Specialization maps for Scholze's category of diamonds

Ian Gleason

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Abstract

The purpose of this article is to define and study the specialization map in the context of Scholze's category of diamonds and to prove some basic results on its behavior. Our specialization map generalizes the classical specialzation map that appears in the theory of formal schemes. Afterwards, as an example of interest, we study the specialization map for p-adic Beilinson-Drinfeld Grassmanians and moduli spaces of mixed-characteristic shtukas associated to reductive groups over $\mathbb{Z}_p$. Finally, as an application, we describe the geometric connected components of moduli spaces of mixed-characteristic shtukas at hyper-special level in terms of the connected components of affine Deligne Lusztig varieties. This generalizes a similar theorem of Chen, Kisin, and Viehmann that describes the connected components of unramified Rapoport-Zink spaces in terms of connected components of affine Deligne Lusztig varieties.

Contents

Introduction

The purpose of this paper is twofold, and each goal corresponds to a chapter. In the first chapter we construct the specialization map in the context of Scholze's category of diamonds, and we study abstract properties and related constructions in a very theoretical framework. In the second chapter, we apply the theory developed in the first chapter to study the specialization map for the p-adic Beilinson-Drinfeld Grassmanians and moduli spaces of mixed-characteristic shtukas. These moduli spaces were introduced in the Berkeley notes ([28]) as some of the most important examples that motivated the development of the theory of diamonds.

To fix ideas let us recall the classical case. Let $\mathcal{X}$ be a separated formal scheme topologically of finite type over $\mathbb{Z}_p$. One can associate to $\mathcal{X}$ a rigid analytic space over $\mathbb{Q}_p$, that we will denote by $X_\eta := \mathcal{X} \times_{\mathbb{Z}_p} \mathbb{Q}_p$. We can also associate to $\mathcal{X}$ a finite type reduced scheme over $\mathbb{F}_p$, that we denote by $\overline{X} := (\mathcal{X} \times_{\mathbb{Z}_p} \mathbb{F}_p)^{\mathrm{red}}$. Now, Huber's theory of adic spaces allows us to consider X_η as an adic space and in particular assign to it a locally spectral topological space $|X_\eta|$. Moreover, one can construct a continuous and spectral map of locally spectral spaces $\mathrm{sp}_{\mathcal{X}} : |X_\eta| \to |\overline{X}|$, where $|\overline{X}|$ is the usual Zariski space underlying $\overline{X}$ (See [3] 7.4.12 or [19] 6.4). It is this specialization map that our work generalizes, we elaborate below.

In [26] Scholze sets foundations for the theory of diamonds which can be defined as certain sheaves on the category of characteristic p perfectoid spaces endowed with a Grothendieck topology called the v-topology. He associates to any pre-adic space X over $\mathbb{Z}_p$ (not necessarily analytic) a v-sheaf $X^\diamond$, and whenever X is analytic he proves that $X^\diamond$ is a (locally spatial) diamond. Moreover, Scholze assigns to any v-sheaf $\mathcal{F}$ an underlying topological space $|\mathcal{F}|$ and whenever $\mathcal{F} = X^\diamond$ he constructs a functorial surjective and continuous map $|\mathcal{F}| \to |X|$. When X is analytic it is proven in [28] that this map is a homeomorphism, but as we will discuss below this fails for non-analtyic pre-adic spaces.

In the first chapter, we take as input what we call below a *specializing* v-sheaf $\mathcal{F}$ and we assign to it: a scheme-theoretic v-sheaf $\mathcal{F}^{\mathrm{red}}$ which is the analogue of the reduced special fiber of a formal scheme, and a continuous map of topological spaces $\mathrm{sp}_{\mathcal{F}} : |\mathcal{F}| \to |\mathcal{F}^{\mathrm{red}}|$ that we call the *specialization map* of $\mathcal{F}$. If $\mathcal{X}$ is a separated formal scheme over $\mathbb{Z}_p$, we can prove that $\mathcal{X}^\diamond$ is a specializing v-sheaf, and in this case we have natural identifications $|\mathcal{X}_\eta| = |\mathcal{X}^\diamond \times_{\mathbb{Z}_p^\diamond} \mathbb{Q}_p^\diamond|$ and $|\overline{X}| = |(\mathcal{X}^\diamond)^{\mathrm{red}}|$ together with a commutative diagram:

$$\begin{tikzcd} {|\, \mathcal{X}^\diamond \times_{\mathbb{Z}_p^\diamond} \mathbb{Q}_p^\diamond \,|} \arrow[r, "\cong"] \arrow[d, "\mathrm{sp}_{\mathcal{X}^\diamond}"] & {|\, \mathcal{X}_\eta \,|} \arrow[d, "\mathrm{sp}_{\mathcal{X}}"] \\ {|\, (\mathcal{X}^\diamond)^{\mathrm{red}} \,|} \arrow[r, "\cong"] & {|\, \overline{X} \,|} \end{tikzcd}$$

It is in this sense that our specialization map generalizes the classical one.

The advantage of working in this broader context is that the categories of diamonds and v-sheaves are much more flexible than those of formal schemes and rigid analytic spaces. This allows us to construct interesting spaces that do not come from applying the $\diamond$-functor to pre-adic spaces. Actually, the main reason the author found the specialization map for diamonds interesting is that it has potential applications to the study of moduli spaces of mixed-characteristic shtukas. Typically, these moduli spaces are locally spatial diamonds that do not come from a pre-adic space. In forthcoming work of the author, we use the tools developed here to describe the profinite set of geometric connected components of moduli spaces of mixed shtukas at any chosen level (including infinite level). This work builds on and generalizes the work of Chen on the geometric connected components of unramified Rapoport-Zink spaces (See [5]).

To describe the main results of our second chapter, we fix some notation. Let $\mathscr{G}$ be a reductive group over $\mathbb{Z}_p$, and denote by G the generic fiber of $\mathscr{G}$ over $\mathbb{Q}_p$. Fix $T \subseteq B \subseteq G$ a maximal $\mathbb{Q}_p$-rationally defined torus and a Borel respectively, and let $\mathfrak{f}$ be an algebraically closed field extension of $\overline{\mathbb{F}}_p$. We let X_*^+ denote the subset of dominant cocharacters in $X_*(T_{\overline{\mathbb{Q}}_p})$, fix a $\mu \in X_*^+$ and an element $b \in G(W(\mathfrak{f})[\frac{1}{p}])$. Let $E := E(\mu)$ be the reflex field of μ. Since $\mathscr{G}$ is reductive over $\mathbb{Z}_p$ this is an unramified extension of $\mathbb{Q}_p$. Let J_b denote the σ-centralizer of b appearing in Kottwitz' theory of isocrystals with G-structure [18]. Let F_1 denote a complete non-Archimedean field extension of E, with ring of integers of O_{F_1} and residue field k_{F_1}. Let F_2 be a complete non-Archimedean field extension of $W(\mathfrak{f})[\frac{1}{p}]$ with residue field k_{F_2}. To this data one can associate the following objects:

a.- A spatial diamond $\mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond}$ proper over ${F_1}^\diamond$, parametrizing μ-bounded B^+_{dR}-lattices with G-structure. Here B^+_{dR} is the de Rham period ring of Fontaine, and this moduli is the B_{dR}-Grassmanian of the Berkeley notes [28].

b.- A perfect scheme $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}$ proper and perfectly finitely presented over $\mathrm{Spec}(k_{F_1})$ parametrizing μ-bounded Witt-vector lattices with $\mathscr{G}$-structure. This is Zhu's Witt-vector Grassmanian [30], [4].

c.- A locally spatial diamond $\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}$ partially proper over $F_2^\diamond$, parametrizing mixed-characteristic shtukas with G-structure that have relative position bounded by μ, and with level structure $\mathscr{G}(\mathbb{Z}_p)$. This is the moduli space of mixed-characteristic shtukas at hyperspecial level that appears in the Berkeley notes [28]. It comes endowed with a continuous $J_b(\mathbb{Q}_p)$-action.

d.- A perfect scheme $X^{\mathscr{G}}_{\leq\mu}(b)$ locally perfectly finitely presented over k_{F_2}, which on geometric points evaluates to affine Deligne-Lusztig sets of Rapoport [22]. This space also comes equipped with a continuous $J_b(\mathbb{Q}_p)$-action.

Fix an algebraically closed non-Archimedean field C over F_1 with ring of integers O_C and let k_C denote the residue field of O_C. In [1], Anschűtz constructs a map going from $\mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond}(C,O_C)$ to $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}(k_C)$ which for now we denote sp_{Ans}. Before this work, the map was only known as a map of sets. Building on the work of Anschűtz we upgrade that specialization map to construct a specialization map of topological spaces $\mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^\diamond_{F_1}}} : |\mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond}| \to |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond}(C,O_C) \arrow[r, "\iota"] \arrow[d, "\mathrm{sp}_{Ans}"] & {|\mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond}|} \arrow[d, "\mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^\diamond_{F_1}}}"] \\
\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}(k_C) \arrow[r, "\iota"] & {|\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|}
\end{tikzcd}$$

Here ι associates to a $\mathrm{Spa}(C,O_C)$-valued point (k_C-valued point respectively) its underlying topological point. We prove the following properties about our specialization map.

Theorem 1. *a) The specialization map*

$$\mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^\diamond_{F_1}}} : |\mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond}| \to |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$$

is a closed and spectral map of spectral topological spaces.

b) Given a closed point $x \in |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$ let $T_x := \mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^\diamond_{F_1}}}{}^{-1}(x)$, then the interior T_x° of T_x in $|\mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond}|$ is a dense subset of T_x.

c) T_x and T_x° are non-empty and connected.

Using a technique that we learned from reading [28] together with the work of Anschűtz, we construct a second specialization map but now its source is a moduli space of mixed-characteristic shtukas at hyperspecial level and the target is the affine Deligne-Lusztig variety associated to $(\mathscr{G}, b, \mu)$.

Theorem 2. *a) There is a continuous specialization map*

$$\mathrm{sp}_{\mathrm{Sht}_{O_{F_2}}^{\mathscr{G}_b,\leq\mu}} : |\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}| \to |X^{\mathscr{G}}_{\leq\mu}(b)|,$$

this map is a specializing and spectral map of locally spectral topological spaces. It is a quotient map and it is $J_b(\mathbb{Q}_p)$-equivariant.

b) Given a closed point $x \in |X^{\mathscr{G}}_{\leq\mu}(b)|$ let $S_x = \mathrm{sp}_{\mathrm{Sht}_{O_{F_2}}^{\mathscr{G}_b,\leq\mu}}{}^{-1}(x)$, then the interior S_x° of S_x as a subspace of $|\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}|$ is dense in S_x.

c) S_x and S_x° are non-empty and connected.

d) The specialization map induces a $J_b(\mathbb{Q}_p)$-equivariant bijection of connected components

$$\mathrm{sp}_{\mathrm{Sht}_{O_{F_2}}^{\mathscr{G}_b,\leq\mu}} : \pi_0(\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}) \to \pi_0(X^{\mathscr{G}}_{\leq\mu}(b))$$

The work of Scholze and Weinstein identifies the diamond associated to moduli spaces of p-divisible groups as special instances of moduli spaces of mixed-characteristic shtukas (See [28] 24.3.5). Under this light, the last part of theorem 2 is a generalization of Theorem 5.1.5.(i) of [6] that describes the connected components of unramified Rapoport-Zink spaces at hyperspecial level. The study of the set of connected components of affine Deligne-Lusztig varieties had a lot of progress in the past 10 years. In the case of unramified groups at hyperspecial level the problem is very well understood, and the reader can find in [20] theorem 1.1, [6] theorem 1.1, [11] theorem 0.1, 0.2 concrete descriptions of these sets.

Our last main result compares the preimages of the specialization map of Grassmanians to those of moduli spaces of shtukas. The statement is philosophically aligned with Grothendieck-Messing's deformation theory of p-divisible groups and it is a key input in the proof of theorem 2. A precise statement is:

Theorem 3. *If we let $F_1 = F_2$, then for a closed point $x \in |X^{\mathscr{G}}_{\leq\mu}(b)|$ there is a closed point $y \in |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_2}}|$ such that S_x° considered as an open subdiamond of $\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}$ is isomorphic to T_y° when considered as an open subdiamond of $\mathrm{Gr}^{G,\leq\mu}_{F_2^\diamond}$. Here S_x and T_y are as in theorems 2 and 1 respectively.*

Now that we have stated our main results we would like to give a short summary of the theory of specialization of the first chapter and provide sketches of the proves of the main results of the second chapter.

Given a Tate Huber pair (A, A^+) with pseudo-uniformizer $\varpi \in A^+$ the specialization map sp_A : $\mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A^+/\varpi)$ assigns to $x \in \mathrm{Spa}(A, A^+)$ the prime ideal $\mathfrak{p}_x$ of those elements $a \in A^+$ for which $|a|_x < 1$. This is a continuous and closed map of spectral topological spaces and the construction is functorial in the category of Tate Huber pairs. The central idea of our theory is that, regardless of the definition, the specialization map for more general objects should also be functorial and should agree with the case of Tate Huber pairs. This desideratum naturally leads to defining the specialization map as the only map (if such a thing exists) that could be functorial. One is then forced to change perspective and to look for hypotheses that would prove that a functorial map exists and for conditions that would make this map unique.

The first question one needs to answer is what should the target and source of the specialization map be? The case of Tate Huber pairs may be a little bit misleading in that Tate Huber pairs come, by dessign, with a canonical "integral model". Namely, the integral model for $\mathrm{Spa}(A, A^+)$ is simply given by $\mathrm{Spa}(A^+, A^+)$. For general locally spatial diamond we do not have a canonical "integral model" (at least to the author's knowledge) and for this reason it is more convenient to attach specialization maps to models rather than to the object that one makes models of. In the case of Tate Huber pairs the specialization map can be promoted to a map $\mathrm{sp}_{A^+} : \mathrm{Spa}(A^+, A^+) \to \mathrm{Spec}(A^+/\varpi)$ whose restriction to $\mathrm{Spa}(A, A^+)$ is the original one. In the general case, we will find integral models for diamonds within the more general category of v-sheaves.

An important result of the Berkeley notes proves that the $\diamond$-functor is a fully-faithful embedding of the category of characteristic p perfect schemes to Scholze's category of v-sheaves. Our observation is that, with the correct setup, this functor admits a right adjoint which we call suggestively the reduction functor. Moreover, we compute directly that if B is a topological ring over $\mathbb{Z}_p$ endowed with the I-adic topology for some finitely generated ideal I, then $(\mathrm{Spd}(B, B))^{\mathrm{red}}$ is given by the perfection of $\mathrm{Spec}(B/I)$. In particular, if (A, A^+) is a uniform Tate Huber pair, then $(\mathrm{Spd}(A^+, A^+))^{\mathrm{red}}$ is given by the perfection of $\mathrm{Spec}(A^+/\varpi)$. This suggests that one can define the target of our specialization to be the result of applying the reduction functor to the integral models that we want to associate a specialization map to. In general, these objects will not be a perfect scheme but it will be what we call below a scheme theoretic v-sheaf, which will come equipped with an underlying topological space that extends the Zariski topology on perfect schemes.

The next step is to construct the specialization map. The key aspect that makes the specialization map for Tate Huber pairs functorial is that every map of Tate Huber pairs $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(B, B^+)$ automatically upgrades to a map of "integral models" $\mathrm{Spa}(A^+, A^+) \to \mathrm{Spa}(B^+, B^+)$. This motivates the following definition: given a v-sheaf $\mathcal{F}$, an affinoid perfectoid space $\mathrm{Spa}(A, A^+)$ and a map $\iota : \mathrm{Spa}(A, A^+) \to \mathcal{F}$, we say that $\mathcal{F}$ *formalizes* ι whenever it factors through a map $f : \mathrm{Spd}(A^+, A^+) \to \mathcal{F}$. We say that $\mathcal{F}$ is v-formalizing if for every ι as above there is a v-cover $g : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ such that $\mathcal{F}$ formalizes $\iota \circ g$. Given a v-formalizing sheaf $\mathcal{F}$ one can try to define the specialization map $\mathrm{sp}_{\mathcal{F}} : |\mathcal{F}| \to |\mathcal{F}^{\mathrm{red}}|$ so that for any "formalized" map $f : \mathrm{Spd}(A^+, A^+) \to \mathcal{F}$ the following diagram is commutative:

$$\begin{tikzcd} \mid \mathrm{Spa}(A, A^+) \mid \arrow[r] \arrow[d, "\mathrm{sp}_A"] & \mid \mathrm{Spd}(A^+, A^+) \mid \arrow[r, "f"] & \mid \mathcal{F} \mid \arrow[d, "\mathrm{sp}_{\mathcal{F}}"] \\ \mid \mathrm{Spec}(A^+/\varpi) \mid \arrow[rr, "f^{\mathrm{red}}"] & & \mid \mathcal{F}^{\mathrm{red}} \mid \end{tikzcd}$$

The recipe to compute the specialization map would then be the following: given $x \in |\mathcal{F}|$ for $\mathcal{F}$ v-formalizing we find an algebraically closed perfectoid field C and an open and bounded valuation ring C^+ together with a map $\iota_x : \mathrm{Spa}(C, C^+) \to \mathcal{F}$ such that the closed point of $\mathrm{Spa}(C, C^+)$ maps to x under ι_x. After replacing $\mathrm{Spa}(C, C^+)$ by a v-cover, we find a formalization $f_x : \mathrm{Spd}(C^+, C^+) \to \mathcal{F}$ of ι_x. We apply the reduction functor to f_x and obtain a map $f_x^{\mathrm{red}} : \mathrm{Spec}(C^+/\varpi) \to \mathcal{F}^{\mathrm{red}}$. Finally, we look at the topological image of the unique closed point of $\mathrm{Spec}(C^+/\varpi)$ under f_x^{red}. We define $\mathrm{sp}_{\mathcal{F}}(x)$ to be this image.

The natural question becomes whether or not this construction is well defined. The problem being that the map $\iota_x : \mathrm{Spa}(C, C^+) \to \mathcal{F}$ might have more than one formalization. The naive guess would be that this doesn't happen when $\mathcal{F}$ is separated as a v-sheaf. Unfortunately, this is false. At the heart of the problem is the following pathology, although $|\mathrm{Spa}(C, C^+)|$ is dense within $|\mathrm{Spa}(C^+, C^+)|$ it is not true that $|\mathrm{Spd}(C, C^+)|$ is dense within $|\mathrm{Spd}(C^+, C^+)|$ whenever the valuation ring C^+ has rank bigger than 1.

Resolving this problem forces us to give a concrete description of the topological spaces of the form $|\mathrm{Spd}(A^+, A^+)|$. To do so, we introduce what we call below the *olivine spectrum* of a Huber pair. After this rather subtle topological discussion, we identify a slightly stronger notion than separatedness that we call *formal separatedness*. The main feature of a formally separated v-sheaves $\mathcal{F}$ is that the maps $\iota : \mathrm{Spa}(A, A^+) \to \mathcal{F}$ as above, have at most one formalization.

Combining these two inputs we say that a v-sheaf is specializing if it is v-formalizing and formally separated. We prove that specializing v-sheaves have a unique map that satisfies the commutative diagrams as above for any formalizable map. We prove that this specialization map is functorial in the full subcategory of specializing v-sheaves and that these specialization maps are continuous.

Although specializing v-sheaves produce the specialization maps that we are interested in, they are too general for practical purposes. For this reason, we focus our attention on a more restrictive class of v-sheaves that will have better behaved specialization maps. The central objects of our theory is what we call below *kimberlites* and *smelted kimberlites*. These will be specific kinds of specializing v-sheaves that satisfy many pleasant properties. For example, the specialization map for kimberlites and smelted kimberlites are spectral maps of locally spectral spaces and they come equipped with a good notion of "analytic locus" that is an open subsheaf and a locally spatial diamond. We will see that kimberlites are abundant. Actually, given a topological ring B endowed with the I-adic topology for some finitely generated ideal $I \subseteq B$ containing p, we prove that $\mathrm{Spd}(B, B)$ is a kimberlite.

Let us move on and discuss the content of the second chapter.

The construction of the specialization maps for the moduli spaces that we study follows from the general formalism that we discuss in the first chapter. To apply the theory one has to find a specializing v-sheaf "interpolating" the source and target of the desired specialization map. The candidates are already provided in the Berkeley notes [28]. More precisely, with the setup as in the beginning, Scholze and Weinstein describe what we call here the p-adic Beilinson-Drinfeld Grassmanians $\mathrm{Gr}^{\mathscr{G}, \leq \mu}_{O_{F_1}}$ as a v-sheaf over $O^\diamond_{F_1}$ whose generic fiber is $\mathrm{Gr}^{G, \leq \mu}_{F^\diamond_1}$. Also, they describe a v-sheaf $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{O_{F_2}}$ over $O^\diamond_{F_2}$ whose generic fiber is $\mathrm{Sht}_{(\mathscr{G}, b, \mu), F^\diamond_2}$, we still call this v-sheaf the moduli space of mixed characteristic shtukas at hyperspecial level.

The proof that these v-sheaves are specializing uses all the machinery of modern p-adic Hodge theory as it is discussed in the Berkeley notes. Some key technical inputs are Kedlaya's GAGA-type theorems [16] and Anschűtz' theorem 1.2 of [1] which allows us to prove that the v-sheaves of the Berkeley notes are v-formalizing. Once we know these v-sheaves are specializing, our theory then produces automatically the specialization map. With more work, we prove that these specializing v-sheaves are even nicer. Namely, we prove that p-adic Beilinson-Drinfeld Grassmanians are kimberlites, and that moduli spaces of shtukas at hyperspecial level are smelted kimberlites. We also prove the identities: $(\mathrm{Gr}^{\mathscr{G}, \leq \mu}_{O_{F_1}})^{\mathrm{red}} = \mathrm{Gr}^{\mathscr{G}, \leq \mu}_{\mathcal{W}, k_{F_1}}$ and $(\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{O_{F_2}})^{\mathrm{red}} = X^{\mathscr{G}}_{\leq \mu}(b)$ which tell us that the targets of the specialization maps that our formalism constructs are the desired ones.

After that work, the difficulty becomes to understand the preimages of the specialization map. To tackle this difficulty we introduce some theoretical tools. In our first chapter, to any kimberlite or smelted kimberlite $\mathcal{F}$ and a closed point $x \in |\mathcal{F}^{\mathrm{red}}|$ we define a tubular neighborhood $\widehat{\mathcal{F}}_{/x}$ of $\mathcal{F}$ at x. Roughly speaking, these tubular neighborhoods are the subsheaves of points that specialize to x. In general $|\widehat{\mathcal{F}}_{/x}| \subseteq {\mathrm{sp}_{\mathcal{F}}}^{-1}(x)$ but the equality usually doesn't hold and one has to explore carefully the relation between these two sets. We identify a class of kimberlites (respectively smelted kimberlites), which we call Orapian kimberlites (respectively Orapian smelted kimberlites), for which tubular neighborhoods behave as nicely as possible. We prove that p-adic Beilinson-Drinfeld Grassmanians and moduli spaces of shtukas at hyperspecial level are Orapian kimberlites and Orapian smelted kimberlites, respectively. Orapianess will imply that $|\widehat{\mathcal{F}}_{/x}|$ is dense within ${\mathrm{sp}_{\mathcal{F}}}^{-1}(x)$, proving the density part of theorems 1 and 2.

Once we know that these v-sheaves are Orapian kimberlites most of the work required to prove theorem 1 and theorem 2 is subsumed by our formalism. The last thing that remains to be proved is that the preimages of the specialization map are non-empty and connected. As we have briefly mentioned, one can apply theorem 3 to reduce the non-empty and connected part of theorem 2 to the similar claim of theorem 1.

To better understand the preimages of the specialization map in the case of theorem 1 we construct a "Demazure resolution" in the spirit of [28] §19.3. Contrary to the case of GL_n, for other reductive groups the subset of dominant minuscule cocharacters doesn't generate the monoid of dominant cocharacters. This failure turns out to be a rather subtle matter and forced us to introduce what we call below "parahoric loop groups" for Chevalley groups. These groups are attached to points in the (Bruhat-Tits) apartment of G associated to T. They are subsheaves of the usual loop group given by the condition that their value on geometric points $\mathrm{Spa}(C^\sharp, {C^\sharp}^+)$ is precisely the corresponding parahoric subgroup of $G(B_{\mathrm{dR}})$ that the Bruhat-Tits theory attaches to the same point in the apartment and the (discrete valuation) period ring B^+_{dR}.

Although it is clear what parahoric loop groups should evaluate to on geometric points, finding the right definition that works for mixed characteristic families is part of the challenge. Our construction relies on a construction of Pappas and Zhu and on theorem 3.1 of [21]. The tradeoff to introducing parahoric loop groups is that now our "Schubert varieties" are indexed by elements of the Iwahori-Weyl group, and they can all be resolved using simple reflections in the affine Weyl group. The functoriality of the specialization map allows us to reduce questions on the target of the resolution to questions on the source of the resolution. The v-sheaves that serve as source of this resolution, which we call Demazure kimberlites, are much easier to understand. We hope that our version of the "Demazure resolution" finds applications in other contexts as well.

Finally, we explain the content of each section:

- In the first section of chapter 1 we give a short review of the theory of diamonds, the v-topology and some facts about spectral topological spaces. We also review Scholze's $\diamond$ functor that takes as input a pre-adic space over $\mathbb{Z}_p$ and returns as output a v-sheaf.
- In the second section of chapter 1 we introduce and study what we call the olivine spectrum of a Huber pair. As we have mentioned already, for a pre-adic space X over $\mathbb{Z}_p$ Scholze constructs a surjective map of topological spaces $|X^\diamond| \to |X|$. This map is a homeomorphism whenever X is

analytic, but the map will not be injective whenever X is not analytic, and in pathological cases not even a quotient map. The olivine spectrum is a very concrete topological space that one can associate to any Huber pair without any mention to the theory of perfectoid spaces or diamonds. The main result of the section is as follows: if (A, A^+) a Huber pair over $\mathbb{Z}_p$ such that A is its own ring of definition and if $X = \mathrm{Spa}(A, A^+)$, then $|X^\diamond|$ can be identified with the olivine spectrum of (A, A^+).

- In the third section of chapter 1 we introduce and study a reduction functor that takes as input a small v-sheaf in the category of characteristic p perfectoid spaces, and returns a small v-sheaf in the category of perfect schemes in characteristic p. This functor generalizes the construction that assigns to a formal scheme topologically of finite type over $\mathbb{Z}_p$ the perfection of its reduced special fiber. To the author's knowledge, although this construction is simple, it had not been considered in the literature before. As we have mentioned, this reduction functor will construct the target of our specialization map.

- In the fourth section of chapter 1 we develop the theoretical framework to study the specialization map. We introduce specializing v-sheaves, kimberlites, and smelted kimberlites. We introduce tubular neighborhoods and relate them with preimages of the specialization map. We define Orapian kimberlites, incorporating some "finiteness" notions that are tailored to control the behavior of the preimages of the specialization map.

- In a lighther note, we provide at the end chapter 1 a small explanation of the unusual terminology we decided to use in our theory of specialization.

- In the first section of chapter 2 we review the main geometric objects of modern p-adic Hodge theory. We review Kedlaya and Liu's theory of vector bundles on adic spaces, Kedlaya's GAGA type theorems and Anschűtz' theorem on extending G-torsors on the punctured spectrum of A_{inf}. One may think of Anschűtz' result as a statement over a point, and Scholze proves in [28] a small improvement to this theorem by considering what we call here a product of points. We review Scholze's proof with our application in mind.

- In the second section of chapter 2 we study the specialization map for p-adic Beilinson-Drinfeld Grassmanians. We construct parahoric loop groups and construct Demazure kimberlites, which are the source of our "Demazure resolution". We prove that Demazure kimberlites are Orapian kimberlites which allows us to prove that p-adic Beilinson-Drinfeld Grassmanians are also Orapian kimberlites with non-empty connected tubular neighborhoods. We prove theorem 1.

- In the third section of chapter 2 we study the specialization map for moduli spaces of mixed-characteristic shtukas. We prove that moduli spaces of mixed characteristic shtukas at hyperspecial level are Orapian smelted kimberlites. We prove theorems 2 and 3.

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Chapter 1

Theoretical aspects of the specialization map

Throughout this article we assume that the reader is familiar with the basic theory of perfectoid spaces as discussed in ([28] §7) or ([26] §3). In most of our proofs we ignore the set-theoretic subtleties that arise from the theory, but we inherit the usage of the term "small" that is used to address such issues. We provide some indications on how to proceed when set-theoretic carefulness is absolutely necessary.

1.1 The v-topology

1.1.1 Diamonds and small v-sheaves

We let Perfd denote the category of perfectoid spaces and Perf the subcategory of perfectoid spaces in characteristic p. The following definition is taken from ([26] 7.8).

Definition 1.1.1. *Given a map of perfectoid spaces $f : Y \to X$ we say:*

1. *f is affinoid pro-étale if $Y = \mathrm{Spa}(S, S^+)$, $X = \mathrm{Spa}(R, R^+)$ and the map f is a small cofiltered limit of maps $f_i : \mathrm{Spa}(S_i, {S_i}^+) \to \mathrm{Spa}(R, R^+)$ where each f_i is étale.*
2. *f is pro-étale if for every $y \in Y$, there is an open neighborhood $V \subseteq Y$ containing y and an open $U \subseteq X$ satisfying $f(V) \subseteq U$ and $f|_V : V \to U$ is affinoid pro-étale.*

We can endow Perfd with two Grothendieck topologies, called the pro-étale topology and v-topology respectively, as follows:

Definition 1.1.2. *(See ([26] 8.1))*

1. *A family $\{f_i : Y_i \to X\}_{i\in I}$ of maps in* Perfd *is a cover for the pro-étale topology if Each f_i is pro-étale and for every quasi-compact open $U \subseteq X$ there is a finite subset $I_U \subseteq I$ and quasi-compact open subsets $V_i \subseteq Y_i$ for all $i \in I_U$, such that $U = \bigcup f_i(V_i)$*
2. *A family $\{f_i : Y_i \to X\}_{i\in I}$ of maps in* Perfd *is a cover for the v-topology if For every quasi-compact open $U \subseteq X$ there is a finite subset $I_U \subseteq I$ and quasi-compact open subsets $V_i \subseteq Y_i$ for all $i \in I_U$, such that $U = \bigcup f_i(V_i)$*

Remark 1.1.3. *To make the pro-étale and v topologies useful, it is important to add the quasi-compactness hypothesis. Indeed, since open embeddings are étale the inclusion of points are pro-étale, but we do not want to consider the collection of inclusions of points as a cover.*

The following example of a cover for the v-topology will be used repeatedly.

Example 1.1.4. *Let $\mathrm{Spa}(A, A^+)$ be an affinoid perfectoid space $\varpi \in A^+$ and a choice of a pseudo-uniformizer, we consider the following construction. For every point $x \in |\mathrm{Spa}(A, A^+)|$ consider the inclusion of affinoid residue field $\iota_x : \mathrm{Spa}(k(x), k(x)^+) \to \mathrm{Spa}(A, A^+)$. Note that by ([24] 6.7) each $\mathrm{Spa}(k(x), k(x)^+)$ is perfectoid. Consider $R^+ := \prod_{x\in|\mathrm{Spa}(A,A^+)|} k(x)^+$ as a topological ring with the ϖ-adic topology and let $R = R^+[\frac{1}{\varpi}]$. We have that $\mathrm{Spa}(R, R^+)$ is perfectoid and that the natural map $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(A, A^+)$ is a cover for the v-topology.*

Definition 1.1.5. *Given a set I and a collection of tuples $\{(C_i, {C_i}^+), \varpi_i\}_{i\in I}$ we construct an adic space $\mathrm{Spa}(R, R^+)$. Here each C_i is an algebraically closed non-Archimedean field, the C_i^+ are open and bounded valuation subrings of C_i, and ϖ_i is a choice of pseudo-uniformizer. We let $R^+ := \prod_{i\in I} C_i^+$, $\varpi = (\varpi_i)_{i\in I}$, we give R^+ the ϖ-adic topolgy and we let $R := R^+[\frac{1}{\varpi}]$. Any space constructed in this way will be called a product of points.*

Remark 1.1.6. *We point out that different choices of pseudo-uniformizers $(\varpi_i)_{i\in I}$ will give rise to different adic spaces. Indeed, in general $R \subseteq \prod_{i\in I} C_i$ but if I is infinite this is a proper inclusion and the image of this inclusion depends of the choice of pseudo-uniformizers.*

Example 1.1.4 proves that products of points form a basis for the v-topology in the category of perfectoid spaces.

Scholze proves that the v-topology (and consequently the pro-étale topology) on Perfd is subcanonical ([26] 8.6). To simplify notation, we denote a perfectoid space and the sheaf it represents with the same letter. In case we need to make a distinction, whenever X is a perfectoid space the sheaf it represents will be denoted by h_X. From now on we will focus most of our attention to the site Perf endowed with either the pro-étale or the v-topology. Let us recall Scholze's category of diamonds.

Definition 1.1.7. *(See [26] 11.1) A pro-étale sheaf Y on* Perf *is a diamond if it can be written as X/R where X and R are representable by perfectoid spaces and $R \subseteq X \times X$ is an equivalence relation for which the two projections to X are pro-étale maps of perfectoid spaces.*

Given a diamond Y we can associate to it a topological space, denoted $|Y|$, as follows:

Definition 1.1.8. *We say that a map $p : \mathrm{Spa}(K, K^+) \to Y$ is a point if K is a perfectoid field in characteristic p and K^+ is an open and bounded valuation subring of K. Two points $p_i : \mathrm{Spa}(K_i, {K_i}^+) \to Y$, $i \in \{1, 2\}$, are equivalent if there is a third point $p_3 : \mathrm{Spa}(K_3, {K_3}^+) \to Y$, and surjective maps $q_i : \mathrm{Spa}(K_3, {K_3}^+) \to \mathrm{Spa}(K_i, {K_i}^+)$ making the following commutative diagram:*

$$\begin{tikzcd}
 & \mathrm{Spa}(K_1, {K_1}^+) \arrow[dr, "p_1"] & \\
\mathrm{Spa}(K_3, {K_3}^+) \arrow[ur, "q_1"] \arrow[rr, "p_3"] \arrow[dr, "q_2"'] & & Y \\
 & \mathrm{Spa}(K_2, {K_2}^+) \arrow[ur, "p_2"'] &
\end{tikzcd}$$

We let $|Y|$ denote the set of equivalence classes of points of Y.

Scholze proves that if Y has a presentation X/R, then there is canonical bijection of sets between $|Y|$ and $|X|/|R|$ (where $|X|$ and $|R|$ are the topological space corresponding to the perfectoid spaces X and R). This gives a surjective map $|X| \to |Y|$ and we give $|Y|$ the quotient topology for such a map.

Proposition 1.1.9. *(See [26] 11.13) Let Y be a diamond. The topology on $|Y|$ is independent of the presentation of Y as a quotient $Y = X/R$, with X and R perfectoid spaces.*

We remark that if X is a perfectoid space, then the sheaf h_X represented by X is a diamond and that $|h_X|$ is canonically homeomorphic to $|X|$.

We refer to sheaves on Perf for the v-topology as v-sheaves and we say that a v-sheaf X is small if it admits a surjective map from a representable sheaf. This is a set theoretic condition.

Proposition 1.1.10. *(See [26] 11.9) Every diamond is a small v-sheaf.*

Recall that any Grothendieck site has an intrinsic notion of quasi-compactness. Quasi-compact v-sheaves are other important examples of small v-sheaves.

We denote by $\widetilde{\mathrm{Perf}}$ the category of small v-sheaves, it may be constructed as follows. Given a cut-off cardinal κ (see [26] §4 and §8 for details) we denote by Perf_κ the category of κ-small perfectoid spaces in characteristic p and by $\widetilde{\mathrm{Perf}}_\kappa$ the topos of sheaves for the v-topology on this category. Objects in this topos will be called κ-small v-sheaves. We have natural fully-faithful embeddings $\widetilde{\mathrm{Perf}}_\kappa \subseteq \widetilde{\mathrm{Perf}}_\lambda$ for $\kappa < \lambda$ and we define $\widetilde{\mathrm{Perf}} = \bigcup_\kappa \widetilde{\mathrm{Perf}}_\kappa$ as a big filtered colimit over all cut-off cardinals κ.

Scholze associates to any small v-sheaf a topological space. The definition is almost identical to 1.1.8, the key point being that if $X \to Y$ is a map of small v-sheaves with X a diamond then $R = X \times_Y X$ is also a diamond and $Y = X/R$ ([26] 12.3). Scholze then defines $|Y|$ as $|X|/|R|$ with the quotient topology.

Proposition 1.1.11. *(See [26] 12.7) Let Y be a small v-sheaf. The set of equivalence classes of points of Y is in canonical bijection with $|X|/|R|$ for any presentation $Y = X/R$ with X and R diamonds. Moreover, the topology induced this way is independent of the presentation.*

Given a topological space T we can consider a presheaf on Perf, denoted $\underline{T}$, defined as

$$\underline{T}(R, R^+) = \{f : |\mathrm{Spa}(R, R^+)| \to T \mid f \textit{ is continuous}\}$$

This forms a v-sheaf but we warn the reader that it might not be small. The functors $(|\cdot|, \underline{\cdot})$ form an adjunction between the category of v-sheaves and the category of topological spaces. Given a small v-sheaf X we get by adjunction a canonical morphism:

$$X \to \underline{|X|}$$

A morphism of small v-sheaves $j : U \to X$ is said to be open if it is relatively representable in perfectoid spaces and after basechange by a perfectoid space it becomes an open embedding of perfectoid spaces. The following proposition shows that this is a purely topological notion.

Proposition 1.1.12. *(See [26] 11.15 and 12.9) Let Y be a small v-sheaf and let $|V|' \subseteq |Y|$ be an open subset. Define V as the cartesian product:*

$$\begin{tikzcd} V \arrow[r] \arrow[d] & \underline{|V|'} \arrow[d] \\ Y \arrow[r] & \underline{|Y|} \end{tikzcd}$$

The following assertions hold:

1. *The map $V \to Y$ is a an open embedding of small v-sheaves.*

2. *The induced map $|V| \to |Y|$ is an open embedding of topological spaces and factors through a homeomorphism to $|V|'$.*

3. *Every open embedding of small v-sheaves is isomorphic to one constructed in this way.*

4. *If Y is a diamond then V is also a diamond.*

1.1.2 Spectral spaces and locally spatial diamonds

The category of diamonds is too general for some purposes and one can construct many "pathological" examples of diamonds that do not arise from an algebro-geometric context. To control this flexibility Scholze considers some restricitions on the underlying topological space of a diamond.

We begin by recalling the basic theory of spectral topological spaces. This material is taken from section §2 of [26] where most of the proves can be found.

Definition 1.1.13. *For topological spaces S, T and a continuous map $f : S \to T$ we say that:*

1. *T is spectral if it is quasi-compact, quasi-separated, and it has a basis of open neighborhoods stable under intersection that consists of quasi-compact and quasi-separated subsets.*

2. *T is locally spectral if it admits an open cover by spectral spaces.*

3. *f is a spectral map of spectral spaces if S and T is are spectral and f is quasi-compact.*

4. *f is a spectral map of locally spectral spaces if S and T are locally spectral and for every quasi-compact open $U \subseteq S$ and quasi-compact open $V \subseteq T$ with $f(U) \subseteq V$ the function $f|_U : U \to V$ is spectral.*

Theorem 1.1.14. *(Hochsteter) For a topological space T the following conditions are equivalent:*

1. *T is spectral.*

2. *T is homeomorphic to the spectrum of a ring.*

3. *T is a projective limit of finite T_0 topological spaces.*

Moreover, the category of spectral topological spaces with spectral maps is equivalent to the pro-category of finite T_0 topological spaces.

Given a spectral space T, we say that a subset S is constructible if it lies in the boolean algebra formed by quasi-compact open subsets of T. For a locally spectral space T, a subset S is constructible if for every quasi-compact open subset $U \subseteq T$ the subset $S \cap U$ is constructible in U. The patch (or constructible) topology on T is the one in which constructible subsets form a basis for the topology. A spectral space is Hausdorff and profinite for its patch topology and a locally spectral space is Hausdorff and locally profinite for the patch topology.

Proposition 1.1.15. *A continuous map of locally spectral spaces $f : S \to T$ is spectral if and only if it is continuous for the patch topology.*

Definition 1.1.16. *1. A map of topological spaces $f : S \to T$ is generalizing if for elements $t_1, t_2 \in T$ and $s_1 \in S$ such that $f(s_1) = t_1$ and t_2 generalizes t_1, there exists an element s_2 generalizing s_1 with $f(s_2) = t_2$.*

2. *A map of topological spaces $f : S \to T$ is specializing if for elements $t_1, t_2 \in T$ and $s_1 \in S$ such that $f(s_1) = t_1$ and t_2 specializes from t_1, there exists an element s_2 specializaing from s_1 with $f(s_2) = t_2$.*

For a locally spectral space T we say that a subset is pro-constructible if it is closed for the patch topology, or equivalently if it is an arbitrary intersection of constructible subsets. The following will be really useful for our purposes.

Proposition 1.1.17. *(See [26] 2.4) Let T be a spectral space and $S \subseteq T$ a pro-constructible subset. The closure $\overline{S}$ of S in T consists of the points that specialize from a point in S.*

Corollary 1.1.18. *Let $f : S \to T$ be a spectral map of spectral spaces. If f is specializing then it is also a closed map.*

We warn the reader that the analogue of 1.1.18 for locally spectral spaces does not hold.

Proposition 1.1.19. *(See [26] 2.5) Let $f : S \to T$ be a spectral map of spectral topological spaces. Assume f is surjective and generalizing, then it is a quotient map.*

One can think of spectral spaces as the topological spaces that arise from an algebro-geometric situation. For this reason we will restrict our attention to diamonds that have this behavior.

Definition 1.1.20. *(See [26] 11.17) Let X be a diamond. We say that X is a spatial diamond if it is quasi-compact, quasi-separated and $|X|$ has a basis of open neighborhoods of the form $|U|$ where $U \subseteq X$ is a quasi-compact open embedding. We say that X is locally spatial if it has an open cover by spatial diamonds.*

As promised, the topology of spatial diamonds is spectral. Neverhteless, we remark that a diamond that has a spectral underlying topological space might not be spatial since the quasi-compactness and quasi-separatedness conditions of definition 1.1.20 are imposed on the topos-theoretic sense.

Proposition 1.1.21. *(See [26] 11.18, 11.19) Let X and Y a be locally spatial diamonds and $f : X \to Y$ a morphism of v-sheaves. The following assertions hold:*

1. *$|X|$ is a locally spectral topological space.*
2. *Any open subfunctor $U \subseteq X$ is a locally spatial diamond.*
3. *$|X|$ is quasi-compact (respectively quasi-separated) as a topological space if and only if X is quasi-compact (respectively quasi-separated) as a v-sheaf.*
4. *The topological map $|f|$ is spectral and generalizing. In particular, if $|X|$ is quasi-compact and $|f|$ is surjective then by proposition 1.1.19 it is also a quotient map.*

1.1.3 Pre-adic spaces as v-sheaves

The theory of diamonds is mainly of "analytic" nature. On the other hand, we will need to consider some spaces that have a scheme-theoretic and formal-scheme-theoretic flavor instead. The category of v-sheaves allows us to consider this three types of spaces at the same time. In what follows, we show (following the Berkeley notes) how to consider any pre-adic space over $\mathbb{Z}_p$ as a v-sheaf. Interestingly, this functor is far from being fully-faithful but one can still recover and then generalize many key aspects of the classical theory.

Let Caff^{op} denote the opposite category to the category of complete Huber pairs. This category can be regarded as a site when we consider the topology generated by rational covers. Although the topology in this site is not subcanonical any Huber pair $(A, A^+) \in \mathrm{Caff}^{op}$ defines a sheaf $\mathrm{Spa}(A, A^+)^Y : \mathrm{Caff}^{op} \to \mathrm{Sets}$ by taking sheafification of the functor $(B, B^+) \mapsto Hom_{\mathrm{Caff}}((A, A^+), (B, B^+))$. Scholze and Weinstein define the category of pre-adic spaces as sheaves $\mathcal{F} : \mathrm{Caff}^{op} \to \mathrm{Sets}$ that are locally isomorphic to $\mathrm{Spa}(A, A^+)^Y$ (See [28] 3.4.1 for details).

Definition 1.1.22. *1. We define the presheaf $\mathbb{Z}_p^\diamond$ on $Perf$ as the moduli of untilts, more precisely:*

$$\mathbb{Z}_p^\diamond(Y) = \{(Y^\sharp, \iota)\}/\cong$$

Where $Y^\sharp$ is a perfectoid space in Perfd *and $\iota : (Y^\sharp)^\flat \to Y$ is an isomorphism of perfectoid spaces in characteristic p.*

2. Given a pre-adic space $X/\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ we define the presheaf $X^\diamond$ on $Perf$ as:

$$X^\diamond(Y) = \{(Y^\sharp, \iota, f)\}/\cong$$

Where $Y^\sharp \in \mathrm{Perfd}$, $\iota : (Y^\sharp)^\flat \to Y$ is an isomorphism of perfectoid spaces in characteristic p, and $f : Y^\sharp \to X$ is a morphism of pre-adic spaces.

Notice there is a canonical morphism $X^\diamond \to \mathbb{Z}_p^\diamond$ given by forgetting the last entry of data.

Proposition 1.1.23. *(See [28] 18.1.1) For any pre-adic space X (not necessarily analytic) over $\mathbb{Z}_p$, the presheaf $X^\diamond$ is a small v-sheaf.*

From now on, given a Huber pair (A, A^+) we will denote the v-sheaf $(\mathrm{Spa}(A, A^+)^Y)^\diamond$ by $\mathrm{Spd}(A, A^+)$. We make some remarks on this construction.

Remark 1.1.24. *1. For any perfectoid space X we have that $X^\diamond \cong h_{X^\flat}$ as v-sheaves. (See [26] 15.2).*

2. For any analytic pre-adic space X over $\mathbb{Z}_p$, the functor $X^\diamond$ is a locally spatial diamond and $|X^\diamond| \cong |X|$. (See [26] 15.6).

3. For any pre-adic space over $\mathbb{Z}_p$ there is a surjective map of topological spaces $|X^\diamond| \to |X|$ (See [28] 18.2.2).

4. If $\mathrm{PreAd}_{\mathbb{Z}_p}$ denotes the category of pre-adic spaces over $\mathbb{Z}_p$ then $\diamond : \mathrm{PreAd}_{\mathbb{Z}_p} \to \widetilde{\mathrm{Perf}}$ commutes with limits and colimits. More precisely, if X_i is a family of pre-adic spaces indexed by small category I and the functor $\varinjlim_{i\in I} X_i$ (respectively $\varprojlim_{i\in I} X_i$) is represented by a pre-adic space X then $X^\diamond = \varinjlim_{i\in I} X_i^\diamond$ (respectively $X^\diamond = \varprojlim_{i\in I} X_i^\diamond$). Indeed, when computing both constructions the only difference is topology that one uses to sheafify, but if a colimit is represented by a pre-adic space by proposition 1.1.23 it is already a v-sheaf.

We will need the following two lemmas later on.

Lemma 1.1.25. *Suppose that (A, A^+) is a perfect Huber pair in characteristic p that has the discrete topology, then there is a unique morphism of v-sheaves $\mathrm{Spd}(A, A^+) \to \mathbb{Z}_p^\diamond$. It is given by the composition $\mathrm{Spd}(A, A^+) \to \mathbb{F}_p^\diamond \to \mathbb{Z}_p^\diamond$.*

Proof. We let $S = \mathrm{Spd}(A, A^+)$, $R = A((t^{\frac{1}{p^\infty}}))$, $R^+ = A^+ + (t^{\frac{1}{p^\infty}})A[[t^{\frac{1}{p^\infty}}]]$, $X = \mathrm{Spa}(R, R^+)$ and $X' = X \times_S X$, notice that X is an affinoid perfectoid space and X' is a perfectoid space. The natural map $X \to \mathrm{Spd}(A, A^+)$ is a surjective map of v-sheaves (or v-cover), and we get an equalizer diagram:

$$0 \to Hom(\mathrm{Spd}(A, A^+), \mathbb{Z}_p^\diamond) \to Hom(X, \mathbb{Z}_p^\diamond) \rightrightarrows Hom(X', \mathbb{Z}_p^\diamond).$$

Recall that specifying an untilt $X^\sharp$ of X amounts to specifying an ideal of $W(R^+)$ generated by an element ξ of the form $\xi = p + [t^{\frac{1}{p^k}}] \cdot \alpha$ with $\alpha \in W(R^+)$. Let $\mathcal{I} \subseteq W(R^+)$ be the ideal corresponding to an untilt in the image of $Hom(S, \mathbb{Z}_p^\diamond)$, then it is stable under the automorphisms of X over S. Let $\phi_n : R \to R$ the automorphism preserving A and mapping $t \mapsto t^n$. If $\mathcal{I}$ is generated by $\xi = p + [t^{\frac{1}{p^k}}] \cdot \alpha$, then $\phi_n(\xi) \in \mathcal{I}$ since ξ and $\phi_n(\xi)$ define the same untilt. Recall that $W(R^+)$ comes equipped with the $(p, [t])$-adic and that $\mathcal{I}$ is closed subset of $W(R^+)$ for this topology. The sequence $\phi_n(\xi) = p + [t^n p^k] \cdot \phi_n(\alpha)$ converges to p in $\mathcal{I}$, which implies that $p \in \mathcal{I}$ and that $X^\sharp$ is the charactetic p untilt. $\square$

Lemma 1.1.26. *Suppose that (A, A^+) and (B, B^+) are two complete Huber pairs over $\mathbb{Z}_p$, such that A is perfect of characteristic p and has the discrete topology, then every morphism of v-sheaves* $\mathrm{Spd}(A, A^+) \to \mathrm{Spd}(B, B^+)$ *comes from a unique morphism of adic spaces* $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(B, B^+)$.

Proof. Let R, R^+, X and X' be as in the proof of lemma 1.1.25. We get again an equalizer diagram:

$$0 \to Hom(\mathrm{Spd}(A, A^+), \mathrm{Spd}(B, B^+)) \to Hom(X, \mathrm{Spd}(B, B^+)) \rightrightarrows Hom(X', \mathrm{Spd}(B, B^+)).$$

Since X is affinoid perfectoid a homomorphism $f : X \to \mathrm{Spd}(B, B^+)$ is given by an untilt $R^\sharp$ and a continuous ring map $f^* : B \to R^\sharp$ such that $f^*(B^+) \subseteq R^{+,\sharp}$. By lemma 1.1.25 the untilt must be R.

A necessary condition for such a morphism to glue, is that $f^*(B)$ must be invariant under any automorphism of R over A. In particular, we may replace t by any topological nilpotent element in R without changing the image of $f^*(B)$. Take an element $b \in B$, we want to show $f^*(b) \in A$. Observe that $t^{p^n} \cdot f^*(b)$ is topologically nilpotent for big enough n. Replacing by $t \mapsto t^{\frac{1}{p^m}}$ we conclude that $t^{p^n} f^*(b)$ is topologically nilpotent for all $n \in \mathbb{Z}$. This proves $f^*(b)$ is power-bounded which gives $f^*(b) \in A[[t^{\frac{1}{p^\infty}}]]$. We can write $f^*(b)$ as $a_0 + t^{\frac{1}{p^m}} q$ with $a_0 \in A$ and $q \in A[[t^{\frac{n}{p^m}}]]$ with $\frac{n}{p^m} \geq 1$. Since the second term converges to 0 under the substitution $t \mapsto t^n$ we see that $f^*(b) = a_0$. This defines a map of rings $B \to A$. Since the subspace topology of A in R is the discrete topology this ring morphism is continuous if and only if the original one was. Finally, we see that B^+ maps to $R^+ \cap A$ which is easily seen to be A^+. $\square$

1.2 The olivine spectrum

As we will see below, given a pre-adic space X over $\mathbb{Z}_p$ the map $|X^\diamond| \to |X|$ of remark 1.1.24 will usually not be injective when X has non-analytic points. Although the map is always surjective, it might not be a quotient map in pathological cases. To develop our theory of specialization map we need a concrete description of $|\mathrm{Spd}(A, A^+)|$ for adic spaces $\mathrm{Spa}(A, A^+)$ over $\mathbb{Z}_p$ that are their own ring of definition. To tackle this difficulty we introduce below what we call the *olivine spectrum* of a Huber pair, which is a very small variation of Huber's adic spectrum. The interest in studying this space is that if (B, B^+) is a complete Huber pair over $\mathbb{Z}_p$ such that B is its own ring of definition, then the topological space $|\mathrm{Spd}(B, B^+)|$ is homeomorphic to the olivine spectrum of (B, B^+). Given the extremely technical nature of this section we recommend the reader to skip most if not all of the proofs on a first read.

For the rest of this section, we fix (B, B^+) a complete Huber pair (not necessarily over $\mathbb{Z}_p$).

1.2.1 Review, terminology and conventions

We assume the reader to be familiar with the construction of Huber's adic spectrum, $\mathrm{Spa}(B, B^+)$, but we review some key aspects and definitions. We also fix some terminology.

1. Given $x \in \mathrm{Spa}(B, B^+)$ we define the support $supp(x) \subseteq B$ as the set of elements $b \in B$ for which $|b|_x = 0$. This is a prime ideal of B.

2. We say that a point $x \in \mathrm{Spa}(B, B^+)$ is non-analytic if *supp*(x) is an open ideal of B, we say it is analytic otherwise.

3. Given an equivalence class of valuations on B, say represented by $|\cdot|_x : B \to \Gamma_x \cup \{0\}$, and a convex subgroup $H \subseteq \Gamma_x$, we define a second equivalence class of valuations represented by $|\cdot|_y : B \to (\Gamma_x/H) \cup \{0\}$ with $|b|_y = |b|_x + H \in \Gamma_x/H$ when $|b|_x \neq 0$ and $|b|_y = 0$ when $|b|_x = 0$. Any equivalence class of valuations constructed in this way is called a *vertical generization* of x.

4. Given a complete Huber pair (B, B^+) and a point $x \in \mathrm{Spa}(B, B^+)$ there is a residue field map of complete Huber pairs $\iota_x^* : (B, B^+) \to (K_x, {K_x}^+)$. In this case K_x is either a discrete field or a complete non-Archimedean field. In both cases, K_x^+ is an open and bounded valuation subring of K_x. The induced map $\iota_x : \mathrm{Spa}(K_x, {K_x}^+) \to \mathrm{Spa}(B, B^+)$ is a homeomorphism onto the subspace of $\mathrm{Spa}(B, B^+)$ consisting of continuous vertical generizations of x. The map satisfies the following universal property: For any map of complete Huber pairs $f^* : (B, B^+) \to (A, A^+)$ such that $f(\mathrm{Spa}(A, A^+)) \subseteq \mathrm{Spa}(B, B^+)$ consists of vertical generizations of x, there is a unique factorization $f^* = g^* \circ \iota_x^*$.

5. Vertical generizations and residue field maps have the following compatibility. Fix $x \in \mathrm{Spa}(B, B^+)$ with residue field $(K_x, {K_x}^+)$, and consider K_x° the subring of power-bounded elements. Given y a continuous vertical generizations of x we can associate a valuation subring K_y^+ by letting $K_y^+ = \{b \in K_x \mid |b|_y \leq 1\}$. This association gives a bijection between the set of continuous vertical generizations of x and valuation subrings of K_x° containing K_x^+. Moreover, in this case the residue field at y is (K_x, K_y^+).

6. We say that a valuation x is *trivial* if it is equivalent to some valuation for which $\Gamma_x = \{1\}$. The residue field of a trivial valuation is discrete.

7. We say that a valuation is *microbial* if it has a non-trivial rank 1 vertical generization.

8. For technical reasons that will become clear to the reader below, we take the convention of considering trivial valuations as rank 1 valuations.

9. Given a valuation $|\cdot|_x$ of B we define the *characteristic subgroup* of $|\cdot|_x$, denoted by $c\Gamma_x$, as the smallest convex subgroup of Γ_x containing all elements of the form $\gamma = |b|_x$ for $b \in B$ with $1 \leq \gamma$.

10. Given an equivalence class of valuations $|\cdot|_x$ and a convex subgroup $H \subseteq \Gamma_x$ containing $c\Gamma_x$, we define a second equivalence class of valuations $|\cdot|_y$ with $|\cdot|_y : B \to H \cup \{0\}$. We let $|b|_y = |b|_x$ if $|b|_x \in H$ and we let $|b|_y = 0$ otherwise. Any equivalence class of valuations constructed in this way is continuous if $|\cdot|_x$ is continuous. An equivalence classes of valuations constructed in this way are called *horizontal specialization* of x.

11. Horizontal specializations and residue field maps have the following compatibility. Fix $x \in \mathrm{Spa}(B, B^+)$ with residue field $(K_x, {K_x}^+)$. We let K_B be the smallest valuation subring of K_x containing K_x^+ and the image of B in K_x. We get a natural map of Huber pairs $(B, B^+) \to (K_B, K_x^+)$, we consider the induced map $f : \mathrm{Spec}(K_B) \to \mathrm{Spec}(B)$. Horizontal specializations of x are in bijection with prime ideals of B that are in the image of f. Given a convex subgroup H containing $c\Gamma_x$ we can describe the prime ideal $\mathfrak{p}_y$ associated to y as the set of elements of B with $|b|_x < \gamma$ for all $\gamma \in H$. We will denote $|\cdot|_y$ as $|\cdot|_x/\mathfrak{p}_y$.

12. Given a topological space T we construct a partial order on elements of T by letting $t_1 \leq t_2$ whenever $t_1 \in \overline{\{t_2\}}$. We call this partial order the *generization pattern* of T.

13. Vertical generizations and horizontal specializations completely describe the generization pattern of $\mathrm{Spa}(B, B^+)$. More precisely, if y is a vertical generization of x then $x \in \overline{\{y\}}$ and we let xRy. If z is a horizontal specialization of x then $z \in \overline{\{x\}}$ and we let zRx. The generization pattern of $\mathrm{Spa}(B, B^+)$ is the transitive closure of the relation R.

1.2.2 Definitions and basic properties

Definition 1.2.1. *1. We let* $\mathrm{Spo}(B, B^+)$, *denote the subset of* $\mathrm{Spa}(B, B^+) \times \mathrm{Spa}(B, B^+)$ *consisting of pairs* $x = (|\cdot|_x^h, |\cdot|_x^a)$ *such that* $|\cdot|_x^a$ *is a rank* 1 *valuation and a vertical generization of* $|\cdot|_x^h$.

2. *Given two elements* $b_1, b_2 \in B$ *we let* $U_{b_1 \leq b_2 \neq 0}$ *to be the set*

$$\{x \in \mathrm{Spo}(B, B^+) \mid |b_1|_x^h \leq |b_2|_x^h \neq 0\},$$

we call such subsets classical localizations.

3. *Given two elements* $b_1, b_2 \in B$ *we let* $N_{b_1 << b_2}$ *to be the set*

$$\{x \in \mathrm{Spo}(B, B^+) \mid |b_1|_x^a < |b_2|_x^a \neq 0\},$$

we call such subsets analytic localizations.

4. *We give* $\mathrm{Spo}(B, B^+)$ *the topology generated by classical and analytic localizations, and we call the resulting topological space the olivine spectrum of* (B, B^+).

We will denote by $h : \mathrm{Spo}(B, B^+) \to \mathrm{Spa}(B, B^+)$ the projection onto the first coordinate. This map is continuous and both $\mathrm{Spo}(-, -^+)$ and h are functorial in the category of Huber pairs.

Definition 1.2.2. *Let* $x \in \mathrm{Spo}(B, B^+)$.

1. *We say that* x *is algebraic if* $|\cdot|_x^h$ *is trivial.*

2. *We say that* x *is non-analytic if* $|\cdot|_x^a$ *is trivial.*

3. *We say that* x *is analytic if* $|\cdot|_x^a$ *is non-trivial.*

4. *We say that* x *is formal analytic if it is analytic and* $|B|_x^a \leq 1$.

5. *We say that* x *is meromorphic analytic if it is analytic and the unique trivial vertical generization of* $|\cdot|_x^a$ *is also continuous.*

Notice that for any point $x \in \mathrm{Spo}(B, B^+)$ the set $h^{-1}(h(x))$ has at most one analytic point and at most one non-analytic point, also the cardinality of $h^{-1}(h(x))$ is either one or two.

We define *supp*(x) to be *supp*$(h(x))$, and if x is formal analytic we let *sp*(x) denote the prime ideal of elements of B for which $|b|_x^a < 1$. Notice that for a formal analytic point x we have $c\Gamma_{x^a} = \{1\}$.

Definition 1.2.3. *Let* x *and* y *be two points in* $\mathrm{Spo}(B, B^+)$.

1. *We say that* y *is a vertical generization of* x *(*x *is a vertical specialization respectively) if* $|\cdot|_x^a = |\cdot|_y^a$ *and* $|\cdot|_y^h$ *is a vertical generization of* $|\cdot|_x^h$ *in* $\mathrm{Spa}(B, B^+)$.

2. *We say that* y *is a meromorphic generization of* x *(meromorphic specialization respectively) if* y *is analytic,* x *is non-analytic and* $h(x) = h(y)$.

3. *We say that* y *is a formal generization of* x *(formal specialization respectively) if* y *is formal analytic,* x *is non-analytic* $sp(y) = supp(x)$ *and* $|\cdot|_x^h = |\cdot|_y^h / sp(y)$.

Given $x \in \mathrm{Spo}(B, B^+)$ let $\mathcal{I}^{\leq}(x)$ denote the set of generizations of x in $\mathrm{Spo}(B, B^+)$ and let $\mathcal{I}^{\leq}_{\overline{ver}}(x)$ denote the set of vertical generizations of x. If the context is clear, for a point $y \in \mathrm{Spa}(B, B^+)$ we will also use $\mathcal{I}^{\leq}_{\overline{ver}}(y)$ to denote the vertical generizations of y in $\mathrm{Spa}(B, B^+)$. Let us make some easy observations and set some convenient notation:

1. If x is non-analytic it has a meromorphic generization if and only if $|\cdot|^h_x$ is a microbial valuation, we denote this generization by x^{mer}.
2. An analytic point x is meromorphic analytic if and only if $h(x)$ is non-analytic in $\mathrm{Spa}(A, A^+)$. In this case it has a unique meromorphic specialization, we denote it by x_{mer}.
3. If x is formal analytic it has a unique formal specialization, we denote it by x_{for}. If x is non-analytic, we let x^{For} denote the set of formal generizations of x.

Example 1.2.4. *If $B = \mathbb{F}_p[[t]]$, the ring of formal power series over $\mathbb{F}_p$ endowed with the trivial topology, then $\mathrm{Spa}(B, B)$ consists of 3 points:*

$$\Big\{\eta = |\cdot|_\eta,\, s = |\cdot|_s,\, t = |\cdot|_t\Big\}$$

Here $|\cdot|_\eta$ is the trivial valuation with residue field $\mathbb{F}_p((t))$, $|\cdot|_s$ is the trivial valuation with residue field $\mathbb{F}_p$ and $|\cdot|_t$ is the (t)-adic valuation on $\mathbb{F}_p[[t]]$ with residue affinoid field $(\mathbb{F}_p((t)), \mathbb{F}_p[[t]])$. All three valuations have rank 1. The only non-trivial vertical generization in $\mathrm{Spa}(B, B)$ goes from $|\cdot|_t$ to $|\cdot|_\eta$.

On the other hand Spo(B) has 4 points:

$$\Big\{\eta = (|\cdot|_\eta, |\cdot|_\eta),\, s = (|\cdot|_s, |\cdot|_s),\, t^a = (|\cdot|_t, |\cdot|_t),\, t^h = (|\cdot|_t, |\cdot|_\eta)\Big\}$$

One can verify directly from the definition that $\{\eta\} = U_{1 \leq t \neq 0}$, $\{\eta, t^h, t^a\} = U_{0 \leq t \neq 0}$, $\{t^a\} = N_{t^2 << t}$ and $\{t^a, s\} = N_{t << 1}$, and that these are the only proper open susbsets.

In this example s and η are algebraic points, t^h is non-analytic and microbial, and t^a is both meromorphic and formal analytic. The generization pattern is as follows: η is a vertical generization of t^h, t^h is the meromorphic specialization of t^a, and s is the formal specialization of t^a.

The following proposition shows that vertical generizations, formal specializations and meromorphic specializations completely describe the generization pattern in $\mathrm{Spo}(B, B^+)$.

Proposition 1.2.5. *Let $x \in \mathrm{Spo}(B, B^+)$.*

1. *If x is analytic then $\mathcal{I}^{\leq}(x) = \mathcal{I}^{\leq}_{\overline{ver}}(x)$.*
2. *If x is non-analytic then $\mathcal{I}^{\leq}(x) = \mathcal{I}^{\leq}_{\overline{ver}}(x) \cup \mathcal{I}^{\leq}_{\overline{ver}}(x^{mer}) \cup (\bigcup_{z \in \mathcal{I}^{\leq}_{\overline{ver}}(x)} z^{For})$.*

Proof. We start by proving the right to left inclusion. Let $y \in \mathcal{I}^{\leq}_{\overline{ver}}(x) \cup \mathcal{I}^{\leq}_{\overline{ver}}(x^{mer}) \cup (\bigcup_{z \in \mathcal{I}^{\leq}_{\overline{ver}}(x)} z^{For})$ if x is non-analytic and let $y \in \mathcal{I}^{\leq}_{\overline{ver}}(x)$ otherwise. Since h is continuous and $h(y)$ is a generization of $h(x)$ in $\mathrm{Spa}(B, B^+)$ we have that y is contained in every classical localization containing x, so it is enough to check on analytic localizations. Suppose that $x \in N_{b_1 << b_2}$, if y is a vertical generization we have that $|\cdot|^a_y = |\cdot|^a_x$ so $y \in N_{b_1 << b_2}$. If x is non-analytic then $|b_1|^a_x = 0$ and $|b_2|^a_x = 1$, this implies that $|b_1|^a_{x^{mer}} = 0$ and that $|b_2|^a_{x^{mer}} \neq 0$, so $x^{mer} \in N_{b_1 << b_2}$ whenever x^{mer} exists. Moreover, for $y \in x^{For}$, we have that $sp(y) = supp(x)$ which gives $|b_1|^a_y < 1$, $|b_2|^a_y = 1$, and $x^{For} \in N_{b_1 << b_2}$.

Now we prove the left to right inclusion, for this take $y \in \mathcal{I}^{\leq}(x)$. Using classical localizations one can deduce that $supp(y) \subseteq supp(x)$, and if x is analytic we claim that $supp(y) = supp(x)$. Indeed, let $b \in B$ such that $|b|^a_x \notin \{0, 1\}$, and let $b_1 \in supp(x)$. If $|b|^a_x < 1$ then $|b|^a_y < 1$, which implies that y is analytic. Additionally, the inequalities $|b_1|^a_y < |b^n|^a_y$ must hold for all n since $x \in N_{b_1 << b^n}$ and $y \in \mathcal{I}^{\leq}(x)$. In a similar way, if $1 < |b|_x^a$ then $1 < |b|_y^a$ and we may look at the inequalities $|b_1 \cdot b^n|_y^a < |b|_y^a$ instead. In both cases the Archimedean property of rank 1 valuations imply that $b_1 \in supp(y)$. Since the only generizations of $h(x)$ in $\mathrm{Spa}(B, B^+)$ that have the same support as $h(x)$ are vertical generizations we must have $h(y) \in \mathcal{I}_{ver}^{\leq}(h(x))$. Consequently, $y \in \mathcal{I}_{ver}^{\leq}(x)$ holds in the analytic case.

Suppose now that x is non-analytic, if $supp(x) = supp(y)$ then we can reason as above to conclude $y \in \mathcal{I}_{ver}^{\leq}(x) \cup \mathcal{I}_{ver}^{\leq}(x^{mer})$. Let us assume there is $b \in supp(x) \setminus supp(y)$. Since $x \in N_{b<<1}$ we have $0 < |b|_y^a < 1$ and that y is analytic. By similar reasoning for all $b_1 \in B$ we have $|b \cdot b_1^n|_y^a < 1$ which implies that y is formal analytic, we also have that $supp(x) \subseteq sp(y)$. Moreover, for elements $b_2 \notin supp(x)$ we have $x \in U_{b \leq b_2^n \neq 0}$ for all n, this implies that $|b_2|_y^a = 1$ so $sp(y) = supp(x)$. If we let $z = y_{for}$ then $supp(z) = supp(x)$ and one can check from the construction of horizontal specializations that z is also a generization of x. As above, we may conclude that $h(z)$ is a vertical generization of $h(x)$, and since both z and x are non-analytic then z is a vertical generization of x. In other words, $z \in \mathcal{I}_{ver}^{\leq}(x)$ and $y \in z^{For}$. □

1.2.3 The olivine spectrum as a quotient space

Proposition 1.2.6. *If R is a Tate Huber pair the projection map $h : \mathrm{Spo}(R, R^+) \to \mathrm{Spa}(R, R^+)$ is a homeomorphism.*

Proof. Since (R, R^+) is Tate there are no trivial continuous valuations in $\mathrm{Spa}(R, R^+)$. In particular every point in $\mathrm{Spo}(R, R^+)$ is analytic and h is injective. If x^a is the maximal generization of x in $\mathrm{Spa}(R, R^+)$ then $h^{-1}(x) = \{(x, x^a)\}$. It is enough to prove that $h(N_{r_1<<r_2})$ is open. If $\varpi \in R$ is a topologically nilpotent unit, then

$$h(N_{r_1<<r_2}) = \bigcup_{0<n} \{z \in \mathrm{Spa}(R, R^+) \mid |r_1^n|_z \leq |r_2^n \varpi|_z \neq 0\}$$

Indeed, a point $x \in \mathrm{Spa}(R, R^+)$ is in $h(N_{r_1<<r_2})$ if $|r_1|_{x^a} < |r_2|_{x^a}$. By the Archimedean property of rank 1 valuations there is $n \in \mathbb{N}$ such that $(\frac{|r_1|_{x^a}}{|r_2|_{x^a}})^n \leq |\varpi|_{x^a}$ since ϖ is a unit and $|\varpi|_{x^a} > 0$. On the other hand, if $|r_1^n|_x \leq |r_2^n \cdot \varpi|_x$ we also have $|r_1^n|_{x^a} \leq |r_2^n \cdot \varpi|_{x^a} < |r_2^n|_{x^a}$ since ϖ is topologically nilpotent. □

If $m^* : (B, B^+) \to (R, R^+)$ is a map of Huber pairs, we denote by $\mathrm{Spo}(m) : \mathrm{Spo}(R, R^+) \to \mathrm{Spo}(B, B^+)$ the corresponding map of olivine spectra. In case (R, R^+) is Tate we have a continuous map $\mathrm{Spo}(m) \circ h^{-1} : \mathrm{Spa}(R, R^+) \to \mathrm{Spo}(B, B^+)$. When the context is clear, we also abbreviate $\mathrm{Spo}(m) \circ h^{-1}$ by $\mathrm{Spo}(m)$.

Let C_i with $i \in \{1, 2\}$ be a pair of algebraically closed non-Archimedean fields and let $C_i^+ \subseteq C_i$ be open and bounded valuation subrings. We will say that two maps $p_i : \mathrm{Spa}(C_i, {C_i}^+) \to \mathrm{Spa}(B, B^+)$ are equivalent if there is an algebraically closed non-Archimedean field C_3 and surjective maps $\pi_i : \mathrm{Spa}(C_3, {C_3}^+) \to \mathrm{Spa}(C_i, {C_i}^+)$ for which $p_1 \circ \pi_1 = p_2 \circ \pi_2$.

Proposition 1.2.7. *Given $(C_i, {C_i}^+)$ as above let $s_i \in \mathrm{Spa}(C_i, {C_i}^+)$ denote the unique closed point of this space. The maps p_i are equivalent if and only if $\mathrm{Spo}(p_1)(s_1) = \mathrm{Spo}(p_2)(s_2)$ in $\mathrm{Spo}(B, B^+)$. In particular, the set of equivalence classes of analytic geometric points as above are in natural bijection with points in $\mathrm{Spo}(B, B^+)$.*

Proof. By functoriality, if the maps are equivalent then $\mathrm{Spo}(p_1)(s_1) = \mathrm{Spo}(p_2)(s_2)$. We prove the converse, let x be the common image and let $(K_{h(x)}, {K_{h(x)}}^+)$ be the affinoid residue field of $h(x)$ in $\mathrm{Spa}(B, B^+)$. We split our analysis in three cases.

Case 1: Suppose that x is analytic and non-meromorphic, in this case $h(x)$ is analytic in $\mathrm{Spa}(B, B^+)$ and we may restrict to the open analytic locus $\mathrm{Spa}(B, B^+)^a$. In this locus h is a homeomorphism and the p_i are analytic geometric points of an analytic pre-adic space over the same underlying topological point. That p_1 and p_2 are equivalent in this case is a standard fact.

Case 2: Suppose that x is analytic meromorphic, we have that $h(x)$ is non-analytic in $\mathrm{Spa}(B, B^+)$. Let $K^\circ_{h(x)} := \{k \in K_{h(x)} \mid |k|^a_x \leq 1\}$ since $|\cdot|^a_x$ is non-trivial $K^\circ_{h(x)}$ is a proper valuation subring of $K_{h(x)}$. Choose $b \in B$ such that either $|b|^a_x < 1$ or $|b|^a_x > 1$, then the subspace topology on $(K^\circ_{h(x)}) \subseteq_{p_i^*} O_{C_i}$ coincides with the (b)-adic topology or, respectively, the $(\frac{1}{b})$-adic topology. Taking the completion with respect to this topology we get a commutative diagram of pre-adic spaces:

$$\begin{tikzcd}
& \mathrm{Spa}(C_1, {C_1}^+) \arrow[d, "p_1'"] \arrow[ddr, bend left, "p_1"] & \\
\mathrm{Spa}(C_2, {C_2}^+) \arrow[r, "p_2'"] \arrow[drr, bend right, "p_2"] & \mathrm{Spa}(\hat{K}_{h(x)}, \hat{K}^+_{h(x)}) \arrow[dr, "\iota_x"] & \\
& & \mathrm{Spa}(K_{h(x)}, {K_{h(x)}}^+)
\end{tikzcd}$$

The maps p_i' are surjective maps of analytic adic spaces and we may conclude that the p_i' are equivalent as in the first case above.

We point out for later reference that in this case $\mathrm{Spo}(\iota_x)$ defines a homeomorphism from $\mathrm{Spa}(\hat{K}_{h(x)}, \hat{K}^+_{h(x)})$ onto $\mathcal{I}^{\leq}_{ver}(x) \subseteq \mathrm{Spo}(B, B^+)$ while the classical map $\iota_x : \mathrm{Spa}(\hat{K}_{h(x)}, \hat{K}^+_{h(x)}) \to \mathrm{Spa}(K_{h(x)}, {K_{h(x)}}^+)$ is injective but not necessarily surjective onto $\mathcal{I}^{\leq}_{ver}(h(x)) \subseteq \mathrm{Spa}(B, B^+)$.

Case 3: Suppose that x is non-analytic, in this case $h(x)$ is non-analytic in $\mathrm{Spa}(B, B^+)$. We have that $(K_{h(x)}, {K_{h(x)}}^+)$ is given the discrete topology and the p_i factor through the natural inclusion $\iota_{h(x)} : \mathrm{Spa}(K_{h(x)}, {K_{h(x)}}^+) \to \mathrm{Spa}(B, B^+)$. Since $|\cdot|^a_x$ is trivial we have that $p_i(K_{h(x)}) \subseteq O_{C_i}$. After choosing pseudo-uniformizers $\varpi_i \in O_{C_i}$ we may extend the p_i to continuous adic maps of topological rings $p_i'^* : K_{h(x)}[[t]] \to O_{C_i}$ where $K_{h(x)}[[t]]$ is given the (t)-adic topology. These induce the following commutative diagram:

$$\begin{tikzcd}
& \mathrm{Spa}(C_1, {C_1}^+) \arrow[d, "p_1'"] \arrow[ddr, bend left, "p_1"] & \\
\mathrm{Spa}(C_2, {C_2}^+) \arrow[r, "p_2'"] \arrow[drr, bend right, "p_2"] & Spa(K_{h(x)}\left((t)\right), K^+_{h(x)} + t \cdot K_{h(x)}\left[[t]\right]) \arrow[dr, "\iota_x"] & \\
& & \mathrm{Spa}(K_{h(x)}, {K_{h(x)}}^+)
\end{tikzcd}$$

The maps p_i' are again surjective maps of analytic pre-adic spaces so we may conclude. In this case as well, the map $\mathrm{Spo}(\iota_x)$ defines a homeomorphism onto $\mathcal{I}^{\leq}_{ver}(x)$. $\square$

Definition 1.2.8. *Whenever x is analytic we let $(K_x, {K_x}^+)$ denote $(\hat{K}_{h(x)}, \hat{K}^+_{h(x)})$, and if x is non-analytic we let $(K_x, {K_x}^+)$ denote $(K_{h(x)}\left((t)\right), K^+_{h(x)} + t \cdot K_{h(x)}\left[[t]\right])$ as in the proof of proposition 1.2.7. In both cases we call $(K_x, {K_x}^+)$ the pseudo-residue field at x.*

Corollary 1.2.9. *For any map of Huber pairs $m^* : (B_1, {B_1}^+) \to (B_2, {B_2}^+)$ the map $\mathrm{Spo}(m)$ is compatible with vertical generization. More precisely, if $x \in \mathrm{Spo}(B_2, {B_2}^+)$, $y = \mathrm{Spo}(m)(x)$ and y' is a vertical generization of y then there exist x', a vertical generization of x, with $\mathrm{Spo}(m)(x') = y'$.*

Proof. Given $x \in \mathrm{Spo}(B_2, {B_2}^+)$ and $y \in \mathrm{Spo}(B_1, {B_1}^+)$ as in the statement we may, after making some choices if necessary, construct the following commutative diagram of pseudo-residue fields:

$$\begin{tikzcd}
\mathrm{Spa}(K_x, {K_x}^+) \arrow[r] \arrow[d] & \mathrm{Spa}(K_y, {K_y}^+) \arrow[d] \\
\mathrm{Spo}(B_2, {B_2}^+) \arrow[r] & \mathrm{Spo}(B_1, {B_1}^+)
\end{tikzcd}$$

Since the map $\mathrm{Spa}(K_x, {K_x}^+) \to \mathrm{Spa}(K_y, {K_y}^+)$, being a map of analytic pre-adic spaces, is generalizing, we may find $K_x^+ \subseteq K_{x'}^+ \subseteq K_x$ such that the closed point of $Spa(K_x, K_{x'}^+)$ maps to the closed point of $Spa(K_y, K_{y'}^+)$. $\square$

Lemma 1.2.10. *Let us denote by* $\mathrm{Spo}(B, B^+)'$ *the set* $\mathrm{Spo}(B, B^+)$ *endowed with the strongest topology making all of the maps* $\mathrm{Spo}(m) \circ h^{-1}$ *continuous, with* $m^* : (B, B^+) \to (R, R^+)$ *ranging over maps from* (B, B^+) *to Tate Huber pairs* (R, R^+)*. The following hold:*

1. *Vertical generizations are generizations in* $\mathrm{Spo}(B, B^+)'$.
2. *Formal and meromorphic specializations are specializations in* $\mathrm{Spo}(B, B^+)'$.
3. $\mathrm{Spo}(B, B^+)$ *and* $\mathrm{Spo}(B, B^+)'$ *have the same generization pattern.*

Proof. For any $x \in \mathrm{Spo}(B, B^+)$ the pseudo-residue field map $\iota_x : \mathrm{Spa}(K_x, {K_x}^+) \to \mathrm{Spa}(B, B^+)$ in the proof of proposition 1.2.7 defines a bijection onto $\mathcal{I}^{\leq}_{ver}(x)$. Since $(K_x, {K_x}^+)$ is Tate, vertical generizations are generizations in $\mathrm{Spo}(B, B^+)'$.

Let $x \in \mathrm{Spo}(B, B^+)$ be analytic and let b such that $|b|^a_x \neq 0$. Let $p : \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(B, B^+)$ be an analytic geometric point mapping to x and let $\varpi \in C^{\circ\circ}$ be either $p^*(b)$ or $\frac{1}{p^*(b)}$. To this choice we will associate two product of points as follows. Let $R^+ = \prod_{i=1}^\infty C^+$, let $\varpi_0 = (\varpi^{\frac{1}{n}})_{n=1}^\infty$ and $\varpi_\infty = (\varpi^n)_{n=1}^\infty$. Let R_0^+ (R_∞^+ repsectively) be R^+ endowed with the ϖ_0-topology (ϖ_∞-topology respectively), and let $R_0 = R_0^+[\frac{1}{\varpi_0}]$ ($R_\infty = R_\infty^+[\frac{1}{\varpi_\infty}]$ respectively). We have diagonal maps of rings $C^+ \to R_\infty^+$ and $C \to R_\infty$, but we warn the reader that this maps are not continuous. On the other hand, the map $C^+ \to R_0^+$ is continuous but ϖ is not invertible in R_0 so the map does not extend to a map $C \to R_0$. Intuitively speaking, the product of points $\mathrm{Spa}(R_\infty, {R_\infty}^+)$ "converges outside" of the locus in which ϖ is topologically nilpotent and the product of points $\mathrm{Spa}(R_0, {R_0}^+)$ "converges outside" of the locus in which ϖ is invertible.

Suppose that x is meromorphic analytic, then the affinoid residue field $(K_{h(x)}, {K_{h(x)}}^+)$ is given the discrete topology. In particular, the diagonal map $f : B \to R_\infty$ is continuous and defines a map $\mathrm{Spa}(R_\infty, {R_\infty}^+) \to \mathrm{Spa}(B, B^+)$. The space of connected components $\pi_0(|\mathrm{Spa}(R_\infty, {R_\infty}^+)|)$ is the Stone-Čech compactification of $\mathbb{N}$ which has as underlying set the the set of ultrafilters of $\mathbb{N}$. Principal ultrafilters $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ define inclusions $\iota_n : \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(R_\infty, {R_\infty}^+)$ that correspond to the nth-projection in the coordinate rings. In particular, the closed point of a principal connected component maps to x under $\mathrm{Spo}(f)$. We claim that the closed point of a non-principal connected component maps to x_{mer}. It is enough to construct a commutative diagram as below:

$$\begin{tikzcd}
\mathrm{Spa}(C_{\mathcal{U}}, {C_{\mathcal{U}}}^+) \arrow[r] \arrow[d] & \mathrm{Spa}(K_{x_{mer}}, {K_{x_{mer}}}^+) \arrow[d] & \\
\mathrm{Spa}(R_\infty, {R_\infty}^+) \arrow[r] & \mathrm{Spa}(K_{h(x)}, {K_{h(x)}}^+) \arrow[r] & \mathrm{Spa}(B, B^+)
\end{tikzcd}$$

We claim that the natural map $K_{h(x)} \to C_{\mathcal{U}}$ maps to $O_{C_{\mathcal{U}}}$. Indeed, it is enough to prove $\varpi_\infty \cdot K_{h(x)} \subseteq O_{C_{\mathcal{U}}}$ since then every element of $K_{h(x)}$ would be power bounded. Clearly $K_{h(x)}^+ \subseteq O_{C_{\mathcal{U}}}$ and since $K_{h(x)} = K_{h(x)}^+[\frac{1}{b}]$ it is enough to prove that $\frac{\varpi_\infty}{\varpi^n}$ for $n \in \mathbb{N}$. Clearly $\frac{\varpi_\infty}{\varpi^n} \in \prod_{i=n+1}^\infty O_C$ and since our ultrafilter is non-principal complements of finite sets are in $\mathcal{U}$, which finishes the proof of the claim.

By letting t map to ϖ_∞ we get a map $K_{h(x)}((t)) \to C_{\mathcal{U}}$, the intersection of $K_{h(x)}[[t]]$ with $C_{\mathcal{U}}^+$ in $O_{C_{\mathcal{U}}}$ is $K_{h(x)}^+ + t \cdot K_{h(x)}[[t]] = K_{x_{mer}}^+$ which gives our factorization. Since the set of points that are contained in a principal connected component and that are closed in $|\mathrm{Spa}(R, R^+)|$ is dense within the set of closed points of $|\mathrm{Spa}(R, R^+)|$, meromorphic specializations in $\mathrm{Spo}(B, B^+)$ are specializations in $\mathrm{Spo}(B, B^+)'$.

Suppose now that x is formal analytic, since $|B|^a_x \leq 1$ the map $(B, B^+) \to (C, C^+)$ factors through a map to (O_C, C^+) and we have $sp(x) = B \cap C^{\circ\circ}$. This allows us to define a map $\mathrm{Spa}(R_0, {R_0}^+) \to$

$\mathrm{Spa}(B, B^+)$. As in the previous case the space of connected components of $\mathrm{Spa}(R_0, {R_0}^+)$ is Stone-Čech compactification of $\mathbb{N}$, principal connected components of $\mathrm{Spa}(R_0, {R_0}^+)$ map to x in $\mathrm{Spo}(B, B^+)$ and we will show that the non-principal ones map to x_{for}.

Let $k = O_C/C^{\circ\circ}$ and $k^+ = C^+/C^{\circ\circ}$, it is enough to prove that the map $(O_C, C^+) \to (C_\mathcal{U}, {C_\mathcal{U}}^+)$ factors as:

$$(O_C, C^+) \to (k, k^+) \to (k((t)), k^+ + t \cdot k[[t]]) \to (C_\mathcal{U}, {C_\mathcal{U}}^+)$$

Now $\frac{\varpi}{\varpi_0^n} \in \prod_{i=n+1}^\infty O_C$ which implies that $|\varpi|_\mathcal{U} \leq |\varpi_0^n|_\mathcal{U}$. Since ϖ_0 is a pseudo-uniformizer in $C_\mathcal{U}$ this implies $|\varpi|_\mathcal{U} = 0$. Clearly $k \subseteq O_{C_\mathcal{U}}$ and we may send t to ϖ_0 to construct our factorization. We may conclude the proof that formal specializations are specializations in $\mathrm{Spo}(B, B^+)'$ as in the previous case. $\square$

In what follows we restrict to Huber pairs that are their own ring of definition. The main technical advantage of restricting to this case is that the open unit ball over $\mathrm{Spa}(B, B^+)$ is easy to describe. Indeed, it is given by the following construction: let $I \subseteq B$ be an ideal of definition, let $B' = B[[t]]$ given the (I, t)-adic topology and let $B'^+ = B^+ + tB[[t]]$. The map $f : \mathrm{Spa}(B', B'^+) \to \mathrm{Spa}(B, B^+)$ that one gets from the continuous inclusion of B into B' represents the open unit ball.

Proposition 1.2.11. *Let (B, B^+) be a Huber pair such that B is its own ring of definition, let $I \subseteq B$ an ideal of definition and let B' be as above:*

1. $\mathrm{Spo}(B, B^+)$ *has the strongest topology making the maps* $\mathrm{Spo}(m)$ *continuous, in other words*

$$\mathrm{Spo}(B, B^+) = \mathrm{Spo}(B, B^+)'.$$

2. *The map* $\mathrm{Spo}(f) : \mathrm{Spo}(B', B'^+) \to \mathrm{Spo}(B, B^+)$ *is a quotient map when we restrict it to the analytic locus,* $h^{-1}(\mathrm{Spa}(B', B'^+)^{an})$.

3. *If* $\mathfrak{S} \subseteq \mathrm{Spa}(B, B^+)$ *is a subset stable under generization in* $\mathrm{Spa}(B, B^+)$*, then* $\mathfrak{S}$ *is open in* $\mathrm{Spa}(B, B^+)$ *if and only if* $h^{-1}(\mathfrak{S})$ *is open in* $\mathrm{Spo}(B, B^+)$.

Proof. The first claim follows from the second claim so we will only prove the later one. Let U be an open subset of $\mathrm{Spo}(B, B^+)'$, let $x \in U$ and let $y \in \mathrm{Spa}(B', B'^+)$ be a point mapping to x, with $|t|_y \neq 0$. We construct a neighborhood of x contained in U that is open in $\mathrm{Spo}(B, B^+)$.

Given a classical localization $U_{b_1 \leq b_2 \neq 0}$ or an analytic localization $N_{b_1 << b_2}$ containing x we choose quasi-compact neighborhoods of y, that we denote $U_{b_1, b_2, y}$ and $N_{b_1, b_2, y}$, whose image in $\mathrm{Spo}(B, B^+)$ is contained in $U_{b_1 \leq b_2 \neq 0}$ and $N_{b_1 << b_2}$ respectively. The construction is as follows, given the classical localization $U_{b_1 \leq b_2 \neq 0}$ we pick a finite set of elements $S \subseteq B$ and a positive integer n such that $|s|_y \leq |b_2|_y$ for $s \in S$, that $|t^n|_y \leq |b_2|_y$, and that the ideal generated by S is open in B. We let $U_{b_1, b_2, y}$ be the rational localization $U(\frac{S, t^n, b_1}{b_2}) \subseteq \mathrm{Spa}(B', B'^+)$. Rational localizations of affinoid adic spaces are always quasi-compact open subsets and clearly $\mathrm{Spo}(f)(U_{b_1, b_2, y}) \subseteq U_{b_1 \leq b_2 \neq 0}$.

Analogously, given $N_{b_1 << b_2}$ we pick a set S and two positive integers, n_1 and n_2, such that $|b_1^{n_1}|_y \leq |b_2^{n_1} \cdot t|_y$, that $|s|_y \leq |b_2^{n_1} \cdot t|_y$ for $s \in S$, that $|t^{n_2}|_y \leq |t \cdot b_2^{n_1}|_y$ and that S generates an open ideal in B, we let $N_{b_1, b_2, y} = U(\frac{S, t^{n_2}, b_1^{n_1}}{b_2^{n_1} \cdot t})$. Since t is topologically nilpotent in B', for any point $z \in \mathrm{Spa}(B', B'^+)$ we must have $|t|_z < 1$, which proves $\mathrm{Spo}(f)(N_{b_1, b_2, y}) \subseteq N_{b_1 << b_2}$.

Let X denote the intersection of all neighborhoods of y of the form $N_{b_1, b_2, y}$ and $U_{b_1, b_2, y}$ that were chosen in this way, then $\mathrm{Spo}(f)(X)$ is contained in $\mathcal{I}^{\leq}(x)$. By lemma 1.2.10 $\mathrm{Spo}(B, B^+)'$ and $\mathrm{Spo}(B, B^+)$ have the same generization pattern, so we also have that $\mathrm{Spo}(f)(X) \subseteq U$. Since $\mathrm{Spo}(f)^{-1}(U)$ is open in $\mathrm{Spa}(B', B'^+)$ and the two families, $U_{b_1, b_2, y}$ and $N_{b_1, b_2, y}$, consist of quasi-compact open subsets the standard compactness argument in the patch topology of $\mathrm{Spa}(B', B'^+)$ will prove that a finite intersection of these neighborhoods is contained in U. We prove below that the image under $\mathrm{Spo}(f)$ of such a finite intersection is open in $\mathrm{Spo}(B, B^+)$.

It is enough to show that if a set Y is a finite intersections of sets of the form

$$U_{b'_1,b'_2} := \{z \in \mathrm{Spa}(B', B'^+) \mid |b'_1|_z \leq |b'_2|_z \neq 0\}$$

where $b'_1 \in B \cup \{t^n\}$ and $b'_2 \in B \cup t \cdot B$, then $\mathrm{Spo}(f)(Y)$ is open in $\mathrm{Spo}(B, B^+)$. Observe that if $b'_1, b'_2 \in B$ then $U_{b'_1,b'_2} = \mathrm{Spo}(f)^{-1}(U_{b'_1 \leq b'_2 \neq 0})$ and that for any Y as above we have $\mathrm{Spo}(f)(Y \cap U_{b'_1,b'_2}) = \mathrm{Spo}(f)(Y) \cap U_{b'_1 \leq b'_2 \neq 0}$. This allow us to reduce to the case in which Y is an intersections of opens such that at least one of $b'_1 = t^n$ or $b'_2 = b_2 \cdot t$ holds.

Let T^n be the subset of B for which either $U_{t^n,b}$ or $U_{t^{n+1},(b\cdot t)}$ appear in the expression of Y as an intersection, we let $T^{<<}$ be the set of pairs $(b_1, b_2) \in B \times B$ such that $U_{b_1,(b_2 \cdot t)}$ appears in the expression of Y as an intersection, and we let T^- and T^+ denote the image of $T^{<<}$ under the projection onto the first and second factors respectively. We claim and prove below that $\mathrm{Spo}(f)(Y)$ is the intersection of all the sets of the form $U_{b_1^n \leq b_2^n \cdot b_3 \neq 0}$ where $(b_1, b_2) \in T^{<<}$ and $b_3 \in T^n$ and all the sets of the form $N_{b_1 << b_2}$, with $(b_1, b_2) \in T^{<<}$. This implies that Y is open as we needed to show.

Let $z \in Y$ with associated rank 1 point $z^a \in Y$, let $w = \mathrm{Spo}(f)(z)$ and fix b_1, b_2 and b_3 as above. By raising to the nth-power we have that $|b_1^n|_z \leq |b_2^n|_z \cdot |t^n|_z$ and $|t^n|_z \leq |b_3|_z$ hold. In particular,

$$|b_1^n|_z = |b_1^n|_w^h \leq |b_2^n|_w^h \cdot |b_3|_w^h = |b_2^n|_z \cdot |b_3|_z$$

holds as well and we can conclude that $w \in U_{b_1^n \leq b_2^n \cdot b_3 \neq 0}$. Similarly, since t is topologically nilpotent we have $|t|_{z^a} < 1$ which implies that $|b_1|_{z^a} \leq |b_2|_{z^a}$ and consequently that $|b_1|_w^a < |b_2|_w^a$. This says that $w \in N_{b_1 << b_2}$.

To prove the converse containment given a point w in the intersection of those sets we need to construct a lift landing in Y. Pick an analytic geometric point $q : \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(B, B^+)$ mapping to w in $\mathrm{Spo}(B, B^+)$, the choice of an element $\varpi \in C^{\circ\circ,\times}$ defines a lift of q to a map $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(B', B'^+)^a$ simply by letting t map to ϖ. If w is non-analytic then $|b_1|_w^a = 0$ for every $b_1 \in T^-$ and $|b_2|_w^a = |b_3|_w^a = 1$ for every $b_2 \in T^+$ and $b_3 \in T^n$. In this case, any choice of ϖ defines a lift landing inside of Y.

If w is analytic ϖ must be chosen more carefully. Since C is algebraically closed we may choose nth-roots of (b_3) for all $b_3 \in T^n$. For a lift of q to land in Y, ϖ must satisfy the following: $|\varpi|_q \leq |(b_3)^{\frac{1}{n}}|_q$ for all $b_3 \in T^n$ and $\frac{|(b_1)|_q}{|(b_2)|_q} \leq |\varpi|_q$ for all $(b_1, b_2) \in T^{<<}$. We let m be the smallest of the values in Γ_q of the form $|b_3^{\frac{1}{n}}|_q$ with $b_3 \in T^n$ and we let M be the largest of the values of the form $|\frac{b_1}{b_2}|_q$ with $(b_1, b_2) \in T^{<<}$. Since $w \in U_{b_1^n \leq b_2^n \cdot b_3 \neq 0}$ we have $M \leq m$. Since $w \in N_{b_1 << b_2}$ for all pairs $(b_1, b_2) \in T^{<<}$ we also have $M < 1$. Any $\varpi \in C$ with $|\varpi|_q < 1$ and $M \leq |\varpi|_q \leq m$ defines a lift of q in Y. This finishes the proof of the first two claims.

The proof of the third claim is very similar. Notice that since $\mathfrak{S}$ is stable under generization in $\mathrm{Spa}(B, B^+)$ the intersection of all the classical localizations that contain a fixed element of $h^{-1}(\mathfrak{S})$ will still be contained in $h^{-1}(\mathfrak{S})$. Let $x \in h^{-1}(\mathfrak{S})$ and $y \in \mathrm{Spa}(B', B'^+)$ a lift of x with $|t|_y \neq 0$. For every classical localization of x we pick rational localizations of y, $U_{b_1,b_2,y}$, as above. To ensure that t is invertible around y we define $N_{0,1}$ to be $U(\frac{S,t}{t})$ where S is a set of generators for I^{n_y} where n_y is chosen suffciently big so that $|s|_y \leq |t|_y$ holds. We let $\mathcal{F}$ be the family of sets $U_{b_1,b_2,y}$ together with $N_{0,1}$. By a compactness argument we may again find a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ such that the image of $\bigcap_{U \in \mathcal{F}'} U$ is contained in $h^{-1}(\mathfrak{S})$. Letting $Y = \bigcap_{U \in \mathcal{F}'} U$, and letting the sets T^n, $T^{<<}$, T^+, and T^- be as above, we find that the only elements of T^- appearing in the expression of Y as an intersection are elements in I^{n_y}. We see from our work above that $\mathrm{Spo}(f)(Y)$ is the intersection of a finite number of classical localizations together with open sets of the form $N_{s<<1}$ with $s \in I^{n_y}$. Since the elements s are topologically nilpotent $N_{s<<1} = \mathrm{Spo}(B, B^+)$, from which we conclude that $h(\mathrm{Spo}(f)(Y))$ is an open containing $h(x)$ and contained in $\mathfrak{S}$, as we needed to show. $\square$

The proof of the next proposition is very technical and although the author finds the result interesting we will not need it. For this reason we advice the reader to skip this proof. We decided to include it for completeness.

Proposition 1.2.12. *Suppose that (B, B^+) is its own ring of definiton. Given a point $x \in \operatorname{Spa}(B, B^+)$ let $\mathcal{I}^{\leq}_{\overline{ver}}(x)^{op}$ be the total order with underlying set $\mathcal{I}^{\leq}_{\overline{ver}}(x)$ and order given by $y \leq z$ if $y \in \mathcal{I}^{\leq}_{\overline{ver}}(z)$. Suppose that for all $x \in \operatorname{Spa}(B, B^+)$ the total order $\mathcal{I}^{\leq}_{\overline{ver}}(x)^{op}$ is well-ordered, then the map*

$$h : \operatorname{Spo}(B, B^+) \to \operatorname{Spa}(B, B^+)$$

is a quotient map.

Proof. We prove that under this technical hypothesis if a subset $\mathfrak{S} \subseteq \operatorname{Spa}(B, B^+)$ satisfies that $h^{-1}(\mathfrak{S})$ is open in $\operatorname{Spo}(B, B^+)$ then $\mathfrak{S}$ is stable under generization in $\operatorname{Spa}(B, B^+)$. The third part of proposition 1.2.11 would allow us to conclude if this was the case. Since vertical generizations are generizations in $\operatorname{Spo}(B, B^+)$ we conclude that $\mathfrak{S}$ is stable under vertical generization. We need to prove that $\mathfrak{S}$ is also stable under arbitrary horizontal generizations. Let $x \in \mathfrak{S}$ and let $y \in \operatorname{Spa}(B, B^+)$ be a horizontal generization of x, suppose to get a contradiction that $y \notin \mathfrak{S}$. We will get our contradiction by making a series of reductions.

Let us first reduce to the case in which B and B^+ are both valuation rings with the same fraction field. Consider $(K_y, {K_y}^+)$, the affinoid residue field of y and let K_B the smallest valuation ring of K_y containing the image of B and containing K_y^+. We get a morphism $\iota_y : \operatorname{Spa}(K_B, K_y^+) \to \operatorname{Spa}(B, B^+)$, such that x and y are in the image of this map and there exist $y' \in \iota_y^{-1}(y)$ and $x' \in \iota_y^{-1}(x)$ with y' is a horizontal generization of x'. If we take the inverse image of $\mathfrak{S}$ in $\operatorname{Spa}(K_B, K_y^+)$ then $x' \in \mathfrak{S}$, $y' \in \mathfrak{S}$ but y' is still a horizontal generization of x'. To finish justifying our reduction we need to prove that for $t \in \operatorname{Spa}(K_B, K_y^+)$ the total order $\mathcal{I}^{\leq}_{\overline{ver}}(t)^{op}$ is well-ordered and that K_B is its own ring of definition, we reason abstractly. Take an arbitrary Huber pair of the form (V_1, V_2) such that V_1 and V_2 are valuation rings with the same fraction field. With little work it can be shown that for every point $q \in \operatorname{Spa}(V_1, V_2)$, there are a pair of prime ideals $\mathfrak{p}^q_i \subseteq V_2$ such that $\mathcal{I}^{\leq}_{\overline{ver}}(q)^{op}$ as a total order is isomorphic to the set of prime ideals $\mathfrak{p} \subseteq V_2$ such that

$$\mathfrak{p}^q_1 \subseteq \mathfrak{p} \subseteq \mathfrak{p}^q_2,$$

when we order it by containment. In our case, for $t \in \operatorname{Spa}(K_B, K_y^+)$ the total order $\mathcal{I}^{\leq}_{\overline{ver}}(t)^{op}$ is isomorphic to an interval contained in $\mathcal{I}^{\leq}_{\overline{ver}}(y)^{op}$, so it is well-ordered. On the other hand, for (V_1, V_2) as above, one can show that V_1 is its own ring of definition as long as V_1 is not a field. But K_B is not a field since it has a non-trivial horizontal specialization.

From now on suppose that B^+ and B are valuation rings with the same fraction field. In this case, horizontal specializations of y are in bijection with prime ideals of B and the bijection is given by the formula $\mathfrak{p} \mapsto |\cdot|_y/\mathfrak{p}$. The observation above proves that the set of prime ideals of B are well-ordered by containmnent. Indeed, since B^+ is a valuation ring with fraction field K_y the total order $\mathcal{I}^{\leq}_{\overline{ver}}(y)^{op} \subseteq \operatorname{Spa}(K_y, B^+)$ is isomorphic to the set of open prime ideals of B^+ ordered by containment. Using that this set is well-ordered, we may take the smallest prime $\mathfrak{p} \subseteq B$ for which the horizontal specialization associated to $\mathfrak{p}$ is in $\mathfrak{S}$. We may replace x by this horizontal specialization of y. Moreover, after localization around x we may assume that x is the unique closed point of $\operatorname{Spa}(B, B^+)$, that x is the horizontal specialization of y that corresponds to the maximal ideal of B and that $\mathfrak{S}$ contains x but does not contain any horizontal generization of x in $\operatorname{Spa}(B, B^+)$. We will get a contradiction in this case.

Take $z \in \operatorname{Spo}(B, B^+)$ with $h(z) = x$, since x admits a non-trivial horizontal generization in $\operatorname{Spa}(B, B^+)$ it is non-analytic and we may take z to be non-analytic in $\operatorname{Spo}(B, B^+)$. Since we picked x to be the closed point in $\operatorname{Spa}(B, B^+)$ classical localizations containing z are equal to $\operatorname{Spo}(B, B^+)$. Also, since

$N_{b_1<<b_2} \subseteq U_{0\leq b_2 \neq 0}$, if $z \in N_{b_1<<b_2}$ then ${b_2}^{-1} \in B$ and we may rewrite $N_{b_1<<b_2}$ as $N_{b<<1}$ with $b = \frac{b_1}{b_2}$. We claim that analytic localizations of the form $N_{b<<1}$ with $b \in B$ are nested. Indeed, if $\frac{b_1}{b_2} \in B^+$ then $N_{b_1<<1} \subseteq N_{b_2<<1}$ and since B^+ is a valuation ring one of $\frac{b_1}{b_2} \in B^+$ or $\frac{b_2}{b_1} \in B^+$ must hold.

Since $h^{-1}(\mathfrak{S})$ is open there is a finite intersection of classical and analytic localization containing z and contained in $h^{-1}(\mathfrak{S})$. Our observations above imply that $z \in N_{b<<1} \subseteq h^{-1}(\mathfrak{S})$ for an element $b \in B$. Since z is non-analytic $b \in supp(z)$ and the ideal generated by b in B is a proper ideal. We may consider $(B_b, {B_b}^+)$ the completion of both B and B^+ with respect to the (b)-adic topology. The valuation induced by the map $Spa(B_b[\frac{1}{b}], B_b^+) \to \mathrm{Spa}(B, B^+)$ is a horizontal specialization of y. It corresponds to the largest prime of B contained in $(b) \subseteq B$. This valuation is in $N_{b<<1} \subseteq h^{-1}(\mathfrak{S})$ which contradicts the assumption that $\mathfrak{S}$ did not contain any horizontal generizations of x. □

Corollary 1.2.13. *If (B, B^+) is its own ring of definition and the Krull dimension of $\mathrm{Spa}(B, B^+)$ is finite, then $h : \mathrm{Spo}(B, B^+) \to \mathrm{Spa}(B, B^+)$ is a quotient map.*

Proof. If the Krull dimension of $\mathrm{Spa}(B, B^+)$ is finite, then $\mathcal{I}_{ver}^{\leq}(x)^{op}$ is finite and well-ordered. We can use proposition 1.2.12 to conclude. □

Remark 1.2.14. *We warn the reader that the map $\mathrm{Spo}(B, B^+) \to \mathrm{Spa}(B, B^+)$ is not always a quotient map. The third part of proposition 1.2.11 proves that a counterexample should come from a subset of $\mathrm{Spa}(B, B^+)$ that is not stable under generization but becomes open after pullback to $\mathrm{Spo}(B, B^+)$.*

1.2.4 Schematic subsets

In this section we prove some results on the behaviour of the olivine spectrum of adic spaces that are coming from schemes.

Proposition 1.2.15. *Let A be a ring endowed with the discrete topology and $f^* : (B, B^+) \to (A, A)$ a map of Huber pairs, then the following hold:*

1. *$\mathrm{Spo}(f)$ is saturated with respet to h, in other words*
$$\mathrm{Spo}(f)(\mathrm{Spo}(A, A)) = h^{-1}(f(\mathrm{Spa}(A, A))).$$
2. *$f(\mathrm{Spa}(A, A))$ is stable under horizontal specializations in $\mathrm{Spa}(B, B^+)$.*
3. *$f(\mathrm{Spa}(A, A))$ is stable under vertical generization in $\mathrm{Spa}(B, B^+)$.*

Proof. To prove the first claim let $y \in \mathrm{Spo}(A, A)$ and let $x = \mathrm{Spo}(f)(y)$. If x is analytic, y must be meromorphic analytic and y_{mer} maps to x_{mer} under $\mathrm{Spo}(f)$, that is $h^{-1}(h(x)) \subseteq Im(\mathrm{Spo}(f))$.

Suppose now that x is non-analytic and that x^{mer} exists in $\mathrm{Spo}(B, B^+)$. In this case, y^{mer} might not exist if $|\cdot|_y^h$ is not microbial, and even when y^{mer} exists it may not be true that y^{mer} maps to x^{mer} under $\mathrm{Spo}(f)$. Indeed, it may happen that y^{mer} maps to x instead, for these reasons we use a different construction.

Consider $h(y) \in \mathrm{Spa}(A, A)$ together with its residue field map $\iota_{h(y)} : \mathrm{Spa}(K_{h(y)}, {K_{h(y)}}^+) \to \mathrm{Spa}(A, A)$. Notice that $\iota_{h(y)}$ factors through a map $g : \mathrm{Spa}(K_{h(y)}^+, K_{h(y)}^+) \to \mathrm{Spa}(A, A)$, so it is enough to prove that x^{mer} is in the image of $\mathrm{Spo}(f \circ g)$. Take an element $b \in B$ with $|b|_{x^{mer}}^a \notin \{0, 1\}$ and replace it by its inverse in $K_{h(y)}$, whenever it is necessary, so that $b \in K_{h(y)}^+$. Define K^+ as the (b)-adic completion of $K_{h(y)}^+$, and let $K = K^+[\frac{1}{b}]$. We get the following commutative diagram,

$$\begin{tikzcd}
\mathrm{Spa}(K, K^+) \arrow[rr] \arrow[dr] & & \mathrm{Spo}(K_{h(y)}^+, K_{h(y)}^+) \arrow[dl] \\
& \mathrm{Spo}(B, B^+) &
\end{tikzcd}$$

and one can verify that the image of the closed point of $\mathrm{Spa}(K, K^+)$ in $\mathrm{Spo}(B, B^+)$ is x^{mer}. This proves that in this case as well $h^{-1}(h(x)) \subseteq Im(\mathrm{Spo}(f))$.

The proof of the second claim also follows from observing that the residue field map $\iota_{h(y)}$ factors through g. Indeed, we get the following commutative diagram of adic spaces:

$$\begin{tikzcd}
\mathrm{Spa}(K_{h(y)}, {K_{h(y)}}^+) \arrow[r] \arrow[d] & \mathrm{Spa}(K^+_{h(y)}, K^+_{h(y)}) \arrow[r] \arrow[d] & \mathrm{Spa}(A, A) \arrow[d] \\
\mathrm{Spa}(K_{h(x)}, {K_{h(x)}}^+) \arrow[r] & \mathrm{Spa}(K^+_{h(x)}, K^+_{h(x)}) \arrow[r] & \mathrm{Spa}(B, B^+)
\end{tikzcd}$$

The vertical map on the right is surjective since $h(x) = f(h(y))$ and one can deduce that the vertical map in the middle column is also surjective because the map of valuation rings is local. A prime ideal of $J \subseteq K^+_{h(x)}$ determines a horizontal specializations of $|\cdot|_{h(x)}$, namely $|\cdot|_{h(x)}/J$, and every horizontal specialization of $h(x)$ can be constructed in this way. For J as above we let $K^+_J = K^+_{h(x)}/J$ and $K_J = Frac(K^+_J)$, we get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spa}(K_J, {K_J}^+) \arrow[r] \arrow[dr, bend right] & \mathrm{Spa}(K^+_J, K^+_J) \arrow[rr] \arrow[d] & & \mathrm{Spa}(K^+_{h(y)}, K^+_{h(y)}) \arrow[dll] \\
& \mathrm{Spa}(B, B^+) & &
\end{tikzcd}$$

The closed point of $\mathrm{Spa}(K_J, {K_J}^+)$ maps to the horizontal specialization of $h(x)$ associated to the ideal J.

The third claim follows from corollary 1.2.9 and from the first claim. $\square$

Definition 1.2.16. *We say that a subset of* $\mathrm{Spo}(B, B^+)$ *is a schematic subset if it is a union of sets of the form* $\mathrm{Spo}(m)(\mathrm{Spo}(A, A))$ *where* (A, A) *is given the discrete topology and* $m^* : (B, B^+) \to (A, A)$ *is a map of Huber pairs.*

Proposition 1.2.17. *Suppose that* $Z \subseteq \mathrm{Spo}(B, B^+)$ *is a schematic closed subset. Let* $\sigma : \mathrm{Spo}(B, B^+) \to \mathrm{Spec}(B)$ *denote the composition of* h *and* $supp$ *where* $supp : \mathrm{Spa}(B, B^+) \to \mathrm{Spec}(B)$ *is the map that sends a valuation to its support. Then* $Z = \sigma^{-1}(V(I))$ *for some prime ideal* $I \subseteq B$ *open for the topology in* B*.*

Proof. Any map $m^* : (B, B^+) \to (A, A)$ with A a discrete ring must factor through $(B/B^{\circ\circ}, B^+/B^{\circ\circ})$, by reducing to this case we may assume that B has the discrete topology. By proposition 1.2.15, $Z = h^{-1}(h(Z))$ and by corollary 1.2.9, Z is closed under vertical generization. Moreover, since Z is closed in $\mathrm{Spo}(B, B^+)$ it is also stable under vertical specialization. This implies that $Z = \sigma^{-1}(\sigma(Z))$. Indeed, any two points $x, y \in \mathrm{Spa}(B, B^+)$ with $supp(x) = supp(y) = \mathfrak{p}$ are vertical specializations of the trivial valuation on B with support $\mathfrak{p}$.

We only have left to prove that $\sigma(Z)$ is closed in the Zariski topology of $\mathrm{Spec}(B)$. Since B has the discrete topology, the support map admits a continuous section $triv : \mathrm{Spec}(B) \to \mathrm{Spa}(B, B^+)$ that assigns to a prime ideal in B the trivial valuation with that support. We have $\sigma(Z) = triv^{-1}(h(Z))$ so we may prove $h(Z)$ is closed instead. By proposition 1.2.15, $h(Z)$ is also closed under horizontal specialization, this gives that the complement of $h(Z)$ in $\mathrm{Spa}(B, B^+)$ is stable under (arbitrary) generization. Since we assumed B to have the discrete topology, B is its own ring of definition and we may use the third part of proposition 1.2.11 to prove that $\mathrm{Spa}(B, B^+) \setminus h(Z)$ is open. $\square$

1.2.5 The olivine spectrum as $|\mathrm{Spd}(B, B^+)|$

In this subsection we will assume that (B, B^+) is a complete Huber pair over $\mathbb{Z}_p$. We define a map of v-sheaves $\mathrm{Spd}(B, B^+) \to \underline{\mathrm{Spo}(B, B^+)}$ on Perf, given by the following rule. A map $\mathrm{Spa}(R, R^+) \to \mathrm{Spd}(B, B^+)$ is given by an untilt $(R^\sharp, f)$ over $\mathrm{Spa}(B, B^+)$. The map $f : \mathrm{Spa}(R^\sharp, R^{\sharp +}) \to \mathrm{Spa}(B, B^+)$ induces a continuous map $\mathrm{Spo}(f) : \mathrm{Spo}(R^\sharp, R^{\sharp +}) \to \mathrm{Spo}(B, B^+)$. On the other hand, since $(R^\sharp, R^{\sharp +})$ is a Tate Huber pair we have an identification $h^{-1} : \mathrm{Spa}(R^\sharp, R^{\sharp +}) = \mathrm{Spo}(R^\sharp, R^{\sharp +})$, and because R is perfectoid we also have $\flat : |\mathrm{Spa}(R, R^+)| \cong |\mathrm{Spa}(R^\sharp, R^{\sharp +})|$ by the tilting equivalence. The map

$$\mathrm{Spo}(f) \circ h^{-1} \circ \flat^{-1} : |\mathrm{Spa}(R, R^+)| \to \mathrm{Spo}(B, B^+)$$

defines a (R, R^+)-point of $\underline{\mathrm{Spo}(B, B^+)}$. By adjunction we get a map of topological spaces $f : |\mathrm{Spd}(B, B^+)| \to \mathrm{Spo}(B, B^+)$.

Proposition 1.2.18. *Suppose that B is its own ring of definiton, then the map of topological spaces $f : |\mathrm{Spd}(B, B^+)| \to \mathrm{Spo}(B, B^+)$ is a homeomorphism.*

Proof. Let B' be as in the proof of proposition 1.2.15, and consider the analytic pre-adic space $X = \mathrm{Spa}(B', B'^+)^a$. By remark 1.1.24, $X^\diamond$ is a locally spatial diamond and the map $|X^\diamond| \to |X|$ is a homeomorphism. The map of v-sheaves $X^\diamond \to \mathrm{Spd}(B, B^+)$ is surjective and we have a commutative diagram:

$$\begin{tikzcd} \mid X^\diamond \mid \arrow[r] \arrow[d] & \mid X \mid \arrow[d] \\ \mid \mathrm{Spd}(B, B^+) \mid \arrow[r, "f"] & \mathrm{Spo}(B, B^+) \end{tikzcd}$$

The top horizontal arrow is a homeomorphism and the the two vertical arrows are quotient maps by propositions 1.2.11 and 1.1.11. This implies that the map $|\mathrm{Spd}(B, B^+)| \to \mathrm{Spo}(B, B^+)$ is a quotient map. To finish the proof we only need to show that this map is injective. Let $p_i \in |\mathrm{Spd}(B, B^+)|$, $i \in \{1, 2\}$ be two elements represented by geometric points $p_i : \mathrm{Spa}(C_i^\sharp, C_i^{\sharp +}) \to \mathrm{Spa}(B, B^+)$ respectively. Working through the definitions we see that $f(p_i)$ is the image of the closed point of $|\mathrm{Spa}(C_i^\sharp, C_i^{\sharp +})|$ under $\mathrm{Spo}(p_i, p_i{}^+)$. If $f(p_1) = f(p_2)$ there is, by proposition 1.2.7, a third algebraically closed non-Archimedean field $\mathrm{Spa}(C_3, C_3{}^+)$ and a commutative diagram:

$$\begin{tikzcd} \mathrm{Spa}(C_3, C_3{}^+) \arrow[r] \arrow[d] & \mathrm{Spa}(C_1^\sharp, C_1^{\sharp +}) \arrow[d] \\ \mathrm{Spa}(C_2^\sharp, C_2^{\sharp +}) \arrow[r] & \mathrm{Spa}(B, B^+) \end{tikzcd}$$

This gives a similar commutative diagram:

$$\begin{tikzcd} \mathrm{Spa}(C_3^\flat, C_3^{\flat +}) \arrow[r] \arrow[d] & \mathrm{Spa}(C_1, C_1{}^+) \arrow[d] \\ \mathrm{Spa}(C_2, C_2{}^+) \arrow[r] & \mathrm{Spd}(B, B^+) \end{tikzcd}$$

which proves that $p_1 = p_2$ and the injectivity of f. $\square$

By [26] 12.9 open subsheaves of $\mathrm{Spd}(B, B^+)$ are in bijective correspondence with open subsets of $\mathrm{Spo}(B, B^+)$. The formation of $\mathrm{Spd}(B, B^+)$ commutes with localization in $\mathrm{Spa}(B, B^+)$, this observation gives a description of the open subsheaf corresponding to classical localizations. The following lemma describes, in some cases, the open subsheaf associated to analytic localizations .

Lemma 1.2.19. *Let (B, B^+) be a complete Huber pair over $\mathbb{Z}_p$ and suppose that B is its own ring of definition with ideal of definition I. Let $b \in B$, let B_b be the completion of B with respect to the (b, I)-adic topology and let B_b^+ be the integral closure of $B^+ + (B_b^+)^{\circ\circ}$ in B_b. Then the open subsheaf associated to $N_{b<<1} \subseteq |\mathrm{Spd}(B, B^+)|$ is represented by $\mathrm{Spd}(B_b, {B_b}^+)$.*

Proof. If a map $f : \mathrm{Spa}(R, R^+) \to \mathrm{Spd}(B, B^+)$ factors through $\mathrm{Spd}(B_b, {B_b}^+)$ then $f^*(b)$ is topologically nilpotent in $R^\sharp$. This implies that $\mathrm{Spo}(f)(\mathrm{Spa}(R, R^+)) \subseteq N_{b<<1}$ and since this happens for every test space $(R, R^+) \in \mathrm{Perf}$, the map of v-sheaves $\mathrm{Spd}(B_b, {B_b}^+) \to \mathrm{Spd}(B, B^+)$ must factors through the subsheaf associated to $N_{b<<1}$. Moreover, since $B \subseteq B_b$ is dense, the maps $f^* : B_b \to R^\sharp$ are determined by their restriction to B, this implies that $\mathrm{Spd}(B_b, {B_b}^+) \to \mathrm{Spd}(B, B^+)$ is an injective map.

We must prove that if $f : \mathrm{Spa}(R^\sharp, {R^\sharp}^+) \to \mathrm{Spa}(B, B^+)$ is such that $f(\mathrm{Spa}(R^\sharp, {R^\sharp}^+)) \subseteq N_{b<<1}$, then it factors through a (unique) map $\mathrm{Spa}(R^\sharp, {R^\sharp}^+) \to \mathrm{Spa}(B_b, {B_b}^+)$. Given a point $x \in \mathrm{Spa}(R^\sharp, {R^\sharp}^+)$ and a pseudo-uniformizer $\varpi \in {R^\sharp}^+$ we let x^a denote the rank 1 generization of x, we have that $|f^*b^n|_x \leq |\varpi|_x$ for some n since by hypothesis $|f^*b|_{x^a} < 1$.

We compute explicitly this rational localization, $U = U(\frac{f^*b^n}{\varpi}) \subseteq \mathrm{Spa}(R^\sharp, {R^\sharp}^+)$. If we let $R_1 = \widehat{(R^{\sharp,+})}[\frac{b^n}{\varpi}]$ where we complete by the ϖ-adic topology, then $R' = H^0(U, \mathcal{O}_X) = R_1[\frac{1}{\varpi}]$ and $R'^+ = H^0(U, \mathcal{O}_X^+)$ is the integral closure of $R^{\sharp,+} + R_1^{\circ\circ}$ in R'. Since $f^*b^n \in R'^{\circ\circ}$ the map $B \to R'$ is continuous when B is given the (I, b)-adic topology. Moreover, since R'° is complete we get a map $B_b \to R'^\circ$. This gives a factorization $\mathrm{Spa}(R', {R'}^+) \to \mathrm{Spa}(B_b, {B_b}^+)$, and a map $\mathrm{Spa}(R', {R'}^+)^\flat \to \mathrm{Spd}(B_b, {B_b}^+)$. We have proved that locally f factors through $\mathrm{Spd}(B_b, {B_b}^+)$, since $\mathrm{Spd}(B_b, {B_b}^+)$ is a v-sheaf and the factorization is unique we may glue this to a map $\mathrm{Spa}(R, R^+) \to \mathrm{Spd}(B_b, {B_b}^+)$. □

1.3 The reduction functor

1.3.1 The v-topology for perfect schemes

This is the only section in which we will be forced to be set-theoretically careful, we advise the reader that does not wish the ignore the set-theoretic subtleties that arise in this section to review the definition and basic properties of cut-off cardinals that are given in [26] §4.

We will denote by $\mathrm{PCAlg}^{op}_{\mathbb{F}_p}$ the category of perfect affine schemes over $\mathbb{F}_p$. If κ is a cut-off cardinal we denote by $\mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$ the category of perfect affine schemes over $\mathbb{F}_p$ whose algebra has cardinality bounded by κ. Given $S = \mathrm{Spec}(A) \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p}$ we associate to it a v-sheaf in Perf given by:

$$S^\diamond((R, R^+)) = \{f : A \to R^+ | f \text{ } is\, a\, morphism\, of\, rings\}$$

Remark 1.3.1. *Notice that* $\mathrm{Spec}(A)^\diamond = \mathrm{Spd}(A, A)$ *when A is given the discrete topology. Later on, we will work with v-sheaves of the form* $\mathrm{Spd}(A, A)$ *where A can be given either the discrete topology or a more interesting topology and we might consider both kind of sheaves at the same time. To avoid having to specify the topology given to A every time, we will use* $\mathrm{Spec}(A)^\diamond$ *whenever A is given the discrete topology and we will use* $\mathrm{Spd}(A, A)$ *when A is given a more interesting topology.*

The following propostion is set-theoretic and may be ignored.

Proposition 1.3.2. *If κ is a cut-off cardinal and $S \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$ then $S^\diamond$ is κ-small v-sheaf.*

Proof. Let $S = \mathrm{Spec}(A)$ and let λ denote the cardinality of A, by hypothesis $\lambda < \kappa$. We claim there exist a cardinal η with $\lambda \leq \eta < \kappa$ such that if $\mathrm{Spa}(R, R^+) \in \mathrm{Perf}$ has arbitrary cardinality and $f : A \to R^+$ is a ring map, then there is a perfectoid space $\mathrm{Spa}(R^\eta, R^{\eta+})$ and two maps $f^\eta : A \to R^\eta$, $\pi_\eta : \mathrm{Spa}(R, R^+) \to \mathrm{Spa}(R^\eta, R^{\eta+})$ such that $\mathrm{Spa}(R^\eta, R^{\eta+})$ is η-small and $f = \pi_\eta^* \circ f^\eta$. Indeed, fix such a map $f : A \to R^+$. We let $R^{\eta,+}$ be the smallest subring of R^+ that contains $f(A)$, that contains a choice of pseudo-uniformizer $\varpi \in R^+$, that is stable under Frobenious operator and that is complete for the ϖ-adic topology. It is not hard to prove that this ring is perfectoid with cardinality at most 2^λ. Indeed, the Frobenious orbit of an element has cardinality bounded by $\aleph_0$ and the cardinality of the completion of a ring with cardinality λ is at most 2^λ. We let $\eta = 2^\lambda$, we let $R^\eta = R^{\eta,+}[\frac{1}{\varpi}]$ and we let the maps f^η and π^η to be the obvious ones.

The family of isomorphism classes of η-small perfectoid spaces over $S^\diamond$ is a κ-small set, denote this set by P_η. For every $q \in P_\eta$ choose a representative $X_q \in \mathrm{Perf}_{\eta,S^\diamond}$, then $X := \coprod_{q\in P_\eta} X_q$ is a κ-small perfectoid space that admits an open and closed immersion from any η-small perfectoid space over $S^\diamond$. The natural map $X \to S^\diamond$ is surjective. Indeed any map $\mathrm{Spa}(R, R^+) \to S^\diamond$ factors as:

$$\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(R^\eta, R^{\eta+}) \to X \to S^\diamond.$$

□

Proposition 1.3.2 gives rise to functors $\diamond_\kappa : \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa} \to \widetilde{\mathrm{Perf}}_\kappa$ that are compatible when we vary κ and give rise to a functor $\diamond : \mathrm{PCAlg}^{op}_{\mathbb{F}_p} \to \widetilde{\mathrm{Perf}}$

Proposition 1.3.3. *The functors $\diamond : \mathrm{PCAlg}^{op}_{\mathbb{F}_p} \to \widetilde{\mathrm{Perf}}$ and $\diamond_\kappa : \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa} \to \widetilde{\mathrm{Perf}}_\kappa$ are fully-faithful and commute with finite limits.*

Proof. The fully-faithful part is a direct consequence of 1.1.26. Now, given affine schemes $S_i = \mathrm{Spec}(A_i)$ with $i \in \{1, 2, 3\}$, and morphisms $\pi_i : S_i \to S_3$ with $i \in \{1, 2\}$ we must show that $(S_1 \times_{S_3} S_2)^\diamond = S_1^\diamond \times_{S_3^\diamond} S_2^\diamond$. Given an affinoid perfectoid $\mathrm{Spa}(R, R^+)$ we have that $(S_1 \times_{S_3} S_2)^\diamond(R, R^+) = Hom_{Ring}(A_1 \otimes_{A_3} A_2, R^+)$ and that

$$S_1^\diamond \times_{S_3^\diamond} S_2^\diamond(R, R^+) = Hom_{Ring}(A_1, R^+) \times_{Hom_{Ring}(A_3,R^+)} Hom_{Ring}(A_2, R^+).$$

The claim follows from the universal property of $A_1 \otimes_{A_3} A_2$.

The claim for $\diamond_\kappa$, is deduced from the other one by observing that $\mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa} \subseteq \mathrm{PCAlg}^{op}_{\mathbb{F}_p}$ is stable under finite limits. □

After embedding $\mathrm{PCAlg}^{op}_{\mathbb{F}_p}$ in $\widetilde{\mathrm{Perf}}$ one can define a Grothendieck topology on $\mathrm{PCAlg}^{op}_{\mathbb{F}_p}$ by considering a small family of maps of affine schemes, $(S_i \to T)_{i\in\mathcal{F}}$, to be a cover if the map $\coprod_{i\in\mathcal{F}} S_i^\diamond \to T^\diamond$ is a surjective map of v-sheaves. However, there is an intrinsic way of defining this topology which we now discuss.

Definition 1.3.4. *(See [4] 2.1)*

1. *A morphisms of qcqs schemes $S \to T$, is said to be universally subtrusive (or a v-cover) if for any valuation ring V and a map $\mathrm{Spec}(V) \to T$ there is an extension of valuation rings $V \subseteq W$ (see [29] 0ASG) and a map $\mathrm{Spec}(W) \to S$ making the following diagram commutative:*

$$\begin{tikzcd} \mathrm{Spec}(W) \arrow[r] \arrow[d] & S \arrow[d] \\ \mathrm{Spec}(V) \arrow[r] & T \end{tikzcd}$$

2. *A small family of morphisms in $\mathrm{PCAlg}^{op}_{\mathbb{F}_p}$, $(S_i \to T)_{i\in\mathcal{F}}$, is said to be universally subtrusive (or a v-cover) if there is a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ for which $\coprod_{i\in\mathcal{F}'} S_i \to T$ is universally subtrusive.*

Lemma 1.3.5. *(See [4] 2.2) A morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes (not necessarily over $\mathbb{F}_p$) is universally subtrusive if and only if the map of topological spaces $|f^{ad}| : |\mathrm{Spa}(B, B)| \to |\mathrm{Spa}(A, A)|$ is surjective.*

Proof. Let $T = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(B)$, $T^{ad} = \mathrm{Spa}(A, A)$ and $S^{ad} = \mathrm{Spa}(B, B)$. Assume f to be universally subtrusive and take $x \in |T^{ad}|$. Taking a representative we can consider x as a valuation $|\cdot|_x : A \to \Gamma_x$, which gives a valuation subring V of $Frac(A/supp(|\cdot|_x))$ together with a map $\mathrm{Spec}(V) \to \mathrm{Spec}(A)$. Since f is universally subtrusive we can take an extension of valuation rings W/V and a map $\mathrm{Spec}(W) \to \mathrm{Spec}(B)$ making diagram 1 above commutative. The map $B \to W$ induces a valuation $|\cdot|_y : B \to \Gamma_y$ and consequently a point $y \in S^{ad}$. Moreover, the composition $|\cdot|_{f(y)} : A \to B \to \Gamma_y$ is equivalent to $|\cdot|_x$ which proves that $|S^{ad}| \to |T^{ad}|$ is surjective. For the converse, given a map $\mathrm{Spec}(V) \to T$ we may consider the induced map $\mathrm{Spa}(K, V) \to T^{ad}$ with $K = Frac(V)$. The closed point of $\mathrm{Spa}(K, V)$ gives a point $x \in T^{ad}$ and by surjectivity of f^{ad} we may pick a point $y \in \mathrm{Spa}(B, B)$ lifting x. Consider the affinoid residue fields $(K_x, {K_x}^+)$ and $(K_y, {K_y}^+)$ at x and y respectively. We get the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Spec}(K_y^+) \arrow[r] \arrow[d] & S \arrow[d] \\
\mathrm{Spec}(V) \arrow[r] & \mathrm{Spec}(K_x^+) \arrow[r] & T
\end{tikzcd}$$

Both K_y^+ and V are valuation extensions of K_x^+, consequently there is a valuation ring W extending both K_y^+ and V making the diagram commute (See [14] 1.1.14-f). This proves that f is universally subtrusive. □

Lemma 1.3.6. *Let $f : S \to T$ be a morphism of perfect affine schemes over $\mathbb{F}_p$. The map $f^\diamond : S^\diamond \to T^\diamond$ is a quasi-compact map of v-sheaves.*

Proof. By writting $B = A[t_i]_{i\in I}/JA[t_i]_{i\in I}$ for some variables t_i and an ideal J we can reduce to the cases where either f is a closed embedding or f is the base change of the structure map $g : \mathrm{Spec}(\mathbb{F}_p[t_i]_{i\in I}) \to \mathrm{Spec}(\mathbb{F}_p)$.

Let $X = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$ it is enough to prove that $X \times_{T^\diamond} S^\diamond$ is quasi-compact. For the later case, the basechage gives the sheaf $X \times \mathrm{Spec}(\mathbb{F}_p[t_i])^\diamond$. This functor is represented by

$$\mathrm{Spa}(R\langle t_i^{\frac{1}{p^\infty}}\rangle_{i\in I}, R^+\langle t_i^{\frac{1}{p^\infty}}\rangle_{i\in I}),$$

which is affinoid perfectoid and consequenlty quasi-compact. For the former case let $B = A/J$, and let $Z = X \times_{T^\diamond} S^\diamond$. For a perfectoid Huber pair (L, L^+) we have:

$$Z(L, L^+) = \{r : (R, R^+) \to (L, L^+) \mid r(R \cdot J) = 0\}$$

This is the definition of a Zariski closed subset of X and by ([25] Lemma II.2.2) representable by an affinoid perfectoid, in particular Z is a quasi-compact v-sheaf. □

Proposition 1.3.7. *1. Let $f : S \to T$ be a morphism of perfect affine schemes over $\mathbb{F}_p$. The map f is universally subtrusive if and only if $f^\diamond : S^\diamond \to T^\diamond$ is a surjective map of v-sheaves.*

2. A family of morphisms $(S_i \to T)_{i\in\mathcal{F}}$ is universally subtrusive if and only if $(\coprod_{i\in\mathcal{F}} S_i) \to T$ is a surjective map of v-sheaves.

Proof. Let $T = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(B)$. Since the map of v-sheaves $f^\diamond : S^\diamond \to T^\diamond$ is quasi-compact, by ([26] 12.11) it is a surjective map of v-sheaves if and only if $|f^\diamond|$ is a surjective map of topological spaces. By proposition 1.2.18 and lemma 1.3.5, it suffices to prove that the map $\mathrm{Spo}(B, B) \to \mathrm{Spo}(A, A)$ is surjective if and only if the map $\mathrm{Spa}(B, B) \to \mathrm{Spa}(A, A)$ is. Functoriality and surjectivtiy of h proves one direction, and the converse direction is a direct consequence of 1.2.15.

For the second claim, it follows easily from above that a universally subtrusive family of maps $(S_i \to T)_{i\in\mathcal{F}}$ induces a surjective map of v-sheaves $(\coprod_{i\in\mathcal{F}} S_i) \to T$, actually a finite subfamily is already surjetive. To prove the converse we have to take a family of maps $(S_i \to T)_{i\in\mathcal{F}}$ such that $(\coprod_{i\in\mathcal{F}} S_i^\diamond) \to T^\diamond$ is a surjective map of v-sheaves and prove there is a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ for which $\coprod_{i\in\mathcal{F}'} S_i^\diamond \to T^\diamond$ is still surjective. Let $S_i = \mathrm{Spec}(R_i)$ and $T = \mathrm{Spec}(P)$ and consider affinoid perfectoid spaces $Y = Spa(P((t^{\frac{1}{p^\infty}})), P[[t^{\frac{1}{p^\infty}}]])$ and $X_i = Spa(R_i((t^{\frac{1}{p^\infty}})), R_i[[t^{\frac{1}{p^\infty}}]])$. The map $(\coprod_{i\in\mathcal{F}} X_i) \to Y$ is surjective and since Y is quasicompact there is a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ such that $(\coprod_{i\in\mathcal{F}'} X_i) \to Y$ is still surjective. An easy argument proves that for $\mathcal{F}'$ chosen in this way $(\coprod_{i\in\mathcal{F}'} S_i^\diamond) \to T^\diamond$ is also surjective. $\square$

Remark 1.3.8. *In this context, one can discuss the analogue of Example 1.1.4. Given an index set I and $\{V_i\}_{i\in I}$ a family of perfect valuation rings over $\mathbb{F}_p$, we construct the ring $R = \prod_{i\in I} V_i$. We call the affine schemes constructed in this way a scheme-theoretic product of points. They form a basis for the v-topology on $\mathrm{PCAlg}_{\mathbb{F}_p}^{op}$ (See [4] 6.2).*

Given a cut-off cardinal κ we let $\widetilde{\mathrm{SchPerf}}_\kappa$ be the topos associated to the site $\mathrm{PCAlg}_{\mathbb{F}_p,\kappa}^{op}$ with the v-topology, and we will refer to an object in this topos as a κ-small scheme-theoretic v-sheaf. For any pair of cut-off cardinals $\kappa < \lambda$ we have a continuous fully-faithful embedding of sites $\iota_{\kappa,\lambda}^* : \mathrm{PCAlg}_{\mathbb{F}_p,\kappa}^{op} \to \mathrm{PCAlg}_{\mathbb{F}_p,\lambda}^{op}$, which induces a morphism of topoi $\iota_{\kappa,\lambda} : \widetilde{\mathrm{SchPerf}}_\lambda \to \widetilde{\mathrm{SchPerf}}_\kappa$.

Proposition 1.3.9. *The functor $\iota_{\kappa,\lambda}^* : \widetilde{\mathrm{SchPerf}}_\kappa \to \widetilde{\mathrm{SchPerf}}_\lambda$ is fully-faithful (See [26] 8.2).*

Proof. It is enough to prove that the adjunction $\mathcal{F} \to \iota_{\kappa,\lambda,*}\iota_{\kappa,\lambda}^*\mathcal{F}$ is an isomorphism. Let

$$\mathcal{G} : \mathrm{PCAlg}_{\mathbb{F}_p,\lambda}^{op} \to \mathrm{Sets}$$

be the presheaf with $S \mapsto \mathcal{G}(S)$ constructed as follows. Let $\mathcal{C}_S^\kappa$ denote the category of maps of affine schemes $S \to T$ with $T \in \mathrm{PCAlg}_{\mathbb{F}_p,\kappa}^{op}$. This category is cofiltered and there is a λ-small set of objects $I_S^\kappa \subseteq \mathcal{C}_S^\kappa$, that is cofinal in $\mathcal{C}_S^\kappa$. We let $\mathcal{G}(S) = \varinjlim_{T\in I_S^\kappa} \mathcal{F}(T)$, for any choice of I_S^κ. Unraveling the definitions we see that $\iota_{\kappa,\lambda}^*\mathcal{F}$ is the sheafification of $\mathcal{G}$.

We claim that $\mathcal{G}$ is already a sheaf. Indeed, since filtered colimits are exact it is enough to prove that any v-cover $S' \to S$ in $\mathrm{PCAlg}_{\mathbb{F}_p,\lambda}^{op}$ can be expressed as a filtered colimit of v-covers in $\mathrm{PCAlg}_{\mathbb{F}_p,\kappa}^{op}$. Let $S = \mathrm{Spec}(A)$ and let $S' = \mathrm{Spec}(B)$, write $A = \varinjlim_{i\in I_S^\kappa} A_i$ and $B = \varinjlim_{j\in I_{S'}^\kappa} B_j$ with A_i and B_j κ-small rings, we may assume that the transition maps are all injective. By lemma 1.3.10 below we may assume that all morphisms $\mathrm{Spec}(A) \to \mathrm{Spec}(A_i)$ are v-covers. Consequently, the composition $S' \to S \to \mathrm{Spec}(A_i)$ is also a v-cover and whenever $S' \to \mathrm{Spec}(A_i)$ factors through a map $\mathrm{Spec}(B_j) \to \mathrm{Spec}(A_i)$ this later one is also a v-cover. We can replace our index sets I_S^κ and $I_{S'}^\kappa$ by a common index set I for which $(\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i))_{i\in I}$ is always defined and is a v-cover. We get the filtered colimit of v-covers required.

We claim that $\mathcal{G}$ is already a sheaf. Indeed, since filtered colimits are exact it is enough to prove that any v-cover $S' \to S$ in $\mathrm{PCAlg}_{\mathbb{F}_p,\lambda}^{op}$ can be expressed as a filtered colimit of v-covers in $\mathrm{PCAlg}_{\mathbb{F}_p,\kappa}^{op}$. Let $S = \mathrm{Spec}(A)$ and let $S' = \mathrm{Spec}(B)$, write $A = \varinjlim_{i\in I_S^\kappa} A_i$ and $B = \varinjlim_{j\in I_{S'}^\kappa} B_j$ with A_i and B_j κ-small rings, we may assume that the transition maps are all injective. By lemma 1.3.10 below we may assume that all morphisms $\mathrm{Spec}(A) \to \mathrm{Spec}(A_i)$ are v-covers. Consequently, the composition $S' \to S \to \mathrm{Spec}(A_i)$ is also a v-cover and whenever $S' \to \mathrm{Spec}(A_i)$ factors through a map $\mathrm{Spec}(B_j) \to \mathrm{Spec}(A_i)$ this later one is also a v-cover. We can replace our index sets I_S^κ and $I_{S'}^\kappa$ by a common index set I and replace the rings B_j by the smallest subring of B containing B_j and A_i for some $i \in I_S^\kappa$ so that we get a family indexed by I for which $(\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i))_{i\in I}$ is always defined and is a v-cover. We get our expression

$$(S' \to S) = \varprojlim_{i\in I} (\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i))_{i\in I}.$$

Once we know $\iota_{\kappa,\lambda}^*\mathcal{F} = \mathcal{G}$, we compute $\iota_{\kappa,\lambda,*}\iota_{\kappa,\lambda}^*\mathcal{F}(S)$ for $S \in \mathrm{PCAlg}_{\mathbb{F}_p,\kappa}^{op}$ to be $\varinjlim_{T\in I_S^\kappa} \mathcal{F}(T)$, but since S is κ-small the identity map is cofinal in $\mathcal{C}_S^\kappa$ and $\varinjlim_{I_S^\kappa} \mathcal{F}(S) = \mathcal{F}(S)$ as we needed to show. $\square$

Lemma 1.3.10. *Let κ be a cut-off cardinal, $S \in \mathrm{PCAlg}_{\mathbb{F}_p}^{op}$ and $T \in \mathrm{PCAlg}_{\mathbb{F}_p,\kappa}^{op}$. Given a morphism $g : S \to T$, there is $T' \in \mathrm{PCAlg}_{\mathbb{F}_p,\kappa}^{op}$ together with morphisms $f : S \to T'$ and $h : T' \to T$ such that f is a v-cover and $g = h \circ f$.*

Proof. This lemma is purely of set-theoretic nature and contentless otherwise. Indeed, if S was κ-small we could simply choose $T' = S$ and f to be the identity. Lets treat the general case, let $S = \mathrm{Spec}(B)$ and $T = \mathrm{Spec}(A)$. By replacing A by its image in B we may assume $g^* : A \to B$ to be injective. We construct a countable sequence of subrings

$$A = A_0 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} \subseteq \ldots B$$

with the property that each A_i is κ-small and that the image of the map $\mathrm{Spa}(B, B) \to \mathrm{Spa}(A_n, A_n)$ coincides with the image of $\mathrm{Spa}(A_{n+1}, A_{n+1}) \to \mathrm{Spa}(A_n, A_n)$. We do this inductively as follows: Assume A_n to be defined and let $Z_n \subseteq \mathrm{Spa}(A_n, A_n)$ be the image of $\mathrm{Spa}(B, B)$ in $\mathrm{Spa}(A_n, A_n)$. If x is an element of $\mathrm{Spa}(A_n, A_n) \setminus Z_n$ the valuation $|\cdot|_x : A_n \to \Gamma_x$ can't be extended to a valuation $|\cdot| : B \to \Gamma$. A compactness argument proves there are finitely many elements $\{a_1, \ldots a_m\}$ such that $|\cdot|_x$ does not extend to $A_n[a_1, \ldots, a_m] \subseteq B$. Since $\mathrm{Spa}(A_n, A_n) \setminus Z_n$ is κ-small, there is $\lambda < \kappa$ and a set $\{a_i\}_{i\in\lambda} \subseteq B$ such that $A_n[a_i]_{i\in\lambda}$ does not extend any $x \in \mathrm{Spa}(A_n, A_n) \setminus Z_n$. We let $A_{n+1} = A_n[a_i^{\frac{1}{p^\infty}}]_{i\in\lambda}$, clearly A_{n+1} satisfies the desired properties.

Let $A_\infty = \varinjlim_{i\in\mathbb{N}} A_i$, we claim that the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A_\infty)$ is a v-cover and that A_∞ is κ-small. Indeed, since each A_i is κ-small and since the cofinality of κ is larger than ω (See [26] 4.1) A_∞ is also κ-small. To prove it is a v-cover, we can use lemma 1.3.5 to prove instead that $\mathrm{Spa}(B, B) \to \mathrm{Spa}(A_\infty, A_\infty)$ is surjective. One verifies that $\mathrm{Spa}(A_\infty, A_\infty) = \varprojlim_{i\in\mathbb{N}} \mathrm{Spa}(A_i, A_i)$ as topological spaces. Given a compatible sequence $x_i \in \mathrm{Spa}(A_i, A_i)$ we define M_i to be the preimage of x_i in $\mathrm{Spa}(B, B)$. This gives a sequence of sets

$$\mathrm{Spa}(B, B) \supseteq M_0 \supseteq M_1 \ldots M_n \supseteq \ldots$$

Since the maps $\mathrm{Spa}(B, B) \to \mathrm{Spa}(A_i, A_i)$ are spectral maps of spectral topological spaces, each of the M_i is closed and compact in the patch topology and their intersection is non-empty. Any element in this intersection will map to the element $x_\infty \in \mathrm{Spa}(A_\infty, A_\infty)$ represented by the compatible sequence x_i. $\square$

We define $\widetilde{\mathrm{SchPerf}}$ as the $\underline{\text{big}}$ colimit $\bigcup_\kappa \widetilde{\mathrm{SchPerf}}_\kappa$ along all cut-off cardinals and the fully-faithful embeddings $\iota_{\kappa,\lambda}^*$. Objects in $\widetilde{\mathrm{SchPerf}}$ are called small scheme-theoretic v-sheaves.

The general formalism of topoi, specifically ([2] IV 4.9.4), allows us to promote $\diamond_\kappa : \mathrm{PCAlg}_{\mathbb{F}_p,\kappa}^{op} \to \widetilde{\mathrm{Perf}}_\kappa$ to a morphism of topoi $f_\kappa : \widetilde{\mathrm{Perf}}_\kappa \to \widetilde{\mathrm{SchPerf}}_\kappa$ for which $f_\kappa^*|_{\mathrm{PCAlg}_{Fp,\kappa}^{op}} = \diamond_\kappa$. Indeed, proposition 1.3.3 shows that $\diamond_\kappa$ is left-exact and proposition 1.3.7 gives us continuity of $\diamond_\kappa$.

Proposition 1.3.11. 1. *Given two cut-off cardinals $\kappa < \lambda$ we have a commutative diagram of morphism of topoi:*

$$\begin{array}{ccc}
\widetilde{\mathrm{Perf}}_\lambda & \xrightarrow{f_\lambda} & \widetilde{\mathrm{SchPerf}}_\lambda \\
\big\downarrow{\scriptstyle \iota_{\kappa,\lambda}} & & \big\downarrow{\scriptstyle \iota_{\kappa,\lambda}} \\
\widetilde{\mathrm{Perf}}_\kappa & \xrightarrow{f_\kappa} & \widetilde{\mathrm{SchPerf}}_\kappa
\end{array}$$

2. *We also have that the natural morphism $\iota_{\kappa,\lambda}^* \circ f_{\kappa,*} \to f_{\lambda,*} \circ \iota_{\kappa,\lambda}^*$ is an isomorpism.*

Proof. The commutativity of morphism of topoi follows formally from the similar commutativity of continuous functors:

$$\begin{tikzcd} \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa} \arrow[r, "\diamond_\kappa"] \arrow[d, "\iota^*_{\kappa,\lambda}"] & \widetilde{\mathrm{Perf}}_\kappa \arrow[d, "\iota^*_{\kappa,\lambda}"] \\ \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\lambda} \arrow[r, "\diamond_\lambda"] & \widetilde{\mathrm{Perf}}_\lambda \end{tikzcd}$$

For the second claim, given an element $S \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\lambda}$ we let I^κ_S be an index set category as in the proof of 1.3.9. If $S = \mathrm{Spec}(A)$ we let $X = Spa(A((t^{\frac{1}{p^\infty}})), A[[t^{\frac{1}{p^\infty}}]])$ and $Y = X \times_{S^\diamond} X$. In a similar way, for $T \in I^\kappa_S$ with $T = \mathrm{Spec}(B)$ we let $X_T = Spa(B((t^{\frac{1}{p^\infty}})), B[[t^{\frac{1}{p^\infty}}]])$ and $Y_T = X_T \times_{T^\diamond} X_T$. The family of perfectoid spaces $(X_T)_{T\in I^\kappa_S}$ ($(Y_T)_{T\in I^\kappa_S}$ respectively) is cofinal in the category $\mathcal{C}^\kappa_X$ of maps $X \to X'$ with X' a κ-small perfectoid space ($\mathcal{C}^\kappa_Y$ respectively). We get the following chain of isomorphisms:

$$\begin{aligned}
\iota^*_{\kappa,\lambda} f_{\kappa,*}\mathcal{F}(S) &= \varinjlim_{T\in I^\kappa_S} Hom_{\widetilde{\mathrm{SchPerf}}_\kappa}(h_T, f_{\kappa,*}\mathcal{F}) && (1.1)\\
&= \varinjlim_{T\in I^\kappa_S} Hom_{\widetilde{\mathrm{Perf}}_\kappa}(f^*_\kappa h_T, \mathcal{F}) && (1.2)\\
&= \varinjlim_{T\in I^\kappa_S} Hom_{\widetilde{\mathrm{Perf}}_\kappa}(T^{\diamond_\kappa}, \mathcal{F}) && (1.3)\\
&= \varinjlim_{T\in I^\kappa_S} Eq_{\widetilde{\mathrm{Perf}}_\kappa}(Hom(X_T,\mathcal{F}) \rightrightarrows Hom(Y_T,\mathcal{F})) && (1.4)\\
&= Eq_{\widetilde{\mathrm{Perf}}_\lambda}(\varinjlim_{T\in I^\kappa_S} Hom(X_T,\mathcal{F}) \rightrightarrows \varinjlim_{T\in I^\kappa_S} Hom(Y_S,\mathcal{F})) && (1.5)\\
&= Eq_{\widetilde{\mathrm{Perf}}_\lambda}(Hom(X_S,\iota^*_{\kappa,\lambda}\mathcal{F}) \rightrightarrows Hom(Y_S,\iota^*_{\kappa,\lambda}\mathcal{F})) && (1.6)\\
&= Hom_{\widetilde{\mathrm{Perf}}_\lambda}(S^{\diamond_\lambda}, \iota^*_{\kappa,\lambda}\mathcal{F}) && (1.7)\\
&= Hom_{\widetilde{\mathrm{SchPerf}}_\lambda}(h_S, f_{\lambda,*}\iota^*_{\kappa,\lambda}\mathcal{F}) && (1.8)\\
&= f_{\lambda,*}\iota^*_{\kappa,\lambda}\mathcal{F}(S) && (1.9)
\end{aligned}$$

□

Recall that a morphism of topoi consists of a pair of adjoint functors (f^*, f_*) such that f^* commutes with finite limits. By proposition 1.3.11 above we can gather all of the morphisms of topoi $f_\kappa : \widetilde{\mathrm{Perf}}_\kappa \to \widetilde{\mathrm{SchPerf}}_\kappa$ into a pair of adjoint functors $(f^*, f_*) : \widetilde{\mathrm{Perf}} \to \widetilde{\mathrm{SchPerf}}$ such that f^* commutes with finite limits. This is not a morphism of topoi because $\widetilde{\mathrm{Perf}}$ and $\widetilde{\mathrm{SchPerf}}$ are not topoi, but they behave as such.

Definition 1.3.12. *Let (f^*, f_*) the pair of adjoint functors described above, given $\mathcal{F} \in \widetilde{\mathrm{SchPerf}}$ we will denote $f^*\mathcal{F}$ by $\mathcal{F}^\diamond$ and given $\mathcal{G} \in \widetilde{\mathrm{Perf}}$ we will denote $f_*\mathcal{G}$ by $(\mathcal{G})^{\mathrm{red}}$. We refer to $(-)^{\mathrm{red}}$ as the reduction functor.*

Remark 1.3.13. *The functor $(-)^{\mathrm{red}}$ will be very important for our purposes. To make this functor explicit take a small v-sheaf $\mathcal{F} \in \widetilde{\mathrm{Perf}}$ and $S \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p}$. By adjunction $\mathcal{F}^{\mathrm{red}}(S) = Hom_{\widetilde{\mathrm{Perf}}}(S^\diamond, \mathcal{F})$.*

We can endow any small scheme-theoretic v-sheaf with a topological space in a similar fashion to definition 1.1.8. Given $\mathfrak{S} \in \widetilde{\mathrm{SchPerf}}$ we let $|\mathfrak{S}|$ denote the set of equivalence classes of maps $\mathrm{Spec}(k) \to \mathfrak{S}$, where k is a perfect field over $\mathbb{F}_p$. Two maps p_1, p_2 are equivalent if we can complete a commutative diagram as below:

$$\begin{tikzcd}
 & \mathrm{Spec}(k_1) \arrow[dr, "p_1"] & \\
\mathrm{Spec}(k_3) \arrow[ur, "q_1"] \arrow[rr, "p_3"] \arrow[dr, "q_2"'] & & \mathfrak{S} \\
 & \mathrm{Spec}(k_2) \arrow[ur, "p_2"'] &
\end{tikzcd}$$

Proposition 1.3.14. *1. Given $\mathfrak{S} \in \widetilde{\mathrm{SchPerf}}$ there is a pair of cut-off cardinals $\kappa < \lambda$ and a λ-small family $\{S_i\}_{i\in I}$ of objects in $\mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$ together with a surjective map $X = (\coprod_{i\in I} S_i) \to \mathfrak{S}$.*

2. *The small scheme-theoretic v-sheaf $R = X \times_{\mathfrak{S}} X$ has a similar cover $Y = (\coprod_{j\in J} T_j) \to R$, there is a natural map $|X| \to |\mathfrak{S}|$ which induces a bijection $|\mathfrak{S}| \cong |X|/|Y|$. We endow $|\mathfrak{S}|$ with the quotient topology induced by this bijection.*

3. *The topology on $|\mathfrak{S}|$ does not depend on the choices of X or Y.*

4. *Any map of small v-sheaves $\mathfrak{S}_1 \to \mathfrak{S}_2$ induces a continuous map of topological spaces $|\mathfrak{S}_1| \to |\mathfrak{S}_2|$.*

Proof. By definition $\mathfrak{S} \in \widetilde{\mathrm{SchPerf}}_\kappa$ for some cut-off cardinal κ, the category $\mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$ is a small category. By cofinality of cut-off cardinals we may pick λ larger than $\sup_{T\in \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}} \mathfrak{S}(T)$. We let $X = (\coprod_{T\in \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}} \coprod_{x\in\mathfrak{S}(T)} T)$ with the evident projection map to $\mathfrak{S}$. We claim this map is surjective. This map is defined in $\widetilde{\mathrm{SchPerf}}_\lambda$ and it is enough to prove surjectivity there. Given $S \in \mathrm{PCAlg}_{\mathbb{F}_p,\lambda}$, we have $\mathfrak{S}(S) = \varinjlim_{T\in I^\kappa_S} \mathfrak{S}(T)$. Since this colimit is filtered, for a fixed map $g : S \to \mathfrak{S}$ we can find $(f : S \to T) \in I^\kappa_S$ and a map $h : T \to \mathfrak{S}$ with $g = h \circ f$. In particular, g factors through a map to X since X contains a copy of T mapping to $\mathfrak{S}$ via h.

We move on to the second claim. Given $x \in |X|$ we take the residue field inclusion $\iota_x : \mathrm{Spec}(k_x) \to X$. The composition $\mathrm{Spec}(k_x) \to X \to \mathfrak{S}$ defines an element of $|\mathfrak{S}|$. Suppose now that $x_1, x_2 \in |X|$, we must show that $(x_1, x_2) \in |X| \times |X|$ is in the image of $|Y|$ if and only if x_1 and x_2 define the same element in $|\mathfrak{S}|$. If the maps ι_{x_1} and ι_{x_2} are equivalent we get a map $\iota_{x_3} : \mathrm{Spec}(k_3) \to X \times_{\mathfrak{S}} X$. By replacing k_3 by a larger field if necessary, we may assume that ι_{x_3} lifts to Y and defines an element $y \in |Y|$. We see that y maps to (x_1, x_2) in $|X| \times |X|$. On the other hand, if there is $y \in Y$ mapping to (x_1, x_2) and $\iota_y : \mathrm{Spec}(k_y) \to Y$ is the residue field map, the compositions $\mathrm{Spec}(k_y) \to Y \xrightarrow{\pi_i} X$ factor through $\iota_{x_i} : \mathrm{Spec}(k_{x_i}) \to X$. This proves that x_1 and x_2 map to the same point $|\mathfrak{S}|$.

For the third claim suppose we are given two covers $X_i \to \mathfrak{S}$ with $i \in \{1, 2\}$, we must show that the two quotient topologies coming from the surjections $|X_i| \to |\mathfrak{S}|$ agree. The small scheme theoretic v-sheaf $R = X_1 \times_{\mathfrak{S}} X_2$ admits a v-cover $X_3 \to R$ by the first claim. By replacing X_2 by X_3 we may assume that we have a commutative diagram of surjective maps:

$$\begin{tikzcd}
X_2 \arrow[dd] \arrow[dr] & \\
 & \mathfrak{S} \\
X_1 \arrow[ur] &
\end{tikzcd}$$

Since $X_2 \to X_1$ is a v-cover we get a quotient map of topological spaces $|X_2| \to |X_1|$. If we give $|\mathfrak{S}|$ the quotient topology coming from the surjection $|X_1| \to |\mathfrak{S}|$ the composition map $|X_2| \to |\mathfrak{S}|$ is also a quotient map. This implies that the two topologies agree.

For the last claim, we may find covers $\mathfrak{S}_1$ and $\mathfrak{S}_2$ by X_1 and X_2 respectively forming the following commutative diagram:

$$\begin{array}{ccc} \mid X_2 \mid & \xrightarrow{q} & \mid \mathfrak{S}_2 \mid \\ \downarrow & & \downarrow \\ \mid X_1 \mid & \xrightarrow{q} & \mid \mathfrak{S}_1 \mid \end{array}$$

Both horizontal maps are quotient maps and the leftmost vertical map is continuos since it is induced by a morphism of unions of affine schemes, this prove the required continuity. □

1.3.2 Reduced v-sheaves

Definition 1.3.15. *We say that a small scheme-theoretic v-sheaf $\mathcal{F}$ is reduced if the adjunction morphism $\mathcal{F} \to (\mathcal{F}^\diamond)^{\mathrm{red}}$ is an isomorphism in $\widetilde{\mathrm{SchPerf}}$.*

We have the following formal consequences of our definition.

Proposition 1.3.16. *1. If $S \in \mathrm{PCAlg}_{\mathbb{F}_p}^{op}$ then h_S reduced.*

2. The functor $\diamond : \widetilde{\mathrm{SchPerf}} \to \widetilde{\mathrm{Perf}}$ is fully-faithful when restricted to small reduced v-sheaves.

Proof. The first claim follows by proposition 1.1.26. The second claim follows from adjunction. Indeed, if $\mathcal{F}$ is reduced and $\mathcal{G} \in \widetilde{\mathrm{SchPerf}}$ then:

$$Hom_{\widetilde{\mathrm{Perf}}}(\mathcal{G}^\diamond, \mathcal{F}^\diamond) = Hom_{\widetilde{\mathrm{SchPerf}}}(\mathcal{G}, (\mathcal{F}^\diamond)^{\mathrm{red}}) = Hom_{\widetilde{\mathrm{SchPerf}}}(\mathcal{G}, \mathcal{F})$$

□

Proposition 1.3.17. *If $\mathcal{F} \in \widetilde{\mathrm{SchPerf}}$ is represented by a perfect scheme over $\mathbb{F}_p$ then $\mathcal{F}$ is reduced. (See [28] 18.3.1)*

Proof. Fix a presentation $\mathcal{F} = X/Y$ where X and Y are disjoint unions of perfect affine schemes. We have that $\mathcal{F}^\diamond = X^\diamond / Y^\diamond$, and that $\mathcal{F}^\diamond$ is the sheafification of the functor that assigns to (R, R^+) the set $X^\diamond(R, R^+)/Y^\diamond(R, R^+)$. Since $\mathcal{F}$ is a scheme it is enough to sheafify with respect to the analytic topology of $\mathrm{Spa}(R, R^+)$. As a consequence to specify a map $\mathcal{G} \to \mathcal{F}^\diamond$ it is enough to cover $\mathcal{G}$ by open sub-v-sheaves and to give a map from this cover to X that glues over Y.

The rest of the proof is a small variation of the proof given in loc. cit. taking advantage of the olvine spectrum for schemes. We prove that if V is a valuation ring then any morphism $\mathrm{Spec}(V)^\diamond \to \mathcal{F}^\diamond$ must factor through an open affine $S^\diamond \subseteq \mathcal{F}^\diamond$. Suppose to get a contradiction that this was not the case and fix an "exotic" map $\mathrm{Spec}(V)^\diamond \to \mathcal{F}^\diamond$ that doesn't factor through an affine subsheaf. Let k denote the residue field of V at the maximal ideal and let K be the fraction field of V, then $\mathrm{Spo}(k, k)$ and $\mathrm{Spo}(K, K)$ consist of one point $s \in \mathrm{Spo}(V, V)$ and $\eta \in \mathrm{Spo}(V, V)$ respectively. The map $\mathrm{Spec}(k)^\diamond \to \mathcal{F}^\diamond$ factors through an affine subset $S^\diamond$, with $S = \mathrm{Spec}(A)$. Let $U_1 \subseteq \mathrm{Spo}(V, V)$ be the open subset associated to the pullback of $S^\diamond$, this is by assumption a proper open subset. Let $Z = \mathrm{Spo}(V, V) \setminus U_1$, it is a quasicompact topological space and we may use [28] 18.3.2 to find a minimal $\mathfrak{p}_m$ point in $supp \circ h(Z) \subseteq \mathrm{Spec}(V)$. Replacing V by $V/\mathfrak{p}_m$ we may assume that all elements of $z \in Z$ satisfy $supp(z) = \{0\}$, and that $Z \subseteq \mathrm{Spo}(K, V)$. Since Z is a closed subset it contains the unique closed point q of $\mathrm{Spo}(K, V)$. Also, U_1 contains every analytic localization of the form $N_{b<<1}$ with $b \neq 0$. Indeed, if $z \in Z \cap U_1$ then $|b|_z^a < 1$ and either z or its formal specialization would have non-trivial support contradicting our assumption.

The composition $\mathrm{Spd}(K, V) \to \mathrm{Spec}(V)^\diamond \to \mathcal{F}^\diamond$ must also factor through some other open affine subsheaf $T^\diamond$ with $T = \mathrm{Spec}(B)$. We let U_2 be the open in $\mathrm{Spo}(V, V)$ associated to the pullback of $T^\diamond$.

Observe that since V is a valuation ring and q is the closed point of $\mathrm{Spo}(K,V)$, whenever $q \in U_{a\leq b\neq 0}$ the identity $U_{a\leq b\neq 0} = U_{0\leq b\neq 0}$ holds. Moreover, analytic localizations containing q are trivial. In summary, all localizations containing q can be expressed as classical localizations of the form $U_{0\leq b\neq 0}$. This family of localizations form a total order by containment with $U_{0\leq b\neq 0} \subseteq U_{0\leq a\neq 0}$ if and only if $\frac{b}{a} \in V$. Consequently, there is a classical localization $U_{0\leq b\neq 0}$ contained in U_2.

We have found neighborhoods $N_{b<<1} \subseteq U_1$ and $U_{0\leq b\neq 0} \subseteq U_2$ with $N_{b<<1} \cap U_{0\leq b\neq 0} \subseteq U_1 \cap U_2$, observe that $\mathrm{Spo}(V,V) = N_{b<<1} \cup U_{0\leq b\neq 0}$. Let V_b denote the (b)-adic completion of V, then lemma 1.2.19 shows that $N_{b<<1}$ is represented by $\mathrm{Spd}(V_b, V_b)$. We also have that $U_{0\leq b\neq 0}$ is represented by $\mathrm{Spd}(V[\frac{1}{b}], V)$ and that the intersection $N_{b<<1} \cap U_{0\leq b\neq 0}$ is represented by $\mathrm{Spa}(V_b[\frac{1}{b}], V_b)$, notice that this last one is a perfectoid field.

Since these morphisms glue, there is an affine open subsheaf $R^\diamond \subseteq S^\diamond \times_{\mathcal{F}} T^\diamond$ with $R = \mathrm{Spec}(D)$ and a map $\mathrm{Spa}(V_b[\frac{1}{b}], V_b) \to R^\diamond$ making the following diagram commutative:

$$\begin{tikzcd}
\mathrm{Spa}(V_b[\frac{1}{b}], V_b) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathrm{Spd}(V[\frac{1}{b}], V) \arrow[d] \\
& R^\diamond \arrow[r] \arrow[d] & T^\diamond \arrow[d] \\
\mathrm{Spd}(V_b, V_b) \arrow[r] & S^\diamond \arrow[r] & \mathcal{F}^\diamond
\end{tikzcd}$$

The maps $\mathrm{Spa}(V_b[\frac{1}{b}], V_b) \to R^\diamond$ and $\mathrm{Spd}(V[\frac{1}{b}], V) \to T^\diamond$ are given by ring maps $D \to V_b$ and $B \to V$, because $\mathrm{Spa}(V_b[\frac{1}{b}], V_b)$ is perfectoid and because of lemma 1.1.26 respectively. This gives the following commutative diagram of morphisms of affine schemes

$$\begin{tikzcd}
\mathrm{Spec}(V_b) \arrow[r] \arrow[d] & \mathrm{Spec}(V) \arrow[d] \\
\mathrm{Spec}(D) \arrow[r] & \mathrm{Spec}(B)
\end{tikzcd}$$

where the map $\mathrm{Spec}(D) \to \mathrm{Spec}(B)$ is an open embedding. Since the closed point of $\mathrm{Spec}(V)$ is in the image of $\mathrm{Spec}(V_b)$ we get a map $\mathrm{Spec}(V) \to \mathrm{Spec}(D)$. This contradicts the assumption that $\mathrm{Spa}(K,V)$ doesn't factor through $S^\diamond$, and finishes the proof that there are no "exotic" maps $\mathrm{Spec}(V)^\diamond \to X$.

Now we prove that if $L = \prod_{i\in I} V_i$ is a schematic-products of points, then there are no "exotic" maps $\mathrm{Spec}(L)^\diamond \to \mathcal{F}^\diamond$ either. Let k_i, be the residue field of V_i. The product of points $\mathrm{Spec}(\prod_{i\in I} k_i)^\diamond$ has underlying topological space the Stone-Čech compactification of I. There is a partition $I = I_1 \coprod I_2 \cdots \coprod I_n$ and open affine subsheaves $S_j^\diamond \subseteq \mathcal{F}^\diamond$ such that $\mathrm{Spec}(\prod_{i\in I_j} k_i)^\diamond \to \mathcal{F}^\diamond$ factors through $S_j^\diamond$. Let U_j be the pullback of $S_j^\diamond$ to $\mathrm{Spec}(L)^\diamond$. We claim that $\mathrm{Spec}(\prod_{i\in I_j} V_i)^\diamond \subseteq U_j$ which would finish the proof. Indeed, the connected components of $\mathrm{Spec}(\prod_{i\in I_j} V_i)^\diamond$ have the form $\mathrm{Spec}(V_{\mathcal{U}})^\diamond$ for some ultrafilter $\mathcal{U}$ of I_j. The point corresponding to the residue field $k_{\mathcal{U}}$ is in $\mathrm{Spec}(\prod_{i\in I_j} k_i)^\diamond$ and by our work above all of $\mathrm{Spec}(V_{\mathcal{U}})^\diamond$ factors through U_j.

We have proven that $(\mathcal{F}^\diamond)^{\mathrm{red}}$ takes the same values as $\mathcal{F}$ when evaluated on product of points, since product of points form a basis for the topology these must be isomorphic. This finishes the proof. $\square$

1.3.3 Reduction functor and formal adicness

Intuitively speaking, the reduction functor kills all topological nilpotents and removes analytic points from our v-sheaf. Below, we try to justify why one can think of this reduction functor as an analogue of taking the underlying reduced subscheme of a formal scheme.

Lemma 1.3.18. *The scheme-theoretic v-sheaf $(\mathbb{Z}_p{}^\diamond)^{\mathrm{red}}$ is represented by $\mathbb{F}_p$.*

Proof. We must prove that for any $S \in \mathrm{PCAlg}_{\mathbb{F}_p}^{op}$ there is a unique map $S^\diamond \to \mathbb{Z}_p^\diamond$. This is a direct consequence of lemma 1.1.25. $\square$

We make some conventions. We say that a Huber pair (A, A) is formal if A is its own ring of definition. For a Huber pair (A, A^+) over $\mathbb{Z}_p$, we let A_{red} denote the perfection of $A/(A \cdot A^{\circ\circ})$ where $A \cdot A^{\circ\circ}$ is the ideal generated by the set of topologically nilpotent elements. The following statement generalizes lemma 1.3.18

Proposition 1.3.19. *Let (A, A^+) be Huber pair over $\mathbb{Z}_p$, then $\mathrm{Spd}(A, A^+)^{\mathrm{red}}$ is represented by $\mathrm{Spec}(A_{\mathrm{red}})$.*

Proof. By lemma 1.1.26 if $S = \mathrm{Spec}(R) \in \mathrm{PCAlg}_{\mathbb{F}_p}^{op}$ then morphisms $S^\diamond \to \mathrm{Spd}(A, A^+)$ are given by maps of adic spaces $f : \mathrm{Spa}(R, R) \to \mathrm{Spa}(A, A^+)$. Since $0 \in R$ is the only topological nilpotent $f(A^{\circ\circ}) = 0$, since R is perfect the map $f^* : A/A \cdot A^{\circ\circ} \to R$ factors uniquely through perfection.

$\square$

The following statement may be thought of as a global version of proposition 1.3.19 and lemma 1.1.26.

Proposition 1.3.20. *If X is a pre-adic sace over $\mathbb{Z}_p$ and $X^{na} \to X$ is the closed embedding associated to the non-analytic locus of X then the following hold:*

1. *Given $S \in \mathrm{PCAlg}_{\mathbb{F}_p}^{op}$, every morphism $S^\diamond \to X^\diamond$ is given by a unique morphism of pre-adic spaces.*
2. *The natural map $(X^{na,\diamond})^{\mathrm{red}} \to (X^\diamond)^{\mathrm{red}}$ is an isomorphism.*

Proof. We claim that for a scheme-theoretic product of points, $\mathrm{Spec}(P)$, every map $\mathrm{Spec}(P)^\diamond \to X^\diamond$ is coming from a map of adic spaces $\mathrm{Spa}(P, P) \to X$ by applying the functor $\diamond$. The statements above follows from this claim since any map $\mathrm{Spa}(P, P) \to X$ must factor through the non-analytic locus, and since product of points form a basis for the topology.

If for any V a perfect valuation ring in characteristic p we can prove that maps $\mathrm{Spec}(V)^\diamond \to X^\diamond$ factor through an open affinoid subsheaf of $X^\diamond$, then lemma 1.1.26 allow us to conclude. We use the same dévissage technique of proposition 1.3.17 to understand the problem. Assume there is an “exotic” map $\mathrm{Spec}(V)^\diamond \to X^\diamond$, that is, a map $\mathrm{Spec}(V)^\diamond \to X^\diamond$ that doesn't factor through an open affinoid subsheaf of X. The key difference with our setup and the one in proposition 1.3.17 comes when we get to the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spa}(V_b[\frac{1}{b}], V_b) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathrm{Spd}(V[\frac{1}{b}], V) \arrow[d] \\
& R^\diamond \arrow[r] \arrow[d] & T^\diamond \arrow[d] \\
\mathrm{Spd}(V_b, V_b) \arrow[r] & S^\diamond \arrow[r] & X^\diamond
\end{tikzcd}$$

In this case $S = \mathrm{Spa}(A, A^+)$, $T = \mathrm{Spa}(B, B^+)$, $R = \mathrm{Spa}(C, C^+)$ and R is a rational localization of both S and T. Since $\mathrm{Spa}(V_b[\frac{1}{b}], V_b)$ is perfectoid and by lemma 1.1.26 the map $\mathrm{Spa}(V_b[\frac{1}{b}], V_b) \to R^\diamond$ and $\mathrm{Spd}(V[\frac{1}{b}], V) \to T^\diamond$ come from morphisms of adic spaces, we get the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Spa}(V_b[\frac{1}{b}], V_b) \arrow[r] \arrow[d] & \mathrm{Spa}(V[\frac{1}{b}], V) \arrow[d] \\
R \arrow[r] & T
\end{tikzcd}$$

Since $R \to T$ is a rational localization, we get that $\mathrm{Spd}(V[\frac{1}{b}], V) \times_T R$ is the open subsheaf associated to a classical localization of $\mathrm{Spo}(V[\frac{1}{b}], V)$ containing $N_{b<<1} = \mathrm{Spa}(V_b[\frac{1}{b}], V_b)$. Since classical localizations are stable under meromorphic specialization, the unique closed point of $\mathrm{Spa}(V[\frac{1}{b}], V)$ must factor through R. But any classical localization containing this closed point contains all of $\mathrm{Spa}(V[\frac{1}{b}], V)$ which contradicts the existance of "exotic" maps $\mathrm{Spec}(V)^\diamond \to X^\diamond$. $\square$

We now justify the intuition behind thinking of diamonds as purely analytic objects.

Proposition 1.3.21. *For a quasi-separated diamond Y the associated reduced functor Y^{red} is the empty-sheaf.*

Proof. We need to prove that for a perfect scheme S there are no morphisms $S^\diamond \to Y$. It is enough to prove this for $S = \mathrm{Spec}(k)$ the spectrum of an algebraically closed field. Suppose there is such a map $f : S^\diamond \to Y$, and let $y \in |Y|$ be the unique point in the image of $|f|$. We consider Y_y the sub-v-sheaf of points of $\mathrm{Spa}(R, R^+) \to Y$ for which $|\mathrm{Spa}(R, R^+)| \to |Y|$ factors through y. The map f factors through Y_y and by ([26] 11.10) it is a quasi-separated diamond with $|Y_y|$ consisting of one point. We can use ([26] 21.9) to write $Y_y = \mathrm{Spa}(C, O_C)/\underline{G}$ with C a non-Archimedean algebraically closed field in characteristic p and $\underline{G}$ a profinite group acting continuously and faithfully on C.

Consider the v-cover $S' = \mathrm{Spa}(K_1, O_{K_1}) \to \mathrm{Spec}(k)^\diamond$ where K_1 is an algebraic closure of $k((t^{\frac{1}{p^\infty}}))$. Similarly, let $T = \mathrm{Spa}(K_2, O_{K_2})$ where K_2 is an algebraically closed non-Archimedean field containing k discretely and whose value group $\Gamma_{K_2} \subseteq \mathbb{R}^{\geq 0}$ has at least two elements that are linearly independent when we treat $\Gamma_{K_2} \setminus \{0\}$ as vector space over $\mathbb{Q}$. By our hypothesis on K_2, we can find two continuous embeddings $\iota_i^* : K_1 \to K_2$ such that $|\iota_1^*(K_1)|_{\Gamma_{K_2}} \cap |\iota_2^*(K_1)|_{\Gamma_{K_2}} = 1$ and in particular, such that $\iota_1^*(K_1) \cap \iota_2^*(K_1) = k$.

The composition of $f : S^\diamond \to Y_y$ with the natural projection $\mathrm{Spa}(K_1, {K_1}^+) \to S^\diamond$ gives a map $[g] : \mathrm{Spa}(K_1, {K_1}^+) \to Y_y$ such that $[g] \circ \iota_1 = [g] \circ \iota_2$. Since both $\mathrm{Spa}(K_1, {K_1}^+)$ and $\mathrm{Spa}(K_2, {K_2}^+)$ are algebraically closed fields the sets of maps to Y_y are given by G-orbits of maps to $\mathrm{Spa}(C, C^+)$, that is $Hom(\mathrm{Spa}(K_i, {K_i}^+), Y_y) = Hom(\mathrm{Spa}(K_i, {K_i}^+), \mathrm{Spa}(C, O_C))/G$. Let $g^* : (C, O_C) \to (K_1, O_{K_1})$ represent $[g]$ in $Hom(\mathrm{Spa}(K_1, {K_1}^+), Y_y)$, we get maps $\iota_i^* \circ g^* : (C, O_C) \to (K_2, O_{K_2})$ and since $[g] \circ \iota_1 = [g] \circ \iota_1$ we have $\iota_1^* \circ g^*(C) = \iota_2^* \circ g^*(C) \subseteq k$. This contradicts that k has the discrete topology and that C is a non-Archimedean field, the contradiction shows that the map $f : S^\diamond \to Y_y$ does not exist.

$\square$

Recall that a morphism of adic spaces $X \to Y$ is said to be adic if the image of an analytic point is again an analytic point. For v-sheaves we can define a related notion.

Definition 1.3.22. *We say that a morpism of v-sheaves $\mathcal{F} \to \mathcal{G}$ is formally adic if the commutative diagram that one obtains from adjunction:*

$$\begin{tikzcd} (\mathcal{F}^{\mathrm{red}})^\diamond \arrow[r] \arrow[d] & (\mathcal{G}^{\mathrm{red}})^\diamond \arrow[d] \\ \mathcal{F} \arrow[r] & \mathcal{G} \end{tikzcd}$$

is a Cartesian diagram.

We warn the reader that although the notion of a morphism of adic spaces to be adic is related to the morphism of v-sheaves being formally adic neither of this notions implies the other.

Example 1.3.23. *Take a perfect field k in characteristic p together with a rank 1 valuation subring $O_k \subseteq k$ with the discrete topology. The morphism of adic spaces $\mathrm{Spa}(k, O_k) \to \mathrm{Spa}(\mathbb{F}_p, \mathbb{F}_p)$ is adic. Nevertheless, the induced morphism $\mathrm{Spd}(k, O_k) \to \mathrm{Spd}(\mathbb{F}_p, \mathbb{F}_p)$ is not formally adic since $\mathrm{Spd}(k, O_k)^{\mathrm{red}}$ is represted by $\mathrm{Spec}(k)$.*

Example 1.3.24. *Take a non-Archimedean perfect field K in characteristic p and consider the morphism $id : Spa(K_1, O_{K_1}) \to Spa(K_2, O_{K_2})$ where $K_2 = K$ given the discrete topology and $K_1 = K$ given the topology induced by the norm. This morphism is not adic, nevertheless the reduction diagram looks like this:*

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \mathrm{Spec}(K_2)^\diamond \arrow[d] \\
\mathrm{Spd}(K_1, O_{K_1}) \arrow[r] & \mathrm{Spd}(K_2, O_{K_2})
\end{tikzcd}$$

Which is Cartesian.

Although the notion of formal adicness does not recover the notion of adicness in general, it will in some important situations:

Proposition 1.3.25. *Let A and B be formal Huber pairs over $\mathbb{Z}_p$ with ideals of definition I_A and I_B respectively. A morphism of adic spaces $\mathrm{Spa}(A, A) \to \mathrm{Spa}(B, B)$ is adic if and only if the corresponding morphism of v-sheaves $\mathrm{Spd}(A, A) \to \mathrm{Spd}(B, B)$ is formally adic.*

Proof. The reduction diagram looks as follows:

$$\begin{tikzcd}
\mathrm{Spec}(A_{\mathrm{red}})^\diamond \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& (\mathrm{Spec}(A/I_B)^{perf})^\diamond \arrow[r] \arrow[d] & \mathrm{Spec}(B_{\mathrm{red}})^\diamond \arrow[d] \\
& \mathrm{Spd}(A, A) \arrow[r] & \mathrm{Spd}(B, B)
\end{tikzcd}$$

Continuity of the morphism $B \to A$ ensures that $I_B^n \subseteq I_A$ for some n. In this context, the morphism is adic if and only if $I_B \cdot A$ is an ideal of definition of A which can only happen if $I_A^m \subseteq I_B$ for some m. If the morphism is adic, then after taking perfection the rings A_{red} and $(A/I_B)^{perf}$ become isomorphic which proves formal adicness. Conversely, if the morphism is formally adic, by hypothesis the rings $(A/I_B)^{perf}$, and A_{red} are isomorphic with the isomorphism being induced by the natural projections from $(A/p)^{perf}$. This implies that the ideals I_A and I_B define the same Zariski closed subset in $\mathrm{Spec}(A)$. In particular, the elements of I_A are nilpotent in A/I_B, and since I_A is finitely generated $I_A^m \subseteq I_B$ for some m. $\square$

Proposition 1.3.26. 1. *If $\mathcal{F} \to \mathcal{H}$ and $\mathcal{H} \to \mathcal{G}$ are formally adic, the composition $\mathcal{F} \to \mathcal{G}$ is formally adic.*

2. *If $\mathcal{F} \to \mathcal{H}$ is formally adic, the basechange $\mathcal{G} \times_{\mathcal{H}} \mathcal{F} \to \mathcal{G}$ is formally adic.*

Proof. The first claim is clear. For the second one we get a Cartesian diagram:

$$\begin{tikzcd}
(\mathcal{G}^{\mathrm{red}})^\diamond \times_{(\mathcal{H}^{\mathrm{red}})^\diamond} (\mathcal{F}^{\mathrm{red}})^\diamond \arrow[r] \arrow[d] & (\mathcal{F}^{\mathrm{red}})^\diamond \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
(\mathcal{G}^{\mathrm{red}})^\diamond \arrow[r] & (\mathcal{H}^{\mathrm{red}})^\diamond \arrow[r] & \mathcal{H}
\end{tikzcd}$$

The functors $(-)^{\mathrm{red}}$ and $(-)^\diamond$ commute with finite limits. This gives $(\mathcal{G}^{\mathrm{red}})^\diamond \times_{(\mathcal{H}^{\mathrm{red}})^\diamond} (\mathcal{F}^{\mathrm{red}})^\diamond = ((\mathcal{G} \times_{\mathcal{H}} \mathcal{F})^{\mathrm{red}})^\diamond$, and proves that

$$\begin{tikzcd}
((\mathcal{G} \times_{\mathcal{H}} \mathcal{F})^{\mathrm{red}})^\diamond \arrow[r] \arrow[d] & \mathcal{G} \times_{\mathcal{H}} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
(\mathcal{G}^{\mathrm{red}})^\diamond \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H}
\end{tikzcd}$$

is also Cartesian. □

Definition 1.3.27. *We say that a v-sheaf $\mathcal{F}$ over $\mathbb{Z}_p^\diamond$ is formally p-adic if the morphism $\mathcal{F} \to \mathbb{Z}_p^\diamond$ is formally adic.*

Over $\mathbb{Z}_p$ the situation of example 1.3.24 does not happen.

Proposition 1.3.28. *Suppose we have a Huber pair (A, A^+) and a map $f : \mathrm{Spa}(A, A^+) \to \mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, if $f^\diamond$ is formally adic then f is adic (as a morphism of adic spaces).*

Proof. Let $U \subseteq \mathrm{Spa}(A, A^+)$ the open subset of analytic points. It is easy to verify that this open embedding is formally adic because $\mathrm{Spec}(A_{\mathrm{red}})^\diamond \to \mathrm{Spd}(A, A^+)$ factors through the complement of $U^\diamond$ and because by proposition 1.3.21 $(U^\diamond)^{\mathrm{red}} = \emptyset$ holds. Since formal adicness is preserved by composition $U^\diamond \to \mathbb{Z}_p^\diamond$ is formally adic. By formal adicness the map $U^\diamond \to \mathbb{Z}_p^\diamond$ must factor through $\mathbb{Q}_p^\diamond$. This proves $f(U) \subseteq \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which proves that f is adic. □

Recall that a v-sheaves $\mathcal{F}$ is said to be separated if the diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is a closed immersion (See [26] 10.7). We need the following related notion:

Definition 1.3.29. *We say that a v-sheaf is formally separated if $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is a formally adic closed immersion.*

Lemma 1.3.30. *The v-sheaf $\mathbb{Z}_p^\diamond$ is formally separated.*

Proof. We need to prove that the diagonal $\mathbb{Z}_p^\diamond \to \mathbb{Z}_p^\diamond \times \mathbb{Z}_p^\diamond$ is a closed immersion of perfectoid spaces after any basechange by maps $\mathrm{Spa}(R, R^+) \to \mathbb{Z}_p^\diamond \times \mathbb{Z}_p^\diamond$, with $\mathrm{Spa}(R, R^+) \in \mathrm{Perf}$. This amounts to proving that the locus on which two untilts agree is closed inside $|\mathrm{Spa}(R, R^+)|$ and representable by a perfectoid space. Now, each untilt is individually cut out of $\mathrm{Spa}(W(R^+), W(R^+)) \setminus \{V([\varpi])\}$ as a closed Cartier divisor (See [28] 11.3.1). We can take the intersection which will define a Zariski closed subset in each of the untilts, but Zariski closed subsets of a perfectoid space are representable by some other perfectoid space. The tilt of such a closed immersion represents this basechange.

To prove the diagonal is formally adic we compute directly $(\mathbb{Z}_p^\diamond \times_{\mathbb{F}_p^\diamond} \mathbb{Z}_p^\diamond)^{\mathrm{red}} = \mathbb{F}_p$ since $(-)^{\mathrm{red}}$ commutes with limits. The basechange $\mathbb{F}_p^\diamond \times_{\mathbb{Z}_p^\diamond \times_{\mathbb{F}_p^\diamond} \mathbb{Z}_p^\diamond} \mathbb{Z}_p^\diamond$ agrees with $\mathbb{F}_p^\diamond$. □

Proposition 1.3.31. *If a v-sheaf $\mathcal{F}$ is formally p-adic, then the diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is formally adic.*

Proof. We have a formally adic map $\mathcal{F} \to \mathbb{Z}_p^\diamond$, and since formal adicness is preserved by basechange we get a formally adic map $\mathcal{F} \times_{\mathbb{Z}_p^\diamond} \mathcal{F} \to \mathbb{Z}_p^\diamond$. By the two out of three property of Cartesian diagrams, the diagonal map $\mathcal{F} \to \mathcal{F} \times_{\mathbb{Z}_p^\diamond} \mathcal{F}$ is also formally adic. Now, $\mathcal{F} \times_{\mathbb{Z}_p^\diamond} \mathcal{F}$ is the basechange of the diagonal map $\mathbb{Z}_p^\diamond \to \mathbb{Z}_p^\diamond \times \mathbb{Z}_p^\diamond$ by the projection map $\mathcal{F} \times \mathcal{F} \to \mathbb{Z}_p^\diamond \times \mathbb{Z}_p^\diamond$. This gives us that $\mathcal{F} \times_{\mathbb{Z}_p^\diamond} \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is also formally adic. Since formal adicness is preserved by composition, $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is also formally adic as we needed to show. □

Lemma 1.3.32. *The diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is formally adic if and only if the adjunction morphim $(\mathcal{F}^{\mathrm{red}})^\diamond \to \mathcal{F}$ is injective. In this case, if (A, A^+) is a perfectoid Huber pair, and $m \in \mathcal{F}(A, A^+)$ then $m \in (\mathcal{F}^{\mathrm{red}})^\diamond(A, A^+)$ if and only if $\mathrm{Spa}(A, A^+)$ admits a v-cover $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(A, A^+)$ and a morphism $\mathrm{Spec}(R^+)^\diamond \to \mathcal{F}$ making the following diagram commutative:*

$$\begin{tikzcd}
\mathrm{Spa}(R,R^+) \arrow[r] \arrow[d] & \mathrm{Spec}(R^+)^\diamond \arrow[d] \\
\mathrm{Spa}(A,A^+) \arrow[r, "m"] & \mathcal{F}
\end{tikzcd}$$

Proof. In general, a map of sheaves $\mathcal{G} \to \mathcal{F}$ is injective if and only if $(\mathcal{G} \times \mathcal{G}) \times_{\mathcal{F}\times\mathcal{F}} \mathcal{F} = \mathcal{G}$. We can apply this reasoning to the map $(\mathcal{F}^{\mathrm{red}})^\diamond \to \mathcal{F}$.

For the second claim let $\mathcal{C}_R$ be the category of maps $\mathrm{Spa}(R,R^+) \to S^\diamond$ with $S \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p}$, this category is cofiltered. Now, $(\mathcal{F}^{\mathrm{red}})^\diamond$ is the sheafification of the functor that assigns to (R,R^+):

$$\varinjlim_{S^\diamond \in \mathcal{C}_R} Hom(S^\diamond, \mathcal{F}).$$

But the evident map $\mathrm{Spa}(R,R^+) \to \mathrm{Spec}(R^+)^\diamond$ is cofinal in $\mathcal{C}_R$. That is, $(\mathcal{F}^{\mathrm{red}})^\diamond$ is the sheafifiaction of the presheaf that assigns $(R,R^+) \mapsto Hom(\mathrm{Spec}(R^+)^\diamond, \mathcal{F})$. The description given in the statement above is what one gets from taking sheafification and assuming injectivity of $(\mathcal{F}^{\mathrm{red}})^\diamond \to \mathcal{F}$. $\square$

The following lemma will be key for our theory of specialization, it roughly says that formally adic closed immersions behave as expected:

Lemma 1.3.33. *Let (A,A^+) be a perfectoid Huber pair and let $\mathcal{F} \to \mathrm{Spd}(A^+,A^+)$ be formally adic and a closed immersion. Then $(\mathcal{F}^{\mathrm{red}})^\diamond = \mathrm{Spec}(A^+/J)^\diamond$ for some open ideal $J \subseteq A^+$.*

Proof. Since $\mathcal{F} \to \mathrm{Spd}(A^+,A^+)$ is a closed immersion, $|\mathcal{F}| \subseteq \mathrm{Spo}(A^+,A^+)$ is a closed subset and we have a Cartesian diagram,

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & \mathrm{Spd}(A^+,A^+) \arrow[d] \\
\underline{|\,\mathcal{F}\,|} \arrow[r] & \underline{\mathrm{Spo}(A^+,A^+)}
\end{tikzcd}$$

By proposition 1.3.19, $(\mathrm{Spd}(A^+,A^+)^{\mathrm{red}})^\diamond = \mathrm{Spec}(A^+_{\mathrm{red}})^\diamond$ which is also a closed subsheaf of $\mathrm{Spd}(A^+,A^+)$. By formal adicness $(\mathcal{F}^{\mathrm{red}})^\diamond$ is a closed subsheaf of $\mathrm{Spd}(A^+,A^+)$ given again by the topological condition $|\mathcal{F}| \cap |\mathrm{Spec}(A^+_{\mathrm{red}})|$. By lemma 1.3.32 a map $\mathrm{Spa}(R,R^+) \to \mathcal{F}$ factors through $(\mathcal{F}^{\mathrm{red}})^\diamond$ if after possibly replacing R by a v-cover it factors through $\mathrm{Spec}(R^+)^\diamond \to \mathcal{F} \cap \mathrm{Spec}(A^+_{\mathrm{red}})^\diamond$. This proves that $|(\mathcal{F}^{\mathrm{red}})^\diamond|$ is a schematic closed subset of $\mathrm{Spo}(A^+,A^+)$ as in definition 1.2.16. By proposition 1.2.17 it is a Zariski closed subset corresponding to an open ideal $J \subseteq A^+$, this proves the claim. $\square$

We will often use implicitly the following easy result.

Lemma 1.3.34. *Let $\mathcal{F}$ and $\mathcal{G}$ be two small v-sheaves, and $f : \mathcal{F} \to \mathcal{G}$ a map between them. Suppose that $\mathcal{F} \times_{\mathcal{G}} (\mathcal{G}^{red})^\diamond$ is representable by a scheme-theoretic v-sheaf, then f is formally adic.*

Proof. Let $T \in \widetilde{\mathrm{SchPerf}}$ be such that $T^\diamond = \mathcal{F} \times_{\mathcal{G}} (\mathcal{G}^{red})^\diamond$. Recall that for any pair of adjoint functors (L,R) the compositions $R \to R \circ L \circ R \to R$ and $L \to L \circ R \circ L \to L$ are the identity. We compute directly:

$$\begin{aligned}
(T^\diamond)^{\mathrm{red}} &= (\mathcal{F} \times_{\mathcal{G}} (\mathcal{G}^{\mathrm{red}})^\diamond)^{\mathrm{red}} \\
&= \mathcal{F}^{\mathrm{red}} \times_{\mathcal{G}^{\mathrm{red}}} ((\mathcal{G}^{\mathrm{red}})^\diamond)^{\mathrm{red}} \\
&= \mathcal{F}^{\mathrm{red}} \times_{\mathcal{G}^{\mathrm{red}}} \mathcal{G}^{\mathrm{red}} \\
&= \mathcal{F}^{\mathrm{red}}
\end{aligned}
\qquad
\begin{aligned}
(\mathcal{F}^{\mathrm{red}})^\diamond &= ((T^\diamond)^{\mathrm{red}})^\diamond \\
&= T^\diamond
\end{aligned}$$

$\square$

1.4 Specialization

In this section we review the specialization map in the context of formal schemes and generalize it to the context of v-sheaves. We identify a class of v-sheaves, that we call kimberlites, whose specialization maps behave like those of formal schemes. We prove some abstract statement on the behavior of the specialization map in this context, which we will later use when we discuss the examples of interest.

1.4.1 Specialization for Tate Huber pairs

Definition 1.4.1. *Given a Tate Huber pair (A, A^+) over $\mathbb{Z}_p$ and a pseudo-uniformizer $\varpi \in A$, we define the specialization map $\mathrm{sp}_A : |\mathrm{Spa}(A, A^+)| \to |\mathrm{Spec}(A^+_{\mathrm{red}})|$ by sending a valuation $|\cdot|_x \in |\mathrm{Spa}(A, A^+)|$ to the ideal $\mathfrak{p} \subseteq A^+$ given by $\mathfrak{p} = \{a \in A^+ \mid |a|_x < 1\}$*

These maps of sets are functorial in the category of Tate Huber pairs.

Remark 1.4.2. *We want to emphasize that the specialization map as defined above is different from the specialization ideal that we considered when we studied the olivine spectrum.*

Proposition 1.4.3. *(See [3] 8.1.2) The specialization map $\mathrm{sp}_A : |\mathrm{Spa}(A, A^+)| \to |\mathrm{Spec}(A^+_{\mathrm{red}})|$ is a continuous, surjective, spectral and closed map of spectral topological spaces.*

We recall the notion of totally disconnected spaces which will allow us to define the specialization map "v-locally".

Definition 1.4.4. *(See [26] 7.1, 7.15, 7.5) An affinoid perfectoid space $\mathrm{Spa}(R, R^+)$ is totally disconnected if it splits every open cover. Moreover, it is strictly totally disconnected if it splits every étale cover.*

We have the following useful criterion:

Proposition 1.4.5. *(See [26] 7.3, 7.16, 11.27) Let Y be a spatial diamond. Y is represented by a strictly totally disconnected space if and only if every connected component of Y is represented by $\mathrm{Spa}(C, C^+)$ for C an algebraically closed field and C^+ an open and bounded valuation subring.*

Strictly totally disconnected spaces form a basis for the pro-étale topology on Perf. In particular, any small v-sheaf admits a surjective map from a union of totally disconnected spaces. Moreover, as the following proposition shows, the specialization map for these spaces is usefully nice.

Proposition 1.4.6. *For a strictly totally disconnected space $\mathrm{Spa}(R, R^+)$, the specialization map sp_R is a homeomorphism.*

Proof. By proposition 1.4.3 the map is surjective and a quotient map so it is enough to prove injectivity. Suppose $x, y \in |\mathrm{Spa}(R, R^+)|$ map to the same point in $|\mathrm{Spec}(R^+_{\mathrm{red}})|$. We claim that x and y are in the same connected component of $|\mathrm{Spa}(R, R^+)|$. Indeed, let π_x and π_y be the connected components of x and y respectively. The closed-open subsets $U \subseteq \mathrm{Spa}(R, R^+)$ are Zariski closed subsets defined by an idempotent $1_U \in R^+$. The ones containing x are precisely those for which $|1_U|_x = 1$ or equivalently for which $1_U \notin \mathrm{sp}_R(x) \subseteq R^+$. By assumption $\mathrm{sp}_R(x) = \mathrm{sp}_R(y)$ so x and y are contained in the same closed-opens, this gives $\pi_x = \pi_y$.

By proposition 1.4.5, π_x is representable by $\mathrm{Spa}(C, C^+)$ for some perfectoid field C and open valuation subring C^+. By functoriality of the specialization map it is enough to prove that the maps sp_C and $|\mathrm{Spec}(C^+/\varpi)| \to |\mathrm{Spec}(R^+/\varpi)|$ are injective. The former is injective by lemma 1.4.7 below. To prove injectivity of the later map we argue as follows: $\pi_x = \bigcap U$ where U ranges over the closed-open subsets of $|\mathrm{Spa}(R, R^+)|$ containing x. Each closed-open $U \subseteq \mathrm{Spa}(R, R^+)$ is of the form $U = \mathrm{Spa}(R_U, {R_U}^+)$ and if U^c denotes the complement of U then $R^+ = R_U^+ \times R_{U^c}^+$ as topological rings. In particular, the map $R^+ \to R_U^+$ is surjective. We have that C^+ is the ϖ-adic completion of $\varinjlim_{x\in U} R_U^+$ which implies that the image of $R^+ \to C^+$ is dense. Consequently, $\operatorname{Spec}(C^+/\varpi) \to \operatorname{Spec}(R^+/\varpi)$ is a closed immersion and injective. $\square$

Lemma 1.4.7. *1. Given a non-Archimedean field K there is an order preserving bijection between open and bounded valuation subrings K^+ of K, and valuation subrings of $O_K/K^{\circ\circ}$, given by $K^+ \mapsto K^+/K^{\circ\circ}$*

2. Given K as above and an open and bounded valuation subring K^+ the specialization map sp_K is a homeomorphism.

Proof. This is well known and the proof is left to the reader.

$\square$

Proposition 1.4.8. *Product of points as in definition 1.1.5 are strictly totally disconnected perfectoid space.*

Proof. Take $R^+ := \prod_{i\in I} C_i^+$ and pseudo-uniformizers $\varpi = (\varpi_i)_{i\in I}$ as in definition 1.1.5. The closed-opens subsets of $\operatorname{Spa}(R, R^+)$ are given by idempotents in R^+, which in turn are given by subsets of I. A connected component $\pi \in \pi_0(\operatorname{Spa}(R, R^+))$ is computed by intersecting $\bigcap_{U\in\mathcal{U}} U$ for some ultrafilter and it is a Zariski closed subsets cut out by the ideal, $I_\pi = \langle 1_V \rangle_{V \notin \mathcal{U}}$, where the idempotents are indexed by the sets that do not belong to the ultrafilter. To compute the structure sheaf of this connected component we have to consider the ϖ-completion of R^+/I_π. Let $V = R^+/I_\pi$ and V' be the completion of V with respect to ϖ.

To prove that $\operatorname{Spa}(R, R^+)$ is a strictly totally disconnected perfectoid space it is enough, by proposition 1.4.5, to prove that V' is a valuation ring with algebraically closed fraction field. In general, if W is a valuation ring with algebraically closed fraction field and if $a \in W$ is not a unit, then the (a)-adic completion of W is also a valuation ring with algebraically closed fraction field. Applying this reasoning to V and V', we see that it is enough to prove V is a valuation ring with algebraically closed fraction field.

To prove that V is a domain take two elements in $v_1, v_2 \in R^+$ with $v_1 \cdot v_2 = 0$. If we let $I_j \subseteq I$ with $j \in \{1, 2\}$ be the subsets of $i \in I$ such that $v_j = 0$ in C_i^+ then $I_1 \cup I_2 = I$ and one of I_1 or $I_2 \setminus I_1$ is in the ultrafilter, this implies that one of v_1 or v_2 equals 0 in V. Take an element of $v \in Frac(V)$, this element may be represented by an element of $\prod_{i\in I} C_i$. Since each entry of the product defining R^+ is a valuation ring one of the sets $\{i \in I \mid v_i \in C_i^+\}$ or $\{i \in I \mid v_i^{-1} \in C_i^+\}$ is in the ultrafilter, this implies $v \in V$ or $v^{-1} \in V$ and that V is a valuation ring. One can prove in a similar way that $Frac(V) = \prod_{i\in I} C_i/I_\pi$. In particular, it is an ultraproduct of algebraically closed fields, so $Frac(V)$ is algebraically closed. $\square$

Remark 1.4.9. *Although the construction of V in the proof above does not depend of the choice of $\varpi = (\varpi_i)_{i\in I}$, the ring V' very much depends of this choice. This is in agreement with remark 1.1.6.*

1.4.2 Specializing v-sheaves

We now discuss the specialization map for v-sheaves.

Definition 1.4.10. *Let $\mathcal{F}$ be a small v-sheaf, $\operatorname{Spa}(A, A^+)$ an affinoid perfectoid space in characteristic p and $f : \operatorname{Spa}(A, A^+) \to \mathcal{F}$ a map of v-sheaves.*

1. We say that $\mathcal{F}$ formalizes f if there exists a commutative diagram as follows:

$$\begin{tikzcd}
\mathrm{Spa}(A,A^+) \arrow[r] \arrow[d, "\iota"] & \mathcal{F} \\
\mathrm{Spd}(A^+,A^+) \arrow[ur] &
\end{tikzcd}$$

2. *We say that $\mathcal{F}$ v-formalizes f if there is a v-cover $g : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ of affinoid perfectoid spaces for which $\mathcal{F}$ formalizes $f \circ g$.*

3. *We say that $\mathcal{F}$ is formalizing if it formalizes any f as above.*

4. *We say that $\mathcal{F}$ is v-formalizing if it v-formalizes any f as above.*

Lemma 1.4.11. *The following statements hold:*

1. *The v-sheaf $\mathbb{Z}_p^\diamond$ is formalizing.*

2. $\mathrm{Spd}(B, B)$ *is formalizing for any formal Huber pair over $\mathbb{Z}_p$.*

Proof. Given an affinoid perfectoid $\mathrm{Spa}(R, R^+)$ in characteristic p and an untilt $\iota : (R^\sharp)^\flat \to R$ we need to produce a natural transformation $\mathrm{Spd}(R^+, R^+) \to \mathbb{Z}_p^\diamond$ for which the composition with the canonical map $\mathrm{Spa}(R, R^+) \to \mathrm{Spd}(R^+, R^+)$ gets mapped to $R^\sharp$. Let $\xi = p + [\varpi]\alpha$ be a generator of the kernel of $W(R^+) \to (R^\sharp)^+$, where $\varpi \in R^+$ denotes a pseudo-uniformizer and $\alpha \in W(R^+)$. Let $\mathrm{Spa}(A, A^+)$ be some other affinoid perfectoid space in characteristic p. Recall that, since R^+ is in characteristic p

$$\mathrm{Spd}(R^+, R^+)(A, A^+) = \{f : \mathrm{Spa}(A, A^+) \to \mathrm{Spa}(R^+, R^+)\}$$

Consider the following construction, take the map of topological rings $f^* : R^+ \to A^+$ defined by f, apply the Witt vector functor to f^* to get $W(f^*) : W(R^+) \to W(A^+)$ and consider the element $W(f^*)(\xi) \in W(A^+)$. We claim that $W(f^*)(\xi)$ is primitive of degree 1 (See [28] 6.2.8) and defines an untilt of $\mathrm{Spa}(A, A^+)$ over $\mathrm{Spa}({R^\sharp}^+, {R^\sharp}^+)$. Indeed $W(f^*)(\xi) = p + [f^*(\varpi)]f^*(\alpha)$ and it is enough to prove that there is a pseudo-uniformizer ϖ_A that divides $f^*(\varpi)$. This follows from the fact that $f^*(\varpi)$ is topologically nilpotent.

For the second claim, if we fix an untilt and a morphism $\mathrm{Spa}(R^\sharp, {R^\sharp}^+) \to \mathrm{Spa}(B, B)$ we can construct for any affinoid perfectoid space over $\mathrm{Spa}(R^{\sharp,+}, R^{\sharp,+})$ a point of $\mathrm{Spd}(B, B)(A, A^+)$ by composing the maps. $\square$

Remark 1.4.12. *We list some properties of these notions that are easy to verify.*

1. *If $f : \mathcal{F} \to \mathcal{G}$ is a surjective map of small v-sheaves and $\mathcal{F}$ is v-formalizing then $\mathcal{G}$ is v-formalizing.*

2. *If $\mathrm{Spec}(R) \in \mathrm{PCAlg}_{\mathbb{F}_p}^{op}$ then $\mathrm{Spec}(R)^\diamond$ is formalizing.*

3. *If $X \in \widetilde{\mathrm{SchPerf}}$ then $X^\diamond$ is v-formalizing by lemma 1.3.32.*

4. *Non-empty v-formalizing v-sheaves have non-empty reduction. Consequently, diamonds are not v-formalizing.*

5. *If $\mathcal{F}$ formalizes $f : \mathrm{Spa}(A, A^+) \to \mathcal{F}$ then $\mathcal{F}$ formalizes $f \circ g$ for any map $g : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$*

Proposition 1.4.13. *Let $\mathcal{F}$ be a small v-sheaf, and $f : \mathrm{Spa}(R, R^+) \to \mathcal{F}$ a map with $\mathrm{Spa}(R, R^+)$ affinoid perfectoid in characteristic p. If $\mathcal{F}$ is formally separated then f admits at most one formalization.*

Proof. Suppose we are given two formalizations $g_i : \mathrm{Spd}(R^+, R^+) \to \mathcal{F}$ that agree on $\mathrm{Spa}(R, R^+)$. We get a map $(g_1, g_2) : \mathrm{Spd}(R^+, R^+) \to \mathcal{F} \times \mathcal{F}$, and we can pullback algong the diagonal $\Delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ to get $\mathcal{G} \subseteq \mathrm{Spd}(R^+, R^+)$ a formally closed subsheaf. We want to prove that $\mathcal{G} = \mathrm{Spd}(R^+, R^+)$, and to do so it is enough to prove the equality at the level of topological spaces, $|\mathcal{G}| = \mathrm{Spo}(R^+, R^+)$. Moreover, since $|\mathrm{Spa}(R, R^+)| \subseteq |\mathcal{G}|$ and $\mathrm{Spo}(R^+, R^+) = \mathrm{Spa}(R, R^+) \cup |\mathrm{Spec}(R^+_{\mathrm{red}})^\diamond|$ it is enough to prove $|(\mathcal{G}^{\mathrm{red}})^\diamond| = |\mathrm{Spec}(R^+_{\mathrm{red}})^\diamond|$. We warn the reader that we can't use a direct density argument because although $\mathrm{Spa}(R, R^+)$ is dense in $\mathrm{Spa}(R^+, R^+)$, it is no longer dense in $\mathrm{Spo}(R^+, R^+)$.

We first deal with the case in which C is a non-Archimedean field and $C^+ \subseteq C$ is an open and bounded valuation subring. Let $k^+ = C^+_{\mathrm{red}}$ and $k = Frac(k^+)$, we have that $\mathrm{Spec}(k^+) = \mathrm{Spd}(C^+, C^+)^{\mathrm{red}}$ and by lemma 1.3.33 $(\mathcal{G}^{\mathrm{red}})^\diamond = \mathrm{Spec}(k^+/I)^\diamond$ for some ideal I. On the other hand, since $\mathrm{Spa}(C, C^+) \subseteq \mathcal{G}$ and $|\mathcal{G}|$ is closed in $\mathrm{Spo}(C^+, C^+)$, $|\mathcal{G}|$ contains the formal specialization of $\mathrm{Spa}(C, O_C)$ in $\mathrm{Spo}(C^+, C^+)$, this corresponds to the image of $\mathrm{Spec}(k)^\diamond$. By formal adicness $|(\mathcal{G}^{\mathrm{red}})^\diamond| = |\mathcal{G}| \cap |\mathrm{Spec}(k^+)^\diamond|$ and we can conclude that $\mathrm{Spec}(k)^\diamond \subseteq (\mathcal{G}^{\mathrm{red}})^\diamond$. This proves that $I = \{0\}$ and that $(\mathcal{G}^{\mathrm{red}})^\diamond = \mathrm{Spec}(k^+)^\diamond$ as we needed to show.

For the general case, we get that for every map $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(R, R^+)$ the canonical formalization $\mathrm{Spd}(C^+, C^+) \to \mathrm{Spd}(R^+, R^+)$ factors through $\mathcal{G}$. In parictular, after taking reduction, the map $\mathrm{Spec}(k^+) \to \mathrm{Spec}(R^+_{\mathrm{red}})$ factors through $\mathcal{G}^{\mathrm{red}}$. This says that $|\mathcal{G}^{\mathrm{red}}|$ contains every point of $|\mathrm{Spec}(R^+_{\mathrm{red}})|$ in the image of the specialization map. By lemma 1.3.33 $\mathcal{G}^{\mathrm{red}} \to \mathrm{Spec}(R^+_{\mathrm{red}})$ is a closed immersion and by proposition 1.4.3 the specialization map is surjective, these two imply that $\mathcal{G}^{\mathrm{red}} = \mathrm{Spec}(R^+_{\mathrm{red}})$. This also shows that $|(\mathcal{G}^{\mathrm{red}})^\diamond| = |\mathrm{Spec}(R^+_{\mathrm{red}})^\diamond|$ and concludes the proof. □

Proposition 1.4.14. *The following statements hold:*

1. *Given two maps of v-sheaves $\mathcal{F} \to \mathcal{H}$, $\mathcal{G} \to \mathcal{H}$ if $\mathcal{F}$ and $\mathcal{G}$ are v-formalizing and $\mathcal{H}$ is formally separated then $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is v-formalizing.*

2. *The subcategory of v-sheaves that are v-formalizing and formally separated is stable under fiber product and contains $\mathbb{Z}_p^\diamond$.*

Proof. Given a map $\mathrm{Spa}(A, A^+) \to \mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ we can find a cover $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ for which the compositions with the projections to $\mathcal{F}$ and $\mathcal{G}$ are both formalizable. By formal separatedness any pair of choices of formalizations $\mathrm{Spd}(B^+, B^+) \to \mathcal{G}$ and to $\mathrm{Spd}(B^+, B^+) \to \mathcal{F}$ define the same formalization to $\mathcal{H}$ and a map to $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$. The second claim follows from the stability of separatedness by basechange and composition, from lemma 1.3.30 and from lemma 1.3.32. Indeed, we need to prove that $(\mathcal{F}^{\mathrm{red}})^\diamond \times_{(\mathcal{H}^{\mathrm{red}})^\diamond} (\mathcal{G}^{\mathrm{red}})^\diamond$ is a subsheaf of $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$, but this follows from knowing that $\mathcal{F}^{\mathrm{red}}$ (respectively $\mathcal{H}$, $\mathcal{G}$) is a subsheaf of $\mathcal{F}$ (respectively $\mathcal{H}$, $\mathcal{G}$). □

Definition 1.4.15. *Let $\mathcal{F}$ be a small v-sheaf, we say that it is specializing if it is formally separated and v-formalizing.*

Definition 1.4.16. *Let $\mathcal{F}$ be a specializing v-sheaf. The specialization map for $\mathcal{F}$, denoted $\mathrm{sp}_{\mathcal{F}}$, is the map of sets $\mathrm{sp}_{\mathcal{F}} : |\mathcal{F}| \to |\mathcal{F}^{\mathrm{red}}|$ defined as follows. Given a point $[x] \in |\mathcal{F}|$ we take a formalizable representative $x : \mathrm{Spa}(K, K^+) \to \mathcal{F}$. We choose a formalization $\mathrm{Spd}(K^+, K^+) \to \mathcal{F}$ and we apply the reduction functor to this map. We obtain a map $\mathrm{Spec}(K^+_{\mathrm{red}}) \to \mathcal{F}^{red}$, and the maximal ideal of K^+_{red} defines a point in $|\mathcal{F}^{\mathrm{red}}|$. We define $\mathrm{sp}_{\mathcal{F}}([x])$ to be this point. We use proposition 1.4.13 to prove that this map of sets is well defined and does not depend on the choices taken.*

Proposition 1.4.17. *For any specializing v-sheaf $\mathcal{F}$ the specialization map $\mathrm{sp}_{\mathcal{F}} : |\mathcal{F}| \to |\mathcal{F}^{\mathrm{red}}|$ is continuous. Moreover, this construction is functorial in the category of specializing v-sheaves.*

Proof. We prove functoriality first, take a map of v-sheaves as above $g : \mathcal{F} \to \mathcal{G}$. Given a point $x : \mathrm{Spa}(K, K^+) \to \mathcal{F}$ the image in $|\mathcal{G}|$ is given by composition. A formalization for x gives a formalization for $g(x)$ and we get maps $\mathrm{Spec}(K^+_{\mathrm{red}}) \to \mathcal{F}^{\mathrm{red}} \to \mathcal{G}^{\mathrm{red}}$, we get that $g^{\mathrm{red}}(\mathrm{sp}_{\mathcal{F}}(x)) = \mathrm{sp}_{\mathcal{G}}(g(x))$. Let us prove continuity, for this take a cover $g : X \to \mathcal{F}$ with $X = \coprod_{i \in I} \mathrm{Spa}(R_i, {R_i}^+)$ and each $\mathrm{Spa}(R_i, {R_i}^+)$ affinoid perfectoid for which the composition $\mathrm{Spa}(R_i, {R_i}^+) \to \mathcal{F}$ is formalizable. If we let $Y = \coprod_{i \in I} \mathrm{Spd}(R_i^+, R_i^+)$ we get the following commutative diagram:

$$\begin{array}{ccc}
\mid X \mid & \xrightarrow{\quad g \quad} & \mid \mathcal{F} \mid \\
\downarrow{\scriptstyle \mathrm{sp}_X} & & \downarrow{\scriptstyle \mathrm{sp}_{\mathcal{F}}} \\
\mid Y^{\mathrm{red}} \mid & \xrightarrow{g^{\mathrm{red}}} & \mid \mathcal{F}^{\mathrm{red}} \mid
\end{array}$$

The map g^{red} is continuous by proposition 1.3.14, the map g is continuous and a quotient map, and the map sp_X is continuous by proposition 1.4.3. Since the diagram is commutative, the map $\mathrm{sp}_{\mathcal{F}}$ is also continuous. □

1.4.3 Kimberlites and tubular neighborhoods

To prove pleasant properties of the specialization map we need to restrict our discussion to certain types of specializing v-sheaves.

Definition 1.4.18. *1. A pre-kimberlite is a small v-sheaf $\mathcal{F}$ such that:*

a) $\mathcal{F}$ is specializing.

b) $\mathcal{F}^{\mathrm{red}}$ is represented by a scheme.

c) The map $(\mathcal{F}^{\mathrm{red}})^{\diamond} \to \mathcal{F}$ coming from adjunction is a closed immersion.

2. For a pre-kimberlite $\mathcal{F}$, we define $\mathcal{F}^{an}$ as the open subsheaf $\mathcal{F} \setminus \mathcal{F}^{\mathrm{red}}$. If $\mathcal{F}^{an}$ is a locally spatial diamond we say that $\mathcal{F}$ is a kimberlite.

3. A smelted kimberlite is a pair $(\mathcal{F}, \mathscr{D})$ where $\mathcal{F}$ is a pre-kimberlite and $\mathscr{D} \subseteq \mathcal{F}^{an}$ is a open subsheaf such that $\mathscr{D}$ is a locally spatial diamond.

4. Morphisms of pre-kimberlites and kimberlites are maps of v-sheaves.

5. A morphism of smelted kimberlites $f : (\mathcal{F}_1, \mathscr{D}_1) \to (\mathcal{F}_2, \mathscr{D}_2)$ is a morphism of v-sheves $f : \mathcal{F}_1 \to \mathcal{F}_2$ such that $f(\mathscr{D}_1) \subseteq \mathscr{D}_2$.

Remark 1.4.19. *Notice that a kimberlite $\mathcal{F}$ gives rise to the smelted kimberlite $(\mathcal{F}, \mathcal{F}^{an})$ but this is not a functor in the category of kimberlites. Indeed, $f(\mathcal{F}^{an}) \subseteq \mathcal{G}^{an}$ if and only if $f : \mathcal{F} \to \mathcal{G}$ is formally adic. Nevertheless, if $\mathcal{F} \to \mathcal{G}$ is a map of kimberlites then $(\mathcal{F}, \mathcal{F}^{an} \times_{\mathcal{G}} \mathcal{G}^{an}) \to (\mathcal{G}, \mathcal{G}^{an})$ is a map of smelted kimberlites.*

Definition 1.4.20. *Given a smelted kimberlite $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ we define the map of topological spaces $\mathrm{sp}_{\mathcal{K}} : |\mathscr{D}| \to |\mathcal{F}^{\mathrm{red}}|$ as the composition of $|\mathscr{D}| \to |\mathcal{F}| \xrightarrow{\mathrm{sp}_{\mathcal{F}}} |\mathcal{F}^{\mathrm{red}}|$. For a kimberlite $\mathcal{F}$ we abbreviate $\mathrm{sp}_{(\mathcal{F}, \mathcal{F}^{an})}$ by $\mathrm{sp}_{\mathcal{F}^{an}}$.*

Proposition 1.4.21. *Let $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ be a smelted kimberlite, then $\mathrm{sp}_{\mathcal{K}}$ is a spectral map of locally spectral spaces. This assignation is functorial in the category of smelted kimberlites.*

Proof. Continuity and functoriality follows directly from proposition 1.4.17. We need to prove that this map is also continuous for the constructible topology. Since it is enough to prove continuity on an open cover of $|\mathscr{D}|$, we may assume that $\mathscr{D}$ is a spatial diamond. We cover $\mathscr{D}$ by an affinoid perfectoid space $X = \mathrm{Spa}(A, A^+)$ as in the proof of proposition 1.4.17. Consider the diagram,

$$\begin{tikzcd}
\mid \mathrm{Spa}(A,A^+) \mid^{cons} \arrow[r, "g"] \arrow[d, "\mathrm{sp}_X"] & \mid \mathscr{D} \mid^{cons} \arrow[d, "\mathrm{sp}_{\mathcal{K}}"] \\
\mid \mathrm{Spec}(A^+_{\mathrm{red}})^{\mathrm{red}} \mid^{cons} \arrow[r, "g^{\mathrm{red}}"] & \mid \mathcal{F}^{\mathrm{red}} \mid^{cons}
\end{tikzcd}$$

where now the spaces are given the constructible topology. Since $\mathcal{F}^{\mathrm{red}}$ is represented by a scheme proposition 1.3.16 implies that g^{red} is continuous for the constructible topology. Indeed, morphisms of schemes induce spectral maps. Similarly, the map sp_X is continuous and since X is a spatial diamond proposition 1.1.21 shows that the map g is also continuous. Moreover, g gives a surjective map of compact spaces and is consequently a quotient map. Since the diagram commutes, $\mathrm{sp}_{\mathcal{K}}$ is continuous for the patch topology. $\square$

The author thinks of kimberlites as a natural category to find “integral models” for diamonds. The following proposition partially justifies this thought.

Proposition 1.4.22. *Let (A, A) be a formal Huber pair over $\mathbb{Z}_p$, then $\mathrm{Spd}(A, A)$ is a kimberlite.*

Proof. By lemma 1.4.11 it is v-formalizing. If I is an ideal of definition, the reduction $\mathrm{Spd}(A, A)^{\mathrm{red}}$ is representable by $\mathrm{Spec}(A/I)^\diamond$ which gives us that the unit of the adjunction is a Zariski closed immersion and representable by a scheme. In particular $\mathrm{Spd}(A, A) \to \mathrm{Spd}(A, A) \times \mathrm{Spd}(A, A)$ is formally adic by lemma 1.3.32. Moreover, if we let $Z \subseteq \mathrm{Spa}(A, A)$ denote the non-analytic locus then $|\mathrm{Spec}(A_{\mathrm{red}})^\diamond| \subseteq \mathrm{Spo}(A, A)$ is $h^{-1}(Z)$. This implies that $\mathrm{Spd}(A, A)^{an} = (\mathrm{Spa}(A, A)^{an})^\diamond$ which is a locally spatial diamond by remark 1.1.24. To prove $\mathrm{Spd}(A, A)$ is separated, it is enough to prove the map $\mathrm{Spd}(A, A) \to \mathbb{Z}_p^\diamond$ is separated. We need to prove that for any affinoid perfectoid space $\mathrm{Spa}(R, R^+)$ and two maps of topological rings $f_i : A \to R^+$ the functor $Z : \mathrm{Perfd} \to \mathrm{Sets}$ that associates to $\mathrm{Spa}(S, S^+)$ the set $\{g : \mathrm{Spd}(S, S^+) \to \mathrm{Spd}(R, R^+) \mid g \circ f_1 = g \circ f_2\}$ is representable by a closed subspace of $\mathrm{Spa}(R, R^+)$. Regardless of the topology on the rings we get a map $A \otimes_{\mathbb{Z}_p} A \to R^+$, and the compositions $g \circ f_i : A \to S^+$ agree if and only if the following diagram of rings is commutative:

$$\begin{tikzcd}
A \otimes_{\mathbb{Z}_p} A \arrow[r] \arrow[d, "\Delta"] & R^+ \arrow[d] \\
A \arrow[r] & S^+
\end{tikzcd}$$

but this is a Zariski closed condition on $\mathrm{Spa}(R, R^+)$ given by the ideal $R \cdot I_\Delta$ where $I_\Delta = Ker(A \otimes_{\mathbb{Z}_p} A \to A)$. $\square$

Definition 1.4.23. *A kimberlite (respectively prekimberlite) $\mathcal{F}$ together with a formally adic map $\mathcal{F} \to \mathbb{Z}_p^\diamond$ is said to be a p-adic kimberlite (respectively prekimberlite). Given a map of sheaves $\mathcal{F} \to \mathbb{Z}_p^\diamond$ we denote by $\mathcal{F}_\eta$ the basechange $\mathcal{F} \times_{\mathbb{Z}_p^\diamond} \mathbb{Q}_p^\diamond$. A p-adically smelted kimberlite is a prekimberlite $\mathcal{F}$ over $\mathbb{Z}_p^\diamond$ for which $(\mathcal{F}, \mathcal{F}_\eta)$ forms a smelted kimberlite.*

The following concept is central to our purposes.

Definition 1.4.24. *Let $\mathcal{F}$ be a prekimberlite and let $S \subseteq \mathcal{F}^{\mathrm{red}}$ be a locally closed immersion of schemes.*

1. *We define the tubular neighborhood of S on $\mathcal{F}$, denoted $\widehat{\mathcal{F}}_{/S}$, as the sub-$v$-sheaf of $\mathcal{F}$ defined by the following cartesian diagram:*

$$\begin{tikzcd}
\widehat{\mathcal{F}}_{/S} \arrow[r] \arrow[dd] & \mathcal{F} \arrow[d] \\
& \underline{|\,\mathcal{F}\,|} \arrow[d, "\mathrm{sp}_{\mathcal{F}}"] \\
\underline{|\,S\,|} \arrow[r] & \underline{|\,\mathcal{F}^{\mathrm{red}}\,|}
\end{tikzcd}$$

2. *If $\mathcal{F}$ comes equipped with a map to $\mathbb{Z}_p^\diamond$ (not necessarily formally adic) we define the p-adic tubular neighborhood of $\mathcal{F}$, denoted $(\widehat{\mathcal{F}}_{/S})_\eta$, as the basechange $\widehat{\mathcal{F}}_{/S} \times_{\mathbb{Z}_p^\diamond} \mathbb{Q}_p^\diamond$.*

3. *If $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ is a smelted kimberlite we let $\widehat{\mathscr{D}}_{/S}$ be $\widehat{\mathcal{F}}_{/S} \cap \mathscr{D}$ and we refer to this sheaf as the smelted tubular neighborhood.*

Intuitively speaking, $\widehat{\mathcal{F}}_{/S}$ is the subsheaf of points whose specialization map factors through S. This notion generalizes completions along a closed subscheme in formal geometry:

Proposition 1.4.25. *Suppose (A, A) is a formal Huber pair over $\mathbb{Z}_p$ with ideal of definition I. Let $J \subseteq A$ be a finitely generated ideal containing I and B the completion of A with respect to J. The closed immersion of schemes $S = \mathrm{Spec}(B_{\mathrm{red}}) \to \mathrm{Spec}(A_{\mathrm{red}})$, induces an identification $\widehat{\mathrm{Spd}(A,A)}_{/S} = \mathrm{Spd}(B,B)$.*

Proof. Let $S = \mathrm{Spec}(B_{\mathrm{red}})$ and $T = \mathrm{Spec}(A_{\mathrm{red}})$. The reduction of the map $\mathrm{Spd}(B,B) \to \mathrm{Spd}(A,A)$ induces the map $S \to T$. Since specialization is functorial any point coming from $\mathrm{Spd}(B,B)$ has to specialize to S. Consequently the map factors as $\mathrm{Spd}(B,B) \to \widehat{\mathrm{Spd}(A,A)}_{/S} \to \mathrm{Spd}(A,A)$. Since A is dense in B, it is easy to see that this map is an injection. To prove surjectivity onto $\widehat{\mathrm{Spd}(A,A)}_{/S}$, suppose we have a map $f : A \to R^+$ for which the induced map $f : \mathrm{Spec}(R^+_{\mathrm{red}}) \to \mathrm{Spec}(A_{\mathrm{red}})$ factors through $|S|$. Then for every $a \in J$ the element $f(a)$ is nilpotent in $\mathrm{Spec}(R^+/\varpi^n)$. Since J is finitely generated there is an m for which $J^m \subseteq (\varpi^n)$ in R^+. This proves that the map $f : A \to R^+$ is continuous for the J-adic topology on A. Since R^+ is complete the map $f : A \to R^+$ factors through B, which proves that any map $\mathrm{Spa}(R, R^+) \to \widehat{\mathrm{Spd}(A,A)}_{/S}$ factors through a map to $\mathrm{Spd}(B,B)$. $\square$

Proposition 1.4.26. *Let $f : \mathcal{G} \to \mathcal{F}$ be a morphism prekimberlites and let $S \subseteq |\mathcal{F}^{\mathrm{red}}|$ a locally closed subscheme. If we define $T = S \times_{\mathcal{F}^{\mathrm{red}}} \mathcal{G}^{\mathrm{red}}$, then $\widehat{\mathcal{F}}_{/S} \times_{\mathcal{F}} \mathcal{G} = \widehat{\mathcal{G}}_{/T}$. In particular, a map of prekimberlites $\mathcal{G} \to \mathcal{F}$ factors through $\widehat{\mathcal{F}}_{/S}$ if and only if $\mathcal{G}^{\mathrm{red}} \to \mathcal{F}^{\mathrm{red}}$ factors through S.*

Proof. Since S is a locally closed immersion we have $|T| = |S| \times_{|\mathcal{F}^{\mathrm{red}}|} |\mathcal{G}^{\mathrm{red}}|$. We can look at the following commutative diagram:

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
\underline{|\,G^{\mathrm{red}}\,|} \arrow[r] & \underline{|\,\mathcal{F}^{\mathrm{red}}\,|}
\end{tikzcd}$$

The first claim follows by basechanging this diagram by the map $\underline{|S|} \to \underline{|\mathcal{F}^{\mathrm{red}}|}$. For the second claim, observe that the map $\mathcal{G} \to \mathcal{F}$ factoring through $\widehat{\mathcal{F}}_{/S}$ is equivalent to $\mathcal{G} \times_{\mathcal{F}} \widehat{\mathcal{F}}_{/S} = \mathcal{G}$. By the first claim this is equivalent to $\widehat{\mathcal{G}}_{/T} = \mathcal{G}$, which happens if and only if $T = \mathcal{G}^{\mathrm{red}}$ and $\mathcal{G}^{\mathrm{red}} \to \mathcal{F}^{\mathrm{red}}$ factors through S. $\square$

Remark 1.4.27. *Let $\mathcal{F}$ be a prekimberlite and $S \subseteq \mathcal{F}^{\mathrm{red}}$ locally closed subset. One can prove that the v-sheaf $\widehat{\mathcal{F}}_{/S}$ is a small v-sheaf but this is not automatic. The problem is that the v-sheaf $\underline{T}$ is not small whenever the topological space T does not satisfy the separation axiom $T1$.*

Proposition 1.4.28. *Let $\mathcal{F}$ be a prekimberlite and let $S \subseteq |\mathcal{F}^{\mathrm{red}}|$ a locally closed subset, then $\widehat{\mathcal{F}}_{/S}$ is a prekimberlite and $(\widehat{\mathcal{F}}_{/S})^{\mathrm{red}} = S$.*

Proof. The formula $(\widehat{\mathcal{F}}_{/S})^{\mathrm{red}} = S$ follows easily from observing that by proposition 1.4.26 a map $\mathrm{Spec}(A)^\diamond \to \mathcal{F}$ factors through $\widehat{\mathcal{F}}_{/S}$ if and only if the map obtained by adjunction $\mathrm{Spec}(A) \to \mathcal{F}^{\mathrm{red}}$ factors through S. Indeed, S and $\widehat{\mathcal{F}}^{\mathrm{red}}_{/S}$ represent the same functor in this case.

For the first claim, since $\widehat{\mathcal{F}}_{/S}$ is a subsheaf of a formally separated v-sheaf it is formally separated as well. To prove it is v-formalizing take a map $\mathrm{Spa}(R, R^+) \to \widehat{\mathcal{F}}_{/S} \subseteq \mathcal{F}$. After replacing $\mathrm{Spa}(R, R^+)$ by a v-cover if necessary we get a formalization $\mathrm{Spd}(R^+, R^+) \to \mathcal{F}$. By proposition 1.4.26 this formalization factors through $\widehat{\mathcal{F}}_{/S}$ if and only if $\mathrm{Spec}(R^+_{\mathrm{red}}) \to \mathcal{F}^{\mathrm{red}}$ factors through S. But this later condition holds since $\mathrm{Spa}(R, R^+) \to \mathcal{F}$ factors through $\widehat{\mathcal{F}}_{/S}$. To finish the proof we need to show that $S^\diamond \to \widehat{\mathcal{F}}_{/S}$ is a closed immersion. Consider the base change $\widehat{\mathcal{F}}_{/S} \times_{\mathcal{F}} (\mathcal{F}^{\mathrm{red}})^\diamond$. On one hand the projection to $\widehat{\mathcal{F}}_{/S}$ is a closed immersion, and on the other hand by proposition 1.4.26 this identifies with $(\widehat{(\mathcal{F}^{\mathrm{red}})^\diamond})_{/S}$. In case S is a closed subscheme of $\mathcal{F}^{\mathrm{red}}$ we have that the map of v-sheaves $S^\diamond \to (\mathcal{F}^{\mathrm{red}})^\diamond$ is proper so that $S^\diamond \to (\widehat{(\mathcal{F}^{\mathrm{red}})^\diamond})_{/S}$ is also a closed immersion. In case S is an open subscheme of $\mathcal{F}^{\mathrm{red}}$, we can verify $(\widehat{(\mathcal{F}^{\mathrm{red}})^\diamond})_{/S} = S^\diamond$. The general case follows from these two cases.

□

Whenever S is a constructible subset we can say more:

Proposition 1.4.29. *Let $\mathcal{F}$ be a prekimberlite, $S \subseteq |\mathcal{F}^{\mathrm{red}}|$ a locally closed constructible subset then:*

1. *The map $\widehat{\mathcal{F}}_{/S} \to \mathcal{F}$ is an open immersion.*
2. *If $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ is a smelted kimberlite, then $\widehat{\mathcal{F}}_{/S} \cap \mathscr{D}$ is the open subsheaf corresponding to the interior of $\mathrm{sp}_{\mathcal{K}}{}^{-1}(S)$ in $|\mathscr{D}|$.*

Proof. For the first claim we begin by observing that the question is Zariski local in $\mathcal{F}^{\mathrm{red}}$. Indeed, an open cover $\coprod_{i\in I} U_i \to \mathcal{F}^{\mathrm{red}}$ induces an open cover $\coprod_{i\in I} \widehat{\mathcal{F}}_{/U_i} \to \mathcal{F}$. After localizing, we may assume that $\mathcal{F}^{\mathrm{red}} = \mathrm{Spec}(A)$ and that S is closed subset of $\mathrm{Spec}(A)$ that is open for the constructible topology. Write $S = \mathrm{Spec}(A/I)$ for $I \subseteq A$ an ideal, since we are only interested in S as a topological space, a compactness argument allows us to assume that I is finitely generated. Let $(i_1, \dots, i_n)$ be a list of generators for I, let (R, R^+) be a perfectoid Huber pair and $\mathrm{Spd}(R^+, R^+) \to \mathcal{F}$ a map. We can describe the basechange $X := \mathrm{Spd}(R^+, R^+) \times_{\mathcal{F}} \widehat{\mathcal{F}}_{/S}$ as follows. Let $\varpi \in R^+$ be a pseudo-uniformizer and $(j_1, \dots, j_n)$ a list of lifts of $(i_1, \dots i_n)$ in R^+_{red}. We claim X is the open subsheaf of $\mathrm{Spd}(R^+, R^+)$ defined by $\bigcap_{k=1}^n N_{j_k << 1}$. Indeed, by proposition 1.4.26 X is given by $\mathrm{Spd}\widehat{(R^+, R^+)}_{/V(I)}$ and by propostion 1.4.25 if we let B^+ be the completion of R^+ by the (I, ϖ)-adic topology then $X = \mathrm{Spd}(B^+, B^+)$. That $\mathrm{Spd}(B^+, B^+) = \bigcap_{k=1}^n N_{j_k << 1}$ is a direct consequence of lemma 1.2.19. Since $\mathcal{F}$ is v-formalizing every map $\mathrm{Spa}(R, R^+) \to \mathcal{F}$ factors through $\mathrm{Spd}(R^+, R^+)$ after replacing $\mathrm{Spa}(R, R^+)$ by a v-cover. In particular, the basechanges $\mathrm{Spa}(R, R^+) \times_{\mathcal{F}} \widehat{\mathcal{F}}_{/S}$ are open after taking a v-cover. By [26] 10.11 $\widehat{\mathcal{F}}_{/S} \to \mathcal{F}$ is open.

For the second claim let $T \subseteq \mathrm{sp}_{\mathcal{K}}{}^{-1}(S)$ be the largest subset stable under generization. We prove that $T \subseteq \widehat{\mathcal{F}}_{/S} \cap \mathscr{D}$ since we already have the a chain of inclusions:

$$\widehat{\mathcal{F}}_{/S} \cap \mathscr{D} \subseteq (\mathrm{sp}_{\mathcal{K}}{}^{-1}(S))^{int} \subseteq T.$$

Take $x \in T$ and a formalizable geometric point $\iota_x : \mathrm{Spa}(C_x, C_x{}^+) \to \mathcal{F}$ over x. Since every generization of x is in $\mathrm{sp}_{\mathcal{K}}{}^{-1}(S)$ the map $\mathrm{Spec}((C_x^+)_{\mathrm{red}}) \to \mathcal{F}^{\mathrm{red}}$ factors through S so that the composition $|\mathrm{Spa}(C_x, C_x{}^+)| \to |\mathscr{D}| \to |\mathcal{F}^{\mathrm{red}}|$ factors through $|S|$, giving that ι_x factors through $\widehat{\mathcal{F}}_{/S}$.

□

Proposition 1.4.30. *Let $f : \mathcal{G} \to \mathcal{F}$ be a formally closed immersion of small v-sheaves. The following hold:*

1. *If $\mathcal{F}$ is a specializing v-sheaf, then $\mathcal{G}$ is a specializing v-sheaf.*
2. *If $\mathcal{F}$ is a prekimberlite, then $\mathcal{G}$ is a prekimberlite.*
3. *If $\mathcal{F}$ is a kimberlite, then $\mathcal{G}$ is a kimberlite.*
4. *If $(\mathcal{F}, \mathscr{D})$ forms a smelted kimberlite then $(\mathcal{G}, \mathcal{G} \cap \mathscr{D})$ forms a smelted kimberlite.*

Proof. Suppose $\mathcal{F}$ is specializing, since $\mathcal{G}$ is a subsheaf of $\mathcal{F}$ it is formally separated. Observe that for a perfectoid Huber pair (R, R^+) and a map $\mathrm{Spd}(R^+, R^+) \to \mathcal{F}$ the basechange $X := \mathcal{G} \times_{\mathcal{F}} \mathrm{Spd}(R^+, R^+)$ is a formally closed subsheaf of $\mathrm{Spd}(R^+, R^+)$. We may reason as in the proof of proposition 1.4.13 to conclude $X = \mathrm{Spd}(R^+, R^+)$ whenever $\mathrm{Spa}(R, R^+) \to \mathcal{F}$ factors through $\mathcal{G}$. This proves that $\mathcal{G}$ is also v-formalizing and a specializing sheaf. Suppose $\mathcal{F}$ is a prekimberlite, we have a commutative diagram:

$$\begin{tikzcd}
(\mathcal{G}^{\mathrm{red}})^{\diamond} \arrow[r] \arrow[d] & (\mathcal{F}^{\mathrm{red}})^{\diamond} \arrow[d] \\
\mathcal{G} \arrow[r] & \mathcal{F}
\end{tikzcd}$$

By formal adicness this diagram is Cartesian which gives that $(\mathcal{G}^{\mathrm{red}})^{\diamond} \to \mathcal{G}$ is a closed immersion. Since the map $(\mathcal{G}^{\mathrm{red}})^{\diamond} \to (\mathcal{F}^{\mathrm{red}})^{\diamond}$ is also a formally closed immersion by lemma 1.3.33 $\mathcal{G}^{\mathrm{red}}$ is represented by a closed subscheme of $\mathcal{F}$, finishing the proof that $\mathcal{G}$ is a prekimberlite. Suppose now that $\mathcal{F}$ is a kimberlite, then $\mathcal{G}^{an} = \mathcal{F}^{an} \times_{\mathcal{F}} \mathcal{G}$ and by ([26] 11.20) it is a locally spatial diamond, so $\mathcal{G}$ is a kimberlite. The same applies for $\mathcal{G} \times_{\mathcal{F}} \mathscr{D}$ in the smelted kimberlite case. □

1.4.4 cJ-diamonds and Orapian kimberlites

Suppose we have a formal scheme $\mathcal{X}$ topologically of finite type over $\mathbb{Z}_p$, suppose we let X_η denote the generic fiber of $\mathcal{X}$ considered as an adic space over $\mathbb{Q}_p$ and suppose we let X^{red} denote the reduced special fiber of $\mathcal{X}$ considered as a scheme over $\mathbb{F}_p$. In this classical situation we have a specialization map $\mathrm{sp}_{X_\eta} : |X_\eta| \to |X^{\mathrm{red}}|$, and for a fixed closed point $x \in |X^{\mathrm{red}}|$ we have the following chain of inclusions $|(\widehat{\mathcal{X}}_{/x})_\eta| \subseteq \mathrm{sp}_{X_\eta}{}^{-1}(x) \subseteq |X_\eta|$. These inclusions satisfy that:

1. $\mathrm{sp}_{X_\eta}{}^{-1}(x)$ is a closed subset.
2. $|(\widehat{\mathcal{X}}_{/x})_\eta|$ is the interior of $\mathrm{sp}_{X_\eta}{}^{-1}(x)$ in $|X_\eta|$.
3. $|(\widehat{\mathcal{X}}_{/x})_\eta|$ is dense in $\mathrm{sp}_{X_\eta}{}^{-1}(x)$

The first two conditions generalize, by proposition 1.4.29, to the case of kimberlites for which closed points are constructible. In this section we give sufficient conditions that make a kimberlite have the third property as well. Before discussing these condition we give an example showing that some sort of finiteness hypothesis need to be imposed for the third property to hold.

Example 1.4.31. *Let C be a p-adic non-Archimedean field and C^+ an open and bounded valuation subring whose rank is strictly larger than 1. We have that sp_C is a homeomorphism between $\mathrm{Spa}(C, C^+)$ and $\mathrm{Spec}(C^+/C^{\circ\circ})$. In particular, if x denotes the closed point of $\mathrm{Spec}(C^+/C^{\circ\circ})$ then $\mathrm{sp}_C{}^{-1}(x)$ is the closed point of $y \in \mathrm{Spa}(C, C^+)$. The interior of $\{y\}$ is empty, therefore it is not a dense subset of $\{y\}$.*

Definition 1.4.32. *We say that a locally spatial diamond X is constructibly Jacobson if the subset of rank 1 points are dense for the constructible topology of $|X|$. Locally spatial diamonds with this property will be called cJ-diamonds.*

Proposition 1.4.33. *Suppose that $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ is a smelted kimberlite with $\mathscr{D}$ a cJ-diamond, let $S \subseteq |\mathcal{F}|$ a constructible subset. Then $|\mathscr{D} \cap \widehat{\mathcal{F}}_{/S}|$ is dense in $\mathrm{sp}_{\mathcal{K}}{}^{-1}(S)$.*

Proof. By the proof of proposition 1.4.29, we know that $|\mathscr{D} \cap \widehat{\mathcal{F}}_{/S}|$ is the largest subset of $\mathrm{sp}_{\mathcal{K}}{}^{-1}(S)$ stable under generization. Since S is constructible and $\mathrm{sp}_{\mathcal{K}}$ is a spectral map, the set $\mathrm{sp}_{\mathcal{K}}{}^{-1}(S)$ is open in the constructible topology of $|\mathscr{D}|$ and rank 1 points contained in this set are dense in it. Since rank 1 points are stable under generization, they belong to $|\mathscr{D} \cap \widehat{\mathcal{F}}_{/S}|$. This proves that $|\mathscr{D} \cap \widehat{\mathcal{F}}_{/S}|$ is dense in $\mathrm{sp}_{\mathcal{K}}{}^{-1}(S)$ for the constructible topology, but the usual topology is coarser so it is dense for the usual topology as well. □

We discuss some properties of this concept.

Proposition 1.4.34. *Let $f : X \to Y$ be a morphsim of locally spatial diamonds the following hold:*

1. *Suppose that $|f|$ is a surjective map of topological spaces and that X is a cJ-diamond, then Y is a cJ-diamond.*
2. *Suppose that f is an open immersion and that Y is a cJ-diamond, then X is a cJ-diamond.*
3. *Suppose that f realizes X as a quasi-pro-étale $\underline{J}$-torsor over Y for some profinite group J and that X is a cJ-diamond, then Y is a cJ-diamond.*
4. *Suppose that f is étale and that Y is a cJ-diamond, then X is a cJ-diamond.*

Proof. Maps of locally spatial diamonds induce continuous spectral maps of locally spectral spaces. Surjective maps send dense subsets to dense subsets. Moreover, maps of locally spatial diamonds are generalizing which implies that rank 1 points can only map to rank 1 points. This proves the first claim.

Suppose now that Y is a cJ-diamond. If f is an open immersion, any open in the patch topology of X is also open in the patch topology of Y and contains a rank 1 point, this proves the second claim. Moreover this allow us to localize in the analytic topology, so we can assume for the rest of the argument X and Y are spatial.

If f is étale, by ([26] 11.31) locally for the analytic topology we can write f as the composition of an open immersion and a finite étale map. The category of finite étale morphisms over a fixed spatial diamond is a Galois category and using the first claim we may reduce to the case in which f is Galois with finite Galois group G. In this way, the fourth claim follows from the third.

In the setup of the third claim, we prove below that f is open for the patch topology which would finish the proof. Let $J = \varprojlim_i J_i$ with J_i a family of finite groups and denote by $f_i : X_i \to Y$ the induced J_i-torsors. We get action maps $J_i \times |X_i| \to |X_i|$ that are continuous for the discrete topology on J_i and the constructible topology on $|X_i|$. Moreover, for any set $S \subseteq |X_i|$ we have that $f_i^{-1}(f_i(S)) = J_i \cdot S$. Now, the formation of the patch topology on a spectral space commutes with limits along spectral maps. This gives an action map $J \times |X| \to |X|$ that is continuous when $|X|$ is given the patch topology and J is given its profinite topology. Let $U \subseteq X$ be open in the constructible topology, then $f^{-1}(f(U)) = J \cdot U$ which is also open. The map $|f|^{cons} : |Y|^{cons} \to |X|^{cons}$ is a surjective continuous map of compact spaces, so it is a quotient map. Since $J \cdot U$ is open and saturated $f(J \cdot U) = f(U)$ is open as we wanted to show. □

Let's recall the following theorem of Huber:

Theorem 1.4.35. *(See [12] Theorem 4.1) Let k be a complete field with respect to a rank 1 valuation, and let A be a k-algebra of topologically finite type over k. Then the subset $Max(A) \subseteq \mathrm{Spa}(A, A^\circ)$ is dense for the constructible topology.*

Huber's statement says something a bit stronger, but this weaker form of the statement is easier to state and the one we will use in applications.

Corollary 1.4.36. *If X is an adic space topologically of finite type over* $\mathrm{Spa}(k, k^\circ)$*, where* $\mathrm{Spa}(k, k^\circ)$ *is a non-Archimedean field over* $\mathbb{Z}_p$*. Then* $X^\diamond$ *is a cJ-diamond.*

Proof. The claim is local on X so we can assume $X = \mathrm{Spa}(A, A^\circ)$ for a Tate algebra, A/k. In this case every point $Max(A)$, when considered as a valuation, is a rank 1 valuation. □

Example 1.4.37. *The perfectoid unit ball* $\mathbb{B}_n = \mathrm{Spa}(C\langle T_1^{\frac{1}{p^\infty}} \dots T_n^{\frac{1}{p^\infty}}\rangle, O_C\langle T_1^{\frac{1}{p^\infty}} \dots T_n^{\frac{1}{p^\infty}}\rangle)$ *over a perfectoid field C of characteristic p, is a cJ-diamond. Indeed, we have the equality of diamonds*

$$\mathrm{Spa}(C\langle T_1 \cdots T_n\rangle, O_C\langle T_1 \cdots T_n\rangle)^\diamond = \mathbb{B}_n,$$

and we may conclude by theorem 1.4.35.

Definition 1.4.38. *Let C be a perfectoid field in characteristic p and X a locally spatial diamond over $Spa(C, O_C)$. We say that X has "enough facets" over C if it admits a v-cover of the form* $\coprod_{i\in I} \mathrm{Spd}(A_i, A_i^+) \to X$ *where each A_i is an algebra topologically of finite type algebra over C.*

Proposition 1.4.39. *Let X and Y be two locally spatial diamonds with enough facets over C, and let $C^\sharp$ denote an untilt of C. The following hold:*

1. *For any morphism of perfectoid fields* $\mathrm{Spa}(C', O_{C'}) \to \mathrm{Spa}(C, O_C)$ *the base change* $X \times_{\mathrm{Spa}(C,O_C)} \mathrm{Spa}(C', O_{C'})$ *has enough facets over C'.*

2. *The fiber product* $X \times_{\mathrm{Spa}(C,O_C)} Y$ *has enough facets over C.*

3. *X is a cJ-diamond.*

4. *If* $X = \mathrm{Spa}(A, A^+)$ *for a smooth and topologically of finite type $C^\sharp$-algebra A, then X has enough facets.*

Proof. Since the property of being topologically of finite type is stable under products and change of the base ground field one can prove easily the first two claims. The third claim follows from corollary 1.4.36 and proposition 1.4.34.

For the last claim, let $\mathbb{T}^n_{C^\sharp}$ denote $\mathrm{Spa}(C^\sharp\langle T_1^\pm, \dots T_n^\pm\rangle, O_{C^\sharp}\langle T_1^\pm, \dots T_n^\pm\rangle)$, and let $\widetilde{\mathbb{T}}^n_{C^\sharp} = \varprojlim_{T_i \mapsto T_i^p} \mathbb{T}^n_{C^\sharp}$ analogously for $\mathbb{T}^n_C$ and $\widetilde{\mathbb{T}}^n_C$. For any point $x \in \mathrm{Spa}(A, A^+)$ we may find an open neighborhood U of x together with an étale map $\eta : U \to \mathbb{T}^n_{C^\sharp}$. Let $\widetilde{U}$ denote the pullback of η along $\widetilde{\mathbb{T}}^n_{C^\sharp} \to \mathbb{T}^n_{C^\sharp}$, we get an étale map $\widetilde{U}^\flat \to \widetilde{\mathbb{T}}^n_C$. By the invariance of the étale site under perfection (see [26] lemma 15.6) $\widetilde{U}^\flat = U'^\diamond$ for an adic space U' that is étale over $\mathbb{T}^n_C$. Now, U' admits an open cover of the form $\coprod_{i\in I} \mathrm{Spa}(A_i, A_i{}^+) \to U'$ with each A_i topologically of finite type over C. This gives a cover,

$$\coprod_{i\in I} \mathrm{Spd}(A_i, A_i{}^+) \to \widetilde{U}^\flat \to U^\diamond.$$

□

We now define Orapian kimberlites, which are some of the kimberlites that will satisfy the third condition we discussed above.

Definition 1.4.40. *Let $\mathcal{F}$ be a prekimberlite and $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ a smelted kimberlite.*

1. *We say that $\mathcal{K}$ is Orapian if the following conditions hold:*

- $\mathscr{D}$ *is a cJ-diamond.*
- $|\mathcal{F}^{\mathrm{red}}|$ *is a locally Noetherian topological space.*
- *The specialization map* $\mathrm{sp}_{\mathcal{K}} : |\mathscr{D}| \to |\mathcal{F}^{\mathrm{red}}|$ *is specializing and a quotient map.*

2. *If* $\mathcal{F}$ *is a kimberlite we say it is Orapian if* $(\mathcal{F}, \mathcal{F}^{an})$ *is Orapian. If* $\mathcal{F}$ *is a p-adically smelted kimberlite we say it is Orapian if* $(\mathcal{F}, \mathcal{F}_\eta)$ *is Orapian.*

Remark 1.4.41. *To the author's knowledge, the theory of diamonds and v-sheaves doesn't have a notion of what it means to be of finite type. Orapianess, is an ad hoc condition that is good enough for the applications that we have in mind. On a different tone, Orapa is a town in Botswana that hosts the largest diamond producing mine in the world.*

The following fact about Orapian smelted kimberlites is a crucial property that we use later on in our applications.

Proposition 1.4.42. *Let* $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ *be an Orapian smelted kimberlite and suppose that for any closed point* $x \in |\mathcal{F}^{\mathrm{red}}|$ *the smelted tubular neighborhood* $\widehat{\mathscr{D}}_{/x}$ *is connected, then* $\pi_0(\mathrm{sp}_{\mathscr{D}}) : \pi_0(|\mathscr{D}|) \to \pi_0(|\mathcal{F}^{\mathrm{red}}|)$ *is a bijection between sets of connected components.*

Proof. Let $U, V \subseteq |\mathscr{D}|$ be two non-empty closed-open subsets with $V \cup U = |\mathscr{D}|$. Since $\mathrm{sp}_{\mathscr{D}}$ is a quotient map the map of connected components is surjective. Suppose now that $\emptyset \neq \mathrm{sp}_{\mathcal{F}}(U) \cap \mathrm{sp}_{\mathcal{F}}(V)$ we want to show that $U \cap V \neq 0$ which implies that $|\mathscr{D}|$ and $|\mathcal{F}^{\mathrm{red}}|$ have the same families of closed-open subsets. Since $\mathrm{sp}_{\mathscr{D}}$ is specializing we can assume there is a closed point $x \in \mathrm{sp}_{\mathscr{D}}(U) \cap \mathrm{sp}_{\mathscr{D}}(V)$. Since $|\mathcal{F}^{\mathrm{red}}|$ is locally Noetherian the closed points are open in the constructible topology. By proposition 1.4.33, $\widehat{\mathscr{D}}_{/x}$ is dense in ${\mathrm{sp}_{\mathscr{D}}}^{-1}(x)$, this implies that ${\mathrm{sp}_{\mathscr{D}}}^{-1}(x)$ is connected. Connectedness gives that $({\mathrm{sp}_{\mathscr{D}}}^{-1}(x) \cap U) \cap ({\mathrm{sp}_{\mathscr{D}}}^{-1}(x) \cap V) \neq \emptyset$ and in particular $U \cap V \neq \emptyset$ which is what we wanted to show. □

We will need the following two technical lemmas later:

Lemma 1.4.43. *Suppose* $\mathcal{F}$ *is a p-adically smelted kimberlite and that* $\mathcal{F}_\eta$ *is partially proper over* $\mathbb{Q}_p^\diamond$, *then:*

1. $\mathrm{sp}_{\mathcal{F}_\eta} : |\mathcal{F}_\eta| \to |\mathcal{F}^{\mathrm{red}}|$ *is specializing.*
2. *If* $\mathrm{sp}_{\mathcal{F}_\eta}$ *is surjective and* $|\mathcal{F}^{\mathrm{red}}|$ *is a locally Noetherian topological space then it is also a quotient map.*

Proof. Take a point $r \in |\mathcal{F}_\eta|$ mapping to $x \in |\mathcal{F}^{\mathrm{red}}|$ and take $y \in |\mathcal{F}^{\mathrm{red}}|$ specializing from x. We need to find q specializing from r that maps to y. Suppose r is represented by a map $f_r : \mathrm{Spa}(C, C^+) \to \mathcal{F}$ and suppose that $\mathcal{F}$ formalizes f_r. Let $K = O_C/C^{\circ\circ}$ and $K^+ = C^+/C^{\circ\circ}$, then x is the image of the maximal ideal of K^+ under the map $f_x : \mathrm{Spec}(K^+) \to \mathcal{F}^{\mathrm{red}}$. Consider the local ring R, constructed from $\mathcal{F}^{\mathrm{red}}$ by taking the reduced subscheme whose underlying topological spaces is the intersection of the closure of x and the localization at y. We let $k = K^+/\mathfrak{m}_{K^+}$, and so we have $R \subseteq k$. By ([29] Tag 00IA), we have a valuation subring $R \subseteq V \subseteq k$ such that $Frac(V) = k$ and V dominates R. This induces a valuation subring $K'^+ \subseteq K^+$ and a map $f_y : \mathrm{Spec}(K'^+) \to \mathcal{F}^{\mathrm{red}}$ whose closed point maps to y. In turn, this induces a valuation subring $C'^+ \subseteq C^+$ with $C'^+/C^{\circ\circ} = K'^+$ by lemma 1.4.7. Since $\mathcal{F}_\eta$ is partially proper, we get a map $f_q : \mathrm{Spa}(C, C'^+) \to \mathcal{F}_\eta$ extending f_r, and the point $q = [f_q] \in |\mathcal{F}_\eta|$ maps to y.

For the second claim, we first prove the case in which $|\mathcal{F}^{\mathrm{red}}|$ is irreducible. Let g be the generic point of $|\mathcal{F}^{\mathrm{red}}|$, and take a rank 1 point in $r \in |\mathcal{F}_\eta|$ mapping to g. Take a map $f_r : \mathrm{Spa}(C, O_C) \to \mathcal{F}_\eta$ representing r, and let C^{min} be the minimal integrally closed subring of C containing $\mathbb{Z}_p$ and $C^{\circ\circ}$, this is the minimal ring of integral elements for C. By partial properness we get a map $\mathrm{Spa}(C, C^{min}) \to \mathcal{F}_\eta$

whose image consists of the set of specializations of x in $|\mathcal{F}_\eta|$. The composition of the map $f^{min}: |\mathrm{Spa}(C, C^{min})| \to |\mathcal{F}^{\mathrm{red}}|$ is specializing, surjective and a spectral map of spectral spaces (surjectivity of this map proves that $|\mathcal{F}^{\mathrm{red}}|$ is also spectral instead of just locally spectral). By corollary 1.1.18 f^{min} is a closed map and consequenlty a quotient map of topological spaces.

The case in which $|\mathcal{F}^{\mathrm{red}}|$ has a finite number of irreducible components is analogous. For the general case, it is enough to prove locally on $|\mathcal{F}^{\mathrm{red}}|$ (for the Zariski topology) that $\mathrm{sp}_{\mathcal{F}_\eta}$ is a quotient map. By assumption around each point $x \in |\mathcal{F}^{\mathrm{red}}|$ there is an open neighborhood $U_x \subseteq |\mathcal{F}^{\mathrm{red}}|$ for which $|U_x|$ is a Noetherian topological space. In particular, U_x has a finite number of irreducible components and the closure $\overline{U}_x \subseteq |\mathcal{F}^{\mathrm{red}}|$ also has a finite number of irreducible components. Let $T = \mathrm{sp}_{\mathcal{F}_\eta}{}^{-1}(\overline{U}_x)$, this set is closed and consequently stable under specialization. If we lift the generic points of $\overline{U}_x$ to T we can argue as above to prove that the map $\mathrm{sp}_{\mathcal{F}_\eta} : T \to \overline{U}_x$ is a quotient map. This finishes the proof. □

Lemma 1.4.44. *Let C be a characteristic zero non-Archimedean algebraically closed field, and let $k = O_C/\mathfrak{m}_C$. Let $\mathcal{F}$ be a p-adically smelted kimberlite over* $\mathrm{Spd}(O_C, O_C)$. *Suppose that for every algebraically closed non-Archimedean field extension C'/C the basechange $\mathcal{F}_{O_{C'}} = \mathcal{F} \times_{\mathrm{Spd}(O_C, O_C)} \mathrm{Spd}(O_{C'}, O_{C'})$ satisfies that for closed point $x \in |\mathcal{F}^{\mathrm{red}}_{O_{C'}}|$ the p-adic tubular neighborhood $(\widehat{\mathcal{F}_{O_{C'}}}_{/x})_\eta$ is non-empty. Then $\mathrm{sp}_{\mathcal{F}_\eta}$ is a surjection.*

Proof. Given a point in $x \in |\mathcal{F}^{\mathrm{red}}|$ we can find a field extension of perfect fields K/k for which $\mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K)$ has a section $y : \mathrm{Spec}(K) \to \mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K)$ mapping to x under $\mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K) \to \mathcal{F}^{\mathrm{red}}$. Since $\mathcal{F}$ is formally separated, $\mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K)$ is also separated and sections to the structure map define closed points. We can construct a non-Archimedean field C' with $C \subseteq C'$ and $W(k)[\frac{1}{p}] \subseteq W(K)[\frac{1}{p}] \subseteq C'$. We get a map of p-adically smelted kimberlites $\mathcal{F}_{O_{C'}} \to \mathcal{F}$, and in $|\mathcal{F}^{\mathrm{red}}_{O_{C'}}|$ there is a closed point y mapping to x. Any point $r \in |\mathcal{F}_{C'}|$ with $\mathrm{sp}_{\mathcal{F}_{C'}}(r) = y$ maps to a point whose image under the specialization map is x. This proves surjectivity. □

A note on the terminology

The author would like to use this paragraph to make a small comment on the terminology. Some of the terms introduced below come with a metaphor. The incorporation of these metaphors into the text is nothing but a playful manner in which the author decided to interact with the minerological history of the field. In particular, they shouldn't be taken seriously for any scientific or mathematical purposes.

The first term that requires some explanation is the olivine spectrum of a Huber pair. Olivine minerals are a series of minerological structures that can be found most commonly in mafic and ultramafic igneous rocks, they are characteristic by their green olive like color. During the formation of a diamond small minerals like olivine, garnet, and chromite among others get surrounded by a host diamond. When these minerals get included in diamonds their morphology changes to ressemble the structure that is found in diamonds. Similarly, the olivine spectrum of a Huber pair is a very small variation of the usual adic specturm that has a subtle diamond-like change.

The second term that requires explanation are kimberlites. In minerology, kimberlites are hybrid rocks that are known to contain diamonds. The formation of diamonds happens in the depths of Earth and through geological processes, kimberlite magma pipes bring the diamonds to the surface. The interest in mining kimberlites comes from the hope of finding diamonds within. Similarly, the author thinks of kimberlites as a natural category for finding integral models of diamonds. The author hopes that these kimberlites can also bring to surface some facts about diamonds.

Chapter 2

Specialization maps for some moduli problems

2.1 G-torsors, lattices and shtukas

In this section we recall the theory of vector bundles over the Fargues-Fontaine curve, and point to the technical statements that allow us to discuss the specialization map for the p-adic Beilinson-Drinfeld Grassmanians and moduli spaces of mixed-characteristic shtukas. Nothing in this section is essentially new and it is all written in some form in ([28], [17], [8], [1]). Nevertheless, we need specific formulations for some of these results that are not explicit in the literature. For the convenience of the reader, we justify how our formulations follow from other (harder) statements that can be explicitly found in the literature.

For the rest of this Chapter we let k be an auxiliary perfect field in characteristic p and we let $\mathscr{G}$ be a parahoric group scheme over $\mathrm{Spec}(W(k))$ with reductive generic fiber G. Depending on the context, we will introduce more notation and add restrictive hypothesis on what $\mathscr{G}$ and k are allowed to be. We will often times abbreviate $\mathrm{Spd}(W(k), W(k))$ by $W(k)^\diamond$, and $\mathrm{Spec}(k)^\diamond$ by $k^\diamond$.

2.1.1 Vector bundles, torsors and meromorphicity

We give a quick review of the theory of vector bundles for adic and perfectoid spaces. Given an analytic Huber pair (A, A^+), and an A module M we can define $\tilde{M}$ as a presheaf on the open sets of $\mathrm{Spa}(A, A^+)$ defined as $\tilde{M}(U) = \varprojlim_{\mathrm{Spa}(B,B^+)\subseteq U} M \otimes_A B$ running over all the rational subsets $\mathrm{Spa}(B, B^+) \subseteq \mathrm{Spa}(A, A^+)$, and where $M \otimes_A B$ refers to the usual tensor product of A-modules ignoring the topology (See [15] 1.3.2). Kedlaya and Liu prove that whenever (A, A^+) is sheafy and M is a finite projective A-module $\tilde{M}$ is an acyclic sheaf.

Definition 2.1.1. *Given an adic space X, a vector bundle of rank n over X is a sheaf of $\mathcal{O}_X$-modules which is locally isomorphic to $\tilde{M}(U_i)$ for some affinoid open cover $U_i = \mathrm{Spa}(A_i, {A_i}^+)$ and rank n projective modules $M(U_i)$ over A_i.*

In what follows we will need to work with categories of vector bundles over adic spaces and over schemes at the same time. The following important result of Kedlaya and Liu makes the bridge between these categories:

Theorem 2.1.2. *(See [15] 1.4.2) Let $X = \mathrm{Spa}(A, A^+)$ be an analytic affinoid adic space, suppose that*

A is sheafy. The functor

$$H^0(\mathrm{Spa}(A,A^+),-): Vec_{\mathrm{Spa}(A,A^+)} \to Vec_{\mathrm{Spec}(A)}$$

from the category of vector bundles over $\mathrm{Spa}(A,A^+)$ *to the category of finite projective A-modules is an exact equivalence of exact categories.*

Remark 2.1.3. *The aciclicity of $\tilde{M}$ proves that $H^0(\mathrm{Spa}(A,A^+),-)$ is exact. The quasi-inverse $(\tilde{-})$ is also exact since $Tor_i(M,B)=0$ for M finite projecitve A-module and $i>0$.*

As in the theory of analytic functions on a complex variable one can introduce the notion of poles and meromorphic functions between vector bundles. We discuss how to do this:

Definition 2.1.4. *(See [28] 5.3.1, 5.3.2, 5.3.7) Given a uniform analytic adic space X, and an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, we say that $\mathcal{I}$ defines a Cartier divisor if $\mathcal{I}$ is a line bundle over X. Let $Z \subseteq X$ denote the support of $\mathcal{O}_X/\mathcal{I}$. We say that $\mathcal{I}$ is a closed Cartier divisor if the topologically ringed topological space equipped with valuations $(Z, \mathcal{O}_X/\mathcal{I}, |\cdot|_{x\in Z})$ is an adic space.*

The data of a Cartier divisor allows us to define the notion of meromorphicity.

Proposition 2.1.5. *(See [28] 5.3.4) Let X be a uniform analytic adic space and $\mathcal{I} \subseteq \mathcal{O}_X$ a Cartier divisor. Let $U = X \setminus V(\mathcal{I})$ be the complement of the Cartier divisor and denote $j : U \subseteq X$ the natural inclusion, we then have inclusions of $\mathcal{O}_X$-modules:*

$$\mathcal{O}_X \subseteq \varinjlim \mathcal{I}^{\otimes(-n)} \subseteq j_*(\mathcal{O}_U)$$

Definition 2.1.6. *Let X be a uniform analytic adic space, let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two vector bundles over X and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a Cartier divisor. Let us denote by U the complement of the support of $\mathcal{I}$. We say that a map in $Hom_U(\mathcal{V}_1,\mathcal{V}_2)$ is meromorphic along $\mathcal{I}$ if it is in*

$$H^0(X, \underline{Hom}(\mathcal{V}_1,\mathcal{V}_2) \otimes (\varinjlim \mathcal{I}^{\otimes(-n)}))$$

where $\underline{Hom}(\mathcal{V}_1,\mathcal{V}_2)$ denotes the internal Hom vector bundle.

Definition 2.1.7. *Let X be a uniform analytic adic space over* $\mathrm{Spa}(W(k)[\frac{1}{p}], W(k))$. *We define a $\mathscr{G}$-torsor over X to be a $\otimes$-exact functor from the category of algebraic representations over finite free $W(k)$-modules, $Rep(\mathscr{G})$, to the category of vector bundles over X, Vec_X.*

We can then generalize the notion of meromorphicity to that of $\mathscr{G}$-torsors.

Definition 2.1.8. *With X as in definition 2.1.6 and definition 2.1.7, we will say that a morphism, $f : \mathscr{T}_{1|_U} \to \mathscr{T}_{2|_U}$, of $\mathscr{G}$-torsors over X, is meromorphic along $\mathcal{I}$ if for all representation $\pi \in Rep(\mathscr{G})$ the corresponding map $f(\pi) : \mathscr{T}_1(\pi)_{|_U} \to \mathscr{T}_2(\pi)_{|_U}$ is meromorphic along $\mathcal{I}$.*

We will often use the following fact.

Theorem 2.1.9. *(See [28] 17.1.8) The category of vector bundles fibered over* Perfd *forms a stack for the v-topology.*

2.1.2 Vector bundles on $\mathcal{Y}$

We defined $\mathbb{Z}_p^\diamond$ as the v-sheaf parametrizing untilts. Although $\mathbb{Z}_p^\diamond$ is not itself represented by an analytic adic space, the product $\mathbb{Z}_p^\diamond \times S$ for any $S \in \mathrm{Perf}$ can be represented by an analytic adic space. Let us recall this construction.

Definition 2.1.10. *Given a perfectoid Huber pair (R, R^+) and a pseudo-uniformizer $\varpi \in R^+$, we define $\mathcal{Y}^{R^+}_{[0,\infty)}$ as $\mathrm{Spa}(W(R^+), W(R^+)) \setminus V([\varpi])$. Where $[\varpi]$ denotes a Teichműller lift of ϖ, and where $W(R^+)$ is given the $(p, [\varpi])$-adic topology. We also define $\mathcal{Y}_{R^+}$ as $\mathrm{Spa}(W(R^+), W(R^+)) \setminus V(p, [\varpi])$.*

Proposition 2.1.11. *(See [16] 3.6, [28] 11.2.1]) For any perfectoid Huber pair (R, R^+) the space $\mathcal{Y}_{R^+}$ has a cover by sheafy Huber pairs. Consequently, $\mathcal{Y}_{R^+}$ and $\mathcal{Y}^{R^+}_{[0,\infty)}$ are adic spaces. Moreover, $(\mathcal{Y}^{R^+}_{[0,\infty)})^\diamond = \mathbb{Z}_p^\diamond \times \mathrm{Spa}(R, R^+)$.*

Let us review the geometry of $\mathcal{Y}_{R^+}$, for this fix a psuedo-uniformizer $\varpi \in R^+$. One defines a continuous map $\kappa_\varpi : |\mathcal{Y}_{R^+}| \to [0, \infty]$ characterized by the property that $\kappa(y) = r$ if and only if for any positive rational number $r \leq \frac{m}{n}$ the inequality $|p|^m_y \leq |[\varpi]|^n_y$ holds and for any positive rational number $\frac{m}{n} \leq r$ the inequality $|[\varpi]|^n_y \leq |p|^m_y$ holds.

Given an interval $I \subseteq [0, \infty]$ we denote by $\mathcal{Y}^{R^+}_I$ the open subset corresponding to the interior of $\kappa_\varpi^{-1}(I)$. For example, $\mathcal{Y}^{R^+}_{(0,\infty]}$ corresponds to the locus in $\mathcal{Y}_{R^+}$ where $|p| \neq 0$ and $\mathcal{Y}^{R^+}_{[0,\infty)}$ corresponds to the locus where $|[\varpi]| \neq 0$. For intervals of the form $[0, \frac{h}{d}]$ where h and d are integers the space $\mathcal{Y}^{R^+}_{[0,\frac{h}{d}]}$ is represented by $\mathrm{Spa}(R', R'^+)$ corresponding to the rational localization,

$$\{x \in \mathrm{Spa}(W(R^+), W(R^+)) \mid |p^h|_x \leq |[\varpi]^d|_x \neq 0\}.$$

In this case, we can compute R'^+ explicitly as the $[\varpi]$-adic completion of $W(R^+)[\frac{p^h}{[\varpi]^d}]$ and R' as $R'^+[\frac{1}{[\varpi]}]$. A direct computation shows that R' does not depend of R^+. In particular, the exact category of vector bundles over $\mathcal{Y}^{R^+}_{[0,\infty)}$ does not depend of the choice of R^+ either.

We will also need to work with an algebraic version of $\mathcal{Y}_{R^+}$, which we will denote Y_{R^+}. This is defined as the scheme $\mathrm{Spec}(W(R^+)) \setminus V(p, [\varpi])$. Since $W(R^+) \subseteq \mathcal{O}_{\mathcal{Y}_{R^+}}$ and since p, $[\varpi]$, do not vanish simultaneously on $\mathcal{Y}_{R^+}$ we get a map of locally ringed spaces $f : \mathcal{Y}_{R^+} \to Y_{R^+} \subseteq \mathrm{Spec}(W(R^+)$.

Recall that given an untilt $R^\sharp$ of R there is a canonical surjection $W(R^+) \to R^{\sharp+}$ whose kernel is generated by an element $\xi \in W(R^+)$ primitive of degree 1 (See [28] 6.2.8). The element ξ defines a closed Cartier divisor over $\mathcal{Y}_{R^+}$ and also defines a Cartier divisor on the scheme Y_{R^+}. In what follows, we compare the categories of vector bundles over $\mathrm{Spec}(W(R^+))$, Y_{R^+} and $\mathcal{Y}_{R^+}$ with morphisms being functions that are meromorphic along the ideal (ξ).

Recall Kedlaya's GAGA-type theorem:

Theorem 2.1.12. *(See [16] 3.8) Suppose (R, R^+) is a perfecotid Huber pair in characteristic p. The natural morphisms of locally ringed spaces $f : \mathcal{Y}_{R^+} \to Y_{R^+}$ gives, via the pullback functor $f^* : Vec_{Y_{R^+}} \to Vec_{\mathcal{Y}_{R^+}}$, an exact equivalence of exact categories.*

Remark 2.1.13. *Although the reference does not explictly claim that this equivalence is exact, one can simply follow the proof loc. cit. exchanging the word "equivalence" by "exact equivalence" since every arrow involved in the proof is an exact functor.*

Corollary 2.1.14. *With the notation as above, the pullback f^* induces an equivalence*

$$f^* : (Vec_{Y^{\xi\neq 0}_{R^+}})^{mer} \to (Vec_{\mathcal{Y}^{\xi\neq 0}_{R^+}})^{mer}$$

between the category whose objects are vector bundles over $\mathcal{Y}_{R^+}$ (respectively vector bundles over Y_{R^+}) and morphisms are functions meromorphic along the ideal (ξ) (respectively functions over $Y_{R^+} \setminus V(\xi)$).

Proof. By theorem 2.1.12 it is enough to prove that f^* is fully-faithful. Using internal $\underline{Hom}$ we can reduce to proving $H^0(\mathcal{Y}_{R^+}, f^*\mathcal{V})^{mer}_{\xi\neq 0} = H^0(Y^{\xi\neq 0}_{R^+}, \mathcal{V})$. For quasi-compact, quasi-separated schemes the global sections of a quasi-coherent sheaf after localizing by a global section of the structure sheaf is given simply by localization. That is $H^0(Y^{\xi\neq 0}_{R^+}, \mathcal{V}) = H^0(Y_{R^+}, \mathcal{V})[\frac{1}{\xi}]$. On the other hand, by definition

$H^0(\mathcal{Y}_{R^+}, f^*\mathcal{V})^{mer}_{\xi\neq 0} = H^0(\mathcal{Y}_{R^+}, f^*\mathcal{V} \otimes \varinjlim(\xi)^{\otimes(-n)})$. Now the ideal sheaf (ξ) is isomorphic to $\mathcal{O}_{\mathcal{Y}_{R^+}}$ since it is a principal Cartier divisor so we can view $f^*\mathcal{V} \otimes \varinjlim(\xi)$ as:

$$f^*\mathcal{V} \xrightarrow{\xi} f^*\mathcal{V} \xrightarrow{\xi} \cdots$$

And since H^0 commutes with filtered colimits we get precisely $H^0(\mathcal{Y}_{R^+}, f^*\mathcal{V})[\frac{1}{\xi}]$. □

Since we defined $\mathscr{G}$-torsors Tannakianly these statements immediately generalize to those for $\mathscr{G}$-torsors. Kedlaya proves another important statement.

Theorem 2.1.15. *(See [16] 2.3, 2.7, 3.11) With notation as above, and letting j be the open embedding, $j : Y_{R^+} \to \mathrm{Spec}(W(R^+))$ the following statements hold:*

1. *The pullback functor $j^* : Vec_{\mathrm{Spec}(W(R^+))} \to Vec_{Y_{R^+}}$ is fully-faithful.*
2. *If R^+ is a valuation ring then j^* is an equivalence.*
3. *Taking categories of quasi-coherent sheaves the adjunction morphism $j^*j_*\mathcal{V} \to \mathcal{V}$ is an isomorphism.*

Remark 2.1.16. *One may think that the third statement together with the first statement of theorem 2.1.15 above would give an equivalence of categories of vector bundles for any ring R^+. This is not the case because even if $\mathcal{V}$ is a vector bundle, $j_*\mathcal{V}$ might not be a vector bundle over $\mathrm{Spec}(W(R^+))$.*

We will need a small modification of theorem 2.1.15. For this we recall a few facts about topological modules on a Tate ring, this material is taken from ([28] 14.2.3). Let A be a complete Tate ring, f a topological nilpotent unit, A_0 a ring of definition, and N a projective module over A. One can endow N with its canonical topology (See [15] 1.1.11).

1. An A_0-submodule $M \subseteq N$ is open if and only if $M[\frac{1}{f}] = N$.
2. An A_0-submodule $M \subseteq N$ is bounded if and only if M is contained in a finitely generated A_0-submodule.
3. If $A \subseteq B$ with the subspace topology, B is Tate and complete for its topology, and $S \subseteq N \otimes_A B$ is bounded, then $S \cap N \subseteq N$ is also bounded.

The following statement is implicitly used and proved in ([28] 25.1.2).

Proposition 2.1.17. *Let $\mathrm{Spa}(R, R^+)$ be the product of points constructed from the family $\{(C_i, {C_i}^+), \varpi_i\}_{i\in I}$ as in definition 1.1.5. The pullback functor $j^* : Vec_{\mathrm{Spec}(W(R^+))} \to Y_{R^+}$ gives an equivalence of categories of vector bundles with fixed rank.*

Proof. We already have a fully-faithful embedding by theorem 2.1.15, so it is enough to prove it is essentially surjective. Let $\mathcal{V}$ be a vector bundle over Y_{R^+} of constant rank n, we let $M' = H^0(Y_{R^+}, \mathcal{V})$ which is a $W(R^+)$-module whose pullback to Y_{R^+} identifies with $\mathcal{V}$ by theorem 2.1.15, we want to prove that M' is a projective module. Let $N = M' \otimes_{W(R^+)} W(R^+)[\frac{1}{p}]$, this module is projective since N is the pullback of $\mathcal{V}$ to $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$.

Define M_i as $H^0(Y_{C_i^+}, \iota_i^*\mathcal{V})$ where $\iota_i : Y_{C_i^+} \to Y_{R^+}$ is the closed embedding produced by the idempotent $1_i \in W(R^+) = \prod_{i\in I} W(C_i^+)$. For each i, this is a free $W(C_i^+)$-module by theorem 2.1.15 and because $W(C_i^+)$ is a local ring. We define $M = \prod_{i\in I} M_i$ which is a free $W(R^+)$-module of constant rank n. Since we have maps of $W(R^+)$ modules

$$M' = H^0(Y_{R^+}, \mathcal{V}) \to H^0(Y_{C_i^+}, \iota_i^*\mathcal{V}) = M_i$$

We get a map $M' \to M$. This map is injective since the family $Y_{C_i^+}$ is dense in the Zariski topology of Y_{R^+}. We want to prove that this map is an isomorphism.

As a first step we prove that the map induces an isomorphism $M'[\frac{1}{p}] \to M[\frac{1}{p}]$. For this we will consider $W(R^+)[\frac{1}{p}]$ as a Tate ring with its p-adic topology, and $W(R^+)$ as a ring of definition. In this context $M' \subseteq N$ is an open subset when N is given its canonical topology. This follows from property 1, since by construction $N = M'[\frac{1}{p}]$. The map of schemes $\mathrm{Spec}(W(R)) \to \mathrm{Spec}(W(R^+))$ factors through Y_{R^+}. This implies that $M' \otimes_{W(R^+)} W(R)$ is a projective module over $W(R)$. The usual map realizes $W(R^+)[\frac{1}{p}]$ as a topological subring of $W(R)[\frac{1}{p}]$. Moreover, $M' \otimes_{W(R^+)} W(R)$ is a bounded subset of $N \otimes_{W(R^+)[\frac{1}{p}]} W(R)[\frac{1}{p}]$ by property 2. On the other hand, property 3 readily implies M' is a bounded subset of N.

We construct an injection $M \subseteq N$ as $W(R^+)$-modules. Consider $N_i = 1_i N$ as a $W(C_i^+)[\frac{1}{p}]$-module but also as a subset of N. We have an injection $N \subseteq \prod_{i\in I} N_i$ and an element $(n_i)_{i\in I}$ is in the image of N if and only if the set $S = \{n_i\}_{i\in I} \subseteq N$ is bounded in N. There is a clear injection $M = \prod_{i\in I} M_i \to \prod_{i\in I} N_i$ and we claim that it factors through N. Let us prove that, first observe that if 1_i^c denotes the complementary idempotent of 1_i then $1_i \cdot M' = M'[\frac{1}{1_i^c}]$. Since taking global sections commutes with localization on qcqs schemes, we have that $M_i = 1_i \cdot M'$. Then the image of any element $m \in M$ in $\prod_{i\in I} N_i$ has the form $(m_i)_{i\in I}$ with $m_i \in 1_i \cdot M'$. Since M' is bounded in N, the set $\coprod_{i\in I} M_i$ is bounded and the map $M \to \prod_{i\in I} N_i$ defines and embedding into N. We have $M' \subseteq M \subseteq N$ and in particular $M'[\frac{1}{p}] = M[\frac{1}{p}]$, which finishes the first step.

We define $\mathcal{V}_2$ to be j^*M, which is a vector bundle over Y_{R^+}. The situation is as follows, we have a morphism of vector bundles $\mathcal{V}_1 \to \mathcal{V}_2$ over Y_{R^+} with $\mathcal{V}_2$ a trivial vector bundle, that becomes an isomorphism over $Y_{C_i^+}$ for every $i \in I$ and also becomes an isomorphism over $\mathrm{Spec}(W(R^+)[\frac{1}{p}]) \subseteq Y_{R^+}$, we wish to prove that it is already an isomorphism over Y_{R^+}. After taking determinant and fixing a trivialization we get a map $\wedge V_1 \to \mathcal{O}_X$, and it is enough to prove this one is an isomorphism.

Upon applying Beauville-Laszlo lemma (See [28] 5.2.9) to $p \in W(R^+)[\frac{1}{[\varpi]}]$ the morphism $\wedge \mathcal{V}_1 \to \mathcal{O}_X$ produces for us a family of lattices over $W(R) = (W(\widehat{R^+)[\frac{1}{[\varpi]}}])_p$ parametrized by $\mathrm{Spec}(R)$. This is the same as a morphism of schemes $\mathrm{Spec}(R) \to \mathrm{Gr}_{\mathcal{W}}^{\mathbb{G}_m}$ to the 1-dimensional Witt-vector Grassmanian (See [4] 8.1). Pullback of the map $\mathcal{V}_1 \to \mathcal{O}_X$ to $W(C_i)$ gives a lattice corresponding to the composition $\mathrm{Spec}(C_i) \to \mathrm{Spec}(R) \to \mathrm{Gr}_{\mathcal{W}}^{\mathbb{G}_m}$, but for each $i \in I$ the restriction of the morphism $\mathcal{V}_1 \to \mathcal{O}_X$ to $W(C_i)$ is an isomorphism. In particular, we get the following commutative diagram,

$$\begin{tikzcd}
\coprod_{i\in I} \mathrm{Spec}(C_i) \arrow[r] \arrow[d] & \mathrm{Spec}(R) \arrow[d] \\
\mathrm{Spec}(\mathbb{F}_p) \arrow[r, "e"] & \mathrm{Gr}_{\mathcal{W}}^{\mathbb{G}_m}
\end{tikzcd}$$

where the map $e : \mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Gr}_{\mathcal{W}}^{\mathbb{G}_m}$ is the one associated to the identity of $\mathcal{O}_X$. The image of $|\coprod_{i\in I} \mathrm{Spec}(C_i)|$ in $|\mathrm{Spec}(R)|$ is dense since the map of rings $R \to \prod_{i\in I} C_i$ is injective. But $\mathrm{Gr}_{\mathcal{W}}^{\mathbb{G}_m}$ is representable by a discrete disjoint union of points of the form $\mathrm{Spec}(\mathbb{F}_p)$. So the map $\mathrm{Spec}(R) \to \mathrm{Gr}_{\mathcal{W}}^{\mathbb{G}_m}$ factors through the identity section which finishes the proof.

□

Given $\xi \in W(R^+)$ primitive of degree 1 as before, observe that since both $\mathrm{Spec}(W(R^+))$ and Y_{R^+} are qcqs schemes the equivalence of vector bundles of proposition 2.1.17 generalizes to the categories where the objects are the same, but morphism are allowed to have poles along ξ on both categories.

Interestingly, extending $\mathscr{G}$-torsors from Y_{R^+} to $\mathrm{Spec}(W(R^+))$ adds yet another layer of complexity. Indeed, the equivalences of theorem 2.1.15 and proposition 2.1.17 are not exact equivalences, so Tannakian formalism can't be used directly. As a matter of fact, only the pullback functor j^* is exact. Anschűtz gives a detailed study of the problem of extending $\mathscr{G}$-torsors along j in [1]. He gives a complete solution for a big family of smooth algebraic groups, $\mathscr{G}'$, defined over the ring of integers of some discretely valued p-adic subfield $E \subseteq C$ that have parahoric reduction over O_E. This family includes all reductive groups over $\mathbb{Z}_p$, which is the only case we will need.

We want to emphasize that, as we discussed in the introduction, the methods of [1] allow Anschűtz to construct a point-wise specialization map for the p-adic Beilinson-Drinfeld Grassmanians attached to any group $\mathscr{G}'$ with parahoric reduction. Proposition 2.1.20 below, which is nothing but a small improvement to theorem 2.1.19, is the main technical input that we will need to upgrade Anschűtz map to a map of topological spaces. When $\mathscr{G}'$ is reductive we will be able to say more.

Theorem 2.1.18. *(See [1] 7.2, 7.3, 7.9, 6.5, 7.6) Let C be an algebraically closed non-Archimedean field over k, let $C^+ \subseteq C$ an open and bounded valuation subring with $k \subseteq C^+$, if $\mathscr{G}$ is a parahoric group scheme over $W(k)$. Then:*

1. *A $\mathscr{G}$-torsor $\mathscr{T}$ over Y_{C^+} extends to $\mathrm{Spec}(W(C^+))$ if and only if $\mathscr{T}$ is trivial when restricted to $\mathrm{Spec}(W(C^+)[\frac{1}{\xi}]) \subseteq Y_{C^+}$, for some $\xi \in W(C^+)$ primitive of degree 1.*
2. *If $\mathscr{G}$ is reductive over $W(k)$ then any $\mathscr{G}$-torsor extends from Y_{C^+} to $\mathrm{Spec}(W(C^+))$.*

Remark 2.1.19. *In the reference, only the case dealt with explicitly is $C^+ = O_C$, but the proof for more general open and bounded valuation subrings C^+ is identical when one makes the observation that $\mathrm{Spec}(W(O_C)[\frac{1}{[\varpi]}]) = \mathrm{Spec}(W(C^+)[\frac{1}{[\varpi]}])$.*

We now state an analogue of proposition 2.1.17 for $\mathscr{G}$-torsors.

Proposition 2.1.20. *Keep the notation as in theorem 2.1.18, and let $\mathrm{Spa}(R, R^+)$ be the product of points over k. A $\mathscr{G}$-torsor $\mathscr{T}$ over Y_{C^+} extends along $j : Y_{R^+} \to \mathrm{Spec}(W(R^+))$ if and only if $\mathscr{T}$ extends along $j_i : Y_{C_i^+} \to \mathrm{Spec}(W(C_i^+))$ for all $i \in I$. In particular the following two statements hold:*

1. *It extends if and only if it is trivial on $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$ for some ξ primitive of degree 1.*
2. *It extends if $\mathscr{G}$ is reductive over $W(k)$.*

Proof. We need to prove that the functor $j_*\mathscr{T} : Rep(\mathscr{G}') \to Vec_{\mathrm{Spec}(W(R^+))}$ is exact, and since the functor j_* is always left-exact we only have to prove right-exactness of $j_*\mathscr{T}$. Suppose we have a morphism of free modules $f : \mathcal{V}_1 \to \mathcal{V}_2$ over $\mathrm{Spec}(W(R^+))$ and we have that the basechange to $\mathrm{Spec}(W(C_i^+))$ is surjective for every $i \in I$, we need to prove that the morphism is surjective. Taking determinant bundles we can reduce to the case that $\mathcal{V}_2$ is free of rank 1. After taking trivializations we have n sections $f_1, \cdots, f_n \in W(R^+)$ and we need to prove that they generate the unit ideal. Consider the family of subsets $\{I_m\}_{1 \leq m \leq n}$ defined by

$$I_m = \{i \in I \mid f_m \in W(C_i^+)^\times\}$$

By construction 1_{I_m} is in the ideal generated by the f_i. Since each $W(C_i^+)$ is a local ring and the $\{f_m\}_{1 \leq m \leq n}$ generate the unit ideal in $W(C_i^+)$ the union $\bigcup I_m$ has to be I. This finishes the proof. ☐

We need the following descent result which is similar to theorem 2.1.9.

Proposition 2.1.21. *(See [28] 19.5.3) Let S be a perfectoid space over k and let $U \subseteq \mathcal{Y}^S_{[0,\infty)}$ be an open subset. For map of perfectoid spaces $f : S' \to S$, let $\mathcal{C}_{S'}$ denote the category of $\mathscr{G}$-torsors over $\mathcal{Y}^{S'}_{[0,\infty)} \times_{\mathcal{Y}^S_{[0,\infty)}} U$. Then the assignment $S' \mapsto \mathcal{C}_{S'}$, as a fibered category over Perf_S, is a v-stack.*

2.1.3 Lattices and shtukas

For this section fix $\mathrm{Spa}(R, R^+)$ an affinoid perfectoid space over k, $\varpi \in R^+$ a choice of pseudo-uniformizer, $R^\sharp$ an untilt of R and $\xi_{R^\sharp}$ a generator for the kernel of the map $W(R^+) \to R^{\sharp,+}$.

Definition 2.1.22. *We define the category $B^+_{dR}(R^\sharp)$-lattices with $\mathscr{G}$-structure to have as objects pairs $(\mathscr{T}, \psi)$ where $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathcal{Y}^{R^+}_{[0,\infty)}$ and $\psi : \mathscr{T} \to \mathscr{G}$ is an isomorphism over $\mathcal{Y}^{R^+}_{[0,\infty)} \setminus V(\xi_{R^\sharp})$ that is meromorphic along $(\xi_{R^\sharp})$. Morphisms are the evident isomorphisms of pairs.*

Given data $(\mathscr{T}, \psi)$ as above we can choose a big enough number $r_\varpi \in \mathbb{R}$ for which $\mathcal{Y}^{R^+}_{[r_\varpi,\infty)}$ is disjoint from $V(\xi_{R^\sharp})$. Over this locus we can glue along ψ to extend $\mathscr{T}$ canonically to a $\mathscr{G}$-torsor over $\mathcal{Y}_{R^+}$. Using Corollary 2.1.14 and Beauville-Laszlo on the scheme $\mathrm{Spec}(W^+_R)[\frac{1}{[\varpi]}]$ we get an equivalence of categories with the category of pairs (Ξ, ψ) where Ξ is $\mathscr{G}$-torsor over $\mathrm{Spec}(B^+_{dR}(R^\sharp))$ and $\psi : \Xi \to \mathscr{G}$ is a trivialization over $\mathrm{Spec}(B_{dR}(R^\sharp))$, where $B^+_{dR}(R^\sharp)$ denotes the completion of $W(R^+)[\frac{1}{[\varpi]}]$ along $\xi_{R^\sharp}$, and $B_{dR}(R^\sharp) = B^+_{dR}(R^\sharp)[\frac{1}{\xi_{R^\sharp}}]$.

Recall that for an algebraically closed non-Archimedean field C the ring $B_{dR}(C^\sharp)$ is a complete discrete valuation field so that the set of isomorphism classes of $\mathscr{G}$-torsors over $\mathrm{Spec}(B^+_{dR}(C^\sharp))$ is in canonical bijection with $\mathscr{G}(B_{dR}(C^\sharp))/\mathscr{G}(B^+_{dR}(C^\sharp))$. In case $\frac{1}{p} \in C^\sharp$ we also have that $\frac{1}{p} \in B^+_{dR}(C^\sharp)$, and we will find that $\mathscr{G}_{B^+_{dR}(C^\sharp)} = G_{B^+_{dR}(C^\sharp)}$ is split reductive. After fixing auxiliary groups $T \subseteq B \subseteq G_{B^+_{dR}(C^\sharp)}$, a maximal torus and a Borel respectively, the Cartan decomposition gives an identification:

$$\mathscr{G}(B^+_{dR}(C^\sharp))\backslash\mathscr{G}(B_{dR}(C^\sharp))/\mathscr{G}(B^+_{dR}(C^\sharp)) = G(B^+_{dR}(C^\sharp))\backslash G(B_{dR}(C^\sharp))/G(B^+_{dR}(C^\sharp)) = X^+_*(T) \tag{2.1}$$

Suppose that B and T are fixed and understood from the context, and let $\mu \in X^+_*(T)$. We say that a $B^+_{dR}(C^\sharp)$-lattice (Ξ, ψ) is of type μ if the isomorphism class of (Ξ, ψ) maps to μ under the identification above.

We now consider mixed-characteristic shtukas. Recall that the spaces $\mathrm{Spec}(W(R^+))$, $\mathcal{Y}^{R^+}_{[0,\infty)}$, Y_{R^+} and $\mathcal{Y}_{R^+}$ come equiped with a Frobenious action. All of this actions are coming from the usual Frobenious action on $W(R^+)$ given by:

$$\phi^*\left(\sum_{i=0}^{\infty}[\alpha_i]p^i\right) = \sum_{i=0}^{\infty}[\alpha_i^p]p^i$$

A computation shows that $\phi(\mathcal{Y}^{R^+}_{[a,b]}) = \mathcal{Y}^{R^+}_{[pa,pb]}$ which proves that all of the loci considered above are preserved by Frobenious.

Definition 2.1.23. *We define the category of shtukas with one paw over $\mathrm{Spa}(R^\sharp, {R^\sharp}^+)$ and $\mathscr{G}$-structure. For this we require that $k = \mathbb{F}_p$ and that $\mathscr{G}$ is defined over $\mathbb{Z}_p$. This category has as objects pairs $(\mathscr{T}, \Phi)$ where $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathcal{Y}^{R^+}_{[0,\infty)}$ and $\Phi : \phi^*\mathscr{T} \to \mathscr{T}$ is an isomorphism over $\mathcal{Y}^{R^+}_{[0,\infty)} \setminus V(\xi_{R^\sharp})$ meromorphic along $(\xi_{R^\sharp})$. Morphisms are the evident isomorphisms of pairs.*

Definition 2.1.24. *Given a ϕ-module with $\mathscr{G}$-structure $(\mathcal{E}, \Phi_{\mathcal{E}})$ over $\mathcal{Y}^{R^+}_{(0,\infty)}$ and a shtuka $(\mathscr{T}, \Phi_{\mathscr{T}})$ we say that $(\mathscr{T}, \Phi_{\mathscr{T}})$ is isogenous to $(\mathcal{E}, \Phi_{\mathcal{E}})$ if there is a number $r \in \mathbb{R}$ (that depends of the choice of ϖ) and a ϕ-equivaraint isomorphism $f : (\mathscr{T}, \Phi_{\mathscr{T}}) \to (\mathcal{E}, \Phi_{\mathcal{E}})$ defined over $\mathcal{Y}^{R^+}_{[r,\infty)}$. We call such a pair (r, f) an isogeny. Two isogenies (r_1, f_1) and (r_2, f_2) are equivalent if there is a third isogeny (r_3, f_3) with $r_3 > r_1, r_2$ and $f_1 = f_3 = f_2$ when restricted to $\mathcal{Y}^{R^+}_{[r_3,\infty)}$. We also refer by isogenies to the elements of the set of equivalence classes of pairs (r, f).*

After the work of Scholze and Weinstein one may think of mixed-characteristic shtukas as a generalization of p-divisible groups (See [28] 14.11,[27] Theorem B). We do not make this precise, but isogenies as defined above are closely related with isogenies of p-divisible groups. In what follows, we prove three technical lemmas that intuitively speaking allow us to "deform" lattices and shtukas with $\mathscr{G}$-structure. Later on it will become clear why we think of these lemmas as "deformation" statements.

For any $r \in [0, \infty)$ let $B^{R^+}_{[r,\infty]} = H^0(\mathcal{Y}^{R^+}_{[r,\infty]}, \mathcal{O}_{\mathcal{Y}^{R^+}_{[r,\infty]}})$, and consider the ring $R^+_{\mathrm{red}} = (R^+/\varpi)^{perf}$. We observe that the universal property of $\mathcal{Y}^{R^+}_{[r,\infty]}$ as a rational subset of $\mathrm{Spa}(W(R^+), W(R^+))$ induces compatible maps of rings $B^{R^+}_{[r,\infty]} \to W(R^+_{\mathrm{red}})[\frac{1}{p}]$ for varying r. We denote this family of reduction maps by $(-_{\mathrm{red}})$.

Lemma 2.1.25. *Let $s \in B^{R^+}_{[r,\infty]}$ and suppose that the reduction s_{red}, originally defined over $W(R^+_{\mathrm{red}})[\frac{1}{p}]$, lies in $W(R^+_{\mathrm{red}})$, then there are: a number r' with $r \leq r'$, elements $a \in W(R^+)$, $b \in B^{R^+}_{[r',\infty]}$ and a pseudo-uniformizer $\varpi_s \in R^+$ such that $s = a + b$ and $b \in [\varpi_s] \cdot B^{R^+}_{[r',\infty]}$.*

Proof. By enlarging r if necessary we can assume $\mathcal{Y}^{R^+}_{[r,\infty]}$ is of the form:

$$\{x \in \mathrm{Spa}(W(R^+), W(R^+)) \mid |[\varpi]|_x \leq |p^m|_x \neq 0\}$$

for some m, we compute $B^{R^+}_{[r,\infty]}$ explicitly. If S^+ denotes the p-adic completion of $W(R^+)[\frac{[\varpi]}{p^m}]$, then $B^{R^+}_{[r,\infty]} = S^+[\frac{1}{p}]$. Any element $s \in B^{R^+}_{[r,\infty]}$ is of the form $s = \frac{1}{p^n} \cdot \Sigma_{i=0}^{\infty} [a_i] x^{m(i)} p^i$ where $a_i \in R^+$, $x = \frac{[\varpi]}{p^m}$, and $m(i)$ denotes a non-negative integer. We can decompose $p^n \cdot s$ as

$$x \cdot \left(\sum_{i=0, m(i)>0}^{\infty} [a_i] x^{m(i)-1} p^i \right) + \sum_{i=0, m(i)=0}^{\infty} [a_i] p^i.$$

Since $x = \frac{[\varpi]}{p^m}$, we have that $[\varpi]$ divides in $B^{R^+}_{[r,\infty]}$ the first term of this decomposition. As long as we pick a ϖ_s that divides ϖ, we may and do reduce to the case $s = \Sigma_{i=0}^{\infty} [a_i] p^{i-n}$. In this case, $s_{\mathrm{red}} = \Sigma_{i=0}^{\infty} [(a_i)_{\mathrm{red}}] p^{i-n}$ and by hypothesis we have that for $i \leq n$ $(a_i)_{\mathrm{red}} = 0$ in R^+_{red}. We can choose a pseudo-uniformizer ϖ_s for which all of a_i, for $i \leq n$, are zero in R^+/ϖ_s. We can take $a = \sum_{i=n}^{\infty} [a_i] p^{i-n}$ and $b = \sum_{i=0}^{n-1} [a_i] p^{i-n}$. These clearly satisfy the properties. $\square$

Lemma 2.1.26. *Suppose that $\mathscr{T}_1$ and $\mathscr{T}_2$ are trivial $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$ and that $\lambda : \mathscr{T}_1 \to \mathscr{T}_2$ is an isomorphism defined over $\mathcal{Y}^{R^+}_{[r,\infty]}$ which upon reduction to $\mathrm{Spec}(W(R^+_{\mathrm{red}})[\frac{1}{p}])$ extends to an isomorphism over $\mathrm{Spec}(W(R^+_{\mathrm{red}}))$. Then, there is an isomorphism $\widetilde{\lambda} : \mathscr{T}_1 \to \mathscr{T}_2$ defined over $\mathrm{Spec}(W(R^+))$, a pseudo-uniformizer $\varpi_\lambda \in R^+$ and a number $r' \in \mathbb{R}$ with $r \leq r'$ such that $\lambda = \widetilde{\lambda}$ in:*

$$Hom_{\mathrm{Spec}(B^{R^+}_{[r',\infty]}/[\varpi_\lambda])}(\mathscr{T}_1, \mathscr{T}_2)$$

Proof. Fix for the rest of the proof trivializations $\iota_i : \mathscr{T}_i \to \mathscr{G}$, and consider $\iota_2 \circ \lambda \circ \iota_1^{-1}$ as an element $g \in H^0(\mathcal{Y}^{R^+}_{[r,\infty]}, \mathscr{G}) \subseteq H^0(\mathcal{Y}^{R^+}_{[r,\infty]}, Gl_n)$ for some n and some embedding $\mathscr{G} \to Gl_n$ defined over $W(k)$. By lemma 2.1.25 we can find ϖ_λ such that one can write g as $M_1 + [\varpi_\lambda] M_2$ where $M_1 \in Gl_n(W(R^+))$ and $M_2 \in M_{n \times n}(B^{R^+}_{[r',\infty]})$. With this setup the reduction of M_1 to $Gl_n(B^{R^+}_{[r',\infty]}/[\varpi_\lambda])$ lies in $\mathscr{G}(W(R^+)/[\varpi_\lambda])$. Moreover, since $\mathscr{G}$ is a smooth group and $W(R^+)$ is $[\varpi_\lambda]$-complete, we can lift this to an element $g' \in \mathscr{G}(W(R^+))$ with $g' = M_1$ in $Gl_n(W(R^+)/[\varpi_\lambda])$. Consequently $g' = g$ in $\mathscr{G}(B^{R^+}_{[r',\infty]}/[\varpi_\lambda])$, and by letting $\widetilde{\lambda} = \iota_2^{-1} \circ g' \circ \iota_1$ we get the desired isomorphism. $\square$

Remark 2.1.27. *In lemmas 2.1.26 and 2.1.25 above one can take $r = r'$ but that would extend the arguments and we will not need this.*

The proof of the following lemma is inspired by the computations that appear in [10] Theorem 5.6, and it is a key input in the proof of theorem 2.3.14.

Lemma 2.1.28 (Unique liftability of isogenies)**.** *Let $\mathscr{T}$ be a trivial $\mathscr{G}$-torsor over* $\mathrm{Spec}(W(R^+))$ *and let $\mathscr{G}_b$ denote the trivial $\mathscr{G}$-torsor endowed with the ϕ-module structure over $\mathcal{Y}^{R^+}_{(0,\infty]}$ given by an element $b \in \mathscr{G}(\mathcal{Y}^{R^+}_{(0,\infty]})$. Let $\Phi : \phi^* \mathscr{T} \to \mathscr{T}$ be an isomorphism defined over* $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$ *and $\lambda : \mathscr{T} \to \mathscr{G}_b$ a ϕ-equivariant isomorphism defined over $B^{R^+}_{[r,\infty]}/[\varpi]$ for some r big enough so that $\xi_{R^\sharp}$ becomes a unit. Then, there is a unique ϕ-equivariant isomorphism $\widetilde{\lambda} : \mathscr{T} \to \mathscr{G}_b$ defined over $\mathcal{Y}^{R^+}_{[r,\infty]}$ such that $\widetilde{\lambda} = \lambda$ in $B^{R^+}_{[r,\infty]}/[\varpi]$.*

Proof. After fixing a trivialization $\iota : \mathscr{T} \to \mathscr{G}$ we may assume, by transport of structure, that $\mathscr{G} = \mathscr{T}$, that Φ is given by an element $\mathscr{G}(W(R^+)[\frac{1}{\xi}])$, and that λ is given by an element $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi])$. We need to find an element $\widetilde{\lambda} \in \mathscr{G}(B^{R^+}_{[r,\infty]})$ reducing to λ and satisfying $\Phi = \widetilde{\lambda}^{-1} \circ b \circ \phi^*(\widetilde{\lambda})$. Choose an arbitrary lift $\lambda_0 \in \mathscr{G}(B^{R^+}_{[r,\infty]})$ of λ, and let $\eta_0 = {\lambda_0}^{-1} \circ b \circ \phi^*(\lambda_0) \circ \Phi^{-1}$. We construct a pair of sequences of maps, $\lambda_i : \mathscr{G} \to \mathscr{G}_b$ and $\eta_i : \mathscr{G} \to \mathscr{G}$ defined recursively as follows:

$$\lambda_{n+1} = \lambda_n \circ \eta_n$$

$$\eta_n = \lambda_n^{-1} \circ b \circ \phi^*(\lambda_n) \circ \Phi^{-1}$$

We make the observation that $\eta_0 = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi])$ and we prove inductively that $\eta_n = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^{p^n}])$. If $g \in \mathscr{G}(B^{R^+}_{[r,\infty]})$ is such that $g = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^{p^n}])$, then $\phi^*(g) = Id$ in

$$\mathscr{G}(B^{R^+}_{[\frac{r}{p},\infty]}/[\varpi^{p^{n+1}}]) \subseteq \mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^{p^{n+1}}]).$$

The induction then follows from the computation:

$$\begin{aligned}
\eta_{n+1} &= \lambda_{n+1}^{-1} \circ b \circ \phi^*(\lambda_{n+1}) \circ \Phi^{-1} && (2.2)\\
&= \eta_n^{-1} \circ \lambda_n^{-1} \circ b \circ \phi^*(\lambda_{n+1}) \circ \Phi^{-1} && (2.3)\\
&= \Phi \circ \phi^*(\lambda_n)^{-1} \circ b^{-1} \circ \lambda_n \circ \lambda_n^{-1} \circ b \circ \phi^*(\lambda_{n+1}) \circ \Phi^{-1} && (2.4)\\
&= \Phi \circ \phi^*(\lambda_n)^{-1} \circ \phi^*(\lambda_{n+1}) \circ \Phi^{-1} && (2.5)\\
&= \Phi \circ \phi^*(\eta_n) \circ \Phi^{-1} && (2.6)
\end{aligned}$$

Since $\phi^*(\eta_n) = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^{p^{n+1}}])$ we also have that $\eta_{n+1} = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^{p^{n+1}}])$.

This let us conclude that η_i converges to Id in $\mathscr{G}(B^{R^+}_{[r,\infty]})$. We define $\widetilde{\lambda} \in \mathscr{G}(B^{R^+}_{[r,\infty]})$ as the limit of the λ_i. Taking limits we deduce the identities $Id = \eta_\infty = \widetilde{\lambda} \circ b \circ \phi^*(\widetilde{\lambda}) \circ \Phi^{-1}$ and $\widetilde{\lambda} = \lambda_0 = \lambda$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi])$ as we needed to show.

Suppose that there are two lifts $\widetilde{\lambda}_i$ of λ with the required properties. We get a ϕ-equivariant automorphism $\widetilde{\lambda}_1 \circ \widetilde{\lambda}_2^{-1}$ of $\mathscr{G}_b$ which we may think of as an element of $g \in \mathscr{G}(B^{R^+}_{[r,\infty]})$ that reduces to the identity in $B^{R^+}_{[r,\infty]}/[\varpi]$. Now, ϕ-equivariance gives $b = g^{-1} \circ b \circ \phi^*(g)$, and since $g = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi])$ then $\phi^*(g) = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^p])$ and we get the identity $b = g^{-1} \circ b \circ Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^p])$. We may proceed inductively to prove that $g = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^{p^n}])$ for every n. Since $B^{R^+}_{[r,\infty]}$ is complete and separated for the $[\varpi]$-adic topology we conclude that $g = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]})$. □

2.2 The specialization map for p-adic Beilinson-Drinfeld Grassmanians

2.2.1 Grassmanians as kimberlites

In the Berkeley notes, Scholze and Weinstein define a p-adic analogue of the Beilison-Drinfeld Grassmanian where the parameter "curve" is given by $\mathbb{Z}_p^\diamond$, or in our case $W(k)^\diamond = \mathrm{Spec}(k)^\diamond \times \mathbb{Z}_p^\diamond$. We will adopt the definition that is the most convenient for studying the specialization map for this object.

Definition 2.2.1. *(See [28] 20.3.1) We let* $\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ *denote the presheaf that assigns to an affinoid perfectoid pair* (R, R^+) *the set:*

$$\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}(R, R^+) = \{(R^\sharp, \iota, f, \mathscr{T}, \psi)\}/\cong$$

Where $(R^\sharp, \iota, f)$ *is an untilt over* $W(k)$ *and* $(\mathscr{T}, \psi)$ *is a lattice with* $\mathscr{G}$*-structure as in definition 2.1.22.*

Whenever $\mathscr{G}$ is reductive over $W(k)$ with quasi-split fibers we fix $\mathfrak{T} \subseteq \mathfrak{B} \subseteq \mathscr{G}$, integrally defined maximal torus and Borel subgroups respectively.

Definition 2.2.2. *Suppose that* $\mathscr{G}$ *is reductive and* $\mu \in X_*^+(\mathfrak{T})$ *is a dominant cocharacter with reflex field* E*. We define* $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E}$ *as the subsheaf of* $\mathrm{Gr}^{\mathscr{G}}_{O_E} := \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond} \times_{W(k)^\diamond} O_E^\diamond$ *that on geometric points evaluates to* $\mathscr{G}$*-lattices* (Ξ, ψ) *whose type is bounded by* μ *in the Bruhat order. We use equation 2.1 to compare* μ_Ξ *with* μ*.*

Recall the following theorem of the Berkeley notes.

Theorem 2.2.3. *(See [28] 20.3.2, 21.2.1) For any parahoric group scheme* $\mathscr{G}$ *over* $W(k)$*, the presheaf* $\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ *is a small* v*-sheaf and ind-proper over* $W(k)^\diamond$*. Moreover, if* $\mathscr{G}$ *is reductive and* $\mu \in X_*(\mathfrak{T})$ *the functor* $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E}$ *is proper and representable in spatial diamonds over* $O_E^\diamond$*. The inclusion of sheaves* $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E} \to \mathrm{Gr}^{\mathscr{G}}_{O_E}$ *is a closed embedding.*

Proposition 2.2.4. *The* v*-sheaf* $\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ *formalizes products of points. In particular, it is* v*-formalizing.*

Proof. Let $\mathrm{Spa}(R, R^+)$ be a product of points and $f : \mathrm{Spa}(R, R^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ a map. By definition, associated to this map we have an untilt $(R^\sharp, \iota, m)$ over $W(k)$ and a $\mathscr{G}$-torsor $\mathscr{T}$ over $\mathcal{Y}^{R^+}_{[0,\infty)}$ together with a trivialization $\psi : \mathscr{T} \to \mathscr{G}$ over $\mathcal{Y}^{R^+}_{[0,\infty)} \setminus V(\xi_{R^\sharp})$ meromorphic along $\xi_{R^\sharp}$. We can use ψ to glue $\mathscr{T}$ and $\mathscr{G}$ along $\mathcal{Y}^{R^+}_{[r,\infty)}$ (for big enough r) and get a $\mathscr{G}$-torsor defined over $\mathcal{Y}_{R^+}$, together with a meromorphic isomorphism over $\mathcal{Y}_{R^+} \setminus V(\xi_{R^\sharp})$ which restricts to the original data. Using corollary 2.1.14, proposition 2.1.20 and the fact that by construction $\mathscr{T}$ is trivial on $Y_{R^+} \setminus V(\xi)$ we can extend $\mathscr{T}$ to a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$ together with a trivialization over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi_{R^\sharp}}])$. We claim that this is enough to define a map $\mathrm{Spd}(R^+, R^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$. Indeed, take a second affinoid perfectoid $\mathrm{Spa}(S, S^+)$ and a map $g : \mathrm{Spa}(S, S^+) \to \mathrm{Spd}(R^+, R^+)$, we want to produce functorially a map $\mathrm{Spa}(S, S^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$. We may construct an untilt $(S^\sharp, \iota, m)$ as in lemma 1.4.11. The map g gives a map $g' : W(R^+) \to W(S^+)$ with $g'(\xi_{R^\sharp}) = \xi_{S^\sharp}$. Basechange along g' gives a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(S^+))$ together with a trivialization over $\mathrm{Spec}(W(S^+)[\frac{1}{g'(\xi)}])$. This restricts to a $\mathscr{G}$-torsor over $\mathcal{Y}^{S^+}_{[0,\infty)}$ and a trivialization over $\mathcal{Y}^{S^+}_{[0,\infty)} \setminus V(g'(\xi))$ that is meromorphic along $g'(\xi)$. This gives our desired natural transformation $\mathrm{Spd}(R^+, R^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$. Clearly the composition $\mathrm{Spa}(R, R^+) \to \mathrm{Spd}(R^+, R^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ agrees with f, so this map is a formalization. ☐

Proposition 2.2.5. *(See [28] Section 20.3) The* v*-sheaf* $\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ *is specializing and formally* p*-adic, and* $(\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond})^{red}$ *is represented by the Witt-vector Grassmanian,* $\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k}$*. Moreover, if* $\mathscr{G}$ *is reductive and* k_E *denotes the residue field of* O_E*, then* $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond}$ *is also formally* p*-adic and* $(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond})^{\mathrm{red}} = \mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E}$*.*

Proof. That $\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ is specializing would follow from proposition 2.2.4, theorem 2.2.3 and proposition 1.3.31 once we establish that it is formally p-adic. We begin by constructing a map $\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k} \to (\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond})^{red}$. Given a map $\mathrm{Spec}(R) \to \mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k}$ we need to produce functorially a map $\mathrm{Spec}(R)^\diamond \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$. The map to $\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k}$ is given by a $\mathscr{G}$-torsor $\mathscr{T}$ over $\mathrm{Spec}(W(R))$ together with a trivialization $\psi : \mathscr{T} \to \mathscr{G}$ over $\mathrm{Spec}(W(R)[\frac{1}{p}])$. Given an affinoid perfectoid $\mathrm{Spa}(S, S^+)$ and a map $f : \mathrm{Spa}(S, S^+) \to \mathrm{Spec}(R)^\diamond$ we need to produce functorially a map $\mathrm{Spa}(S, S^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$. The morphism f induces the ring map $f' : W(R) \to W(S^+)$. We can assign to f the characteristic p untilt and assign the $\mathscr{G}$-bundle

$f'^*\mathscr{T}$ over $\mathcal{Y}^{S^+}_{[0,\infty)}$ with trivialization $f'^*\psi$, using corollary 2.1.14 we see that it is meromoprhic along p. This construction is clearly functorial and gives the desired map.

Now, by Beauville-Laszlo theorem, we may also think of $(f'^*\mathscr{T}, f'^*\psi)$ as a pair (Ξ_S, ψ_S) with Ξ_S a $\mathscr{G}$-torsor over $\mathrm{Spec}(B^+_{dR}(S))$ and $\psi_S : \Xi_S \to \mathscr{G}$ a trivialization over $\mathrm{Spec}(B_{dR}(S))$. Since S is the characteristic p untilt, we have $B^+_{dR}(S) = W(S)$ and $B_{dR}(S) = W(S)[\frac{1}{p}]$. In this case, (Ξ_S, ψ_S) is simply the pullback of $(\mathscr{T}, \psi)$ along $W(R) \to W(S)$. In particular, if $\mathscr{G}$ is reductive and the type of $(\mathscr{T}, \psi)$ is pointwise bounded by $\mu \in X^+_*(\mathfrak{T})$, then the type of (Ξ_S, ψ_S) is also pointwise bounded by μ. This last observation gives us a commutative diagram which we will use later in the proof:

$$\begin{tikzcd} (\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E})^\diamond \arrow[r] \arrow[d] & (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\diamond \arrow[d] \\ \mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond} \arrow[r] & \mathrm{Gr}^{\mathscr{G}}_{O_E^\diamond} \end{tikzcd} \tag{2.7}$$

For the moment, let us move on and prove explicitly that for any (R, R^+) we have bijection of sets:

$$(\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k})^\diamond(R, R^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond} \times_{W(k)^\diamond} \mathrm{Spec}(k)^\diamond(R, R^+).$$

By lemma 1.3.34, this would give that $\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond} \to W(k)^\diamond$ is formally adic and would prove $(\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond})^{\mathrm{red}} = \mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k}$. To prove injectivity, suppose we are given two maps $g_i : \mathrm{Spa}(R, R^+) \to (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k})^\diamond$ in characteristic p whose composition agree. It is enough to prove that $g_1 = g_2$ after taking a v-cover of $\mathrm{Spa}(R, R^+)$. Locally for the v-topology we can assume that both maps factor through morphisms $g'_i : \mathrm{Spec}(R^+) \to \mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k}$ given by pairs $(\mathscr{T}_i, \psi_i)$. Since the compositions agree, these pairs become isomorphic over $\mathcal{Y}^{R^+}_{[0,\infty)}$. Since both $\mathscr{T}_i$ are defined over $\mathrm{Spec}(W(R^+))$ and the pullback functor $j^* : Vec_{\mathrm{Spec}(W(R^+))} \to Vec_{Y_{R^+}}$ of theorem 2.1.15 is fullyfaithful (even when it is not an equivalence), we can conclude that $g'_1 = g'_2$.

To prove surjectivity take a map $f : \mathrm{Spa}(R, R^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond} \times_{W(k)^\diamond} \mathrm{Spec}(k)^\diamond$. Since surjectivity can be checked v-locally we can assume that $\mathrm{Spa}(R, R^+)$ is a product of points. By the proof of proposition 2.2.4 we get a $\mathscr{G}$-torsor $\mathscr{T}$ over $\mathrm{Spec}(W(R^+))$ and a trivialization over $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$ which gives a map $\mathrm{Spec}(R^+) \to \mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k}$ and consequently the required lift to our original map $\mathrm{Spa}(R, R^+) \to (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k})^\diamond$.

The second claim will follow from proving that the commutative diagram 2.7 is Cartesian. Indeed, that would prove that the closed immersion $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond} \to \mathrm{Gr}^{\mathscr{G}}_{O_E^\diamond}$ is formally adic, and that $(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond})^{\mathrm{red}} = \mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E}$. All of the morphisms in diagram 2.7 are closed immersions, so it is enough to check that the diagram is Cartesian on (C, O_C)-points. Suppose we have a map $p : \mathrm{Spa}(C, O_C) \to \mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond} \cap (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\diamond$, this factors through a map $p' : \mathrm{Spec}(O_C)^\diamond \to (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\diamond$. The map is given by a lattice $(\mathscr{T}_{O_C}, \psi_{O_C})$ over $W(O_C)$ whose basechange to $W(C)$ is bounded by μ. By proposition 1.3.17 p' is coming from a map $\mathrm{Spec}(O_C) \to \mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E}$ for which the composition $\mathrm{Spec}(C) \to \mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E}$ factors through $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E}$. Since $\mathrm{Spec}(C)$ is Zariski dense in $\mathrm{Spec}(O_C)$, p' factors through $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E}$ and p factors through $(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E})^\diamond$. $\square$

Remark 2.2.6. *We want to remark that although the v-sheaf $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond}$ is formally p-adic, the similarly defined moduli $\mathrm{Gr}^{\mathscr{G},\mu}_{O_E^\diamond}$ of B^+_{dR}-lattices of type exactly μ is not formally p-adic if μ is not miniscule. Indeed, in that case there are points $p : \mathrm{Spa}(C, O_C) \to (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\diamond$ with formalization $m : \mathrm{Spd}(O_C, O_C) \to (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\diamond$ such that p factors through $\mathrm{Gr}^{\mathscr{G},\mu}_{O_E^\diamond}$ but m doesn't. This implies that $\mathrm{Gr}^{\mathscr{G},\mu}_{O_E^\diamond} \cap (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\diamond$ is not v-formalizing and consequently not represented by a scheme-theoretic v-sheaf. Similarly, this proves that if μ is not miniscule $\mathrm{Gr}^{\mathscr{G},\mu}_{O_E^\diamond} \cap (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\diamond$ properly contains $(\mathrm{Gr}^{\mathscr{G},\mu}_{\mathcal{W},k_E})^\diamond$ as open subsheaves of $(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E})^\diamond$.*

Corollary 2.2.7. *If $\mathscr{G}$ is reductive over $W(k)$ and $\mu \in X^+_*(\mathfrak{T})$, then the v-sheaf $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E}$ is a p-adic kimberlite.*

Proof. We know $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond}$ is separated by theorem 2.2.3. Since $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond}$ is formally p-adic, by proposition 1.3.31 it is also formally separated. The map $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond} \to \mathrm{Gr}^{\mathscr{G}}_{O_E^\diamond}$ is a formally closed immersion and $\mathrm{Gr}^{\mathscr{G}}_{O_E^\diamond}$ is specializing so by propositon 1.4.30 $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond}$ is also specializing. The morphism $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond} \to O_E^\diamond$ is formally adic which implies that the adjunction morphism $((\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond})^{red})^\diamond \to \mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond}$ is a closed embedding. By proposition 2.2.5 $(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond})^{red} = \mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E}$, which is represented by a scheme (See [4] Theorem 8.3). We also have $(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond})^{an} = \mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond} \times_{O_E^\diamond} E^\diamond$, which is represented by a spatial diamond by theorem 2.2.3. $\square$

The purpose of the rest of this section is to upgrade corollary 2.2.7 and prove that $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E}$ is Orapian and that its p-adic tubular neighborhoods are connected.

2.2.2 Twisted loop sheaves

We begin by discussing two constructions that are related to twisted loop sheaves and that we will use below. Given an affine scheme $X = \mathrm{Spec}(A)$ of finite type over $W(k)$ with structure morphism $\pi : X \to \mathrm{Spec}(W(k))$, we can associate to it two v-sheaves over $W(k)^\diamond$ which we will denote by $X(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+})$. Here $X(\mathcal{O}^\sharp) : \mathrm{Perf_k} \to \mathrm{Sets}$ is defined to be the presheaf that assigns to $\mathrm{Spa}(R, R^+)$ the set of triples $(R^\sharp, \iota, f)$ where $(R^\sharp, \iota)$ is an untilt and $f \in Hom_{W(k)}(A, R^\sharp)$ is a $W(k)$-algebra homomorphism. On the other hand, $X(\mathcal{O}^{\sharp,+})$ assigns triples $(R^\sharp, \iota, f)$ with $f \in Hom_{W(k)}(A, R^{\sharp,+})$. Notice that we have an open inclusion of v-sheaves $X(\mathcal{O}^{\sharp,+}) \subseteq X(\mathcal{O}^\sharp)$. Both of these functors glue to give a construction that is now defined for every scheme X locally of finite type over $\mathrm{Spec}(W(k))$. Visibly, these two constructions are very related to the functor $\diamond : \mathrm{PreAd}_{W(k)} \to \widetilde{\mathrm{Perf}}$, we make this explicit below.

We still assume $X = \mathrm{Spec}(A)$, and we let X_p denote the p-adic completion of X. Now, X_p is a p-adic Noetherian formal scheme that we may regard it as an affinoid adic space $\mathrm{Spa}(A_p, A_p)$. Since for any untilt of R the ring $R^{\sharp,+}$ is p-adically complete, we have an identification $X_p^\diamond = X(\mathcal{O}^{\sharp,+})$. Also, if $Y \to X$ is an open cover of the form $Y = \coprod_{i=1}^n \mathrm{Spec}(A[\frac{1}{f_i}])$ with $f_i \in A$, then $Y_p \to X_p$ is also an open cover of adic spaces. Indeed, $\mathrm{Spec}(A[\frac{1}{f_i}])_p$ corresponds to the open subset of X_p where $1 \leq |f_i|$.

The construction of $X(\mathcal{O}^\sharp)$ is a little more elaborate. Given an adic space S (thought of as a triple $(|S|, \mathcal{O}_S, \{v_s : s \in |S|\})$ in Huber's category $\mathscr{V}$ see [13]), we let S^H denote the topologically ringed space $(|S|, \mathcal{O}_S)$ that is obtained from S by forgetting the last entry of data. Suppose we are given a morphism of schemes $f : X \to Y$ that is locally of finite type and a morphism $g : S^H \to Y$ of locally ringed spaces where S is an adic space for which every point $s \in S$ has an affinoid open neighborhood with Noetherian ring of definition. In [13] (proposition 3.8) Huber constructs an adic space "$S \times_Y X$" together with a map of adic spaces p_1 : "$S \times_Y X$" $\to S$ and a map of locally ringed spaces p_2 : ("$S \times_Y X$")$^H \to X$ with the following universal property. If T is an adic space, $\pi_1 : T \to S$ is a map of adic spaces and $\pi_2 : T^H \to X$ is a map of locally ringed spaces such that $f \circ \pi_1 = g \circ \pi_2^H$, then there is a unique map $\pi : T \to$ "$S \times_Y X$" such that $p_1 \circ \pi = \pi_1$ and $p_2 \circ \pi^H = \pi_2$.

With this adic space at hand we can let $Y = \mathrm{Spec}(W(k))$, $S = \mathrm{Spa}(W(k), W(k))$ and X the finite type scheme over Y that we started with and define X^{ad} as ("$S \times_Y X$"). With this definition we have $X(\mathcal{O}^\sharp) = (X^{ad})^\diamond$. Moreover, if $X = \mathrm{Spec}(A)$ and $X_f = \mathrm{Spec}(A[\frac{1}{f}])$ for $f \in A$ we can see from the universal property that X^{ad}_f is the open locus of X^{ad} where $f \neq 0$.

The advantage of these two-step constructions is that it makes it clear, by proposition 1.1.23, that $X(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+})$ are small v-sheaves and it also clarifies the glueing process for $X(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+})$ when X is an arbitrary finite type scheme.

We will later on use the following facts about these constructions:

Proposition 2.2.8. *If $X \to \mathrm{Spec}(W(k))$ is a proper map of schemes, then the natural map $X(\mathcal{O}^{\sharp,+}) \to X(\mathcal{O}^\sharp)$ is an isomorphism.*

Proof. By [13] remark 4.6.(iv).d we have an isomorphism of adic spaces $X_p \to X^{ad}$ where X_p and X^{ad} are as above. Since $X(\mathcal{O}^\sharp) = (X^{ad})^\diamond$ and $X(\mathcal{O}^{\sharp,+}) = X_p^\diamond$ the conclusion holds. □

Proposition 2.2.9. *Suppose that X and Y are qcqs finite type schemes over* $\mathrm{Spec}(W(k))$ *and that $X \to Y$ is universally subtrusive as in definition 1.3.4, then $X(\mathcal{O}^\sharp) \to Y(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+}) \to Y(\mathcal{O}^{\sharp,+})$ are surjective maps of v-sheaves.*

Proof. Replacing Y by an open cover we may assume that $Y = \mathrm{Spec}(A)$ for a ring A of finite type over $W(k)$. By [23] theorem 3.12 we may assume that $X \to Y$ factors as $X \to Y' \to Y$ with $Y' \to Y$ proper and surjective and $X \to Y'$ a quasi-compact open covering. Since open covers of adic spaces induce surjective maps of v-sheaves we only need to deal with the proper case. Moreover, by Chow's lemma ([29] Tag 0200) we may assume $Y' \to Y$ is projective. We claim that both maps of v-sheaves $Y'(\mathcal{O}^\sharp) \to Y(\mathcal{O}^\sharp)$ and $Y'(\mathcal{O}^{\sharp,+}) \to Y(\mathcal{O}^{\sharp,+})$ are quasi-compact. Indeed, they are both the composition of a closed immersion and the first projection map of $(\mathbb{P}^n_{W(k)})^\diamond \times_{W(k)^\diamond} Y(\mathcal{O}^\sharp)$ and $(\mathbb{P}^n_{W(k)})^\diamond \times_{W(k)^\diamond} Y(\mathcal{O}^{\sharp,+})$ respectively. By ([26] 12.11) we may check surjectivity at a topological level. Take an algebraically closed non-Archimedean field C with open and bounded valuation subring $C^+ \subseteq C$, and consider ring maps $r^* : A \to C$ and $s^* : A \to C^+$ representing (C, C^+)-valued points in $r \in Y(\mathcal{O}^\sharp)$ and $s \in Y(\mathcal{O}^{\sharp,+})$ respectively. Since $Y' \to Y$ is proper and surjective the map of schemes $\mathrm{Spec}(C) \times_Y Y' \to \mathrm{Spec}(C)$ admits a section which induces a lift of r to $Y'(\mathcal{O}^\sharp)$. Analogously, the composition $s^* : A \to C^+ \to C$ admits a lift to $Y'(\mathcal{O}^\sharp)$ and by the valuative criterion of properness such lift also defines an element of $Y'(\mathcal{O}^{\sharp,+})$ lifting s.

□

Perhaps unsurprisingly, for a scheme X over $\mathrm{Spec}(W(k))$ the reduction functor applied to $X(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+})$ give the same scheme-theoretic v-sheaf.

Proposition 2.2.10. *Given X a scheme locally of finite type over* $\mathrm{Spec}(W(k))$ *we have identifications in* $\widetilde{\mathrm{SchPerf}}$*:*

$$X(\mathcal{O}^\sharp)^{\mathrm{red}} \cong X \times_{W(k)} \mathrm{Spec}(k) \cong X(\mathcal{O}^{\sharp,+})^{\mathrm{red}}$$

Proof. Both identifications follow from proposition 1.3.20. By the construction of X_p as a p-adic completion in the case of $X(\mathcal{O}^{\sharp,+})$, and by the universal property of X^{ad} in the category of adic spaces in the case of $X(\mathcal{O}^\sharp)$.

□

We move on to discuss twisted loop sheaves. For the rest of this section we let C be an algebraically closed non-Archimedean field over k with ring of integers O_C and residue field k_C. We fix a characteristic 0 untilt $C^\sharp$ and we pick $\xi \in W(O_C)$ a generator for the kernel of $W(O_C) \to O_{C^\sharp}$. The choice of untilt determines a unique map $O_C^\diamond \to \mathbb{Z}_p^\diamond$ that we also fix throughout this section.

Definition 2.2.11.
1. *We let* $W^+(\mathcal{O}) : \mathrm{Perf}_{O_C^\diamond} \to \mathrm{Sets}$ *denote the presheaf that assigns to* $\mathrm{Spa}(R, R^+) \to O_C^\diamond$ *the ring $W(R^+)$.*
2. *We let* $B^+_{dR}(\mathcal{O}^\sharp) : \mathrm{Perf}_{O_C^\diamond} \to \mathrm{Sets}$ *denote the presheaf that assigns to* $\mathrm{Spa}(R, R^+) \to O_C^\diamond$ *the ring $B^+_{dR}(R^\sharp)$ where $R^\sharp$ is the untilt associated with our fixed choice of $\xi \in W(O_C)$.*
3. *We let* $W(\mathcal{O}) : \mathrm{Perf}_{O_C^\diamond} \to \mathrm{Sets}$ *denote the presheaf that assigns to* $\mathrm{Spa}(R, R^+) \to O_C^\diamond$ *the ring $W(R^+)[\frac{1}{\xi}]$.*
4. *We let* $B_{dR}(\mathcal{O}^\sharp) : \mathrm{Perf}_{O_C^\diamond} \to \mathrm{Sets}$ *denote the presheaf that assigns to* $\mathrm{Spa}(R, R^+) \to O_C^\diamond$ *the ring $B_{dR}(R^\sharp) := B^+_{dR}(R^\sharp)[\frac{1}{\xi}]$.*

Proposition 2.2.12. *The presheaves $W^+(\mathcal{O})$, $B^+_{dR}(\mathcal{O}^\sharp)$, $W(\mathcal{O})$, and $B_{dR}(\mathcal{O}^\sharp)$ are small v-sheaves.*

Proof. Ignoring the ring structure, we see that the Teichműller expansion of $W(R^+)$ gives a bijection to $(R^+)^{\mathbb{N}}$ which is a small v-sheaf. We can prove inductively that $B^+_{dR}(\mathcal{O}^\sharp)/\xi^n$ is a small v-sheaf. Indeed, it sits in the exact sequence of presheaves:

$$0 \to B^+_{dR}(\mathcal{O}^\sharp)/\xi^{n-1} \xrightarrow{\cdot\xi} B^+_{dR}(\mathcal{O}^\sharp)/\xi^n \to \mathcal{O}^\sharp \to 0$$

By induction the leftmost term is a small v-sheaf and we already know that the rightmost term is a small v-sheaf. A diagram chase gives that the middle one is also a small v-sheaf. Since $B^+_{dR}(\mathcal{O}^\sharp) = \varprojlim_n B^+_{dR}(\mathcal{O}^\sharp)/\xi^n$ this other one is also a small v-sheaf.

Now $W(\mathcal{O}) = \varinjlim(W^+(\mathcal{O}) \xrightarrow{\xi} W^+(\mathcal{O}) \xrightarrow{\xi} \dots)$ and $B_{dR}(\mathcal{O}^\sharp) = \varinjlim(B^+_{dR}(\mathcal{O}^\sharp) \xrightarrow{\xi} B^+_{dR}(\mathcal{O}^\sharp) \xrightarrow{\xi} \dots)$. Since these are filtered colimit of sheaves they define small v-sheaves as well. □

Notice that $W^+(\mathcal{O})$ and $B^+_{dR}(\mathcal{O}^\sharp)$ come equipped with reduction maps

$$red : W^+(\mathcal{O}) \to W^+(\mathcal{O})/\xi = \mathcal{O}^{\sharp,+}$$

and

$$red : B^+_{dR}(\mathcal{O}^\sharp) \to B^+_{dR}(\mathcal{O}^\sharp)/\xi = \mathcal{O}^\sharp.$$

Definition 2.2.13. *Let H be a finite type affine scheme over $\mathrm{Spec}(W(k)[t,t^{-1}])$, and let $(\mathcal{H},\rho)$ be a finite type affine scheme over $\mathrm{Spec}(W(k)[t])$ together with an isomorphism $\rho : \mathcal{H} \times_{\mathbb{A}^1_{W(k)}} \mathbb{G}_m \to H$. To this setup we associate the following presheaves over $O_C^\diamond$:*

1. *$W^+\mathcal{H}$ assigns to $\mathrm{Spa}(R,R^+) \to O_C^\diamond$ the set of sections $\mathrm{Spec}(W(R^+)) \to \mathcal{H} \times_{\mathbb{A}^1_{W(k)}} \mathrm{Spec}(W(R^+))$.*
2. *WH assigns to $\mathrm{Spa}(R,R^+) \to O_C^\diamond$ the set of sections $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}]) \to H \times_{\mathbb{G}_{m,W(k)}} \mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$.*
3. *$L^+\mathcal{H}$ assigns to $\mathrm{Spa}(R,R^+) \to O_C^\diamond$ the set of sections $\mathrm{Spec}(B^+_{dR}(R^\sharp)) \to \mathcal{H} \times_{\mathbb{A}^1_{W(k)}} \mathrm{Spec}(B^+_{dR}(R^\sharp))$.*
4. *LH assigns to $\mathrm{Spa}(R,R^+) \to O_C^\diamond$ the set of sections $\mathrm{Spec}(B_{dR}(R^\sharp)) \to H \times_{\mathbb{G}_{m,W(k)}} \mathrm{Spec}(B_{dR}(R^\sharp))$.*

where the base change in all cases is given by the usual map on $W(k)$ deduced from the composition $k \to O_C \to R^+$ and given by $t \mapsto \xi$.

Proposition 2.2.14. *With the notation as above $W^+\mathcal{H}$, WH, $L^+\mathcal{H}$ and LH are small v-sheaves.*

Proof. Let $\mathcal{R} \in \{W^+(\mathcal{O}), W(\mathcal{O}), B^+_{dR}(\mathcal{O}^\sharp), B_{dR}(\mathcal{O}^\sharp)\}$ denote one of the sheaves of rings of definition 2.2.11. Suppose that $\mathcal{H} = \mathrm{Spec}(W(k)[t][x_1, \dots x_n]/(f_1(t,\overline{x}), \dots, f_m(t,\overline{x}))$. Notice there is a map of v-sheaves $O_C^\diamond \xrightarrow{0} \mathcal{R}$ corresponding to the constant 0 section. Consider the following basechange diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & O_C^\diamond \arrow[d, "0"] \\ \mathcal{R}^n \arrow[r, "F"] & \mathcal{R}^m \end{tikzcd}$$

Where $F(\overline{r}) = (f_1(\xi,\overline{r}), \dots, f_m(\xi,\overline{r}))$. Whenever $\mathcal{R}$ is $W^+(\mathcal{O})$, $W(\mathcal{O})$, $B^+_{dR}(\mathcal{O}^\sharp)$, or $B_{dR}(\mathcal{O}^\sharp)$ then X is isomorphic as presheaves to $W^+\mathcal{H}$, WH, $L^+\mathcal{H}$, or LH respectively. From this diagram, it is clear that X is a small v-sheaf. □

In our setup, ρ will induce maps of v-sheaves $L^+\mathcal{H} \xrightarrow{\rho} LH$ and $W^+\mathcal{H} \xrightarrow{\rho} WH$. We get the following diagrams of inclusions:

$$\begin{tikzcd}
 & W(\mathcal{O}) \arrow[dr, hook] & \\
W^+(\mathcal{O}) \arrow[ur, hook] \arrow[dr, hook] & & B_{dR}(\mathcal{O}^\sharp) \\
 & B_{dR}^+(\mathcal{O}^\sharp) \arrow[ur, hook] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & WH \arrow[dr] & \\
W^+\mathcal{H} \arrow[ur, hook, "\rho"] \arrow[dr] & & LH \\
 & L^+\mathcal{H} \arrow[ur, hook, "\rho"] &
\end{tikzcd}$$

Moreover, if we let $\overline{\mathcal{H}}$ denote the basechange $\mathcal{H} \times_{\mathbb{A}^1_{W(k)}} \mathrm{Spec}(W(k))$ at $t = 0$ we get reduction morphisms $W^+\mathcal{H} \to \overline{\mathcal{H}}(\mathcal{O}^{\sharp,+})_{O_C^\diamond}$ and $L^+\mathcal{H} \to \overline{\mathcal{H}}(\mathcal{O}^\sharp)_{O_C^\diamond}$.

Proposition 2.2.15. *If $\mathcal{H}$ is smooth over $\mathrm{Spec}(W(k)[t])$ then the reduction maps $W^+\mathcal{H} \to \overline{\mathcal{H}}(\mathcal{O}^{\sharp,+})_{O_C^\diamond}$ and $L^+\mathcal{H} \to \overline{\mathcal{H}}(\mathcal{O}^\sharp)_{O_C^\diamond}$ are surjective maps of v-sheaves.*

Proof. We claim that the map is surjective even at the level of presheaves. The (R, R^+)-valued points of $\overline{\mathcal{H}}(\mathcal{O}^\sharp)$ and $\overline{\mathcal{H}}(\mathcal{O}^{\sharp,+})$ can be seen as maps $\mathrm{Spec}(R^\sharp) \to \mathcal{H}_{B_{dR}(R^\sharp)}$ and $\mathrm{Spec}(R^{\sharp,+}) \to \mathcal{H}_{W(R^+)}$ whose composition with the projections to $\mathrm{Spec}(B_{dR}(R^\sharp))$ and $\mathrm{Spec}(W(R^+))$ are the usual closed embeddings. By smoothness of $\mathcal{H}$, for any $n \in \mathbb{N}$ the maps can be lifted to maps $\mathrm{Spec}(B_{dR}(R^\sharp)/\xi^n) \to \mathcal{H}_{B_{dR}(R^\sharp)}$ and $\mathrm{Spec}(W(R^+)/\xi^n) \to \mathcal{H}_{W(R^+)}$ respectively. Since $\mathcal{H}$ is an affine scheme and since both $B_{dR}(R^\sharp)$ and $W(R^+)$ are (ξ)-adically complete we may pass to the inverse limit by choosing compatible lifts. □

Definition 2.2.16.
1. *With the setup as above, consider the ring k_C with the discrete topology, we let $\mathcal{W}^+_{\mathrm{red}}\mathcal{H} \in \widetilde{\mathrm{SchPerf}}_{k_C}$ be the scheme-theoretic v-sheaf that assigns to $\mathrm{Spec}(R) \in \mathrm{PCAlg}^{op}_{k_C}$ sections $\mathrm{Spec}(W(R)) \to \mathcal{H} \times_{\mathbb{A}^1_{W(k)}} \mathrm{Spec}(W(R))$ where the basechange is given by $t \mapsto p$.*
2. *We let $\mathcal{W}^+_{\mathrm{red}}(\mathcal{O}) : \mathrm{PCAlg}^{op}_{k_C} \to \mathrm{Sets}$ denote sheaf that sends $\mathrm{Spec}(R)$ to $W(R)$. This sheaf can also be expressed as $\mathcal{W}^+_{\mathrm{red}}\mathbb{A}^1_{W(k)[t]}$.*

Remark 2.2.17. *The scheme-theoretic v-sheaves $\mathcal{W}^+_{\mathrm{red}}\mathcal{H}$ of definition 2.2.16 are the v-sheaves that Zhu calls p-adic jet spaces in [30] 1.1.1. These sheaves are represented by perfect affine schemes.*

Proposition 2.2.18. *With the notation as above the v-sheaf $W^+\mathcal{H}$ is a p-adic kimberlite and $(W^+\mathcal{H})^{\mathrm{red}} = (\mathcal{W}^+_{\mathrm{red}}\mathcal{H})$.*

Proof. Observe that $W^+(\mathcal{O})$ is represented by $\mathrm{Spd}(O_C\langle T_n\rangle_{n\in\mathbb{N}}, O_C\langle T_n\rangle_{n\in\mathbb{N}})$, by proposition 1.4.22 and proposition 1.3.25 $W^+(\mathcal{O})$ is a p-adic kimberlite. Moreover, by proposition 1.3.19 $W^+(\mathcal{O})^{\mathrm{red}}$ is represented by $\mathrm{Spec}(k_C[T_n]_{n\in\mathbb{N}})$ which is $\mathcal{W}^+_{\mathrm{red}}(\mathcal{O})$. Lets move on to the general case, recall from the proof of proposition 2.2.14 that if $\mathcal{H} = \mathrm{Spec}(A)$ is presented as $A = W(k)[t][\overline{x}]/I$ with $I = (f_1(t, \overline{x}), \ldots, f_m(t, \overline{x}))$. Then $W^+\mathcal{H}$ fits in the commutative diagram with Cartesian square:

$$\begin{tikzcd}
W^+\mathcal{H} \arrow[r] \arrow[d] & O_C^\diamond \arrow[d, "0"] \arrow[dr, "id"] & \\
W^+(\mathcal{O})^n \arrow[r, "F"] & W^+(\mathcal{O})^m \arrow[r] & O_C^\diamond
\end{tikzcd}$$

We claim that all of these maps are formally adic, and in particular $W^+\mathcal{H}$ is formally p-adic. This follows from the fact that formal adicness is stable under basechange, that it has the 2-out-of-3 property and that, as we proved above, $W^+(\mathcal{O}) \to O_C^\diamond$ is formally adic. Since $W^+(\mathcal{O})$ is separated over $O_C^\diamond$ the section $O_C^\diamond \xrightarrow{0} W^+(\mathcal{O})^m$ is a formally adic closed immersion. We can conclude that $W^+\mathcal{H}$ is a p-adic kimberlite by using lemma 1.4.30. Finally, since we can basechange by $\mathrm{Spec}(k_C)^\diamond \to \mathrm{Spd}(O_C, O_C)$ to compute reductions we get the following Cartesian diagram:

$$\begin{tikzcd} W^+\mathcal{H}^{\mathrm{red}} \arrow[r] \arrow[d] & \mathrm{Spec}(k_C) \arrow[d, "0"] \\ \mathcal{W}^+_{\mathrm{red}}(\mathcal{O})^n \arrow[r, "F"] & \mathcal{W}^+_{\mathrm{red}}(\mathcal{O})^m \end{tikzcd}$$

which gives the isomorphism $W^+\mathcal{H}^{\mathrm{red}} = \mathcal{W}^+_{\mathrm{red}}\mathcal{H}$.

□

2.2.3 Demazure kimberlites

In this subsection we use twisted loop sheaves to construct a family of kimberlites that will allow us to understand how the specialization map for the p-adic Beilinson-Drinfeld Grassmanians behave. We change the setup a little bit and fix some notation first:

1. Let H be a split reductive group over $W(k)$, let $T \subseteq B \subseteq H$ a choice of maximal split torus and a Borel respectively.
2. Let $(X^*, \Phi, X_*, \Phi^\vee)$ be the root datum associated to (H, T).
3. We let $\langle \cdot, \cdot \rangle : X^* \times X_* \to \mathbb{Z}$ denote perfect pairing between roots and coroots.
4. Φ^+ the positive roots associated to B.
5. N the normalizer of T in H.
6. $W = N/T$ the Weyl group of H.
7. We let $\mathcal{A} = \mathcal{A}(H, T)$ denote $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
8. We let $\Psi = \{\alpha + n \mid \alpha \in \Phi, n \in \mathbb{Z}\}$ denote the set of affine functionals on $\mathcal{A}$ coming from the natural perfect pairing $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$. We call them affine roots.
9. Given a point $q \in \mathcal{A}$ we let $\Phi_q = \{\alpha \in \Phi \mid \alpha(q) \in \mathbb{Z}\}$ this is clearly a closed sub-root system. We let M_q be the the generalized Levi subgroup of H containing T with root datum given by $(X^*, \Phi_q, X_*, \Phi_q^\vee)$.
10. Ψ defines a hyperplane structure on $\mathcal{A}$, and for any point $q \in \mathcal{A}$ we can associate a polysimplicial closed region of $\mathcal{A}$ that we will denote by F_q. In case H is semisimple this region is bounded and forms a polytope. We let o denote the vertex associated to the origin in $\mathcal{A}$ and $\mathcal{C}$ the unique alcove containing o and contained in the Bruhat chamber associated to B.
11. We denote by $\mathbb{S}$ the set of reflexions along the walls of $\mathcal{C}$, we let W^{aff} the the affine Weyl group generated by $\mathbb{S}$. Given any facet $\mathcal{F} \subseteq \mathcal{C}$ we let $\mathbb{S}_{\mathcal{F}}$ be the subset of elements of $\mathbb{S}$ fixing $\mathcal{F}$ and we let $W_{\mathcal{F}}$ be the subgroup of W^{aff} generated by $\mathbb{S}_{\mathcal{F}}$.
12. We let $\tilde{W}$ denote the Iwahori-Weyl group of H. Recall that $W^{aff} \subseteq \tilde{W}$ and that if we let $\Omega_H = \pi_1(H^{der})$ we have a decomposition $\tilde{W} = W^{aff} \rtimes \Omega_H$.

Fix a point $q \in \mathcal{A}$. In ([21] §3) Pappas and Zhu use Bruhat-Tits theory and dilatation techniques to construct smooth affine algebraic groups $\mathcal{H}_q$ over $\mathrm{Spec}(W(k)[t])$ together with an isomorphism ρ from $\mathcal{H}_q \times_{W(k)[t]} \mathrm{Spec}(W(k)[t, t^{-1}])$ to $H \times_{W(k)} \mathrm{Spec}(W(k)[t, t^{-1}])$ with the following list of properties:

a) For any discrete valuation ring V and a map $W(k)[t] \to V$ given by $t \mapsto \pi$ with $\pi \in V$ a uniformizer, the basechange $\mathcal{H}_q \times_{\mathbb{A}^1_{W(k)}} \mathrm{Spec}(V)$ is the parahoric group scheme associated to $q \in \mathcal{A}(H, V[\frac{1}{\pi}])$ by Bruhat-Tits theory under the identification $\mathcal{A}(H, V[\frac{1}{\pi}]) = X^*(T_{V[\frac{1}{\pi}]}) \otimes \mathbb{R} = X^*(T) \otimes \mathbb{R}$.

b) For any root $\alpha \in \Phi$ there are smooth connected closed subgroups $\mathcal{U}_\alpha^q \subseteq \mathcal{H}_q$ (respectively $\mathcal{T} \subseteq \mathcal{H}_q$) extending the usual root subgroup $U_\alpha \subseteq H$ (respectively extending the torus $T \subseteq H$). Over $W(k)[t]$, the groups $\mathcal{U}_\alpha^q$ are isomorphic to $\mathbb{G}_a$ and $\mathcal{T}$ is isomorphic to $\mathbb{G}_m^n$ for some n.

c) There is an open cell decomposition:

$$\mathcal{V}_q := \prod_{\alpha\in\Phi^-} \mathcal{U}_\alpha^q \times \mathcal{T} \times \prod_{\alpha\in\Phi^+} \mathcal{U}_\alpha^q \to \mathcal{H}_q.$$

This map forms an open embedding onto a fiberwise Zariski-dense neighborhood of the identity section.

d) The group multiplication map $\mathcal{V}_q \times \mathcal{V}_q \xrightarrow{\mu} \mathcal{H}_q$ is smooth and surjective.

e) The basechange $\overline{\mathcal{H}}_q := \mathcal{H}_q \times_{\mathbb{A}^1_{W(k)}} \operatorname{Spec}(W(k))$ along $t = 0$ supports a split reductive quotient $\overline{\mathcal{H}}_q^{Red}$ over $W(k)$ with root datum canonically identified with $(X^*, \Phi_q, X_*, \Phi_q^\vee)$. In particular we can identify M_q with $\overline{\mathcal{H}}_q^{Red}$.

f) If $\alpha \in \Phi_q$ the composition $\overline{\mathcal{U}}_\alpha^q \to \overline{\mathcal{H}}_q^{Red}$ at $t = 0$ defines an isomorphism onto the root group of $\overline{\mathcal{H}}_q^{Red}$ corresponding to α. On the other hand, if $\alpha \in \Phi \setminus \Phi_q$ then the composition $\overline{\mathcal{U}}_\alpha^q \to \overline{\mathcal{H}}_q^{Red}$ factors through the identity section.

g) We have a commutative diagram of open cell decompostion:

$$\begin{tikzcd}
\prod_{\alpha\in\Phi^-} \overline{\mathcal{U}}_\alpha \times \overline{\mathcal{T}} \times \prod_{\alpha\in\Phi^+} \overline{\mathcal{U}}_\alpha \arrow[r, "\pi"] \arrow[d, "\mu"] & \prod_{\alpha\in\Phi_q^-} \overline{\mathcal{U}}_\alpha \times \overline{\mathcal{T}} \times \prod_{\alpha\in\Phi_q^+} \overline{\mathcal{U}}_\alpha \arrow[d, "\mu"] \\
\overline{\mathcal{H}}_q \arrow[r] & \overline{\mathcal{H}}_q^{Red}
\end{tikzcd}$$

where π is the projection map.

If we are given two points $q_1, q_2 \in \mathcal{A}$ such that $F_{q_1} \subseteq F_{q_2}$ we also get a map of algebraic groups $f : \mathcal{H}_{q_2} \to \mathcal{H}_{q_1}$. This map has the following properties:

a) $\rho_1 \circ f = \rho_2$ over $W(k)[t, t^{-1}]$.

b) The composition $\overline{\mathcal{H}}_{q_2} \to \overline{\mathcal{H}}_{q_1}^{Red}$ surjects onto the parabolic subgroup of $\overline{\mathcal{H}}_{q_1}^{Red}$ associated to the closed sub-root system given by $\Phi q_1.q_2 := \{\alpha \in \Phi_{q_1} \mid \lfloor \alpha(q_2) \rfloor = \alpha(q_1)\}$. Moreover, the kernel of this map is fiberwise a vector group.

We are now prepared to define "parahoric" versions of the positive loop groups which we will use to define Demazure kimberlites.

Definition 2.2.19. *1. We define the loop group LH to be as in definition 2.2.13 when we consider H as a scheme over $W(k)[t, t^{-1}]$ by taking the appropriate basechage.*

2. Given a point $q \in \mathcal{A}$ define the parahoric loop group to be $L^+\mathcal{H}_q$ as in definition 2.2.13.

3. Associated to the same point we also define the formal parahoric loop group to be $W^+\mathcal{H}_q$.

Notice that we have injective maps of v-sheaves $W^+\mathcal{H}_q \subseteq L^+\mathcal{H}_q \overset{\rho}{\subseteq} LH$.

Proposition 2.2.20. *With the notation as above, for any point $q \in \mathcal{A}$ we have surjective morphisms of v-sheaves in groups:*

$$L^+\mathcal{H}_q \to \overline{\mathcal{H}}_q^{Red}(\mathcal{O}_X^\sharp) = M_q(\mathcal{O}_X^\sharp)$$
$$W^+\mathcal{H}_q \to \overline{\mathcal{H}}_q^{Red}(\mathcal{O}_X^{\sharp,+}) = M_q(\mathcal{O}_X^{\sharp,+})$$

Proof. This is a direct consequence of proposition 2.2.15 and proposition 2.2.9 since the map $\overline{\mathcal{H}} \to \overline{\mathcal{H}}^{Red}$ is smooth surjective and consequently universally subtrusive. □

We let $L^u\mathcal{H}_q$ and $W^u\mathcal{H}_q$ denote respectively the kernels of the morphisms of proposition 2.2.20 above.

Proposition 2.2.21. *If $q_1, q_2 \in \mathcal{A}$ are such that $F_{q_1} \subseteq F_{q_2}$, then we get inclusions of v-sheaves in groups:*

$$L^u\mathcal{H}_{q_1} \subseteq L^u\mathcal{H}_{q_2} \subseteq L^+\mathcal{H}_{q_2} \subseteq L^+\mathcal{H}_{q_1} \subseteq LH$$
$$W^u\mathcal{H}_{q_1} \subseteq W^u\mathcal{H}_{q_2} \subseteq W^+\mathcal{H}_{q_2} \subseteq W^+\mathcal{H}_{q_1} \subseteq WH$$

Moreover, the map from $L^+\mathcal{H}_{q_2}$ to $M_{q_1}(\mathcal{O}^\sharp)$ surjects onto $P_{\Phi_{q_1,q_2}}(\mathcal{O}^\sharp) \subseteq M_{q_1}(\mathcal{O}^\sharp)$. Analogously, $W^+\mathcal{H}_{q_2}$ surjects onto $P_{\Phi_{q_1,q_2}}(\mathcal{O}^{\sharp,+}) \subseteq M_{q_1}(\mathcal{O}^{\sharp,+})$.

Proof. We will deal only with the case of parahoric loop groups since the other case is completely analogous. Recall that we have a morphism of algebraic groups over $W(k)[t]$, $f : \mathcal{H}_{q_2} \to \mathcal{H}_{q_1}$, such that $\rho_1 \circ f = \rho_2$. Functoriality of L^+, gives us maps $L^+\mathcal{H}_{q_2} \to L^+\mathcal{H}_{q_1} \to LH$, since $L^+\mathcal{H}_{q_2} \to LH$ is an injection, then $L^+\mathcal{H}_{q_2} \to L^+\mathcal{H}_{q_1}$ is also injective.

Now, since the map of affine schemes $\overline{\mathcal{H}}_{q_2} \to P_{\Phi_{q_1,q_2}}$ is faithfully flat of finite presentation it is universally subtrusive. This implies, by proposition 2.2.15 and proposition 2.2.9, that the composition of $L^+\mathcal{H}_{q_2} \to \overline{\mathcal{H}}_{q_2}(\mathcal{O}^\sharp)$ with $\overline{\mathcal{H}}_{q_2}(\mathcal{O}^\sharp) \to P_{\Phi_{q_1,q_2}}(\mathcal{O}^\sharp)$ is surjective.

Finally, we claim that any map $g : \mathrm{Spec}(B^+_{dR}(R^\sharp)) \to \mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}$ whose reduction $\mathrm{Spec}(R^\sharp) \to M_q$ factors through the identity section lifts to a map $\mathrm{Spec}(B^+_{dR}(R^\sharp)) \to \mathcal{H}_{q_2,B^+_{dR}(R^\sharp)}$ which would give the inclusion $L^u\mathcal{H}_{q_1} \subseteq L^u\mathcal{H}_{q_2}$ and finish the proof. Consider the truncations $g_n : \mathrm{Spec}(B^+_{dR}(R^\sharp)/\xi^n) \to \mathcal{H}_{q_1}$ obtained from g. Since $\mathcal{V}_{q_1}$ is an open neighborhood of the identity each of the g_n has the form $(\prod_{\alpha\in\Phi^-_{q_1}} u_\alpha(g_n)) \cdot t(g_n) \cdot (\prod_{\alpha\in\Phi^+_{q_1}} u_\alpha(g_n))$ each of which reduces to the identity. Taking limits we get a point $\mathrm{Spec}(B^+_{dR}(R^\sharp)) \to \mathcal{V}_{q_1}$. Since $\mathcal{H}_{q_1}$ is separated and the maps $\mathrm{Spec}(B^+_{dR}(R^\sharp)/\xi^n) \to \mathrm{Spec}(B^+_{dR}(R^\sharp))$ are scheme-theoretically dense this is the original map so that g has an open cell decomposition of the form $(\prod_{\alpha\in\Phi^-_{q_1}} u_\alpha(g)) \cdot t(g) \cdot (\prod_{\alpha\in\Phi^+_{q_1}} u_\alpha(g))$ with each of the factors reducing to the identity.

We can verify directly from the construction of the map $\mathcal{H}_{q_2} \to \mathcal{H}_{q_1}$ and the fact that $F_{q_1} \subseteq F_{q_2}$ that each of this elements lifts uniquely to an element in $\mathcal{V}_{q_2}$. Indeed, on the torus $\mathcal{T}$ and on $\mathcal{U}_\alpha$ with $\alpha \in \Phi_{q_1,q_2}$ f induces an isomorphism. For $\Phi \setminus \Phi_{q_1,q_2}$ we may, after making some choices, write $\mathcal{U}^{q_1}_\alpha$ as $\mathrm{Spec}(W(k)[t,u])$ and $\mathcal{U}^{q_2}_\alpha$ as $\mathrm{Spec}(W(k)[t,\frac{u}{t}])$. In this case f restricted to $\mathcal{U}^{q_2}_\alpha$ is given by the natural inclusion of rings. The map of rings $u_\alpha(g)^* : W(k)[t,u] \to B^+_{dR}(R^\sharp)$ with $t \mapsto \xi$ extends to a map $W(k)[t,\frac{u}{t}] \to B^+_{dR}(R^\sharp)$ whenever ξ divides the image of u, but this happens whenever $u_\alpha(g)$ reduces to identity. □

Proposition 2.2.22. *Let $q_1, q_2 \in \mathcal{A}$ such that $F_{q_1} \subseteq F_{q_2}$, then we have isomorphisms of quotient v-sheaves $L^+\mathcal{H}_{q_1}/L^+\mathcal{H}_{q_2} = W^+\mathcal{H}_{q_1}/W^+\mathcal{H}_{q_2}$. Moreover, we can identify both of these quotients with $(Fl_{q_1,q_2,O_C})^\diamond$, where Fl_{q_1,q_2} denotes the flag variety $M_{q_1}/P_{\Phi_{q_1,q_2}}$ when thought of as a p-adic formal scheme.*

Proof. We have sequence of equalities:

$$\begin{aligned}
L^+\mathcal{H}_{p_1}/L^+\mathcal{H}_{p_2} &= (L^+\mathcal{H}_{p_1}/L^u\mathcal{H}_{p_1})/(L^+\mathcal{H}_{p_2}/L^u\mathcal{H}_{p_1}) \\
&= M_{p_1}(\mathcal{O}^\sharp)/P_{\Phi_{p_1,p_2}}(\mathcal{O}^\sharp) \\
&= Fl_{p_1,p_2}(\mathcal{O}^\sharp) \\
&= Fl^\diamond_{p_1,p_2}
\end{aligned}$$

The last equality follows from proposition 2.2.8. Analogously:

$$\begin{aligned}
W^+\mathcal{H}_{p_1}/W^+\mathcal{H}_{p_2} &= (W^+\mathcal{H}_{p_1}/W^u\mathcal{H}_{p_1})/(W^+\mathcal{H}_{p_2}/W^u\mathcal{H}_{p_1}) \\
&= M_{p_1}(\mathcal{O}^{\sharp,+})/P_{\Phi_{p_1,p_2}}(\mathcal{O}^{\sharp,+}) \\
&= Fl^\diamond_{p_1,p_2}
\end{aligned}$$

□

Lemma 2.2.23. *1. Let $\mathcal{F}$ be a locally spatial diamond with a map $\mathcal{F} \to O_C^\diamond$ and fix two points $q_1, q_2 \in \mathcal{A}$ with $F_{q_1} \subseteq F_{q_2}$. The natural map $L^+\mathcal{H}_{q_1} \times_{O_C^\diamond} \mathcal{F} \to Fl^\diamond_{q_1,q_2} \times_{O_C^\diamond} \mathcal{F}$ admits pro-étale locally a section.*

2. If $\mathrm{Spa}(R,R^+)$ is affinoid perfectoid and we are given a map $\mathrm{Spa}(R,R^+) \to Fl^\diamond_{q_1,q_2}$ then the pullback $L^+\mathcal{H}_{q_1} \times_{Fl^\diamond_{q_1,q_2}} \mathrm{Spa}(R,R^+) \to \mathrm{Spa}(R,R^+)$ admits a section locally on the analytic topology of $\mathrm{Spa}(R,R^+)$

Proof. We may reduce the first claim to the second one by [26] 11.24. Indeed, by proposition 2.2.22 the map in question forms a $L^+\mathcal{H}_{q_2}$-torsor so it is enough to prove it is pro-étale locally the trivial torsor. Since the map $Fl^\diamond_{q_1,q_2,} \to O_C^\diamond$ is representable in spatial diamonds we can find a pro-étale cover $\mathrm{Spa}(R,R^+) \to Fl^\diamond_{q_1,q_2} \times_{O_C^\diamond} \mathcal{F}$ with $\mathrm{Spa}(R,R^+)$ affinoid perfectoid.

Let us prove the second claim. The obstruction to the triviality of the $L^+\mathcal{H}_{q_2}$-torsor over $\mathrm{Spa}(R,R^+)$ is an element $\mathfrak{obs}$ in $H^1_v(\mathrm{Spa}(R,R^+), L^+\mathcal{H}_{q_2})$. We prove that this obstruction vanishes after a localization in the analytic topology, recall the following two sequences of maps:

$$(L^+\mathcal{H}_{p_1}) \to M_{p_1}(\mathcal{O}^\sharp_X) \to Fl^\diamond_{p_1,p_2}$$

$$e \to L^u\mathcal{H}_{q_1} \to L^+\mathcal{H}_{q_2} \to P_{\Phi_{q_1,q_2}}(\mathcal{O}^\sharp_X) \to e$$

The map $M_{q_1}(\mathcal{O}^\sharp_X) \to Fl^\diamond_{q_1,q_2}$ is a $P_{\Phi_{q_1,q_2}}(\mathcal{O}^\sharp_X)$-torsor with obstruction to triviality lying in

$$H^1_v(Fl^\diamond_{q_1,q_2}, P_{\Phi_{q_1,q_2}}(\mathcal{O}^\sharp_X)).$$

Since the map of schemes $M_{q_1} \to Fl_{q_1,q_2}$ admits Zariski locally a section we may replace $\mathrm{Spa}(R,R^+)$ by an analytic cover for which $\mathfrak{obs}$ in $H^1_v(\mathrm{Spa}(R,R^+), P_{\Phi_{q_1,q_2}}(\mathcal{O}^\sharp_X))$ is trivial. Observe that in this case $\mathfrak{obs}$ comes from an element of $H^1(\mathrm{Spa}(R,R^+), L^u\mathcal{H}_{q_1})$, we prove below that $\mathfrak{obs}$ is already trivial. By definition, $(L^u\mathcal{H}_{q_1})(R,R^+)$ is the kernel of the map

$$\mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}(B^+_{dR}(R^\sharp)) \to \mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}(B^+_{dR}(R^\sharp)/\xi)$$

We can construct a family of groups $L^{u,n}$ filtering $L^u\mathcal{H}_{q_1,R}$ that we define as:

$$L^{u,n}(R,R^+) := Ker(\mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}(B^+_{dR}(R^\sharp)) \to \mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}(B^+_{dR}(R^\sharp)/\xi^n))$$

Succesive quotients $L^{u,n}/L^{u,n+1}$ get identified with the sheaves assigning to $\mathrm{Spa}(R,R^+)$ the groups:

$$Ker(\mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}(B^+_{dR}(R^\sharp)/\xi^{n+1}) \to \mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}(B^+_{dR}(R^\sharp)/\xi^n))$$

Since $\mathrm{Spec}(B^+_{dR}(R^\sharp)/\xi^n) \to \mathrm{Spec}(B^+_{dR}(R^\sharp)/\xi^{n+1})$ is a first order nilpotent thickening, deformation theory gives:

$$L^{u,n}/L^{u,n+1} = Hom(e^*\Omega^1_{\mathcal{H}_{q_1}} \otimes_{W(k)[t]} B^+_{dR}(R^\sharp), (\xi^n \cdot B^+_{dR}(R^\sharp)/\xi^{n+1})) = Hom(e^*\Omega^1_{\mathcal{H}_{q_1}} \otimes R^\sharp, R^\sharp)$$

Since $\mathcal{H}_{q_1}$ has an open cell decomposition we can see explicitly that $e^*\Omega^1_{\mathcal{H}_{q_1}/W(k)[t]}$ is a finite free module over $W(k)[t]$ (a priori it is only projective), and after fixing a basis we get an identification $L^{u,n}/L^{u,n+1} = (\mathcal{O}^\sharp)^n$. By ([26] 8.8) the cohomology group $H^1_v(\mathrm{Spa}(R,R^+),\mathcal{O}^\sharp) = 0$. Let $I^{u,n}$ be the image of $H^1_v(\mathrm{Spa}(R,R^+),L^{u,n})$ in $H^1_v(\mathrm{Spa}(R,R^+),L^u\mathcal{H}_{q_1})$. The argument above shows that $\mathfrak{obs} \in \bigcap_{n\in\mathbb{N}} I^{u,n}$. On the other hand,

$$\mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}(B^+_{dR}(R^\sharp)) = \varprojlim \mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}(B^+_{dR}(R^\sharp)/\xi^n)$$

which proves that any element in the intersection of these pointed sets must be trivial. □

Definition 2.2.24. *Let $\sigma_r := \{r_i\}_{1\leq i\leq n}$ and $\sigma_q := \{q_i\}_{1\leq i\leq n}$ denote a pair of sequences of points in $\mathcal{A}$ such that $F_{r_i}, F_{r_{i+1}} \subseteq F_{q_i}$, and let σ denote the tuple (σ_r,σ_q). To each σ of this form we associate a v-sheaf that we call the Demazure kimberlite of σ. We define them as the contracted group product:*

$$D(\sigma) = L^+\mathcal{H}_{r_1} \overset{L^+\mathcal{H}_{q_1}}{\times_{O^\diamond_C}} L^+\mathcal{H}_{r_2} \overset{L^+\mathcal{H}_{q_2}}{\times_{O^\diamond_C}} \dots \overset{L^+\mathcal{H}_{r_{n-1}}}{\times_{O^\diamond_C}} L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n}$$

The purpose of the remaining of this subsection is to prove that for any σ as above the $D(\sigma)$ are p-adic Orapian kimberlites that are proper and smooth over $O^\diamond_C$.

Proposition 2.2.25. *The map of v-sheaves $D(\sigma) \to O^\diamond_C$ is representable in spatial diamonds, proper and ℓ-cohomologically smooth for any $\ell \neq p$.*

Proof. Let σ be as above and let $\sigma' = (\{r_i\}_{1\leq i\leq n-1}, \{q_i\}_{1\leq i\leq n-1})$ be the subsequence of the first $n-1$ points of σ. We have a projection morphism of v-sheaves $f : D(\sigma) \to D(\sigma')$ given by forgetting the last entry corresponding to r_n. It is enough to inductively show that this map satisfies all of the properties in the hypothesis. Since the definition of $D(\sigma)(R,R^+)$ is independent of R^+ to prove the map is proper it is enough to prove it is quasi-compact and separated over $D(\sigma')$. By ([26] 23.15, 10.11, 13.4) separatedness, quasi-compactness and almost all of the requirements that a map need to satisfy to be ℓ-cohomologically smooth ([26] 23.8) can be checked v-locally. The following diagram is Cartesian with surjective horizontal arrows:

$$\begin{tikzcd}
L^+\mathcal{H}_{r_1}\times_{O^\diamond_C}\cdots\times_{O^\diamond_C} L^+\mathcal{H}_{r_{n-1}} \times_{O^\diamond_C} (L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n}) \arrow[r] \arrow[d] & D(\sigma) \arrow[d] \\
L^+\mathcal{H}_{r_1}\times_{O^\diamond_C}\cdots\times_{O^\diamond_C} L^+\mathcal{H}_{r_{n-1}} \arrow[r] & D(\sigma')
\end{tikzcd}$$

By proposition 2.2.22 we have that $L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n} = Fl^\diamond_{r_n,q_n,O_C}$ which is proper, representable in spatial diamonds and ℓ-cohomologically smooth over $O^\diamond_C$ for any $\ell \neq p$. This proves the map in the left column of the diagram is representable in spatial diamonds, ℓ-cohomologically smooth and proper. It also proves that the map in the right column of the diagram satisfy the properties that can be checked v-locally.

We now verify the finer properties that cannot be checked v-locally. Namely, we need to check that $f : D(\sigma) \to D(\sigma')$ is representable in spatial diamonds and that f has bounded topological trascendence degree. By ([26] 13.4) the first property can be checked pro-étale locally. Given a map from a spatial diamond $\mathcal{F} \to D(\sigma')$ we let $X = \mathcal{F} \times_{D(\sigma')} D(\sigma)$. Applying lemma 2.2.23 repeatedly to the quotients $L^+\mathcal{H}_{r_k}/L^+\mathcal{H}_{q_k}$ we get that pro-étale locally on $\mathcal{F}$, X is of the form $\mathcal{F} \times_{O_C^\diamond} Fl^\diamond_{r_n,q_n}$ which is a spatial diamond. Moreover, if $\mathcal{F} = \mathrm{Spa}(C', C'^+)$ with C' algebraically closed non-Archemedean field and C'^+ an open and bounded valuation subring then $X = Fl^\diamond_{r_n,q_n,C'}$ and $dim.trg.(f) = dim(Fl_{r_n,q_n}) < \infty$ (See [26] 21.7).

□

Proposition 2.2.26. *Fix σ as above. The projection map $\pi : W^+\mathcal{H}_{p_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} W^+\mathcal{H}_{p_n} \to D(\sigma)$ induced from the family of injections $W^+\mathcal{H}_{p_i} \subseteq L^+\mathcal{H}_{p_i}$ is a surjective map of v-sheaves. It induces an identification:*

$$\iota : D(\sigma) \cong W^+\mathcal{H}_{p_1} \overset{W^+\mathcal{H}_{q_1}}{\times_{O_C^\diamond}} \cdots \overset{W^+\mathcal{H}_{q_{n-1}}}{\times_{O_C^\diamond}} W^+\mathcal{H}_{p_n}/W^+\mathcal{H}_{q_n}$$

Consequently, $D(\sigma)$ is v-formalizing.

Proof. Consider the following basechange diagram:

$$\begin{array}{ccccc} W^+\mathcal{H}_{p_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} (L^+\mathcal{H}_{p_n}/L^+\mathcal{H}_{q_n}) & \longrightarrow & L^+\mathcal{H}_{p_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} (L^+\mathcal{H}_{p_n}/L^+\mathcal{H}_{q_n}) & \longrightarrow & D(\sigma) \\ \downarrow & & \downarrow & & \downarrow \\ W^+\mathcal{H}_{p_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} W^+\mathcal{H}_{p_{n-1}} & \longrightarrow & L^+\mathcal{H}_{p_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} L^+\mathcal{H}_{p_{n-1}} & \longrightarrow & D(\sigma') \end{array} \tag{2.8}$$

Proposition 2.2.22 gives the equality $W^+\mathcal{H}_{r_n}/W^+\mathcal{H}_{q_n} = L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n}$ and allow us to conclude surjectivity by induction. Assume that we have an identification:

$$\iota' : D(\sigma') \cong W^+\mathcal{H}_{p_1} \overset{W^+\mathcal{H}_{q_1}}{\times_{O_C^\diamond}} \cdots \overset{W^+\mathcal{H}_{q_{n-2}}}{\times_{O_C^\diamond}} W^+\mathcal{H}_{p_{n-1}}/W^+\mathcal{H}_{q_{n-1}}$$

Sice $W^+\mathcal{H}_{q_k} \subseteq L^+\mathcal{H}_{q_k}$ the map ι is defined and surjective, we need to prove that ι is also injective. Let $[g_1]$ and $[g_2]$ be two maps

$$[g_1], [g_2] : \mathrm{Spa}(R, R^+) \to W^+\mathcal{H}_{r_1} \overset{W^+\mathcal{H}_{q_1}}{\times_{O_C^\diamond}} \cdots \overset{W^+\mathcal{H}_{q_{n-1}}}{\times_{O_C^\diamond}} W^+\mathcal{H}_{r_n}/W^+\mathcal{H}_{q_n}$$

and suppose that they get identified after mapping to $D(\sigma)(R, R^+)$. By our inductive hypothesis on $D(\sigma')$ we may locally for the v-topology find representatives g_1 and g_2 of $[g_1]$ and $[g_2]$ whose projection to the first $n-1$ coordinates is the same. That is g_i are of the form $(g_i^1, \dots, g_i^n)$ in $W^+\mathcal{H}_{r_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} W^+\mathcal{H}_{r_n}$ with $g_1^j = g_2^j$ for $j \in \{1, \dots, n-1\}$. Since $[g_1]$ and $[g_2]$ get identified in $D(\sigma)$ we must have that (v-locally) g_1 and g_2 are on the same $L^+\mathcal{H}_{q_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} L^+\mathcal{H}_{q_n}$-orbit. Since g_1 and g_2 share all of their entries except possibly the last we have that g_1^n and g_2^n are in the same $L^+\mathcal{H}_{q_n}$-orbit. Since $g_1^n, g_2^n \in W^+\mathcal{H}_{r_n}$ and $W^+\mathcal{H}_{q_n} = W^+\mathcal{H}_{r_n} \cap L^+\mathcal{H}_{q_n}$ they are in the same $W^+\mathcal{H}_{q_n}$-orbit which proves $[g_1] = [g_2]$.

Finally, by proposition 2.2.18 each $W^+\mathcal{H}_{r_i}$ is formalizing, proposition 1.4.14 implies the same for the product, and since $D(\sigma)$ is the quotient of a v-formalizing sheaf it is also v-formalizing. □

Lemma 2.2.27. *Given two points $r_1, r_2 \in \mathcal{A}$ with $F_{r_1} \subseteq F_{r_2}$ the projection map of perfect schemes $\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_1} \to \mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_1}/\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_2} = Fl^{perf}_{r_1,r_2}$ admits Zariski locally a section.*

Proof. Let $\mathfrak{obs} \in H^1_{v-Sch}(Fl^{perf}_{r_1,r_2}, \mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_2})$ be the obstruction to finding a section. Consider the reduction morphism exact sequence:

$$e \to \mathcal{W}\mathcal{H}^u_{r_1} \to \mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_2} \to P^{perf}_{\Phi_{r_1,r_2}} \to e$$

We also have $Fl^{perf}_{r_1,r_2} = M^{perf}_{r_1}/P^{perf}_{\Phi_{r_1,r_2}}$ and the cohomology class in $H^1_{v-Sch}(Fl^{perf}_{r_1,r_2}, P^{perf}_{\Phi_{r_1,r_2}})$ associated to the $P^{perf}_{\Phi_{r_1,r_2}}$-torsor $M^{perf}_{r_1} \to Fl^{perf}_{r_1,r_2}$ is the image of $\mathfrak{obs}$. Zariski locally on $Fl^{perf}_{r_1,r_2}$ the $P^{perf}_{\Phi_{r_1,r_2}}$-torsor is trivial. This is known for the classical flag variety Fl_{r_1,r_2} over $\mathrm{Spec}(k_C)$ and the result will follow from taking perfection. Indeed, consider a commutative diagram trivializing the $P_{\Phi_{r_1,r_2}}$-torsor:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\iota"] \arrow[dr, "s"'] & Fl_{p_1,p_2} \\ & M_{p_1} \arrow[u] \end{tikzcd}$$

Any such diagram having ι as Zariski open cover will produce a similar diagram after we take perfections and will trivialize the $P^{perf}_{\Phi_{r_1,r_2}}$-torsor.

Fix an affine cover $\mathrm{Spec}(R) \to Fl^{perf}_{r_1,r_2}$ trivializaing $red(\mathfrak{obs})$. We claim that $\mathrm{Spec}(R)$ has a section to $\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_1} \times_{Fl^{perf}_{r_1,r_2}} \mathrm{Spec}(R)$. By construction, we know that $\mathfrak{obs} \in H^1_{v-Sch}(\mathrm{Spec}(R), \mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_2})$ is in the image of $H^1_{v-Sch}(\mathrm{Spec}(R), \mathcal{W}^u\mathcal{H}_{r_1})$, but this pointed set is trivial. Indeed, we have that $H^1_{v-Sch}(\mathrm{Spec}(R), \mathbb{G}_a) = \{e\}$ which is a particular case of theorem 4.1 in [4]. One can finish the proof by using the argument given in lemma 2.2.23. $\square$

Proposition 2.2.28. *The map $D(\sigma) \to O_C^\diamond$ is formally adic. Moreover, $D(\sigma)^{\mathrm{red}}$ is represented by a qcqs scheme that is perfectly finitely presented and proper over* $\mathrm{Spec}(k_C)$ *(See [4] 3.11 and 3.14 for definitions).*

Proof. Let $\sigma = (\{r_i\}_{1\leq i\leq n}, \{q_i\}_{1\leq i\leq n})$ and let $\sigma' = (\{r_i\}_{1\leq i\leq n-1}, \{q_i\}_{1\leq i\leq n-1})$. In any Grothendieck topos pullback commutes with finite limits and colimits, so by proposition 2.2.18 we have:

$$D(\sigma) \times_{O_C^\diamond} \mathrm{Spec}(k_C)^\diamond = (\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_1})^\diamond \overset{(\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_1})^\diamond}{\times_{k_C^\diamond}} \dots \overset{(\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_{n-1}})^\diamond}{\times_{k_C^\diamond}} (\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_n})^\diamond / (\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_n})^\diamond$$

Since the functor $(\cdot)^\diamond$ is a left adjoint it commutes with colimits, so we get:

$$D(\sigma) \times_{O_C^\diamond} \mathrm{Spec}(k_C)^\diamond = \left(\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{p_1} \overset{\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_1}}{\times_{k_C}} \dots \overset{\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_{n-1}}}{\times_{k_C}} \mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{p_n} / \mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_n} \right)^\diamond$$

Lemma 1.3.34 proves that $D(\sigma) \to O_C^\diamond$ is formally adic and that $D(\sigma)^{\mathrm{red}} = D(\sigma) \times_{O_C^\diamond} \mathrm{Spec}(k_C)^\diamond$. In particular $\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_k}/\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_k} = Fl_{r_k,q_k}$.

We prove inductively that $D(\sigma)^{\mathrm{red}}$ is represented by a qcqs scheme perfectly finitely presented and proper over $\mathrm{Spec}(k_C)$. Iterating lemma 2.2.27 we see that the map $D(\sigma)^{\mathrm{red}} \to D(\sigma')^{\mathrm{red}}$ is Zariski locally a trivial $Fl^{perf}_{r_k,q_k}$-bundle. Now, quasi-compactness, quasi-separatedness, separatedness, representability in schemes, being perfectly of finite presentation and being proper can all be checked Zariski locally on the target and are stable under basechange and composition (See [29] Tag 02YJ). By induction, $D(\sigma')$ enjoys all of these properties over $\mathrm{Spec}(k)$ and $Fl_{r_n,q_n} \times_{k_C} \mathrm{Spec}(A)$ enjoys them over $\mathrm{Spec}(A)$ for any affine open $\mathrm{Spec}(A) \subseteq D(\sigma')^{\mathrm{red}}$. This proves that $D(\sigma)$ also enjoys them over $\mathrm{Spec}(k_C)$, which finishes the proof. $\square$

Proposition 2.2.29. *For any σ and any geometric point* $\mathrm{Spa}(C', O_{C'}) \to O_C^\diamond$ *the base change* $D(\sigma) \times_{O_C^\diamond} \mathrm{Spa}(C', O_{C'})$ *is a cJ-diamond.*

Proof. We prove by induction that $D(\sigma)$ has enough facets, let $\sigma = (\{r_i\}_{1\leq i\leq n}, \{q_i\}_{1\leq i\leq n})$ and let $\sigma' = (\{r_i\}_{1\leq i\leq n-1}, \{q_i\}_{1\leq i\leq n-1})$. Suppose that $D(\sigma')_{C'}$ has enough facets, let $B := \coprod_{i\in I} \mathrm{Spd}(B_i, {B_i}^+)$ with each B_i a topologically of finite type C'-algebra and let $f : B \to D(\sigma')_{C'}$ be a surjective map. Let $\mathcal{F} = D(\sigma)_{C'} \times_{D(\sigma')_{C'}} B$, we prove that $\mathcal{F}$ has a enough facets. Analytically locally on B the projection map $\mathcal{F} \to B$ is a trivial $(Fl_{r_n,q_n,C'^\sharp})^\diamond$-fibration. The proof of this claim is an iteration of the argument given on lemma 2.2.23 together with the observation that B is already a disjoint union of affinoid perfectoid spaces. We may replace B by an analytic cover B' so that we get the expression:

$$\mathcal{F}' = D(\sigma)_{C'} \times_{D(\sigma')_{C'}} B' = \coprod \mathrm{Spd}(B'_i, B'^{+}_i) \times_{Spa(C', O_{C'})} (Fl_{r_n, q_n, C'^{\sharp}})^{\diamond}$$

By proposition 1.4.39 having enough facets is stable under products so $\mathcal{F}'$ has enough facets and consequently $D(\sigma)$ as well. □

We can summarize this subsection with the following theorem:

Theorem 2.2.30. *For any σ as in definition 2.2.24 the Demazure kimberlite $D(\sigma)$ is an Orapian p-adic kimberlite. The p-adic tubular neighborhoods are non-empty and connected, and the structure morphism $D(\sigma) \to O_C^{\diamond}$ is proper and ℓ-cohomologically smooth.*

Proof. Separatedness and the properties of the structure morphism were proven on proposition 2.2.25. That it is v-formalizing is proven in proposition 2.2.26. We have

$$D(\sigma)^{\mathrm{red}} = D(\sigma) \times_{O_C^{\diamond}} \mathrm{Spec}(k_C)^{\diamond}$$

this implies that the adjunction morphism $D(\sigma)^{\mathrm{red}} \to D(\sigma)$ is a closed embedding, also by proposition 1.3.31 $D(\sigma)$ is formally separated and specializaing. By proposition 2.2.28 $D(\sigma)^{\mathrm{red}}$ is represented by a scheme. At this point we have proved that $D(\sigma)$ is a p-adic kimberlite.

Since $D(\sigma)^{\mathrm{red}}$ is represented by a proper perfectly finitely presented scheme over k_C then $|D(\sigma)^{\mathrm{red}}|$ is a Noetherian and spectral topological space. The analytic locus $D(\sigma)^{an}$ coincides with the generic fiber $D(\sigma) \times_{O_C^{\diamond}} \mathrm{Spa}(C, O_C)$ and by proposition 2.2.29 this is a cJ-diamond. One can easily prove inductively over the map $D(\sigma) \to D(\sigma')$ that the specialization map is surjective on closed points. By lemmas 1.4.43 and 1.4.44 it is a quotient map. This finishes the proof that $D(\sigma)$ is an Orapian kimberlite.

The connectedness of tubular neighborhoods will follow from lemma 2.2.31 below. Indeed, we have already verified that all but conditions 4 and 5 hold. Condition 5 holds by induction over the maps $D(\sigma) \to D(\sigma')$ and condition 4 follows from the diagram 2.8 since each of the $W^+\mathcal{H}_{r_i}$ is formalizing and basechanges along maps that factor through $W^+\mathcal{H}_{r_1} \times_{O_C^{\diamond}} \cdots \times_{O_C^{\diamond}} W^+\mathcal{H}_{r_{n-1}}$ will give a trivial bundle. □

Lemma 2.2.31. *Let $f : \mathcal{F} \to \mathcal{G}$ be a map of kimberlites over $O_C^{\diamond}$, let $X \to \mathrm{Spec}(O_{C^{\sharp}})$ be a smooth projective scheme. Suppose the following properties hold:*

1. *f is ℓ-cohomologically smooth for some $\ell \neq p$.*

2. *f is proper.*

3. *$f : \mathcal{F} \to \mathcal{G}$ is a $X(\mathcal{O}^{\sharp,+})$-bundle locally trivial for the v-topology.*

4. *For any non-Archimedean field C' in characteristic 0 and any map $t : Spa(C', O_{C'}) \to \mathcal{G}$ there is a v-cover $r : Spa(C'', O_{C''}) \to Spa(C', O_{C'})$ such that $\mathcal{G}$ formalizes $t \circ r$ and the base change $\mathcal{F} \times_{\mathcal{G}} O_{C''}^{\diamond}$ is isomorphic to $X(\mathcal{O}^{\sharp,+}) \times_{O_C^{\diamond}} O_{C''}^{\diamond}$.*

5. *For any closed point $x \in |\mathcal{G}^{\mathrm{red}}|$ the p-adic tubular neighborhood $(\widehat{\mathcal{G}}_{/x})_{\eta}$ is connected.*

6. *That $\mathcal{G}^{\mathrm{red}}$ and $\mathcal{F}^{\mathrm{red}}$ are perfectly finitely presented (See [4] 3.10) over $\mathrm{Spec}(k_C)$.*

Then, for any closed point $y \in |\mathcal{F}^{\mathrm{red}}|$ the p-adic tubular neighborhood $(\widehat{\mathcal{F}}_{/y})_{\eta}$ is also connected.

Proof. We begin by making the observation that f is open and closed since it is ℓ-cohomologically smooth and proper (See [26] 23.11). Take a closed point $y \in |\mathcal{F}^{\mathrm{red}}|$ with $x = f(y)$ and consider the map $f : (\widehat{\mathcal{F}}_{/y})_\eta \to (\widehat{\mathcal{G}}_{/x})_\eta$. We assume for now that given C' an arbitrary algebraically closed non-Archimedean field and a map of the form $\mathrm{Spa}(C', O_{C'}) \to (\widehat{\mathcal{G}}_{/x})_\eta$ the base change $(\widehat{\mathcal{F}}_{/y})_\eta \times_{(\widehat{\mathcal{G}}_{/x})_\eta} \mathrm{Spa}(C', O_{C'})$ is connected and non-empty, let us finish the proof under this assumption. Observe that the map of topological spaces $|(\widehat{\mathcal{F}}_{/y})_\eta| \to |(\widehat{\mathcal{G}}_{/x})_\eta|$ is specializing, and by assumption surjective on rank 1 points.

Take two non-empty closed-opens U and V with $U \cup V = |(\widehat{\mathcal{F}}_{/y})_\eta|$. Then $f(U) \cup f(V) = |(\widehat{\mathcal{G}}_{/x})_\eta|$ and consequently $f(U) \cap f(V) \neq \emptyset$. Since f is an open map $f(U)$ and $f(V)$ must meet in a rank 1 point, this implies that U and V also meet which finishes the proof under our assumption.

Let us prove our assumption holds. Take a map $t : \mathrm{Spa}(C', O_{C'}) \to (\widehat{\mathcal{G}}_{/x})_\eta$ and after replacing $\mathrm{Spa}(C', O_{C'})$ by a v-cover we can assume $\mathcal{G}$ formalizes the composition $\mathrm{Spa}(C', O_{C'}) \to \mathcal{G}$ and has the base change property of condition 4 with respect to the unique formalization $O_{C'}^\diamond \to \mathcal{G}$. We get a Cartesian diagram:

$$\begin{tikzcd}
\widehat{\mathcal{F}}_{/y} \times_{\mathcal{G}} O_{C'}^\diamond \arrow[r] \arrow[d] & X(\mathcal{O}^{\sharp,+}) \times_{O_C^\diamond} O_{C'}^\diamond \arrow[r] \arrow[d] & O_{C'}^\diamond \arrow[d] \\
\widehat{\mathcal{F}}_{/y} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G}
\end{tikzcd}$$

After taking reduction functor of this diagram we get the following Cartesian diagram:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & X \times \mathrm{Spec}(k') \arrow[r] \arrow[d] & \mathrm{Spec}(k') \arrow[d] \\
\mathrm{Spec}(k(y)) \arrow[r, "y"] & \mathcal{F}^{\mathrm{red}} \arrow[r] & \mathcal{G}^{\mathrm{red}}
\end{tikzcd}$$

Since $\mathcal{F}^{\mathrm{red}} \to \mathcal{G}^{\mathrm{red}}$ is perfectly finitely presented and k is algebraically closed we have that $k = k(y) = k(x)$ and the composition $y : \mathrm{Spec}(k) \to \mathcal{G}^{\mathrm{red}}$ is the closed immersion corresponding to the point x. Consequently $Z \to \mathrm{Spec}(k')$ is a closed immersion, therefore an isomorphism. We have that

$$\widehat{\mathcal{F}}_{/y} \times_{\mathcal{G}} O_{C'}^\diamond = X(\mathcal{O}^{\sharp,+})_{O_{C'}} \times_{\mathcal{F}} \widehat{\mathcal{F}}_{/y} = (X(\widehat{\mathcal{O}^{\sharp,+})_{O_{C'}}})_{/Z}$$

by proposition 1.4.26, but $Z \to X \times \mathrm{Spec}(k')$ is a closed point, so $(X(\widehat{\mathcal{O}^{\sharp,+})_{O_{C'}}}_{/Z})_\eta$ is isomorphic to an open unit ball $\mathbb{B}_n^{<1}$ over $C'^\sharp$, where $n = dim(X)$ and $C'^\sharp$ is the untilt determined by $O_{C^\sharp}$. We have proved that the fibers are non-empty and connected. □

2.2.4 Resolution of p-adic Beilinson-Drinfeld Grassmanians

In this subsection we discuss an analogue of the Demazure resolution for split reductive groups in the context of v-sheaves (also known as Bott-Samelson resolution). We keep the notation from the begining of the previous subsection and we restrict our attention to parahoric loop groups associated to points contained in our chosen alcove $\mathcal{C}$. Given $s_j \in \mathbb{S}$ we denote by $L^+\mathcal{H}_{s_j}$ the parahoric loop group associated to the wall F_{s_j} in $\mathcal{C}$ corresponding to the reflexion s_j. For a point $r \in \mathcal{C}$ we let $J_r \subseteq \mathbb{S}$ denote the set $\{s_j \mid r \in F_{s_j}\}$. We will denote by L^+B the parahoric loop group associated $\mathcal{C}$.

By functoriality of $L(-)$ we we cand define a loop group version of the Weyl group by the formula $LW := LN/LT$, we can also define the Iwahori-Weyl group as $L\tilde{W} := LN/L^+\mathcal{T}$. We have an exact sequence of v-sheaves in groups:

$$e \to LT/L^+\mathcal{T} \to L\tilde{W} \to LW \to e$$

One can prove by a direct computation that $LW = L(N/T) = \underline{W} \times O_C^\diamond$ and that $LT/L^+\mathcal{T} = \underline{X_*(T)} \times O_C^\diamond$ by using the Cartan decomposition. These two imply that $L\tilde{W} = \underline{\tilde{W}} \times O_C^\diamond$.

Since H is a split reductive group over $W(k)[t, t^{-1}]$, for any element $w \in W$ we can find a section $n_w : \mathrm{Spec}(W(k)[t, t^{-1}]) \to N$ whose projection to W is w (See [7] 5.1.11). This allow us to define a similar section $n_w : O_C^\diamond \to LN \subseteq LH$. Also for any $\mu \in X_*(T)$ and any $\mathrm{Spa}(R, R^+) \to O_C^\diamond$ we can consider the element $\xi^\mu \in T(B_{dR}(R^\sharp))$. This assignation is functorial and defines a section $O_C^\diamond \to LT$ mapping to μ. In particular, for any element $\tilde{w} \in \tilde{W}$ there is a section $n_{\tilde{w}} : O_C^\diamond \to LN$ projecting to $\tilde{w}$ in $L\tilde{W}$. We can use $n_{\tilde{w}}$ to construct an automorphism $n_{\tilde{w}} : \mathrm{Gr}^H_{O_C^\diamond} \to \mathrm{Gr}^H_{O_C^\diamond}$ with

$$n_{\tilde{w}}(x \cdot L^+H) := n_{\tilde{w}} \cdot x \cdot L^+H.$$

We will use this discussion in the proof of theorem 2.2.33.

Proposition 2.2.32. *Let $\sigma = (\sigma_r, \sigma_q)$ with σ as in the previous subsection except that we require $\sigma_r, \sigma_q \subseteq \mathcal{C}$. Suppose that $L^+\mathcal{H}_{q_n} = L^+\mathcal{H}_{r_n} = L^+H$ then the multiplication map $\mu : D(\sigma) \to \mathrm{Gr}^H_{O_C} = LH/L^+H$ has geometrically connected fibers.*

Proof. This proof follows the classical one. The key inputs are as follows, the basechange of $D(\sigma) \to O_C^\diamond$ by geometric points are proper spatial diamonds, rank 1 points are dense for any spatial diamond and the group of rank 1 geometric points of a parahoric loop group coincide with the "parabolic subgroups" of a Tits-systems (or BN-pair). These two observations together with ([26] 12.11) reduces the proof to the classical combinatorial case. Indeed, properness (which includes quasi-compactness) will allow us to prove all surjectivity claims at the level of rank 1 geometric points. We provide further details below for the convenience of the reader.

Fix a geometric point $\mathrm{Spa}(C', O_{C'}) \to O_C^\diamond$, all of the objects considered in our argument below are considered over $\mathrm{Spa}(C', O_{C'})$ but we omit the basechange from the notation. Let us start by making some reductions, observe that since we are assuming that $\sigma_q \subseteq \mathcal{C}$ we have $L^+B \subseteq L^+\mathcal{H}_{q_i}$ so we get a surjective map:

$$D(\tau) := L^+\mathcal{H}_{r_1} \overset{L^+B}{\times} \dots \overset{L^+B}{\times} L^+\mathcal{H}_{r_n}/L^+B \to D(\sigma)$$

Surjectivity allows us to replace $D(\sigma)$ for $D(\tau)$ so we may assume $L^+\mathcal{H}_{q_i} = L^+B$ for all $i \leq n$. Now the flag varieties $L^+\mathcal{H}_{r_i,}/L^+B$ admit a surjective map from a finite contracted product of the form:

$$L^+\mathcal{H}_{s_{j_1},} \overset{L^+B}{\times} \dots \overset{L^+B}{\times} L^+\mathcal{H}_{s_{j_m},}/L^+B \to L^+\mathcal{H}_{r_1,}/L^+B$$

Where $s_{j_k} \in J_{r_i}$ and the product $s_{j_1} \cdot \dots \cdot s_{j_m}$ is a reduced expression for the longest word in the finite Coxeter group generated by J_{r_i}. This lets us reduce to the case in which for all $i \leq n$, $L^+\mathcal{H}_{r_i} = L^+\mathcal{H}_{s_j}$ for some j. Moreover, in this case the map $D(\tau) \to \mathrm{Gr}^H_{C'}$ factors through $LH/L^+B \to \mathrm{Gr}^H_{C'}$ which is a L^+H/L^+B-bundle.

We prove inductively that $D(\tau) \to LH/L^+B$ has connected geometric fibers. Write $S(\tau)$ for the image of $D(\tau)$ in LH/L^+B. The multiplication map factors as:

$$D(\tau) = L^+\mathcal{H}_{s_{j_1},} \overset{L^+B}{\times} D(\tau') \to L^+\mathcal{H}_{s_{j_1},} \overset{L^+B}{\times} S(\tau') \to S(\tau)$$

If we assume inductively that the map $D(\tau') \to S(\tau')$ has connected geometric fibers, then it suffices to prove that $L^+\mathcal{H}_{s_{j_1},} \overset{L^+B}{\times} S(\tau') \to S(\tau)$ also has connected geometric fibers.

Notice that by construction $S(\tau') \subseteq LH/L^+B$ is a closed subsheaf that is stable under the action of L^+B. As in the classical case the L^+B-orbits of geometric points in LH/L^+B are indexed $\tilde{W}$. Given an element $w \in \tilde{W}$ we can consider $C(w)$ the locally-closed subsheaf of LH/L^+B associated to this

L^+B-orbit and we can let $S(w) = \bigcup_{w' \leq w} C(w')$ where '$\leq$' denotes the Bruhat order of the quasi-Coxeter system $\mathbb{S} \subseteq W^{aff} \subseteq \tilde{W}$. We also assume in our inductive hypothesis that $S(\tau') = S(w)$ for $w \in W^{aff}$ of the form $w = s_{j_{k_1}} \dots s_{j_{k_l}}$ where s_{j_k} is a subsequence of elements in $\mathbb{S}$ of the sequence appearing in the definition of $D(\tau')$.

For the induction step we have two cases, either $s_{j_1} \cdot w < w$ or $s_{j_1} \cdot w > w$. In the first case we will have that the action of $L^+\mathcal{H}_{s_{j_1}}$ on LH/L^+B stabilizes $S(w)$ so that the multiplication map

$$L^+\mathcal{H}_{s_{j_1}} \overset{L^+B}{\times} S(w) \to S(w)$$

decomposes as the composition of an isomorphism $L^+\mathcal{H}_{s_{j_1}} \overset{L^+B}{\times} S(w) \to L^+\mathcal{H}_{s_{j_1}}/L^+B \times S(w)$ followed by the second projection. In this case geometric fibers are isomorphic to $L^+\mathcal{H}_{s_{j_1},C''}/L^+B_{C''} \cong (\mathbb{P}^1_{C''^\sharp})^\diamond$, and we have that $S(\tau) = S(\tau') = S(w)$ which is of the form assumed in our inductive hypothesis.

On the other hand, if $s_{j_1} \cdot w > w$ we can consider the collection T of $w' < w$ for which $s \cdot w' < w'$ we have that the multiplication map

$$L^+\mathcal{H}_{s_{j_1}} \overset{L^+B}{\times} S(w) \setminus \bigcup_{w' \in T} S(w') \to S(s_{j_1} \cdot w) \setminus \bigcup_{w' \in T} S(w')$$

is an isomorphism while the map

$$L^+\mathcal{H}_{s_{j_1}} \overset{L^+B}{\times} \bigcup_{w' \in T} S(w') \to \bigcup_{w' \in T} S(w')$$

has geometric fibers as in the previous case since this set is also $L^+\mathcal{H}_{s_{j_1}}$-stable. Moreover, we have $S(\tau) = S(s_{j_1} \cdot w)$ which is again of the form assumed in our induction hypothesis, this finishes inductive step and the proof. $\square$

Theorem 2.2.33. *Let $\mathscr{G}$ be a quasi-split reductive group over $W(k)$, $\mathfrak{T} \subseteq \mathfrak{B} \subseteq \mathscr{G}$ a Borel and a maximal torus in $\mathscr{G}$ defined over $W(k)$ and take a cocharacter $\mu \in X_*(\mathfrak{T})$ defined over an algebraic closure of $W(k)[\frac{1}{p}]$. Let F be a non-Archimedean field extension of $W(k)[\frac{1}{p}]$ containing $E(\mu)$ the reflex field of μ. We let O_F the ring of integers of F and the residue field k_F, assume that F is complete for the p-adic topology and that k_F is perfect. Then $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_F^\diamond}$ is an Orapian p-adic kimberlite over $O_F^\diamond$. Moreover, the p-adic tubular neighborhoods of $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_F^\diamond}$ at closed points are non-empty and connected.*

Proof. In corollary 2.2.7 we show that $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_F^\diamond}$ is a p-adic kimberlite so the only thing left to prove are the statements related to the structure of p-adic tubular neighborhoods. We first prove the case in which F is a complete algebraically closed extension of $W(k)[\frac{1}{p}]$ which we will denote instead by C. In this case $\mathscr{G} \times_{W(k)} W(k_C)$ is isomorphic to a split reductive group, and since the functor $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_C^\diamond}$ only depends on the isomorphism class of $\mathscr{G}_{W(k_C)}$, we may assume $\mathscr{G} = H$ with H split reductive. Furthermore, we discuss first the case in which H is semisimple and simply connected, in this case $\tilde{W} = W^{aff}$.

Recall that we have inclusions $X_*^+(\mathfrak{T}) \subseteq X_*(\mathfrak{T}) \subseteq \tilde{W}$ so we may think of μ as an element of the Iwahori-Weyl group. By definition, $\mathrm{Gr}^{H,\leq\mu}_{O_C^\diamond}(R,R^+)$ consists of those elements in $\mathrm{Gr}^{H}_{O_C^\diamond}(R,R^+)$ satisfying that for any geometric point $q : \mathrm{Spa}(C',C'^+) \to \mathrm{Spa}(R,R^+)$ the type of q, μ_q, is in the double coset

$$H(B^+_{dR}(C'^\sharp))\backslash H(B_{dR}(C'^\sharp))/H(B^+_{dR}(C'^\sharp)) = X_*^+(T) = W_o\backslash W^{aff}/W_o$$

satisfies that $\mu_q \leq \mu$ in the Bruhat order. Now given any element $w \in \tilde{W}$ we may consider the subsheaf $\mathrm{Gr}^{G,\leq w}_{O_C^\diamond} \subseteq \mathrm{Gr}^{G}_{O_C^\diamond}$ given instead by the property that on a geometric point $q : \mathrm{Spa}(C',C'^+) \to \mathrm{Spa}(R,R^+)$ the type of q, $[w_q]$, in the double coset

$$\mathfrak{B}(B^+_{dR}(C'^\sharp))\backslash H(B_{dR}(C'^\sharp))/H(B^+_{dR}(C'^\sharp)) = W^{aff}/W_o$$

satisfies that $[w_q] \leq [w]$ in the Bruhat order. The projection map $\pi : W^{aff}/W_o \to W_o\backslash W^{aff}/W_o$ is order preserving and $\pi^{-1}(\mu)$ has a unique element $[w_\mu]$ of largest length, it has the property that $v \leq w_\mu$ if and only if $\pi(v) \leq \mu$. In particular, we have equalities of sheaves $\mathrm{Gr}^{H,\leq w_\mu}_{O_C^\diamond} = \mathrm{Gr}^{H,\leq \mu}_{O_C^\diamond}$. We prove that for any word $w \in W^{aff}$ the v-sheaf $\mathrm{Gr}^{H,\leq w}_{O_C^\diamond}$ satisfies the conclusions of the theorem. If we find a reduced expression for $w = s_{j_1} \dots s_{j_n}$ we can use the theory of BN-pairs to construct a Demazure kimberlite

$$D(w) := L^+H_{s_{j_1}} \overset{L^+\mathfrak{B}}{\times_{O_C^\diamond}} \dots L^+H_{s_{j_n}}/L^+H$$

for which the multiplication map $m : D(w) \to \mathrm{Gr}^H_{O_C^\diamond}$ factors through $\mathrm{Gr}^{H,\leq w}_{O_C^\diamond}$ and surjects onto it at the level of rank 1 geometric points. But m is a proper map so that by ([26] 12.11) it is actually a surjection of v-sheaves. Moreover, this also proves that $\mathrm{Gr}^{H,\leq w}_{O_C^\diamond}$ is a closed subsheaf of $\mathrm{Gr}^H_{O_C^\diamond}$. Theorem 2.2.30 and proposition 2.2.32 combined with lemma 2.2.34 below allow us to conclude in this case.

Suppose now that H is an arbitrary split reductive group. In this case, $\mu \in \tilde{W}$ can be expressed as $\mu = (w, \omega)$ with $w \in W^{aff}$ and $\omega \in \Omega_H$ for the decomposition $\tilde{W} = W^{aff} \rtimes \Omega_H$. We may find a section $n_\omega : O_C^\diamond \to LN$ projecting to (e, ω) in $L\tilde{W}$, this section induces an isomorphism between $\mathrm{Gr}^{H,\leq \mu}_{O_C^\diamond}$ and $\mathrm{Gr}^{H,\leq (w,e)}_{O_C^\diamond}$. But for any $w \in W^{aff}$ the v-sheaf $\mathrm{Gr}^{H,\leq (w,e)}_{O_C^\diamond}$ admits a surjective map by a Demazure kimberlite as in the previous case.

Finally, let us deal with the general case in which F is not assumed to be algebraically closed. Let C be the completion of an algebraic closure of F and F' the completion of the maximal unramified subextension of F inside C. We have surjective maps of v-sheaves:

$$\mathrm{Gr}^{\mathscr{G},\leq \mu}_{O_C^\diamond} \to \mathrm{Gr}^{\mathscr{G},\leq \mu}_{O_{F'}^\diamond} \to \mathrm{Gr}^{\mathscr{G},\leq \mu}_{O_F^\diamond}$$

Lemma 2.2.34 implies that $\mathrm{Gr}^{\mathscr{G},\leq \mu}_{O_F^\diamond}$ and $\mathrm{Gr}^{\mathscr{G},\leq \mu}_{O_{F'}^\diamond}$ are Orapian kimberlites. Moreover, we can infer by proposition 1.4.26 that $\mathrm{Gr}^{\mathscr{G},\leq \mu}_{O_{F'}^\diamond}$ has connected p-adic tubular neighborhoods since we have an identification $(\mathrm{Gr}^{\mathscr{G},\leq \mu}_{O_C^\diamond})^{\mathrm{red}} = (\mathrm{Gr}^{\mathscr{G},\leq \mu}_{O_{F'}^\diamond})^{\mathrm{red}}$. On the other hand, the map $\mathrm{Gr}^{G,\leq \mu}_{O_{F'}^\diamond} \to \mathrm{Gr}^{\mathscr{G},\leq \mu}_{O_F^\diamond}$ is a $\underline{\pi_1}(\mathrm{Spec}(O_F))$-torsor and for any closed point $x \in |\mathrm{Gr}^{\mathscr{G},\leq \mu^{\mathrm{red}}}_{O_F^\diamond}|$ the action of $\pi_1(\mathrm{Spec}(O_F))$ will permute transitively the closed points $y \in |\mathrm{Gr}^{\mathscr{G},\leq \mu^{\mathrm{red}}}_{O_{F'}^\diamond}|$ over x. In particular, the action permutes transitively the p-adic tubular neighborhoods associated to such y. This proves that the tubular neighborhood over x is also connected. □

Lemma 2.2.34. *Let $f : \mathcal{F} \to \mathcal{G}$ be a map of p-adic kimberlites over $\mathbb{Z}_p^\diamond$. Suppose that f is surjective, that $\mathcal{F}$ is an Orapian kimberlite, that $|\mathcal{G}^{\mathrm{red}}|$ is locally Noetherian and that f^{red} is a specializing map. Then:*

1. *$\mathcal{G}$ is Orapian.*

2. *If $\mathcal{F}$ has connected p-adic tubular neighborhoods and f^{red} has connected geometric fibers then $\mathcal{G}$ has connected p-adic tubular neighborhoods.*

Proof. Since the map $\mathcal{F}_\eta \to \mathcal{G}_\eta$ is surjective we have that, by proposition 1.4.34, $\mathcal{G}_\eta$ is a cJ-diamond. Since we assumed the kimberlites to be p-adic the map $\mathcal{F}^{\mathrm{red}} \to \mathcal{G}^{\mathrm{red}}$ is surjective, $|\mathcal{F}^{\mathrm{red}}| \to |\mathcal{G}^{\mathrm{red}}|$ is a quotient map by proposition 1.3.14 and a specializing map by hypothesis. Since we assumed that $|\mathcal{G}^{\mathrm{red}}|$ is locally Noetherian we only need to prove that $\mathrm{sp}_\mathcal{G}$ is specializing and a quotient map. Observe that the composition $f^{\mathrm{red}} \circ \mathrm{sp}_\mathcal{F}$ is specializing and quotient map, from which we can conclude.

For the claim on p-adic tubular neighborhoods pick a closed point $x \in |\mathcal{G}^{\mathrm{red}}|$, by proposition 1.4.26 $(\widehat{\mathcal{G}}_{/x})_\eta \times_\mathcal{G} \mathcal{F} = (\widehat{\mathcal{F}}_{/S})_\eta$ with $S = |f^{\mathrm{red}}|^{-1}(x)$. One can easily deduce from the hypothesis on geometric fibers that S is connected which implies by propositions 1.4.42 and 1.4.29 that $(\widehat{\mathcal{F}}_{/S})_\eta$ is also connected. Since f is surjective $(\widehat{\mathcal{G}}_{/x})_\eta$ is also connected. □

We will finish this section with the proof of theorem 1 which is just a rephrasing of theorem 2.2.33 in less technical language. For the convenience of the reader we write the statement again.

Theorem 2.2.35. *With notation as in the introduction the following holds:*

a) The specialization map

$$\mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^{\diamond}_{F_1}}} : |\mathrm{Gr}^{G,\leq\mu}_{F_1^{\diamond}}| \to |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$$

is a closed and spectral map of spectral topological spaces.

b) Given a closed point $x \in |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$ let $T_x := \mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^{\diamond}_{F_1}}}{}^{-1}(x)$, then the interior T_x° of T_x in $|\mathrm{Gr}^{G,\leq\mu}_{F_1^{\diamond}}|$ is a dense subset of T_x.

c) T_x and T_x° are connected.

Proof of theorem 1. We may apply theorem 2.2.33 and proposition 2.2.5 to the case in which $k = \mathbb{F}_p$ to conclude that $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^{\diamond}_{F_1}}$ is a p-adic Orapian kimberlite with generic fiber $\mathrm{Gr}^{G,\leq\mu}_{F_1^{\diamond}}$ and with reduction $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}$. Since $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^{\diamond}_{F_1}}$ is a kimberlite by proposition 1.4.21 the specialization map $\mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^{\diamond}_{F_1}}} : |\mathrm{Gr}^{G,\leq\mu}_{F_1^{\diamond}}| \to |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$ is a spectral map of locally spectral spaces. Since $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^{\diamond}_{F_1}}$ is Orapian, the map is surjective and specializing, and since $\mathrm{Gr}^{G,\leq\mu}_{F_1^{\diamond}}$ is quasi-compact proposition 1.1.17 gives that the specialization map is closed, this finishes the proof of the first claim. For the second claim let $x \in |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$, we can use proposition 1.4.29 to identify T_x° with $|(\widehat{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_{F_1}}}_{/x})_{\eta}|$. Since $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_{F_1}}$ is Orapian we can apply proposition 1.4.33 to prove that T_x° is dense in T_x giving the second claim. By 2.2.33 T_x° is connected and since it is dense in T_x this later one is also connected. $\square$

2.3 Specialization for moduli of mixed characteristic shtukas

For the rest of this section we will assume that $k = \mathbb{F}_p$ and that $\mathscr{G}$ is a reductive group over $\mathbb{Z}_p$. We fix a torus and a Borel $\mathfrak{T} \subseteq \mathfrak{B} \subseteq \mathscr{G}$. We fix $\mathfrak{f}$ an algebraically closed field extension of $\overline{\mathbb{F}}_p$ and we let $K_0 = W(\mathfrak{f})[\frac{1}{p}]$, we fix an element $b \in \mathscr{G}(K_0)$ and we let $\mathscr{G}_b : \mathrm{Rep}^{\mathscr{G}}_{\mathbb{Z}_p} \to \mathrm{IsoCrys}_{K_0}$ denote the $\otimes$-exact functor from the category of algebraic representations of $\mathscr{G}$ to the category of isocrystals over K_0 associated to b.

Definition 2.3.1. *We define the moduli space of mixed characteristic shtukas associated to $\mathscr{G}_b$, which we denote by $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$, as the functor $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} : \mathrm{Perf} \to \mathrm{Sets}$:*

$$\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R, R^+) = \{(R^\sharp, \iota, f), \mathscr{T}, \Phi, \lambda\}/\cong$$

Where $(R^\sharp, \iota, f)$ denotes an untilt of R over $\mathrm{Spa}(W(\mathfrak{f}), W(\mathfrak{f}))$, the pair $(\mathscr{T}, \Phi)$ is a shtuka as in definition 2.1.23 and $\lambda : \mathscr{T} \to \mathscr{G}_b|_{\mathcal{Y}^{R^+}_{[r,\infty)}}$ is an equivalence class of isogenies as in definition 2.1.24. Here $\mathscr{G}_b|_{\mathcal{Y}^{R^+}_{[r,\infty)}}$ denotes the pullback along the natural map of locally ringed spaces $\mathcal{Y}^{R^+}_{[r,\infty)} \to \mathrm{Spec}(K_0)$ induced by f.

As with p-adic Beilinson-Drinfeld Grassmanians, moduli spaces of shtukas admit bounded versions. Given a geometric point of our moduli $\mathrm{Spa}(C, C^+) \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ the torsor $\mathscr{T}$ can be glued with $\mathscr{G}_b$ along $\mathcal{Y}^{C^+}_{[0,\infty)}$ to extend it to $\mathcal{Y}_{C^+}$. This gives a $\mathscr{G}$-torsor over Y_{C^+} by theorem 2.1.12. One can basechange this torsor to $B^+_{dR}(C^\sharp)$ where we can choose a trivialization of $\tau : \mathscr{T} \to \mathscr{G}$. The morphism $\tau \circ \Phi : \phi^*\mathscr{T} \to \mathscr{G}$ defines an element of $\mathscr{G}(B_{dR}(C^\sharp))/\mathscr{G}(B^+_{dR}(C^\sharp))$ whose image, $\mu_{(\mathscr{T},\Phi)}$, in the double coset

$$\mathscr{G}(B^+_{dR}(C^\sharp))\backslash\mathscr{G}(B_{dR}(C^\sharp))/\mathscr{G}(B^+_{dR}(C^\sharp)) = X^+_*(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$$

does not depend on the choice of τ. We call $\mu_{(\mathscr{T},\Phi)}$ the relative position of the shtuka at that geometric point.

Definition 2.3.2. *Let $\mu \in X_*^+(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$. We define the moduli space of mixed characteristic shtukas associated to $\mathscr{G}_b$ and bounded by μ, which we denote by $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$, as the functor $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} : \mathrm{Perf} \to \mathrm{Sets}$:*

$$\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}(R,R^+) = \{(R^\sharp,\iota,f),\mathscr{T},\Phi,\lambda\}/\cong$$

Where $(R^\sharp,\iota,f)$ denotes an untilt of R over $\mathrm{Spa}(W(\mathfrak{f}),W(\mathfrak{f}))$, the pair $(\mathscr{T},\Phi)$ is a shtuka whose relative position is point-wise bounded by μ in the Bruhat order and $\lambda : \mathscr{T} \to \mathscr{G}_b|_{\mathcal{Y}^{R^+}_{[r,\infty)}}$ is an (equivalence class of) isogenies.

Remark 2.3.3. *In definition 2.3.2, let $E(\mu)$ denote the reflex field of μ. Since $\mathscr{G}$ is reductive over $\mathbb{Z}_p$, $E(\mu)$ is an unramified extension of $\mathbb{Q}_p$. Moreover, since $\mathfrak{f}$ commes equipped with an inclusion $\overline{\mathbb{F}}_p \to \mathfrak{f}$ we get an inclusion $E(\mu) \to W(\mathfrak{f})[\frac{1}{p}]$. We are implicitly using this inclusion of fields to compare the relative positions.*

The purpose of this section is to prove that the $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ are p-adically smelted Orapian kimberlites that have connected p-adic tubular neighborhoods.

2.3.1 Moduli spaces of shtukas are kimberlites

In this subsection we verify that the map $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} \to W(\mathfrak{f})^\diamond$ forms a p-adically smelted kimberlite. We will need to define auxiliary spaces to simplify some of the arguments below:

Definition 2.3.4. *We let $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ denote the functor $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} : \mathrm{Perf} \to \mathrm{Sets}$:*

$$\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R,R^+) = \{(R^\sharp,\iota,f),M,\lambda\}$$

Where the triple $(R^\sharp,\iota,f)$ denotes an untilt over $\mathrm{Spa}(W(\mathfrak{f}),W(\mathfrak{f}))$, $M \in \mathscr{G}(W(R^+)[\frac{1}{\xi_{R^\sharp}}])$ and $\lambda : \mathscr{G}_M \to \mathscr{G}_b$ is an equivalence class of isogenies defined over $\mathcal{Y}^{R^+}_{[r,\infty]}$ for some r. Here $\mathscr{G}_M$ denotes the tuple $(\mathscr{G},\Phi_M)$ with $\Phi_M : \phi^*\mathscr{G} \to \mathscr{G}$ an isomorphism given by M and defined over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi_{R^\sharp}}])$.

Notice that there is a natural map $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ given by restriction and assigning $(M,\lambda) \mapsto (\mathscr{G},\Phi_M,\lambda)$. We denote by $\mathbb{W}^+\mathscr{G}$ the sheaf in groups $\mathbb{W}^+\mathscr{G}(R,R^+) = \mathscr{G}(W(R^+))$, notice that $(\mathbb{W}^+\mathscr{G})_{O_C} = W^+(\mathscr{G} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[t])$ as in definition 2.2.13.

Proposition 2.3.5. *1. The functors $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ and $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ are small v-sheaves.*

2. The map $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is a $\mathbb{W}^+\mathscr{G}$-torsor for the v-topology.

3. $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is formalizing and $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is v-formalizing.

Proof. To prove that it is a v-sheaf one has to prove that each of the entries descend. A standard argument using 2.1.21 repeatedly proves this. Given $N \in \mathbb{W}^+\mathscr{G}(R,R^+)$ and $(M,\lambda) \in \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R,R^+)$ we let $N \cdot (M,\lambda) = (NM\phi(N)^{-1},\lambda \circ N)$. This specifies an action of $\mathbb{W}^+\mathscr{G}$ on $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ that makes the map $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ equivariant when the target is endowed with the trivial action. It is enough to prove that the basechange of $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ along product of points gives a trivial $\mathbb{W}^+\mathscr{G}$-torsor.

Let $\mathrm{Spa}(R,R^+)$ be a product of points, and take $(\mathscr{T},\Phi,\lambda) \in \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R,R^+)$. We can glue $\mathscr{T}$ along λ over $\mathcal{Y}^{R^+}_{[r,\infty)}$ to extend to a $\mathscr{G}$-bundle over $\mathcal{Y}_{R^+}$, a meromorphic isomorphism Φ over $\mathcal{Y}_{R^+} \setminus V(\xi_{R^\sharp})$ and an isogeny λ over $\mathcal{Y}^{R^+}_{[r,\infty]}$. We can use theorem 2.1.12 and theorem 2.1.18 to get a $\mathscr{G}$-bundle over

$\mathrm{Spec}(W(R^+))$ with a meromorphic Φ that restrict to the previous ones. Since R^+ is a product of valuation rings with algebraically closed fraction field any $\mathscr{G}$-bundle on $\mathrm{Spec}(W(R^+))$ is trivial. This is the case because $\mathrm{Spec}(W(R^+))$ splits every étale cover. The choice of a trivialization specifies a section $(M, \lambda) \in \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R, R^+)$ and after chasing definitions one can see that the natural action of $\mathbb{W}^+\mathscr{G}$ on the set of trivialization acts compatibly with the action specified above.

We prove that $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is formalizing, this already implies that $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is v-formalizing since the map $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is surjective. Let $\mathrm{Spa}(S, S^+) \in \mathrm{Perf}$, fix $\varpi_S \in S^+$ a pseudo-uniformizer and take

$$((S^\sharp, \iota, f), M, \lambda) \in \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(S, S^+)$$

Given a map $f : \mathrm{Spa}(L, L^+) \to \mathrm{Spd}(S^+, S^+)$ we get a map of rings $f : W(S^+)[\frac{1}{\xi_{S^\sharp}}] \to W(L^+)[\frac{1}{\xi_{L^\sharp}}]$, and we can let M_L be $f(M)$. Moreover, fix a pseudo-uniformizer $\varpi_L \in L^+$, we claim that for any such choice and for any $r \in \mathbb{R}$ there is a large enough $r' \in \mathbb{R}$ for which the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{Y}^{L^+}_{[r',\infty]} \arrow[r] \arrow[d] & \mathcal{Y}^{R^+}_{[r,\infty]} \arrow[d] \\
\mathrm{Spa}(W(L^+), W(L^+)) \arrow[r] & \mathrm{Spa}(W(S^+), W(S^+))
\end{tikzcd}$$

This map allows us to pullback the isogeny λ to $\mathrm{Spa}(L, L^+)$. The equivalence class of isogenies constructed this way does not depend of the choices of ϖ_S, ϖ_L, r or r' and the construction is functorial, so it defines a map $\mathrm{Spd}(S^+, S^+) \to \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$. $\square$

Moduli spaces of shtukas satisfy the valuative criterion for partial properness over $W(\mathfrak{f})^\diamond$ since the definition of all of the data involved (via Tannakian formalism) takes place in the exact category of vector bundles over $\mathcal{Y}^{R^+}_{[0,\infty)}$ which is equivalent (by an exact equivalence) to the category of vector bundles over $\mathcal{Y}^{R^\circ}_{[0,\infty)}$.

Lemma 2.3.6. *Let $\mathscr{G}_1 \to \mathscr{G}_2$ be a closed embeddings of reductive groups over $\mathbb{Z}_p$ and $\mathscr{G}_b$ an isocrystal with $\mathscr{G}_1$ structure. Let $\mathscr{G}'_b = \mathscr{G}_b \overset{\mathscr{G}_1}{\times} \mathscr{G}_2$, the induced map $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{LSht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}$ is a closed immersion.*

Proof. It is enough to prove that the basechange by any totally disconnected perfectoid space is a closed immersion. Let $\mathrm{Spa}(S, S^+)$ in Perf be totally disconnected, and let $(M, \lambda) \in \mathrm{LSht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}(S, S^+)$. Abusing notation we let λ denote a choice of representative of the equivalence class of isogenies and we let $r \in \mathbb{R}$ such that λ is defined over $\mathcal{Y}^{S^+}_{[r,\infty]}$. By unraveling the definitions we can think of M as a ring map $\mathcal{O}_{\mathscr{G}_2} \to W(S^+)[\frac{1}{\xi_{S^\sharp}}]$ and we think of λ as a ring map $\mathcal{O}_{\mathscr{G}_2} \to B^{[r,\infty]}_{S^+}$ (with the notation as in lemma 2.1.25). Since $\mathscr{G}_1 \to \mathscr{G}_2$ is a closed embedding of affine algebraic groups we have that $\mathcal{O}_{\mathscr{G}_1} = \mathcal{O}_{\mathscr{G}_2}/I$ for some finitely generated ideal $I \subseteq \mathcal{O}_{\mathscr{G}_2}$. The basechange

$$\mathrm{Spa}(S, S^+) \times_{\mathrm{LSht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}} \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$$

is representing the moduli of maps $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(S, S^+)$ for which the compositions:

$$M : \mathcal{O}_{\mathscr{G}_2} \to W(S^+)[\frac{1}{\xi_{S^\sharp}}] \to W(R^+)[\frac{1}{\xi_{R^\sharp}}]$$

$$\lambda : \mathcal{O}_{\mathscr{G}_2} \to B^{S^+}_{[r,\infty]} \to B^{R^+}_{[r,\infty]}$$

map elements of I to 0.

Let us prove that for any element $t \in W(S^+)[\frac{1}{\xi_{S^\sharp}}]$ (or $t \in B^{[r,\infty]}_{S^+}$) the moduli of points in $\mathrm{Spa}(S, S^+)$ where t is identically 0 forms a closed immersion. Fix $t \in W(S^+)[\frac{1}{\xi_{S^\sharp}}]$, since $\xi_{S^\sharp}$ is not a zero-divisor the moduli of points where t is 0 is the same as that of $\xi^n \cdot t$ so we may assume $t \in W(S^+)$. Using the Teichmüller expansion we have $t \in (S^+)^{\mathbb{N}}$ and t is 0 if and only if each entry is 0. This defines a Zariski closed subset of $\mathrm{Spa}(S, S^+)$. We prove the other case, fix $t \in B^{[r,\infty]}_{S^+} \subseteq B^{[r,\infty)}_{S^+}$ and let $Z \subseteq |\mathcal{Y}^{R^+}_{[r,\infty)}|$ be the set of valuations with $|t|_z = 0$. We have a projection map of diamonds $\pi : (\mathcal{Y}^{S^+}_{[r,\infty)})^\diamond \to \mathrm{Spd}(S, S^+)$ which is ℓ-cohomologically smooth and consequently universally open (See [26] 24.5). The moduli of points we are considering is given by maps to $\mathrm{Spa}(S, S^+)$ that factor through $Z' = |\mathrm{Spa}(S, S^+)| \setminus \pi(|\mathcal{Y}^{R^+}_{[r,\infty)}| \setminus Z)$ which is a closed subset. Since the subset of interest is closed and generalizing, it defines a closed immersion of $\mathrm{Spa}(S, S^+)$ (See [26] 7.6). $\square$

Proposition 2.3.7. *With notation as in lemma 2.3.6 the map* $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}$ *is a closed immersion. In particular, if we let* $\mathscr{G}_2 = \mathscr{G}_1 \times_{\mathbb{Z}_p} \mathscr{G}_1$ *and we apply the result to the diagonal embedding* $\Delta : \mathscr{G}_1 \to \mathscr{G}_2$ *we deduce that* $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ *is separated over* $W(\mathfrak{f})^\diamond$.

Proof. We begin by proving that the map is injective. For this consider two sets of triples

$$t_i = (\mathscr{T}_i, \Phi_i, \lambda_i) \in \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R, R^+) \, with \, i \in \{1, 2\}$$

and suppose that the $t_i \overset{\mathscr{G}_1}{\times} \mathscr{G}_2 := (\mathscr{T}_i \overset{\mathscr{G}_1}{\times} \mathscr{G}_2, \Phi_i, \lambda_i)$ become isomorphic, we need to prove $t_1 \cong t_2$. Since products of points form a basis for the v-topology we can assume $\mathrm{Spa}(R, R^+)$ to be a product of points. For a product of points any map $\mathrm{Spa}(R, R^+) \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ factors through $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$. Let $T_i \in \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R, R^+)$ factoring t_i with $T_i := (M_i, \lambda_i)$. The set of choices for T_i mapping to t_i forms a $\mathbb{W}^+\mathscr{G}_1(R, R^+)$-torsors. Since $t_1 \overset{\mathscr{G}_1}{\times} \mathscr{G}_2 \cong t_2 \overset{\mathscr{G}_1}{\times} \mathscr{G}_2$ we have that $T_1 \overset{\mathscr{G}_1}{\times} \mathscr{G}_2$ and $T_2 \overset{\mathscr{G}_1}{\times} \mathscr{G}_2$ are in the same $\mathbb{W}^+\mathscr{G}_2(R, R^+)$-orbit. But $\lambda_i \in \mathscr{G}_1(B^{[r,\infty)}_{R^+})$ so that $\lambda_1 \circ \lambda_2^{-1} \in \mathscr{G}_1(B^{[r,\infty)}_{R^+}) \cap \mathscr{G}_2(W(R^+))$, since $W(R^+) \to B^{[r,\infty)}_{R^+}$ is injective this intersection is $\mathscr{G}_1(W(R^+))$. This, together with the injectivity of lemma 2.3.6, proves that T_1 and T_2 are in the same $\mathbb{W}^+\mathscr{G}_1$-orbit, which proves $t_1 = t_2$.

Once we know $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}$ is injective we may prove that the map is proper for it to be a closed immersion. Injectivity implies the map is a separated map of v-sheaves and since each of them satisfies the valuative criterion of partial properness over $\mathbb{F}_p^\diamond$, the map between them is a partially proper map. We only have left to prove that the map $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}$ is quasi-compact. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} & \longrightarrow & \mathrm{LSht}^{\mathscr{G}'_b}_{W(\mathfrak{f})} \\ \downarrow & & \downarrow \\ \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} & \longrightarrow & \mathrm{Sht}^{\mathscr{G}'_b}_{W(\mathfrak{f})} \end{array}$$

The composition $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}$ is a quasi-compact map, and the map $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is surjective which formally implies that the map $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}$ is quasi-compact. $\square$

Proposition 2.3.8. *For any* $\mu \in X^+_*(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$ *we have that* $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ *is a closed immersion. Moreover,* $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$ *is* v*-formalizing.*

Proof. Let $\mathrm{LSht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$ denote the basechange of $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ by $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$. Given an element $(M, \lambda) \in \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R, R^+)$, M naturally defines an (R, R^+)-valued point of $\mathrm{Gr}^{\mathscr{G}}_{W(\mathfrak{f})}$ when we think of M as an element of $\mathscr{G}(B_{dR}(R^\sharp))$.

We have the following pair of Cartesian diagrams:

$$\begin{tikzcd}
\mathrm{LSht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} \arrow[r] \arrow[d] & \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \arrow[d] \\
\mathrm{Gr}^{\mathscr{G},\leq\mu}_{W(\mathfrak{f})} \arrow[r] & \mathrm{Gr}^{\mathscr{G}}_{W(\mathfrak{f})}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{LSht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} \arrow[r] \arrow[d] & \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \arrow[d] \\
\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} \arrow[r] & \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}
\end{tikzcd}$$

Since being a closed immersion can be checked v-locally on the target (See [26] 10.11), and since $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is surjective $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is a closed immersion. Moreover, by 2.2.5 the map $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{W(\mathfrak{f})} \to \mathrm{Gr}^{\mathscr{G}}_{W(\mathfrak{f})}$ is formally adic which implies that $\mathrm{LSht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is formalizing and consequently that $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is v-formalizing. $\square$

In what follows we will prove that the functors $(\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})})^{\mathrm{red}}$ are represented by affine Deligne Lusztig varieties. We warn the reader that the definition that we take of affine Deligne Lusztig varieties is not the standard one. Nevertheless, it is well known and easy to establish that the definition we take defines the same objects as the standard definition.

Definition 2.3.9. *Let $\mathscr{G}_b$ be an isocrystal with $\mathscr{G}$-structure and $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}} \to \mathfrak{T}_{\overline{\mathbb{Q}}_p}$ a cocharacter. We define the v-sheaf $X^{\mathscr{G}_b}_{\leq\mu} : \mathrm{PCAlg}^{op}_{/\mathfrak{f}} \to \mathrm{Sets}$ as:*

$$X^{\mathscr{G}_b}_{\leq\mu}(R) = \{(\mathscr{T}, \Phi, \lambda)\}/\cong$$

Where $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R))$, $\Phi : \phi^*\mathscr{T} \to \mathscr{T}$ is an isomorphism over $\mathrm{Spec}(W(R)[\frac{1}{p}])$ of relative position bounded by μ and $\lambda : \mathscr{T} \to \mathscr{G}_b$ is a ϕ-equivariant isomorphism over $\mathrm{Spec}(W(R)[\frac{1}{p}])$

Proposition 2.3.10. *We have an identification $X^{\mathscr{G}_b}_{\leq\mu} = (\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})})^{\mathrm{red}}$. Moreover, the map $(X^{\mathscr{G}_b}_{\leq\mu})^\diamond \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is injective and $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is a specializing v-sheaf.*

Proof. Given a map $\mathrm{Spec}(R) \to X^{\mathscr{G}_b}_{\leq\mu}$ we construct functorially a map $\mathrm{Spec}(R)^\diamond \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ in what follows. The untilt is always the characteristic p untilt. For any perfectoid space $f : \mathrm{Spa}(S, S^+) \to \mathrm{Spec}(R)^\diamond$ we get a triple $(f^*\mathscr{T}, f^*\Phi, f^*\lambda)$ coming from the map of rings $f : W(R) \to W(S^+)$ and by restriction to the appropriate loci $\mathcal{Y}^{S^+}_{[0,\infty)}$, $\mathcal{Y}^{S^+}_{[0,\infty)} \setminus V(p)$ and $\mathcal{Y}^{S^+}_{[r,\infty)}$ respectively. This data defines functorially a map $\mathrm{Spa}(S, S^+) \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$, consequently a map $\mathrm{Spec}(R)^\diamond \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$. The construction of $\mathrm{Spec}(R)^\diamond \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is clearly functorial in $\mathrm{PCAlg}^{op}_{/k}$. By adjunction, this gives a map $X^{\mathscr{G}_b}_{\leq\mu} \to (\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})})^{\mathrm{red}}$, we claim this map is an isomorphism.

We begin by proving it is injective, since $X^{\mathscr{G}_b}_{\leq\mu}$ is represented by a perfect scheme $((X^{\mathscr{G}_b}_{\leq\mu})^\diamond)^{\mathrm{red}} = X^{\mathscr{G}_b}_{\leq\mu}$ and we may prove the map $(X^{\mathscr{G}_b}_{\leq\mu})^\diamond \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is injective instead. Take two arbitrary maps $g_i : \mathrm{Spa}(R, R^+) \to (X^{\mathscr{G}_b}_{\leq\mu})^\diamond$, it is enough to prove injectivity v-locally so we may assume the maps factor through maps of the form $g'_i : \mathrm{Spec}(R^+)^\diamond \to (X^{\mathscr{G}_b}_{\leq\mu})^\diamond$. The g'_i are given by data $(\mathscr{T}_i, \Phi_i, \lambda_i)$ over $\mathrm{Spec}(W(R^+))$, $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$ and $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$ respectively and the g_i are given by restricting these data to $\mathcal{Y}^{R^+}_{[0,\infty)}$, $\mathcal{Y}^{R^+}_{[0,\infty)} \setminus V(p)$ and $\mathcal{Y}^{R^+}_{[r,\infty)}$ respectively. Nevertheless, we can recover g'_i from g_i since we can use λ_i to glue back the restricted data as in the proof of proposition 2.3.5.

Let us now prove surjectivity. Let $f : \mathrm{Spec}(A)^\diamond \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ and $g : \mathrm{Spa}(R, R^+) \to \mathrm{Spec}(A)^\diamond$ be a map with $\mathrm{Spa}(R, R^+)$ a product of points and $A \in \mathrm{PCAlg}^{op}_{/\mathfrak{f}}$. We will show below how to construct the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spa}(R, R^+) \arrow[r] \arrow[d] & \mathrm{Spec}(A)^\diamond \arrow[d] \\
(X^{\mathscr{G}_b}_{\leq\mu})^\diamond \arrow[r] & \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}
\end{tikzcd}$$

Since products of points are a basis for the topology, and since $(X^{\mathscr{G}_b}_{\leq\mu})^\diamond \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is injective this defines a map $\mathrm{Spec}(A)^\diamond \to (X^{\mathscr{G}_b}_{\leq\mu})^\diamond$ factoring our original map to $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ and proves the desired surjectivity.

Fix a pseudo-uniformizer $\varpi \in R^+$, we let $\mathrm{Spa}(R_\infty, {R_\infty}^+)$ be a second product of points defined as follows: $R^+_\infty = \prod_{i=1}^\infty R^+$ with pseudo-uniformizer now given by $\varpi_{R_\infty} = (\varpi^i)_{i=1}^\infty$. The product of points $\mathrm{Spa}(R_\infty, {R_\infty}^+)$ comes equipped with a family of closed embeddings $\iota_i : \mathrm{Spa}(R, R^+) \to \mathrm{Spa}(R_\infty, {R_\infty}^+)$ given in coordinates by the projection onto the ith-factor. The diagonal ring map $\Delta_g : A \to \prod_{i=1}^\infty R^+$ induces a map $\Delta_g : \mathrm{Spa}(R_\infty, {R_\infty}^+) \to \mathrm{Spec}(A)^\diamond$ with the property that $\Delta_g \circ \iota_i = g$ for every i. Since $\mathrm{Spa}(R_\infty, {R_\infty}^+)$ is a product of points, by proposition 2.1.20, the map $f \circ \Delta_g$ can be represented by a triple $(\mathcal{G}_{R_\infty}, \Phi_{R_\infty}, \lambda_{R_\infty})$ with $\mathcal{G}_{R_\infty}$ trivial. After choosing a trivialization for $\mathcal{G}_{R_\infty}$ we can think of λ_{R_∞} as a ring map $\mathcal{O}_{\mathscr{G}} \to B^{{R_\infty}^+}_{[r,\infty]}$. Moreover, since $f \circ \Delta_g \circ \iota_i = f \circ \Delta_g \circ \iota_j$ we have that for all i the composition

$$\lambda_i : \mathcal{O}_{\mathscr{G}} \to B^{{R_\infty}^+}_{[r,\infty]} \to B^{R_i^+}_{[r_i,\infty]} = B^{R^+}_{[r_i,\infty]}$$

lies in the same $\mathscr{G}(W(R^+))$-orbit. Clearly $\mathscr{G}(W({R_\infty}^+)) = \prod_{i=1}^\infty \mathscr{G}(W(R^+))$ so after a change of trivialization we may assume that $r_i = r_j$ and that $\lambda_i = \lambda_j =: \lambda_R$ for all $1 \leq i, j < \infty$. We claim that λ_R factors through the inclusion of rings $W(R^+)[\frac{1}{p}] \subseteq B^{R^+}_{[r,\infty]}$. Take an element $t \in \mathcal{O}_{\mathscr{G}}$ and consider $s = \lambda_{R_\infty}(t) \in B^{{R_\infty}^+}_{[r,\infty]}$, after replacing r by a larger number if necessary we may assume $r = n \in \mathbb{N}$. In particular, $p^k \cdot s$ lies in the p-adic completion of $W({R_\infty}^+)[\frac{[\varpi_{R_\infty}]}{p^n}]$ for some large enough $k \in \mathbb{N}$. Let us write $p^k \cdot s$ as $\sum_{j=0}^\infty x^{n(j)}[\alpha_j]p^j$ where x denotes $\frac{[\varpi_{R_\infty}]}{p^n}$, $0 \leq n(j)$ is a multiplicity, and $\alpha_j \in R^+_\infty$. We have that $\iota_i(p^k \cdot s) = \sum_{j=0}^\infty (\frac{[\varpi]^i}{p^n})^{n(j)}[\iota_i(\alpha_j)]p^j$ with $\iota_i(\alpha_j) \in R^+$. In particular,

$$p^k \cdot \lambda_R(t) \in \bigcap_{i\in\mathbb{N}} (H^0(\mathcal{Y}^{R^+}_{[\frac{n}{i},\infty]}, \mathcal{O}^+)),$$

but this intersection is $W(R^+)$ proving the claim.

Since the elements of the triple $(\mathcal{G}_{R_\infty}, \Phi_{R_\infty}, \lambda_{R_\infty})$ are defined over $\mathrm{Spec}(W(R^+))$ and $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$ they define a map to $\mathrm{Spec}(R^+) \to X^{\mathscr{G}_b}_{\leq\mu}$. The composition $\mathrm{Spa}(R, R^+) \to \mathrm{Spec}(R^+)^\diamond \to (X^{\mathscr{G}_b}_{\leq\mu})^\diamond$ gives the factorization we were looking for.

That $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is formally separated follows from lemma 1.3.32 and proposition 2.3.7, that $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is v-formalizing follows from proposition 2.3.8.

□

Lemma 2.3.11. *The adjunction map $(X^{\mathscr{G}_b}_{\leq\mu})^\diamond \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ arising from the identification of proposition 2.3.10 is a closed immersion.*

Proof. We will use that $X^{\mathscr{G}_b}_{\leq\mu}$ admits a closed immersion into the Witt vector Grassmanian $Gr^{\mathscr{G},}_{\mathcal{W},\mathfrak{f}}$. We have that

$$(X^{\mathscr{G}_b}_{\leq\mu})^\diamond = \bigcup_{\nu\in X_*(\mathfrak{T}_{\overline{\mathbb{Q}}_p})} (X^{\mathscr{G}_b}_{\leq\mu} \cap Gr^{\mathscr{G},\leq\nu}_{\mathcal{W},\mathfrak{f}})^\diamond$$

and since each of this subsheaves are coming from a perfectly finitely presented proper scheme over $\mathfrak{f}$, they are proper as v-sheaves over $\mathfrak{f}^\diamond$. Consequently, the map $(X^{\mathscr{G}_b}_{\leq\mu} \cap Gr^{\mathscr{G},\leq\nu}_{\mathcal{W},\mathfrak{f}})^\diamond \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is proper and since it is injective a closed immersion.

Now, $X^{\mathscr{G}_b}_{\leq\mu}$ is a scheme which is locally perfectly of finite type (See [9] theorem 1.1), and in particular each point admits an open neighborhood that is spectral and Noetherian as a topological space. Using a compactness argument in the patch topology, to every point $x \in |X^{\mathscr{G}_b}_{\leq\mu}|$ we may associate an open neighborhood $U_x \subseteq X^{\mathscr{G}_b}_{\leq\mu}$ and finite number of $\nu_i \in X^+_*(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$ for which $U = U \cap (\bigcup_{i\in I_x} Gr^{\mathscr{G},\leq\nu_i}_{\mathcal{W},\mathfrak{f}})$. Indeed, if U_x is Noetherian every closed subset is open in the constructible topology.

In proposition 2.3.10, we proved that $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is a specializing v-sheaf, so by proposition 1.4.17 we get a specialization map $\mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}} : |\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}| \to |X^{\mathscr{G}_b}_{\leq\mu}|$. We let $V_x = (\mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}})^{-1}(U_x)$ for $x \in |X^{\mathscr{G}_b}_{\leq\mu}|$ and U_x as above, this forms an open cover of $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$. Since being a closed immersion is v-local on the target and $V_x \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is a formally adic open immersion it is enough to verify that $(V_x^{\mathrm{red}})^\diamond \to V_x$ is a closed immersion. But the adjunction map $(U_x)^\diamond \to V_x$ fits in the following Cartesian diagram:

$$\begin{tikzcd} U_x^\diamond \arrow[r, "Id"] \arrow[d] & U_x^\diamond \arrow[r] \arrow[d] & V_x \arrow[d] \\ (\bigcup_{i\in I_x} Gr^{\mathscr{G},\leq\nu_i}_{\mathcal{W},\mathfrak{f}} \cap X^{\mathscr{G}_b}_{\leq\mu})^\diamond \arrow[r] & (X^{\mathscr{G}_b}_{\leq\mu})^\diamond \arrow[r] & \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} \end{tikzcd}$$

Since the union of a finite number of closed immersions still defines a closed immersion we can conclude by basechange that $U_x^\diamond \to V_x$ is also a closed immersion.

□

Proposition 2.3.12. *With the notation as above the map $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} \to W(\mathfrak{f})^\diamond$ is a p-adically smelted kimberlite as in definition 1.4.23.*

Proof. Proposition 2.3.10 proves that $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is a specializing v-sheaf, in proposition 2.3.10 we proved that $(\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})})^{\mathrm{red}}$ is represented by a scheme and by lemma 2.3.11 the adjunction map $((\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})})^{\mathrm{red}})^\diamond \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is a closed immersion which finished the proof that $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is prekimberlite. Theorem 23.1.4 of [28] proves that $(\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})})_\eta$ is a locally spatial diamond. □

2.3.2 Comparison of tubular neighborhoods

Recall that in this section $\mathscr{G}$ is a reductive group over $\mathrm{Spec}(\mathbb{Z}_p)$, let $\mathcal{D} = (\mathcal{D}, \Phi)$ be a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(\mathfrak{f}))$ together with an isomorphism $\Phi_{\mathcal{D}} : \phi^*\mathcal{D} \to \mathcal{D}$ defined over $\mathrm{Spec}(W(\mathfrak{f})[\frac{1}{p}])$, and fix $\mu \in X_*^+(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$. We can define some objects associated to this data, which are nothing but "coordinate-free" versions of the moduli we defined in the previous sections:

Definition 2.3.13. *1. We denote the functor $\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})} : \mathrm{Perf}_{\mathfrak{f}} \to \mathrm{Sets}$ with*

$$\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}(R, R^+) = \{((R^\sharp, \iota, f), \mathscr{T}, \psi)\}/\cong$$

Where $(R^\sharp, \iota, f)$ is an untilt over $W(\mathfrak{f})$, $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathcal{Y}_{R^+}$ and $\psi : \mathscr{T} \to \mathcal{D}$ is an isomorphism defined over $\mathcal{Y}_{R^+} \setminus V(\xi_{R^\sharp})$ that is meromorphic along $\xi_{R^\sharp}$.

2. We denote the functor $\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})} : \mathrm{Perf}_{\mathfrak{f}} \to \mathrm{Sets}$ with

$$\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}(R, R^+) = \{((R^\sharp, \iota, f), \mathscr{T}, \Phi, \lambda)\}/\cong$$

Where $(R^\sharp, \iota, f)$ is an untilt over $W(\mathfrak{f})$, $(\mathscr{T}, \Phi)$ is a shtuka with $\mathscr{G}$-structure, and $\lambda : \mathscr{T} \to \mathcal{D}$ is an isogeny.

The functors $\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ and $\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}$ come with a canonical section $can_{\mathcal{D}} : \mathrm{Spec}(\mathfrak{f})^\diamond \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ given by the data $(\phi^*\mathcal{D}, \Phi_{\mathcal{D}})$ and $can_{\mathcal{D}} : \mathrm{Spec}(\mathfrak{f})^\diamond \to \mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}$ given by $(\mathcal{D}, \Phi_{\mathcal{D}}, Id)$ respectively. We point out that if we fix an isomorphism $\tau : \mathcal{D} \to \mathscr{G}$ we get isomorphisms $\tau : \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})} \to \mathrm{Gr}^{\mathscr{G}}_{W(\mathfrak{f})}$, and $\tau : \mathrm{Gr}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})} \to \mathrm{Gr}^{\mathscr{G},\leq\mu}_{W(\mathfrak{f})}$. Moreover, if we are given a section $\sigma : \mathrm{Spec}(\mathfrak{f})^\diamond \to \mathrm{Gr}^{\mathscr{G}}_{W(\mathfrak{f})}$ we can construct a pair $(\mathcal{D}, \Phi_{\mathcal{D}})$ and an isomorphism $\tau : \mathcal{D} \to \mathscr{G}$ such that the following diagram is commutative:

$$\begin{tikzcd} & \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})} \arrow[dd, "\tau"] \\ \mathrm{Spec}(\mathfrak{f})^{\diamond} \arrow[ur, "can_{\mathcal{D}}"] \arrow[dr, "\sigma"'] & \\ & \mathrm{Gr}^{\mathscr{G}}_{W(\mathfrak{f})} \end{tikzcd}$$

Analogously, if we find a ϕ-equivariant isomorphism $\tau : \mathcal{D} \to \mathscr{G}_b$ over $\mathrm{Spec}(W(\mathfrak{f})[\frac{1}{p}])$ we get an isomorphism $\tau : \mathrm{Sht}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$, and given a section $\sigma : \mathrm{Spec}(\mathfrak{f})^{\diamond} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ we can construct $(\mathcal{D}, \Phi_{\mathcal{D}})$ and τ making the following diagram commutative:

$$\begin{tikzcd} & \mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})} \arrow[dd, "\tau"] \\ \mathrm{Spec}(\mathfrak{f})^{\diamond} \arrow[ur, "can_{\mathcal{D}}"] \arrow[dr, "\sigma"'] & \\ & \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \end{tikzcd}$$

Since $\mathfrak{f}$ is algebraically closed every tubular neighborhood of Grassmanians and moduli of mixed characteristic shtukas at closed points are coming from the canonical one associated to some pair $(\mathcal{D}, \Phi_{\mathcal{D}})$. Indeed, every closed point of $X^{\mathscr{G}_b}_{\leq\mu}$ and $\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},\mathfrak{f}}$ is the image of a section since the bounded version of these ind-schemes are locally perfectly of finite presentation over $\mathrm{Spec}(\mathfrak{f})$.

Theorem 2.3.14. *Given $(\mathcal{D}, \Phi_{\mathcal{D}})$ and $\mu \in X_*(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$ as above, we have canonical identifications of v-sheaves $\widehat{\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}} \cong \widehat{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ and $\widehat{\mathrm{Sht}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}} \cong \widehat{\mathrm{Gr}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$.*

Before proving the theorem and describing the map identifying these tubular neighborhoods we will need some preparation.

Definition 2.3.15. *1. We let $\widehat{LG}_{\mathcal{D}}$ denote the sheaf of groups over $W(\mathfrak{f})^{\diamond}$ given by $\widehat{LG}_{\mathcal{D}}(R, R^+) = \{((R^\sharp, \iota, f), g)\}$. Where $(R^\sharp, \iota, f)$ is an untilt over $W(\mathfrak{f})$ and $g : \mathcal{D} \to \mathcal{D}$ is an automorphism of $\mathscr{G}$-torsors defined over $\mathrm{Spec}(W(R^+))$ for which there is a pseudouniformizer $\varpi_g \in R^+$, depending of g, such that g restricts to the identity over $\mathrm{Spec}(W(R^+)/[\varpi_g])$. We define $\widehat{LG}_{\phi^*\mathcal{D}}$ in a similar way exchanging the role of $\mathcal{D}$ for that of $\phi^*\mathcal{D}$.*

2. We let $\widehat{LGr}_{\mathcal{D}}$ be the v-sheaf over $W(\mathfrak{f})^{\diamond}$ assigning

$$\widehat{LGr}_{\mathcal{D}}(R, R^+) = \{(R^\sharp, \iota, f), \mathscr{T}, \psi, \sigma\}/\cong$$

Where $(R^\sharp, \iota, f)$ is an untilt over $W(\mathfrak{f})$, $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$, $\psi : \mathscr{T} \to \mathcal{D}$ is an isomorphism over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$ and $\sigma : \mathscr{T} \to \phi^*\mathcal{D}$ is an isomorphism of $\mathscr{G}$-torsors over $\mathrm{Spec}(W(R^+))$ such that there is a pseudouniformizer $\varpi \in R^+$ depending on the data for which $\Phi_{\mathcal{D}} \circ \sigma = \psi$ when restricted to $\mathrm{Spec}(W(R^+)/[\varpi])$.

3. We let $\widehat{\mathrm{LSht}}_{\mathcal{D}}$ be the v-sheaf over $W(\mathfrak{f})^{\diamond}$ assigning

$$\widehat{\mathrm{LSht}}_{\mathcal{D}}(R, R^+) = \{(R^\sharp, \iota, f), \mathscr{T}, \Phi, \lambda, \sigma\}/\cong$$

Where $(R^\sharp, \iota, f)$ is an untilt over $W(\mathfrak{f})$, $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$, $\Phi : \phi^*\mathscr{T} \to \mathscr{T}$ is an isomorphism over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$, $\lambda : \mathscr{T} \to \mathcal{D}$ is an isogeny over $\mathcal{Y}^{R^+}_{[r,\infty]}$ and $\sigma : \mathscr{T} \to \mathcal{D}$ is an isomorphism of $\mathscr{G}$-torsors over $\mathrm{Spec}(W(R^+))$ such that there is a pseudouniformizer $\varpi \in R^+$ depending on the data for which $\sigma = \lambda$ when restricted to $\mathrm{Spec}(B^{R^+}_{[r,\infty]}/[\varpi])$.

Standard arguments using proposition 2.1.21 will prove that the objects in definition 2.3.15 are v-sheaves. Notice though, that the category of vector bundles over $\mathrm{Spec}(W(R^+))$ fibered over Perf does not form a stack for the v-topology. Nevertheless, the category fibered over Perf that assigns to $\mathrm{Spa}(R, R^+)$ the category of pairs $(\mathscr{T}, \sigma)$ where $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$ and $\sigma : \mathscr{T} \to \mathscr{G}$ is a trivialization does form a stack for the v-topology on Perf.

There is a natural map $\widehat{LGr}_{\mathcal{D}} \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ that takes a triple $(\mathscr{T}, \psi, \sigma)$ and assigns the pair $(\mathscr{T}, \psi)$ restricted to $\mathcal{Y}^{R^+}_{[0,\infty)}$ and $\mathcal{Y}^{R^+}_{[0,\infty)} \setminus V(\xi)$ respectively. This map is $\widehat{LG}_{\phi^*\mathcal{D}}$-equivariant when we consider the left action $\widehat{LG}_{\phi^*\mathcal{D}} \times \widehat{LGr}_{\mathcal{D}} \to \widehat{LGr}_{\mathcal{D}}$ sending an element $(g, (\mathscr{T}, \psi, \sigma))$ to $(\mathscr{T}, \psi, g \circ \sigma)$ and $\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ is given the trivial $\widehat{LG}_{\phi^*\mathcal{D}}$-action.

Lemma 2.3.16. *The natural map $\widehat{LGr}_{\mathcal{D}} \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ factors through $\widehat{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$. Moreover, the map $\widehat{LGr}_{\mathcal{D}} \to \widehat{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ is a $\widehat{LG}_{\phi^*\mathcal{D}}$-torsor.*

Proof. We begin by proving that $\widehat{LGr}_{\mathcal{D}}$ formalizes any affinoid perfectoid $\mathrm{Spa}(A, A^+)$. Indeed, take a map $\mathrm{Spa}(A, A^+) \to \widehat{LGr}_{\mathcal{D}}$ given by an untilt and a triple $(\mathscr{T}, \psi, \sigma)$ and take a map $f : \mathrm{Spa}(B, B^+) \to \mathrm{Spd}(A^+, A^+)$, we have to define functorially a map $\mathrm{Spa}(B, B^+) \to \widehat{LGr}_{\mathcal{D}}$. We can construct an untilt for B functorially as in lemma 1.4.11. We have a map of affine schemes $f : \mathrm{Spec}(W(B^+)) \to \mathrm{Spec}(W(A^+))$ along which we can pullback to get the triple $(f^*\mathscr{T}, f^*\psi, f^*\sigma)$ where in this case $f^*\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(B^+))$, $f^*\psi : f^*\mathscr{T} \to \mathcal{D}$ is an isomorphism over $\mathrm{Spec}(W(B^+)[\frac{1}{f(\xi)}]$ and $f^*\sigma : f^*\mathscr{T} \to \phi^*\mathcal{D}$ is an isomorphism over $\mathrm{Spec}(W(B^+))$. We need to verify that this triple satisfies the constraints. Take a pseudouniformizer $\varpi_A \in A^+$ for which $\Phi_{\mathcal{D}} \circ \sigma = \psi$ in $\mathrm{Spec}(W(A^+)/[\varpi_A])$. By continuity, $f(\varpi_A)$ is topologically nilpotent and there is a pseudo-uniformizer $\varpi_B \in B^+$ with $f(\varpi_A) = \varpi_B \cdot t$ for some $t \in B^+$. We have $\Phi_{\mathcal{D}} \circ f^*\sigma = f^*\psi$ over $\mathrm{Spec}(W(B^+)/[\varpi_B])$ proving the constraint holds.

To prove that $\widehat{LGr}_{\mathcal{D}} \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ factors through $\widehat{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ it is enough to prove that for any map $\mathrm{Spd}(R^+, R^+) \to \widehat{LGr}_{\mathcal{D}}$ the map of reductions $(\mathrm{Spd}(R^+, R^+))^{\mathrm{red}} = \mathrm{Spec}(R^+_{\mathrm{red}})^\diamond \to (\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})})^{\mathrm{red}}$ factors through the canonical map $can_{\mathcal{D}} : \mathrm{Spec}(\mathfrak{f})^\diamond \to (\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})})^{\mathrm{red}}$. After restricting the data $(\mathscr{T}, \psi, \sigma)$ to $\mathrm{Spec}(W(R^+_{\mathrm{red}}))$ we get the identity $\Phi_{\mathcal{D}} \circ \sigma = \psi$. After pullback, the map $\mathrm{Spec}(R^+_{\mathrm{red}})^\diamond \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ is given by the tuple $(\mathscr{T}, \psi)$. Since this data is isomorphic via σ to $(\phi^*\mathcal{D}, \Phi_{\mathcal{D}})$, the map factors through $can_{\mathcal{D}} : \mathrm{Spec}(\mathfrak{f})^\diamond \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$.

We now prove that $\widehat{LGr}_{\mathcal{D}} \to \widehat{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ is surjective. It is enough to prove this for a product of points which we denote $\mathrm{Spa}(R, R^+)$, with pseudo-uniformizer $\varpi \in R^+$. In this case, by proposition 2.1.20, a (R, R^+)-valued point is given by $(\mathscr{T}, \psi)$ with $\mathscr{T}$ defined over $\mathrm{Spec}(W(R^+))$ and $\psi : \mathscr{T} \to \mathcal{D}$ defined over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$, with the aditional condition that $(\mathscr{T}, \psi)$ is isomorphic to $(\phi^*\mathcal{D}, \Phi_{\mathcal{D}})$ when restricted to $W(R^+_{\mathrm{red}})$ and $W(R^+_{\mathrm{red}})[\frac{1}{p}]$. Such an isomorphism $\sigma_{\mathrm{red}} : (\mathscr{T}, \psi) \to (\phi^*\mathcal{D}, \Phi_{\mathcal{D}})$ is unique and has to be given by $\sigma_{\mathrm{red}} = \Phi_{\mathcal{D}}^{-1} \circ \psi_{\mathrm{red}}$, since it has to satisfy the commutative diagram:

$$\begin{tikzcd}
\mathscr{T} \arrow[dr, "\psi"] \arrow[dd, "\sigma_{\mathrm{red}}"'] & \\
& \mathcal{D} \\
\phi^*\mathcal{D} \arrow[ur, "\Phi_{\mathcal{D}}"] &
\end{tikzcd}$$

The morphism $\widetilde{\sigma} = \Phi_{\mathcal{D}}^{-1} \circ \psi : \mathscr{T} \to \phi^* D$ is defined over $\mathcal{Y}^{R^+}_{[r,\infty]}$ for r sufficiently big (so that it avoids $V(\xi)$) and it restricts to σ_{red}. We can use lemma 2.1.26 to construct an isomorphism $\sigma : \mathscr{T} \to \phi^* D$ such that $\sigma = \widetilde{\sigma}$ when restricted to $\mathrm{Spec}(B^{R^+}_{[r,\infty]}/[\varpi'])$ for some pseudo-uniformizer $\varpi' \in R^+$. In particular

$\Phi_{\mathcal{D}} \circ \sigma = \psi$ over $\mathrm{Spec}(W(R^+)/[\varpi'])$. The data $(\mathscr{T}, \psi, \sigma)$ constructs a map $\mathrm{Spa}(R, R^+) \to \widehat{LGr}_{\mathcal{D}}$ which evidently composes to the original map $\mathrm{Spa}(R, R^+) \to \widehat{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$.

Finally, we need to prove $\widehat{LGr}_{\mathcal{D}} \times_{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}} \widehat{LGr}_{\mathcal{D}} \cong \widehat{LG}_{\phi^*\mathcal{D}} \times_{W(\mathfrak{f})^\diamond} \widehat{LGr}_{\mathcal{D}}$. Take two sets of data $(\mathscr{T}_i, \psi_i, \sigma_i)$ over $\mathrm{Spa}(A, A^+)$ and suppose that $(\mathscr{T}_1|_{\mathcal{Y}_{A^+}}, \psi_1|_{\mathcal{Y}_{A^+}\setminus V(\xi)}) \cong (\mathscr{T}_2|_{\mathcal{Y}_{A^+}}, \psi_2|_{\mathcal{Y}_{A^+}\setminus V(\xi)})$. The isomorphism must be given by $\psi_1^{-1} \circ \psi_2 : \mathscr{T}_2 \to \mathscr{T}_1$ and by the fully-faithfulness part of theorem 2.1.15 the isomorphism extends over $\mathrm{Spec}(W(A^+))$. Moreover, we can define $g = \sigma_1 \circ \psi_1^{-1} \circ \psi_2 \circ \sigma_2^{-1} : \phi^*\mathcal{D} \to \phi^*\mathcal{D}$. By hypothesis, $\sigma_1 \circ \psi_1^{-1} = \Phi_{\mathcal{D}}^{-1}$ and $\psi_2 \circ \sigma_2^{-1} = \Phi_{\mathcal{D}}$ on $\mathrm{Spec}(W(A^+)/[\varpi_A]$ for some suitable choice of pseudouniformizer $\varpi_A \in A^+$. Consequently we can associate to the original data the tuple $(g, \mathscr{T}_2, \psi_2, \sigma_2) \in \widehat{LG}_{\phi^*\mathcal{D}} \times_{W(\mathfrak{f})^\diamond} \widehat{LGr}_{\mathcal{D}}(A, A^+)$. On the other hand, to a tupple $(g, \mathscr{T}, \psi, \sigma)$ the action of $\widehat{LG}_{\phi^*\mathcal{D}}$ associates the pair of tuples $(\mathscr{T}, \psi, g \circ \sigma)$ and $(\mathscr{T}, \psi, \sigma)$. Since these two constructions are functorial and compose to the identity they define isomorphisms. $\square$

For moduli spaces of shtukas we have a very similar story. We have a projection map $\pi : \widehat{\mathrm{LSht}}_{\mathcal{D}} \to \mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}$, which we can construct by assigning to a tuple $(\mathscr{T}, \Phi, \lambda, \sigma)$ the tuple $(\mathscr{T}|_{\mathcal{Y}^{R^+}_{[0,\infty)}}, \Phi|_{\mathcal{Y}^{R^+}_{[0,\infty)}\setminus V(\xi)}, \lambda)$. Moreover, this projection is $\widehat{LG}_{\mathcal{D}}$-equivariant when we endow $\widehat{\mathrm{LSht}}_{\mathcal{D}}$ with the left action $\widehat{LG}_{\mathcal{D}} \times \widehat{\mathrm{LSht}}_{\mathcal{D}} \to \widehat{\mathrm{LSht}}_{\mathcal{D}}$ sending the tuple $(g, (\mathscr{T}, \Phi, \lambda, \sigma))$ to the tuple $(\mathscr{T}, \Phi, \lambda, g \circ \sigma)$ and when $\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}$ is given the trivial action.

Lemma 2.3.17. *The natural map $\widehat{\mathrm{LSht}}_{\mathcal{D}} \to \mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}$ factors through $\widehat{\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$. Moreover, the map $\widehat{\mathrm{LSht}}_{\mathcal{D}} \to \widehat{\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ is a $\widehat{LG}_{\mathcal{D}}$-torsor.*

Proof. The proves that $\widehat{\mathrm{LSht}}_{\mathcal{D}}$ formalizes any map $\mathrm{Spa}(A, A^+) \to \widehat{\mathrm{LSht}}_{\mathcal{D}}$ with $\mathrm{Spa}(A, A^+) \in \mathrm{Perf}$, that the map $\widehat{\mathrm{LSht}}_{\mathcal{D}} \to \mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}$ factors through $\widehat{\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ and that this later map is surjective in that locus follow very similar arguments to those given in the proof of lemma 2.3.16. We omit the details.

Let us prove that $\widehat{\mathrm{LSht}}_{\mathcal{D}} \times_{\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}} \widehat{\mathrm{LSht}}_{\mathcal{D}} \cong \widehat{LG}_{\mathcal{D}} \times_{W(\mathfrak{f})^\diamond} \widehat{\mathrm{LSht}}_{\mathcal{D}}$. Take two sets of data $(\mathscr{T}_i, \Phi_i, \lambda_i, \sigma_i)$ over $\mathrm{Spa}(A, A^+)$ and suppose that $(\mathscr{T}_1|_{\mathcal{Y}^{A^+}_{[0,\infty)}}, \Phi_1|_{\mathcal{Y}^{A^+}_{[0,\infty)}\setminus V(\xi)}, \lambda_1) \cong (\mathscr{T}_2|_{\mathcal{Y}^{A^+}_{[0,\infty)}}, \Phi_2|_{\mathcal{Y}^{A^+}_{[0,\infty)}\setminus V(\xi)}, \lambda_2)$. The isomorphism must be the unique lift of $\lambda_1^{-1} \circ \lambda_2 : \mathscr{T}_2 \to \mathscr{T}_1$ to $\mathcal{Y}^{A^+}_{[0,\infty)}$. Glueing along the λ_i we can also define a lift to $\mathcal{Y}_{A^+}$. Since the $\mathscr{T}_i$ are defined over $\mathrm{Spec}(W(A^+))$ and by the fully-faithfulness part of theorem 2.1.15 the isomorphism extends to $\mathrm{Spec}(W(A^+)$. Moreover, we can define $g = \sigma_1 \circ \lambda_1^{-1} \circ \lambda_2 \circ \sigma_2^{-1} : \mathcal{D} \to \mathcal{D}$. By hypothesis, $\sigma_1 \circ \lambda_1^{-1} = Id = \lambda_2 \circ \sigma_2^{-1}$ over $\mathrm{Spec}(B^{A^+}_{[r,\infty]}/[\varpi_A])$ for some suitable choice of pseudouniformizer $\varpi_A \in A^+$. Consequently we can associate to the original data the tuple $(g, \mathscr{T}_2, \Phi_2, \lambda_2, \sigma_2) \in \widehat{LG}_{\mathcal{D}} \times_{W(\mathfrak{f})^\diamond} \widehat{\mathrm{LSht}}_{\mathcal{D}}(A, A^+)$. The action map gives back isomorphic data. $\square$

We can now prove the theorem.

Proof. (of theorem 2.3.14). For this proof we define φ to be the inverse of Frobenious, $\varphi = \phi^{-1}$. We begin by observing there is an isomorphism $\theta : \widehat{LG}_{\mathcal{D}} \to \widehat{LG}_{\phi^*\mathcal{D}}$ given by sending $g \in \widehat{LG}_{\mathcal{D}}(R, R^+)$ with $g : \mathcal{D} \to \mathcal{D}$ to $\phi^* g : \phi^*\mathcal{D} \to \phi^*\mathcal{D}$. By definition of $\widehat{LG}_{\mathcal{D}}$, there is a pseudo-uniformizer $\varpi_g \in R^+$ for which $g = Id$ in $\mathrm{Spec}(W(R^+)/[\varpi_g])$. One can verify that $\phi^* g = Id$ in $\mathrm{Spec}(W(R^+)/[\varpi^p])$ so that $\phi^* g \in \widehat{LG}_{\phi^*\mathcal{D}}$ the inverse of this group homomorphism is of course given by sending $h \in \widehat{LG}_{\phi^*\mathcal{D}}(R, R^+)$ to $\varphi^* h$. Using θ one can then endow $\widehat{LGr}_{\mathcal{D}}$ with a $\widehat{LG}_{\mathcal{D}}$ action for which the projection $\pi : \widehat{LGr}_{\mathcal{D}} \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ of lemma 2.3.16 is a $\widehat{LG}_{\mathcal{D}}$-torsor.

In what follows we construct a $\widehat{LG}_{\mathcal{D}}$-equivariant isomorphism between $\widehat{LGr}_{\mathcal{D}} \to \widehat{\mathrm{LSht}}_{\mathcal{D}}$. Take a perfectoid Huber pair (A, A^+) and tuple $(\mathscr{T}, \psi, \sigma) \in \widehat{LGr}_{\mathcal{D}}(A, A^+)$. Consider the $\mathscr{G}$-torsor $\varphi^*\mathscr{T}$, and consider the map $\Phi : \mathscr{T} \to \varphi^*\mathscr{T}$ defined by $\Phi = (\varphi^*\sigma)^{-1} \circ \psi$. We now construct a ϕ-equivariant map $\lambda : \varphi^*\mathscr{T} \to \mathcal{D}$. Consider the following (non-commutative!!!) diagram:

$$\begin{tikzcd} \mathscr{T} \arrow[r, "\sigma"] \arrow[d, "\Phi"] & \phi^*\mathcal{D} \arrow[d, "\Phi_{\mathcal{D}}"] \\ \varphi^*\mathscr{T} \arrow[r, "\varphi^*\sigma"] & \mathcal{D} \end{tikzcd}$$

Each of the arrows of the diagram is defined over $\mathcal{Y}^{A^+}_{[r,\infty]}$ for big enough r avoiding $V(\xi)$. Moreover, by hypothesis there is a pseudo-uniformizer $\varpi \in A^+$ for which $\psi = \Phi_{\mathcal{D}} \circ \sigma$ over $\mathrm{Spec}(W(R^+)/[\varpi])$. We can see that $\varphi^*\sigma \circ \Phi = \Phi_{\mathcal{D}} \circ \sigma$ over $\mathrm{Spec}(B^{A^+}_{[r,\infty]}/[\varpi])$ and in particular the morphism $\varphi^*\sigma : \varphi^*\mathscr{T} \to \mathcal{D}$ is ϕ-equivariant over this locus. By lemma 2.1.28 there is a unique isogeny over $\mathcal{Y}^{A^+}_{[r,\infty]}$ denoted $\lambda : \varphi^*\mathscr{T} \to \mathcal{D}$ such that $\lambda = \varphi^*\sigma$ when restricted to $\mathrm{Spec}(B^{A^+}_{[r,\infty]}/[\varpi])$. We can associate to our original data:

$$(\mathscr{T}, \psi, \sigma) \mapsto (\varphi^*\mathscr{T}, \Phi, \lambda, \varphi^*\sigma)$$

This construction is functorial when we let (A, A^+) vary by the uniqueness of λ. This gives a map $\Theta : \widehat{LGr}_{\mathcal{D}} \to \widehat{\mathrm{LSht}}_{\mathcal{D}}$. Moreover, we have $g \cdot (\varphi^*\mathscr{T}, \Phi, \tau, \varphi^*\sigma) = (\varphi^*\mathscr{T}, \Phi, \tau, g \circ \varphi^*\sigma)$ and $g \cdot (\mathscr{T}, \lambda, \sigma) = (\mathscr{T}, \lambda, \phi^* g \circ \sigma)$ so the map Θ is $\widehat{LG}_{\mathcal{D}}$-equivariant.

We construct explicitly the inverse Θ^{-1}. Given a tuple $(\mathscr{T}, \Phi, \lambda, \sigma) \in \widehat{\mathrm{LSht}}_{\mathcal{D}}(A, A^+)$ we can assign:

$$(\mathscr{T}, \Phi, \lambda, \sigma) \mapsto (\phi^*\mathscr{T}, \sigma \circ \Phi, \phi^*\sigma)$$

this construction is clearly functorial in (A, A^+), and if $\varpi_A \in A^+$ is such that $\lambda = \sigma$ over $B^{A^+}_{[r,\infty]}/[\varpi_A]$ then $\Phi_{\mathcal{D}} \circ \phi^*\sigma = \sigma \circ \Phi$ over $\mathrm{Spec}(W(A^+)/[\varpi_A])$ since λ is ϕ-equivariant. This gives a map $\Omega : \widehat{\mathrm{LSht}}_{\mathcal{D}} \to \widehat{LGr}_{\mathcal{D}}$ the composition $\Omega \circ \Theta$ is clearly the identity. One can verify directly that $\Theta \circ \Omega(\mathscr{T}, \Phi, \lambda, \sigma) = (\mathscr{T}, \Phi, \lambda', \sigma)$ for some λ' nevertheless $\lambda' = \sigma = \lambda$ over $B^{A^+}_{[r,\infty]}/[\varpi]$ as ϕ-equivariant maps for some $\varpi \in A^+$. By the unicity part of lemma 2.1.28 we have $\lambda = \lambda'$.

Since Θ gives a $\widehat{LG}_{\mathcal{D}}$-equivariant isomorphism we also get isomorphisms

$$\Theta : \widehat{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}} = \widehat{LGr}_{\mathcal{D}} / \widehat{LG}_{\mathcal{D}} \cong \widehat{\mathrm{LSht}}_{\mathcal{D}} / \widehat{LG}_{\mathcal{D}} = \widehat{\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$$

One can also verify directly by the construction of Θ that it preserves the boundedness condition so that $\Theta : \widehat{\mathrm{Gr}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}} \to \widehat{\mathrm{Sht}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ is also an isomorphism. $\square$

Let us prove that moduli spaces of mixed characteristic shtukas are Orapian p-adically smelted kimberlites.

Theorem 2.3.18. *With the notation as in the beginning of this section we have that the map* $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} \to W(\mathfrak{f})^\diamond$ *forms an Orapian p-adically smelted kimberlite with connected p-adic tubular neighborhoods.*

Proof. Proposition 2.3.12 proves this map forms a p-adically smelted kimberlite. In [28] 23.3.3 it is proven that the period morphism $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} \to \mathrm{Gr}^{\mathscr{G},\leq\mu}_{W(\mathfrak{f})}$ is étale. By proposition 1.4.34 and theorem 2.2.33 we know that $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})[\frac{1}{p}]}$ is a cJ-diamond.

By theorem 1.1 of [9] we know that $X^{\mathscr{G}_b}_{\leq\mu}$ is locally Noetherian. By lemmas 1.4.43 and 1.4.44 to prove that the specialization map is a quotient and specializing map we only need to prove that for any non-Archimedean field extension $C/W(\mathfrak{f})[\frac{1}{p}]$ with C algebraically closed the specialization map of the base change $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_C}$ is surjective on closed points. It is then enough to prove that for any such C the p-adic tubular neighborhoods of $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_C}$ are non-empty and connected.

Take a closed point $x \in |(\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_C})^{\mathrm{red}}|$ we claim that we have an isomorphism $(\widehat{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_C}}_{/x})_\eta \cong (\widehat{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_C}}_{/y})_\eta$ for some $y \in |(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_C})^{\mathrm{red}}|$. Indeed, if $\mathfrak{f}_C$ denotes the residue field of O_C we may apply theorem 2.3.14 to compare $(\widehat{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f}_C)}}_{/x})_\eta$ with $(\widehat{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{W(\mathfrak{f}_C)}}_{/y})_\eta$ for some y. Since $C/W(\mathfrak{f}_C)[\frac{1}{p}]$ is purely ramified we have identifications $|(\mathrm{Sht}_{O_C}^{\mathscr{G}_b,\leq\mu})^{\mathrm{red}}| = |(\mathrm{Sht}_{W(\mathfrak{f}_C)}^{\mathscr{G}_b,\leq\mu})^{\mathrm{red}}|$ and $|(\mathrm{Gr}_{O_C}^{\mathscr{G},\leq\mu})^{\mathrm{red}}| = |(\mathrm{Gr}_{W(\mathfrak{f}_C)}^{\mathscr{G},\leq\mu})^{\mathrm{red}}|$. For any $x \in |(\mathrm{Sht}_{O_C}^{\mathscr{G}_b,\leq\mu})^{\mathrm{red}}|$ and any $y \in |(\mathrm{Gr}_{O_C}^{\mathscr{G},\leq\mu})^{\mathrm{red}}|$ we have the identities $(\widehat{\mathrm{Sht}_{O_C}^{\mathscr{G}_b,\leq\mu}}_{/x})_\eta = (\widehat{\mathrm{Sht}_{W(\mathfrak{f}_C)}^{\mathscr{G}_b,\leq\mu}}_{/x})_\eta \times_{W(\mathfrak{f})^\diamond} O_C^\diamond$, and $(\widehat{\mathrm{Gr}_{O_C}^{\mathscr{G},\leq\mu}}_{/y})_\eta = (\widehat{\mathrm{Gr}_{W(\mathfrak{f}_C)}^{\mathscr{G},\leq\mu}}_{/y})_\eta \times_{W(\mathfrak{f})^\diamond} O_C^\diamond$ which finishes the proof of the claim. Finally, by theorem 2.2.33 $(\widehat{Gr_{O_C}^{\mathscr{G},\leq\mu}}_{/y})_\eta$ is non-empty and connected as we wanted to show. $\square$

We finish this section with the proof of theorem 2 which is a rephrasing of 2.3.18 in less technical language. For the convenience of the reader we write the statement again.

Theorem 2.3.19. *With notation as in the introduction the following holds:*

a) There is a continuous specialization map

$$\mathrm{sp}_{\mathrm{Sht}_{O_{F_2}}^{\mathscr{G}_b,\leq\mu}} : |\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}| \to |X_{\leq\mu}^{\mathscr{G}}(b)|,$$

this map is a specializing and spectral map of locally spectral topological spaces. It is a quotient map and $J_b(\mathbb{Q}_p)$-equivariant.

b) Given a closed point $x \in |X_{\leq\mu}^{\mathscr{G}}(b)|$ let $S_x = \mathrm{sp}_{\mathrm{Sht}_{O_{F_2}}^{\mathscr{G}_b,\leq\mu}}{}^{-1}(x)$, then the interior S_x° of S_x as a subspace of $|\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}|$ is dense in S_x.

c) S_x and S_x° are non-empty and connected.

d) The specialization map induces a $J_b(\mathbb{Q}_p)$-equivariant bijection of connected components

$$\mathrm{sp}_{\mathrm{Sht}_{O_{F_2}}^{\mathscr{G}_b,\leq\mu}} : \pi_0(\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}) \to \pi_0(X_{\leq\mu}^{\mathscr{G}}(b))$$

Proof of theorem 1. We may apply theorem 2.3.18 and proposition 2.3.10 to conclude that the pair $(\mathrm{Sht}_{O_{F_2}^\diamond}^{\mathscr{G}_b,\leq\mu}, \mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond})$ is an Orapian smelted kimberlite with reduction $X_{\leq\mu}^{\mathscr{G}}(b)$. This implies by proposition 1.4.21 that the specialization map $\mathrm{sp}_{\mathrm{Sht}_{O_{F_2}}^{\mathscr{G}_b,\leq\mu}} : |\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}| \to |X_{\leq\mu}^{\mathscr{G}}(b)|$ is a spectral map of locally spectral spaces. Now, $\mathrm{Sht}_{O_{F_2}^\diamond}^{\mathscr{G}_b,\leq\mu}$ is Orapian, by definition the specialization map is specializing and a quotient map. Moreover, $J_b(\mathbb{Q}_p)$ acts on $\mathrm{Sht}_{O_{F_2}}^{\mathscr{G}_b,\leq\mu}$ by ϕ-equivariant automorphisms of $\mathscr{G}_b$, since the construction of the specialization map is functorial in the category of smelted kimberlites the map is equivariant, this finishes the proof of the first clam. Let $x \in |X_{\leq\mu}^{\mathscr{G}}(b)|$, we can use proposition 1.4.29 to identify S_x° with $|(\widehat{\mathrm{Sht}_{O_{F_2}}^{\mathscr{G}_b,\leq\mu}}_{/x})_\eta|$. Since $\mathrm{Sht}_{O_{F_2}}^{\mathscr{G}_b,\leq\mu}$ is Orapian we can apply proposition 1.4.33 to prove that S_x° is dense in S_x giving the second claim. By theorem 2.3.18 S_x° is connected and since it is dense in S_x this later one is also connected giving the third claim. For the last claim we may apply proposition 1.4.42 to any connected component of $|X_{\leq\mu}^{\mathscr{G}}(b)|$. $\square$

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