

$$(\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{\mathrm{ad}}(R, R^+) = \left\{ \mathrm{Fil}^{\bullet}(\omega_{b,R}) \mid [\mu_{\mathrm{Fil}^{\bullet}(\omega)}(x)] = [\mu] \, for \, all \, x \in \mathrm{Spa}(R, R^+) \right\}$$

In particular, if K/K_0 is a complete nonarchimedean field extension then

$$(\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{\mathrm{ad}}(K, O_K) = \mathscr{F}l^{\omega_b}_{E_0,[\mu]}(K).$$

Just as $[\mu]$ allows us to define $\mathscr{F}l^{\omega_b}_{E_0,[\mu]}$ it also allows us to discuss boundedness conditions on affine B_{dR}-Grassmannians.

We can define subsheaves

$$\mathrm{Gr}^{[\mu]}_{G,E} \subseteq \mathrm{Gr}^{\leq[\mu]}_{G,E} \subseteq \mathrm{Gr}_G \times \mathrm{Spd}(E),$$

given by the condition that for every geometric point, the pullback $x^* m$ has relative position $[\mu]$ (bounded by $[\mu]$ respectively). The space $\mathrm{Gr}^{\leq[\mu]}_{G,E}$ is spatial diamond that is proper over $\mathrm{Spd}(E)$ and $\mathrm{Gr}^{[\mu]}_{G,E} \subseteq \mathrm{Gr}^{\leq[\mu]}_{G,E}$ is an open subdiamond.

We can now compare the affine B_{dR}-Grassmannian to the flag variety. Recall that there is a Tannakianly defined Białynicki-Birula map [SW20, Proposition 19.4.2],

$$\pi^{[\mu]}_{BB} : \mathrm{Gr}^{[\mu]}_{G,E} \to (\mathscr{F}l^{\omega_{\mathrm{can}}}_{E,[-\mu]})^{\diamond}.$$

We emphasize that there is a change of signs which is a consequence of the change of signs that appeared in Remark 1.2 and of our convention on filtrations.

One can also construct the following variation of the Białynicki-Birula map

$$\pi^{[\mu]}_{BB} : \mathrm{Gr}^{[\mu]}_{E_0}(\mathcal{E}_b) \to \mathscr{F}l^{\omega_b}_{E_0,[-\mu]}.$$

This allows the following group-theoretically enhanced rephrasing of Proposition 1.1.

Proposition 1.9. *With notation as above and letting K/E_0 be a finite field extension. Then, the Białynicki-Birula map induces a bijection*

$$\pi^{[\mu]}_{BB} : \mathrm{Gr}^{[\mu]}(\mathcal{E}_b)(K, O_K) \cong (\mathscr{F}l^{\omega_b}_{E_0,[-\mu]})^{\diamond}(K, O_K),$$

of $\mathrm{Spd}(K)$-valued points.

Proof. We omit the details. $\qquad\square$

Let $\mathrm{Rep}^{\mathrm{cont}}_{\Gamma_K}(\mathbb{Q}_p)$ denote the category of continuous Galois representations. It is a neutral Tannakian category with canonical fiber functor $\omega^{\Gamma_K}_{\mathrm{can}}(W, \tau) = W$. Recall that by the Tannakian formalism to specify a continuous representation $\rho : \Gamma_K \to G(\mathbb{Q}_p)$ (up to $G(\mathbb{Q}_p)$-conjugation) it is sufficient to specify a $\otimes$-exact functor $\mathcal{F} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \mathrm{Rep}^{\mathrm{cont}}_{\Gamma_K}(\mathbb{Q}_p)$ for which $\omega^{\Gamma_K}_{\mathrm{can}} \circ \mathcal{F}$ is isomorphic to ω_{can}. Now, the full subcategory $\mathrm{Rep}^{\mathrm{crys}}_{\Gamma_K}(\mathbb{Q}_p)$ of crystalline representations is Tannakian and we can define crystalline representations with G-structure as those $\otimes$-exact functors $\mathcal{F} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \mathrm{Rep}^{\mathrm{cont}}_{\Gamma_K}(\mathbb{Q}_p)$ such that $\mathcal{F}(V, \rho)$ is crystalline for all $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$.

Given a pair (b, μ) with $b \in G(K_0)$ and $\mu : \mathbb{G}_{m,K} \to G_K$ we can construct a filtered isocrystal with G-structure by defining a functor

$$\mathcal{F}_{b,\mu} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \Phi-\mathrm{ModFil}_{K/K_0}$$

such that

$$\mathcal{F}_{b,\mu}(V, \rho) = (D_{b,\rho}, \Phi_{b,\rho}, \mathrm{Fil}_\mu^\bullet)$$

with

$$\mathrm{Fil}_\mu^i(D_{b,\rho} \otimes K) = \oplus_{i \leq n}(V \otimes K)^{(\rho \circ \mu(t) \cdot v = t^n \cdot v)}.$$

Definition 1.10. ([RZ96, Definition 1.18]). We say that a pair (b, μ) with $b \in G(K_0)$ and $\mu : \mathbb{G}_m \to G_K$ is admissible if the functor $\mathcal{F}_{b,\mu}$ only takes values on weakly admissible filtered isocrystals.

In general, even if (b, μ) is admissible the functor $V_{cris} \circ \mathcal{F}_{b,\mu}$ might not define a crystalline representation with G-structure. Indeed, the composition $\omega_{\mathrm{can}}^{\Gamma_K} \circ V_{cris} \circ \mathcal{F}_{b,\mu}$ might fail to be isomorphic to ω_{can}. Nevertheless, this issue goes away if we impose that $[b]$, the φ-conjugacy class of b in $G(K_0)$, lies on the Kottwitz set $B(G, \mu)$ [DOR10, Proposition 11.4.3].

Associated to the admissible pair (b, μ) there is a map $y_{b,\mu} : \mathrm{Spd}(K) \to \mathscr{F}l_{E_0,[-\mu]}^{\omega_b}$ defined by the filtration $\mathrm{Fil}_\mu^\bullet$ on ω_b, and we can let $x_{b,\mu} : \mathrm{Spd}(K) \to \mathrm{Gr}_{E_{0\mu}}^{[\mu]}(\mathcal{E}_b)$ denote the unique lift of $y_{b,\mu}$ of Proposition 1.9. The following is a group-theoretic refinement of Proposition 1.4 and it is one of the key inputs from modern p-adic Hodge theory that we will need later on.

Proposition 1.11. *Suppose that (b, μ) is an admissible pair with $[b] \in B(G, \mu)$, then the map $x_{b,\mu} : \mathrm{Spd}(K) \to \mathrm{Gr}_{E_0}^{[\mu]}(\mathcal{E}_b)$ factors through the admissible locus $\mathrm{Gr}_{E_0}^{[\mu]}(\mathcal{E}_b)^{\mathrm{adm}}$. Moreover, if $\underline{\mathbb{L}}$ denotes the pro-étale $G(\mathbb{Q}_p)$-torsor on $\mathrm{Gr}(\mathcal{E}_b)^{\mathrm{adm}}$ then $x_{b,\mu}^*\mathbb{L}$ agrees with the crystalline representation with G-structure defined by the functor $V_{cris} \circ \mathcal{F}_{b,\mu}$.*

Proof. We omit the details. $\square$

1.1.5. *M. Chen's result on p-adic Hodge Theory.* In this subsection, we assume that G is an unramified reductive group over $\mathbb{Q}_p$, this implies the group is quasi-split.

Definition 1.12. ([Che14, Définition 5.0.4], [CKV15, Theorem 2.5.6]) Recall the notation of Definition 1.7. We say that a pair $([b], [\mu])$ with $[b] \in B(G, \mu)$ and $\mu \in X_*(T)$ is HN-irreducible if all the coefficients of $\overline{\mu} - \nu_b^{\mathrm{dom}}$ as a $\mathbb{Q}$-linear combination of simple coroots are strictly positive.

The following result of M. Chen is a key ingredient to our computation.

Theorem 1.13. ([Che14, Théorème 5.0.6])

Let $\mu : \mathbb{G}_m \to G_K$ be a morphism and let $b \in G(K_0)$ be a decent element such that $[b] \in B(G, \mu)$ and $[\mu]$ has reflex field E. Suppose that the map $\mathrm{Spec}(K) \to \mathscr{F}l_{E_0,[-\mu]}^{\omega_b}$ induced by the filtration defined by μ maps to the generic point of $|\mathscr{F}l_{E,[-\mu]}^{\omega_{\mathrm{can}}}|$ under the map

$$\mathscr{F}l_{E_0,[-\mu]}^{\omega_b} = \mathscr{F}l_{E,[-\mu]}^{\omega_{\mathrm{can}}} \times_E E_0 \to \mathscr{F}l_{E,[-\mu]}^{\omega_{\mathrm{can}}},$$

induced from the canonical isomorphism $\omega_{\mathrm{can}} \otimes_{\mathbb{Q}_{p^s}} K_0 \cong \omega_b$. Assume further that the pair $([b], [\mu])$ is HN-irreducible, then the following hold:

(1) *The pair (b, μ) is admissible and defines a crystalline representation $\xi_{b,\mu} : \Gamma_K \to G(\mathbb{Q}_p)$, well-defined up to conjugation.*
(2) *The Zariski closure of $\xi_{b,\mu}(\Gamma_K) \subseteq G$ contains G^{der} and $\xi_{b,\mu}(\Gamma_K)$ contains an open subgroup of $G^{\mathrm{der}}(\mathbb{Q}_p)$.*

Remark 1.14. M. Chen's result is slightly stronger, but this is the formulation that we will use below. Observe that K has infinite transcendence degree over E, so it makes sense for a K-point to lie topologically over the generic point of $\mathscr{F}l_{E,[-\mu]}^{\omega_{\mathrm{can}}}$.

Combining Proposition 1.11 with Chen's Theorem 1.13 and using the fact that every element $b \in G(K_0)$ is φ-conjugate to a decent one we can deduce the following statement.

Corollary 1.15. *Let $b \in G(K_0)$ and $\mu \in X_*^+(T)$. Suppose that $[b] \in B(G, \mu)$ and that $([b], [\mu])$ is HN-irreducible. For every finite extension K/K_0 there is a map $x : \mathrm{Spd}(K) \to \mathrm{Gr}_E^{[\mu]}(\mathcal{E}_b)^{\mathrm{adm}}$ such that if $\rho_x : \Gamma_K \to G(\mathbb{Q}_p)$ denotes the Galois representation associated to $x^*\mathbb{L}$, then $\rho_x(\Gamma_K) \cap G^{\mathrm{der}}(\mathbb{Q}_p)$ is open in $G^{\mathrm{der}}(\mathbb{Q}_p)$.*

1.2. **The three actions.**

1.2.1. *The action of $G(\mathbb{Q}_p)$.* We fix $b \in G(K_0)$, $[\mu] \in \mathrm{Hom}(\mathbb{G}_m, G_{\overline{\mathbb{Q}}_p})$ and we let $E_0 = K_0 \cdot E$ denote the field of definition of $[\mu]$ over K_0. Let $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ denote an open compact subgroup, recall the moduli space of p-adic shtukas that appears in the Berkeley notes.

Definition 1.16. ([SW20, Proposition 23.3.1]) We define $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} : \mathrm{Perf}_k \to \mathrm{Sets}$ as the presheaf that assigns to $S \in \mathrm{Perf}_k$ isomorphism classes of tuples

$$((S^\sharp, f), \mathcal{E}, \alpha, \mathbb{P}_\mathcal{K}, \iota)$$

such that:

(1) $(S^\sharp, f)$ is an untilt of S over E_0.
(2) $\mathcal{E}$ is a G-bundle on the relative Fargues-Fontaine $\mathcal{X}_{FF,S}$ curve whose fibers on geometric points of S are isomorphic to the trivial G-torsor.
(3) $\alpha : \mathcal{E} \dashrightarrow \mathcal{E}_b$ is a modification of G-bundles defined over $\mathcal{X}_{FF,S} \setminus S^\sharp$ meromorphic along $S^\sharp$ and whose type is bounded by $[\mu]$ on geometric points.
(4) $\mathbb{P}_\mathcal{K}$ is a pro-étale $\underline{\mathcal{K}}$-torsor and ι is an identification of $\mathbb{P}_\mathcal{K} \times^{\mathcal{K}} \underline{G(\mathbb{Q}_p)}$ with the pro-étale $\underline{G(\mathbb{Q}_p)}$-torsor that $\mathcal{E}$ defines under the equivalence of [SW20, Theorem 22.5.2].

It is proven in [SW20] that the presheaves $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}}$ are locally spatial diamonds over $\mathrm{Spd}(E_0)$, and that whenever μ is a minuscule conjugacy class of cocharacters then $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}}$ is represented by the diamond associated to a smooth rigid-analytic space over $\mathrm{Spa}(E_0)$.

Scholze and Weinstein construct a family of "Grothendieck–Messing" period morphisms

$$\pi_{GM,\mathcal{K}} : \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \to \mathrm{Gr}_{E_0}^{\leq [\mu]}(\mathcal{E}_b)^{\mathrm{adm}}$$

given by the formula:

$$((S^\sharp, f), \mathcal{E}, \alpha, \mathbb{P}_\mathcal{K}, \iota) \mapsto ((S^\sharp, f), \mathcal{E}, \alpha)$$

For every $\mathcal{K}$ this gives a surjective étale morphism of locally spatial diamonds. Moreover, this family is functorial on $\mathcal{K}$. That is, if $\mathcal{K}_1 \subseteq \mathcal{K}_2$ are two compact and open subsets then we get a commutative diagram of étale maps,

$$
\begin{array}{ccc}
\mathrm{Sht}_{G,b,[\mu],\mathcal{K}_1} & \xrightarrow{\quad \pi_{\mathcal{K}_1,\mathcal{K}_2} \quad} & \mathrm{Sht}_{G,b,[\mu],\mathcal{K}_2} \\
& \searchrow{\pi_{GM,\mathcal{K}_1}} \quad \swarrow{\pi_{GM,\mathcal{K}_2}} & \\
& \mathrm{Gr}_{E_0}^{\leq [\mu]}(\mathcal{E}_b)^{\mathrm{adm}} &
\end{array}
$$

where the transition map $\pi_{\mathcal{K}_1,\mathcal{K}_2}$ is the one deduced from assigning to $\mathbb{P}_{\mathcal{K}_1}$ the corresponding $\underline{\mathcal{K}_2}$-torsor $\mathbb{P}_{\mathcal{K}_1} \times^{\mathcal{K}_1} \mathcal{K}_2$. Also, if $\mathcal{K}_1 \subseteq \mathcal{K}_2$ is normal of finite index then the transition maps $\pi_{\mathcal{K}_1,\mathcal{K}_2}$ are surjective and finite étale.

The flexibility of the category of diamonds allows us to define moduli spaces of p-adic shtukas associated to an arbitrary compact subgroup $\mathcal{K}' \subseteq G(\mathbb{Q}_p)$ including the case $\mathcal{K}' = \{e\}$ (which is usually referred to as the infinite level). Indeed, the set of compact open subgroups $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ containing $\mathcal{K}'$ is co-filtered and has intersection equal to $\mathcal{K}'$. We may define the limit of diamonds $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}'} = \varprojlim_{\mathcal{K}' \subseteq \mathcal{K}} \mathrm{Sht}_{G,b,[\mu],\mathcal{K}}$, together with a period map

$$\pi_{GM,\mathcal{K}'} : \mathrm{Sht}_{G,b,[\mu],\mathcal{K}'} \to \mathrm{Gr}_{E_0}^{\leq [\mu]}(\mathcal{E}_b)^{\mathrm{adm}}.$$

This sheaf has the structure of a locally spatial diamond. Moreover, although the period map in general might not be étale it is always a quasi-proétale map [Sch17, Definition 10.1].

Moduli spaces of shtukas at infinite level ($K' = \{e\}$) have the following pleasant description,

$$\mathrm{Sht}_{G,b,[\mu],\infty}(S) = \{(S^\sharp, f), \alpha : G \dashrightarrow \mathcal{E}_b\}$$

where $(S^\sharp, f)$ denotes an untilt of S over E_0, G denotes the trivial G-bundle over $\mathcal{X}_{FF,S}$ and α is a modification of G-bundles over $\mathcal{X}_{FF,S} \setminus S^\sharp$, meromorphic along $S^\sharp$ and whose type is bounded by $[\mu]$ on geometric points. The natural action of $G(\mathbb{Q}_p)$ on the trivial torsor G induces a right action of $\underline{G(\mathbb{Q}_p)}$ on $\mathrm{Sht}_{G,b,[\mu],\infty}$.

1.2.2. *Weil descent.* Recall that we defined $W_E^{K_0}$ as the subset of continuous automorphisms of C_p that act as $\hat{\varphi} := \mathrm{Id}_E \otimes \varphi^{n \cdot s}$ on $E_0 = E \cdot K_0$. It evidently contains Γ_{E_0} and we may topologize $W_E^{K_0}$ so that $\Gamma_{E_0} \hookrightarrow W_E^{K_0}$ is a topological immersion and an open map. We get a strict exact sequence of topological groups

$$e \to \Gamma_{E_0} \to W_E^{K_0} \to \hat{\varphi}^{\mathbb{Z}} \to e.$$

Definition 1.17. Let $\mathcal{G}$ be a v-sheaf over $\mathrm{Spd}(E_0)$, a Weil descent datum for $\mathcal{G}$ is an isomorphism $\tau : \mathcal{G} \to \hat{\varphi}^{\mathrm{op},*}\mathcal{G}$ over $\mathrm{Spd}(E_0)$.

Weil descent datum provide us with actions by $W_{E_0}^{\mathrm{op}}$ instead of only $\Gamma_{E_0}^{\mathrm{op}}$. But we need to endow our spaces with continuous actions rather than plain actions by an abstract group. An efficient way to endow a v-sheaf with a continuous action is to endow it with the action of the group sheaf $\underline{W_{E_0}^{\mathrm{op}}}$ that parametrizes continuous maps $|\mathrm{Spa}(R, R^+)| \to W_{E_0}^{\mathrm{op}}$.

Lemma 1.18. *Suppose we are given a right $\underline{\Gamma_{E_0}}$-action on a v-sheaf,*

$$m : \mathcal{F} \times \underline{\Gamma_{E_0}} \to \mathcal{F},$$

and suppose we are given a group homomorphism $\theta : W_{E_0}^{\mathrm{op}} \to \mathrm{Aut}(\mathcal{F})$ such that $\theta(\gamma^{\mathrm{op}}) = m(-, \gamma)$ for all constant elements $\gamma \in \Gamma_{E_0} \subseteq \underline{\Gamma_{E_0}}$. Then there is a unique right $\underline{W_E^{K_0}}$-action $m' : \mathcal{F} \times \underline{W_E^{K_0}} \to \mathcal{F}$ with $m'_{|\underline{\Gamma_{E_0}}} = m$ and $\theta(\gamma^{\mathrm{op}}) = m'(-, \gamma)$ for all constant elements $\gamma \in W_E^{K_0}$.

Proof. We omit the details. $\square$

Proposition 1.19. *If $(\mathcal{G}, \tau)$ is a v-sheaf over $\mathrm{Spd}(E_0)$ equipped with a Weil-descent datum then $\mathcal{G} \times_{E_0} \mathrm{Spd}(C_p)$ comes equipped with a right action by $\underline{W_E^{K_0}}$.*

Given two diamonds with Weil descent datum $(\mathcal{G}_i, \tau_i)$ over $\mathrm{Spd}(E_0)$ and a map $f : \mathcal{G}_1 \to \mathcal{G}_2$ compatible with τ_i, then the corresponding map $f : \mathcal{G}_1 \times_{E_0} \mathrm{Spd}(C_p) \to \mathcal{G}_2 \times_{E_0} \mathrm{Spd}(C_p)$ is $\underline{W_E^{K_0}}$-equivariant.

Proposition 1.20. *There are canonical isomorphisms of v-sheaves over $\mathrm{Spd}(E_0)$ compatible with the inclusion and the period morphism.*

(1) $\hat{\varphi}^{\mathrm{op},*}\mathrm{Gr}_{E_0}^{\leq[\mu]}(\mathcal{E}_b) = \mathrm{Gr}_{E_0}^{\leq[\mu]}(\mathcal{E}_{\varphi^s(b)})$.
(2) $\hat{\varphi}^{\mathrm{op},*}\mathrm{Gr}_{E_0}^{\leq[\mu]}(\mathcal{E}_b)^{\mathrm{adm}} = \mathrm{Gr}_{E_0}^{\leq[\mu]}(\mathcal{E}_{\varphi^s(b)})^{\mathrm{adm}}$.
(3) $\hat{\varphi}^{\mathrm{op},*}\mathrm{Sht}_{G,b,[\mu],\infty} = \mathrm{Sht}_{G,\varphi^s(b),[\mu],\infty}$

Proof. We omit the details. $\square$

Observe that b and $\varphi(b)$ are φ-conjugate by b. This induces an isomorphism of G-bundles $\Phi_b : \mathcal{E}_{\varphi(b)} \to \mathcal{E}_b$ and allows us to endow our moduli of interest with Weil descent datum. Using Proposition 1.19 we can endow $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)$ with a right $\underline{W_E^{K_0}}$-action. Moreover, the space $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)$ with its right $\underline{W_E^{K_0}}$-action is independent of the choice of $b \in [b]$.

1.2.3. *The action of $J_b(\mathbb{Q}_p)$.* In ([Kot97, A.2]) Kottwitz shows how to associate to the $\otimes$-functor $\mathcal{F}_b : \mathrm{Rep}_G(\mathbb{Q}_p) \to \Phi\text{-}\mathrm{Mod}_{K_0}$ a connected reductive group J_b over $\mathbb{Q}_p$ whose group of $\mathbb{Q}_p$-valued points is the φ-centralizer of b,

$$J_b(\mathbb{Q}_p) = \left\{ g \in G(K_0) \mid g^{-1} \cdot b \cdot \varphi(g) = b \right\}.$$

Let us recall this construction. For any $\mathbb{Q}_p$-algebra R we let $\Phi\text{-}\mathrm{Mod}_{K_0} \otimes_{\mathbb{Q}_p} R$ denote the category whose objects are the same as in $\Phi\text{-}\mathrm{Mod}_{K_0}$ and morphisms are

$$\mathrm{Hom}_R((D_1, \Phi_1), (D_2, \Phi_2)) := \mathrm{Hom}_{\Phi\text{-}\mathrm{Mod}_{K_0}}((D_1, \Phi_1), (D_2, \Phi_2)) \otimes_{\mathbb{Q}_p} R$$

There is a natural $\otimes$-functor $\beta_R : \Phi\text{-}\mathrm{Mod}_{K_0} \to \Phi\text{-}\mathrm{Mod}_{K_0} \otimes_{\mathbb{Q}_p} R$ and $J_b(R)$ is defined as $\mathrm{Aut}^{\otimes}(\beta_R \circ \mathcal{F}_b)$. With J_b defined in this way we have

$$J_b(\mathbb{Q}_p) = \mathrm{Aut}^{\otimes}(\mathcal{F}_b) \subseteq \mathrm{Aut}^{\otimes}(\mathrm{Forg} \circ \mathcal{F}_b) = G(K_0).$$

Moreover, recall that the slope decomposition produces a map $\nu_b : \mathbb{D} \to G_{K_0}$, if we denote M_b the centralizer of ν_b in G_{K_0} then $(J_b)_{K_0}$ is isomorphic to M_b. Since the elements of $J_b(\mathbb{Q}_p)$ act on $\mathcal{F}_b$ then we get a homomorphism of abstract groups $J_b(\mathbb{Q}_p) \to \mathrm{Aut}(\mathcal{E}_{b,S})$ this already gives an action of $J_b(\mathbb{Q}_p)$ on $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)$, but from this description it is not clear, for example, if this action is continuous with respect to the p-adic topology on $J_b(\mathbb{Q}_p)$. A better approach is to endow our moduli spaces with an action of $\underline{J_b(\mathbb{Q}_p)}$. This can be done following [FS21, Proposition III.4.7].

1.2.4. *Group functoriality.* As we have discussed $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)$ comes equipped naturally with a left action by $J_b(\mathbb{Q}_p)$ and right actions by $G(\mathbb{Q}_p)$ and $W_E^{K_0}$. Moreover, these three actions commute. Replacing the left $\underline{J_b(\mathbb{Q}_p)}$-action by a right $\underline{J_b(\mathbb{Q}_p)}$-action, we can say that $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)$ comes equipped with a right action by $\underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \times \underline{W_E^{K_0}}$.

Fix a morphism $f : G \to H$ of reductive groups over $\mathbb{Q}_p$. Let $b_H = f(b) \in H(L)$ and let $[\mu_H] = [f \circ \mu]$. This defines a morphism $f_{\infty,\infty} : \mathrm{Sht}_{G,b,[\mu],\infty} \to \mathrm{Sht}_{H,b_H,[\mu_H],\infty} \times_{E(\mu_H)} E(\mu)$ with $[\alpha : G \dashrightarrow \mathcal{E}_b] \mapsto [f_*\alpha : H \dashrightarrow \mathcal{E}_{b_H}]$.

Associated to b_H we can form $J_{b_H} = \mathrm{Aut}^\otimes(\mathcal{F}_{b_H})$ and we get a morphism of algebraic groups $f : J_b \to J_{b_H}$. Now, if we endow $\mathrm{Sht}_{H,b_H,[\mu_H],\infty}$ with the action induced by $f : \underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \to \underline{H(\mathbb{Q}_p)} \times \underline{J_{b_H}(\mathbb{Q}_p)}$ then $f_{\infty,\infty} \times C_p$ is equivariant with respect to the $\underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \times \underline{W_E^{K_0}}$-action.

We may also impose a level structure $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ to get a family of morphisms $f_{\mathcal{K},f(\mathcal{K})} : \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \to \mathrm{Sht}_{H,b_H,[\mu_H],f(\mathcal{K})} \times_{E(\mu_H)} E(\mu)$.

1.3. Geometric connected components in the case of tori.
In this section we study the case in which G is a torus, we change our notation and let $G = T$. We remark that this case was tackled by M. Chen in [Che13] and it is also discussed in [Far16]. We recall the story in a different language.

By the work of Kottwitz we know that every element of $B(T)$ is basic and that the Kottwitz map $\kappa_T : B(T) \to \pi_1(T)_{\Gamma_{\mathbb{Q}_p}} = X_*(T_{\overline{\mathbb{Q}}_p})_{\Gamma_{\mathbb{Q}_p}}$ is a bijection. The sets $B(T,\mu)$ are singletons and are determined by the image of μ in $\pi_1(T)_{\Gamma_{\mathbb{Q}_p}}$. In this case, moduli spaces of p-adic shtukas are 0-dimensional.

Proposition 1.21. *If $b \in B(T,\mu)$ then all the maps in the following diagram are isomorphisms:*

$$
\begin{array}{ccccc}
\mathrm{Gr}_{E_0}^{[\mu]}(\mathcal{E}_b)^{\mathrm{adm}} & \longrightarrow & \mathrm{Gr}_{E_0}^{[\mu]}(\mathcal{E}_b) & \longrightarrow & \mathrm{Gr}_{E_0}^{\leq[\mu]}(\mathcal{E}_b) \\
 & \searrow & \downarrow{\scriptstyle\pi_{BB}} & & \downarrow \\
 & & (\mathscr{F}l_{E_0,[-\mu]}^{\omega_b})^\diamond & \longrightarrow & \mathrm{Spd}(E_0)
\end{array}
$$

Proof. We omit the details. $\qquad\square$

In particular, on geometric points the situation is very simple. Indeed, the structure map $\mathrm{Gr}_{C_p}^{\leq\mu}(\mathcal{E}_b)^{\mathrm{adm}} \to \mathrm{Spd}(C_p)$ is an isomorphism and

$$\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p \cong \underline{T(\mathbb{Q}_p)} \times \mathrm{Spd}(C_p),$$

since every right $\underline{T(\mathbb{Q}_p)}$-torsor is trivial on $\mathrm{Spd}(C_p)$. It becomes more interesting when we compare the action of $\underline{J_b(\mathbb{Q}_p)}$ and $\underline{W_E^{K_0}}$ to that of $\underline{T(\mathbb{Q}_p)}$.

We begin by discussing the action of $\underline{J_b(\mathbb{Q}_p)}$. Recall that if b is basic then J_b is an inner form of T, and that since T is commutative we must have $T = J_b$. More precisely we have a canonical inclusion $J_b(\mathbb{Q}_p) \subseteq T(K_0)$ that induces an isomorphism onto $T(\mathbb{Q}_p)$, we denote by j_b this identification.

Proposition 1.22. *The action of $\underline{T(\mathbb{Q}_p)}$ and $\underline{J_b(\mathbb{Q}_p)}$ are inverse to each other. In other words, if $S \in \mathrm{Perf}_{C_p}$, $f : |S| \to J_b(\mathbb{Q}_p)$ is a continuous map, and $\alpha \in \mathrm{Sht}_{T,b,[\mu],\infty} \times C_p$ then*

$$\alpha \cdot_{J_b(\mathbb{Q}_p)} f = \alpha \cdot_{T(\mathbb{Q}_p)} j_b(f^{-1}).$$

Proof. We omit the details. $\qquad\square$

Let us study the Weil group action. In contrast to the actions of $J_b(\mathbb{Q}_p)$ and $T(\mathbb{Q}_p)$ the action of $W_E^{K_0}$ on $\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p$ is not C_p-linear. In particular, we can only compare the actions of $W_E^{K_0}$ and $T(\mathbb{Q}_p)$ on those invariants of $\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p$ that do not depend on the structure morphism to $\mathrm{Spd}(C_p)$. In our case we compare the continuous actions on the topological space of connected components. As we have seen above this topological space is a topological right $T(\mathbb{Q}_p)$-torsor. Let $x \in \pi_0(\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p)$ and $\gamma \in W_E^{K_0}$. We have

$$x \cdot_{W_E^{K_0}} \gamma = x \cdot_{G(\mathbb{Q}_p)} g_{\gamma,x}$$

for a unique element $g_{\gamma,x} \in T(\mathbb{Q}_p)$. Since the actions of $W_E^{K_0}$ and $T(\mathbb{Q}_p)$ commute we get a group homomorphism $g_{-,x} : W_{E_0/E}^{\mathrm{op}} \to T(\mathbb{Q}_p)$. Since $T(\mathbb{Q}_p)$ is commutative this morphism is independent of x.

Moreover, the naive map of sets $\gamma \mapsto g_{\gamma,x}$ which would usually not be a group homomorphism is a group homomorphism again by the commutativity of $T(\mathbb{Q}_p)$. We denote this later group homomorphism by

$$m_{T,\mu} : W_E^{K_0} \to T(\mathbb{Q}_p).$$

The following line of reasoning is taken from [RZ96, Lemma 1.22], which in turn is an elaboration of an argument in [Kot92, page 413/41]. Let E denote a finite field extension of $\mathbb{Q}_p$ let $\{\mathrm{Tori}_{\mathbb{Q}_p}\}$ denote the category of tori defined over $\mathbb{Q}_p$. Recall the functor $X_*(-) : \{\mathrm{Tori}_{\mathbb{Q}_p}\} \to$ Sets given by the set of maps $\mathbb{G}_m \to T_{\overline{\mathbb{Q}}_p}$. Consider the subfunctor $X_*^E \subseteq X_*$ given by the subset of maps $\mathbb{G}_m \to T_E$ that are already defined over E. This functor is representable by $\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m$ and comes equipped with a universal cocharacter $\mu_u \in X_*^E(\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m)$. In other words, given a torus $T \in \{\mathrm{Tori}_{\mathbb{Q}_p}\}$ and $\mu \in X_*^E(T)$ there is a unique map $\mathrm{Nm}_\mu : \mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m \to T$ of algebraic groups over $\mathbb{Q}_p$ such that $\mathrm{Nm}_\mu \circ \mu_u = \mu$ in $X_*(T)$. The universal cocharacter can be expressed on E-points as follows:

$$E^\times \xrightarrow{e \mapsto e \otimes e} (E \otimes E)^\times.$$

Associated to μ_u there is a unique element of $[b_u] \in B(\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m, \mu_u)$. We fix a representative $b_u \in \mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m(\breve{\mathbb{Q}}_p)$ and we abbreviate by m_{E,μ_u} the map $m_{(\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m, \mu_u)}$ previously constructed.

We compute the $W_E^{K_0}$-action on $|\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p|$ by reduction to the universal case. Suppose we are given $\mu \in X_*^E(T)$ and $b \in T(K_0)$ with $[b] \in B(T, \mu)$, then automatically (b, μ) is admissible as in Definition 1.10 and from the functoriality of the Kottwitz map we have that $[\mathrm{Nm}_\mu(b_u)] = [b]$ in $B(T)$. We may replace b by $\mathrm{Nm}_\mu(b_u)$ and we get a norm morphism

$$\mathrm{Nm}_\mu : \mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m), b_u, [\mu_u], \infty} \times C_p \to \mathrm{Sht}_{T,b,[\mu],\infty} \times C_p.$$

This map is $E^\times \times W_E^{K_0}$-equivariant when the right space is endowed with the action induced from the map $\mathrm{Nm}_\mu : \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)(\mathbb{Q}_p) = E^\times \to T(\mathbb{Q}_p)$. We can deduce the following.

Proposition 1.23. *Let the notation be as above, for all $T \in \{\mathrm{Tori}_{\mathbb{Q}_p}\}$ and $\mu \in X_*^E(T)$ we have*

$$m_{T,\mu} = \mathrm{Nm}_\mu \circ m_{E,\mu_u}$$

as maps $W_E^{K_0} \to T(\mathbb{Q}_p)$.

Proof. Fix $x \in \pi_0(\mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m), b_u, [\mu_u], \infty} \times C_p)$ with image $y \in \pi_0(\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p)$ and $\gamma \in W_E^{K_0}$. The equivariance of the norm map with respect to $E^\times$ and $W_E^{K_0}$ allow us to compute:

$$\begin{aligned}
y \cdot_{T(\mathbb{Q}_p)} m_{T,\mu}(\gamma) &= y \cdot_{W_E^{K_0}} \gamma \\
&= \mathrm{Nm}_\mu(x \cdot_{W_E^{K_0}} \gamma) \\
&= \mathrm{Nm}_\mu(x \cdot_{E^\times} m_{E,\mu_u}(\gamma)) \\
&= y \cdot_{T(\mathbb{Q}_p)} \mathrm{Nm}_\mu(m_{E,\mu_u}(\gamma))
\end{aligned}$$

$\square$

In turn, one can do an intricate but explicit computation using local class field theory to show $m_{E,\mu_u} = \mathrm{Art}_E$ the Artin reciprocity character.

The following statement summarizes the results discussed on this section:

Theorem 1.24. (Compare with [Che13, Proposition 4.1]) *Let T be a torus over $\mathbb{Q}_p$, $b \in T(K_0)$, $\mu \in X_*(T)$ with $[b] \in B(T, \mu)$. Let $E \subseteq C_p$ be the field of definition of μ, let $\mathrm{Art}_E : W_E \to E^\times$ be Artin's reciprocity character of local class field theory, let $\mathrm{Nm}_\mu : \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m) \to T$ be as above and let $\mathrm{Art}_{K_0,E}$ denote the composition $\mathrm{Art}_{K_0,E} : W_E^{K_0} \to W_E \xrightarrow{\mathrm{Art}_E} E^\times$, induced by the inclusion of fields $E \subseteq E_0 \subseteq C_p$. Then the following hold:*

(1) $\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p$ is a trivial right $\underline{T(\mathbb{Q}_p)}$-torsor over $\mathrm{Spd}(C_p)$.

(2) If $s \in \pi_0(\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p)$ and $(g, j, \gamma) \in T(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E^{K_0}$ then

$$s \cdot (g, j, \gamma) = s \cdot (g \cdot j_b(j^{-1}) \cdot (\mathrm{Nm}_\mu \circ \mathrm{Art}_{K_0,E}(\gamma)))$$

where $j_b : J_b(\mathbb{Q}_p) \to T(\mathbb{Q}_p)$ is the isomorphism specified by regarding $J_b(\mathbb{Q}_p)$ as a subgroup of $T(K_0)$.

Since we have a full description of the Galois action we can easily compute from Theorem 1.24 the connected components of $\mathrm{Sht}_{T,b,[\mu],\infty}$ as a space over $\mathrm{Spd}(E_0)$. The computation is easier to explain with the following lemma:

Lemma 1.25. *Let $\mathcal{K}$ be a locally profinite group, let L a p-adic field with Galois group Γ_L and $\mathbb{L}_{\mathcal{K}}$ a pro-étale $\underline{\mathcal{K}}$-torsor over $\mathrm{Spd}(L)$. Define $\mathrm{Triv}(\mathbb{L}_{\mathcal{K}})$ as the moduli of trivializations of $\mathbb{L}_{\mathcal{K}}$. Then:*

(1) If C is the p-adic completion of an algebraic closure of L, then the choice of a map $\alpha : \mathrm{Spd}(C) \to \mathrm{Triv}(\mathbb{L}_K)$ determines a group homomorphism $\rho_\alpha : \Gamma_L^{\mathrm{op}} \to \mathcal{K}$.

(2) For any $k \in \mathcal{K}$ we have $\rho_{\alpha \cdot k} = k^{-1} \cdot \rho_\alpha \cdot k$.

(3) The right action of $\mathcal{K}$ on $\pi_0(\mathrm{Triv}(\mathbb{L}_{\mathcal{K}}))$ is transitive.

(4) If $\pi_0(\alpha)$ denotes the unique connected component to which $|\alpha|$ maps to, then the stabilizer subgroup is given by the formula $\mathcal{K}_{\pi_0(\alpha)} = \rho_\alpha(\Gamma_L^{\mathrm{op}})$.

Proof. We omit the details. $\square$

Proposition 1.26. *Let $\mathcal{K} \subseteq T(\mathbb{Q}_p)$ denote the largest compact subgroup, the following statements hold.*

(1) $\pi_0(\mathrm{Sht}_{T,b,[\mu],\infty})$ is a free right $T(\mathbb{Q}_p)/\mathrm{Nm}_\mu(\mathrm{Art}_{K_0,E}(\Gamma_{E_0}))$-torsor.

(2) $\pi_0(\mathrm{Sht}_{T,b,[\mu],\mathcal{K}}) = \pi_0(\mathrm{Sht}_{T,b,[\mu],\mathcal{K}} \times C_p)$ and it is a free right $T(\mathbb{Q}_p)/\mathcal{K}$-torsor.

Proof. The first statement follow directly from Lemma 1.25 and Theorem 1.24. The second statement follows from the fact that the action of Γ_{E_0} is continuous so the action of this compact group factors through the maximal compact subgroup. $\square$

1.4. Geometric connected components in the case of unramified groups. In this section we compute $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ together with its right action by $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E^{K_0}$-action under the assumption that G is an unramified reductive group and that (b, μ) is HN-irreducible (Definition 1.12). Recall that in this case the reflex field is of the form $E = \mathbb{Q}_{p^s}$ for some $s \in \mathbb{N}$ and consequently $E_0 = K_0$. Nevertheless, with the notation we have chosen, $W_E^{K_0}$ is the subgroup of W_{K_0} of those automorphisms of C_p that lift a power of $\varphi^s : K_0 \to K_0$. Recall that if G is an unramified group then there is a connected reductive group over $\mathbb{Z}_p$ whose generic fiber is isomorphic to G. We let $\mathscr{G}$ be such a model, and by abuse of notation we let $G = \mathscr{G}$. We let $\mathcal{K} = G(\mathbb{Z}_p)$ and we let $\breve{\mathcal{K}} = G(W(k))$.

1.4.1. Connected components of affine Deligne–Lusztig Varieties. As we prove in the second chapter the moduli spaces of p-adic shtukas at parahoric level are closely related to a corresponding affine Deligne–Lusztig variety of the same level. In this section we recall what is known about the connected components of the later when G is unramified and $\mathscr{G} = G$ is hyperspecial.

Since we are assuming $k = \bar{k}$, the group G_{K_0} is split over K_0 and we have by the Cartan decomposition a bijection $\breve{\mathcal{K}} \backslash G(K_0)/\breve{\mathcal{K}} = X_*^+(T)$ given by $\mu \mapsto p^\mu := \mu(p) \in T(K_0)$. There is a map $\kappa_G : G(K_0) \to \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$ constructed as follows. For an element $b \in G(K_0)$ there is a unique $\mu' \in X_*^+(T)$ with $b \in \breve{\mathcal{K}} \backslash p^{\mu'}/\breve{\mathcal{K}}$. Then $\kappa_G(b)$ is defined to be $[\mu']$, the induced class of μ' in $\pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$. This map is a group homomorphism that is well-defined on φ-conjugacy classes. Moreover, the map constructed in this way descends to the Kottwitz map $\kappa_G : B(G) \to \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$ that we discussed above.

Recall that associated to a pair (b, μ) one can associate an affine Deligne–Lusztig variety $X_G^{\leq \mu}(b)$. This is a perfect scheme ([BS17]) over $\mathrm{Spec}(k)$ whose k-valued points can be described as:

$$X_G^{\leq \mu}(b)(k) = \left\{ g \cdot \breve{\mathcal{K}} \in G(K_0)/\breve{\mathcal{K}} \mid g^{-1} \cdot b \cdot \varphi(g) \in \breve{\mathcal{K}} \backslash p^{\mu'}/\breve{\mathcal{K}} \text{ with } \mu' \leq \mu \right\}$$

In [CKV15], [Nie18] [HZ20], the problem of determining connected components of affine Deligne–Lusztig varieties is thoroughly discussed. Although the description in full generality is complicated, in our situation ($\mathscr{G}$ reductive and $\breve{\mathcal{K}}$ hyperspecial) the problem is completely settled. In the references provided above, the connected components are described in three steps. The first step is to pass to the case of a simple adjoint group and it is done as follows:

Theorem 1.27. *([CKV15, Corollary 2.4.2]) Let G^{ad} denote the adjoint quotient of G, then there are natural maps w_G and $w_{G^{\mathrm{ad}}}$ and elements $c_{b,\mu} \in \pi_1(G)$ ($c_{b_{\mathrm{ad}},\mu_{\mathrm{ad}}} \in \pi_1(G^{\mathrm{ad}})$ respectively) well-defined up to multiplication by $\pi_1(G)^{\Gamma_{\mathbb{Q}_p}}$ (respectively $\pi_1(G^{\mathrm{ad}})^{\Gamma_{\mathbb{Q}_p}}$) making the following diagram commutative and Cartesian:*

$$
\begin{array}{ccc}
X_G^{\leq \mu}(b) & \longrightarrow & X_{G^{\mathrm{ad}}}^{\leq \mu_{\mathrm{ad}}}(b_{\mathrm{ad}}) \\
\downarrow{\scriptstyle w_G} & & \downarrow{\scriptstyle w_{G^{\mathrm{ad}}}} \\
c_{b,\mu}\pi_1(G)^{\Gamma_{\mathbb{Q}_p}} \times \mathrm{Spec}(k) & \longrightarrow & c_{b_{\mathrm{ad}},\mu_{\mathrm{ad}}}\pi_1(G^{\mathrm{ad}})^{\Gamma_{\mathbb{Q}_p}} \times \mathrm{Spec}(k)
\end{array}
$$

In the statement above the two sets that appear on the lower horizontal arrow should be interpreted as discrete topological groups so that the product is a disjoint union of copies of $\mathrm{Spec}(k)$. Once one reduces the problem to the adjoint case, one can further simplify to the simple adjoint case by observing that if $G = G_1 \times G_2$ then we get a decomposition $X_G^{\leq \mu}(b) = X_{G_1}^{\leq \mu_1}(b_1) \times_k X_{G_2}^{\leq \mu_2}(b_2)$. This is how the first step is completed in the references.

The second step of the strategy is to reduce the general simple adjoint group case to the case in which (b, μ) is HN-indecomposable. In this work we only consider the case in which (b, μ) is already HN-irreducible which is a stronger condition to being indecomposable. For this reason we do not review this step.

The third and final step is the determination of $\pi_0(X_G^{\leq \mu}(b))$ when G is simple adjoint and (b, μ) is HN-irreducible or when it is HN-indecomposable, but not HN-irreducible. Again, we only review the HN-irreducible case.

Theorem 1.28. *(*[Nie18, Theorem 1.1]*,* [CKV15, Theorem 1.1]*,* [HZ20, Theorem 8.1]*) If (b, μ) is HN-irreducible and $G = G^{\mathrm{ad}}$ is simple and adjoint then $w_G : \pi_0(X_G^{\leq \mu}(b)) \to c_{b,\mu} \pi_1(G^{\mathrm{ad}})^{\Gamma_{\mathbb{Q}_p}}$ is a bijection.*

We can rephrase these results on connected components in a more geometric form. Let G^{der} denote the derived subgroup of G, let $G^{\mathrm{ab}} := G/G^{\mathrm{der}}$ the maximal abelian quotient and denote by $\det : G \to G/G^{\mathrm{der}}$ the quotient map.

Corollary 1.29. *If G^{der} is simply connected and (b, μ) is HN-irreducible, the natural map $\det : X_G^{\leq \mu}(b) \to X_{G^{\mathrm{ab}}}^{\leq \mu_{\mathrm{ab}}}(b_{\mathrm{ab}})$ induced from $\det : G \to G^{\mathrm{ab}}$ induces a bijection of connected components $\pi_0(X_G^{\leq \mu}(b)) \cong \pi_0(X_{G^{\mathrm{ab}}}^{\leq \mu_{\mathrm{ab}}}(b_{\mathrm{ab}}))$.*

Remark 1.30. Since $X_{G^{\mathrm{ab}}}^{\leq \mu_{\mathrm{ab}}}(b_{\mathrm{ab}})$ is a disjoint union of copies of $\mathrm{Spec}(k)$ and k is algebraically closed, we could say instead that the map $X_G^{\leq \mu}(b) \to X_{G^{\mathrm{ab}}}^{\leq \mu_{\mathrm{ab}}}(b_{\mathrm{ab}})$ has geometrically connected fibers.

1.4.2. *The case $G^{\mathrm{der}} = G^{\mathrm{sc}}$.* In this subsection, we compute $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty})$ under the assumption that G is unramified, G^{der} is simply connected and (b, μ) is HN-irreducible.

Proposition 1.31. *Let G be as above, the determinant map induces a surjective map of locally spatial diamonds*

$$\det : \mathrm{Sht}_{G,b,[\mu],\infty} \to \mathrm{Sht}_{G^{\mathrm{ab}},b^{\mathrm{ab}},[\mu^{\mathrm{ab}}],\infty}$$

Proof. The key point is that since G^{der} is simply connected by Kneser's theorem [Kne65] the map of groups $G(\mathbb{Q}_p) \to G^{\mathrm{ab}}(\mathbb{Q}_p)$ is surjective. We omit the details. $\square$

Lemma 1.32. *Let G and (b, μ) be as above, let $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ be a hyperspecial subgroup. Then*

$$\det : \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \to \mathrm{Sht}_{G^{\mathrm{ab}},b^{\mathrm{ab}},[\mu^{\mathrm{ab}}],\det(\mathcal{K})}$$

has geometrically connected fibers.

Proof. We can construct an exact sequence

$$e \to \mathcal{G}^{\mathrm{der}} \to \mathcal{G} \xrightarrow{\det} \mathcal{G}^{\mathrm{ab}} \to e$$

of reductive groups over $\mathbb{Z}_p$. An application of Lang's theorem proves that $\det(\mathcal{K}) = \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ which is the maximal bounded subgroup of G^{ab}. By functoriality of the specialization map, see [Gle22], we have a commutative diagram:

$$
\begin{array}{ccc}
| \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times C_p | & \xrightarrow{\det} & | \mathrm{Sht}_{G^{\mathrm{ab}},b^{\mathrm{ab}},[\mu^{\mathrm{ab}}],\det(\mathcal{K})} \times C_p | \\
\downarrow{\mathrm{Sp}_{\mathcal{G}}} & & \downarrow{\mathrm{Sp}_{\mathcal{G}^{\mathrm{ab}}}} \\
| X_G^{\leq \mu}(b) | & \xrightarrow{\det} & | X_{G^{\mathrm{ab}}}^{\leq \mu_{\mathrm{ab}}}(b_{\mathrm{ab}}) |
\end{array}
$$

The vertical maps give bijections of connected components by Theorem 2.38 and the lower horizontal map induces a bijection of connected components by Corollary 1.29. $\square$

The following proposition is a particular case of an unpublished result of Hansen and Weinstein that follows from the work done in [Han21]. We provide an alternative proof that follows the steps of the analogous statement in [Che14, Lemme 6.1.3].

Proposition 1.33. *Let G be as above and let (b, μ) be HN-irreducible. Then $\mathrm{Gr}_{K_0}^{\leq[\mu]}(\mathcal{E}_b)^{\mathrm{adm}}$ is geometrically connected over $\mathrm{Spd}(K_0)$.*

Proof. Let X denote a geometric connected component of $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}}$ for $\mathcal{K} = G(\mathbb{Z}_p)$. By étaleness of $\pi_{GM,\mathcal{K}}$ the set $U := \pi_{GM,\mathcal{K}}(X)$ is a connected open subset of $\mathrm{Gr}_{C_p}^{\leq[\mu]}(\mathcal{E}_b)^{\mathrm{adm}}$. We claim, this open subset doesn't depend on the choice of X. This immediately implies $\mathrm{Gr}_{C_p}^{\leq[\mu]}(\mathcal{E}_b)^{\mathrm{adm}} = \pi_{GM,\mathcal{K}}(X)$ and in particular that it is connected.

To prove the claim, take a connected component X_∞ of $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)$ mapping to X. Note that $\pi_{\infty,\mathcal{K}}(X_\infty) = X$ since for groups $\mathcal{K}' \subseteq \mathcal{K}$ of finite index the transition maps $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}'} \times \mathrm{Spd}(C_p) \to \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C_p)$ are finite étale and surjective. This implies $U = \pi_{GM,\infty}(X_\infty)$. By Lemma 1.32 $\pi_0(\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C_p)) \to \pi_0(\mathrm{Sht}_{G^{\mathrm{ab}},b^{\mathrm{ab}},[\mu^{\mathrm{ab}}],\det(\mathcal{K})} \times C_p)$ is a bijection. Let X' denote some other connected component, and let z and z' denote the elements defined by X and X' in $\pi_0(\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C_p))$. Now, $G^{\mathrm{ab}}(\mathbb{Q}_p)$ and $G^{\mathrm{ab}}(\mathbb{Q}_p)/\det(\mathcal{K})$ act transitively on $\pi_0(\mathrm{Sht}_{G^{\mathrm{ab}},b^{\mathrm{ab}},[\mu^{\mathrm{ab}}],\infty} \times \mathrm{Spd}(C_p))$ and $\pi_0(\mathrm{Sht}_{G^{\mathrm{ab}},b^{\mathrm{ab}},[\mu^{\mathrm{ab}}],\det(\mathcal{K})} \times \mathrm{Spd}(C_p))$ respectively. This allow us to find an element $g \in G(\mathbb{Q}_p)$ with $\det(z) \cdot \det(g) = \det(z')$. Now $\pi_{\infty,\mathcal{K}}(X_\infty \cdot g) = X'$, which proves $\pi_{GM,\mathcal{K}}(X) = \pi_{GM,\mathcal{K}}(X')$ by equivariance of $\pi_{GM,\infty}$. $\qquad\square$

Lemma 1.34. *Let $\mathcal{K}$ be a hyperspecial subgroup of $G(\mathbb{Q}_p)$ and let $\mathcal{K}^{\mathrm{der}} = \mathcal{K} \cap G^{\mathrm{der}}(\mathbb{Q}_p)$. Let $m \in \pi_0(\mathrm{Sht}_{G^{\mathrm{ab}},b^{\mathrm{ab}},[\mu^{\mathrm{ab}}],\infty} \times \mathrm{Spd}(C_p))$ and let $X_m = \det^{-1}(m)$. Then $\mathcal{K}^{\mathrm{der}}$ acts transitively on $\pi_0(X_m)$.*

Proof. Since $\mathrm{Sht}_{G^{\mathrm{ab}},b^{\mathrm{ab}},[\mu^{\mathrm{ab}}],\infty} \times \mathrm{Spd}(C_p)$ is 0-dimensional, the space X_m is the collection of connected components of $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)$ that map to m. Let $x, y \in \pi_0(X_m)$, using Lemma 1.32 we see that $\pi_{\infty,\mathcal{K}}(x) = \pi_{\infty,\mathcal{K}}(y)$, we let z denote this connected component. Since $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)$ is a $\underline{\mathcal{K}}$-torsor over $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C_p)$, $\mathcal{K}$ acts transitively on the set of connected components of $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)$ over z. In particular, there is an element $g \in \mathcal{K}$ with $x \cdot g = y$. Since $\det(x) = \det(y)$ we must have that $m \cdot \det(g) = m$, but the action of $G^{\mathrm{ab}}(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p))$ is simple so $\det(g) = e$ and $g \in G^{\mathrm{der}}(\mathbb{Q}_p)$ as we wanted to show. $\qquad\square$

We can now describe connected components at infinite level.

Theorem 1.35. *Suppose G is an unramified group over $\mathbb{Q}_p$, that G^{der} is simply connected and that (b, μ) is HN-irreducible, then the determinant map*

$$\det_{\infty,\infty} : \mathrm{Sht}_{G,b,[\mu],\infty} \to \mathrm{Sht}_{G^{\mathrm{ab}},b^{\mathrm{ab}},[\mu^{\mathrm{ab}}],\infty}$$

has connected geometric fibers.

Proof. Since $\mathrm{Sht}_{G^{\mathrm{ab}},b^{\mathrm{ab}},[\mu^{\mathrm{ab}}],\infty} \times \mathrm{Spd}(C_p)$ is isomorphic to $\underline{G^{\mathrm{ab}}(\mathbb{Q}_p)} \times \mathrm{Spd}(C_p)$, we may prove instead that the determinant map induces a bijection

$$\pi_0(\det) : \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)) \to \pi_0(\mathrm{Sht}_{G^{\mathrm{ab}},b^{\mathrm{ab}},[\mu^{\mathrm{ab}}],\infty} \times \mathrm{Spd}(C_p)).$$

Let $x \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p))$. Given K a finite extension of K_0 we let x_K denote the image of x on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K))$ and let $f : \mathrm{Spd}(K) \to \mathrm{Gr}^{\leq[\mu]}(\mathcal{E}_b)^{\mathrm{adm}}$ be a point whose associated crystalline representation is as in Corollary 1.15. Let $S_f := \mathrm{Triv}(f^*(\mathbb{L}))$ the geometric realization of $f^*\mathbb{L}$. This space is also the fiber over f of the infinite level Grothendieck–Messing period map. Let $s \in \pi_0(S_f)$ be an element mapping to x_K. In summary we have taken a commutative diagram as follows:

$$
\begin{array}{ccc}
* & \xrightarrow{\ x\ } & \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)) \\
\downarrow{\scriptstyle s} & \searrow{\scriptstyle x_K} & \downarrow \\
\pi_0(S_f) & \xrightarrow{\ f\ } & \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K))
\end{array}
$$

We let G_x^{der} (respectively $G_{x_K}^{\mathrm{der}}$ and G_s^{der}) denote the stabilizer in $G^{\mathrm{der}}(\mathbb{Q}_p)$ of its action on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p))$ (respectively $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K))$ and $\pi_0(S_f)$).

We have inclusions $G_x^{\mathrm{der}}, G_s^{\mathrm{der}} \subseteq G_{x_K}^{\mathrm{der}}$ and by Chen's Theorem 1.13 (Lemma 1.25) G_s^{der} is an open subgroup of $G^{\mathrm{der}}(\mathbb{Q}_p)$. By Lemma 1.34, $G_x^{\mathrm{der}} \cdot \mathcal{K}^{\mathrm{der}} = G^{\mathrm{der}}(\mathbb{Q}_p)$ which implies that $G_{x_K}^{\mathrm{der}} \cdot \mathcal{K}^{\mathrm{der}} = G^{\mathrm{der}}(\mathbb{Q}_p)$ as well. In particular, the projection map $\mathcal{K}^{\mathrm{der}} \to G^{\mathrm{der}}(\mathbb{Q}_p)/G_{x_K}^{\mathrm{der}}$ is surjective. Since $G^{\mathrm{der}}(\mathbb{Q}_p)/G_{x_K}^{\mathrm{der}}$ has the discrete topology and $\mathcal{K}^{\mathrm{der}}$ is compact, we get that $G_{x_K}^{\mathrm{der}}$ is closed and of finite index within $G^{\mathrm{der}}(\mathbb{Q}_p)$. Moreover, since G^{der} is quasi-split (even unramified) all of the simple factors of G^{der} are isotropic. By

Margulis theorem [Mar91, Chapter II, Theorem 5.1] we can conclude that $G^{\mathrm{der}}_{x_K} = G^{\mathrm{der}}(\mathbb{Q}_p)$. Since the argument doesn't depend on the choice of x the action of $G^{\mathrm{der}}(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K))$ is trivial.

Now, $\mathrm{Spd}(C_p) = \varprojlim \mathrm{Spd}(K)$ and we may use [Sch17, Lemma 11.22] to compute the action map

$$|\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)| \times G^{\mathrm{der}}(\mathbb{Q}_p) \to |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)|$$

as the limit of the action maps

$$\varprojlim_{K \subseteq C_p} [|\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K)| \times G^{\mathrm{der}}(\mathbb{Q}_p) \to |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K)|].$$

Since in the transition maps $|\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K_1)| \to |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K_2)|$ every connected component on the source surjects onto a connected component on the target we get $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)) = \varprojlim \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K))$. This proves that $G^{\mathrm{der}}(\mathbb{Q}_p)$ acts trivially on the set of connected components and defines a transitive action of $G^{\mathrm{ab}}(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p))$. In turn this proves $\pi_0(\det)$ is bijective.

$\square$

Corollary 1.36. *For G, b and μ as in Theorem 1.35 and any compact subgroup $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ the map*

$$\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \to \mathrm{Sht}_{G^{\mathrm{ab}},b^{\mathrm{ab}},[\mu^{\mathrm{ab}}],\det(\mathcal{K})}$$

has non-empty connected geometric fibers.

Proof. This follows from the identity $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} = \mathrm{Sht}_{G,b,[\mu],\infty}/\underline{\mathcal{K}}$ and that π_0 is a left adjoint. $\square$

Using functoriality and equivariance for the three actions we can describe the actions by the three groups on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ in the spirit of Theorem 1.24.

Theorem 1.37. *(Compare with [Che13, Proposition 4.1]) Let G, b and μ as in Theorem 1.35. Let $E \subseteq C_p$ be the field of definition of $[\mu]$, let $\mathrm{Art}_{K_0,E} : W_E^{K_0} \to E^\times$ be as in Theorem 1.24, let $\mathrm{Nm}_{\mu^{\mathrm{ab}}} : \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m) \to G^{\mathrm{ab}}$ be the norm map associated to μ^{ab} then:*

(1) *The $G(\mathbb{Q}_p)$ right action on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ makes it a trivial right $G^{\mathrm{ab}}(\mathbb{Q}_p)$-torsor.*

(2) *If $s \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ and $j \in J_b(\mathbb{Q}_p)$ then*

$$s \cdot_{J_b(\mathbb{Q}_p)} j = s \cdot_{G^{\mathrm{ab}}(\mathbb{Q}_p)} \det(j^{-1}))$$

where $\det = j_{b^{\mathrm{ab}}} \circ \det_b$ with $\det_b : J_b(\mathbb{Q}_p) \to J_{b^{\mathrm{ab}}}(\mathbb{Q}_p)$ the map obtained from functoriality of the formation of J_b, respectively $J_{b^{\mathrm{ab}}}$, and where the map $j_{b^{\mathrm{ab}}}$ is the isomorphism $j_{b^{\mathrm{ab}}} : J_{b^{\mathrm{ab}}}(\mathbb{Q}_p) \cong G^{\mathrm{ab}}(\mathbb{Q}_p)$ obtained from regarding $J_{b^{\mathrm{ab}}}(\mathbb{Q}_p)$ as a subgroup of $G^{\mathrm{ab}}(K_0)$.

(3) *If $s \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ and $\gamma \in W_E^{K_0}$ then*

$$s \cdot_{W_E^{K_0}} \gamma = s \cdot_{G^{\mathrm{ab}}(\mathbb{Q}_p)} [\mathrm{Nm}_{\mu^{\mathrm{ab}}} \circ \mathrm{Art}_{K_0,E}(\gamma)].$$

1.4.3. *z-extensions.* In this subsection, we extend Theorem 1.35 to the case in which G^{der} is not necessarily simply connected, but we still assume that G is unramified and (b,μ) is HN-irreducible. In what follows, we will denote by G^{sc} the central simply connected cover of G^{der} and we denote by $G^\circ = G(\mathbb{Q}_p)/\mathrm{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$. Notice that when G^{der} is simply connected $G^\circ = G^{\mathrm{ab}}(\mathbb{Q}_p)$. In general, G° surjects onto $G^{\mathrm{ab}}(\mathbb{Q}_p)$ and the kernel is a finite group.

Recall the following definition used extensively by Kottwitz:

Definition 1.38. *A map of connected reductive groups $f : G' \to G$ is a z-extension if: f is surjective, $Z = \ker(f)$ is central in G', Z is isomorphic to a product of tori of the form $\mathrm{Res}_{F_i/\mathbb{Q}_p}\mathbb{G}_m$ for some finite extensions $F_i \subseteq \overline{\mathbb{Q}}_p$ and G' has simply connected derived subgroup.*

By [Kot82, Lemma 1.1] whenever G is an unramified group over $\mathbb{Q}_p$ that splits over $\mathbb{Q}_{p^s}$, there exists a z-extension $G' \to G$ with Z isomorphic to a product of tori of the form $\mathrm{Res}_{\mathbb{Q}_{p^s}/\mathbb{Q}_p}\mathbb{G}_m$. In particular, it is unramified as well.

In [Kot97] Kottwitz proves that for any reductive group G and cocharacter μ the natural morphism $B(G) \to B(G^{\mathrm{ad}})$ induces a bijection $B(G,\mu) \cong B(G^{\mathrm{ad}}, \mu^{\mathrm{ad}})$. From here one can deduce the following statement.

Lemma 1.39. *Let $A \subseteq T \subseteq B \subseteq G$ as in the notation section. Assume that $\mathbb{Q}_{p^s}$ is a splitting field for G. Let $\mu \in X_*^+(T)$, $[b] \in B(G,\mu)$, and $f : G' \to G$ a z-extension with $Z = \ker(f)$ isomorphic to a finite product of copies of $\mathrm{Res}_{\mathbb{Q}_{p^s}/\mathbb{Q}_p}\mathbb{G}_m$. Let $T' = f^{-1}(T)$ denote the maximal torus of G' projecting onto T. Then:*

(1) *For any choice of $\mu' \in X_*(T')^+$ lifting μ there is a unique lift $[b'] \in B(G')$ lifting $[b]$ with $[b'] \in B(G', \mu')$.*

(2) *For b' and μ' as in the previous claim (b, μ) is HN-irreducible if and only if (b', μ') is HN-irreducible.*

(3) *If E is the field of definition of μ with $\mathbb{Q}_p \subseteq E \subseteq \mathbb{Q}_{p^s}$ then there is a lift $\mu' \in X_*(T')^+$ with field of definition E.*

Proof. The first claim follows directly from the identifications $B(G, \mu) = B(G^{\mathrm{ad}}, \mu^{\mathrm{ad}}) = B(G', \mu')$. The second claim follows from the first claim, from the fact that $Z := ker(f)$ is central and from the fact that HN-irreducibility can be checked on the adjoint quotient. For the third claim consider the exact sequence of $\Gamma_{\mathbb{Q}_p}$-modules:

$$e \to X_*(Z) \to X_*(T') \to X_*(T) \to e$$

One can use Shapiro's lemma to prove $X_*(T')^{\Gamma_E} \to X_*(T)^{\Gamma_E}$ is surjective.

$\square$

Proposition 1.40. *Suppose that G' is an unramified group, (b', μ') a pair with $[b'] \in B(G', \mu')$, suppose that $Z \subseteq G'$ is a central torus, and let $G = G'/Z$ with projection map $f : G' \to G$. Let $b = f(b')$ and $\mu' = f \circ \mu$ the following hold:*

(1) $\mathrm{Gr}^{\leq[\mu']}(\mathcal{E}_{b'}) \to \mathrm{Gr}^{\leq[\mu]}(\mathcal{E}_b)$ *is an isomorphism.*

(2) $\mathrm{Gr}^{\leq[\mu']}(\mathcal{E}_{b'})^{\mathrm{adm}} \to \mathrm{Gr}^{\leq[\mu]}(\mathcal{E}_b)^{\mathrm{adm}}$ *is an isomorphism.*

(3) *If $\mathbb{L}_{G'}$ (respectively $\mathbb{L}_G$) denotes the pro-étale $\underline{G'(\mathbb{Q}_p)}$-torsor (respectively $\underline{G(\mathbb{Q}_p)}$-torsor) then $\mathbb{L}_G = f_* \mathbb{L}_{G'}$.*

Proof. We omit the details. $\square$

Proposition 1.41. *If (b, μ) is HN-irreducible then the following hold:*

(1) $\mathrm{Gr}^{\leq[\mu]}(\mathcal{E}_b)^{\mathrm{adm}} \times \mathrm{Spd}(C_p)$ *is connected*

(2) *The right action of $G(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p))$ makes this set into a G°-torsor.*
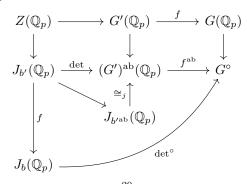
Proof. Using Lemma 1.39 we may find a z-extension $f : G' \to G$ and lift (b, μ) to a pair (b', μ') over G' which is also HN-irreducible. The first claim now follows from Proposition 1.40 and by Proposition 1.33 applied to G'.

Let $Z = Ker(f)$, the map $f : G'(\mathbb{Q}_p) \to G(\mathbb{Q}_p)$ is surjective. This together with Proposition 1.40 gives that $f : \mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C_p) \to \mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)$ is a $\underline{Z(\mathbb{Q}_p)}$-torsor. In particular, the map of sets of connected components is also surjective. Since $\mathrm{Gr}^{\leq[\mu]}(\mathcal{E}_b)^{\mathrm{adm}}$ is connected the action of $G(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p))$ is transitive. Let $x \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p))$ and denote by G_x the stabilizer of x in $G(\mathbb{Q}_p)$. Let $y \in \pi_0(\mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C_p))$ a lift of x, we have $\mathrm{Im}(G_y) = \mathrm{Im}(G_y \cdot Z(\mathbb{Q}_p)) = G_x$.

By Theorem 1.35 the stabilizer of y in $G'(\mathbb{Q}_p)$ is $(G')^{\mathrm{der}}(\mathbb{Q}_p)$, so $G_x = \mathrm{Im}((G')^{\mathrm{der}}(\mathbb{Q}_p)) = \mathrm{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$. $\square$

We describe the action of $J_b(\mathbb{Q}_p)$ and $W_E^{K_0}$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p))$ in terms of the action of G°. We begin with $J_b(\mathbb{Q}_p)$. We first construct a map $\det^\circ : J_b(\mathbb{Q}_p) \to G^\circ$ generalizing the determinant map $\det : J_b(\mathbb{Q}_p) \to G^{\mathrm{ab}}(\mathbb{Q}_p)$ of Theorem 1.37 as follows. Given G and $b \in G(K_0)$ we choose an unramified z-extension $f : G' \to G$ and a lift $b' \in G'(K_0)$ with $f(b') = b$. Let $Z = Ker(f)$. We get a sequence of maps of reductive groups

$$e \to Z \to J_{b'} \to J_b \to e.$$

By Shapiro's lemma $J_{b'}(\mathbb{Q}_p) \to J_b(\mathbb{Q}_p)$ is surjective. We can construct the following commutative diagram of topological groups:

$$
\begin{array}{ccccc}
Z(\mathbb{Q}_p) & \longrightarrow & G'(\mathbb{Q}_p) & \xrightarrow{f} & G(\mathbb{Q}_p) \\
\downarrow & & \downarrow & & \downarrow \\
J_{b'}(\mathbb{Q}_p) & \xrightarrow{\det} & (G')^{\mathrm{ab}}(\mathbb{Q}_p) & \xrightarrow{f^{\mathrm{ab}}} & G^\circ \\
\downarrow{\scriptstyle f} & & {\scriptstyle \cong_j}\uparrow & & \\
& & J_{b'^{\mathrm{ab}}}(\mathbb{Q}_p) & & \\
\downarrow & & & & \\
J_b(\mathbb{Q}_p) & & \xrightarrow{\det^\circ} & &
\end{array}
$$

Now, $\det^\circ$ is defined as the unique morphism that could make this diagram commutative. More explicitly, if $j \in J_b(\mathbb{Q}_p)$ we pick a lift $j' \in J_{b'}(\mathbb{Q}_p)$, and we define $\det^\circ(j) := f^{\mathrm{ab}}(\det(j'))$. One can verify this doesn't depend on any of the choices made.

Let $x \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p))$ and let $y \in \pi_0(\mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C_p))$ be a lift of x. Let $j \in J_b(\mathbb{Q}_p)$, and let $j' \in J_{b'}(\mathbb{Q}_p)$ be an element lifting j. We compute:

$$x \cdot_{J_b(\mathbb{Q}_p)} j = f(y \cdot_{J_{b'}(\mathbb{Q}_p)} j') = f(y \cdot_{G'(\mathbb{Q}_p)} j_{b'}(\det_{b'}(j^{-1}))) = x \cdot_{G^\circ} \det^\circ(j^{-1})$$

We now describe the action of $W_E^{K_0}$, we need a variant of the norm map discussed for tori. Given a connected reductive group G and a conjugacy class of cocharacters $[\mu]$ with reflex field E we define a norm map $\mathrm{Nm}_{[\mu]}^\circ : E^\times \to G^\circ$ as follows. Since is G is quasi-split we may fix $\mathbb{Q}_p$-rationally defined Borel a maximal torus $T \subseteq B \subseteq G$ and the unique dominant cocharacter $\mu \in X_*^+(T)$ representing $[\mu]$ and defined over E. We get a norm map $\mathrm{Nm}_\mu : E^\times \to T(\mathbb{Q}_p)$ and we may define $\mathrm{Nm}_{[\mu]}^\circ$ as the composition:

$$\mathrm{Nm}_{[\mu]}^\circ : E^\times \xrightarrow{\mathrm{Nm}_\mu} T(\mathbb{Q}_p) \to G(\mathbb{Q}_p) \to G^\circ.$$

Proposition 1.42. *With notation as in Proposition 1.41 the action of $W_E^{K_0}$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p))$ is given by the map $\mathrm{Nm}_{[\mu]}^\circ \circ \mathrm{Art}_{K_0,E} : W_E^{K_0} \to G^\circ$. More precisely, if $x \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p))$ and $\gamma \in W_E^{K_0}$ then:*

$$x \cdot_{W_E^{K_0}} \gamma = x \cdot_{G^\circ} \mathrm{Nm}_{[\mu]}^\circ(\mathrm{Art}_{K_0,E}(\gamma)).$$

Proof. Let $f : G' \to G$ be a z-extension, let (b', μ') be a pair lifting (b, μ), and let $Z = ker(f)$. By Lemma 1.39 we can choose G' and μ' so that μ' has the same field of definition as μ. Choose $A \subseteq T \subseteq B \subseteq G$ as above and let $T' = f^{-1}(T)$. Consider the following commutative diagram of spaces.

$$
\begin{array}{ccccc}
\mathrm{Sht}_{G',b',[\mu'],\infty} & \longrightarrow & \mathrm{Sht}_{(G')^{\mathrm{ab}},b'^{\mathrm{ab}},[\mu'],\infty} & \longleftarrow & \mathrm{Sht}_{T',b_{\mu'},[\mu'],\infty} \\
\downarrow & & & & \downarrow \\
\mathrm{Sht}_{G,b,[\mu],\infty} & & & & \mathrm{Sht}_{T,b_\mu,[\mu],\infty}
\end{array}
$$

Since G' is simply connected we get an equivariant bijection of geometric connected components

$$\pi_0(\mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C_p)) \to \pi_0(\mathrm{Sht}_{(G')^{\mathrm{ab}},b'^{\mathrm{ab}},[\mu'^{\mathrm{ab}}],\infty} \times \mathrm{Spd}(C_p)).$$

After forming geometric connected components and choosing a base point $x \in \pi_0(\mathrm{Sht}_{T',b_{\mu'},[\mu'],\infty} \times \mathrm{Spd}(C_p))$ the above diagram looks like this:

$$
\begin{array}{ccccc}
x \cdot G'^{\mathrm{ab}}(\mathbb{Q}_p) & \xrightarrow{\cong} & x \cdot G'^{\mathrm{ab}}(\mathbb{Q}_p) & \longleftarrow & x \cdot T'(\mathbb{Q}_p) \\
\downarrow & & & & \downarrow \\
x \cdot G^\circ & & & & x \cdot T(\mathbb{Q}_p)
\end{array}
$$

All of the maps are equivariant with respect to the groups involved. Since the map $T'(\mathbb{Q}_p) \to G^\circ$ factors through the map $T'(\mathbb{Q}_p) \to T(\mathbb{Q}_p)$, we get a canonical surjective and $W_E^{K_0}$-equivariant map

$$\pi_0(\mathrm{Sht}_{T,b_\mu,[\mu],\infty} \times \mathrm{Spd}(C_p)) \to \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p)).$$

By Theorem 1.24, the action on $\pi_0(\mathrm{Sht}_{T,b_\mu,[\mu],\infty})$ is through $\mathrm{Nm}_\mu \circ \mathrm{Art}_{K_0,E}$. By definition of $\mathrm{Nm}_{[\mu]}^\circ$ the action of $W_E^{K_0}$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p))$ is through $\mathrm{Nm}_{[\mu]}^\circ \circ \mathrm{Art}_{K_0,E}$. $\square$

2. The specialization map for moduli spaces of p-adic shtukas

For some background on specialization maps for v-sheaves we refer the reader to [Gle22]. We will freely use some of the terminology defined in that work.

2.1. G-torsors, lattices and shtukas.
In this section we recall the integral theory of vector bundles over the Fargues-Fontaine curve, and point to the technical statements that allow us to discuss the specialization map for the p-adic Beilinson–Drinfeld Grassmannians and moduli spaces of p-adic shtukas. Nothing in this subsection is new and it is all written in some form in [SW20], [KL15], [FF18], [Ans22]. Nevertheless, we need specific formulations for some of these results that are not explicit in the literature.

2.1.1. *Vector bundles on $\mathcal{Y}$.*

Definition 2.1. Given a perfectoid Huber pair (R, R^+) and a pseudo-uniformizer $\varpi \in R^+$, we define $\mathcal{Y}_{R^+}^{[0,\infty)}$ as $\mathrm{Spa}(W(R^+)) \setminus V([\varpi])$. Here $[\varpi]$ denotes a Teichmüller lift of ϖ, and $W(R^+)$ is given the $(p, [\varpi])$-adic topology. We let $\mathcal{Y}_{R^+}$ denote $\mathrm{Spa}(W(R^+)) \setminus V(p, [\varpi])$.

We review the geometry of $\mathcal{Y}_{R^+}$, fix a pseudo-uniformizer $\varpi \in R^+$. One defines a continuous map $\kappa_\varpi : |\mathcal{Y}_{R^+}| \to [0, \infty]$ characterized by the property that $\kappa(y) = r$ if and only if for any positive rational number $r \leq \frac{m}{n}$ the inequality $|p|_y^m \leq |[\varpi]|_y^n$ holds and for any positive rational number $\frac{m}{n} \leq r$ the inequality $|[\varpi]|_y^n \leq |p|_y^m$ holds. Given an interval $I \subseteq [0, \infty]$ we denote by $\mathcal{Y}_{R^+}^I$ the open subset corresponding to the interior of $\kappa_\varpi^{-1}(I)$. For intervals of the form $[0, \frac{h}{d}]$ where h and d are integers the space $\mathcal{Y}_{R^+}^{[0, \frac{h}{d}]}$ is represented by $\mathrm{Spa}(R', R'^+)$ corresponding to the rational localization, $\{x \in \mathrm{Spa}(W(R^+)) \mid |p^h|_x \leq |[\varpi]|^d|_x \neq 0\}$. In this case, we can compute R'^+ explicitly as the $[\varpi]$-adic completion of $W(R^+)[\frac{p^h}{[\varpi]^d}]$ and R' as $R'^+[\frac{1}{[\varpi]}]$. A direct computation shows that R' does not depend of R^+. In particular, the exact category of vector bundles over $\mathcal{Y}_{R^+}^{[0,\infty)}$ does not depend of the choice of R^+ either.

Recall the algebraic version of $\mathcal{Y}_{R^+}$, which we will denote Y_{R^+} and define as $\mathrm{Spec}(W(R^+)) \setminus V(p, [\varpi])$. Since $W(R^+) \subseteq \mathcal{O}_{\mathcal{Y}_{R^+}}$ and since $p, [\varpi]$, do not vanish simultaneously on $\mathcal{Y}_{R^+}$ we get a map of locally ringed spaces $f : \mathcal{Y}_{R^+} \to Y_{R^+} \subseteq \mathrm{Spec}(W(R^+))$.

Recall that given an untilt $R^\sharp$ of R there is a canonical surjection $W(R^+) \to R^{\sharp+}$ whose kernel is generated by an element $\xi \in W(R^+)$ primitive of degree 1 [SW20, Lemma 6.2.8]. The element ξ defines a closed Cartier divisor over $\mathcal{Y}_{R^+}$ and also defines a Cartier divisor on the scheme Y_{R^+}.

Recall the GAGA-type theorem of Kedlaya and Liu:

Theorem 2.2. ([Ked20, Theorem 3.8]) *Suppose (R, R^+) is a perfectoid Huber pair in characteristic p. The natural morphisms of locally ringed spaces $f : \mathcal{Y}_{R^+} \to Y_{R^+}$ gives, via the pullback functor $f^* : \mathrm{Vec}_{Y_{R^+}} \to \mathrm{Vec}_{\mathcal{Y}_{R^+}}$, an exact equivalence of exact categories.*

Remark 2.3. Although the reference does not explicitly claim that this equivalence is exact, one can simply follow the proof loc. cit. exchanging the word "equivalence" by "exact equivalence" since every arrow involved in the proof is an exact functor.

Corollary 2.4. *With the notation as above, the pullback f^* induces an equivalence*

$$f^* : (\mathrm{Vec}_{Y_{R^+}^{\xi \neq 0}})^{\mathrm{mer}} \to (\mathrm{Vec}_{\mathcal{Y}_{R^+}^{\xi \neq 0}})^{\mathrm{mer}}$$

between the category whose objects are vector bundles over $\mathcal{Y}_{R^+}$ (respectively vector bundles over Y_{R^+}) and morphisms are functions meromorphic along the ideal (ξ) (respectively functions over $Y_{R^+} \setminus V(\xi)$).

Since one can define $\mathscr{G}$-torsors Tannakianly these statements immediately generalize to those for $\mathscr{G}$-torsors. Kedlaya proves another important statement.

Theorem 2.5. ([Ked20, Lemma 2.3, Theorem 2.7, Remark 3.11]) *With notation as above, and letting j be the open embedding, $j : Y_{R^+} \to \mathrm{Spec}(W(R^+))$ the following statements hold:*

(1) The pullback functor $j^* : \mathrm{Vec}_{\mathrm{Spec}(W(R^+))} \to \mathrm{Vec}_{Y_{R^+}}$ is fully-faithful.

(2) If R^+ is a valuation ring then j^* is an equivalence.

(3) Taking categories of quasi-coherent sheaves the adjunction morphism $j^* j_* \mathcal{V} \to \mathcal{V}$ is an isomorphism.

We will need a small modification of Theorem 2.5.

Definition 2.6. Given a set I and a collection of tuples $\{(C_i, C_i^+), \varpi_i\}_{i \in I}$ we construct an adic space $\mathrm{Spa}(R, R^+)$. Here each C_i is an algebraically closed nonarchimedean field, the C_i^+ are open and bounded valuation subrings of C_i, and ϖ_i is a choice of pseudo-uniformizer. We let $R^+ := \prod_{i \in I} C_i^+$, we let $\varpi = (\varpi_i)_{i \in I}$, we endow R^+ with the ϖ-adic topology and we let $R := R^+[\frac{1}{\varpi}]$. Any space constructed in this way will be called a product of points.

The following statement is implicitly used and proved in ([SW20, Theorem 25.1.2]).

Proposition 2.7. *Let $\mathrm{Spa}(R, R^+)$ be the product of points associated to $\{(C_i, C_i^+), \varpi_i\}_{i \in I}$ as in Definition 2.6. The pullback functor $j^* : \mathrm{Vec}_{\mathrm{Spec}(W(R^+))} \to Y_{R^+}$ gives an equivalence of categories of vector bundles with fixed rank.*

Given $\xi \in W(R^+)$ primitive of degree 1 as before, observe that since both $\mathrm{Spec}(W(R^+))$ and Y_{R^+} are qcqs schemes the equivalence of vector bundles of Proposition 2.7 generalizes to the categories where the objects are the same, but morphism are allowed to have poles along ξ on both categories.

Interestingly, extending $\mathscr{G}$-torsors from Y_{R^+} to $\mathrm{Spec}(W(R^+))$ adds yet another layer of complexity. Indeed, the equivalences of Theorem 2.5 and Proposition 2.7 are not exact equivalences, so Tannakian formalism can't be used directly. As a matter of fact, only the pullback functor j^* is exact. J. Anschütz gives a detailed study of the problem of extending $\mathscr{G}$-torsors along j in [Ans22].

Theorem 2.8. ([Ans22, Proposition 11.5]) *Let $\mathrm{Spa}(R, R^+)$ be a product of points over k. Every $\mathscr{G}$-torsor $\mathcal{T}$ over Y_{R^+} extends along $j : Y_{R^+} \to \mathrm{Spec}(W(R^+))$ to a torsor $\mathscr{G}$ torsor over $\mathrm{Spec}(W(R^+))$.*

We use the following descent result repeatedly.

Proposition 2.9. ([SW20, Proposition 19.5.3]) *Let S be a perfectoid space over k and let $U \subseteq \mathcal{Y}_S^{[0,\infty)}$ be an open subset. For map of perfectoid spaces $f : S' \to S$, let $\mathcal{C}_{S'}$ denote the category of $\mathscr{G}$-torsors over $\mathcal{Y}_{S'}^{[0,\infty)} \times_{\mathcal{Y}_S^{[0,\infty)}} U$. Then the assignment $S' \mapsto \mathcal{C}_{S'}$, as a fibered category over Perf_S, is a v-stack.*

2.1.2. *Lattices and shtukas.* For this section, fix $\mathrm{Spa}(R, R^+)$ an affinoid perfectoid space over k, $\varpi \in R^+$ a choice of pseudo-uniformizer, $R^\sharp$ an untilt of R and $\xi_{R^\sharp}$ a generator for the kernel of the map $W(R^+) \to R^{\sharp,+}$.

Definition 2.10. We define the groupoid of $B_{\mathrm{dR}}^+(R^\sharp)$-lattices with $\mathscr{G}$-structure to have as objects pairs $(\mathcal{T}, \psi)$ where $\mathcal{T}$ is a $\mathscr{G}$-torsor over $\mathcal{Y}_{R^+}^{[0,\infty)}$ and $\psi : \mathcal{T} \to \mathscr{G}$ is an isomorphism over $\mathcal{Y}_{R^+}^{[0,\infty)} \setminus V(\xi_{R^\sharp})$ that is meromorphic along $(\xi_{R^\sharp})$. Isomorphisms are the evident ones.

We now consider p-adic shtukas. Recall that the spaces $\mathrm{Spec}(W(R^+))$, $\mathcal{Y}_{R^+}^{[0,\infty)}$, Y_{R^+} and $\mathcal{Y}_{R^+}$ come equipped with a Frobenius action which we denote by φ^{op}, induced from the arithmetic Frobenius ring homomorphism $\varphi : W(R^+) \to W(R^+)$.

Definition 2.11. We define the groupoid of shtukas with one paw over $\mathrm{Spa}(R^\sharp, R^{\sharp,+})$ and $\mathscr{G}$-structure. Objects are pairs $(\mathcal{T}, \Phi)$ where $\mathcal{T}$ is a $\mathscr{G}$-torsor over $\mathcal{Y}_{R^+}^{[0,\infty)}$ and $\Phi : \varphi^{\mathrm{op},*}\mathcal{T} \to \mathcal{T}$ is an isomorphism over $\mathcal{Y}_{R^+}^{[0,\infty)} \setminus V(\xi_{R^\sharp})$ meromorphic along $(\xi_{R^\sharp})$. Isomorphisms being evident.

Definition 2.12. Given a φ-module with $\mathscr{G}$-structure $(\mathcal{E}, \Phi_\mathcal{E})$ over $\mathcal{Y}_{R^+}^{(0,\infty)}$ and a shtuka $(\mathcal{T}, \Phi_\mathcal{T})$ an isogeny is an equivalence class of pairs (r, f) with $r \in \mathbb{R}$ and $f : (\mathcal{T}, \Phi_\mathcal{T}) \to (\mathcal{E}, \Phi_\mathcal{E})$ a φ-equivariant isomorphism defined over $\mathcal{Y}_{R^+}^{[r,\infty)}$. Two pairs (r_1, f_1) and (r_2, f_2) are equivalent if there is a third pair (r_3, f_3) with $r_3 > r_1, r_2$ and $f_1 = f_3 = f_2$ when restricted to $\mathcal{Y}_{R^+}^{[r_3,\infty)}$.

In what follows, we prove three technical lemmas that, intuitively speaking, allow us to "deform" lattices and shtukas with $\mathscr{G}$-structure.

For any $r \in [0, \infty)$ let $B_{R^+}^{[r,\infty]} = H^0(\mathcal{Y}_{R^+}^{[r,\infty]}, \mathcal{O}_{\mathcal{Y}_{R^+}^{[r,\infty]}})$, and consider the ring $R_{\mathrm{red}}^+ = (R^+/\varpi)^{\mathrm{perf}}$. Observe that the universal property of $\mathcal{Y}_{R^+}^{[r,\infty]}$ as a rational subset of $\mathrm{Spa}(W(R^+))$ induces compatible ring maps $B_{R^+}^{[r,\infty]} \to W(R_{\mathrm{red}}^+)[\frac{1}{p}]$ for varying r. We denote this family of reduction maps by $(-_{\mathrm{red}})$.

Lemma 2.13. *Let $s \in B_{R^+}^{[r,\infty]}$ and suppose that the reduction s_{red}, originally defined over $W(R_{\mathrm{red}}^+)[\frac{1}{p}]$, lies in $W(R_{\mathrm{red}}^+)$, then there is a tuple (r', a, b, ϖ_s) with r' a number $r \leq r'$, $a \in W(R^+)$, $b \in B_{R^+}^{[r',\infty]}$ and a pseudo-uniformizer $\varpi_s \in R^+$ such that $s = a + b$ and $b \in [\varpi_s] \cdot B_{R^+}^{[r',\infty]}$.*

Proof. By enlarging r if necessary we can assume $\mathcal{Y}_{R^+}^{[r,\infty]}$ is of the form:

$$\{x \in \mathrm{Spa}(W(R^+)) \mid |[\varpi]|_x \leq |p^m|_x \neq 0\}$$

for some m, we compute $B_{R^+}^{[r,\infty]}$ explicitly. If S^+ denotes the p-adic completion of $W(R^+)[\frac{[\varpi]}{p^m}]$, then $B_{R^+}^{[r,\infty]} = S^+[\frac{1}{p}]$. Any element $s \in B_{R^+}^{[r,\infty]}$ is of the form $s = \frac{1}{p^n} \cdot \Sigma_{i=0}^\infty [a_i] x^{m(i)} p^i$ where $a_i \in R^+$, $x = \frac{[\varpi]}{p^m}$, and $m(i)$ denotes a non-negative integer. We can decompose $p^n \cdot s$ as

$$x \cdot \left(\sum_{i=0,m(i)>0}^\infty [a_i] x^{m(i)-1} p^i \right) + \sum_{i=0,m(i)=0}^\infty [a_i] p^i.$$

Since $x = \frac{[\varpi]}{p^m}$, we have that $[\varpi]$ divides in $B_{R^+}^{[r,\infty]}$ the first term of this decomposition. As long as we pick a ϖ_s that divides ϖ, we may and do reduce to the case $s = \Sigma_{i=0}^{\infty}[a_i]p^{i-n}$. In this case, $s_{\mathrm{red}} = \Sigma_{i=0}^{\infty}[(a_i)_{\mathrm{red}}]p^{i-n}$ and by hypothesis we have that for $i < n$ $(a_i)_{\mathrm{red}} = 0$ in R_{red}^+. We can choose a pseudo-uniformizer ϖ_s for which all of a_i, for $i < n$, are zero in R^+/ϖ_s. We can take $a = \sum_{i=n}^{\infty}[a_i]p^{i-n}$ and $b = \sum_{i=0}^{n-1}[a_i]p^{i-n}$. These clearly satisfy the properties. $\qquad\square$

Lemma 2.14. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be trivial $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$ and let $\lambda : \mathcal{T}_1 \to \mathcal{T}_2$ be an isomorphism over $\mathcal{Y}_{R^+}^{[r,\infty]}$ whose reduction to $\mathrm{Spec}(W(R_{\mathrm{red}}^+)[\frac{1}{p}])$ extends $\mathrm{Spec}(W(R_{\mathrm{red}}^+))$. Then, there is an isomorphism $\widetilde{\lambda} : \mathcal{T}_1 \to \mathcal{T}_2$ over $\mathrm{Spec}(W(R^+))$, a pseudo-uniformizer $\varpi_\lambda \in R^+$ and a number $r \leq r'$ such that $\lambda = \widetilde{\lambda}$ in $\mathrm{Hom}_{\mathrm{Spec}(B_{R^+}^{[r',\infty]}/[\varpi_\lambda])}(\mathcal{T}_1, \mathcal{T}_2)$.*

Proof. Fix trivializations $\iota_i : \mathcal{T}_i \to \mathscr{G}$, and consider $\iota_2 \circ \lambda \circ \iota_1^{-1}$ as an element $g \in H^0(\mathcal{Y}_{R^+}^{[r,\infty]}, \mathscr{G}) \subseteq H^0(\mathcal{Y}_{R^+}^{[r,\infty]}, \mathrm{GL}_n)$ for some n and some embedding $\mathscr{G} \to \mathrm{GL}_n$ defined over $W(k)$. By Lemma 2.13 we can find ϖ_λ such that $g = M_1 + [\varpi_\lambda]M_2$ with $M_1 \in \mathrm{GL}_n(W(R^+))$ and $M_2 \in \mathrm{M}_{n \times n}(B_{R^+}^{[r',\infty]})$. Since $W(R^+)/[\varpi_\lambda] \subseteq B_{R^+}^{[r',\infty]}/[\varpi_\lambda]$ the reduction of M_1 to $\mathrm{GL}_n(B_{R^+}^{[r',\infty]}/[\varpi_\lambda])$ lies in $\mathscr{G}(W(R^+)/[\varpi_\lambda])$. Moreover, since $\mathscr{G}$ is a smooth group and $W(R^+)$ is $[\varpi_\lambda]$-complete, we can lift this to an element $g' \in \mathscr{G}(W(R^+))$ with $g' = M_1$ in $\mathrm{GL}_n(W(R^+)/[\varpi_\lambda])$. Consequently $g' = g$ in $\mathscr{G}(B_{R^+}^{[r',\infty]}/[\varpi_\lambda])$, and by letting $\widetilde{\lambda} = \iota_2^{-1} \circ g' \circ \iota_1$ we get the desired isomorphism. $\qquad\square$

The proof of the following lemma is inspired by the computations that appear in [HV11, Theorem 5.6], and it is a key input in the proof of Theorem 2.33.

Lemma 2.15 (Unique liftability of isogenies)**.** *Let $\mathcal{T}$ be a trivial $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$ and let $\mathscr{G}_b$ denote the trivial $\mathscr{G}$-torsor endowed with the φ-module structure over $\mathcal{Y}_{R^+}^{(0,\infty]}$ given by an element $b \in \mathscr{G}(\mathcal{Y}_{R^+}^{(0,\infty]})$. Let $\Phi : \varphi^{\mathrm{op},*}\mathcal{T} \to \mathcal{T}$ be an isomorphism defined over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$ and $\lambda : \mathcal{T} \to \mathscr{G}_b$ a φ-equivariant isomorphism defined over $B_{R^+}^{[r,\infty]}/[\varpi]$ for some r big enough so that $\xi_{R^\sharp}$ becomes a unit. Then, there is a unique φ-equivariant isomorphism $\widetilde{\lambda} : \mathcal{T} \to \mathscr{G}_b$ defined over $\mathcal{Y}_{R^+}^{[r,\infty]}$ such that $\widetilde{\lambda} = \lambda$ in $B_{R^+}^{[r,\infty]}/[\varpi]$.*

Proof. By transport of structure, we assume that $\mathscr{G} = \mathcal{T}$, that $\Phi \in \mathscr{G}(W(R^+)[\frac{1}{\xi}])$, and that $\lambda \in \mathscr{G}(B_{R^+}^{[r,\infty]}/[\varpi])$. It suffices to find $\widetilde{\lambda} \in \mathscr{G}(B_{R^+}^{[r,\infty]})$ reducing to λ with $\Phi = \widetilde{\lambda}^{-1} \circ b \circ \varphi(\widetilde{\lambda})$. Choose an arbitrary lift $\lambda_0 \in \mathscr{G}(B_{R^+}^{[r,\infty]})$ of λ, and let $\eta_0 = \lambda_0^{-1} \circ b \circ \varphi(\lambda_0) \circ \Phi^{-1}$. We construct a pair of sequences of maps, $\lambda_i : \mathscr{G} \to \mathscr{G}_b$ and $\eta_i : \mathscr{G} \to \mathscr{G}$ defined recursively by the relations $\lambda_{n+1} = \lambda_n \circ \eta_n$ and $\eta_n = \lambda_n^{-1} \circ b \circ \varphi(\lambda_n) \circ \Phi^{-1}$. Observe that $\eta_0 = \mathrm{Id}$ in $\mathscr{G}(B_{R^+}^{[r,\infty]}/[\varpi])$. We prove inductively that $\eta_n = \mathrm{Id}$ in $\mathscr{G}(B_{R^+}^{[r,\infty]}/[\varpi^{p^n}])$. Now, when $g \in \mathscr{G}(B_{R^+}^{[r,\infty]})$ with $g = \mathrm{Id}$ in $\mathscr{G}(B_{R^+}^{[r,\infty]}/[\varpi^{p^n}])$, then $\varphi(g) = \mathrm{Id}$ in $\mathscr{G}(B_{R^+}^{[\frac{r}{p},\infty]}/[\varpi^{p^{n+1}}]) \subseteq \mathscr{G}(B_{R^+}^{[r,\infty]}/[\varpi^{p^{n+1}}])$.

The induction then follows from the computation:

$$\eta_{n+1} = \lambda_{n+1}^{-1} \circ b \circ \varphi(\lambda_{n+1}) \circ \Phi^{-1} = \eta_n^{-1} \circ \lambda_n^{-1} \circ b \circ \varphi(\lambda_{n+1}) \circ \Phi^{-1} \tag{1}$$

$$= \Phi \circ \varphi(\lambda_n)^{-1} \circ b^{-1} \circ \lambda_n \circ \lambda_n^{-1} \circ b \circ \varphi(\lambda_{n+1}) \circ \Phi^{-1} \tag{2}$$

$$= \Phi \circ \varphi(\lambda_n)^{-1} \circ \varphi(\lambda_{n+1}) \circ \Phi^{-1} = \Phi \circ \varphi(\eta_n) \circ \Phi^{-1} \tag{3}$$

Since $\varphi(\eta_n) = \mathrm{Id}$ in $\mathscr{G}(B_{R^+}^{[r,\infty]}/[\varpi^{p^{n+1}}])$ we also have that $\eta_{n+1} = \mathrm{Id}$ in $\mathscr{G}(B_{R^+}^{[r,\infty]}/[\varpi^{p^{n+1}}])$.

This let us conclude that η_i converges to Id in $\mathscr{G}(B_{R^+}^{[r,\infty]})$. Define $\widetilde{\lambda} \in \mathscr{G}(B_{R^+}^{[r,\infty]})$ as the limit of the λ_i. Taking limits we get $\mathrm{Id} = \eta_\infty = \widetilde{\lambda} \circ b \circ \varphi(\widetilde{\lambda}) \circ \Phi^{-1}$ and $\widetilde{\lambda} = \lambda_i = \lambda$ in $\mathscr{G}(B_{R^+}^{[r,\infty]}/[\varpi])$.

Let us prove uniqueness. Given two lifts $\widetilde{\lambda}_i$ of λ we let $g = \widetilde{\lambda}_1 \circ \widetilde{\lambda}_2^{-1}$ with $g \in \mathscr{G}(B_{R^+}^{[r,\infty]})$. Now, φ-equivariance gives $b = g^{-1} \circ b \circ \varphi(g)$, and since $g = \mathrm{Id}$ in $\mathscr{G}(B_{R^+}^{[r,\infty]}/[\varpi])$ then $\varphi(g) = \mathrm{Id}$ in $\mathscr{G}(B_{R^+}^{[r,\infty]}/[\varpi^p])$. From the identity $b = g^{-1} \circ b \circ \mathrm{Id}$ in $\mathscr{G}(B_{R^+}^{[r,\infty]}/[\varpi^p])$ we proceed inductively to prove that $g = \mathrm{Id}$ in $\mathscr{G}(B_{R^+}^{[r,\infty]}/[\varpi^{p^n}])$ for every n and by separatedness also in $\mathscr{G}(B_{R^+}^{[r,\infty]})$. $\qquad\square$

2.2. **Specialization map for p-adic Beilinson–Drinfeld Grassmannians.** We recall the definition of the p-adic Beilinson–Drinfeld Grassmannian that is most suitable to study its specialization map.

Definition 2.16. ([SW20, Definition 20.3.1]) We let $\mathrm{Gr}^{\mathscr{G}}_{W(k)}$ denote the v-sheaf $\mathrm{Gr}^{\mathscr{G}}_{W(k)}(R, R^+) = \{(R^\sharp, \iota, f, \mathcal{T}, \psi)\}_{\cong}$ with $(R^\sharp, \iota, f)$ an untilt over $W(k)$ and $(\mathcal{T}, \psi)$ is a lattice with $\mathscr{G}$-structure as in Definition 2.10.

By Beauville–Laszlo glueing this agrees with the loop group description.

Proposition 2.17. *With terminology as in* [Gle22, Definition 4.6] *the v-sheaf* $\mathrm{Gr}^{\mathscr{G}}_{W(k)}$ *formalizes products of points. In particular, it is v-formalizing.*

Proof. Let $\mathrm{Spa}(R, R^+)$ be a product of points and $f : \mathrm{Spa}(R, R^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)}$ a map. By definition, associated to this map we have an untilt $(R^\sharp, \iota, m)$ over $W(k)$ and a $\mathscr{G}$-torsor $\mathcal{T}$ over $\mathcal{Y}^{[0,\infty)}_{R^+}$ together with a trivialization $\psi : \mathcal{T} \to \mathscr{G}$ over $\mathcal{Y}^{[0,\infty)}_{R^+} \setminus V(\xi_{R^\sharp})$ meromorphic along $\xi_{R^\sharp}$. We use ψ to glue $\mathcal{T}$ and $\mathscr{G}$ along $\mathcal{Y}^{[r,\infty)}_{R^+}$ to get a $\mathscr{G}$-torsor defined over $\mathcal{Y}_{R^+}$. Using Corollary 2.4, Theorem 2.8 and the fact that by construction $\mathcal{T}$ is trivial on $Y_{R^+} \setminus V(\xi)$ we can extend $\mathcal{T}$ to a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$ together with a trivialization over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi_{R^\sharp}}])$. This is enough to define a map $\mathrm{Spd}(R^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)}$ that restricts to the original one. $\square$

Proposition 2.18. ([SW20, §20.3]) *With terminology as in* [Gle22, Definition 4.11, Definition 3.20, Definition 3.12] *the v-sheaf* $\mathrm{Gr}^{\mathscr{G}}_{W(k)}$ *is specializing, formally p-adic, and* $(\mathrm{Gr}^{\mathscr{G}}_{W(k)})^{\mathrm{red}}$ *is represented by the Witt-vector affine flag variety,* $\mathcal{Fl}^{\mathscr{G}}_{\mathcal{W},k}$.

Proof. We need to prove that $\mathrm{Gr}^{\mathscr{G}}_{W(k)}$ is separated, v-formalizing and that the diagonal map is formally adic. The first two properties follow respectively from [SW20, Theorem 20.3.2, Theorem 21.2.1] and Proposition 2.17. By [Gle22, Proposition 3.29], it is enough to prove that $\mathrm{Gr}^{\mathscr{G}}_{W(k)}$ is formally p-adic. This follows from the fact that $\mathrm{Gr}^{\mathscr{G}}_{W(k)} \times_{\mathrm{Spd}(W(k))} \mathrm{Spec}(k)^\diamond = (\mathcal{Fl}^{\mathscr{G}}_{\mathcal{W},k})^\diamond$, from [Gle22, Lemma 3.32] and the fact that $\mathcal{Fl}^{\mathscr{G}}_{\mathcal{W},k}$ is ind-representable by a perfect scheme. Indeed, ind-representability proves that $\mathcal{Fl}^{\mathscr{G}}_{\mathcal{W},k}$ is a reduced scheme-theoretic v-sheaf as in [Gle22, Definition 3.15]. $\square$

Recall that given $\mu \in X^+_*(T)$ with field of definition E we may define a "local model" v-sheaf $\mathcal{M}^{\mathscr{G}, \leq \mu}_{O_E}$ over $\mathrm{Spd}(O_E)$. This is defined as the v-sheaf closure of $\mathrm{Gr}^{G, \leq \mu}_E$ in $\mathrm{Gr}^{\mathscr{G}}_{\mathbb{Z}_p} \times_{\mathbb{Z}_p} O_E$ [SW20, §21.4]. In [AGLR22], our collaboration with Anschütz, Lourenço, and Richarz, we prove the following statement.

Theorem 2.19. ([AGLR22]) *With terminology as in* [AGLR22]. *If $\mathscr{G}$ is parahoric and $\mu \in X^+_*(T)$, then $\mathcal{M}^{\mathscr{G}, \leq \mu}_{O_E}$ is a flat and rich p-adic kimberlite. Moreover,* $(\mathcal{M}^{\mathscr{G}, \leq \mu}_{O_E})^{\mathrm{red}} = \mathcal{A}_{\mathscr{G}, \mu}$, *the μ-admissible locus in* $\mathcal{Fl}^{\mathscr{G}}_{\mathcal{W},k}$.

When $\mathscr{G}$ is reductive we can say a bit more. Indeed, in this case one can use a Demazure resolution as in [Gle22] to prove the following statement.

Theorem 2.20. ([Gle22, Theorem 5.1]) *Let F be a nonarchimedean field extension of E. If $\mathscr{G}$ is reductive and $\mu \in X^+_*(T)$, then $\mathcal{M}^{\mathscr{G}, \leq \mu}_{O_F}$ has geometrically connected tubular neighborhoods.*

2.3. **Moduli spaces of shtukas are smelted kimberlites.** Fix an element $b \in G(K_0)$ and let $\mathscr{G}_b : \mathrm{Rep}^{\mathscr{G}}_{\mathbb{Z}_p} \to \mathrm{IsoCrys}_{K_0}$ denote the associated isocrystal with $\mathscr{G}$-structure.

Definition 2.21. The moduli space of p-adic shtukas associated to $\mathscr{G}_b$, which we denote by $\mathrm{Sht}^{\mathscr{G}_b}_{W(k)}$, is the functor $\mathrm{Sht}^{\mathscr{G}_b}_{W(k)}(R, R^+) = \{(R^\sharp, \iota, f), \mathcal{T}, \Phi, \lambda\}/_{\cong}$ with $(R^\sharp, \iota, f)$ untilt over $W(k)$, $(\mathcal{T}, \Phi)$ is a shtuka as in Definition 2.11 and $\lambda : \mathcal{T} \to \mathscr{G}_b|_{\mathcal{Y}^{[r,\infty)}_{R^+}}$ an isogeny as in Definition 2.12.

We consider the following auxiliary space.

Definition 2.22. Let $\mathrm{WSht}^{\mathscr{G}_b}_{W(k)}$ denote the functor $\mathrm{WSht}^{\mathscr{G}_b}_{W(k)}(R, R^+) = \{(R^\sharp, \iota, f), M, \lambda\}$ with $(R^\sharp, \iota, f)$ an untilt over $W(k)$, $M \in \mathscr{G}(W(R^+)[\frac{1}{\xi_{R^\sharp}}])$ and $\lambda : \mathscr{G}_M \to \mathscr{G}_b$ an isogeny. Here $\mathscr{G}_M := (\mathscr{G}, \Phi_M)$ with $\Phi_M : \varphi^{\mathrm{op},*}\mathscr{G} \to \mathscr{G}$ given by M.

We denote by $\mathbb{W}^+\mathscr{G}$ the sheaf in groups $\mathbb{W}^+\mathscr{G}(R, R^+) = \mathscr{G}(W(R^+))$.

Proposition 2.23. *The following hold:*

(1) *The functors* $\mathrm{Sht}^{\mathscr{G}_b}_{W(k)}$ *and* $\mathrm{WSht}^{\mathscr{G}_b}_{W(k)}$ *are small v-sheaves.*

(2) *The natural map* $\mathrm{WSht}^{\mathscr{G}_b}_{W(k)} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(k)}$ *is a* $\mathbb{W}^+\mathscr{G}$-*torsor for the v-topology.*

(3) $\mathrm{WSht}^{\mathscr{G}_b}_{W(k)}$ is formalizing and $\mathrm{Sht}^{\mathscr{G}_b}_{W(k)}$ is v-formalizing.

Proof. Standard argument using Proposition 2.9 proves the first claim. Given $N \in \mathbb{W}^+\mathscr{G}(R, R^+)$ and $(M, \lambda) \in \mathrm{WSht}^{\mathscr{G}_b}_{W(k)}(R, R^+)$ let $N \cdot (M, \lambda) = (N^{-1}M\varphi(N), \lambda \circ N)$. This action on $\mathrm{WSht}^{\mathscr{G}_b}_{W(k)}$ makes the map $\mathrm{WSht}^{\mathscr{G}_b}_{W(k)} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(k)}$ equivariant for the trivial action on the target. It suffices to prove that the basechange of the map along product of points is the trivial $\mathbb{W}^+\mathscr{G}$-torsor.

Let $\mathrm{Spa}(R, R^+)$ be a product of points, and let $(\mathcal{T}, \Phi, \lambda) \in \mathrm{Sht}^{\mathscr{G}_b}_{W(k)}(R, R^+)$. Similarly to the proof of Proposition 2.17, we can glue $\mathcal{T}$ along λ over $\mathcal{Y}^{[r,\infty)}_{R^+}$ and use Theorem 2.8 to get a $\mathscr{G}$-bundle over $\mathrm{Spec}(W(R^+))$ with a meromorphic Φ that restrict to the previous one. Now, any $\mathscr{G}$-bundle on $\mathrm{Spec}(W(R^+))$ is trivial. Indeed, $\mathrm{Spec}(W(R^+))$ splits every étale cover. The choice of a trivialization specifies a section $(M, \lambda) \in \mathrm{WSht}^{\mathscr{G}_b}_{W(k)}(R, R^+)$ and after chasing definitions one can see that the natural action of $\mathbb{W}^+\mathscr{G}$ on the set of trivialization acts compatibly with the action specified above.

Let us prove that $\mathrm{WSht}^{\mathscr{G}_b}_{W(k)}$ is formalizing. From this and surjectivity of $\mathrm{WSht}^{\mathscr{G}_b}_{W(k)} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(k)}$ it follows that $\mathrm{Sht}^{\mathscr{G}_b}_{W(k)}$ is v-formalizing. Let $\mathrm{Spa}(S, S^+) \in \mathrm{Perf}_k$, and $\varpi_S \in S^+$ a pseudo-uniformizer. Let $((S^\sharp, \iota, f), M, \lambda) \in \mathrm{WSht}^{\mathscr{G}_b}_{W(k)}(S, S^+)$, we construct a natural transformation $\mathrm{Spd}(S^+) \to \mathrm{WSht}^{\mathscr{G}_b}_{W(k)}$. A map $f : \mathrm{Spa}(L, L^+) \to \mathrm{Spd}(S^+)$ induces $f : W(S^+)[\frac{1}{\xi_{S^\sharp}}] \to W(L^+)[\frac{1}{\xi_{L^\sharp}}]$, we let $M_L = f(M)$. Fix a pseudo-uniformizer $\varpi_L \in L^+$, there is a large enough $r' \in \mathbb{R}$ for which the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{Y}^{[r',\infty]}_{L^+} & \longrightarrow & \mathcal{Y}^{[r,\infty]}_{R^+} \\
\downarrow & & \downarrow \\
\mathrm{Spa}(W(L^+)) & \longrightarrow & \mathrm{Spa}(W(S^+))
\end{array}
$$

This map allows us to pullback the isogeny λ to $\mathrm{Spa}(L, L^+)$. The isogeny constructed this way does not depend of the choices of ϖ_S, ϖ_L, r or r'. $\qquad\square$

Recall that moduli spaces of shtukas satisfy the valuative criterion for partial properness over $\mathrm{Spd}(W(k))$.

Lemma 2.24. *Let $\mathscr{G}_1 \to \mathscr{G}_2$ be a closed embeddings of parahoric group schemes over $\mathbb{Z}_p$ and $\mathscr{G}_b$ an isocrystal with $\mathscr{G}_1$ structure. Let $\mathscr{G}'_b = \mathscr{G}_b \overset{\mathscr{G}_1}{\times} \mathscr{G}_2$, the induced map $\mathrm{WSht}^{\mathscr{G}_b}_{W(k)} \to \mathrm{WSht}^{\mathscr{G}'_b}_{W(k)}$ is a closed immersion.*

Proof. Let $\mathrm{Spa}(S, S^+)$ in Perf_k be totally disconnected, and let $(M, \lambda) \in \mathrm{WSht}^{\mathscr{G}'_b}_{W(k)}(S, S^+)$. It suffices to prove that the basechange along S is a closed immersion. Abusing notation, we let (r, λ) represent the isogeny. By unraveling definitions we think of M and λ as ring maps $\mathcal{O}_{\mathscr{G}_2} \to W(S^+)[\frac{1}{\xi_{S^\sharp}}]$ and $\mathcal{O}_{\mathscr{G}_2} \to B^{S^+}_{[r,\infty]}$ with $\mathcal{O}_{\mathscr{G}_1} = \mathcal{O}_{\mathscr{G}_2}/I$. The basechange $\mathrm{Spa}(S, S^+) \times_{\mathrm{WSht}^{\mathscr{G}'_b}_{W(k)}} \mathrm{WSht}^{\mathscr{G}_b}_{W(k)}$ represents the maps $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(S, S^+)$ such that the induced morphisms $M : \mathcal{O}_{\mathscr{G}_2} \to W(R^+)[\frac{1}{\xi_{R^\sharp}}]$ and $\lambda : \mathcal{O}_{\mathscr{G}_2} \to B^{[r,\infty]}_{R^+}$ map elements of I to 0.

Since I is finitely generated, it suffices to prove that if $t \in W(S^+)[\frac{1}{\xi_{S^\sharp}}]$ (or $t \in B^{S^+}_{[r,\infty]}$) the subfunctor of points in $\mathrm{Spa}(S, S^+)$ that map t to 0 forms a closed immersion. Fix $t \in W(S^+)[\frac{1}{\xi_{S^\sharp}}]$, replacing t by $\xi^n \cdot t$ we may assume $t \in W(S^+)$. Using the Teichmüller expansion we have $t \in (S^+)^{\mathbb{N}}$ and t restricts to 0 if and only if each entry restricts to 0. This defines a Zariski closed subset of $\mathrm{Spa}(S, S^+)$.

Now fix $t \in B^{S^+}_{[r,\infty]} \subseteq B^{S^+}_{[r,\infty)}$ and let $Z \subseteq |\mathcal{Y}^{[r,\infty)}_{R^+}|$ be the set of valuations with $|t|_z = 0$. The structure map $\pi : (\mathcal{Y}^{[r,\infty)}_{S^+})^\diamond \to \mathrm{Spd}(S, S^+)$ is ℓ-cohomologically smooth and universally open [Sch17, Proposition 24.5]. The subfunctor of points we consider are those maps that factor through $Z' = |\mathrm{Spa}(S, S^+)| \setminus \pi(|\mathcal{Y}^{[r,\infty)}_{R^+}| \setminus Z)$ which is a closed subset. This set is closed and generalizing, so it defines a closed immersion of $\mathrm{Spa}(S, S^+)$ ([Sch17, Lemma 7.6]). $\qquad\square$

Proposition 2.25. *With notation as in Lemma 2.24 the map $\mathrm{Sht}^{\mathscr{G}_b}_{W(k)} \to \mathrm{Sht}^{\mathscr{G}'_b}_{W(k)}$ is a closed immersion. Now, using diagonal embeddings, it follows that $\mathrm{Sht}^{\mathscr{G}_b}_{W(k)} \to \mathrm{Spd}(W(k))$ is separated.*

Proof. For injectivity let $t_i = (\mathcal{T}_i, \Phi_i, \lambda_i) \in \mathrm{Sht}_{W(k)}^{\mathscr{G}_b}(R, R^+)$ with $i \in \{1, 2\}$ such that $t_i \overset{\mathscr{G}_1}{\times} \mathscr{G}_2 :=$ $(\mathcal{T}_i \overset{\mathscr{G}_1}{\times} \mathscr{G}_2, \Phi_i, \lambda_i)$ are isomorphic. We can assume $\mathrm{Spa}(R, R^+)$ to be a product of points, in this case $\mathrm{Spa}(R, R^+) \to \mathrm{Sht}_{W(k)}^{\mathscr{G}_b}$ the t_i's lift to $\mathrm{WSht}_{W(k)}^{\mathscr{G}_b}$, say given by $T_i \in \mathrm{WSht}_{W(k)}^{\mathscr{G}_b}(R, R^+)$ with $T_i := (M_i, \lambda_i)$. Since $t_1 \overset{\mathscr{G}_1}{\times} \mathscr{G}_2 \cong t_2 \overset{\mathscr{G}_1}{\times} \mathscr{G}_2$ then $T_1 \overset{\mathscr{G}_1}{\times} \mathscr{G}_2$ and $T_2 \overset{\mathscr{G}_1}{\times} \mathscr{G}_2$ are in the same $\mathscr{G}(W(R^+))$-orbit. Now, $\lambda_i \in \mathscr{G}_1(B_{[r,\infty)}^{R^+})$ so $\lambda_1 \circ \lambda_2^{-1} \in \mathscr{G}_1(B_{[r,\infty)}^{R^+}) \cap \mathscr{G}_2(W(R^+))$, this intersection is $\mathscr{G}_1(W(R^+))$. This and Lemma 2.24 proves that T_1 and T_2 are in the same $\mathbb{W}^+\mathscr{G}_1$-orbit, which proves $t_1 = t_2$.

Let us prove $\mathrm{Sht}_{W(k)}^{\mathscr{G}_b} \to \mathrm{Sht}_{W(k)}^{\mathscr{G}_b'}$ is proper. By injectivity the map is a separated, and since each of them satisfies the valuative criterion of partial properness over $\mathrm{Spd}(W(k))$, it is also partially proper. The only thing left to prove is quasi-compactness. Now, the composition $\mathrm{WSht}_{W(k)}^{\mathscr{G}_b} \to \mathrm{WSht}_{W(k)}^{\mathscr{G}_b'} \to \mathrm{Sht}_{W(k)}^{\mathscr{G}_b'}$ is a quasi-compact map, and $\mathrm{WSht}_{W(k)}^{\mathscr{G}_b} \to \mathrm{Sht}_{W(k)}^{\mathscr{G}_b}$ is surjective. It follows that $\mathrm{Sht}_{W(k)}^{\mathscr{G}_b} \to \mathrm{Sht}_{W(k)}^{\mathscr{G}_b'}$ is quasi-compact. $\qquad\square$

As with p-adic Beilinson–Drinfeld Grassmannians, moduli spaces of shtukas admit bounded versions. Given a geometric point $x : \mathrm{Spa}(C, C^+) \to \mathrm{Sht}_{W(k)}^{\mathscr{G}_b}$ the torsor $\mathcal{T}$ is trivial and we can choose a trivialization of $\tau : \mathcal{T} \to \mathscr{G}$. The morphism $\tau \circ \Phi : \varphi^{\mathrm{op},*}\mathcal{T} \to \mathscr{G}$ defines a map $y_{\tau,x} : \mathrm{Spa}(C, C^+) \to \mathrm{Gr}_{W(k)}^{\mathscr{G}}$ and we say x has relative position bounded by μ if $y_{\tau,x}$ factors through $\mathcal{M}_{O_{E_0}}^{\mathscr{G}, \leq \mu}$. Since $\mathcal{M}_{O_{E_0}}^{\mathscr{G}, \leq \mu}$ is stable under the action of the loop group this condition doesn't depend on the choice of τ.

Definition 2.26. We let $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu} \subseteq \mathrm{Sht}_{W(k)}^{\mathscr{G}_b} \times_{W(k)} O_{E_0}$ denote the subfunctor of tuples $\{(R^\sharp, \iota, f), \mathcal{T}, \Phi, \lambda\}$ for which the shtuka $(\mathcal{T}, \Phi)$ is point-wise bounded by μ.

Remark 2.27. Whenever $\mathscr{G}$ is reductive over $\mathbb{Z}_p$, $E(\mu)$ is an unramified extension of $\mathbb{Q}_p$, so $O_{E_0} = W(k)$.

Proposition 2.28. *If $\mu \in X_*^+(T_{\overline{\mathbb{Q}}_p})$ then $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu} \to \mathrm{Sht}_{W(k)}^{\mathscr{G}_b} \times_{W(k)} O_{E_0}$ is a closed immersion. Moreover, $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ is v-formalizing.*

Proof. Let $\mathrm{WSht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ denote the basechange of $\mathrm{WSht}_{O_{E_0}}^{\mathscr{G}_b} \to \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b}$ by $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$. We have a pair of Cartesian diagrams:

$$
\begin{array}{ccc}
\mathrm{WSht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu} \longrightarrow \mathrm{WSht}_{O_{E_0}}^{\mathscr{G}_b} \qquad & \mathrm{WSht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu} \longrightarrow \mathrm{WSht}_{O_{E_0}}^{\mathscr{G}_b} \\
\downarrow \qquad\qquad \downarrow \qquad & \downarrow \qquad\qquad \downarrow \\
\mathcal{M}_{O_{E_0}}^{\mathscr{G}, \leq \mu} \longrightarrow \mathrm{Gr}_{O_{E_0}}^{\mathscr{G}} \qquad & \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu} \longrightarrow \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b}
\end{array}
$$

Since being a closed immersion can be checked v-locally on the target ([Sch17, Proposition 10.11]), and since $\mathrm{WSht}_{O_{E_0}}^{\mathscr{G}_b} \to \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b}$ is surjective $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu} \to \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b}$ is a closed immersion. Moreover, by Proposition 2.18 and Theorem 2.19 the map $\mathcal{M}_{O_{E_0}}^{\mathscr{G}, \leq \mu} \to \mathrm{Gr}_{O_{E_0}}^{\mathscr{G}}$ is formally adic which implies that $\mathrm{WSht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ is formalizing and consequently that $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ is v-formalizing. $\qquad\square$

Let us prove that $(\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu})^{\mathrm{red}}$ is represented by an affine Deligne–Lusztig variety as in [HZ20]. We recall one of the ways to define these spaces.

Definition 2.29. Let $\mathscr{G}_b$ the isocrystal with $\mathscr{G}$-structure associated to b and μ a cocharacter. Let $X_{\mathscr{G}}^{\leq \mu}(b) : \mathrm{PCAlg}_{/k}^{\mathrm{op}} \to \mathrm{Sets}$ be the functor on perfect k-algebras R given by $X_{\mathscr{G}}^{\leq \mu}(b)(R) = \{(\mathcal{T}, \Phi, \lambda)\}/_{\cong}$ with $\mathcal{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R))$, $\Phi : \varphi^{\mathrm{op},*}\mathcal{T} \to \mathcal{T}$ is an isomorphism over $\mathrm{Spec}(W(R)[\frac{1}{p}])$ that is point-wise bounded by μ (i.e. lying over $\mathcal{A}_{\mathscr{G}, \mu}$) and $\lambda : \mathcal{T} \to \mathscr{G}_b$ is a φ-equivariant isomorphism over $\mathrm{Spec}(W(R)[\frac{1}{p}])$

Proposition 2.30. *The identity $X_{\mathscr{G}}^{\leq \mu}(b) = (\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu})^{\mathrm{red}}$ holds, the adjunction map $(X_{\mathscr{G}}^{\leq \mu}(b))^{\diamond} \to \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ is injective. Moreover, $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ is a specializing v-sheaf.*

Proof. To construct $j : X_{\mathscr{G}}^{\leq \mu}(b) \to (\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu})^{\mathrm{red}}$, by adjunction we construct $h : (X_{\mathscr{G}}^{\leq \mu}(b))^{\diamond} \to \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ instead. Before sheafification, a map $\mathrm{Spa}(S, S^+) \to (X_{\mathscr{G}}^{\leq \mu}(b))^{\diamond}$ is given by data $(\mathcal{T}, \Phi, \lambda)$ over S^+ as

in Definition 2.29. Restricting to the appropriate loci defines a map $\mathrm{Spa}(S, S^+) \to \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$. Since $X_{\mathscr{G}}^{\leq \mu}(b)$ is representable, $((X_{\mathscr{G}}^{\leq \mu}(b))^\diamond)^{\mathrm{red}} = X_{\mathscr{G}}^{\leq \mu}(b)$, and to prove r is injective it suffices to prove that h is. Take maps $g_1, g_2 : \mathrm{Spa}(R, R^+) \to (X_{\mathscr{G}}^{\leq \mu}(b))^\diamond$, injectivity can be proved v-locally so we assume the maps factor through $g_1', g_2' : \mathrm{Spec}(R^+)^\diamond \to (X_{\mathscr{G}}^{\leq \mu}(b))^\diamond$. The g_i' are given $(\mathcal{T}_i, \Phi_i, \lambda_i)$ over $\mathrm{Spec}(W(R^+))$, $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$ and $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$ and the $h \circ g_i$ are given by restriction to $\mathcal{Y}_{R^+}^{[0,\infty)}$, $\mathcal{Y}_{R^+}^{[0,\infty)} \setminus V(p)$ and $\mathcal{Y}_{R^+}^{[r,\infty)}$ respectively. Nevertheless, from $h \circ g_1 = h \circ g_2$ and Theorem 2.5 it follows that $g_1' = g_2'$.

To prove surjectivity of j, let $f : \mathrm{Spec}(A)^\diamond \to \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ and $g : \mathrm{Spa}(R, R^+) \to \mathrm{Spec}(A)^\diamond$ with $\mathrm{Spa}(R, R^+)$ a product of points and $A \in \mathrm{PCAlg}_{/k}^{\mathrm{op}}$. We construct below a unique f' fitting in:

$$\begin{array}{ccc} \mathrm{Spa}(R, R^+) & \xrightarrow{\ g\ } & \mathrm{Spec}(A)^\diamond \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (X_{\mathscr{G}}^{\leq \mu}(b))^\diamond & \xrightarrow{\ j\ } & \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu} \end{array}$$

Since products of points are a basis for the topology, and since j is injective the diagram gives $\mathrm{Spec}(A)^\diamond \to (X_{\mathscr{G}}^{\leq \mu}(b))^\diamond$ factoring our original map to $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ and proves the desired surjectivity.

Fix a pseudo-uniformizer $\varpi \in R^+$, we let $\mathrm{Spa}(R_\infty, R_\infty^+)$ be a second product of points defined by $R_\infty^+ = \prod_{i=1}^\infty R^+$ with pseudo-uniformizer $\varpi_{R_\infty} = (\varpi^i)_{i=1}^\infty$. This product of points comes with a family of closed embeddings $\iota_i : \mathrm{Spa}(R, R^+) \to \mathrm{Spa}(R_\infty, R_\infty^+)$ given in coordinates by the projections onto the ith-factor. The diagonal $\Delta_g : A \to \prod_{i=1}^\infty R^+$ induces $\Delta_g : \mathrm{Spa}(R_\infty, R_\infty^+) \to \mathrm{Spec}(A)^\diamond$ with $\Delta_g \circ \iota_i = g$ for every i. Since $\mathrm{Spa}(R_\infty, R_\infty^+)$ is a product of points, by Theorem 2.8, the map $f \circ \Delta_g$ can be represented by a triple $(\mathcal{T}_{R_\infty}, \Phi_{R_\infty}, \lambda_{R_\infty})$ with $\mathcal{T}_{R_\infty}$ trivial. After choosing a trivialization λ_{R_∞} is given by a map $\mathcal{O}_{\mathscr{G}} \to B_{R_\infty^+}^{[r,\infty]}$. Moreover, since $f \circ \Delta_g \circ \iota_i = f \circ \Delta_g \circ \iota_j$ for all i and j, the $\lambda_i^* : \mathcal{O}_{\mathscr{G}} \to B_{R_\infty^+}^{[r,\infty]} \xrightarrow{\iota_i} B_{R^+}^{[r_i,\infty]}$ lie in the same $\mathscr{G}(W(R^+))$-orbit. Clearly, $\mathscr{G}(W(R_\infty^+)) = \prod_{i=1}^\infty \mathscr{G}(W(R^+))$. By changing the trivialization, we may assume that $r_i = r_j =: r$ and $\lambda_i^* = \lambda_j^* =: \lambda_R^*$ for all $1 \leq i, j < \infty$. We claim that λ_R^* factors through $W(R^+)[\frac{1}{p}]$. Take $t \in \mathcal{O}_{\mathscr{G}}$ and consider $s = \lambda_{R_\infty}^*(t) \in B_{R_\infty^+}^{[r,\infty]}$. After replacing r, we may assume $r = n \in \mathbb{N}$. In particular, $p^k \cdot s$ lies in the p-adic completion of $W(R_\infty^+)[\frac{[\varpi_{R_\infty}]}{p^n}]$ for some k. Write $p^k \cdot s$ as $\sum_{j=0}^\infty x^{n(j)}[\alpha_j]p^j$ with $x = \frac{[\varpi_{R_\infty}]}{p^n}$, $0 \leq n(j)$ a multiplicity, and $\alpha_j \in R_\infty^+$. Then, $\iota_i(p^k \cdot s) = \sum_{j=0}^\infty (\frac{[\varpi]^i}{p^n})^{n(j)}[\iota_i(\alpha_j)]p^j$ with $\iota_i(\alpha_j) \in R^+$. In particular, $p^k \cdot \lambda_R^*(t) \in \bigcap_{i \in \mathbb{N}}(H^0(\mathcal{Y}_{R^+}^{[\frac{n}{i},\infty]}, \mathcal{O}^+))$, but this intersection is $W(R^+)$ proving the claim. Since the triple $(\mathcal{T}_{R_\infty}, \Phi_{R_\infty}, \lambda_{R_\infty})$ is defined over $\mathrm{Spec}(W(R^+))$ and $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$ it defines a map $\mathrm{Spec}(R^+) \to X_{\mathscr{G}}^{\leq \mu}(b)$. The composition, $\mathrm{Spa}(R, R^+) \to \mathrm{Spec}(R^+)^\diamond \to (X_{\mathscr{G}}^{\leq \mu}(b))^\diamond$, defines f'.

To prove $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ is specializing we need to prove it is formally separated and v-formalizing. The first follows from [Gle22, Lemma 3.30] and Proposition 2.25, the second follows from Proposition 2.28. $\qquad\square$

Lemma 2.31. *The adjunction map* $(X_{\mathscr{G}}^{\leq \mu}(b))^\diamond \to \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ *arising from the identification of Proposition 2.30 is a closed immersion. In particular,* $(\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}, \mathrm{Sht}_{E_0}^{\mathscr{G}_b, \leq \mu})$ *is a smelted kimberlite.*

Proof. Recall that $X_{\mathscr{G}}^{\leq \mu}(b)$ admits a closed immersion into $\mathcal{F}l_{\mathcal{W}, k}^{\mathscr{G}}$. We write $(X_{\mathscr{G}}^{\leq \mu}(b))^\diamond = \bigcup_{\nu \in \tilde{W}}(X_{\mathscr{G}}^{\leq \mu}(b) \cap \mathcal{F}l_{\mathcal{W}, k}^{\mathscr{G}, \leq \nu})^\diamond$ where $\tilde{W}$ is the Iwahori-Weyl group. Each term in the union is proper over $\mathrm{Spd}(k)$, since they come from a proper perfectly finitely presented schemes over k. Consequently, $(X_{\mathscr{G}}^{\leq \mu}(b) \cap \mathcal{F}l_{\mathcal{W}, k}^{\mathscr{G}, \leq \nu})^\diamond \to \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ is a closed immersion. Now, $X_{\mathscr{G}}^{\leq \mu}(b)$ is locally perfectly of finite type ([HV19, Theorem 1.1]). In particular, each point admits an open neighborhood that is spectral and Noetherian. For all $x \in |X_{\mathscr{G}}^{\leq \mu}(b)|$, there is an open $x \in U_x \subseteq X_{\mathscr{G}}^{\leq \mu}(b)$ and a finite set $I_x \subseteq \tilde{W}$ with $U = (\bigcup_{\nu \in I_x} U \cap \mathcal{F}l_{\mathcal{W}, k}^{\mathscr{G}, \leq \nu})$. Indeed, if U_x is Noetherian every Zariski closed subset is constructible, and we conclude by compactness.

By Proposition 2.30 and [Gle22, Proposition 4.14] we have a specialization map $\mathrm{sp}_{\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}} : |\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}| \to |X_{\mathscr{G}}^{\leq \mu}(b)|$. Let $V_x = (\mathrm{sp}_{\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}})^{-1}(U_x)$ for $x \in |X_{\mathscr{G}}^{\leq \mu}(b)|$ and U_x as above, this forms an open cover of $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$. Being a closed immersion is v-local on the target. Since $V_x \to \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b, \leq \mu}$ is a formally adic, it suffices to see that $(V_x^{\mathrm{red}})^\diamond \to V_x$ is a closed immersion. Now, the adjunction map $(U_x)^\diamond \to V_x$ fits in:

$$\begin{array}{ccccc}
U_x^\diamond & \xrightarrow{\quad\mathrm{Id}\quad} & U_x^\diamond & \longrightarrow & V_x \\
\downarrow & & \downarrow & & \downarrow \\
(\bigcup_{\nu\in I_x}\mathcal{F}l_{\mathcal{W},\overline{k}}^{\mathscr{G};\le\nu}\cap X_{\overline{\mathscr{G}}}^{\le\mu}(b))^\diamond & \longrightarrow & (X_{\overline{\mathscr{G}}}^{\le\mu}(b))^\diamond & \longrightarrow & \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b,\le\mu}
\end{array}$$

Moreover, the above diagram is Cartesian, which proves $U_x^\diamond \to V_x$ is a closed immersion.

Proposition 2.30 proves that $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b,\le\mu}$ is a specializing v-sheaf, in Proposition 2.30 we proved that $(\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b,\le\mu})^{\mathrm{red}}$ is represented by a scheme and by the argument above the map $((\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b,\le\mu})^{\mathrm{red}})^\diamond \to \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b,\le\mu}$ is a closed immersion, this finishes the proof that $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b,\le\mu}$ is a prekimberlite as in [Gle22, Definition 4.15]. Since $\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b,\le\mu} \to \mathrm{Spd}(O_{E_0})$ is partially proper, by [Gle22, Proposition 4.32, Definition 4.30] it is a valuative prekimberlite. Finally, by [SW20, Theorem 23.1.4] $\mathrm{Sht}_{E_0}^{\mathscr{G}_b,\le\mu}$ is a locally spatial diamond. Consequently, $(\mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b,\le\mu}, \mathrm{Sht}_{E_0}^{\mathscr{G}_b,\le\mu})$ is a smelted kimberlite. $\qquad\square$
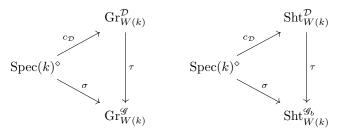
2.4. **Tubular neighborhoods and their local model diagram.** Let $\mathcal{D} = (\mathcal{D}, \Phi_{\mathcal{D}})$ with $\mathcal{D}$ a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(k))$ and $\Phi_{\mathcal{D}} : \varphi^{\mathrm{op},*}\mathcal{D} \to \mathcal{D}$ an isomorphism over $\mathrm{Spec}(W(k)[\frac{1}{p}])$, and fix $\mu \in X_*^+(T)$. We can define "coordinate-free" versions of some moduli spaces that we studied in the previous sections:

Definition 2.32. We denote functors

$$\mathrm{Gr}_{W(k)}^{\mathcal{D}}, \mathrm{Sht}_{W(k)}^{\mathcal{D}}, \mathcal{M}_{O_{E_0}}^{\mathcal{D},\le\mu}, \mathrm{Sht}_{O_{E_0}}^{\mathcal{D},\le\mu} : \mathrm{Perf}_k \to \mathrm{Sets}$$
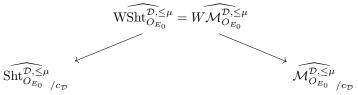
(1) With $\mathrm{Gr}_{W(k)}^{\mathcal{D}}(R,R^+) = \{((R^\sharp,\iota,f),\mathcal{T},\psi)\}_{/\cong}$ where $\mathcal{T}$ is a $\mathscr{G}$-torsor over $\mathcal{Y}_{R^+}$ and $\psi : \mathcal{T} \to \mathcal{D}$ is an isomorphism defined over $\mathcal{Y}_{R^+} \setminus V(\xi_{R^\sharp})$ that is meromorphic along $\xi_{R^\sharp}$.

(2) With $\mathrm{Sht}_{W(k)}^{\mathcal{D}}(R,R^+) = \{((R^\sharp,\iota,f),\mathcal{T},\Phi,\lambda)\}_{/\cong}$ where $(\mathcal{T},\Phi)$ is a shtuka with $\mathscr{G}$-structure, and $\lambda : \mathcal{T} \to \mathcal{D}$ is an isogeny.

(3) With $\mathcal{M}_{O_{E_0}}^{\mathcal{D},\le\mu}$ and $\mathrm{Sht}_{O_{E_0}}^{\mathcal{D},\le\mu}$ denoting the evident bounded versions.

The functors $\mathrm{Gr}_{W(k)}^{\mathcal{D}}$ and $\mathrm{Sht}_{W(k)}^{\mathcal{D}}$ come with canonical sections $c_{\mathcal{D}} : \mathrm{Spec}(k)^\diamond \to \mathrm{Gr}_{W(k)}^{\mathcal{D}}$ and $c_{\mathcal{D}} : \mathrm{Spec}(k)^\diamond \to \mathrm{Sht}_{W(k)}^{\mathcal{D}}$ given by $(\varphi^{\mathrm{op},*}\mathcal{D}, \Phi_{\mathcal{D}})$ and $(\mathcal{D}, \Phi_{\mathcal{D}}, \mathrm{Id})$ respectively. Fixing an isomorphism $\tau : \mathcal{D} \cong \mathscr{G}$ we get isomorphisms $\tau : \mathrm{Gr}_{W(k)}^{\mathcal{D}} \cong \mathrm{Gr}_{W(k)}^{\mathscr{G}}$, and $\tau : \mathcal{M}_{O_{E_0}}^{\mathcal{D},\le\mu} \cong \mathcal{M}_{O_{E_0}}^{\mathscr{G},\le\mu}$. Analogously, given a φ-equivariant isomorphism $\tau : \mathcal{D} \cong \mathscr{G}_b$ over $\mathrm{Spec}(W(k)[\frac{1}{p}])$ we get isomorphisms $\tau : \mathrm{Sht}_{W(k)}^{\mathcal{D}} \cong \mathrm{Sht}_{W(k)}^{\mathscr{G}_b}$ and $\tau : \mathrm{Sht}_{O_{E_0}}^{\mathcal{D},\le\mu} \cong \mathrm{Sht}_{O_{E_0}}^{\mathscr{G}_b,\le\mu}$. Moreover, if we are given a section $\sigma : \mathrm{Spec}(k)^\diamond \to \mathrm{Gr}_{W(k)}^{\mathscr{G}}$ (respectively $\sigma : \mathrm{Spec}(k)^\diamond \to \mathrm{Sht}_{W(k)}^{\mathscr{G}_b}$) we can find $(\mathcal{D}, \Phi_{\mathcal{D}})$ and an isomorphism $\tau : \mathcal{D} \cong \mathscr{G}$ (respectively φ-isomorphism) making the diagrams below commutative:

$$\begin{array}{ccc}
 & \mathrm{Gr}_{W(k)}^{\mathcal{D}} & \\
{\scriptstyle c_{\mathcal{D}}}\nearrow & & \downarrow{\scriptstyle \tau} \\
\mathrm{Spec}(k)^\diamond & & \\
{\scriptstyle \sigma}\searrow & & \\
 & \mathrm{Gr}_{W(k)}^{\mathscr{G}} &
\end{array}
\qquad
\begin{array}{ccc}
 & \mathrm{Sht}_{W(k)}^{\mathcal{D}} & \\
{\scriptstyle c_{\mathcal{D}}}\nearrow & & \downarrow{\scriptstyle \tau} \\
\mathrm{Spec}(k)^\diamond & & \\
{\scriptstyle \sigma}\searrow & & \\
 & \mathrm{Sht}_{W(k)}^{\mathscr{G}_b} &
\end{array}$$

Since k is algebraically closed every tubular neighborhood of the Beilinson–Drinfeld Grassmannian and of moduli spaces of p-adic shtukas at closed points are isomorphic to the canonical one associated to some pair $(\mathcal{D}, \Phi_{\mathcal{D}})$.

Theorem 2.33. *Given $(\mathcal{D}, \Phi_{\mathcal{D}})$ and $\mu \in X_*(T_{\overline{\mathbb{Q}}_p})$ as above, and with notation as below we have a local model diagram:*

$$\begin{array}{ccc}
 & \widehat{\mathrm{WSht}_{O_{E_0}}^{\mathcal{D},\le\mu}} = \widehat{W\mathcal{M}_{O_{E_0}}^{\mathcal{D},\le\mu}} & \\
\swarrow & & \searrow \\
\widehat{\mathrm{Sht}_{O_{E_0}}^{\mathcal{D},\le\mu}}{}_{/c_{\mathcal{D}}} & & \widehat{\mathcal{M}_{O_{E_0}}^{\mathcal{D},\le\mu}}{}_{/c_{\mathcal{D}}}
\end{array}$$

Moreover, both arrows are $\widehat{L_W^+\mathscr{G}}_{\mathcal{D}}$-torsors. In particular, $(\mathrm{Sht}_{O_{E_0}}^{\mathcal{D},\leq\mu})^{\circledcirc}_{/c_{\mathcal{D}}}$ is non-empty if and only if $(\mathcal{M}_{O_{E_0}}^{\mathcal{D},\leq\mu})^{\circledcirc}_{/c_{\mathcal{D}}}$ is and we have a canonical bijection

$$\pi_0((\mathrm{Sht}_{O_{E_0}}^{\mathcal{D},\leq\mu})^{\circledcirc}_{/c_{\mathcal{D}}}) \cong \pi_0((\mathcal{M}_{O_{E_0}}^{\mathcal{D},\leq\mu})^{\circledcirc}_{/c_{\mathcal{D}}}).$$

Before proving the theorem we need some preparation.

Definition 2.34. We denote functors

$$\widehat{L_W^+\mathscr{G}}_{\mathcal{D}}, \widehat{W\mathrm{Gr}}_{\mathcal{D}}, \widehat{W\mathcal{M}}_{O_{E_0}}^{\mathcal{D},\leq\mu}, \widehat{W\mathrm{Sht}}_{\mathcal{D}}, \widehat{W\mathrm{Sht}}_{O_{E_0}}^{\mathcal{D},\leq\mu} : \mathrm{Perf}_{W(k)} \to \mathrm{Sets}^3.$$

(1) With $\widehat{L_W^+\mathscr{G}}_{\mathcal{D}}(R,R^+) = \{((R^\sharp,\iota,f),g)\}$ where $g : \mathcal{D} \to \mathcal{D}$ is an automorphism over $\mathrm{Spec}(W(R^+))$ for which there is a pseudo-uniformizer $\varpi_g \in R^+$, depending of g, with $g = \mathrm{Id}$ over $\mathrm{Spec}(W(R^+)/[\varpi_g])$. We define $\widehat{L_W^+\mathscr{G}}_{\varphi^{\mathrm{op},*}\mathcal{D}}$ exchanging $\mathcal{D}$ for $\varphi^{\mathrm{op},*}\mathcal{D}$.

(2) With $\widehat{W\mathrm{Gr}}_{\mathcal{D}}(R,R^+) = \{(R^\sharp,\iota,f),\mathcal{T},\psi,\sigma\}_{/\cong}$ where $\mathcal{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$, $\psi : \mathcal{T} \to \mathcal{D}$ is an isomorphism over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$ and $\sigma : \mathcal{T} \to \varphi^{\mathrm{op},*}\mathcal{D}$ is an isomorphism over $\mathrm{Spec}(W(R^+))$ such that there is a pseudo-uniformizer $\varpi \in R^+$ depending on the data for which $\Phi_{\mathcal{D}} \circ \sigma = \psi$ when restricted to $\mathrm{Spec}(W(R^+)/[\varpi])$.

(3) With $\widehat{W\mathrm{Sht}}_{\mathcal{D}}(R,R^+) = \{(R^\sharp,\iota,f),\mathcal{T},\Phi,\lambda,\sigma\}_{/\cong}$ where $\mathcal{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$, $\Phi : \varphi^{\mathrm{op},*}\mathcal{T} \to \mathcal{T}$ is an isomorphism over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$, $\lambda : \mathcal{T} \to \mathcal{D}$ is an isogeny over $\mathcal{Y}_{R^+}^{[r,\infty]}$ and $\sigma : \mathcal{T} \to \mathcal{D}$ is an isomorphism over $\mathrm{Spec}(W(R^+))$ such that there is a pseudo-uniformizer $\varpi \in R^+$ depending on the data for which $\sigma = \lambda$ when restricted to $\mathrm{Spec}(B_{R^+}^{[r,\infty]}/[\varpi])$.

(4) Adding a boundedness conditions on ψ or Φ we obtain $\widehat{W\mathcal{M}}_{O_{E_0}}^{\mathcal{D},\leq\mu}$ or $\widehat{W\mathrm{Sht}}_{O_{E_0}}^{\mathcal{D},\leq\mu}$ respectively.

Standard arguments using Proposition 2.9 will prove that the objects in Definition 2.34 are v-sheaves. There are natural maps $\widehat{W\mathrm{Gr}}_{\mathcal{D}} \to \mathrm{Gr}_{W(k)}^{\mathcal{D}}$ (respectively $\pi : \widehat{W\mathrm{Sht}}_{\mathcal{D}} \to \mathrm{Sht}_{W(k)}^{\mathcal{D}}$) that take a tuple $(\mathcal{T},\psi,\sigma)$ (respectively $(\mathcal{T},\Phi,\lambda,\sigma)$) and assign $(\mathcal{T},\psi)$ (respectively $(\mathcal{T},\Phi,\lambda)$) restricted to the appropriate loci. These maps are $\widehat{L_W^+\mathscr{G}}_{\varphi^{\mathrm{op},*}\mathcal{D}}$-equivariant (respectively $\widehat{L_W^+\mathscr{G}}_{\mathcal{D}}$-equivariant) when we endow $\widehat{W\mathrm{Gr}}_{\mathcal{D}}$ (respectively $\mathrm{Sht}_{W(k)}^{\mathcal{D}}$) with the action sending $(g,(\mathcal{T},\psi,\sigma))$ to $(\mathcal{T},\psi,g \circ \sigma)$ (respectively $(g,(\mathcal{T},\Phi,\lambda,\sigma))$ to $(\mathcal{T},\Phi,\lambda,g \circ \sigma)$) and $\mathrm{Gr}_{W(k)}^{\mathcal{D}}$ (respectively $\mathrm{Sht}_{W(k)}^{\mathcal{D}}$) are given the trivial action.

Lemma 2.35. *The map* $\widehat{W\mathrm{Gr}}_{\mathcal{D}} \to \mathrm{Gr}_{W(k)}^{\mathcal{D}}$ *factors surjectively onto* $\widehat{\mathrm{Gr}_{W(k)}^{\mathcal{D}}}_{/c_{\mathcal{D}}}$. *Moreover,* $\widehat{W\mathrm{Gr}}_{\mathcal{D}} \to \widehat{\mathrm{Gr}_{W(k)}^{\mathcal{D}}}_{/c_{\mathcal{D}}}$ *is a* $\widehat{L_W^+\mathscr{G}}_{\varphi^{\mathrm{op},*}\mathcal{D}}$*-torsor.*

Proof. Observe that $\widehat{W\mathrm{Gr}}_{\mathcal{D}}$ formalizes points valued on affinoid perfectoid. Indeed, for $(\mathcal{T},\psi,\sigma) \in \widehat{W\mathrm{Gr}}_{\mathcal{D}}(A,A^+)$ and a map $f : \mathrm{Spa}(B,B^+) \to \mathrm{Spd}(A^\sharp)$, we form $(f^*\mathcal{T},f^*\psi,f^*\sigma)$ with $f^*\mathcal{T}$ defined over $\mathrm{Spec}(W(B^+))$, $f^*\psi : f^*\mathcal{T} \to \mathcal{D}$ defined over $\mathrm{Spec}(W(B^+)[\frac{1}{f(\xi)}])$ and $f^*\sigma : f^*\mathcal{T} \to f^*\varphi^{\mathrm{op},*}\mathcal{D}$ defined over $\mathrm{Spec}(W(B^+))$. To prove this triple satisfies the constraints, take $\varpi_A \in A^+$ with $\Phi_{\mathcal{D}} \circ \sigma = \psi$ in $\mathrm{Spec}(W(A^+)/[\varpi_A])$. Then $f(\varpi_A)$ is topologically nilpotent and any pseudo-uniformizer $\varpi_B \in B^+$ dividing $f(\varpi_A)$ will satisfy $\Phi_{\mathcal{D}} \circ f^*\sigma = f^*\psi$ over $\mathrm{Spec}(W(B^+)/[\varpi_B])$.

To prove that $\widehat{W\mathrm{Gr}}_{\mathcal{D}} \to \mathrm{Gr}_{W(k)}^{\mathcal{D}}$ factors through $\widehat{\mathrm{Gr}_{W(k)}^{\mathcal{D}}}_{/c_{\mathcal{D}}}$ it suffices to show that for $\mathrm{Spd}(R^+) \to \widehat{W\mathrm{Gr}}_{\mathcal{D}}$ the reduction $(\mathrm{Spd}(R^+))^{\mathrm{red}} \to (\mathrm{Gr}_{W(k)}^{\mathcal{D}})^{\mathrm{red}}$ factors through $c_{\mathcal{D}} : \mathrm{Spec}(k)^\diamond \to (\mathrm{Gr}_{W(k)}^{\mathcal{D}})^{\mathrm{red}}$. Let $R_{\mathrm{red}}^+ = (R^+/\varpi)^{\mathrm{perf}}$. After restricting $(\mathcal{T},\psi,\sigma)$ to $\mathrm{Spec}(W(R_{\mathrm{red}}^+))$ we get $\Phi_{\mathcal{D}} \circ \sigma = \psi$, and $\mathrm{Spec}(R_{\mathrm{red}}^+)^\diamond \to \mathrm{Gr}_{W(k)}^{\mathcal{D}}$ is given by $(\mathcal{T},\psi)$. Now, σ is a witness that $(\mathcal{T},\psi) \cong (\varphi^{\mathrm{op},*}\mathcal{D},\Phi_{\mathcal{D}})$ in this locus, so the map factors through $c_{\mathcal{D}} : \mathrm{Spec}(k)^\diamond \to \mathrm{Gr}_{W(k)}^{\mathcal{D}}$.

To prove $\widehat{W\mathrm{Gr}}_{\mathcal{D}} \to \widehat{\mathrm{Gr}_{W(k)}^{\mathcal{D}}}_{/c_{\mathcal{D}}}$ is surjective it suffices to lift maps valued on product of points. In this case, by Theorem 2.8, such a (R,R^+)-valued point is given by $(\mathcal{T},\psi)$ with $\mathcal{T}$ defined over $\mathrm{Spec}(W(R^+))$ and $\psi : \mathcal{T} \to \mathcal{D}$ defined over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$, with $(\mathcal{T},\psi)$ isomorphic to $(\varphi^{\mathrm{op},*}\mathcal{D},\Phi_{\mathcal{D}})$ when restricted to $W(R_{\mathrm{red}}^+)$ and $W(R_{\mathrm{red}}^+)[\frac{1}{p}]$. Such an isomorphism $\sigma_{\mathrm{red}} : (\mathcal{T},\psi) \to (\varphi^{\mathrm{op},*}\mathcal{D},\Phi_{\mathcal{D}})$ is unique and given by $\sigma_{\mathrm{red}} = \Phi_{\mathcal{D}}^{-1} \circ \psi$. We define $\tilde{\sigma} := \Phi_{\mathcal{D}}^{-1} \circ \psi : \mathcal{T} \to \varphi^{\mathrm{op},*}D$ over $\mathcal{Y}_{R^+}^{[r,\infty]}$ for r sufficiently big (avoiding

$V(\xi)$), clearly $\widetilde{\sigma}$ restricts to σ_{red}. Using Lemma 2.14 we find $\sigma : \mathcal{T} \to \varphi^{\mathrm{op},*}D$ such that $\sigma = \widetilde{\sigma}$ when restricted to $\mathrm{Spec}(B_{R^+}^{[r,\infty]}/[\varpi'])$ for some pseudo-uniformizer $\varpi' \in R^+$. In particular $\Phi_{\mathcal{D}} \circ \sigma = \psi$ over $\mathrm{Spec}(W(R^+)/[\varpi'])$. The data $(\mathcal{T}, \psi, \sigma)$ defines $\mathrm{Spa}(R, R^+) \to \widehat{W\mathrm{Gr}}_{\mathcal{D}}$ lifting $\mathrm{Spa}(R, R^+) \to \widehat{\mathrm{Gr}^{\mathcal{D}}_{W(k)}}_{/c_{\mathcal{D}}}$.

To prove $\widehat{W\mathrm{Gr}}_{\mathcal{D}} \times_{\mathrm{Gr}^{\mathcal{D}}_{W(k)}} \widehat{W\mathrm{Gr}}_{\mathcal{D}} \cong \widehat{L_W^+ \mathscr{G}}_{\varphi^{\mathrm{op},*}\mathcal{D}} \times_{\mathrm{Spd}(W(k))} \widehat{W\mathrm{Gr}}_{\mathcal{D}}$, take two sets of data $(\mathcal{T}_i, \psi_i, \sigma_i)$ over $\mathrm{Spa}(A, A^+)$ with $(\mathcal{T}_1|_{\mathcal{Y}_{A^+}}, \psi_1|_{\mathcal{Y}_{A^+} \setminus V(\xi)}) \cong (\mathcal{T}_2|_{\mathcal{Y}_{A^+}}, \psi_2|_{\mathcal{Y}_{A^+} \setminus V(\xi)})$. The isomorphism must be given by $\psi_1^{-1} \circ \psi_2 : \mathcal{T}_2 \to \mathcal{T}_1$ and by the full-faithfulness of Theorem 2.5 it extends to $\mathrm{Spec}(W(A^+))$. Let $g = \sigma_1 \circ \psi_1^{-1} \circ \psi_2 \circ \sigma_2^{-1} : \varphi^{\mathrm{op},*}\mathcal{D} \to \varphi^{\mathrm{op},*}\mathcal{D}$. By hypothesis, $\sigma_i \circ \psi_i^{-1} = \Phi_{\mathcal{D}}^{-1}$ on $\mathrm{Spec}(W(A^+)/[\varpi_i])$ for suitable choices of $\varpi_i \in A^+$. Consequently, $(g, \mathcal{T}_2, \psi_2, \sigma_2) \in \widehat{L_W^+ \mathscr{G}}_{\varphi^{\mathrm{op},*}\mathcal{D}} \times_{\mathrm{Spd}(W(k))} \widehat{W\mathrm{Gr}}_{\mathcal{D}}(A, A^+)$, giving the left to right map. On the other hand, to $(g, \mathcal{T}, \psi, \sigma)$ we associate the pair of tuples $(\mathcal{T}, \psi, g \circ \sigma)$ and $(\mathcal{T}, \psi, \sigma)$. These constructions are inverse to each other. $\square$

Lemma 2.36. *The map* $\widehat{W\mathrm{Sht}}_{\mathcal{D}} \to \mathrm{Sht}^{\mathcal{D}}_{W(k)}$ *factors surjectively onto* $\widehat{\mathrm{Sht}^{\mathcal{D}}_{W(k)}}_{/c_{\mathcal{D}}}$. *Moreover,* $\widehat{W\mathrm{Sht}}_{\mathcal{D}} \to \widehat{\mathrm{Sht}^{\mathcal{D}}_{W(k)}}_{/c_{\mathcal{D}}}$ *is a* $\widehat{L_W^+ \mathscr{G}}_{\mathcal{D}}$-*torsor.*

Proof. Proving that $\widehat{W\mathrm{Sht}}_{\mathcal{D}} \to \mathrm{Sht}^{\mathcal{D}}_{W(k)}$ factors surjectively onto $\widehat{\mathrm{Sht}^{\mathcal{D}}_{W(k)}}_{/c_{\mathcal{D}}}$ follows closely the argument of Lemma 2.35, and we omit the details.

To prove that $\widehat{W\mathrm{Sht}}_{\mathcal{D}} \times_{\mathrm{Sht}^{\mathcal{D}}_{W(k)}} \widehat{W\mathrm{Sht}}_{\mathcal{D}} \cong \widehat{L_W^+ \mathscr{G}}_{\mathcal{D}} \times_{\mathrm{Spd}(W(k))} \widehat{W\mathrm{Sht}}_{\mathcal{D}}$, take two sets of data $(\mathcal{T}_i, \Phi_i, \lambda_i, \sigma_i)$ over $\mathrm{Spa}(A, A^+)$ with $(\mathcal{T}_1|_{\mathcal{Y}^{[0,\infty)}_{A^+}}, \Phi_1|_{\mathcal{Y}^{[0,\infty)}_{A^+} \setminus V(\xi)}, \lambda_1) \cong (\mathcal{T}_2|_{\mathcal{Y}^{[0,\infty)}_{A^+}}, \Phi_2|_{\mathcal{Y}^{[0,\infty)}_{A^+} \setminus V(\xi)}, \lambda_2)$. The isomorphism must be the unique lift of $\lambda_1^{-1} \circ \lambda_2 : \mathcal{T}_2 \to \mathcal{T}_1$ to $\mathcal{Y}^{[0,\infty)}_{A^+}$. Glueing along the λ_i and by the fully-faithfulness part of Theorem 2.5 $\lambda_1^{-1} \circ \lambda_2$ extends to $\mathrm{Spec}(W(A^+))$. Moreover, letting $g = \sigma_1 \circ \lambda_1^{-1} \circ \lambda_2 \circ \sigma_2^{-1} : \mathcal{D} \to \mathcal{D}$ we have $\sigma_1 \circ \lambda_1^{-1} = \mathrm{Id} = \lambda_2 \circ \sigma_2^{-1}$ over $\mathrm{Spec}(B_{A^+}^{[r,\infty]}/[\varpi_A])$ for suitable $\varpi_A \in A^+$. We associate to the original data the tuple $(g, \mathcal{T}_2, \Phi_2, \lambda_2, \sigma_2) \in \widehat{L_W^+ \mathscr{G}}_{\mathcal{D}} \times_{\mathrm{Spd}(W(k))} \widehat{W\mathrm{Sht}}_{\mathcal{D}}(A, A^+)$ giving the left to right map. One constructs the inverse using the action map. $\square$

We can now prove Theorem 2.33.

Proof. (of Theorem 2.33). Let $\phi = (\varphi^{\mathrm{op}})^{-1}$. Observe that $\theta : \widehat{L_W^+ \mathscr{G}}_{\mathcal{D}} \to \widehat{L_W^+ \mathscr{G}}_{\varphi^{\mathrm{op},*}\mathcal{D}}$ given by $g \mapsto \varphi^{\mathrm{op},*}g$ is an isomorphism with inverse $h \mapsto \phi^* h$. Using θ we can endow $\widehat{W\mathrm{Gr}}_{\mathcal{D}}$ with a $\widehat{L_W^+ \mathscr{G}}_{\mathcal{D}}$ action, and the projection $\pi : \widehat{W\mathrm{Gr}}_{\mathcal{D}} \to \mathrm{Gr}^{\mathcal{D}}_{W(k)}$ of Lemma 2.35 is a $\widehat{L_W^+ \mathscr{G}}_{\mathcal{D}}$-torsor.

We construct an isomorphism $\Theta : \widehat{W\mathrm{Gr}}_{\mathcal{D}} \to \widehat{W\mathrm{Sht}}_{\mathcal{D}}$, given on (A, A^+)-valued points by

$$(\mathcal{T}, \psi, \sigma) \mapsto (\phi^*\mathcal{T}, \Phi, \lambda, \phi^*\sigma).$$

Here $\Phi : \mathcal{T} \to \phi^*\mathcal{T}$ is defined by $\Phi = (\phi^*\sigma)^{-1} \circ \psi$, and $\lambda : \phi^*\mathcal{T} \to \mathcal{D}$ is constructed as follows. Consider the following (non-commutative!!!) diagram,

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\ \sigma\ } & \varphi^{\mathrm{op},*}\mathcal{D} \\
\downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi_{\mathcal{D}}} \\
\phi^*\mathcal{T} & \xrightarrow{\ \phi^*\sigma\ } & \mathcal{D}
\end{array}
$$

defined over $\mathcal{Y}^{[r,\infty]}_{A^+}$ for big enough r avoiding $V(\xi)$. By hypothesis, there is $\varpi \in A^+$ with $\psi = \Phi_{\mathcal{D}} \circ \sigma$ over $\mathrm{Spec}(W(R^+)/[\varpi])$. Consequently, $\phi^*\sigma \circ \Phi = \Phi_{\mathcal{D}} \circ \sigma$ over $\mathrm{Spec}(B_{A^+}^{[r,\infty]}/[\varpi])$. By Lemma 2.15, we can construct λ as the unique isogeny over $\mathcal{Y}^{[r,\infty]}_{A^+}$ lifting $\phi^*\sigma$ with $\lambda = \phi^*\sigma$ over $\mathrm{Spec}(B_{A^+}^{[r,\infty]}/[\varpi])$. The uniqueness of λ makes this construction functorial so that $\Theta : \widehat{W\mathrm{Gr}}_{\mathcal{D}} \to \widehat{W\mathrm{Sht}}_{\mathcal{D}}$ is well-defined.

The inverse $\Omega = \Theta^{-1}$ is given on (A, A^+)-valued points by

$$(\mathcal{T}, \Phi, \lambda, \sigma) \mapsto (\varphi^{\mathrm{op},*}\mathcal{T}, \sigma \circ \Phi, \varphi^{\mathrm{op},*}\sigma).$$

Direct computations show $\Omega \circ \Theta = \mathrm{Id}$, and that $\Theta \circ \Omega(\mathcal{T}, \Phi, \lambda, \sigma) = (\mathcal{T}, \Phi, \lambda', \sigma)$ for some λ' with $\lambda' = \sigma = \lambda$ over $B_{A^+}^{[r,\infty]}/[\varpi]$. The uniqueness part of Lemma 2.15 proves $\lambda = \lambda'$ and $\Theta \circ \Omega = \mathrm{Id}$.

By inspection, one shows that Θ and Ω that it preserve the boundedness conditions so that $\Theta : \widehat{W\mathcal{M}^{\mathcal{D},\leq\mu}_{O_{E_0}}} \to \widehat{\mathrm{WSht}^{\mathcal{D},\leq\mu}_{O_{E_0}}}$ is also an isomorphism. Finally,

$$\pi_0((\mathrm{Sht}^{\mathcal{D},\leq\mu}_{O_{E_0}})^{\circledcirc}_{/c_{\mathcal{D}}}) = \pi_0((\widehat{\mathrm{WSht}^{\mathcal{D},\leq\mu}_{O_{E_0}}})_\eta) = \pi_0((\widehat{W\mathcal{M}^{\mathcal{D},\leq\mu}_{O_{E_0}}})_\eta) = \pi_0((\mathcal{M}^{\mathcal{D},\leq\mu}_{O_{E_0}})^{\circledcirc}_{/c_{\mathcal{D}}})$$

since the v-sheaf in groups $\widehat{L^+_W\mathscr{G}}_{\mathcal{D}}$ is connected. $\qquad\square$

We can now prove that moduli spaces of p-adic shtukas are rich.

Theorem 2.37. $\mathcal{K} = (\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{E_0}}, \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{E_0})$ *is a rich smelted kimberlite. If $\mathscr{G}$ is reductive, $\mathcal{K}$ is topologically normal as in* [Gle22, *Definition 4.52*].

Proof. Lemma 2.31 proves this map is a smelted kimberlite. In [SW20, Proposition 23.3.3] it is proven that the period morphism $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{E_0} \to \mathcal{M}^{\mathscr{G},\leq\mu}_{E_0}$ is étale. By [Gle22, Proposition 4.46] and Theorem 2.19, we know that $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{E_0}$ is a cJ-diamond. By [HV19, Theorem 1.1], we know that $X^{\leq\mu}_{\mathscr{G}}(b)$ is locally Noetherian. By [Gle22, Lemma 5.23], to prove that the specialization map is surjective we only need to prove that for any nonarchimedean field extension $C/W(k)[\frac{1}{p}]$ with C algebraically closed, the specialization map of the base change $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_C}$ is surjective on closed points. It is then enough to prove that for any such C the p-adic tubular neighborhoods of $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_C}$ is non-empty and that when $\mathscr{G}$ is reductive that these are connected. These follow from Theorem 2.33, Theorem 2.19 and Theorem 2.20. $\qquad\square$

We finish this section with the proof of Theorem 2.

Theorem 2.38. *For every nonarchimedean field extension F of E_0:*

  a) *There is a continuous specialization map*

$$\mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_F}} : |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(F)| \to |X^{\leq\mu}_{\mathscr{G}}(b)|,$$

  this map is a specializing and spectral map of locally spectral topological spaces. It is a quotient map and $J_b(\mathbb{Q}_p)$-equivariant.

  b) *If $\mathscr{G}$ is reductive the specialization map induces a bijection of connected components*

$$\mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_F}} : \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(F)) \to \pi_0(X^{\leq\mu}_{\mathscr{G}}(b))$$

Proof. By Theorem 2.37 and Proposition 2.30 $(\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_F}, \mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(F))$ is a rich smelted kimberlite with reduction $X^{\leq\mu}_{\mathscr{G}}(b)$. This implies by [Gle22, Theorem 4.40] that the specialization map $\mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_F}} : |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(F)| \to |X^{\leq\mu}_{\mathscr{G}}(b)|$ is a specializing spectral map of locally spectral spaces. Moreover, by [Gle22, Lemma 4.53] it is a quotient map. Moreover, $J_b(\mathbb{Q}_p)$-equivariance follows from functoriality of the specialization map. For the last claim we may apply [Gle22, Proposition 4.55] and the second part of Theorem 2.37. $\qquad\square$

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Mathematisches Institut der Universität Bonn, Endenicher Allee 60, Bonn, Germany

Email address: igleason@uni-bonn.de