Intro:

What are Kimberlites?

$$\begin{tikzcd}
\left\{\begin{matrix}\text{rigid}\\ \text{spaces}\end{matrix}\right\}_{/\mathbb{Q}_p} \arrow[r, hook, squiggly, "\diamond"'] & \left\{\begin{matrix}\text{diamonds}\\ /\mathrm{Spd}\,\mathbb{Q}_p\end{matrix}\right\} \ni \begin{matrix}B_{dR}^{+}\text{-}\\ \text{Grassmannian}\\ \mathrm{Gr}_{dR}\end{matrix} \\
\left\{\begin{matrix}\text{formal}\\ \text{schemes}\end{matrix}\right\}_{/\mathbb{Z}_p} \arrow[u, "\text{generic fiber}"'] \arrow[r, squiggly, "\diamond"'] & \left\{\begin{matrix}?\\ /\mathrm{Spd}\,\mathbb{Z}_p\end{matrix}\right\} \ni \begin{matrix}\text{Families}\\ \text{of}\\ B_{dR}^{+}\text{-Grassmannians}\end{matrix} \arrow[u, "\times_{\mathbb{Z}_p} \mathbb{Q}_p"']
\end{tikzcd}$$

$? \subseteq$ v-sheaves

Goal: find ? that retains key properties of formal schemes but also contains $\mathrm{Gr}_{G, \mathrm{Spd}\,\mathbb{Z}_p}$.

Talk 1:

Fix (A, A^+) a Huber pair over $\mathbb{Z}_p$. (Not necessarily analytic!).

We attach a v-sheaf

$$\mathrm{spd}(A, A^+) : \{\mathrm{Perf}\} \longrightarrow \mathrm{Sets}$$

char p perfectoid

by formula

$$\mathrm{spd}(A, A^+)[R, R^+] = \{ (R^\#, f) \mid f : (A, A^+) \to (R^\#, R^{\#,+}) \}$$

This extends to a functor

$$\left\{ \begin{matrix} \text{Adic} \\ \text{spaces}/\mathbb{Z}_p \end{matrix} \right\} \xrightarrow{\diamond} \{ \text{v-sheaves} \}$$

This is the diamond functor for adic spaces.

Variants:

$$\underbrace{\diamond, \blacklozenge, \dagger, /\circ\circ}_{\text{several}} : \mathrm{Sch}/\mathbb{Z}_p \longrightarrow \{v\text{-sheaves}\}$$

$$\mathrm{Sch}/\mathbb{Z}_p \overset{?}{\longrightarrow} \{\text{"Adic spaces"}\} \longrightarrow \{v\text{-sheaves}\}$$

$\diamond$ } only one

$$\mathrm{Spec}(A)^{?}(R, R^+) = \{(R^\#, f)\}$$

$f : A \to R^{\#,}$!

Main:

$$\mathrm{Spec}(A)^{\diamond} \sim f : A \to R^{\#+}$$

$$\mathrm{Spec}(A)^{\blacklozenge} \sim f : A \to R^{\#}$$

Also important:

$$\mathrm{Spec}(A)^{\dagger} \sim f : A \to R^{\#,\circ}$$

$$\mathrm{Spec}(A)^{/\circ\circ} \sim f : A \to R^+/R^{\circ\circ}$$

Example: $A = \mathbb{Z}_p[T]$

$$\mathrm{Spec}(A)^{\mathrm{an}} = \mathbb{B}^1_{\mathbb{Z}_p}, \quad \mathrm{Spec}(A)^{\diamond} = \mathbb{A}^1_{\mathbb{Z}_p}$$

$$\mathrm{Spec}(A)^{\dagger} = \overline{\mathbb{B}^1_{\mathbb{Z}_p}}, \quad \mathrm{Spec}(A)^{\mathrm{an}/\infty} = \mathbb{B}^1_{\mathbb{Z}_p} / \mathbb{B}^{<1}_{\mathbb{Z}_p}$$

Topological Spaces:

If X is a v-sheaf we let $|X| = \{ f : \mathrm{Spa}(C, C^+) \to X \} / \sim$

$f \sim g$ if $\exists\ h_1, h_2$

$$\begin{tikzcd} & \mathrm{Spa}(C, C^+) \arrow[dr, "f"] & \\ \mathrm{Spa}(L, L^+) \arrow[ur, "h_1"] \arrow[dr, "h_2"'] & & X \\ & \mathrm{Spa}(K, K^+) \arrow[ur, "g"'] & \end{tikzcd}$$

Question: what is the topological space of $|\mathrm{Spec}(A)^{\diamond}|$ or $|\mathrm{Spec}(A)^{\diamond}|$.

More general: what is $|\mathrm{Spd}(A, A^+)|$?

There's a continuous map
$$h : |\mathrm{Spd}(A, A^+)| \longrightarrow \mathrm{Spa}(A, A^+).$$

First guess: $\mathrm{Spa}(A, A^+) = |\mathrm{Spd}(A, A^+)|$?

- This true "essentially" only when (A, A^+) is analytic.
- We need to work with non-analytic spaces as well!!

Two key failures:

a) $\mathrm{Spd}(A, A^+)$ has more points.

b) $\mathrm{Spd}(A, A^+)$ has more opens.

About b):

$$\mathrm{Spa}(\mathbb{F}_p[[t]]) \longrightarrow \mathrm{Spa}(\mathbb{F}_p[t])$$
$$\mathbb{B}^{<1}_{\mathbb{F}_p} \subseteq \mathbb{B}^1_{\mathbb{F}_p}$$

open in the world of v-sheaves.

About a)

Take V a char p DVR with uniformizer $\pi \in V$, endowed with the discrete topology. Let $K = V[\frac{1}{\pi}]$

$$\mathrm{Spec}(V) = \left\{ \begin{matrix} \eta \\ v \\ s \end{matrix} \right\}$$

generic

special

$$\mathrm{Spa}(V, V) = \left\{ \begin{matrix} \eta \\ v \\ t > s \end{matrix} \right\}$$

generic

π-adic

special

vertical specialization

horizontal specialization

$|\cdot|_t : V \longrightarrow \mathbb{R}$, characterized by $0 < |\pi|_t < 1$

$$\phi \to \mathrm{Spa}(K((t)), V + t \cdot K[[t]]) \to \mathrm{Spd}(V, V)$$

π infinitesimally smaller than 1

$$\phi \to \mathrm{Spa}(\hat{V}_\pi[\tfrac{1}{\pi}], V_\pi) \to \mathrm{Spd}(V, V)$$

π topologically smaller than 1.

Both map to $|\cdot|_t$!!

Key failures a), b) is the only obstruction.

Fix: a) Remember the topology of valuations

b) Add opens

$$\begin{tikzcd} U_{f<<1} \arrow[r, phantom, "\subseteq"] \arrow[d, "f"] & \mathrm{Spa}(A, A^+) \arrow[d, "f"] \\ \mathrm{Spa}(\mathbb{F}_p[[t]]) \arrow[r, phantom, "\subseteq"] & \mathrm{Spa}(\mathbb{F}_p[t]) \end{tikzcd}$$

Olivine spectrum:

$$\mathrm{Spo}(A, A^+) \subseteq \mathrm{Spa}(A, A^+)^2$$

$$(\underbrace{|\cdot|_h}_{\text{Huber component}}, \underbrace{|\cdot|_b}_{\text{Berkovich component}})$$

Two conditions:

1) $|\cdot|_b$ is vertical generization of $|\cdot|_h$

2) $|\cdot|_b$ is either trivial or rank 1.

Basis of topology:

- classical Localization

$$\mathrm{Spa}(A,A^+) \xrightarrow{h} \mathrm{Spa}(A_g,A^+)$$
$$(|\cdot|_h, |\cdot|_b) \longmapsto |\cdot|_h$$

- we ask that h is continuous:

$$U_{f\leq g\neq 0} = h^{-1}\left(U\left(\frac{f}{g}\right)\right)$$

- Analytic Localizations:

$$N_{f<<g} = \{ (|\cdot|_h, |\cdot|_b) \mid |f|_b < |g|_h \}$$

* If g is a unit this is the same as asking $\left|\frac{f}{g}\right| << 1$

Thm There's a bijective canonical continuous map $|\mathrm{Spd}(A,A^+)| \longrightarrow \mathrm{Spa}(A_b,A^+)$.

It is a homeomorphism if:

a) the finiteness condition * holds.

b) A, A^+ are valuation rings.

* A and A^+ are topologically of finite type over a ring of definition

Corollary:

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(X^{\diamond}, Y^{\diamond})$$

whenever X is a perfect non-analytic adic space in char p and Y is any adic space over Spd $\mathbb{Z}_p$

Example:

Take V a char p DVR with uniformizer $\pi \in V$, endowed with the discrete topology. Let $K = V[\frac{1}{\pi}]$

$$\mathrm{Spec}\, V = \left\{ \begin{matrix} k \\ v \\ s \end{matrix} \right\} \qquad \mathrm{Spa}\, V = \left\{ \begin{matrix} k \\ v \\ t > s \end{matrix} \right\}$$

$\mathrm{Spo}(K,K)$ $\qquad \{1 \leq \pi\}$

$$\mathrm{Spo}\, V = \left\{ \begin{matrix} (k,k) & & \\ \vee & & (t,t) \\ (t,k) & \angle & \vee \\ & & (s,s) \end{matrix} \right\}$$

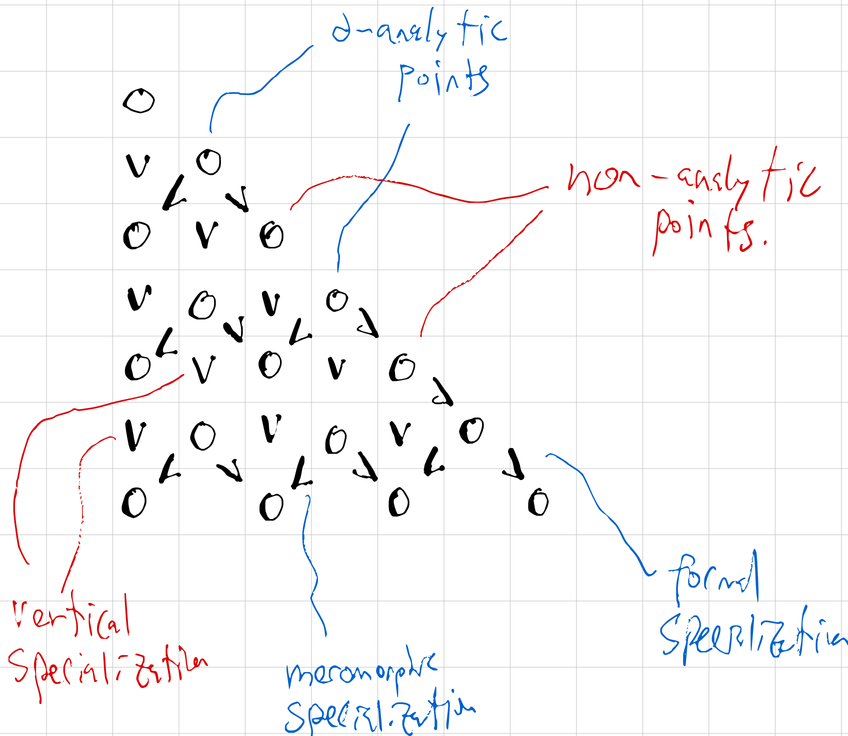
$\{\pi << 1\}$

$\quad \| $

$\mathrm{Spo}(\hat{V}_\pi, \hat{V}_\pi)$

$$\mathrm{Spo}(K,V) = \{\pi \neq 0\} = \{\pi^2 \leq \pi \neq 0\}$$

Rank n case:



Important observation:

Although $\mathrm{spa}(C, C^+)$ is dense in $\mathrm{spa}(C^+, C^+)$, $\mathrm{spo}(C, C^+)$ is not dense in $\mathrm{spo}(C^+, C^+)$.

This is key for next talk!!!

Let G be quasi-split reductive group $/\mathbb{Q}_p$

$\underline{\mathrm{Bun}_G}$ vs $\underline{\mathrm{Isoc}_G}$

G-bundles on FF curve

G-isocrystals

$$|\mathrm{Bun}_G| = B(G)$$

$$|\mathrm{Isoc}_G| = B(G)$$

but for different topologies!

Newton point

$$\nu : B(G) \longrightarrow \mathcal{N}(G) = \left(X_*(T)^+_{\mathbb{Q}}\right)^{\Gamma}$$

$$\kappa : B(G) \longrightarrow \pi_1(G)_{\Gamma}$$

partial order in $B(G)$

$b_1 \le b_2$ if $\kappa(b_1) = \kappa(b_2)$

and

$$\nu_{b_2} - \nu_{b_1} = \mathbb{Q}_{\geq 0}\check{\Delta}_G$$

We can topologize $B(G)$ with the <u>upper</u> or <u>Lower</u> semicontinuous topologies.

$$B(G)^{\text{upper}} : \{ b' \mid b' \leq b \} = I_{\leq b}$$

basis for topology

$$B(G)^{\text{Lower}} : \{ b' \mid b \leq b' \} = I_{b \leq}$$

basis for topology

Thm (Viehmann) $|\mathrm{Bun}_G| = B(G)^{\text{upper}}$

Thm (Rapoport-Richartz, He) $|\mathrm{Isoc}_G| = B(G)^{\text{Lower}}$

Topologies get reversed!

"Explanation" (not a proof).

Suppose $b_1 < b_2$ and take V DVR

$\mathrm{Spec}(V) \longrightarrow \mathrm{Isoc}_G$ with $\eta \rightsquigarrow b_2$

$s \rightsquigarrow b_1$

Generic point

special point.

This induces a map

$$\mathrm{Spd}(V,V) \longrightarrow \mathrm{Bun}_G$$

and we have

$$\begin{array}{ccc} (h,h) & & (t,t) \\ \vee & < & \vee \\ (t,h) & & (s,s) \end{array}$$

b_2 b_1

by partial properness

By equivalence

$$\{\mathrm{Spd}(\mathcal{O}_C) \to \mathrm{Bun}_G\} \cong \{\mathrm{Spa}(C,\mathcal{O}_C) \to \mathrm{Bun}_G\}$$

$$\begin{array}{ccc} \wr\wr\wr & & \wr\wr\wr \\ \varphi\text{-modules} & & \varphi\text{-modules} \\ \mathcal{Y}_{(0,\infty]} & & \mathcal{Y}_{(0,\infty)} \end{array}$$

$(t,t) > (t,h)$ gives

$b_1 > b_2$ in Bun_G

whereas

$h > s$ gives

$b_2 > b_1$ in Isoc_G.

One can prove

$\mathrm{Isoc}_{\leq b} \underset{\text{closed}}{\hookrightarrow} \mathrm{Isoc}$ [RR]

$\mathrm{Bun}_{G, \leq b} \underset{\text{open}}{\hookrightarrow} \mathrm{Bun}_G$ [FS]

and $b_2 \in \overline{\{b_1\}}$ in Bun_G

<u>iff</u> $\exists$ $\mathrm{Spec}(V) \to \mathrm{Isoc}$

with $\eta \rightsquigarrow b_2$

$s \rightsquigarrow b_1$

Needs some work!

Take $b_1 \in \mathrm{Isoc}_{\leq b_2}$

then $b_1 \in \underline{\mathrm{Bun}}_{G, \leq b_2}$

so $b_2 \in \overline{\{b_1\}}$ in Bun_G [Vieh]

$\therefore$ $\exists$ $\mathrm{Spec}\, V \to \mathrm{Isoc}$

$\eta \rightsquigarrow b_2$

$s \rightsquigarrow b_1$

$\therefore$ $b_1 \in \overline{\{b_2\}}$.