Lecture 3: Local models of moduli spaces of shtukas.

Let $\mathcal{G}$ parahoric group scheme over $\breve{\mathbb{Z}}_p = W(\bar{\mathbb{F}}_p)$. $G = \mathcal{G} \times_{\breve{\mathbb{Z}}_p} \mathbb{Q}_p$ reductive and $\mu : \mathbb{G}_m \to G_{\bar{\mathbb{Q}}_p}$ conjugacy class of cocharacters.

Loop group:

$$\begin{tikzcd} L\mathcal{G} \arrow[r] & \mathrm{Spd}\,\breve{\mathbb{Z}}_p \\ L^+\mathcal{G} \arrow[u, phantom, "\cup"] \arrow[ur] & \end{tikzcd}$$

$$L^+\mathcal{G}(R,R^+) = \left\{ (R^\#, u) \;\middle|\; u \in \mathcal{G}(B^+_{dR}(R^\#)) \right\}$$

$$L\mathcal{G}(R,R^+) = \left\{ (R^\#, u) \;\middle|\; u \in \mathcal{G}(B_{dR}(R^\#)) \right\}$$

$$Gr_{\mathcal{G}} = L\mathcal{G} / L^+\mathcal{G} \xrightarrow{\text{formally adic}} \mathrm{Spd}\,\breve{\mathbb{Z}}_p .$$

$\therefore$ formally separated.

$$Gr_{\mathcal{G},\bar{\mathbb{F}}_p} = \underbrace{\mathcal{F}\ell_{\mathcal{G}}}_{\text{Witt vector affine flag variety}}{}^{\diamond} = \mathcal{F}\ell_{\mathcal{G}}^{\diamond}$$

Fact: Gr_G is specializing.

Key input: Extending torsors over the punctured spectrum of $A_{\inf}$.

Sketch: Gr_G is small

$$\coprod_{i\in I} \underbrace{\mathrm{Spd}(R_i, R_i^+)}_{\text{product of points}} \longrightarrow \mathrm{Gr}_G \quad \text{cover}$$

$$R^+ = \prod_{j\in J} C_j^+$$

Extending a torsor gives a map:

$$\coprod_{i\in I} \mathrm{Spd}(R_i^+, R_i^+) \longrightarrow \mathrm{Gr}_G$$

Hecke stack: $\mathrm{Hk}_G = L^+G \backslash \mathrm{Gr}_G = L^+G \backslash LG / L^+G$

In char 0: $|\mathrm{Hk}_G| = G(B_{dR}^+) \backslash G(B_{dR}) / G(B_{dR}^+)$

$= X_*^+(T)$ dominant cocharacters.

In char p: (when G Iwahori) $|\mathrm{Hk}_G| = G(W(C)) \backslash G(W(C)[\frac{1}{p}]) / G(W(C))$

$= \tilde{W}$ extended affine Weyl group

$\mathrm{Gr}_{G,\mu} \subseteq \mathrm{Gr}_{G,E}$ schubert variety

reflex field $[E:\breve{\mathbb{Q}}_p]<\infty$

Definition: The local model $\mathcal{M}_{(G,\mu)}$ is the closure of $\mathrm{Gr}_{G,\mu}$ inside of $\mathrm{Gr}_{\mathcal{G}/O_E}$.

Theorem (AGLR, GL22)

: a) $\mathcal{M}_{(\mathcal{G},\mu)}$ is a flat p-adic kimberlite, proper over spd $\breve{\mathbb{Z}}_p$

b) $\mathcal{M}^{red}_{(\mathcal{G},\mu)} = \mathcal{A}_{\mathcal{G},\mu} \rightsquigarrow \mathrm{Adm}(\mu)$ $\subseteq \tilde{W}$

c) $\mathcal{M}_{(\mathcal{G},\mu)}(\mathbb{C}_p) = \mathcal{M}_{(\mathcal{G},\mu)}(O_{\mathbb{C}_p})$.

d) $\mathcal{M}_{(\mathcal{G},\mu)}$ is geometrically unibranch.

e) when μ is minuscule,

$\mathcal{M}_{(\mathcal{G},\mu)} = \mathcal{M}^{sch,\diamond}_{(\mathcal{G},\mu)}$ for a flat normal $\breve{\mathbb{Z}}_p$-scheme with reduced special fiber.

Fast facts

- Properness proven in Berkeley notes.

- $Gr^{red}_{\mathcal{G},\mu} = Fl_S \quad S \subseteq \tilde{W}$

  for some finite set S

  follows from $Gr_{\mathcal{G},\mu} \longrightarrow Ht_{\mathcal{G}}$,

  "smooth".

* Proving $Adm(\mu) \subseteq S$ is easy, $S \subseteq Adm(\mu)$ is harder.

- Properness + Surjectivity => Flatness
  sp

- By continuity of sp for constructible topology

  surjectivity of sp is reduced to closed points.

- $x \in Fl_S \quad \widehat{\mathcal{M}_{(\mathcal{G},\mu),x}} \underset{\text{open}}{\subseteq} \mathcal{M}_{(\mathcal{G},\mu)}$

  $\mathcal{M}^{\circ}_{(\mathcal{G},\mu),x} = \phi \Rightarrow x \notin \overline{Gr_{\mathcal{G},\mu}}$.

- Anschütz Theorem immediately implies
$$\mathcal{M}_{G,\mu}(\mathbb{C}_p) = \mathcal{M}_{G,\mu}(O_{\mathbb{C}_p}).$$

- Left: Finer statement of b) d) and e).

Normality and tubular neighborhoods:

Proposition: Let A be a flat, weakly normal, topologically of finite type p-adically complete domain over $\breve{\mathbb{Z}}_p$. Suppose that $A[\frac{1}{p}]$ is normal and that for all $x \in |\mathrm{Spec}(A/p)|$ the tubular neighborhood $\mathrm{Spd}(A)^{\circ}_x$ is connected. Then A is normal.

Definition: A rich* Kimberlite$/\breve{\mathbb{Z}}_p$ is "topologically normal" or unibranch if tubular neighborhoods at closed points are connected.

*rich: sp is surjective + finiteness conditions

Proposition: If X is a unibranch kimberlite then
$$sp: \pi_0(X^{an}) \longrightarrow \pi_0(X^{red})$$ is bijective.

Nearby cycles: Let X be a kimberlite.

$$Y := X^{an} \xhookrightarrow{j} X \xhookleftarrow{i} X^{red,\diamond} =: Z$$

$$\begin{tikzcd}
D_{et}(Y) \arrow[r, "j_*"] \arrow[rr, bend right, "\Psi_{an}"'] \arrow[rrd, bend right, "\Psi_{for}"'] & D_{et}(X) \arrow[r, "i^*"] & D_{et}(Z) \arrow[d, shift left, "c_*"] \\
& & D_{et}(X_{red}) \arrow[u, shift left, "c^*"]
\end{tikzcd}$$

Comparison Thm:

a) $c_* \Psi_{an} = \Psi_{for}$

b) If $\Psi_{an}(A) = c^* B$ for $B \in D_{et}(X_{red}) \Rightarrow B \cong \Psi_{for}$

Corollary: Suppose X is a kimberlite and Ψ_{an} He is classical.

TFAE:

a) X is unibranch

b) $\forall\, \mathcal{U} \xrightarrow[\text{étale}]{} X$ with $\mathcal{U}$ connected $X^{\circ}_{\mathcal{U}}$ is connected.

$M^{\mathrm{ref}}_{G,\mu} = A_{G,\mu}$:

Recall FS:

$$\{\mathrm{Rep}\, G\} \cong \left\{\begin{array}{c} L^+G\text{-equivariant} \\ \text{Perverse sheaves} \\ Gr_G \end{array}\right\} \cong \left\{\begin{array}{c} \text{Perverse Sheaves on} \\ HK_G \end{array}\right\}$$

Highest weight rep μ $\longmapsto IC_\mu$

$$\Psi_G : D_{et}(HK_G) \longrightarrow D_{et}(HK_G^{\diamond}) \cong D_{et}(HK_{G_1})$$

$$\mathrm{support}(\Psi_G\, IC_\mu) = M^{\mathrm{red}}_{G,\mu}$$

"stalk at $x \cong R\Gamma^{\bullet}(\hat{M}_{G,\mu,x}, IC_\mu)$"

Recall $CT_B : D_{et}(Hk_G) \to D_{et}(Hk_\mu)$

hyperbolic localization functor.

<u>Fact</u>: Nearby cycles commutes with hyperbolic localization.

$$CT_B \circ \Psi_G = \underbrace{\Psi_\mu \circ CT_B}_{\substack{\text{Quite explicit} \\ \text{in terms of Rep theory}}}.$$

This allow us to prove that stalks vanish away from $\mathrm{Adm}(\mu)$.

$M^{reg}_{(G,\mu)} = Fl_S \quad S = \mathrm{Adm}(\mu)$.

<u>Unibranchness of $M_{(G,\mu)}$</u>:

Argue on stack $A_{(G,\mu)}$.

Codim 0 : $x \in W \cdot \mu$, smooth locus

$(R\Psi\, IC_\mu)_x = \mathbb{F}_\ell$

Codim 1: $x \in \tilde{W}$ $\quad \begin{matrix} \mu_1 & & \mu_2 \\ \cup\!\vert & & \cup\!\vert \\ & x & \end{matrix}$ $\quad \mu_1, \mu_2 \in W\mu$ only two above x

$\Psi_{G_J}(IC_\mu) = Z_\mu$ is central

By Wakimoto sheaves theory
(Anschütz - Lourenço - Wu - Yu)

Z_μ is filtered by Wakimoto sheaves of the form $J_\nu \otimes V_\mu(w_0\bar{\nu})$

Only $J_{\bar{\mu}_1} \otimes V_{\mu_1}(w_0\bar{\mu}_1)$ and $J_{\bar{\mu}_2} \otimes V_{\mu_2}(w_0\bar{\mu}_2)$.

contribute to $(Z_\mu)_x$

Choosing B wisely only one contributes, but $\dim_{\mathbb{F}_\ell} V_\mu(w_0\bar{\mu}_1) = 1$ since $\mu_1 \in W\mu$

and $H^0_x(J_{\bar{\mu}_2}[-\ell(\mu_2)]) = \begin{cases} \mathbb{F}_\ell \\ 0 \end{cases}$ or

$\forall x \in Fl$.

Codim ≥ 2 : If $\mathcal{U}^{\odot}_{(G_{J,\mu})_x}$ is disconnected there is $\mathcal{U} \xrightarrow{\text{étale}} A_{G_{J,\mu}}$ connected with $X^{\odot}_{/\mathcal{U}}$ disconnected. Say $X^{\odot}_{\mathcal{U}} = Y_1 \coprod Y_2$

$\underbrace{Sp(Y_2) \subseteq \text{codim } 2}_{\text{By codim 1 case!}}$. Contradicts perversity.

Generic fiber

<u>Representability</u>: $Gr_{G,\mu} = Fl^{\diamond}_{G,\mu} = G/P_\mu^{\diamond}$

More classical local models candidates

$N^{sch}_{G,\mu}$ constructed from classical Grassmannian over $\breve{\mathbb{Z}}_p[[t]]$.

We prove isomorphism of triples:

$$(N^{sch,\diamond}_{G,\mu,\breve{\mathbb{Q}}_p},\ N_{G,\mu,\mathbb{F}_p},\ sp) \simeq (Fl_{G,\mu},\ A_{(G,\mu)},\ sp)$$

- Generic fibers easily identify G-equivariantly in unique way. both equal to G/P_μ.
- Special fiber: Fl^{witt}_{G} compares to $Fl^{\mathbb{F}_p((t))}$ on depth 0 strata. (Action L^+G factors through G.)
- $N_{G,\mu,\mathbb{F}_p}$ and $A_{G,\mu}$ correspond to $Adm(\mu)$.

Computing the specialization map:

Compute sp on $Gr_{G,\mu}(K)$ for $[K:\breve{\mathbb{Q}}_p] < \infty$ (are dense for constructible topology, determines the rest).

Iwasawa Decomposition: $Fl_{G,\mu}(\breve{\mathbb{Q}}_p) = \bigcup_{\lambda} G(\breve{\mathbb{Z}}_p) \cdot \lambda$

[corresponds to G-orbit, smooth locus]

Two tools:

- points that come from tori (explicit).
- Enlarging by G°-action.
- Not enough!

Example: $G = GL_2$, $\mu = (1,0)$, $\mathcal{G}$ = Iwahori

$$Fl \simeq \mathbb{P}^1_{\breve{\mathbb{Q}}_p} \qquad A_{\mathcal{G},\mu} \cong \mathbb{P}^1 \underset{*}{\sqcup} \mathbb{P}^1, \quad M_{\mathcal{G},\mu} \subseteq \mathbb{P}^2_{\breve{\mathbb{Z}}_p}$$

$$\mathrm{Adm}(\mu) = \{-\mu, w, \mu\} \qquad \ell(w)=0 \qquad \underbrace{XY}_{\text{ramifies.}} = p$$

$$\mathcal{I}\begin{pmatrix}p&0\\0&1\end{pmatrix}\mathcal{I}, \quad \mathcal{I}\begin{pmatrix}1&0\\0&p\end{pmatrix}\mathcal{I}, \quad \mathcal{I}\begin{pmatrix}0&1\\p&0\end{pmatrix}\mathcal{I}$$

"Enlarge the smooth b(τ)"

$$\mathcal{G} \rightsquigarrow \mathrm{Res}_{O_E/\breve{\mathbb{Z}}_p}(\mathcal{G}_{O_E}) = \mathcal{H}$$

$$\begin{array}{ccc} M_{(\mathcal{G},\mu)} & \longrightarrow & M_{(\mathcal{H},\mu)} \\ & & \big\uparrow \text{ resolve by convolution.} \\ & & M_{(\mathcal{H},\mu_\bullet)} \end{array}$$

For ramified extensions $E/\breve{\mathbb{Q}}_p$

$$\left.\begin{array}{c} Fl_{G,\mu}(E) \subseteq Fl_{\mathrm{Res}_{E/\breve{\mathbb{Q}}_p} G,\mu}(E) \\ \\ \mathbb{P}^1_{\breve{\mathbb{Q}}_p} \longrightarrow \mathbb{P}^1_E \end{array}\right\} \text{This can be accessed by convolution.}$$