THE CONNECTED COMPONENTS OF AFFINE DELIGNE–LUSZTIG VARIETIES

IAN GLEASON, DONG GYU LIM, YUJIE XU

ABSTRACT. We compute the connected components of arbitrary parahoric level affine Deligne–Lusztig varieties and local Shimura varieties, thus resolving the conjecture raised in [He18] in full generality (even for non-quasisplit groups). We achieve this by relating them to the connected components of infinite level moduli spaces of p-adic shtukas, where we use v-sheaf-theoretic techniques such as the specialization map of *kimberlites*. Along the way, we give a p-adic Hodge-theoretic characterization of HN-irreducibility.

As applications, we obtain many results on the geometry of integral models of Shimura varieties at arbitrary parahoric levels. In particular, we deduce new CM lifting results on integral models of Shimura varieties for quasi-split groups at arbitrary connected parahoric levels.

CONTENTS

1. INTRODUCTION

1.1. **Background.** In [Rap05], Rapoport introduced certain geometric objects called affine Deligne–Lusztig varieties (ADLVs), to study mod p reduction of Shimura varieties. Since then, ADLVs have played a prominent role in the geometric study of: Shimura varieties, Rapoport–Zink spaces, local Shimura varieties and moduli spaces of local shtukas. Moreover, results on connected components of affine Deligne–Lusztig varieties have found remarkable applications to Kottwitz' conjecture and Langlands–Rapoport conjecture, which describe mod p points of Shimura varieties in relation to L-functions, as part of the Langlands program (for more background on this, see for example [Kis17]).

Although there have been many successful approaches [Vie08, CKV15, Nie18, HZ20, Ham20, Nie21] to computing connected components of ADLVs in the past decade, as far as the authors know, the current article is the first one that approaches the problem using p-adic analytic geometry *à la Scholze*. As it turns out, the p-adic approach proves the most general case of Conjecture 1.1 of [He18] and gives a new and uniform proof to all previously known cases. More precisely, we use a combination of Scholze's theory of diamonds [Sch17], the theory of *kimberlites* due to the first author [Gle22b], the connectedness of p-adic period domains [GL22a], and the normality of the local models [AGLR22, GL22b] to compute the connected components of ADLVs. Just as diamonds are generalizations of rigid analytic spaces, kimberlites and prekimberlites are the v-sheaf-theoretic generalizations of formal schemes. Roughly speaking, they are diamondifications of formal schemes.

As is well-known to experts, affine Deligne–Lusztig varieties parametrize (at-p) isogeny classes on integral models of Shimura varieties. As an application of our main theorems, we deduce the isogeny lifting property for integral models for Shimura varieties at parahoric levels constructed in [KP18]. Moreover, we give a new CM lifting result on integral models for Shimura varieties–which is a generalization of the classical Honda-Tate theory–for quasi-split groups at p at connected parahoric levels. This improves on previous CM lifting results, which were proved either assuming (1) $G_{\mathbb{Q}_p}$ residually split, or assuming (2) G unramified, or assuming that (3) the parahoric level is very special.

As a further application, we prove that the Newton strata of the integral models for Shimura varieties at parahoric level constructed in [PR21] satisfy p-adic uniformization, and that the Rapoport–Zink spaces considered *loc.cit.* agree with the moduli spaces of p-adic shtukas of [SW20] associated to the same data.

1.2. **Notations.** To not overload the introduction, we use common terms whose rigorous definitions we postpone till later (§2).

We denote by φ the lift of arithmetic Frobenius to $\breve{\mathbb{Q}}_p$. Let $\mathcal{I}$ and $\mathcal{K}_p$ be $\mathbb{Z}_p$-parahoric group schemes with common generic fiber a reductive group G. We let $K_p = \mathcal{K}_p(\mathbb{Z}_p)$, $\breve{\mathcal{I}} = \mathcal{I}(\breve{\mathbb{Z}}_p)$ and $\breve{K}_p := \mathcal{K}_p(\breve{\mathbb{Z}}_p)$. We require that $\mathcal{I}(\mathbb{Z}_p) \subseteq K_p$ and that $\mathcal{I}$ is an Iwahori subgroup of G.

Fix $S \subseteq G$, a $\mathbb{Q}_p$-torus that is maximally split over $\breve{\mathbb{Q}}_p$. Let $T = Z_G(S)$ be the centralizer of S, by Steinberg's theorem it is a maximal $\mathbb{Q}_p$-torus. Let $B \subseteq G_{\breve{\mathbb{Q}}_p}$ be a Borel containing $T_{\breve{\mathbb{Q}}_p}$, which may be defined only over $\breve{\mathbb{Q}}_p$. Let $\mu \in X_*^+(T)$ be a B-dominant cocharacter, and let $b \in G(\breve{\mathbb{Q}}_p)$. Let $\widetilde{W}$ be the Iwahori–Weyl group of G over $\breve{\mathbb{Q}}_p$. Let $\mathrm{Adm}(\mu) \subseteq \widetilde{W} = \breve{\mathcal{I}} \backslash G(\breve{\mathbb{Q}}_p)/\breve{\mathcal{I}}$ denote the μ-admissible set of Kottwitz–Rapoport [KR00].

The (closed) affine Deligne–Lusztig variety associated to (G, b, μ), denoted as $X_\mu(b)$, is a locally perfectly finitely presented $\bar{\mathbb{F}}_p$-scheme (see [Zhu17]), with $\bar{\mathbb{F}}_p$-valued points given by:

$$X_\mu(b) = \{g\breve{\mathcal{I}} \mid g^{-1}b\varphi(g) \in \breve{\mathcal{I}}\operatorname{Adm}(\mu)\breve{\mathcal{I}}\}. \tag{1.1}$$

By definition, $X_\mu(b)$ embeds into the Witt vector affine flag variety $\mathcal{F}\ell_{\breve{\mathcal{I}}}$, whose $\bar{\mathbb{F}}_p$-valued points are the cosets $G(\breve{\mathbb{Q}}_p)/\breve{\mathcal{I}}$. We also consider the $\mathcal{K}_p$-version $X_\mu^{\mathcal{K}_p}(b)$ with $\bar{\mathbb{F}}_p$-points:

$$X_\mu^{\mathcal{K}_p}(b) = \{g\breve{K}_p \mid g^{-1}b\varphi(g) \in \breve{K}_p\operatorname{Adm}(\mu)\breve{K}_p\}. \tag{1.2}$$

Let $\mathbf{b} \in B(G)$ be the φ-conjugacy class of b, and let $\boldsymbol{\mu}$ be the conjugacy class of μ. Assume $\mathbf{b}$ lies in the Kottwitz set $B(G,\boldsymbol{\mu})$. Let $\mu^\diamond \in X_*(T)^+_{\mathbb{Q}}$ denote the "twisted Galois average" of μ (see (2.7)), and let $\boldsymbol{\nu}_{\mathbf{b}} \in X_*(T)^+_{\mathbb{Q}}$ denote the dominant Newton point. Recall that $\mathbf{b} \in B(G,\boldsymbol{\mu})$ implies that $\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}}$ is a non-negative sum of simple positive coroots with rational coefficients. We say that $(\mathbf{b},\boldsymbol{\mu})$ with $\mathbf{b} \in B(G,\boldsymbol{\mu})$ is *Hodge–Newton irreducible* (HN-irreducible) if all simple positive coroots have non-zero coefficient in $\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}}$.

Let Γ and I denote the Galois groups of $\mathbb{Q}_p$ and $\breve{\mathbb{Q}}_p$ respectively. Recall that the Kottwitz map [Kot97, 7.4]

$$\kappa_G : G(\breve{\mathbb{Q}}_p) \to \pi_1(G)_I \tag{1.3}$$

induces bijections $\pi_0(\mathcal{F}\ell_{\breve{\mathcal{I}}}) \cong \pi_0(\mathcal{F}\ell_{\breve{K}_p}) \cong \pi_1(G)_I$. Moreover, it is known that the map induced by κ_G on connected components of ADLV,

$$\omega_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to \pi_1(G)_I, \tag{1.4}$$

factors surjectively onto $c_{b,\mu}\pi_1(G)_I^\varphi \subseteq \pi_1(G)_I$ for a unique coset element $c_{b,\mu} \in \pi_1(G)_I/\pi_1(G)_I^\varphi$ (see for example [HZ20, Lemma 6.1]).

1.3. **Main Results.** In his ICM talk [He18], X. He underlines the study of connected components as an important open problem in the study of the geometric properties of ADLVs. Moreover, He suggests the following conjecture.

Conjecture 1.1. *If $(\mathbf{b},\boldsymbol{\mu})$ is HN-irreducible, the following map is bijective*

$$\omega_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$$

Our main theorem is the following (see Theorem 6.2).

Theorem 1.2. *For all p-adic shtuka datum (G,b,μ) and all parahoric subgroups $\mathcal{K}_p \subseteq G(\mathbb{Q}_p)$, Conjecture 1.1 holds.*

To state the applications to the geometry of Shimura varieties, we shall also need the following notations. Let $(\mathbf{G},X)$ be a Shimura datum of Hodge type. Suppose $\mathbf{G}$ splits over a tamely ramified extension[1]. We shall always assume $p > 2$ and $p \nmid |\pi_1(\mathbf{G}^{\mathrm{der}})|$.[2] Let $K_p \subseteq \mathbf{G}(\mathbb{Q}_p)$ be a connected parahoric

[1]we expect that this condition can be relaxed using [KZ21].

[2]we expect the same results to hold for $p = 2$, using similar ideas from [KMP16], which only addressed the hyperspecial level integral models.

subgroup[3]. By [KP18][4], there is a normal integral model $\mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)$, for the Shimura variety $\mathrm{Sh}_{\mathcal{K}_p}(\mathbf{G}, X)$.

The following Corollary 1.3 is the parahoric analogue of [Kis17, Proposition 1.4.4] and can be obtained by combining our Theorem 1.2 with [Zho20, Proposition 6.5]. This is a generalization of existing results in literature to $\mathcal{K}_p$ arbitrary connected parahoric. Recall that to any $x \in \mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)(\bar{\mathbb{F}}_p)$, one can associate a $b \in G(\breve{\mathbb{Q}}_p)$ as in [Kis17, Lemma 1.1.12].

Corollary 1.3. *Let $\mathcal{K}_p$ be a connected parahoric. For any $x \in \mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)(\bar{\mathbb{F}}_p)$, there exists a map of perfect schemes*

$$\iota_x : X_\mu^{\mathcal{K}_p}(b) \to \mathscr{S}^{perf}_{\mathcal{K}_p, \bar{\mathbb{F}}_p}(\mathbf{G}, X) \tag{1.5}$$

preserving crystalline tensors and equivariant with respect to the geometric r-Frobenius.

The following Corollary 1.4(1) (resp. Corollary 1.4(2)) is a parahoric analogue to [Kis17, Proposition 2.1.3] (resp. [Kis17, Theorem 2.2.3]) when $\mathbf{G}_{\mathbb{Q}_p}$ is quasi-split, and can be obtained by combining our Theorem 1.2 with [Zho20, Proposition 9.1] (resp. [Zho20, Theorem 9.4]). Notations as *loc.cit.*

Corollary 1.4. *Let $\mathbf{G}$ be quasisplit at p. Let $\mathcal{G} := \mathcal{K}_p$ be a connected parahoric. Let $k \subseteq \bar{\mathbb{F}}_p$ be a finite field extension of $\mathbb{F}_p$.*

(1) *the map ι_x in (1.5) induces an injective map*

$$\iota_x : I_x(\mathbb{Q}) \backslash X_\mu^{\mathcal{K}_p}(b)(\bar{\mathbb{F}}_p) \times G(\mathbb{A}_f^p) \to \mathscr{S}_{\mathcal{K}_p}(G, X)(\bar{\mathbb{F}}_p), \tag{1.6}$$

where I_x is a subgroup of the automorphism group of the abelian variety (base changed to $\bar{\mathbb{F}}_p$) associated to x fixing the Hodge tensors[5].

(2) *The isogeny class $\iota_x(X_\mu^{\mathcal{K}_p}(b)(\bar{\mathbb{F}}_p)) \times \mathbf{G}(\mathbb{A}_f^p))$ contains a point which lifts to a special point on $\mathscr{S}_{\mathcal{K}_p}(G, X)$.*

Theorem 1.2 together with [Zho20, Theorem 8.1(2)] finish the verification of the He–Rapoport axioms [HR17] for integral models of Shimura varieties.

Corollary 1.5. *The He–Rapoport axioms hold for $\mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)$.*

Moreover, we also obtain the following corollary by combining [HK19, Theorem 2] with Corollary 1.3, which allow us to verify Axiom A *loc.cit.*.

Corollary 1.6. *Let $\mathbf{G}$ be quasisplit at p. Let $\mathcal{K}_p$ be a connected parahoric. The "almost product structure" of the Newton strata in $\mathscr{S}_{\mathcal{K}_p, \bar{\mathbb{F}}_p}(G, X)$ holds.*

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[3]Following [Zho20], we say that $\mathcal{K}_p$ is a connected parahoric if it agrees with the stabilizer group scheme of a facet in the Bruhat–Tits building.

[4]In [KP18], the authors construct parahoric integral models assuming that $\mathbf{G}_{\mathbb{Q}_p}$ splits over a tamely ramified extension. We expect that some of the technical conditions of our corollaries can be relaxed using the constructions in [KZ21] or in [PR21].

[5]As is standard in the theory of Shimura varieties, a Shimura variety of Hodge type carries a collection of Hodge tensors that "cut out" the reductive group $\mathbf{G}$. See 5 for more details

We refer the reader to [HK19, Theorem 2] for the precise formulation of the almost product structure of Newton strata.

As yet another corollary, we remove a technical assumption from the following theorem originally due to the third author [Xu21, Main Theorem][6].

Corollary 1.7. *Let $\mathbf{G}$ be quasisplit at p. Let $\mathcal{K}_p$ be a connected parahoric. The normalization step in the construction of $\mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)$ is unnecessary, and the closure model $\mathscr{S}^-_{\mathcal{K}_p}(\mathbf{G}, X)$ is already normal. Therefore we obtain closed embeddings $\mathscr{S}_{\mathcal{K}_p\mathcal{K}^p}(\mathbf{G}, X) \hookrightarrow \mathscr{S}_{\mathcal{K}'_p\mathcal{K}'^p}(\mathrm{GSp}, S^{\pm})$.*
As a further consequence, the analogous statement holds for toroidal compactifications of integral models, for suitably chosen cone decompositions.

Also, we obtain the following corollary by combining Corollary 1.5 with [SYZ21, Theorem C]. See *loc.cit.* for the definition of EKOR strata.

Corollary 1.8. *Every EKOR stratum in $\mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)_{\overline{\mathbb{F}}_p}$ is quasi-affine.*

Remark 1.9. Our main theorem 1.2 is independent of the integral models of Shimura varieties that one works with. For this reason, we expect our main theorems to have similar applications as the above corollaries to the more general setup considered by Pappas–Rapoport [PR21].

Finally, we deduce that the Newton strata of the integral models of Shimura varieties considered by Pappas–Rapoport [PR21, Theorem 4.10.6] satisfy p-adic uniformization with respect to the local Shimura varieties $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ of [SW20, Definition 25.1.1]. Let $(p, \mathbf{G}, X, \mathbf{K})$ be a tuple of global Hodge type [PR21, §1.3], let $\mathscr{S}_{\mathbf{K}}$ denote the integral model of [PR21, Theorem 1.3.2], let k an algebraically closed field in characteristic p and let $x_0 \in \mathscr{S}_{\mathbf{K}}(k)$. Pappas and Rapoport consider a map of v-sheaves $c : \mathrm{RZ}^{\diamondsuit}_{\mathcal{G},\mu,x_0} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ [PR21, Lemma 4.1.0.2], where the source is a Rapoport–Zink space. Let the notations be as in [PR21, Theorem 4.10.6, §4.10.2]. We verify Conjecture (U_x) in [PR21, §4.10.2] and obtain the following.

Corollary 1.10. *(Corollary 6.3) The map $c : \mathrm{RZ}^{\diamondsuit}_{\mathcal{G},\mu,x_0} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is an isomorphism. Thus, $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is representable by a formal scheme $\mathscr{M}_{\mathcal{G},b,\mu}$, and we obtain a p-adic uniformization isomorphism of $O_{\breve{E}}$-formal schemes*

$$I_x(\mathbb{Q})\backslash(\mathscr{M}_{\mathcal{G},b,\mu} \times \mathbf{G}(\mathbb{A}^p_f)/\mathbf{K}^p) \to (\mathscr{S}_{\mathbf{K}} \widehat{\otimes_{O_E}} O_{\breve{E}})_{/\mathcal{I}(x)}. \tag{1.7}$$

[6]The original version of this theorem is stated assuming $\mathbf{G}_{\mathbb{Q}_p}$ residually split for integral models at parahoric levels; at hyperspecial levels, this assumption is not necessary. We are now able to relax "$\mathbf{G}_{\mathbb{Q}_p}$ residually split" to "$\mathbf{G}_{\mathbb{Q}_p}$ quasi-split" thanks to our main theorem 1.2.

1.4. **Rough Sketch of the argument.** Many cases of Conjecture 1.1 have been proved in literature under various additional assumptions[7], see for example [Vie08, Theorem 2], [CKV15, Theorem 1.1], [Nie18, Theorem 1.1], [HZ20, Theorem 0.1], [Ham20, Theorem 1.1(3)], [Nie21, Theorem 0.2].

Previous attempts in literature used characteristic p perfect geometry and combinatorial arguments to construct enough "curves" connecting the components of the ADLV. In our approach, we use the theory of *kimberlites* and their specialization maps [Gle22b], and the general kimberlite-theoretic unibranchness result for the local models considered by Scholze–Weinstein (see [SW20, § 21.4]) recently established in [GL22b, Theorem 1], to turn the problem of computing $\pi_0(X_\mu^{\mathcal{K}_p}(b))$ into the characteristic-zero question of computing $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)})$. We remark that when μ is non-minuscule, diamond-theoretic considerations are necessary, since the spaces $\mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)}$ are not rigid-analytic spaces. Moreover, even when μ is minuscule, the theory of kimberlites is necessary here because: although $\mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)}$ is representable by a rigid-analytic space, its canonical integral model is not known to be representable by a formal scheme.

Once in characteristic zero, we are now able to exploit Fontaine's classical p-adic Hodge theory. In our approach, the role of "connecting curves" is played by "*generic* crystalline representations", inspired by the ideas in [Che14] (see §5.1). Intuitively speaking, the action of Galois groups can be interpreted as "analytic paths" in the moduli spaces of p-adic shtukas.

More precisely, the infinite level moduli space $\mathrm{Sht}_{G,b,\mu,\infty}$ of p-adic shtukas can be realized as the moduli space of trivializations of the universal crystalline $G(\mathbb{Q}_p)$-torsor[8] over the b-admissible locus Gr_μ^b of the affine Grassmannian [SW20]. Then rational points of Gr_μ^b give rise to loops in Gr_μ^b, which produce "connecting paths" inside any covering space over Gr_μ^b (in particular the covering space $\mathrm{Sht}_{G,b,\mu,\infty}$). Thus it suffices to prove that the universal crystalline representation has enough monodromy to "connect" $\mathrm{Sht}_{G,b,\mu,\infty}$. We can then deduce our main theorem 1.2 at finite level $\mathrm{Sht}_{G,b,\mu,\mathcal{K}_p}$ from the analogous result at infinite level.

1.5. **More on the arguments.** We now dig in a bit deeper into the strategy for our main theorem 1.2, and sketch a few more results that led to our main theorem.

To each $(G, b, \mu, \mathcal{I}(\mathbb{Z}_p))$, one can associate a diamond $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}$, which is the moduli space of p-adic shtukas at level $\mathcal{I}(\mathbb{Z}_p)$ defined in [SW20]. In

[7]When G is split and $\mathcal{K}_p$ is hyperspecial, [Vie08, Theorem 2] applies. When G is unramified, $\mathcal{K}_p$ is hyperspecial and $\boldsymbol{\mu}$ is minuscule, [CKV15, Theorem 1.1] applies. When G is unramified, $\mathcal{K}_p$ is hyperspecial and $\boldsymbol{\mu}$ is general, [Nie18, Theorem 1.1] applies. When G is residually split or when $\mathbf{b}$ is basic [HZ20, Theorem 0.1] applies. When G is quasi-split and $\mathcal{K}_p$ is very special, [Ham20, Theorem 1.1(3)] applies. When G is unramified and $\mathcal{K}_p$ is arbitrary, [Nie21, Theorem 0.2] applies.

[8]For local Shimura varieties coming from Rapoport–Zink spaces, this torsor corresponds to the local system defined by the p-adic Tate module of the universal p-divisible group.

[Gle22a], the first author constructed a specialization map

$$\mathrm{sp} : |\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}| \to |X_\mu(b)|. \tag{1.8}$$

By the unibranchness result of the first author joint with Lourenço [GL22b, Theorem 1.3], and the construction of certain v-sheaf local model correspondence due to the first author [Gle22a, Theorem 3], the specialization map induces an isomorphism of sets

$$\mathrm{sp} : \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \cong \pi_0(X_\mu(b)). \tag{1.9}$$

Therefore we have now turned the question on $\pi_0(X_\mu(b))$ into a characteristic zero question on the connected components $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))})$ of the diamond $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}$, which we now compute.

For this purpose, we make use of the infinite level moduli space $\mathrm{Sht}_{(G,b,\mu,\infty)}$ of p-adic shtukas. Since $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} = \mathrm{Sht}_{(G,b,\mu,\infty)}/\mathcal{I}(\mathbb{Z}_p)$, we have

$$\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}) = \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)})/\mathcal{I}(\mathbb{Z}_p). \tag{1.10}$$

Let G^{ad} denote the adjoint group of G. Our main theorem 1.2 follows directly from the following Theorem 1.11 whenever G^{ad} does not have anisotropic factors. When G^{ad} is anisotropic, we give a separate argument (see the proof of Theorem 6.2). Let $G^\circ := G(\mathbb{Q}_p)/\operatorname{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$ denote the maximal abelian quotient of $G(\mathbb{Q}_p)$.

Theorem 1.11. *(Theorem 6.1) Suppose that* $\mathbf{b} \in B(G, \boldsymbol{\mu})$ *and that* G^{ad} *does not have anisotropic factors. The following statements are equivalent:*

(1) *The map* $\omega_G : \pi_0(X_\mu(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ *is bijective.*
(2) *The map* $\omega_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ *is bijective.*
(3) *The pair* $(\mathbf{b}, \boldsymbol{\mu})$ *is HN-irreducible.*
(4) *There exists a field extension* K *of finite index over* $\breve{\mathbb{Q}}_p$*, and a crystalline representation* $\xi : \Gamma_K \to G(\mathbb{Q}_p)$ *with invariants* $(\mathbf{b}, \boldsymbol{\mu})$ *for which* $G^{\mathrm{der}}(\mathbb{Q}_p) \cap \xi(\Gamma_K) \subseteq G^{\mathrm{der}}(\mathbb{Q}_p)$ *is open.*
(5) *The action of* $G(\mathbb{Q}_p)$ *on* $\mathrm{Sht}_{(G,b,\mu,\infty)}$ *makes* $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ *into a* G°*-torsor.*

Remark 1.12. The implication (3) $\implies$ (5) of Theorem 1.11 confirms almost all cases (excluding the anisotropic groups) of a conjecture of Rapoport–Viehmann [RV14, Conjecture 4.30]. Moreover, we generalize the statement to moduli spaces of p-adic shtukas, instead of only for local Shimura varieties as *loc.cit.*

Remark 1.13. The implication (3) $\implies$ (5) of Theorem 1.11 is a more general version of the main theorem of [Gle22a], where the first author proved the statement for unramified G, and computed the Weil group and $J_b(\mathbb{Q}_p)$-actions on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$. One should be able to combine the methods of our current paper with those *loc.cit.* to compute the Weil group and $J_b(\mathbb{Q}_p)$-actions in the more general setup of Theorem 1.11.

1.5.1. *Loop of the argument for Theorem 1.11.* We now discuss the proof of Theorem 1.11. Using ad-isomorphisms and z-extensions (see §3.7), we reduce all statements of Theorem 1.11 to the case where G^{der}–the derived subgroup of G–is simply connected (see Proposition 6.7). In this case, $G^\circ = G^{\mathrm{ab}}(\mathbb{Q}_p)$ and, using that G^{ad} has only isotropic factors, we prove the implications

$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (1).$$

Let us explain the chain of implications. The implication $(1) \implies (2)$ follows from [He16, Theorem 1.1], which says that the map $X_\mu(b) \to X_\mu^{\mathcal{K}_p}(b)$ is surjective. We give a new and simple proof of this result in Theorem 6.8, by observing that $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \to \mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)}$ is automatically surjective. This again exemplifies the advantage of working on the generic fiber (of the v-sheaf $\mathrm{Sht}_\mu^{\mathcal{K}_p}(b)$). For more details, see § 3.4. The implication $(2) \implies (3)$ follows from the HN-decomposition (Theorem 4.3) and group-theoretic manipulations (Proposition 4.10).

The implication $(3) \implies (4)$ follows from an explicit construction that goes back to [Che14, Théorème 5.0.6] when G is unramified (Proposition 5.8). In §5.1, we push the methods *loc.cit.* and generalize the result to arbitrary reductive groups G (see also §1.5.3 in this introduction).

The implication $(5) \implies (1)$ follows from (1.9) (see also Proposition 3.7) and the identification $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}) = \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)})/\mathcal{I}(\mathbb{Z}_p)$. Indeed, using the map $\det : G \to G^{\mathrm{ab}} := G/G^{\mathrm{der}}$, combined with Lang's theorem, we reduce $(5) \implies (1)$ to the tori case which can be handled directly (see §6.1). For more details, see §6.6.

1.5.2. *Proof for* $(4) \implies (5)$ *in Theorem 1.11.* The core of the argument lies in $(4) \implies (5)$. For simplicity, we only discuss the case where G is semisimple and simply connected in the introduction (see §6.7 for the general argument). In this (simplified) case, G° is trivial, thus it suffices to show that $\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p$ is connected. The first step is to prove that $G(\mathbb{Q}_p)$ acts transitively on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ (Proposition 3.10). This follows from the main theorem of [GL22a] (see Theorem 3.9).[9] Let G_x denote the stabilizer of $x \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$. Since $G(\mathbb{Q}_p)$ acts transitively, it suffices to prove $G_x = G(\mathbb{Q}_p)$.

For this, it suffices to show that: (i) G_x is open (see Lemma 6.11); (ii) the normalizer N_x of G_x in $G(\mathbb{Q}_p)$ is of finite index in $G(\mathbb{Q}_p)$ (see Lemma 6.12). Indeed, since we assumed that G is semisimple and simply connected with only isotropic factors, a standard fact from [Mar91, Chapter II, Theorem 5.1] shows that $G(\mathbb{Q}_p)$ does not have finite index subgroups. Thus (ii) allows us to conclude that G_x is normal in $G(\mathbb{Q}_p)$. Moreover, the same standard fact *loc.cit.* shows that $G(\mathbb{Q}_p)$ does not have non-trivial open normal subgroups, therefore (i) implies $G_x = G(\mathbb{Q}_p)$.

[9] Before [GL22a] was available, the argument for Theorem 1.11 relied on the results of [Ham20] which are only available when G is quasi-split.

To prove that $G_x \subseteq G(\mathbb{Q}_p)$ is open, we use the Bialynicki-Birula map (3.17) and the "admissible=weakly admissible" theorem [CF00]. Now, for (ii) we exploit that the actions of $J_b(\mathbb{Q}_p)$ and $G(\mathbb{Q}_p)$ commute. This, together with the key bijection of (1.9), allow us to translate the general finiteness results of [HV20] into the finiteness of $[G_x : N_x]$.

1.5.3. *The Mumford-Tate group of "generic crystalline representations".* Let us give more detail on the construction used to prove the implication (3) $\implies$ (4) from §1.5.1. Fix a finite extension $K/\breve{\mathbb{Q}}_p$ with Galois group $\Gamma_K := \mathrm{Gal}(\overline{K}/K)$, and let $\xi : \Gamma_K \to G(\mathbb{Q}_p)$ denote a conjugacy class of p-adic Hodge–Tate representations.

Definition 1.14. Let MT_ξ denote the connected component of the Zariski closure of the image of ξ in $G(\mathbb{Q}_p)$. This is the p-adic Mumford–Tate group attached to ξ which is well-defined up to conjugation.

It follows from results of Serre [Ser79, Théorème 1] and Sen [Sen73, §4, Théorème 1] (see also [Che14, Proposition 3.2.1]) that $\xi(\Gamma_K) \cap \mathrm{MT}_\xi(\mathbb{Q}_p)$ is open in $\mathrm{MT}_\xi(\mathbb{Q}_p)$. Let $\mu^\eta : \mathbb{G}_m \to G_K$ be a cocharacter conjugate to μ. Suppose that (b, μ^η) defines an admissible pair in the sense of [RZ96, Definition 1.18]. Since $\mathbf{b} \in B(G, \boldsymbol{\mu})$, it induces a conjugacy class of crystalline representations $\xi_{(b,\mu^\eta)} : \Gamma_K \to G(\mathbb{Q}_p)$, and a p-adic Mumford–Tate group $\mathrm{MT}_{(b,\mu^\eta)}$ attached to $\xi_{(b,\mu^\eta)}$ (See Definition 5.1).

Let $\mathrm{Fl}_\mu := G/P_\mu$ denote the generalized flag variety. We say that μ^η is *generic* if the map $\mathrm{Spec}\,(K) \to \mathrm{Fl}_\mu$ induced by μ^η lies over the generic point.[10] Our third main theorem is the following generalization of [Che14, Théorème 5.0.6] to arbitrary reductive groups.

Theorem 1.15. *(Theorem 5.7) Let G be a reductive group over $\mathbb{Q}_p$. Let $b \in G(\breve{\mathbb{Q}}_p)$ and $\mu^\eta : \mathbb{G}_m \to G_K$ as above. Suppose that b is decent, that μ^η is generic and that $\mathbf{b} \in B(G, \boldsymbol{\mu})$. The following hold:*

1. *(b, μ^η) is admissible.*
2. *If $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, then $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}.*

Now, Theorem 1.11 has as corollary a converse to Theorem 1.15. The following gives a p-adic Hodge-theoretic characterization of HN-irreducibility.

Corollary 1.16. *(Proposition 5.10) Assume that G^{ad} has only isotropic factors. If $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}, then $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*

Remark 1.17. Our Corollary 1.16 confirms the expectation in [Che14, Remarque 5.0.5] that, at least when G has only isotropic factors, HN-irreducibility is equivalent to having full monodromy.

[10] this is possible since $\breve{\mathbb{Q}}_p$ has infinite transcendence degree over $\mathbb{Q}_p$.

1.6. **Organization.** Finally, let us describe the organization of the paper.

§2 is a preliminary section. We start by collecting general notation and standard definitions that we omitted in this introduction. We recall the group-theoretic setup of [GHN19] necessary to discuss the Hodge-Newton decomposition for general reductive groups, and its relation to the connected components of affine Deligne–Lusztig varieties.

In §3 we give a brief intuitive account of the theory of kimberlites. We also review the geometry of affine Deligne–Lusztig varieties and their relation to moduli spaces of p-adic shtukas. Moreover, we discuss ad-isomorphisms, z-extensions and compatibility with products (which will be used in §6 to reduce the proofs of Theorem 1.2 and Theorem 1.11 to the key cases).

In §4 we discuss the Hodge-Newton decomposition and use it to prove the implication (2) $\implies$ (3) in Theorem 1.11.

In §5, we discuss Mumford–Tate groups. We review [Che14] and discuss the modifications needed to prove Theorem 1.15. We deduce the implication (3) $\implies$ (4) in Theorem 1.11.

In §6, we give a new proof of [He18, Theorem 7.1] (see Theorem 6.8), and complete proofs of our main results such as Theorem 1.2 and Theorem 1.11.

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2. Group-theoretic setup

Given a group G, let G^{der} denote its derived subgroup, G^{sc} the simply connected cover of G^{der}, and $G^{\mathrm{ab}} := G/G^{\mathrm{der}}$. Since G^{ab} is a torus, it admits a unique parahoric model denoted by $\mathcal{G}^{\mathrm{ab}}$.

We continue the notation from §1.2. Recall that S is a maximal split $\mathbb{Q}_p$-torus of G. Let $\mathcal{N} = N_G(S)$ be the normalizer of S in G. Let $W_0 := \mathcal{N}(\breve{\mathbb{Q}}_p)/T(\breve{\mathbb{Q}}_p)$ be the relative Weyl group. Recall that $T = Z_G(S)$ is the centralizer of S. Let $\mathcal{T}$ denote its unique parahoric model[11]. Denote by

[11]This is the identity component of the locally of finite type Néron model of T.

$\widetilde{W}$ the Iwahori-Weyl group $\mathcal{N}(\breve{\mathbb{Q}}_p)/\mathcal{T}(\breve{\mathbb{Z}}_p)$. There is a φ-equivariant exact sequence ([HR08]):

$$0 \to X_*(T)_I \to \widetilde{W} \to W_0 \to 1 \tag{2.1}$$

Let $\mathcal{A}$ denote the apartment in the Bruhat–Tits building of $G_{\breve{\mathbb{Q}}_p}$ corresponding to S. Let $\mathbf{a} \subseteq \mathcal{A}$ denote the φ-invariant alcove determined by $\mathcal{I}(\mathbb{Z}_p)$. We choose a special vertex $\mathbf{o} \in \mathbf{a}$, and identify $\mathcal{A}$ with $X_*(T)^I \otimes \mathbb{R} = X_*(T)_I \otimes \mathbb{R}$ by sending the origin to $\mathbf{o}$. Let B be the Borel subgroup attached to $\mathbf{a}$ under this identification. Observe that the natural linear action of φ on $X_*(T)^I$ is the gradient of the affine action of φ on $\mathcal{A}$. Let $\Delta \subseteq \Phi^+ \subseteq \Phi \subseteq X^*(T)$ denote the set of simple positive roots, positive roots and roots attached to B, respectively.

The choice of $\mathbf{o}$ defines a splitting $W_0 \to \widetilde{W}$, which may not be φ-equivariant. Let $\bar{\mu}$ denote the image of μ in $X_*(T)_I$. For every element $\lambda \in X_*(T)_I$, let t_λ be its image in $\widetilde{W}$ under (2.1). Let $\mathbb{S}$ be the set of reflections along the walls of $\mathbf{a}$. Let W^{a} be the affine Weyl group generated by $\mathbb{S}$. It is a Coxeter group. There is a φ-equivariant exact sequence ([HR08, Lemma 14]):

$$1 \to W^{\mathrm{a}} \to \widetilde{W} \to \pi_1(G)_I \to 0 \tag{2.2}$$

This sequence splits and we can write $\widetilde{W} = W^{\mathrm{a}} \rtimes \pi_1(G)_I$. We can extend the Bruhat order $\preceq$ given on W^{a} to the one on $\widetilde{W}$ as follows: for elements $(w_i, \tau_i) \in \widetilde{W}$ with $i = 1, 2$, where $w_i \in W^{\mathrm{a}}$ and $\tau_i \in \pi_1(G)_I$, we say

$$(w_1, \tau_1) \preceq (w_2, \tau_2) \tag{2.3}$$

if $w_1 \preceq w_2$ in W^{a} and $\tau_1 = \tau_2 \in \pi_1(G)_I$. By [Hai18, Theorem 4.2], we can define the Kottwitz–Rapoport admissible set as

$$\mathrm{Adm}(\mu) = \{\tilde{w} \in \widetilde{W} \mid \tilde{w} \preceq t_\lambda \,\text{with}\, t_\lambda = t_{w(\bar{\mu})} \,\text{for}\, w \in W_0\}. \tag{2.4}$$

1. Let $\widetilde{W}^{\mathrm{ad}}$ denote the Iwahori–Weyl group of G^{ad}. By [HR08, Lemma 15][12], there exists an element $w^{\mathrm{ad}} \in \widetilde{W}^{\mathrm{ad}}$ such that $w^{\mathrm{ad}} \cdot \varphi(\mathbf{o}) = \mathbf{o}$ and $w^{\mathrm{ad}} \cdot \varphi(\mathbf{a}) = \mathbf{a}$. Conjugation by a lift of w^{ad} to $G^{\mathrm{ad}}(\breve{\mathbb{Q}}_p)$ gives the quasisplit inner form of G, which we denote by G^*. This defines a second action φ_0 on $G(\breve{\mathbb{Q}}_p)$ (called the L-action), whose fixed points are $G^*(\mathbb{Q}_p)$ and that satisfies $\varphi_0(\mathcal{A}) = \mathcal{A}$, $\varphi_0(\mathbf{o}) = \mathbf{o}$, $\varphi_0(B) = B$.

2. Let $\mu \in X_*(T)^+$ be a dominant cocharacter. Denote by $\mu^\natural \in \pi_1(G)_\Gamma$ the image of μ under the natural projection $X_*(T) \to \pi_1(G)_\Gamma$. As in [Kot97, (6.1.1)], we define

$$\mu^\diamond \coloneqq \frac{1}{[\Gamma : \Gamma_\mu]} \sum_{\gamma \in \Gamma/\Gamma_\mu} \gamma(\mu) \in X_*(T)^+_{\mathbb{Q}}, \tag{2.5}$$

[12] More precisely, $P^\vee$ *loc.cit.* acts transitively on the set of special vertices and σ sends a special vertex $\mathbf{o}$ to a special vertex. Thus $P^\vee$ and W_0 together make it possible to find this element w^{ad}.

where the Galois action on $X_*(T)$ is the one coming from G^*. Via the isomorphism $X_*(T)_I \otimes \mathbb{Q} \simeq (X_*(T) \otimes \mathbb{Q})^I$ given by $[\mu] \mapsto \frac{1}{[I:I_\mu]} \sum_{\gamma \in I/I_\mu} \gamma(\mu)$, we may write $\mu^\diamond$ as follows (see [HN18, A.4]):

$$\underline{\mu} \coloneqq \frac{1}{[I : I_\mu]} \sum_{\gamma \in I/I_\mu} \gamma(\mu) \tag{2.6}$$

$$\mu^\diamond = \frac{1}{N} \sum_{i=0}^{N-1} \varphi_0^i(\underline{\mu}) \tag{2.7}$$

Here N is any integer such that $\varphi_0^N(\underline{\mu}) = \underline{\mu}$, and I_μ is the stabilizer of μ associated to the action by the inertia group. Alternatively,

$$\mu^\diamond = \frac{1}{N} \sum_{i=0}^{N-1} \varphi^i(\mu)^{\mathrm{dom}}. \tag{2.8}$$

Here λ^{dom} denotes the unique B-dominant conjugate of λ for $\lambda \in X_*(T) \otimes \mathbb{Q}$.

3. Recall that attached to b, there is a slope decomposition map

$$\nu_b : \mathbb{D} \to G_{\breve{\mathbb{Q}}_p}, \tag{2.9}$$

where $\mathbb{D}$ is the pro-torus with $X^*(\mathbb{D}) = \mathbb{Q}$. We let the *Newton point*, denoted as $\boldsymbol{\nu}_{\mathbf{b}}$, be the unique conjugate in $X_*(T)^+_{\mathbb{Q}}$ of (2.9). Recall that there is a Kottwitz map $\kappa_G : B(G) \to \pi_1(G)_\Gamma$ [Kot85, Kot97].

Definition 2.1. Let $\mathbf{b} \in B(G)$.

(1) We write $\mathbf{b} \in B(G, \boldsymbol{\mu})$ if $\mu^\natural = \kappa_G(\mathbf{b})$ and $\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} = \sum_{\alpha \in \Delta} c_\alpha \alpha^\vee$ with $c_\alpha \in \mathbb{Q}$ and $c_\alpha \geq 0$.
(2) We say $(\mathbf{b}, \boldsymbol{\mu})$ is *HN-irreducible (Hodge–Newton irreducible)* if $\mathbf{b} \in B(G, \boldsymbol{\mu})$ and $c_\alpha \neq 0$ for all $\alpha \in \Delta$.

Definition 2.2. [RZ96, Definition 1.8] Let $s \in \mathbb{N}$. We say that $b \in G(\breve{\mathbb{Q}}_p)$ is *s-decent* if $s \cdot \nu_b$ factors through a map $\mathbb{G}_m \to G_{\breve{\mathbb{Q}}_p}$, and the decency equation $(b\varphi)^s = s \cdot \nu_b(p)\varphi^s$ is satisfied in $G(\breve{\mathbb{Q}}_p) \rtimes \langle \varphi \rangle$. If the context is clear, we say that b is *decent* if it is s-decent for some s.

4. If b is s-decent, then $b \in G(\mathbb{Q}_{p^s})$ and ν_b is also defined over $\mathbb{Q}_{p^s}$, where $\mathbb{Q}_{p^s}$ is the degree s unramified extension of $\mathbb{Q}_p$. Moreover, for all $\mathbf{b} \in B(G)$, there exists an $s \in \mathbb{N}$ and an s-decent representative $b \in G(\mathbb{Q}_{p^s})$ of $\mathbf{b}$, such that $\nu_b = \boldsymbol{\nu}_{\mathbf{b}}$. Indeed, by [RZ96, 1.11], every $\mathbf{b}$ has a decent representative. Moreover, we can choose s large enough such that G is quasisplit over $\mathbb{Q}_{p^s}$, and then take an arbitrary s-decent element. Now, replacing b by a φ-conjugate in $G(\mathbb{Q}_{p^s})$ preserves decency and conjugates the map ν_b, thus we can assume without loss of generality that ν_b is dominant.

Remark 2.3. One can define affine Deligne–Lusztig varieties over any local field F, and the statement of Theorem 1.2 is conjectured to hold in this generality. Our Theorem 1.2 holds when F is a finite extension of $\mathbb{Q}_p$, via a standard restriction of scalars argument (see for example [DOR10, §5&§8]). It is not clear if our method goes through in the equal characteristic case.

5. To conclude this background section, we briefly recall some notations (see for example [Kis17, (1.3.6)], [Zho20, Xu21]) for Corollary 1.4. Let $k \subset \overline{\mathbb{F}}_p$ be a finite extension of the residue field κ_E of $E_{(v)}$, where $E = E(G, X)$ is the reflex field. For $x \in \mathscr{S}_{K_p}(G, X)(k)$, we denote by $I_x \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x \otimes \overline{\mathbb{F}}_p)$ (resp. $I_{/k} \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$) the subgroup fixing the Hodge tensors $s_{\alpha,\ell,x}$ for all $\ell \neq p$ and the crystalline tensor $s_{\alpha,0,x}$. Let $\overline{x}$ be the $\overline{\mathbb{F}}_p$-point associated to x. Recall that the ℓ-adic tensors $s_{\alpha,\ell,x} \in H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Q}_\ell)^{\otimes}$ cut out a group inside $\mathrm{GL}(H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Q}_\ell))$ that is identifiable to $G_{\mathbb{Q}_\ell}$ via the level structure ϵ^p. Since the tensors $s_{\alpha,\ell,x}$ are fixed by the action of the geometric Frobenius γ_ℓ on $H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Q}_\ell)$, we can view γ_ℓ as an element of $G(\mathbb{Q}_\ell)$. We denote by $I_{\ell/k}$ the centralizer of γ_ℓ in $G(\mathbb{Q}_\ell)$ and by I_ℓ the centralizer of γ_ℓ^n for sufficiently large n (recall from [Kis17, 2.1.2] that the centralizers of γ_ℓ^n form and increasing sequence and stabilizes for large enough n).

On the other hand, let $\mathscr{G}_x$ be the p-divisible group associated to x. By [KP18], the Frobenius on $\mathbb{D}(\mathscr{G}_x)$ is of the form $\varphi = \delta\varphi$ for some $\delta \in G(K_0)$ let $I_{p/k}$ be the group over $\mathbb{Q}_p$ whose R-points are given by $I_{p/k}(R) := \{g \in G(W(k)[\frac{1}{p}] \otimes_{\mathbb{Q}_p} R) | g^{-1}\delta\sigma(g) = \delta\}$. The following Corollary 1.4 is a parahoric analogue to [Kis17, Propositions 2.1.5] when $\mathbf{G}_{\mathbb{Q}_p}$ is quasi-split.

Corollary 2.4. *Let $H^p = \prod_{\ell \neq p} I_{\ell/k}(\mathbb{Q}_\ell) \cap K^p$ and $H_p = I_{p/k}(\mathbb{Q}_p) \cap \mathcal{G}(W(k))$. Then the map* (1.6) *induces an injective map*

$$I_{/k}(\mathbb{Q}) \backslash \prod_\ell I_{\ell/k}(\mathbb{Q}_\ell)/H_p \times H^p \to \mathscr{S}_K(G, X)(k), \tag{2.10}$$

where $I_{/k}$ is the analogue of I_x for the abelian variety over k.
In particular, the left hand side of (2.10) *is finite.*

Proof. It follows by combining our Theorem 1.2 with [Zho20, Prop 9.1]. □

Corollary 2.5. *For some prime $\ell \neq p$, $I_{x,\mathbb{Q}_\ell} = I_x \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ contains the connected component of the identity in I_ℓ. In particular, the ranks of I_x and G are equal.*

Proof. Using Corollary 2.4, this follows similarly as [Kis17, 2.1.7]. □

Corollary 2.6. *(Corollary 1.4(2)) The isogeny class $\iota_x(X_\mu^{\mathcal{K}_p}(b)(\bar{\mathbb{F}}_p)) \times \mathbf{G}(\mathbb{A}_f^p))$ contains a point which lifts to a special point on $\mathscr{S}_{\mathcal{K}_p}(G, X)$.*

Proof. This follows the outline from the proof of [Kis17, Theorem 2.2.3], and can be directly obtained by combining our Theorem 1.2 with [Zho20, Theorem 9.4]. Note that Corollary 2.5 is the parahoric version of an ingredient crucially used in [Kis17, Theorem 2.2.3]. □

3. Geometric background

3.1. **v-sheaf-theoretic setup.** We work within Scholze's framework of diamonds and v-sheaves [Sch17]. More precisely, we consider geometric objects that are functors

$$\mathcal{F} : \mathrm{Perf}_{\mathbb{F}_p} \to \mathrm{Sets}, \tag{3.1}$$

where $\mathrm{Perf}_{\mathbb{F}_p}$ is the site of affinoid perfectoid spaces in characteristic p, endowed with the v-topology (see [Sch17, Definition 8.1]). Recall that given a topological space T, we can define a v-sheaf $\underline{T}$ whose value on (R, R^+)-points is the set of continuous maps $|\operatorname{Spa}(R, R^+)| \to T$. We will mostly use this notation $\underline{T}$ for topological groups T.

Example 3.1. $\underline{\mathcal{I}(\mathbb{Z}_p)}$ and $\underline{G(\mathbb{Q}_p)}$ are the v-sheaf group objects attached to the topological groups $\mathcal{I}(\mathbb{Z}_p)$ and $G(\mathbb{Q}_p)$.

Conversely, to any diamond or v-sheaf $\mathcal{F}$, by [Sch17, Proposition 12.7], one can attach an underlying topological space that we denote by $|\mathcal{F}|$.

6. Recall that in the more classical setup of Rapoport–Zink spaces [RZ96], affine Deligne–Lusztig varieties arise, via Dieudonne theory, as the perfection of special fibers of Rapoport-Zink spaces. Moreover, the rigid generic fiber of such a Rapoport-Zink space is a special case of the so called *local Shimura varieties* [RV14]. In this way, Rapoport-Zink spaces (formal schemes) interpolate between local Shimura varieties and their corresponding affine Deligne–Lusztig varieties. Or in other words, Rapoport-Zink spaces serve as *integral models* of local Shimura varieties whose perfected special fibers are ADLVs. Moreover, by [SW20], the diamondification functor

$$\begin{aligned} \diamondsuit : \{\text{Adic Spaces}/\operatorname{Spa}\mathbb{Z}_p\} &\longrightarrow \{\text{v-sheaves}/\operatorname{Spd}\mathbb{Z}_p\} \\ X &\longmapsto X^{\diamondsuit} \end{aligned}$$

applied to a local Shimura variety is a locally spatial diamond that can be identified with a moduli space of p-adic shtukas (see §3.4).

Alternatively, one could consider the diamondification functor applied to the entire formal schemes (such as Rapoport-Zink spaces), rather than only their rigid generic fibres. The diamondification functor naturally takes values in v-sheaves, but contrary to the rigid-analytic case, these v-sheaves are no longer diamonds. Nevertheless, the v-sheaf associated to a formal scheme still has a lot of structure. Indeed, they are what the first author calls *kimberlites* [Gle22a, Definition 4.35], i.e. we have a commutative diagram

$$\begin{tikzcd} \{\text{Adic Spaces}/\operatorname{Spa}\mathbb{Z}_p\} \arrow[r, "\diamondsuit"] & \{\text{v-sheaves}/\operatorname{Spd}\mathbb{Z}_p\} \\ \{\text{Formal Schemes}/\operatorname{Spf}\mathbb{Z}_p\} \arrow[u, hook] \arrow[r, "\diamondsuit"] & \{\text{Kimberlites}/\operatorname{Spd}\mathbb{Z}_p\} \arrow[u, hook] \end{tikzcd} \tag{3.2}$$

Kimberlites share with formal schemes many pleasant properties that general v-sheaves do not. Let us list the main ones. Let $\mathfrak{X}$ be a kimberlite.

(1) Each kimberlite has an open analytic locus $\mathfrak{X}^{\mathrm{an}}$ (which is a locally spatial diamond by definition), and a reduced locus $\mathfrak{X}^{\mathrm{red}}$ (which is by definition a perfect scheme).
(2) Each kimberlite has a continuous "specialization map" whose source is $|\mathfrak{X}^{\mathrm{an}}|$ and whose target is $|\mathfrak{X}^{\mathrm{red}}|$ (see 7 for details).
(3) Kimberlites have a formal étale site and a formal nearby-cycles functor $R\Psi^{\mathrm{for}} : D_{\mathrm{\acute{e}t}}(\mathfrak{X}^{\mathrm{an}}, \Lambda) \to D_{\mathrm{\acute{e}t}}(\mathfrak{X}^{\mathrm{red}}, \Lambda)$ [GL22b].

Although we expect that every local Shimura variety admits a formal scheme "integral model" (see [PR22] for the strongest result on this direction), this is not known in full generality. Nevertheless, as the first author proved, every local Shimura variety (even the more general moduli spaces of p-adic shtukas) is modeled by a *prekimberlite*[13] whose perfected special fiber is the corresponding ADLV (see Theorem 3.5). We shall return to this discussion in §3.4.

7. Recall that given a formal scheme $\mathcal{X}$, one can attach a specialization triple $(\mathcal{X}_\eta, \mathcal{X}^{\mathrm{red}}, \mathrm{sp})$, where $\mathcal{X}_\eta$ is a rigid analytic space (the Raynaud generic fiber), $\mathcal{X}^{\mathrm{red}}$ is a reduced scheme (the reduced special fiber) and

$$\mathrm{sp} : |\mathcal{X}_\eta| \to |\mathcal{X}^{\mathrm{red}}| \tag{3.3}$$

is a continuous map.

Analogously, to a prekimberlite $\mathfrak{X}$ [Gle22b, Definition 4.15] over $\mathrm{Spd}(\mathbb{Z}_p)$, one can attach a specialization triple $(\mathfrak{X}_\eta, \mathfrak{X}^{\mathrm{red}}, \mathrm{sp})$ where

- $\mathfrak{X}_\eta$ is the generic fiber (which is an open subset of the analytic locus $\mathfrak{X}^{\mathrm{an}}$ [Gle22b, Definition 4.15] of $\mathfrak{X}$).
- $\mathfrak{X}^{\mathrm{red}}$ is a perfect scheme over $\mathbb{F}_p$ (obtained via the reduction functor [Gle22b, §3.2]) and
- sp is a continuous map [Gle22b, Proposition 4.14] analogous to (3.3).

For example, if $\mathfrak{X} = \mathcal{X}^\diamondsuit$ for a formal scheme $\mathcal{X}$, then $\mathfrak{X}$ is a kimberlite, and we have $\mathfrak{X}_\eta = \mathcal{X}_\eta^\diamondsuit$, $\mathfrak{X}^{\mathrm{red}} = (\mathcal{X}^{\mathrm{red}})^{\mathrm{perf}}$ and the specialization maps attached to $\mathcal{X}$ and $\mathfrak{X}$ agree, i.e. we have the following commutative diagram:

$$\begin{tikzcd} \mid \mathcal{X}_\eta \mid \arrow[r, "\cong"] \arrow[d, "\mathrm{sp}"'] & \mid \mathfrak{X}_\eta \mid \arrow[d, "\mathrm{sp}"] \\ \mid \mathcal{X}^{\mathrm{red}} \mid \arrow[r, "\cong"] & \mid \mathfrak{X}^{\mathrm{red}} \mid \end{tikzcd} \tag{3.4}$$

8. A *smelted kimberlite* is a pair $(\mathfrak{X}, X)$ where $\mathfrak{X}$ is a prekimberlite and $X \subseteq \mathfrak{X}^{\mathrm{an}}$ is an open subsheaf of the analytic locus, subject to some technical conditions. This is mainly used when $X = \mathfrak{X}^{\mathrm{an}}$ or when X is the generic fiber of a map to $\mathrm{Spd}\,\mathbb{Z}_p$ that is not p-adic.

[13] In fact, we expect moduli spaces of p-adic shtukas to be modeled by kimberlites, but for our purposes this difference is minor, as the specialization map is defined for both kimberlites and prekimberlites.

Given a smelted kimberlite $(\mathfrak{X}, X)$ and a closed point $x \in |\mathfrak{X}^{\mathrm{red}}|$, one can define the tubular neighborhood $X_x^{\circledcirc}$ ([Gle22b, Definition 4.38]). It is an open subsehaf of X which, roughly speaking, is given as the locus in X of points that specialize to x.

3.2. B_{dR}^+-Grassmannians and local models.

Let Gr_G be the B_{dR}^+-Grassmannian attached to G [SW20, §19, 20]. This is an ind-diamond over $\mathrm{Spd}\,\breve{\mathbb{Q}}_p$. We omit G from the notation from now on, and denote by Gr_μ the Schubert variety [SW20, Definition 20.1.3] attached to G and μ. This is a spatial diamond over $\mathrm{Spd}\,\breve{E}$ where $\breve{E} = E \cdot \breve{\mathbb{Q}}_p$ and E is the field of definition of $\boldsymbol{\mu}$. Now, Gr_μ contains the Schubert cell attached to μ, which we denote by Gr_μ°. This is an open dense subdiamond of Gr_μ.

Let $\mathrm{Gr}_{\mathcal{K}_p}$ be the Beilinson–Drinfeld Grassmannian attached to $\mathcal{K}_p$. This is a v-sheaf that is ind-representable in diamonds over $\mathrm{Spd}\,\breve{\mathbb{Z}}_p$, whose generic fiber is Gr_G, and whose reduced special fiber is $\mathcal{F}\ell_{\breve{\mathcal{K}}_p}$. Let $\mathcal{M}_{\mathcal{K}_p,\mu}$ be the local models first introduced in [SW20, Definition 25.1.1] for minuscule μ and later extended to non-minuscule μ in [AGLR22, Definition 4.11].

A priori, these local models are defined only as v-sheaves over $\mathrm{Spd}\,O_{\breve{E}}$, but when μ is minuscule, $\mathcal{M}_{\mathcal{K}_p,\mu}$ is representable by a normal scheme flat over $\mathrm{Spec}\,O_{\breve{E}}$ by [AGLR22, Theorem 1.1] and [GL22b, Corollary 1.4][14]. Moreover, in the general case, i.e. μ not necessarily minuscule, $\mathcal{M}_{\mathcal{K}_p,\mu}$ is a kimberlite by [AGLR22, Proposition 4.14], and it is unibranch by [GL22b, Theorem 1.2]. Let $\mathcal{A}_{\mathcal{K}_p,\mu}$ denote the μ-admissible locus inside $\mathcal{F}\ell_{\breve{\mathcal{K}}_p}$ (see for example [AGLR22, Definition 3.11]). This is a perfect scheme whose $\bar{\mathbb{F}}_p$-valued points agree with $\breve{\mathcal{K}}_p\,\mathrm{Adm}(\mu)\breve{\mathcal{K}}_p/\breve{\mathcal{K}}_p$. The generic fiber of $\mathcal{M}_{\mathcal{K}_p,\mu}$ is Gr_μ and the reduced special fiber is $\mathcal{A}_{\mathcal{K}_p,\mu}$ by [AGLR22, Theorem 1.5].

3.3. **Functoriality of affine Deligne–Lusztig varieties.** The formation of affine Deligne–Lusztig varieties is functorial with respect to morphisms of tuples $(G_1, b_1, \mu_1, \mathcal{K}_{1,p}) \to (G_2, b_2, \mu_2, \mathcal{K}_{2,p})$. More precisely, we have the following lemma.

Lemma 3.2. *Let $f : G_1 \to G_2$ be a group homomorphism such that $b_2 = f(b_1)$, $\mu_2 = f \circ \mu_1$ and $f(\mathcal{K}_{1,p}) \subseteq \mathcal{K}_{2,p}$. Then we have a map $X_{\mu_1}^{\mathcal{K}_{1,p}}(b_1) \to X_{\mu_2}^{\mathcal{K}_{2,p}}(b_2)$ that fits in the following commutative diagram:*

$$\begin{tikzcd} X_{\mu_1}^{\mathcal{K}_{1,p}}(b_1) \arrow[r] \arrow[d] & X_{\mu_2}^{\mathcal{K}_{2,p}}(b_2) \arrow[d] \\ \mathcal{F}\ell_{\breve{K}_{1,p}} \arrow[r] & \mathcal{F}\ell_{\breve{K}_{2,p}} \end{tikzcd} \tag{3.5}$$

Proof. This follows directly from the definitions and from Lemma 3.3. □

[14] Representability is proved in full generality in [AGLR22] and normality is proven when $p \geq 5$. In [GL22b] normality is proved even when $p < 5$.

Lemma 3.3. $f(\breve{\mathcal{K}}_{1,p}\operatorname{Adm}(\mu_1)\breve{\mathcal{K}}_{1,p}) \subseteq \breve{\mathcal{K}}_{2,p}\operatorname{Adm}(\mu_2)\breve{\mathcal{K}}_{2,p}$.

Proof. We give a geometric argument. Let $\mathcal{M}_{\mathcal{K}_{1,p},\mu_1}$ and $\mathcal{M}_{\mathcal{K}_{2,p},\mu_2}$ denote the v-sheaf local models in [AGLR22, Definition 4.11]. Since $f(\mathcal{K}_{1,p}) \subseteq \mathcal{K}_{2,p}$, we have a morphism of parahoric group schemes $\mathcal{K}_{1,p} \to \mathcal{K}_{2,p}$. By the functoriality result of v-sheaf local models [AGLR22, Proposition 4.16], we obtain a morphism $\mathcal{M}_{\mathcal{K}_{1,p},\mu_1} \to \mathcal{M}_{\mathcal{K}_{2,p},\mu_2}$ of v-sheaves. Moreover, by [AGLR22, Theorem 6.16], we know that $\mathcal{M}_{\mathcal{K}_{i,p},\mu_i,\bar{\mathbb{F}}_p} \subseteq \mathcal{F}\ell_{\breve{K}_{i,p}}$ consists of Schubert cells parametrized by $\operatorname{Adm}(\mu_i)$. More precisely, $\mathcal{M}_{\mathcal{K}_{i,p},\mu_i,\bar{\mathbb{F}}_p}(\bar{\mathbb{F}}_p) = \breve{\mathcal{K}}_{i,p}\operatorname{Adm}(\mu_i)\breve{\mathcal{K}}_{i,p}/\breve{\mathcal{K}}_{i,p}$. Therefore, the existence of the map of perfect schemes $\mathcal{M}_{\mathcal{K}_{1,p},\mu_1,\bar{\mathbb{F}}_p} \to \mathcal{M}_{\mathcal{K}_{2,p},\mu_2,\bar{\mathbb{F}}_p}$ immediately implies that $f(\breve{\mathcal{K}}_{1,p}\operatorname{Adm}(\mu_1)\breve{\mathcal{K}}_{1,p}) \subseteq \breve{\mathcal{K}}_{2,p}\operatorname{Adm}(\mu_2)\breve{\mathcal{K}}_{2,p}$. □

Lemma 3.2 is most relevant in the following situations:

(1) When $G_1 = G_2$, $f = \mathrm{id}$, and $\mathcal{K}_{1,p} \subseteq \mathcal{K}_{2,p}$.
(2) When $G_2 = G_1^{\mathrm{ab}}$ and $\mathcal{K}_{2,p}$ is the only parahoric of the torus G_1^{ab}.
(3) When $G_2 = G_1/Z$, where Z a central subgroup of G_1 and $\mathcal{K}_{2,p} = f(\mathcal{K}_{1,p})$.

To simplify certain proofs, we will also need the following statement.

Lemma 3.4. *Suppose $G = G_1 \times G_2$, $b = (b_1, b_2)$, $\mu = (\mu_1, \mu_2)$ and $\mathcal{K}_p = \mathcal{K}_p^1 \times \mathcal{K}_p^2$. Then $X_\mu^{\mathcal{K}_p}(b) = X_{\mu_1}^{\mathcal{K}_p^1}(b_1) \times X_{\mu_2}^{\mathcal{K}_p^2}(b_2)$.*

Proof. This follows directly from the definition. □

3.4. **Moduli spaces of p-adic shtukas.**

9. Recall from [SW20, §23] that to each (G, b, μ) and a closed subgroup $K \subseteq G(\mathbb{Q}_p)$, one can attach a locally spatial diamond $\mathrm{Sht}_{(G,b,\mu,K)}$ over $\operatorname{Spd}\breve{E}$, where $\breve{E} = \breve{\mathbb{Q}}_p \cdot E$ and E is the reflex field of $\boldsymbol{\mu}$, i.e. $\mathrm{Sht}_{(G,b,\mu,K)}$ is the moduli space of p-adic shtukas with level K.

This association is functorial in the tuple (G, b, μ, K), i.e. if $f : G \to H$ is a morphism of groups, we let $b_H := f(b)$, $\mu_H := f \circ \mu$ and we assume $f(K) \subseteq K_H$, then we have a morphism of diamonds

$$\mathrm{Sht}_{(G,b,\mu,K)} \to \mathrm{Sht}_{(H,b_H,\mu_H,K_H)}. \tag{3.6}$$

(1) When $H = G^{\mathrm{ab}}$, $f = \det : G \to G^{\mathrm{ab}}$ is the natural quotient map, and $K_H = \det(K) =: K^{\mathrm{ab}}$, we let $b^{\mathrm{ab}} := \det(b)$, $\mu^{\mathrm{ab}} := \det \circ \mu$, and the morphism (3.6) in this case is called the "determinant map"

$$\det : \mathrm{Sht}_{(G,b,\mu,K)} \to \mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},K^{\mathrm{ab}})}. \tag{3.7}$$

(2) When $H = G$, $f = \mathrm{id}$, and the inclusion $K_1 \subseteq K_2$ is proper, we have a change-of-level-structures map:

$$\mathrm{Sht}_{(G,b,\mu,K_1)} \to \mathrm{Sht}_{(G,b,\mu,K_2)} \tag{3.8}$$

10. For parahoric levels K_p, $\mathrm{Sht}_{(G,b,\mu,K_p)}$ is the generic fiber of a canonical[15] integral model, which is a v-sheaf $\mathrm{Sht}_{\mu}^{\mathcal{K}_p}(b)$ over $\mathrm{Spd}\,\mathcal{O}_{\breve{E}}$ defined in [SW20, §25]. In [Gle22a, Theorem 2], the first author proved that $\mathrm{Sht}_{\mu}^{\mathcal{K}_p}(b)$ is a prekimberlite (see [Gle22b, Definition 4.15]). Moreover, by [Gle22a, Proposition 2.30], its reduction (or its reduced special fiber in the sense of [Gle22b, §3.2]) can be identified with $X_{\mu}^{\mathcal{K}_p}(b)$. Furthermore, the formalism of kimberlites developed in [Gle22b] gives a continuous specialization map which turns out to be surjective (on the underlying topological spaces).

Theorem 3.5. [Gle22a, Theorem 2] *The pair $(\mathrm{Sht}_{\mu}^{\mathcal{K}_p}(b), \mathrm{Sht}_{(G,b,\mu,K_p)})$ is a rich smelted kimberlite*[16]*. Moreover, $\mathrm{Sht}_{\mu}^{\mathcal{K}_p}(b)^{\mathrm{red}} = X_{\mu}^{\mathcal{K}_p}(b)$. In particular, we have a surjective and continuous specialization map.*

$$\mathrm{sp} : |\mathrm{Sht}_{(G,b,\mu,K_p)}| \to |X_{\mu}^{\mathcal{K}_p}(b)|. \tag{3.9}$$

11. Now we recall the infinite-dimensional local model diagram of [Gle22a, Theorem 3][17]. It has the form

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "f"] & \\ (\mathrm{Sht}_{\mu}^{\mathcal{K}_p}(b) \times \mathrm{Spd}\,\mathcal{O}_F)^{\circledcirc}_{x} & & (\mathcal{M}_{\mathcal{K}_p,\mu} \times \mathrm{Spd}\,\mathcal{O}_F)^{\circledcirc}_{y} \end{tikzcd} \tag{3.10}$$

where $x \in X_{\mu}^{\mathcal{K}_p}(b)(k_F)$, $y \in \mathcal{A}_{\mathcal{K}_p,\mu}(k_F)$, and the maps f and g are $\widehat{L_W^+ G}$-torsors for a certain infinite-dimensional connected group v-sheaf $\widehat{L_W^+ G}$.

Theorem 3.6. [GL22b, Theorem 1.3] *For any parahoric $K_p \subseteq G(\mathbb{Q}_p)$ and any field extension $\breve{E} \subseteq F \subseteq \mathbb{C}_p$, the tubular neighborhoods of $(\mathcal{M}_{\mathcal{K}_p,\mu} \times \mathrm{Spd}\,\mathcal{O}_F, \mathrm{Gr}_{\mu} \times \mathrm{Spd}\,F)$ are connected.*

Using (3.10) and Theorem 3.6, one can show that the specialization map (3.9) induces a map $\pi_0(\mathrm{sp})$ on connected components.

Proposition 3.7. *For any parahoric $K_p \subseteq G(\mathbb{Q}_p)$ and any field extension $\breve{E} \subseteq F \subseteq \mathbb{C}_p$, the map*

$$\pi_0(\mathrm{sp}) : \pi_0(\mathrm{Sht}_{(G,b,\mu,K_p)} \times \mathrm{Spd}\,F) \xrightarrow{\sim} \pi_0(X_{\mu}^{\mathcal{K}_p}(b)) \tag{3.11}$$

is bijective.

[15] canonical in the sense that $\mathrm{Sht}_{\mu}^{\mathcal{K}_p}(b)$ represents a moduli problem.

[16] The term "rich" refers to some technical finiteness assumption that ensures that the specialization map can be controlled by understanding the preimage of the closed points in the reduced special fiber.

[17] We warn the reader that this local model correspondence does not agree with the more classical local model diagrams considered in the literature.

Proof. Recall that by [Gle22b, Lemma 4.55], whenever $(\mathfrak{X}, X)$ is a rich smelted kimberlite, to prove that

$$\pi_0(\mathrm{sp}) : \pi_0(X) \to \pi_0(\mathfrak{X}^{\mathrm{red}}) \tag{3.12}$$

is bijective, it suffices to prove that $(\mathfrak{X}, X)$ is unibranch[18] (in the sense of [Gle22b, Definition 4.52]), i.e. tubular neighborhoods are connected. By Theorem 3.5, $(\mathrm{Sht}_\mu^{\mathcal{K}_p}(b), \mathrm{Sht}_{(G,b,\mu,K_p)})$ is a rich smelted kimberlite, and thus it suffices to prove that $(\mathrm{Sht}_\mu^{\mathcal{K}_p}(b) \times \mathrm{Spd}\, \mathcal{O}_F, \mathrm{Sht}_{(G,b,\mu,K_p)} \times \mathrm{Spd}\, F)$ is unibranch, i.e. their tubular neighborhoods are connected.

By (3.10), it suffices to prove that the tubular neighborhoods of $(\mathcal{M}_{\mathcal{K}_p,\mu} \times \mathrm{Spd}\, \mathcal{O}_F, \mathrm{Gr}_\mu \times \mathrm{Spd}\, F)$ are connected, which follows from Theorem 3.6. □

With a similar argument as in Lemma 3.3, one can prove that the formation of $\mathrm{Sht}_\mu^{\mathcal{K}_p}(b)$ is also functorial in tuples $(G, b, \mu, \mathcal{K}_p)$.

Lemma 3.8. *Let $f : G_1 \to G_2$ be a group homomorphism such that $b_2 = f(b_1)$, $\mu_2 = f \circ \mu_1$ and $f(\mathcal{K}_{1,p}) \subseteq \mathcal{K}_{2,p}$. Then we have a map*

$$\mathrm{Sht}_{\mu_1}^{\mathcal{K}_{1,p}}(b_1) \to \mathrm{Sht}_{\mu_2}^{\mathcal{K}_{2,p}}(b_2) \tag{3.13}$$

of v-sheaves. Moreover, taking the reduction functor [Gle22b, §3.2] *of map* (3.13) *induces the map $X_{\mu_1}^{\mathcal{K}_{1,p}}(b_1) \to X_{\mu_2}^{\mathcal{K}_{2,p}}(b_2)$ of Lemma 3.2.*

Proof. The first statement follows from the definition of $\mathrm{Sht}_{\mu_1}^{\mathcal{K}_{1,p}}(b_1)$ (see for example [Gle22a, Definition 2.26]) and from Lemma 3.3. By [Gle22a, Proposition 2.30], we have the identity $X_{\mu_i}^{\mathcal{K}_{i,p}}(b_i) = \mathrm{Sht}_{\mu_i}^{\mathcal{K}_{i,p}}(b_i)^{\mathrm{red}}$, with $i \in \{1, 2\}$, where the right-hand side is the reduced special fiber (more precisely, it is the image under the reduction functor defined *loc.cit.*). □

As a special case, if we fix a datum (G, b, μ) and two parahorics $K_1 \subseteq K_2$ of $G(\mathbb{Q}_p)$, we have a map

$$\mathrm{Sht}_\mu^{\mathcal{K}_1}(b) \to \mathrm{Sht}_\mu^{\mathcal{K}_2}(b) \tag{3.14}$$

of v-sheaves. On the generic fiber, the map (3.14) gives the change-of-level-structures map of (3.8). After applying the reduction functor to the map (3.14), we recover the map $X_\mu^{\mathcal{K}_1}(b) \to X_\mu^{\mathcal{K}_2}(b)$ from Lemma 3.2 applied to scenario (1).

3.5. **The Grothendieck–Messing period map.** Recall that given a triple (G, b, μ), there is a quasi-pro-étale *Grothendieck–Messing period morphism* (see for example [SW20, §23]):

$$\pi_{\mathrm{GM}} : \mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, \mathbb{C}_p \to \mathrm{Gr}_\mu \times \mathrm{Spd}\, \mathbb{C}_p. \tag{3.15}$$

Now, the b-admissible locus $\mathrm{Gr}_\mu^b \subseteq \mathrm{Gr}_\mu$, can be defined as the image of π_{GM}. Note that $\mathrm{Gr}_\mu^b \times \mathrm{Spd}\, \mathbb{C}_p \subseteq \mathrm{Gr}_\mu \times \mathrm{Spd}\, \mathbb{C}_p$ is a dense open subset. Moreover,

[18]The definition of unibranchness, or alternatively *topological normality*, for smelted kimberlites is inspired by a useful criterion for the unibranchness of a scheme (see [AGLR22, Proposition 2.38]).

there is a (universal) $\underline{G(\mathbb{Q}_p)}$-torsor $\mathbb{L}_b$ over Gr^b_μ, such that for each finite extension K over $\breve{E}$ and $x \in \mathrm{Gr}^b_\mu(K)$, $x^*\mathbb{L}_b$ is a crystalline representation associated to the isocrystal with G-structure defined by b (for more details see for example [Gle21, §2.2-2.4]). The map in (3.15) can be then constructed as the geometric $\underline{G(\mathbb{Q}_p)}$-torsor attached to $\mathbb{L}_b$, i.e. $\mathrm{Sht}_{(G,b,\mu,\infty)}$ is the moduli space of trivializations of $\mathbb{L}_b$. The first author together with Lourenço prove the following theorem using diamond-theoretic techniques.

Theorem 3.9 ([GL22a]). *Let (G, b, μ) be a p-adic shtuka datum with $\mathbf{b} \in B(G, \boldsymbol{\mu})$. The b-admissible locus $\mathrm{Gr}^b_\mu \times \mathrm{Spd}\,\mathbb{C}_p$ is connected and dense within $\mathrm{Gr}_\mu \times \mathrm{Spd}\,\mathbb{C}_p$.*

From this we deduce the following.

Proposition 3.10. *The $G(\mathbb{Q}_p)$-action on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ is transitive.*

Proof. Note that π_0 commutes with colimits. This gives an identification $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)/G(\mathbb{Q}_p) = \pi_0(\mathrm{Gr}^b_\mu \times \mathrm{Spd}\,\mathbb{C}_p)$. From which we deduce that $G(\mathbb{Q}_p)$ acts transitively on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$. □

3.6. **The Bialynicki-Birula map.** Recall the Bialynicki-Birula map (see [SW20, Proposition 19.4.2]) from the Schubert cell Gr°_μ to the generalized flag variety $\mathrm{Fl}_\mu := G/P_\mu$

$$\mathrm{BB} : \mathrm{Gr}^\circ_\mu \to \mathrm{Fl}_\mu\,. \tag{3.16}$$

In general, the map (3.16) is not an isomorphism (it is an isomorphism only when μ is minuscule), but it always induces a bijection on classical points, i.e. finite extensions F of $\breve{E}$ (see for example [Vie21, Theorem 5.2]).

Let $\mathrm{Fl}^{\mathrm{adm}}_\mu \subseteq \mathrm{Fl}_\mu$ denote the weakly admissible (or equivalently, semistable) locus inside the flag variety [DOR10, §5], and let $\mathrm{Gr}^{\circ,b}_\mu := \mathrm{Gr}^\circ_\mu \cap \mathrm{Gr}^b_\mu$. By [CF00], we have a bijection $\mathrm{BB} : \mathrm{Gr}^{\circ,b}_\mu(F) \cong \mathrm{Fl}^{\mathrm{adm}}_\mu(F)$ for all finite extensions F of $\breve{E}$. Moreover, (3.16) fits in the following commutative diagram:

$$\begin{tikzcd} \mathrm{Gr}^{\circ,b}_\mu \arrow[r] \arrow[d, "\mathrm{BB}"'] & \mathrm{Gr}^\circ_\mu \arrow[d, "\mathrm{BB}"] \\ \mathrm{Fl}^{\mathrm{adm}}_\mu \arrow[r] & \mathrm{Fl}_\mu. \end{tikzcd} \tag{3.17}$$

3.7. **Ad-isomorphisms and z-extensions.**

Definition 3.11. [Kot85, §4.8] A morphism $f : G \to H$ is called an *ad-isomorphism* if f sends the center of G to the center of H and induces an isomorphism of adjoint groups.

An important example of ad-isomorphisms are z-extensions.

Definition 3.12. [Kot82, §1] A map of connected reductive groups $f : G' \to G$ is a *z-extension* if: f is surjective, $Z = \mathrm{Ker}(f)$ is central in G', Z is isomorphic to a product of tori of the form $\mathrm{Res}_{F_i/\mathbb{Q}_p} \mathbb{G}_m$ for some finite extensions $F_i \subseteq \overline{\mathbb{Q}}_p$, and G' has simply connected derived subgroup.

Lemma 3.13. *Let $f : \tilde{G} \to G$ be a z-extension and $\mathbf{b} \in B(G, \boldsymbol{\mu})$.*
(1) There exist a conjugacy class of cocharacters $\tilde{\boldsymbol{\mu}}$ and an element $\tilde{\mathbf{b}} \in B(\tilde{G}, \tilde{\boldsymbol{\mu}})$ which, under the map $B(\tilde{G}, \tilde{\boldsymbol{\mu}}) \to B(G, \boldsymbol{\mu})$, map to $\boldsymbol{\mu}$ and $\mathbf{b}$, respectively.
(2) $c_{\tilde{b},\tilde{\mu}}\pi_1(\tilde{G})_I^{\varphi} \to c_{b,\mu}\pi_1(G)_I^{\varphi}$ is surjective.

Proof. (1) Let $T \subseteq G$ be a maximal torus and $\tilde{T} \subseteq \tilde{G}$ its preimage under f. Let $Z = \mathrm{Ker}(f)$. We have an exact sequence

$$0 \to Z \to \tilde{T} \to T \to 0 \tag{3.18}$$

Since Z is a torus, we have an exact sequence:

$$0 \to X_*(Z) \to X_*(\tilde{T}) \to X_*(T) \to 0 \tag{3.19}$$

In particular, we can lift $\boldsymbol{\mu}$ to an arbitrary $\tilde{\boldsymbol{\mu}} \in X_*(\tilde{T})$. To lift $\tilde{\mathbf{b}}$ compatibly, it suffices to recall from [Kot97, (6.5.1)] that

$$B(G, \boldsymbol{\mu}) \cong B(G^{\mathrm{ad}}, \boldsymbol{\mu}^{\mathrm{ad}}) \cong B(\tilde{G}, \tilde{\boldsymbol{\mu}}). \tag{3.20}$$

(2) Recall that the map $G(\mathbb{Q}_p) \to \pi_1(G)_I^{\varphi}$ is surjective (see for example [Zho20, Lemma 5.18]). Indeed, this follows from the exact sequence

$$0 \to \mathcal{T}(\breve{\mathbb{Z}}_p) \to T(\breve{\mathbb{Q}}_p) \to \pi_1(G)_I \to 0 \tag{3.21}$$

and the group cohomology vanishing $H^1(\mathbb{Z}, \mathcal{T}(\breve{\mathbb{Z}}_p)) = 0$, where $\mathcal{T}$ is the unique parahoric of T and the $\mathbb{Z}$-action on $\mathcal{T}(\breve{\mathbb{Z}}_p)$ is given by the Frobenius φ. Consider the following commutative diagram:

$$\begin{tikzcd} \tilde{G}(\mathbb{Q}_p) \arrow[r] \arrow[d] & \pi_1(\tilde{G})_I^{\varphi} \arrow[d] \\ G(\mathbb{Q}_p) \arrow[r] & \pi_1(G)_I^{\varphi} \end{tikzcd} \tag{3.22}$$

The horizontal arrows in (3.22) are surjective. Since Z is an induced torus, $H^1_{\text{ét}}(\operatorname{Spec} \mathbb{Q}_p, Z) = 0$. Thus by the exact sequence of pointed sets that

$$0 \to Z \to \tilde{G} \to G \to 0, \tag{3.23}$$

induces, the map $\tilde{G}(\mathbb{Q}_p) \to G(\mathbb{Q}_p)$ is surjective. Therefore $\pi_1(\tilde{G})_I^{\varphi} \to \pi_1(G)_I^{\varphi}$ is surjective. Finally, since $\tilde{\mathbf{b}}$ and $\tilde{\boldsymbol{\mu}}$ map to $\mathbf{b}$ and $\boldsymbol{\mu}$, the coset $c_{\tilde{b},\tilde{\mu}}\pi_1(\tilde{G})_I^{\varphi}$ also maps to the coset $c_{b,\mu}\pi_1(\tilde{G})_I^{\varphi}$. □

Assume that f is an ad-isomorphism for the rest of this subsection. Let $b_H := f(b)$ and $\mu_H := f \circ \mu$. Let $\mathcal{K}_p^H$ denote the unique parahoric of H that corresponds to the same point in the Bruhat–Tits building as $\mathcal{K}_p$.

Proposition 3.14. *The following diagram is Cartesian:*

$$\begin{tikzcd} \pi_0(X_\mu^{\mathcal{K}_p}(b)) \arrow[r] \arrow[d] & c_{b,\mu}\pi_1(G)_I^\varphi \arrow[d] \\ \pi_0(X_\mu^{\mathcal{K}_p^H}(b)) \arrow[r] & c_{b_H,\mu_H}\pi_1(H)_I^\varphi \end{tikzcd} \tag{3.24}$$

Proof. This is a consequence of [PR22, Lemma 5.4.2], which is a generalization of [CKV15, Corollary 2.4.2] for arbitrary parahorics. □

4. Hodge–Newton decomposition

We can classify elements in $B(G,\boldsymbol{\mu})$ into two kinds: Hodge-Newton decomposable or indecomposable.

Definition 4.1 (Hodge-Newton Decomposability)**.** Assume $\mathbf{b} \in B(G,\boldsymbol{\mu})$. We say $\mathbf{b}$ is *Hodge-Newton decomposable* (with respect to M) in $B(G,\boldsymbol{\mu})$ if there exists a φ_0-stable standard Levi subgroup M containing $M_{\boldsymbol{\nu}_{\mathbf{b}}}$, and

$$\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} \in \mathbb{Q}_{\geq 0}\Delta_M^\vee. \tag{4.1}$$

If no such M exists, $\mathbf{b}$ is said to be *Hodge-Newton indecomposable* in $B(G,\boldsymbol{\mu})$.

Example 4.2. A basic element $\mathbf{b}$ is always HN-indecomposable in $B(G,\boldsymbol{\mu})$ since $M_{\boldsymbol{\nu}_{\mathbf{b}}} = G$.

For a HN-decomposable $\mathbf{b}$ in $B(G,\boldsymbol{\mu})$, affine Deligne–Lusztig varieties admit a decomposition theorem (Theorem 4.3). More precisely, suppose $\mathbf{b}$ is HN-decomposable with respect to a Levi subgroup M. Let P be the standard parabolic subgroup containing M and B. As in [GHN19, 4.4], let $\mathfrak{P}^\varphi$ be the set of φ-stable parabolic subgroups containing the maximal torus T and conjugate to P. Given $P' \in \mathfrak{P}^\varphi$, let N' be the unipotent radical, and M' the Levi subgroup containing T such that $P' = M'N'$. We let $\mathcal{K}_p^{M'}$ denote the parahoric group scheme of M' such that $\mathcal{K}_p^{M'}(\breve{\mathbb{Q}}_p) = K_p \cap M'(\breve{\mathbb{Q}}_p)$. Let W_K be the subgroup of W_0 generated by the set of simple reflections corresponding to $\mathcal{K}_p$. Let W_K^φ be the φ-invariant elements of W_K. We have the following.

Theorem 4.3 ([GHN19, Theorem A])**.** *Let $\mathbf{b} \in B(G,\boldsymbol{\mu})$ be HN-decomposable with respect to $M \subset G$. Then there is an isomorphism*

$$X_\mu^{\mathcal{K}_p}(b) \simeq \bigsqcup_{P'=M'N'} X_{\mu_{P'}}^{\mathcal{K}_p^{M'}}(b_{P'}), \tag{4.2}$$

where P' ranges over the set $\mathfrak{P}^\varphi/W_K^\varphi$.

Note that the natural embedding

$$\phi_{P'} : X_{\mu_{P'}}^{\mathcal{K}_p^{M'}}(b_{P'}) \hookrightarrow X_\mu^{\mathcal{K}_p}(b) \tag{4.3}$$

is the composite of the closed immersion $\mathcal{F}\ell_{\mathcal{K}_p^{M'}} \hookrightarrow \mathcal{F}\ell_{\mathcal{K}_p}$ of affine flag varieties and the map $g\breve{\mathcal{K}}_p \mapsto h_{P'}g\breve{\mathcal{K}}_p$, where $h_{P'} \in G(\breve{\mathbb{Q}}_p)$ satisfies $b_{P'} = h_{P'}^{-1}b\sigma(h_{P'})$ ([GHN19, 4.5]).

By the following lemma, we may assume–without loss of generality in the proof of Proposition 4.10–that each $(b_{P'}, \mu_{P'})$ is HN-indecomposable.

Lemma 4.4 ([Zho20, Lemma 5.7])**.** *There exists a unique φ_0-stable $M \subset G$ such that, for each P' appearing in decomposition (4.2), $b_{P'}$ is HN-indecomposable in $B(M', \boldsymbol{\mu}_{P'})$.*

Example 4.5. When G^{ad} is simple, $\mathbf{b}$ is basic and μ is not central, then $\mathbf{b}$ is Hodge-Newton irreducible (Definition 2.1) in $B(G, \boldsymbol{\mu})$ because if a linear combination of coroots is dominant then all the coefficients are positive.

Example 4.5 shows that, except for the "central cocharacter" case, HN-indecomposability is the same as HN-irreducibility whenever $\mathbf{b}$ is basic. The general version of this phenomena is Proposition 4.6 below, which asserts that the gap between HN-indecomposable and HN-irreducible elements consists only of central elements.

Proposition 4.6 (cf. [Zho20, Lemma 5.3])**.** *Suppose that $G = G^{\mathrm{ad}}$ and that G is $\mathbb{Q}_p$-simple. Let $b \in G(\breve{\mathbb{Q}}_p)$ and μ a dominant cocharacter, such that $\mathbf{b} \in B(G, \boldsymbol{\mu})$. Suppose $(\mathbf{b}, \boldsymbol{\mu})$ is HN-indecomposable. Then either $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible or b is φ-conjugate to some $\dot{t}_{\bar{\mu}}$ with $\bar{\mu} \in X_*(T)_I$ central.*

Moreover, when b is φ-conjugate to $\dot{t}_{\bar{\mu}}$ for a central μ, the connected components of affine Deligne–Lusztig varieties have been computed in Proposition 4.7 below. Note that if μ is central, there is a unique $\mathbf{b} \in B(G, \boldsymbol{\mu})$. Moreover, this $\mathbf{b}$ is basic and represented by $\dot{t}_{\bar{\mu}}$, which is a lift of $t_{\bar{\mu}}$ to $N(\breve{\mathbb{Q}}_p)$. We can then apply the following result.

Proposition 4.7 ([HZ20, Theorem 0.1 (1)])**.** *Suppose that G^{ad} is $\mathbb{Q}_p$-simple. Let $b \in G(\breve{\mathbb{Q}}_p)$ be a representative for a basic element $\mathbf{b} \in B(G)$. If μ is central and $\mathbf{b} \in B(G, \boldsymbol{\mu})$, then $X_\mu^{\mathcal{K}_p}(b)$ is discrete and*

$$X_\mu^{\mathcal{K}_p}(b) \simeq G(\mathbb{Q}_p)/\mathcal{K}_p(\mathbb{Z}_p). \tag{4.4}$$

12. Next we show that HN-irreducibility is preserved under ad-isomorphisms and taking projection onto direct factors. Let $f : G \to H$ be an ad-isomorphism. Let $b_H := f(b)$ and $\mu_H = \mu \circ f$. Let T_H denote a maximal torus containing $f(T)$. By functoriality, we have commutative diagrams

$$\begin{tikzcd} X_*(T) \arrow[r, "f_*"] \arrow[d] & X_*(T_H) \arrow[d] \\ \pi_1(G)_\Gamma \arrow[r] & \pi_1(H)_\Gamma \end{tikzcd} \tag{4.5}$$

and

$$
\begin{array}{ccc}
B(G) & \longrightarrow & B(H) \\
\downarrow & & \downarrow \\
\pi_1(G)_\Gamma & \longrightarrow & \pi_1(H)_\Gamma
\end{array}
\qquad
\begin{array}{ccc}
B(G) & \longrightarrow & B(H) \\
\downarrow & & \downarrow \\
X_*(T)^+_{\mathbb{Q}} & \longrightarrow & X_*(T)^+_{\mathbb{Q}}
\end{array}
\tag{4.6}
$$

We have the following.

Proposition 4.8. *Let $\mathbf{b} \in B(G, \boldsymbol{\mu})$ and let f be an ad-isomorphism. Then $(\mathbf{b}_H, \boldsymbol{\mu}_H)$ is HN-irreducible if and only if $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*

Proof. Since $\mathbf{b} \in B(G, \boldsymbol{\mu})$, we have $\kappa_G(\mathbf{b}) = \mu^\natural$ (see 2). By (4.5) and (4.6), we have $\kappa_H(\mathbf{b}_H) = \mu^\natural_H$. Moreover, we can write

$$\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} = \sum_{\alpha\in\Phi^+} c_\alpha \alpha^\vee, \tag{4.7}$$

where $c_\alpha \geq 0$. On the other hand, note that $f_*(\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}}) = \mu^\diamond_H - \boldsymbol{\nu}_{\mathbf{b}_H}$. Since f is an ad-isomorphism, $f_*(\alpha^\vee) = \alpha^\vee$. Thus we have $\mu^\diamond_H - \boldsymbol{\nu}_{\mathbf{b}_H} = \sum_{\alpha\in\Phi^+} c_\alpha \alpha^\vee$, and hence $\mathbf{b}_H \in B(H, \boldsymbol{\mu}_H)$. Now, each $(\mathbf{b}_H, \boldsymbol{\mu}_H)$ is HN-irreducible if and only if $(\mathbf{b}, \boldsymbol{\mu})$ is, since this is in turn equivalent to $c_\alpha > 0$ for all $\alpha \in \Phi^+_G$. ☐

13. Let $G = G_1 \times G_2$, then $T = T_1 \times T_2$, $B(G) = B(G_1) \times B(G_2)$, $\pi_1(G)_\Gamma = \pi_1(G_1)_\Gamma \times \pi_1(G_2)_\Gamma$ and $X_*(T) = X_*(T_1) \times X_*(T_2)$. In this case, the Kottwitz and Newton maps 3 can be computed coordinatewise.

Proposition 4.9. *The following hold:*

1. *$\mathbf{b} \in B(G, \boldsymbol{\mu})$ if and only if each $\mathbf{b}_i \in B(G_i, \boldsymbol{\mu}_i)$ for $i \in \{1, 2\}$.*
2. *$(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible if and only if each $(\mathbf{b}_i, \mu_i)$ is HN-irreducible for $i \in \{1, 2\}$.*

Proof. The condition $\kappa_G(\mathbf{b}) = \mu^\natural$ can be checked component-wise. Moreover, since $\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} = (\mu^\diamond_1 - \boldsymbol{\nu}_{\mathbf{b}1}, \mu^\diamond_2 - \boldsymbol{\nu}_{\mathbf{b}2})$, verifying whether it is a non-negative (resp. positive) sum of positive coroots (see Definition 2.1) can also be done component-wise. ☐

Proposition 4.10. *Assume G^{ad} has only isotropic factors. If the Kottwitz map $\omega : \pi_0(X_\mu(b)) \to c_{b,\mu}\pi_1(G)^\varphi_I$ is a bijection, then $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*

Proof. By Proposition 4.8, Proposition 4.9, Lemma 3.4 and Proposition 3.14, we may assume without loss of generality that G is adjoint and $\mathbb{Q}_p$-simple. We prove by contradiction and assume that $(\mathbf{b}, \boldsymbol{\mu})$ is not HN-irreducible.

(I) If $\mathbf{b}$ is HN-decomposable in $B(G, \boldsymbol{\mu})$, then by Theorem 4.3, we have

$$\pi_0(X_\mu(b)) = \bigsqcup_{P' \in \mathfrak{P}^\varphi / W^\varphi_K} \pi_0(X^{M'}_{\mu_{P'}}(b_{P'})). \tag{4.8}$$

Thus by Lemma 4.4, we may assume that each $b_{P'}$ is HN-indecomposable in $B(M', \mu_{P'})$. Recall from (4.3) that for each $P' \in \mathfrak{P}^\varphi / W^\varphi_K$ we have an embedding $\phi_{P'} : X^{M'}_{\mu_{P'}}(b_{P'}) \hookrightarrow X_\mu(b)$, which induces a map

$$\pi_0(\phi_{P'}) : \pi_0(X^{M'}_{\mu_{P'}}(b_{P'})) \hookrightarrow \pi_0(X_\mu(b)). \tag{4.9}$$

The disjoint union over $P' \in \mathfrak{P}^\varphi / W_K^\varphi$ in (4.9) gives the bijection (4.8).

Consider $\iota : M'(\breve{F}) \to G(\breve{F})$. Let $\iota_I : \pi_1(M')_I \to \pi_1(G)_I$ be the induced map, which then induces a map $\iota_I^\varphi : \pi_1(M')_I^\varphi \to \pi_1(G)_I^\varphi$. By [Kot97, 7.4], the following diagram commutes:

$$\begin{tikzcd} M'(\breve{\mathbb{Q}}_p) \arrow[r, "\kappa_{M'}"] \arrow[d, "\iota"'] & \pi_1(M')_I \arrow[d, "\iota_I"] \\ G(\breve{\mathbb{Q}}_p) \arrow[r, "\kappa_G"] & \pi_1(G)_I \end{tikzcd} \tag{4.10}$$

Denote by $+_{h_{P'}} : \pi_1(G)_I \to \pi_1(G)_I$ the addition-by-$\kappa_G(h_{P'})$ map. Then (4.10) shows that $+_{h_{P'}} \circ \iota_I$ sends $c_{b_{P'},\mu_{P'}} \pi_1(M')_I^\varphi$ to $c_{b,\mu}\pi_1(G)_I^\varphi$. Moreover, we have the following commutative diagram

$$\begin{tikzcd} \pi_0(X^{M'}_{\mu_{P'}}(b_{P'})) \arrow[r, two heads, "\omega_{M'}"] \arrow[d, hook, "\pi_0(\phi_{P'})"'] & c_{b_{P'},\mu_{P'}}\pi_1(M')_I^\varphi \arrow[d, "+_{h_{P'}}\circ\iota_I"] \\ \pi_0(X_\mu(b)) \arrow[r, "\omega_G", "\cong"'] & c_{b,\mu}\pi_1(G)_I^\varphi \end{tikzcd} \tag{4.11}$$

Here the surjectivity of $\omega_{M'}$ follows from [HZ20, Lemma 6.1]. Now, if the lower horizontal arrow ω_G is a bijection, then the upper horizontal arrow $\omega_{M'}$ should also be a bijection. Moreover, this implies that $+_{h_{P'}} \circ \iota_I$ is injective, which then implies that $\iota_I^\varphi : \pi_1(M')_I^\varphi \to \pi_1(G)_I^\varphi$ is injective. This contradicts Lemma 4.11.

(II) If $\mathbf{b}$ is HN-indecomposable in $B(G, \boldsymbol{\mu})$, by Proposition 4.6, we may assume that μ is central and $b = \dot{t}_{\bar{\mu}}$. We now show that $\pi_0(X_\mu(b)) \to \pi_1(G)_I^\varphi$ is not bijective.

By Proposition 4.7, there is a bijection $\pi_0(X_\mu(b)) \simeq G(\mathbb{Q}_p)/\mathcal{K}_p(\mathbb{Z}_p)$. Since G is not anisotropic, there exists a non-trivial $\mathbb{Q}_p$-split torus S, and we can consider the composition of maps

$$S(\mathbb{Q}_p) \hookrightarrow G(\mathbb{Q}_p) \twoheadrightarrow G(\mathbb{Q}_p)/\mathcal{K}_p(\mathbb{Z}_p). \tag{4.12}$$

Since $S(\mathbb{Q}_p) \cap \mathcal{K}_p(\mathbb{Z}_p)$ is compact, we have $S(\mathbb{Q}_p) \cap \mathcal{K}_p(\mathbb{Z}_p) \subseteq S(\mathbb{Z}_p)$. Therefore, we obtain an injective homomorphism

$$X_*(S) \cong S(\mathbb{Q}_p)/S(\mathbb{Z}_p) \hookrightarrow G(\mathbb{Q}_p)/\mathcal{K}_p(\mathbb{Z}_p). \tag{4.13}$$

Since G is adjoint, $\pi_1(G)_I^\varphi$ is finite. However, $X_*(S)$ is infinite, thus the map $\omega_G : \pi_0(X_\mu(b)) \to \pi_1(G)_I^\varphi$ cannot be bijective. We have a contradiction. $\square$

Now we finish the proof of Proposition 4.10 by proving the following lemma.

Lemma 4.11. *Let G be adjoint and $\mathbb{Q}_p$-simple. Let $P \subseteq G$ be a proper parabolic defined over $\mathbb{Q}_p$ with Levi factor M. The natural map $\iota_I^{\varphi} : \pi_1(M)_I^{\varphi} \to \pi_1(G)_I^{\varphi}$ is not injective.*

Proof. Recall that $(\pi_1(G)_I)_{\hat{\mathbb{Z}}} \simeq \pi_1(G)_\Gamma$. We prove by contradiction and assume that the natural map $\iota_I^{\varphi} : \pi_1(M)_I^{\varphi} \to \pi_1(G)_I^{\varphi}$ is injective. In particular, $\pi_1(M)_I^{\varphi} \otimes \mathbb{Q} \hookrightarrow \pi_1(G)_I^{\varphi} \otimes \mathbb{Q}$ is also injective. Via the "average map" under φ-action, we have

$$\pi_1(-)_I^{\varphi} \otimes \mathbb{Q} \simeq (\pi_1(-)_I)_{\langle\varphi\rangle} \otimes \mathbb{Q} \simeq \pi_1(-)_\Gamma \otimes \mathbb{Q} \simeq \pi_1(-)^\Gamma \otimes \mathbb{Q}. \tag{4.14}$$

If ι_I^{φ} is injective, we deduce that the natural map

$$\pi_1(M)^\Gamma \otimes \mathbb{Q} \to \pi_1(G)^\Gamma \otimes \mathbb{Q} \tag{4.15}$$

is injective. Let $M \subseteq P \subseteq G$ be the corresponding parabolic subgroup. Let $\theta_P = \sum_{\alpha \in \Phi_P} \alpha^\vee \in X_*(T)$ denote the sum of coroots of P. Now, θ_P is Γ-stable since P is defined over $\mathbb{Q}_p$. Moreover, its image under the natural projection map $q_M : X_*(T) \to \pi_1(M)$ is Γ-stable. One can check that $q_M(\theta) \neq 0$ in $\pi_1(M)^\Gamma \otimes \mathbb{Q}$. Since $q_G(\theta_P) = 0$ in $\pi_1(G)$, the map in (4.15) is not injective. We have a contradiction, this proves that ι_I^{φ} is not injective. $\square$

5. Generic Mumford–Tate groups

5.1. **Mumford–Tate groups of crystalline representations.** We will use the theory of *crystalline representations with G-structures* (see for example [DOR10]). Let Rep_G be the category of algebraic representations of G in $\mathbb{Q}_p$-vector spaces. Let Isoc be the category of isocrystals over $\bar{\mathbb{F}}_p$.

Fix a finite extension K of $\breve{\mathbb{Q}}_p$. Let $\mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K}$ be the category of crystalline representations of Γ_K on finite-dimensional $\mathbb{Q}_p$-vector spaces. Let $\omega : \mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K} \to \mathrm{Vec}_{\mathbb{Q}_p}$ be the forgetful fibre functor. Let $\mathrm{IsocFil}_{K/\breve{\mathbb{Q}}_p}$ be the category of filtered isocrystals whose objects are pairs of an isocrystal N and a decreasing filtration of $N \otimes K$. Furthermore, let $\mathrm{IsocFil}^{\mathrm{ad}}_{K/\breve{\mathbb{Q}}_p}$ be Fontaine's subcategory of weakly admissible filtered isocrystals [Fon94]. This is a $\mathbb{Q}_p$-linear Tannakian category, which is equivalent to $\mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K}$ through Fontaine's functor V_{cris} [CF00].

14. Fix a pair (b, μ^η) with $b \in G(\breve{\mathbb{Q}}_p)$ and $\mu^\eta : \mathbb{G}_m \to G_K$ a group homomorphism over K. This defines a $\otimes$-functor

$$\mathcal{G}_{(b,\mu^\eta)} : \mathrm{Rep}_G \to \mathrm{IsocFil}_{K/\breve{\mathbb{Q}}_p} \tag{5.1}$$

sending $\rho : G \to \mathrm{GL}(V)$ to the filtered isocrystal $(V \otimes \breve{\mathbb{Q}}_p, \rho(b)\sigma, \mathrm{Fil}^\bullet_{\mu^\eta} V \otimes K)$, where the filtration on $V \otimes K$ is the one induced by μ^η. The pair is called *admissible* [RZ96, Definition 1.18], if the image of $\mathcal{G}_{(b,\mu^\eta)}$ lies in $\mathrm{IsocFil}^{\mathrm{ad}}_{K/\breve{\mathbb{Q}}_p}$. Moreover, when $\mathbf{b} \in B(G, \boldsymbol{\mu})$, $V_{\mathrm{cris}} \circ \mathcal{G}_{(b,\mu^\eta)}$ defines a conjugacy class of crystalline representations $\xi_{(b,\mu^\eta)} : \Gamma_K \to G(\mathbb{Q}_p)$ [DOR10, Proposition 11.4.3].

Definition 5.1. With notation as above, let $\mathrm{MT}_{(b,\mu^\eta)}$ denote the identity component of the Zariski closure of $\xi_{(b,\mu^\eta)}(\Gamma_K)$ in $G(\mathbb{Q}_p)$. This is the *Mumford–Tate group* attached to (b,μ^η).

Theorem 5.2. *(*[Ser79, Théorème 1],[Sen73, §4, Théorème 1]*,* [Che14, Proposition 3.2.1]*) The image of $\xi_{(b,\mu^\eta)}$ contains an open subgroup of $\mathrm{MT}_{(b,\mu^\eta)}$.*

15. As in [Che14, §3], we let $\mathscr{T}^{\mathrm{cris}}_{(b,\mu^\eta)} := \mathcal{G}_{(b,\mu^\eta)}(\mathrm{Rep}_G)$ and $\mathscr{T}_{(b,\mu^\eta)} := V_{\mathrm{cris}} \circ \mathcal{G}_{(b,\mu^\eta)}(\mathrm{Rep}_G)$ be the images of Rep_G. Then $\mathrm{MT}_{(b,\mu^\eta)}$ is the Tannakian group for the fiber functor $\omega : \mathscr{T}_{(b,\mu^\eta)} \to \mathrm{Vec}_{\mathbb{Q}_p}$ by [Che14, Proposition 3.2.3].

In [Che14, §3], there is a fiber functor $\omega_s : \mathscr{T}^{\mathrm{cris}}_{(b,\mu^\eta)} \to \mathrm{Vec}_{\mathbb{Q}_{p^s}}$ for s sufficiently large[19], with Tannakian group $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)} := \mathrm{Aut}^{\otimes}\omega_s$ as in [Che14, Définition 3.3.1]. When b is s-decent (see Definition 2.2), there is a canonical embedding $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)} \subseteq G_{\mathbb{Q}_{p^s}}$ [Che14, Lemme 3.3.2]. Moreover, $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$ and $\mathrm{MT}_{(b,\mu^\eta)} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^s}$ are pure inner forms of each other [Che14, Proposition 3.3.3]. Both claims follow immediately using Tannakian formalism. In particular, to prove that $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}, it suffices to prove $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$ contains $G^{\mathrm{der}}_{\mathbb{Q}_{p^s}}$ (since G^{der} is normal).

16. In fact, there is a more concrete description of $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$ given as follows.

Let $(V,\rho) \in \mathrm{Rep}_G$. The μ^η-filtration of V_K induces a degree function

$$\deg_{\mu^\eta} : V \setminus \{0\} \to \mathbb{Z}, \tag{5.2}$$

where $\deg_{\mu^\eta}(v) = i$ if $v \in \mathrm{Fil}^i_{\mu^\eta} V \setminus \mathrm{Fil}^{i+1}_{\mu^\eta} V$. We shall consider a subset $V^{s,k}_{(b,\mu^\eta)} \subseteq V \otimes \mathbb{Q}_{p^s}$ of elements that satisfy a certain "Newton equation" (5.3) and a certain "Hodge equation" (5.4) with respect to k.

Let $T^{s\cdot\nu_b}_\rho : V \otimes \mathbb{Q}_{p^s} \to V \otimes \mathbb{Q}_{p^s}$ be the operator with formula

$$T^{s\cdot\nu_b}_\rho := \rho \circ [s\cdot\nu_b](p). \tag{5.3}$$

Consider also the function $d^s_{\rho,\mu^\eta} : V \otimes \mathbb{Q}_{p^s} \setminus \{0\} \to \mathbb{Z}$ where

$$d^s_{\rho,\mu^\eta}(v) = \sum_{i=0}^{s-1} \deg_{\mu^\eta}([\rho(b)\varphi]^i(v)). \tag{5.4}$$

We consider the following subset of $V \otimes \mathbb{Q}_{p^s}$ given by

$$V^{s,k}_{(b,\mu^\eta)} := \{v \in V \otimes \mathbb{Q}_{p^s} \mid T^{s\cdot\nu_b}_\rho(v) = p^k v,\, d^s_{\rho,\mu^\eta}(v) = k\} \tag{5.5}$$

By [Che14, Proposition 3.3.6], $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$ consists of those elements $g \in G_{\mathbb{Q}_{p^s}}$ such that: for any $(V,\rho) \in \mathrm{Rep}_G$ and any $k \in \mathbb{Z}$, all of the elements $v \in V^{s,k}_{(b,\mu^\eta)}$ are eigenvectors of $\rho(g)$. In particular, to prove $G^{\mathrm{der}}_{\mathbb{Q}_{p^s}} \subseteq \mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$, it suffices to prove that $G^{\mathrm{der}}_{\mathbb{Q}_{p^s}}$ acts trivially on $V^{s,k}_{(b,\mu^\eta)}$ for all V and k.

[19]Note that our notation ω_s differs from the notations *loc.cit.*, where the notation $\omega^{\mathrm{cris},s}_{b,\mu}$ is used instead.

5.2. **Generic filtrations.**

17. As in [Che14, §4], we give a representation-theoretic formula for d^s_{ρ,μ^η} when μ^η is generic. In our case, G is not assumed to be neither unramified nor quasisplit.

We first recall some generalities, which we will apply later to $G_{\mathbb{Q}_{p^s}}$ for s-sufficiently large such that $G_{\mathbb{Q}_{p^s}}$ is quasisplit. Until further notice, K will denote an arbitrary field of characteristic 0, G a quasisplit reductive group over K, and $\boldsymbol{\mu}$ a conjugacy class of group homomorphisms $\mu : \mathbb{G}_m \to G_{\bar{K}}$. Let E/K be the reflex field of $\boldsymbol{\mu}$. Since G is quasisplit, we can choose a representative $\mu \in \boldsymbol{\mu}$ defined over E such that it is dominant for a choice of K-rational Borel $B \subseteq G$. To this data, we can associate a flag variety $\mathrm{Fl}_\mu := G_E/P_\mu$ over $\mathrm{Spec}\,(E)$ as in (3.16). It parametrizes filtrations of Rep_G of type $\boldsymbol{\mu}$. Given a field extension K'/K, $x \in \mathrm{Fl}_\mu(K')$ and $(V,\rho) \in \mathrm{Rep}_G$, we obtain a filtration $\mathrm{Fil}^\bullet_x V_{K'}$ as in [Che14, Définition 4.1.1].

Definition 5.3 ([Che14, Définition 4.2.1])**.** With the setup as above, let

$$\overline{\mathrm{Fil}}^\bullet_\mu V_E := \left(\bigcap_{x\in \mathrm{Fl}_\mu(E)} \mathrm{Fil}^\bullet_x V_E \right) \tag{5.6}$$

$$\overline{\mathrm{Fil}}^\bullet_{\boldsymbol{\mu}} V := V \cap \left(\bigcap_{x\in \mathrm{Fl}_\mu((E)} \mathrm{Fil}^\bullet_x V_E \right). \tag{5.7}$$

We refer to (5.6) (resp. (5.7)) as the *generic filtration* of V_E (resp. V) attached to μ (resp. $\boldsymbol{\mu}$).

18. Each step of $\overline{\mathrm{Fil}}^i_{\boldsymbol{\mu}} V$ is a subrepresentation of V. Moreover,

$$\overline{\mathrm{Fil}}^\bullet_{\boldsymbol{\mu}} V = V \cap \left(\bigcap_{g\in G(E)} \rho(g)\,\mathrm{Fil}^\bullet_\mu V_E \right). \tag{5.8}$$

This filtration $\overline{\mathrm{Fil}}^i_{\boldsymbol{\mu}} V$ gives rise to a degree function $\overline{\deg}_{\boldsymbol{\mu}} : V \setminus \{0\} \to \mathbb{Z}$ which can be computed as:

$$\overline{\deg}_{\boldsymbol{\mu}}(v) = \inf_{g\in G(E)} \deg_\mu(\rho(g)\cdot v). \tag{5.9}$$

Let K' be an arbitrary extension of K.

Definition 5.4. We say that a map $\mathrm{Spec}(K') \to \mathrm{Fl}_\mu$ is *generic* if, at the level of topological spaces $|\mathrm{Spec}\,(K')| \to |\,\mathrm{Fl}_\mu\,|$, the image of the unique point on the left is the generic point of Fl_μ.

The following statement relates $\overline{\mathrm{Fil}}^\bullet_{\boldsymbol{\mu}} V$ (see (5.7) or (5.8)) to the generic points of Fl_μ in the sense of Definition 5.4.

Proposition 5.5 ([Che14, Lemme 4.2.2])**.** *Let $\mu^\eta : \operatorname{Spec}(K') \to \mathrm{Fl}_\mu$ be generic (in the sense of Definition 5.4). Then for all $i \in \mathbb{Z}$, we have*

$$\overline{\mathrm{Fil}}^i_{\boldsymbol{\mu}} V = V \cap \mathrm{Fil}^i_{\mu^\eta} V_{K'}, \tag{5.10}$$

where the inclusion $V \subseteq V \otimes_K E \subseteq V \otimes_K K'$ is the natural one.

Proof. The following proof is in [Che14, 4.2.2]. We recall the argument for the convenience of the reader. Note that we do not assume G split over K, which is the running assumptions in *loc.cit.*

Let $\widetilde{\mathcal{Y}}_\mu$ be the the universal P_μ-bundle over $\mathrm{Fl}_\mu = G_E/P_\mu$ coming from the natural map to $[*/P_\mu]$. Consider the vector bundle $\mathcal{E} := \widetilde{\mathcal{Y}}_\mu \times_{P_{\mu,\rho}} V$, with a filtration

$$\cdots \supseteq \mathrm{Fil}^0 \mathcal{E} \supseteq \mathrm{Fil}^1 \mathcal{E} \supseteq \cdots \supseteq \mathrm{Fil}^n \mathcal{E} \supseteq \dots$$

of locally free locally direct factors of the form $\widetilde{\mathcal{Y}}_\mu \times_{P_{\mu,\rho}} \mathrm{Fil}^\bullet V$, where $\mathrm{Fil}^\bullet V$ is the natural filtration of V by subrepresentations of P_μ.

We may regard elements $v \in V$ as global sections of $\mathcal{E}$, and we have that

$$v \in \mathrm{Fil}^i_x V \Leftrightarrow v \in \ker\left(\Gamma(\mathrm{Fl}_\mu, \mathcal{E}/\mathrm{Fil}^i \mathcal{E}) \to \Gamma(\operatorname{Spec}\kappa(x), \mathcal{E}/\mathrm{Fil}^i \mathcal{E})\right).$$

The vanishing locus of such an element is a Zariski closed subset, and it contains the generic point if and only if it contains all the E-rational points. Thus $\overline{\mathrm{Fil}}^i_{\boldsymbol{\mu}} V = V \cap \mathrm{Fil}^i_{\mu^\eta} V_{K'}$. □

19. We need a more easily computable description of $\overline{\mathrm{Fil}}^\bullet_{\boldsymbol{\mu}} V$. In [Che14, Proposition 4.3.2], there is such a description assuming that G is split over K. We now prove a generalization in the quasisplit case.

Let Γ_K denote the Galois group of K. We fix K-rational tori $S \subseteq T \subseteq B \subseteq G$ where S is maximally split and T is the centralizer of S. Recall that, by combining the theory of highest weights and Galois theory, one can classify all irreducible representations of a quasisplit group by the Galois orbits $\mathcal{O} \subseteq X_*(T)^+$ of dominant weights. Given $\lambda \in X_*(T)^+$, let $\mathcal{O}_\lambda := \Gamma_K \cdot \lambda$ denote its Galois orbit. We also consider $\mathcal{O}^E_\lambda := \Gamma_E \cdot \lambda$. Given a Γ_K-Galois orbit (resp. Γ_E-Galois orbit) $\mathcal{O} \subseteq X_*(T)^+$ (resp. $\mathcal{O}^E \subseteq X_*(T)^+$), let $V_{\mathcal{O}}$ (resp. $V_{\mathcal{O}^E}$) denote the $\mathcal{O}$-isotypic (resp. $\mathcal{O}^E$-isotypic) direct summand of V (resp. V_E). We have

$$V_{\mathcal{O}} \otimes_K \bar{K} = \bigoplus_{\lambda \in \mathcal{O}} V^\lambda_{\bar{K}}. \tag{5.11}$$

$$V_{\mathcal{O}^E} \otimes_E \bar{K} = \bigoplus_{\lambda \in \mathcal{O}^E} V^\lambda_{\bar{K}}. \tag{5.12}$$

Where $V^\lambda_{\bar{K}}$ is the λ-isotypic part of $V_{\bar{K}}$. Let $\underline{\mathcal{O}} \in (X_*(T)^+_{\mathbb{Q}})^{\Gamma_K}$ (resp. $(X_*(T)^+_{\mathbb{Q}})^{\Gamma_E}$) be given by $\underline{\mathcal{O}} := \frac{1}{|\mathcal{O}|} \sum_{\lambda \in \mathcal{O}} \lambda$. When $\mathcal{O}_\lambda = \Gamma_K \cdot \lambda$, we have

$$\underline{\mathcal{O}_\lambda} = \frac{1}{[\Gamma_K : \Gamma_\lambda]} \sum_{\gamma \in \Gamma_K/\Gamma_\lambda} \gamma(\lambda) \tag{5.13}$$

Analogously, we have $\underline{\mathcal{O}_\lambda^E} = \frac{1}{[\Gamma_E:\Gamma_\lambda]} \sum_{\gamma \in \Gamma_E/\Gamma_\lambda} \gamma(\lambda))$. Let $\mathcal{W}$ denote the absolute Weyl group of G. Let $w_0 \in \mathcal{W}$ be the longest element, which is Γ_K-invariant.

Proposition 5.6. *Let the setup be as above. For any $(V,\rho) \in \mathrm{Rep}_G$, the generic filtration attached to μ is given by the formula:*

$$\overline{\mathrm{Fil}}^k_{\boldsymbol{\mu}} V = \bigoplus_{\substack{\lambda \in X_*(T)^+ \\ \langle \underline{\mathcal{O}^E_{\tau(\lambda)}}, w_0.\mu\rangle \geq k \\ \tau \in \mathrm{Gal}(E/K)}} V_{\mathcal{O}_\lambda} \tag{5.14}$$

Proof. Since $\overline{\mathrm{Fil}}^k_{\boldsymbol{\mu}} V$ consists of subrepresentations, it suffices to show that

$$V_{\mathcal{O}_\lambda} \subseteq \overline{\mathrm{Fil}}^k_{\boldsymbol{\mu}} V \iff k \leq \langle \underline{\mathcal{O}^E_{\tau(\lambda)}}, w_0.\mu\rangle \quad \forall \tau \in \mathrm{Gal}(E/K). \tag{5.15}$$

Let us first prove "$\Longrightarrow$". Let $V = \bigoplus_{\sigma \in \mathrm{Irrep}(T)} V_\sigma$ be the decomposition of $\rho|_T$. Over an algebraic closure, each V_σ decomposes as $V_\sigma = \bigoplus_{\chi' \in \mathcal{O}_\chi} V_{\chi'}$ for some $\chi \in X^*(T)$.

Observe that $\overline{\mathrm{Fil}}^k_{\boldsymbol{\mu}} V \subseteq \mathrm{Fil}^k_{\tau(\mu)} V_E$ for all $\tau \in \mathrm{Gal}(E/K)$, and by definition we have

$$\mathrm{Fil}^k_{\tau(\mu)} V_{\bar{K}} = \bigoplus_{\substack{\langle \chi, \tau(\mu)\rangle \geq k, \\ \chi \in X^*(T)}} V_{\bar{K},\chi}. \tag{5.16}$$

In particular, the anti-dominant weights appearing in $V_{\mathcal{O}_\lambda}$ pair with $\tau(\mu)$ to a number greater than or equal to k. In other words, $k \leq \langle w_0.\xi, \tau(\mu)\rangle$ for $w_0.\xi \in \mathcal{O}_{w_0.\lambda}$, but then pairing $\tau(\mu)$ with their Γ_E-average $w_0.\underline{\mathcal{O}^E_\lambda}$ will still be greater than or equal to k, i.e. $k \leq \langle w_0.\underline{\mathcal{O}^E_\lambda}, \tau(\mu)\rangle = \langle \underline{\mathcal{O}^E_{\tau(\lambda)}}, w_0.\mu\rangle$.

Thus

$$\overline{\mathrm{Fil}}^k_{\boldsymbol{\mu}} V \subset \bigoplus_{\substack{\lambda \in X^*(T)^+ \\ \langle \underline{\mathcal{O}^E_{\tau(\lambda)}}, w_0.\mu\rangle \geq k \\ \tau \in \mathrm{Gal}(E/K)}} V_{\mathcal{O}_\lambda}. \tag{5.17}$$

Let us now prove "$\Longleftarrow$". Suppose $k \leq \langle w_0.\overline{\mathcal{O}^E_\lambda}, \tau(\mu)\rangle$ for all $\tau \in \mathrm{Gal}(E/K)$, this implies that for at least one $w_0.\xi \in \mathcal{O}^E_{w_0.\lambda}$, we have $k \leq \langle w_0.\xi, \tau(\mu)\rangle$. Since $\langle \cdot, \cdot\rangle$ is Γ_E-equivariant, $k \leq \langle w_0.\xi, \tau(\mu)\rangle$ for all $w_0.\xi \in \mathcal{O}^E_{w_0.\lambda}$. We can view $w_0.\xi$ as a cocharacter of T and $V_{w_0.\xi} \subseteq \mathrm{Fil}^k_{\tau(\mu)} V$. Consider $W_\xi := V^\xi_{\bar{K}}$ the isotypic part of $V_{\bar{K}}$ associated to the highest weight representation of ξ on an algebraic closure of K. If χ is a weight appearing in W_ξ, then

$k \leq \langle w_0.\xi, \tau(\mu)\rangle \leq \langle \chi, \tau(\mu)\rangle$, and thus $W_\xi \subseteq \mathrm{Fil}^k_{\tau(\mu)} V_{\bar{K}}$. In particular,

$$W_E := \Bigg(\bigoplus_{w_0.\xi \in \mathcal{O}^E_{w_0\lambda}} W_\xi \Bigg)^{\Gamma_E} \tag{5.18}$$

is a subrepresentation of $\mathrm{Fil}^k_{\tau(\mu)} V_E$ defined over E. Thus $W_E \subseteq \overline{\mathrm{Fil}}^k_{\tau(\mu)} V_E$ for all $\tau \in \mathrm{Gal}(E/K)$. Then $W := \bigoplus_{\tau \in \mathrm{Gal}(E/K)} \tau(W_E)$ is contained in

$$\bigcap_{\tau \in \mathrm{Gal}(E/K)} \overline{\mathrm{Fil}}^k_{\tau(\mu)} V_E, \tag{5.19}$$

and the $\mathrm{Gal}(E/K)$-fixed points of W are contained in

$$\overline{\mathrm{Fil}}^k_{\mu} V = V \cap \bigcap_{\tau \in \mathrm{Gal}(E/K)} \overline{\mathrm{Fil}}^k_{\tau(\mu)} V_E.$$

But $W^{\mathrm{Gal}(E/K)} = V_{\mathcal{O}_\lambda}$, proving the claim. □

5.3. **Mumford–Tate group computations.** The goal of this section is to prove Theorem 5.7 (or Theorem 1.15 in the introduction).

Let G be a reductive group over $\mathbb{Q}_p$. Let K be a finite extension of $\breve{\mathbb{Q}}_p$. Let $b \in G(\breve{\mathbb{Q}}_p)$ be decent (Definition 2.2) and $\mu^\eta : \mathbb{G}_m \to G_K$ be generic (Definition 5.4) with $\mu^\eta \in \boldsymbol{\mu}$. As before, let $\mu \in X_*(T)^+$ be the unique B-dominant cocharacter of $\boldsymbol{\mu}$.

Theorem 5.7. *Suppose that b is decent, μ^η is generic and $\mathbf{b} \in B(G, \boldsymbol{\mu})$. The following hold:*

1. *(b, μ^η) is admissible.*
2. *If $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, then $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}.*

Proof. We fix s large enough so that b is s-decent, G is quasisplit over $\mathbb{Q}_{p^s}$ and splits over a totally ramified extension of $\mathbb{Q}_{p^s}$ that we denote by L. Recall that replacing b by $g^{-1}b\varphi(g)$ and μ^η by $g^{-1}\mu^\eta g$ gives isomorphic fiber functors $\mathcal{G}_{(b,\mu^\eta)}$ (see (5.1)). Moreover, via this kind of replacement, we can arrange that $\nu_b = \boldsymbol{\nu}_{\mathbf{b}}$ as in 4. Note that this replacement preserves genericity of μ^η.

(1) The argument in [Che14, Théorème 5.0.6.(1)] goes through in our setting. Indeed, the only part in the proof *loc.cit.* using that G is unramified is to justify that $\mathrm{Fl}^{\mathrm{ad}}_\mu \neq \emptyset$ whenever $\mathbf{b} \in B(G, \boldsymbol{\mu})$, but this is true by [DOR10, Theorem 9.5.10] in full generality.

(2) Let $(V, \rho) \in \mathrm{Rep}_G$ and let $v \in V^{s,k}_{(b,\mu^\eta)}$ as in (5.5). By 16, it suffices to show that $\rho(g)v = v$ for all $g \in G^{\mathrm{der}}_{\mathbb{Q}_{p^s}}$. Over L, we can write $v = \sum_{\lambda \in \Lambda_v} v_\lambda$ where $\Lambda_v \subseteq X^*(T)^+$, $v_\lambda \in V^\lambda$ and $v_\lambda \neq 0$. Since v is defined over $\mathbb{Q}_{p^s}$, we have $\gamma(v_\lambda) = v_{\gamma(\lambda)}$ for $\gamma \in \Gamma_{\mathbb{Q}_{p^s}}$.

Given $\mathcal{O} \in \mathrm{Irrep}_{G_{\mathbb{Q}_{p^s}}}$, let $v_{\mathcal{O}} = \sum_{\lambda \in \mathcal{O} \subseteq \Lambda_v} v_\lambda$. We have $v_{\mathcal{O}} \in V_{\mathbb{Q}_{p^s}}$. By Proposition 5.6 and Proposition 5.5, we can write

$$\begin{aligned}
\deg_{\mu^\eta}(v) &= \overline{\deg}_{\boldsymbol{\mu}}(v) && (5.20)\\
&= \inf_{\lambda \in \Lambda_v} \overline{\deg}_{\boldsymbol{\mu}}(v_{\mathcal{O}_\lambda}) && (5.21)\\
&\leq \overline{\deg}_{\boldsymbol{\mu}}(v_{\mathcal{O}_\lambda}) && (5.22)\\
&= \inf_{\tau \in \mathrm{Gal}(E/K)} \langle \underline{\mathcal{O}^E_{\tau(\lambda)}}, w_0 \cdot \mu \rangle && (5.23)\\
&\leq \langle \underline{\mathcal{O}_\lambda}, w_0 \cdot \mu \rangle && (5.24)\\
&= \langle w_0 \cdot \lambda, \underline{\mu} \rangle, && (5.25)
\end{aligned}$$

Here (5.20) follows from Proposition 5.5. Since each step of $\overline{\mathrm{Fil}}^{\bullet}_{\boldsymbol{\mu}} V$ is a subrepresentation of V, in order for $v \in \overline{\mathrm{Fil}}^{k}_{\boldsymbol{\mu}} V$, each $v_{\mathcal{O}_\lambda}$ has to be in $\overline{\mathrm{Fil}}^{\bullet}_{\boldsymbol{\mu}} V$, and hence (5.21). Inequality (5.22) follows from the definition of infimum. (5.23) follows from Proposition 5.6, and the fact that

$$v_{\mathcal{O}_\lambda} = \sum_{\tau \in \mathrm{Gal}(E/K)} v_{\mathcal{O}^E_{\tau(\lambda)}}. \tag{5.26}$$

Since the average is smaller than the infimum, (5.24) follows. Finally, (5.25) follows from equivariance of the pairing $\langle \cdot, \cdot \rangle$ with respect to the Γ_K-action, and invariance of the pairing under the w_0-action.

Write $v^i = (\rho(b)\varphi)^i v$. Therefore, we have the following formula

$$\begin{aligned}
d^s_{\rho,\mu^\eta}(v) &= \sum_{i=0}^{s-1} \deg_{\mu^\eta}((\rho(b)\varphi)^i v) && (5.27)\\
&= \sum_{i=0}^{s-1} \inf_{\lambda \in \Lambda_{v^i}} \overline{\deg}_{\boldsymbol{\mu}}(v^i_{\mathcal{O}_\lambda}) && (5.28)\\
&\leq \sum_{i=0}^{s-1} \langle \underline{\mathcal{O}_{\varphi^i(\lambda)^{\mathrm{dom}}}}, w_0 \cdot \mu \rangle && (5.29)\\
&= \sum_{i=0}^{s-1} \langle w_0 \cdot \varphi^i(\lambda)^{\mathrm{dom}}, \underline{\mu} \rangle && (5.30)\\
&= \sum_{i=0}^{s-1} \langle w_0 \cdot \varphi_0^i(\lambda), \underline{\mu} \rangle && (5.31)\\
&= \sum_{i=0}^{s-1} \langle w_0 \cdot \lambda, \varphi_0^i(\underline{\mu}) \rangle && (5.32)\\
&= s \cdot \langle w_0 \cdot \lambda, \mu^\diamond \rangle && (5.33)
\end{aligned}$$

Equality (5.27) follows from the definition in (5.4). By Proposition 5.6, we obtain (5.28). Inequality (5.29) follows from the inequalites (5.20) through

(5.25) above. Since $\lambda \in \Lambda_v$, we have $\varphi^i(\lambda)^{\mathrm{dom}} \in \Lambda_{v^i}$. Equality (5.30) follows from equivariance of $\langle , \rangle$ under the Galois action and w_0-action. Equality (5.31) follows from the definition of φ_0 in (1). Since T is φ_0-stable, (5.32) follows from equivariance of $\langle , \rangle$ under the φ_0-action. Equality (5.33) follows from the definition of $\mu^\diamond$ (see (2.7)).

Since $v \in V^{s,k}_{(b,\mu^\eta)}$, by (5.27) through (5.33), we have $\frac{k}{s} \leq \langle w_0 \cdot \lambda, \mu^\diamond \rangle$ for all $\lambda \in \Lambda_v$. On the other hand, over L, we have a decomposition $v = \sum_{\chi \in X^*(T)} v_\chi$. Since we have arranged that $\nu_b = \boldsymbol{\nu}_{\mathbf{b}}$, by (5.3) and (5.5) we have

$$T_\rho^{s\cdot\nu_b}(v) = \sum_{\chi \in X^*(T)} T_\rho^{s\cdot\nu_b}(v_\chi) = \sum_{\chi \in X^*(T)} p^{\langle \chi, s\cdot\boldsymbol{\nu}_{\mathbf{b}}\rangle} v_\chi. \tag{5.34}$$

The assumption $v \in V^{s,k}_{(b,\mu^\eta)}$ forces χ to satisfy $\langle \chi, s \cdot \boldsymbol{\nu}_{\mathbf{b}} \rangle = k$ for all χ where $v_\chi \neq 0$. In particular, since $w_0 \cdot \lambda \leq \chi$ when $V_L^\chi \subseteq V_L^\lambda$, we have $\langle w_0 \cdot \lambda, \boldsymbol{\nu}_{\mathbf{b}} \rangle \leq \frac{k}{s}$ for all $\lambda \in \Lambda_v$. Therefore $\langle w_0 \cdot \lambda, \mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} \rangle \leq 0$. Since $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, we have $\langle w_0 \cdot \lambda, \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta$. Therefore, the action of G_L^{der} on V^λ is trivial for all $\lambda \in \Lambda_v$. Thus we are done with the proof of (2) in Theorem 1.15. □

Proposition 5.8. *Let (G, b, μ) be a local shtuka datum over $\mathbb{Q}_p$ with $(\mathbf{b}, \boldsymbol{\mu})$ HN-irreducible. There exists a finite extension K over $\breve{\mathbb{Q}}_p$ containing the reflex field of $\boldsymbol{\mu}$, and a point $x \in \mathrm{Gr}^b_\mu(K)$ whose induced (conjugacy class of) crystalline representation(s)*

$$\rho_x : \Gamma_K \to G(\mathbb{Q}_p)$$

satisfies that $\rho_x(\Gamma_K) \cap G^{\mathrm{der}}(\mathbb{Q}_p)$ is open in $G^{\mathrm{der}}(\mathbb{Q}_p)$.

Proof. Recall from [Gle22a, Proposition 2.12] (see also [Vie21, Theorem 5.2]) that the Bialynicki-Birula map BB in (3.16) induces a bijection of classical points. Therefore it suffices to construct the image $\mathrm{BB}(x) \in \mathrm{Fl}_\mu$, which corresponds to constructing a weakly admissible filtered isocrystal with G-structure.

By Lemma 5.9, we can take $\mathrm{BB}(x) = \mu^\eta$ to be generic (Definition 5.4). By Theorem 5.7(2), $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}. By Theorem 5.2, the image of the generic crystalline representation $\xi_{(b,\mu^\eta)}$ contains an open subgroup of $\mathrm{MT}_{(b,\mu^\eta)}$, thus containing an open subgroup of G^{der}. □

Lemma 5.9. *There exist a finite extension K over $\breve{\mathbb{Q}}_p$ and a map μ^η : $\mathrm{Spec}(K) \to \mathrm{Fl}_\mu$ such that $|\mu^\eta| : \{*\} \to |\,\mathrm{Fl}_\mu\,|$ maps to the generic point.*

Proof. Recall from [Che14, Proposition 2.0.3] that the transcendence degree of $\breve{\mathbb{Q}}_p$ over $\mathbb{Q}_p$ is infinite. By the structure theorem of smooth morphisms [Sta18, Tag 054L], one can find an open neighborhood $U \to \mathrm{Fl}_\mu$ that is étale over $\mathbb{A}^n_{\breve{\mathbb{Q}}_p}$. On the other hand, one can always find a map $\mathrm{Spec}(\breve{\mathbb{Q}}_p) \to \mathbb{A}^n_{\breve{\mathbb{Q}}_p}$ mapping to the generic point by choosing n trascendentally independent elements of $\breve{\mathbb{Q}}_p$. Its pullback to U is an étale neighborhood of $\mathrm{Spec}\,(\breve{\mathbb{Q}}_p)$ that consists of a finite disjoint union of finite extensions K of $\breve{\mathbb{Q}}_p$. Any of these components will give a map to the generic point of Fl_μ. □

The following is a partial converse to Theorem 1.15, and it follows directly from Theorem 6.1.

Proposition 5.10 (Proposition 5.10)**.** *Assume that G^{ad} has only isotropic factors. If $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}, then $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*

Proof. If $G^{\mathrm{der}} \subseteq \mathrm{MT}_{(b,\mu^\eta)}$, then by Theorem 5.2 there exists a finite field extension K over $\breve{\mathbb{Q}}_p$, and a crystalline representation $\xi : \Gamma_K \to G(\mathbb{Q}_p)$ with invariants $(\mathbf{b}, \boldsymbol{\mu})$ whose image in $G^{\mathrm{der}}(\mathbb{Q}_p)$ is open. Indeed, we can let K be generic as in Definition 5.4. The result follows from the equivalence (3) $\iff$ (4) in Theorem 6.1. □

6. Proof of main theorems

The first goal in this section is to prove the following main theorem:

Theorem 6.1. *Suppose that $\mathbf{b} \in B(G, \boldsymbol{\mu})$ and that $G^{\mathrm{ad}} \neq \{e\}$ does not have anisotropic factors. The following statements are equivalent:*

(1) *The map $\omega_G : \pi_0(X_\mu(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ is bijective.*
(2) *The map $\omega_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ is bijective.*
(3) *The pair $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*
(4) *There exists a finite field extension K over $\breve{\mathbb{Q}}_p$, and a crystalline representation $\xi : \Gamma_K \to G(\mathbb{Q}_p)$ with invariants $(\mathbf{b}, \boldsymbol{\mu})$ whose image in $G^{\mathrm{der}}(\mathbb{Q}_p)$ is open.*
(5) *The action of $G(\mathbb{Q}_p)$ on $\mathrm{Sht}_{(G,b,\mu,\infty)}$ makes $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ into a G°-torsor.*

The second goal in this section is to prove the following corollary of Theorem 6.1.

Theorem 6.2. *Let G be arbitrary. Suppose that $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, then the Kottwitz map $\omega_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ is bijective.*

The proof of the above main theorems proceeds as follows and will occupy the rest of section 6. We first prove a modified version of the statement in the case of tori (see §6.1, Proposition 6.4, Lemma 6.5). We then use z-extensions and ad-isomorphisms to reduce the proof of Theorem 6.1 and Theorem 6.2 to the case where $G^{\mathrm{der}} = G^{\mathrm{sc}}$ (see Proposition 6.7). We prove the circle of implications of Theorem 6.1 in this case. Then, we deduce Theorem 6.2 from Theorem 6.1 whenever G has no anisotropic factors. Finally, we deduce Theorem 6.2 in the anisotropic case.

Before we dive into the proofs of Theorem 1.2, we deduce Corollary 6.3 below. Let $(p, \mathbf{G}, X, \mathbf{K})$ be a tuple of global Hodge type [PR21, §1.3], let $\mathscr{S}_{\mathbf{K}}$ denote the integral model of [PR21, Theorem 1.3.2], let k an algebraically closed field in characteristic p and let $x_0 \in \mathscr{S}_{\mathbf{K}}(k)$. Pappas and Rapoport consider a map of v-sheaves $c : \mathrm{RZ}^{\diamondsuit}_{\mathcal{G},\mu,x_0} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ [PR21, Lemma 4.1.0.2], where the source is a Rapoport–Zink space and the target is another name for $\mathrm{Sht}^{\mathcal{G}}_{\mu}(b)$ i.e. $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu} = \mathrm{Sht}^{\mathcal{G}}_{\mu}(b)$. Let the notations be as in [PR21, Theorem 4.10.6, §4.10.2].

Corollary 6.3. *The map $c : \mathrm{RZ}^{\diamondsuit}_{\mathcal{G},\mu,x_0} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is an isomorphism. Thus, $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is representable by a formal scheme $\mathscr{M}_{\mathcal{G},b,\mu}$, and we obtain a p-adic uniformization isomorphism of $O_{\breve{E}}$-formal schemes*

$$I_x(\mathbb{Q})\backslash(\mathscr{M}_{\mathcal{G},b,\mu} \times \mathbf{G}(\mathbb{A}^p_f)/\mathbf{K}^p) \to (\mathscr{S}_{\mathbf{K}} \widehat{\otimes_{O_E}} O_{\breve{E}})_{/\mathcal{I}(x)}. \tag{6.1}$$

Proof. It suffices to verify condition (U_x) in [PR21, §4.10.2]. Throughout the argument we let the notation be as in [PR21, §4.8]. By [PR21, Proposition 4.10.3 and Lemma 4.10.2.b)], $c : \mathrm{RZ}^{\diamondsuit}_{\mathcal{G},\mu,x_0} \subseteq \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is an open and closed immersion, and it is enough to justify that for every $x \in \pi_0(\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu})$ there is $y \in \pi_0(\mathrm{RZ}^{\diamondsuit}_{\mathcal{G},\mu,x_0})$ with $c(y) = x$. Let $\tilde{x}_0 \in \mathscr{S}_{\mathbf{K}}(\breve{E}_{\tilde{x}_0})$ be a $\breve{E}_{\tilde{x}_0}$-valued point of $\mathscr{S}_{\mathbf{K}}$ specializing to x_0 with $[\breve{E}_{\tilde{x}_0} : \breve{E}] < \infty$. Such a point exists by flatness of $\mathscr{S}_{\mathbf{K}}$ over $\mathbb{Z}_{(p)}$. By Serre–Tate, $\tilde{x}_0$ induces a canonical point in $\tilde{y}_0 \in \mathrm{RZ}_{\mathcal{H},\iota(x_0)}(\breve{E}_{\tilde{x}_0})$, which overall induces a point $\tilde{z}_0 \in \mathrm{RZ}^{\diamondsuit}_{\mathcal{G},\mu,x_0}(\breve{E}_{\tilde{x}_0})$. Recall that to any element of $G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p)$ we may attach a point in $g \cdot \tilde{x}_0 \in \mathscr{S}_{\mathbf{K}}(\mathbb{C}_p)$ by acting through at-p G-isogenies. Analogously, for every $h \in \mathcal{H}(\mathbb{Q}_p)/\mathcal{H}(\mathbb{Z}_p)$ we get an element $h \cdot \tilde{y}_0 \in \mathrm{RZ}_{\mathcal{H},\iota(x_0)}(\mathbb{C}_p)$, and we get a commutative diagram:

$$\begin{tikzcd}
G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \arrow[r, "g\cdot\tilde{z}_0"] \arrow[d] \arrow[rr, bend left, "g\cdot\tilde{x}_0"] & \mathrm{RZ}_{\mathcal{G},\mu,x_0}(\mathbb{C}_p) \arrow[r] \arrow[d] & \mathscr{S}_{\mathbf{K}}(\mathbb{C}_p) \arrow[d] \\
H(\mathbb{Q}_p)/\mathcal{H}(\mathbb{Z}_p) \arrow[r, "h\cdot\tilde{y}_0"] & \mathrm{RZ}_{\mathcal{H},\iota(x_0)}(\mathbb{C}_p) \arrow[r] & \mathcal{A}_{K^\flat}(\mathbb{C}_p)
\end{tikzcd}$$

Moreover, we get a further compatibility

$$\begin{tikzcd}
G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \arrow[r, "\mathrm{GM}_{\tilde{z}_0}"] \arrow[d, "g\cdot\tilde{z}_0"] & \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu} \\
\mathrm{RZ}^{\diamondsuit}_{\mathcal{G},\mu,x_0} \arrow[ur, "c"] &
\end{tikzcd}$$

where $\mathrm{GM}_{\tilde{z}_0}$ is the map $G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}(\mathbb{C}_p)$ induced from choosing an identification of the fibers of the Grothendieck–Messing period morphism of section 3.5 $G(\mathbb{Q}_p) = \pi^{-1}_{GM}(\pi_{GM}(\tilde{z}_0))$.

It suffices to prove that $\mathrm{GM}_{\tilde{z}_0} : G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \to \pi_0(\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu})$ is surjective, but this follows directly from Theorem 3.9, Proposition 3.10 and Proposition 3.7. Indeed, since $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu} = \mathrm{Sht}^{\mathcal{G}}_{\mu}(b)$ is a prekimberlite and by Proposition 3.7, $\pi_0(\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}) = \pi_0((\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu})^{\mathrm{red}}) = \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{G}(\mathbb{Z}_p))})$. Moreover, we get a commutative diagram:

$$\begin{tikzcd} G(\mathbb{Q}_p) \arrow[r, "\pi_{GM}^{-1}(\pi_{GM}(\tilde{z}_0))"] \arrow[d] & \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \arrow[d] \\ G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \arrow[r, "\mathrm{GM}_{\tilde{z}_0}"] & \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{G}(\mathbb{Z}_p)} \times \mathrm{Spd}\,\mathbb{C}_p)) \end{tikzcd}$$

and the top arrow is surjective by Proposition 3.10. □

6.1. **The tori case.** When $G = T$ is a torus, there is only one parahoric model that we denote by $\mathcal{T}$. The tori analogue of Theorem 6.1 is as follows.

Proposition 6.4. *Suppose that* $\mathbf{b} \in B(T, \boldsymbol{\mu})$. *The following hold:*

(1) *The map* $\omega_T : \pi_0(X^{\mathcal{T}}_{\mu}(b)) \to c_{b,\mu}\pi_1(T)^{\varphi}_I$ *is bijective.*
(2) *The action of* $T(\mathbb{Q}_p)$ *on* $\mathrm{Sht}_{(T,b,\mu,\infty)}$ *makes* $\pi_0(\mathrm{Sht}_{(T,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ *into a* T°*-torsor.*

In this case, both $X^{\mathcal{T}}_{\mu}(b)$ and $\mathrm{Sht}_{(T,b,\mu,\mathcal{T})} \times \mathrm{Spd}\,\mathbb{C}_p$ are zero-dimensional. Since we are working over algebraically closed fields, they are of the form $\coprod_J \mathrm{Spec}\,\bar{\mathbb{F}}_p$ and $\coprod_I \mathrm{Spd}\,\mathbb{C}_p$ for some index sets I and J, respectively. Moreover, by Proposition 3.7, the specialization map (1.8) induces a bijection $\pi_0(\mathrm{sp}) : I \cong J$. Also, $T^\circ = T(\mathbb{Q}_p)$ and $\mathrm{Sht}_{(T,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p$ is a $\underline{T(\mathbb{Q}_p)}$-torsor over $\mathrm{Spd}\,\mathbb{C}_p$ (see for example [Gle22a, Theorem 1.24]). In particular, $\pi_0(\mathrm{Sht}_{(T,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ is a $T(\mathbb{Q}_p)$-torsor and Proposition 6.4 (5) holds. The content of Proposition 6.4 (1) becomes the following lemma.

Lemma 6.5. *Let T be a torus. We have a $T(\mathbb{Q}_p)$-equivariant commutative diagram, where the horizontal arrows are isomorphisms:*

$$\begin{tikzcd} \pi_0(\mathrm{Sht}_{(T,b,\mu,\mathcal{T})} \times \mathrm{Spd}\,\mathbb{C}_p) \arrow[r, "\cong"] & \pi_0(X^{\mathcal{T}}_{\mu}(b)) \arrow[r, "\cong"] \arrow[d] & c_{b,\mu}\pi_1(T)^{\varphi}_I \arrow[d] \\ & \pi_0(\mathcal{F}\ell_{\mathcal{T}}) \arrow[r, "\cong"] & \pi_1(T)_I \end{tikzcd}$$

Proof. Upon fixing an element of $\mathrm{Sht}_{(T,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p$, we can identify $\pi_0(\mathrm{Sht}_{(T,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \cong T(\mathbb{Q}_p)$ (see for example [Gle22a, Theorem 1.24]), which then gives an identification $\mathrm{Sht}_{(T,b,\mu,\mathcal{T})} \times \mathrm{Spd}\,\mathbb{C}_p \cong T(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p) \cong T(\breve{\mathbb{Q}}_p)^{\varphi=\mathrm{id}}/\mathcal{T}(\breve{\mathbb{Z}}_p)^{\varphi=\mathrm{id}}$. Since $H^1_{\acute{e}t}(\mathrm{Spec}\,\mathbb{Z}_p, \mathcal{T})$ vanishes, we can write

$$T(\breve{\mathbb{Q}}_p)^{\varphi=\mathrm{id}}/\mathcal{T}(\breve{\mathbb{Z}}_p)^{\varphi=\mathrm{id}} \cong (T(\breve{\mathbb{Q}}_p)/\mathcal{T}(\breve{\mathbb{Z}}_p))^{\varphi=\mathrm{id}}, \tag{6.2}$$

where the right-hand side is $X_*(T)_I^{\varphi} = \pi_1(T)_I^{\varphi}$. Therefore the $T(\mathbb{Q}_p)$-action makes $\pi_0(X_\mu^{\mathcal{T}}(b))$ and $\pi_0(\mathrm{Sht}_{(T,b,\mu,\mathcal{T})} \times \mathrm{Spd}\,\mathbb{C}_p)$ into $\pi_1(T)_I^{\varphi}$-torsors (via the specialization map (1.8)). Thus by equivariance of $\pi_1(T)_I^{\varphi}$-action, $\pi_0(\mathrm{Sht}_{(T,b,\mu,\mathcal{T})} \times \mathrm{Spd}\,\mathbb{C}_p)$ and $\pi_0(X_\mu^{\mathcal{T}}(b))$ can be identified with a unique coset $c_{b,\mu}\pi_1(T)_I^{\varphi} \subseteq \pi_1(T)_I$ (by the definition of $c_{b,\mu}$). □

6.2. **Reduction to the $G^{\mathrm{der}} = G^{\mathrm{sc}}$ case.** For the rest of this subsection, assume that f is an ad-isomorphism. Let $b_H := f(b)$ and $\mu_H := f \circ \mu$. Let $\mathcal{K}_p^H$ denote the unique parahoric of H that corresponds to the same point in the Bruhat–Tits building as $\mathcal{K}_p$.

Proposition 6.6. *(1) We have a canonical identification of diamonds*

$$\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \cong \mathrm{Sht}_{(G,b,\mu,\infty)} \times^{\underline{G(\mathbb{Q}_p)}} \underline{H(\mathbb{Q}_p)}. \tag{6.3}$$

(2) In particular, if $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ is a G°-torsor, then

$$\pi_0(\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \tag{6.4}$$

is a H°-torsor.

Proof. (1) A version of (6.3) was proven in [Gle21, Proposition 4.15], where the result is phrased in terms of the torsor $\mathbb{L}_b$ from §3.5. [20] We sketch the proof for the reader's convenience:

Step 1. $\mathrm{Gr}_\mu = \mathrm{Gr}_{\mu_H}$: there is an obvious proper map $\mathrm{Gr}_\mu \to \mathrm{Gr}_{\mu_H}$ of spatial diamonds. Therefore, to prove that it is an isomorphism, it suffices to prove bijectivity on points, which can be done as in the classical Grassmannian case (see [AGLR22, Proposition 4.16] for a stronger statement).

Step 2. $\mathrm{Gr}_\mu^b = \mathrm{Gr}_{\mu_H}^{b_H}$: the b-admissible and b_H-admissible loci are open subsets of $\mathrm{Gr}_\mu = \mathrm{Gr}_{\mu_H}$. To prove that they agree, we can prove it on geometric points. This ultimately boils down to the fact that an element $e \in B(G)$ is basic if and only if $f(e) \in B(H)$ is basic, which holds because centrality of the Newton point ν_e can be checked after applying an ad-isomorphism.

Step 3. $\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \cong \mathrm{Sht}_{(G,b,\mu,\infty)} \times^{\underline{G(\mathbb{Q}_p)}} \underline{H(\mathbb{Q}_p)}$: recall from §3.5 that the Grothendieck–Messing period map (3.15) realizes $\mathrm{Sht}_{(G,b,\mu,\infty)}$ (respectively $\mathrm{Sht}_{(H,b_H,\mu_H,\infty)}$) as a $\underline{G(\mathbb{Q}_p)}$-torsor (respectively an $\underline{H(\mathbb{Q}_p)}$-torsor) over $\mathrm{Gr}_\mu^b = \mathrm{Gr}_{\mu_H}^{b_H}$. Since the $\underline{G(\mathbb{Q}_p)}$-equivariant map $\mathrm{Sht}_{(G,b,\mu,\infty)} \to \mathrm{Sht}_{(H,b_H,\mu_H,\infty)}$ extends to a map of $\underline{H(\mathbb{Q}_p)}$-torsors

$$\mathrm{Sht}_{(G,b,\mu,\infty)} \times^{\underline{G(\mathbb{Q}_p)}} \underline{H(\mathbb{Q}_p)} \to \mathrm{Sht}_{(H,b_H,\mu_H,\infty)}, \tag{6.5}$$

and any map of torsors is an isomorphism, the conclusion follows.

[20]Although [Gle21, Proposition 4.15] only considers unramified groups G (since this was the ongoing assumption in *loc.cit.*), the proof goes through without this assumption.

A more detailed proof of Proposition 6.6 (1) can also be found in [PR22, Proposition 5.2.1], which was obtained independently as *loc.cit.*

(2) Recall that since $G \to H$ is an ad-isomorphism, we have an isomorphism $G^{\mathrm{sc}} \to H^{\mathrm{sc}}$. Recall $G^\circ := G(\mathbb{Q}_p)/\operatorname{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$ and $H^\circ := H(\mathbb{Q}_p)/\operatorname{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$. By (6.3), we have a canonical isomorphism

$$\pi_0(\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \times \operatorname{Spd}\mathbb{C}_p) \cong \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p)\times^{G(\mathbb{Q}_p)} H(\mathbb{Q}_p). \tag{6.6}$$

The right-hand side of (6.6) is by definition

$$\left(\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p)\times H(\mathbb{Q}_p)\right)/G(\mathbb{Q}_p), \tag{6.7}$$

where the quotient is via the diagonal action. Since $G^{\mathrm{sc}}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p)$, quotienting (6.7) by $G^{\mathrm{sc}}(\mathbb{Q}_p)$ first gives

$$\begin{aligned} &\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p)\times^{G(\mathbb{Q}_p)} H(\mathbb{Q}_p) && (6.8)\\ &\cong \left(\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p)\times (H(\mathbb{Q}_p)/\operatorname{Im} G^{\mathrm{sc}}(\mathbb{Q}_p))\right)/G^\circ, && (6.9)\end{aligned}$$

which simplifies, via (6.6) and since $G^{\mathrm{sc}}(\mathbb{Q}_p) = H^{\mathrm{sc}}(\mathbb{Q}_p)$, to

$$\pi_0(\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \times \operatorname{Spd}\mathbb{C}_p) = \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p) \times^{G^\circ} H^\circ. \tag{6.10}$$

The right-hand side of (6.10) is clearly an H°-torsor. □

Proposition 6.7. *If Theorem 6.1 holds for $G^{\mathrm{der}} = G^{\mathrm{sc}}$, then it holds in general as well.*

Proof. For each item $i \in \{1, \dots, 5\}$, we show that if (i) holds for $G^{\mathrm{der}} = G^{\mathrm{sc}}$, then (i) also holds for general G. Consider an arbitrary z-extension $\tilde{G} \to G$ (see Definition 3.12). By definition of z-extensions, $\tilde{G}^{\mathrm{der}} = \tilde{G}^{\mathrm{sc}}$. By Lemma 3.13 (1), we may choose a conjugacy class of cocharacters $\tilde{\boldsymbol{\mu}}$ and an element $\tilde{\mathbf{b}} \in B(\tilde{G}, \tilde{\boldsymbol{\mu}})$ that map to $\boldsymbol{\mu}$ and $\mathbf{b}$, respectively, under the map $B(\tilde{G}, \tilde{\boldsymbol{\mu}}) \to B(G, \boldsymbol{\mu})$.

We first justify (1) and (2) of Theorem 6.1. Recall that by Lemma 3.13 (2), $c_{\tilde{b},\tilde{\mu}}\pi_1(\tilde{G})_I^\varphi \to c_{b,\mu}\pi_1(G)_I^\varphi$ is surjective. We apply Proposition 3.14 to the ad-isomorphism $\tilde{G} \to G$. Since the top horizontal arrow in (3.24) is a bijection (of sets), the bottom horizontal arrow in (3.24) is also a bijection of sets, as it is the pullback of the top horizontal arrow under a surjective map. Now, (3) of Theorem 6.1 is a direct consequence of Proposition 4.8.

For (4) recall that the map $\tilde{G}^{\mathrm{der}} \to G^{\mathrm{der}}$ is surjective with finite kernel. In particular, it is an open map. Finally for (5) we use Proposition 6.6 (2). □

6.3. **Argument for** (1) $\implies$ (2)**.**

We start by giving a new proof to [He18, Theorem 7.1].

Theorem 6.8 (He)**.** *The map $X_\mu^{\mathcal{I}}(b) \to X_\mu^{\mathcal{K}_p}(b)$ is surjective.*

Proof. By functoriality of the specialization map [Gle22b, Proposition 4.14] applied to $\mathrm{Sht}_\mu^{\mathcal{I}}(b) \to \mathrm{Sht}_\mu^{\mathcal{K}}(b)$ from (3.14), we get a commutative diagram:

$$\begin{tikzcd} \mid \mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \mid \arrow[r] \arrow[d, "\mathrm{sp}"'] & \mid \mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)} \mid \arrow[d, "\mathrm{sp}"] \\ \mid X_\mu^{\mathcal{I}}(b) \mid \arrow[r] & \mid X_\mu^{\mathcal{K}_p}(b) \mid \end{tikzcd}$$

The top arrow is given by (3.8). By [SW20, Proposition 23.3.1], it is a $\mathcal{K}_p/\mathcal{I}(\mathbb{Z}_p)$-torsor and thus surjective. It then suffices to prove that the specialization map is surjective, which follows directly from [Gle22a, Theorem 2 b)]. ☐

Now, Theorem 6.8 implies the (1) $\implies$ (2) part of Theorem 6.1: by Lemma 3.2, we have the following commutative diagram:

$$\begin{tikzcd} \pi_0(X_\mu^{\mathcal{I}}(b)) \arrow[r] \arrow[d] & \pi_0(X_\mu^{\mathcal{K}_p}(b)) \arrow[d] \\ \pi_0(\mathcal{F}\ell_{\breve{\mathcal{I}}}) \arrow[r, "\cong"] & \pi_0(\mathcal{F}\ell_{\breve{\mathcal{K}}_p}) \end{tikzcd} \tag{6.11}$$

For the bijection of the lower horizontal arrow, see for example [AGLR22, Lemma 4.17]. The left downward arrow is injective by assumption (1), and the top arrow is surjective by Theorem 6.8. Thus the right downward arrow is also injective.

6.4. **Argument for** (2) $\implies$ (3)**.** This is the content of Proposition 4.10.

6.5. **Argument for** (3) $\implies$ (4)**.** This is the content of Proposition 5.8.

6.6. **Argument for** (5) $\implies$ (1)**.**

Proposition 6.9. (5) $\implies$ (1) *in Theorem 6.1.*

Proof. Consider the map $\det : G \to G^{\mathrm{ab}}$ where $G^{\mathrm{ab}} = G/G^{\mathrm{der}}$. Let $\mathcal{I}^{\mathrm{der}}$ denote the Iwahori subgroup of G^{der} attached to our alcove $\mathbf{a}$ (see §2). Let $\mathcal{G}^{\mathrm{ab}}$ be the unique parahoric group scheme of G^{ab}. We have an exact sequence:

$$e \to \mathcal{I}^{\mathrm{der}} \to \mathcal{I} \to \mathcal{G}^{\mathrm{ab}} \to e, \tag{6.12}$$

which induces maps $\mathrm{Sht}_\mu^{\mathcal{I}}(b) \to \mathrm{Sht}_{\mu^{\mathrm{ab}}}^{\mathcal{G}^{\mathrm{ab}}}(b^{\mathrm{ab}})$ and $X_\mu(b) \to X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}})$ by (3.13) and Lemma 3.2, respectively. Recall that by Proposition 6.7, it suffices to assume $G^{\mathrm{der}} = G^{\mathrm{sc}}$. When $G^{\mathrm{der}} = G^{\mathrm{sc}}$, we automatically have $G^\circ = G^{\mathrm{ab}}(\mathbb{Q}_p)$ and $\pi_1(G) = X_*(G^{\mathrm{ab}})$, which induces an isomorphism $\pi_1(G)_I = X_*(G^{\mathrm{ab}})_I$. In this case, by functoriality of the Kottwitz map κ, we have the following commutative diagram

$$\begin{tikzcd} \pi_0(X_\mu(b)) \arrow[r, "\omega_G"] \arrow[d, "\pi_0(\det)"'] & \pi_1(G)_I \arrow[d, "\cong"] \\ X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}}) \arrow[r, "\omega_{G^{\mathrm{ab}}}"] & X_*(G^{\mathrm{ab}})_I. \end{tikzcd} \tag{6.13}$$

Which fits into the following diagram.

$$\begin{tikzcd} \pi_0(X_\mu(b)) \arrow[r, "\omega_G"] \arrow[d, "\pi_0(\det)"'] & c_{b,\mu}\pi_1(G)_I^\varphi \arrow[r, hook] \arrow[d, "\cong"] & \pi_1(G)_I \arrow[d, "\cong"] \\ X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}}) \arrow[r, "\omega_{G^{\mathrm{ab}}}", "\cong"'] & c_{b^{\mathrm{ab}},\mu^{\mathrm{ab}}}X_*(G^{\mathrm{ab}})_I^\varphi \arrow[r, hook] & X_*(G^{\mathrm{ab}})_I. \end{tikzcd} \tag{6.14}$$

In particular, it suffices to prove that left-hand side arrow is a bijection. By functoriality of the specialization map [Gle22b, Proposition 4.14] and Proposition 3.7, we have

$$\begin{tikzcd} \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}\times \operatorname{Spd}\mathbb{C}_p) \arrow[r, "\pi_0(\mathrm{sp})", "\cong"'] \arrow[d, "\pi_0(\det)"'] & \pi_0(X_\mu(b)) \arrow[d, "\pi_0(\det)"] \\ \mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}})}\times \operatorname{Spd}\mathbb{C}_p \arrow[r, "\pi_0(\mathrm{sp})", "\cong"'] & X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}}) \end{tikzcd} \tag{6.15}$$

Note that we have the following identification

$$\begin{aligned} \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}\times \operatorname{Spd}\mathbb{C}_p) &= \pi_0\left(\mathrm{Sht}_{(G,b,\mu,\infty)}\times \operatorname{Spd}\mathbb{C}_p/\,\underline{\mathcal{I}(\mathbb{Z}_p)}\right) \\ &= \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)}\times \operatorname{Spd}\mathbb{C}_p)/\mathcal{I}(\mathbb{Z}_p) \end{aligned}$$

Since $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)}\times \operatorname{Spd}\mathbb{C}_p)$ is a G^{ab}-torsor (i.e. assumption (5) of Theorem 6.1), up to choosing an $x\in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)}\times \operatorname{Spd}\mathbb{C}_p)$, we have compatible identifications $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)}\times \operatorname{Spd}\mathbb{C}_p)\cong G^{\mathrm{ab}}(\mathbb{Q}_p)$ and

$$\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}\times \operatorname{Spd}\mathbb{C}_p)\cong G^{\mathrm{ab}}(\mathbb{Q}_p)/\det(\mathcal{I}(\mathbb{Z}_p)). \tag{6.16}$$

Analogously, taking $x^{\mathrm{ab}}\in \pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty)}\times \operatorname{Spd}\mathbb{C}_p)$ as $x^{\mathrm{ab}}=\pi_0(\det(x))$, we obtain a compatible identification $\pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}})}\times \operatorname{Spd}\mathbb{C}_p) = G^{\mathrm{ab}}(\mathbb{Q}_p)/\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ by Lemma 6.5. Moreover, the map $\det:\pi_0(\mathrm{Sht}_{G,b,\mu,\infty}\times \operatorname{Spd}\mathbb{C}_p)\to \pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty)}\times \operatorname{Spd}\mathbb{C}_p)$ is equivariant with respect to the $G(\mathbb{Q}_p)$-action on the left and the $G^{\mathrm{ab}}(\mathbb{Q}_p)$-action on the right. Thus we have the following commutative diagram:

$$\begin{tikzcd} G^{\mathrm{ab}}(\mathbb{Q}_p)/\det(\mathcal{I}(\mathbb{Z}_p)) \arrow[r, "\cong"] \arrow[d, "\det"'] & \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}\times \operatorname{Spd}\mathbb{C}_p) \arrow[r, "\cong", "\pi_0(\mathrm{sp})"'] \arrow[d, "\det"] & \pi_0(X_\mu(b)) \arrow[d, "\det"] \\ G^{\mathrm{ab}}(\mathbb{Q}_p)/\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \arrow[r, "\cong"] & \mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}})}\times \operatorname{Spd}\mathbb{C}_p \arrow[r, "\cong", "\pi_0(\mathrm{sp})"'] & X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}}) \end{tikzcd}$$

Thus in order to prove that the vertical arrow on the left-hand side is a bijection, it suffices to show that $\mathcal{I}\to\mathcal{G}^{\mathrm{ab}}$ is surjective on the level of $\mathbb{Z}_p$-points. But this follows from Lang's theorem. $\square$

6.7. **Argument for** (4) $\implies$ (5)**.**

Proposition 6.10. (4) $\implies$ (5) *in Theorem 6.1.*

Proof. As seen earlier (for example in §6.6), the map

$$\det : \pi_0(\mathrm{Sht}_{G,b,\mu,\infty} \times \mathrm{Spd}\,\mathbb{C}_p) \to \pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \tag{6.17}$$

is equivariant with respect to the $G(\mathbb{Q}_p)$-action on the source and the $G^{\mathrm{ab}}(\mathbb{Q}_p)$-action on the target. By the assumption that $G^{\mathrm{der}} = G^{\mathrm{sc}}$ (in particular $G^\circ = G^{\mathrm{ab}}(\mathbb{Q}_p)$), it suffices to show that

$$\pi_0(\det) : \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \to \mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty)} \times \mathrm{Spd}\,\mathbb{C}_p \tag{6.18}$$

is bijective. Since the map $G(\mathbb{Q}_p) \to G^{\mathrm{ab}}(\mathbb{Q}_p)$ is surjective, by equivariance of the respective group actions, the map (6.18) is always surjective. By Proposition 3.10, $G(\mathbb{Q}_p)$ acts transitively on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$, thus up to picking an $x \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ we have an identification of sets $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \cong G(\mathbb{Q}_p)/H_x$ for some subgroup $H_x := \mathrm{Stab}(x)$. To prove (4), it suffices to show that $H_x = G^{\mathrm{der}}(\mathbb{Q}_p)$. Firstly, it is easy to see that $H_x \subseteq G^{\mathrm{der}}(\mathbb{Q}_p)$: take any $g \in H_x$, we have $g \cdot x = x$; thus $\deg(g) \cdot \det(x) = \det(g \cdot x) = \det(x)$; by the tori case (see §6.1), $\det(g)$ is trivial, thus $g \in G^{\mathrm{der}}(\mathbb{Q}_p)$.

We now prove the other inclusion, i.e. that $G^{\mathrm{der}}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$. We may argue over finite extensions of $\breve{\mathbb{Q}}_p$.

Indeed, recall from [Sch17, Lemma 12.17] that, the underlying topological space of a cofiltered inverse limit of locally spatial diamonds along qcqs[21] transitions maps is the limit of the underlying topological spaces. Thus it suffices to prove that $G^{\mathrm{der}}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,K)$ for all finite degree extensions K over $\breve{\mathbb{Q}}_p$. For any fixed $x \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,K)$, we denote by $G_x \subseteq G(\mathbb{Q}_p)$ the stabilizer of x. Let $G_x^{\mathrm{der}} := G_x \cap G^{\mathrm{der}}(\mathbb{Q}_p)$. It suffices to prove that $G_x^{\mathrm{der}} = G^{\mathrm{der}}(\mathbb{Q}_p)$, which is shown in Lemma 6.13. $\square$

Lemma 6.11. *G_x^{der} is open in $G^{\mathrm{der}}(\mathbb{Q}_p)$.*

Proof. For any $y \in \mathrm{Gr}_\mu^b(K)$, let $\mathcal{T}_y := \mathrm{Sht}_{(G,b,\mu,\infty)} \times_{\mathrm{Gr}_\mu^b} \mathrm{Spd}\,K$ be the fiber of y under the Grothendieck–Messing period morphism. Take an arbitrary $w \in \pi_0(\mathcal{T}_y)$, by Proposition 3.10, we assume without loss of generality that $w \mapsto x$ under the surjection $\pi_0(\mathcal{T}_y) \to \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,K)$. Consider $G_w^{\mathrm{der}} := G_w \cap G^{\mathrm{der}}(\mathbb{Q}_p)$, and the inclusion of groups $G_w^{\mathrm{der}} \subseteq G_x^{\mathrm{der}} \subseteq G^{\mathrm{der}}(\mathbb{Q}_p)$. It suffices to find a $y \in \mathrm{Gr}_\mu^b(K)$, such that

$$(*) \quad \text{there exists a } w \in \pi_0(\mathcal{T}_y) \text{ with } G_w^{\mathrm{der}} \text{ open in } G^{\mathrm{der}}(\mathbb{Q}_p).$$

Recall the $\underline{G(\mathbb{Q}_p)}$-torsor $\mathbb{L}_b$ over Gr_μ^b from § 3.5. Let $y^*\mathbb{L}_b$ be the corresponding torsor over $\mathrm{Spd}\,K$, which induces a crystalline representation $\rho_y : \Gamma_K \to G(\mathbb{Q}_p)$, well-defined up to conjugacy. We claim that G_w is equal to $\rho_y(\Gamma_K)$ up to $G(\mathbb{Q}_p)$-conjugacy. We now justify the claim. Consider the pullback $\mathcal{T}_t$ of $\mathcal{T}_y$ under the geometric point $t : \mathrm{Spd}\,\mathbb{C}_p \to \mathrm{Spd}\,K$. Thus $\mathcal{T}_t$

[21] i.e. quasi-compact quasi-separated

is a trivial torsor that gives a section $s : \operatorname{Spd}\mathbb{C}_p \to \mathcal{T}_t$. The Galois action of Γ_K on $\mathcal{T}_t$ defines a representative of the crystalline representation ρ_y. The orbit $\Gamma_K \cdot s$ descends to a unique component $w_s \in \pi_0(\mathcal{T}_y)$. Therefore, for any $g \in G(\mathbb{Q}_p)$ such that $g \cdot s \in \Gamma_K \cdot s$, we have $g \cdot w_s = w_s$. This gives us the desired claim. By Proposition 5.8 (which is a consequence of our Theorem 1.15), any generic y satisfies property $(*)$. □

Lemma 6.12. *Assuming hypothesis (4) in Theorem 6.1. Let N_x denote the normalizer of G_x in $G(\mathbb{Q}_p)$. Then N_x has finite index in $G(\mathbb{Q}_p)$. In particular, N_x contains $G^{\mathrm{der}}(\mathbb{Q}_p)$.*

Proof. Let S be the set of orbits of $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \operatorname{Spd} K)$ under the $J_b(\mathbb{Q}_p)$-action. By [HV20, Theorem 1.2], S is finite. For each $s \in S$, we choose a representative $x_s \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd} K)$ that maps to s under the map $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd} K) \to \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \operatorname{Spd} K) \to S$. We can always arrange that x is in this set of representatives for some s. By Proposition 3.10, we can find an element $h_s \in G(\mathbb{Q}_p)$ such that $x_s \cdot h_s = x$, for each $s \in S$. We construct a surjection $\coprod_{s\in S} \mathcal{I}(\mathbb{Z}_p) \cdot h_s \twoheadrightarrow G(\mathbb{Q}_p)/N_x$. We do this in two steps. The first step is to construct, for any $g \in G(\mathbb{Q}_p)$, a triple (i, j, s) where $i \in \mathcal{I}(\mathbb{Z}_p)$, $j \in J_b(\mathbb{Q}_p)$ and $s \in S$ such that

$$j \cdot (x \cdot g) \cdot i = x_s \tag{6.19}$$

(note that s is uniquely determined by x and g). We do this by choosing j so that $j \cdot (x \cdot g)$ and x_s map to the same element in $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \operatorname{Spd} K)$. Since $\mathcal{I}(\mathbb{Z}_p)$ acts transitively on the fibers of the map $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd} K) \to \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \operatorname{Spd} K)$, there exists an i satisfying (6.19). Thus we have

$$j \cdot x \cdot (gih_s) = x \tag{6.20}$$

The second step is to eliminate j from (6.20). By Proposition 3.10, there exists an $n \in G(\mathbb{Q}_p)$ such that $x \cdot n = j \cdot x$. We now show that $n \in N_x$. Indeed, $n^{-1} G_x n = G_{x \cdot n} = G_{j \cdot x} = G_x$ since the actions of $J_b(\mathbb{Q}_p)$ and $G(\mathbb{Q}_p)$ commute. Thus we have $(x \cdot n) \cdot (gih_s) = x$. Since $G_x \subseteq N_x$, in particular $n \cdot (gih_s) \in G_x \subseteq N_x$. Thus $g \cdot i \cdot h_s \in N_x$, and we have a surjection:

$$\coprod_{s \in S} \mathcal{I}(\mathbb{Z}_p) \cdot h_s \twoheadrightarrow G(\mathbb{Q}_p)/N_x. \tag{6.21}$$

The target of (6.21) is discrete, and the source is compact. Thus the index of N_x in $G(\mathbb{Q}_p)$ is finite.

Recall that $G^{\mathrm{der}} = G^{\mathrm{sc}}$. Since G^{der} only has $\mathbb{Q}_p$-simple isotropic factors, and $N_x \cap G^{\mathrm{der}}(\mathbb{Q}_p)$ has finite index in $G^{\mathrm{der}}(\mathbb{Q}_p)$, we have $N_x \cap G^{\mathrm{der}}(\mathbb{Q}_p) = G^{\mathrm{der}}(\mathbb{Q}_p)$. Indeed, it is a standard fact that $G^{\mathrm{der}}(\mathbb{Q}_p)$ has no open subgroups of finite index [Mar91, Chapter II, Theorem 5.1], thus we are done. □

Lemma 6.13. $G_x^{\mathrm{der}} = G^{\mathrm{der}}(\mathbb{Q}_p)$.

Proof. By Lemma 6.11 and Lemma 6.12, $G_x^{\mathrm{der}} \subseteq G^{\mathrm{der}}(\mathbb{Q}_p)$ is open and normal. This already implies $G_x^{\mathrm{der}} = G^{\mathrm{der}}(\mathbb{Q}_p)$, since $G^{\mathrm{der}}(\mathbb{Q}_p)$ does not have open normal subgroups (recall that $G^{\mathrm{der}} = G^{\mathrm{sc}}$). □

This finishes the argument for (4) $\implies$ (5).

Proof of Theorem 6.1. We have now justified the circle of implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (1), i.e. Theorem 6.1 holds in the case $G^{\mathrm{der}} = G^{\mathrm{sc}}$. But by Proposition 6.7 the general case follows. □

Proof of Theorem 6.2. (i.e. also Proof of Theorem 1.2) Using z-extension ad-isomorphisms and decomposition into products (Proposition 4.8, Proposition 4.9, Proposition 3.14 and Lemma 3.4), we may assume without loss of generality that $G^{\mathrm{der}} = G^{\mathrm{sc}}$ and that G^{ad} is $\mathbb{Q}_p$-simple. We split into two cases: (1) when G^{ad} is isotropic, and (2) when G^{ad} is anisotropic. The first case holds from the equivalence (2) $\iff$ (3) of Theorem 6.1.

We now consider the case where G^{ad} is anisotropic. Recall that

$$\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p = \mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p / G(\mathbb{Q}_p). \tag{6.22}$$

When G^{ad} is $\mathbb{Q}_p$-simple and anisotropic, we have that $\mathcal{I}(\mathbb{Z}_p)$ is normal in $G(\mathbb{Q}_p)$, contains $G^{\mathrm{sc}}(\mathbb{Q}_p)$, and $G(\mathbb{Q}_p)/\mathcal{I}(\mathbb{Z}_p) = G^{\mathrm{ab}}(\mathbb{Q}_p)/\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) = \pi_1(G)_I^\varphi$ (see § 6.1 for the last identification). Since $\mathcal{I}(\mathbb{Z}_p)$ is normal in $G(\mathbb{Q}_p)$, $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p$ becomes a $\pi_1(G)_I^\varphi$-torsor over $\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p$. Moreover, the map

$$\det : \mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p \to \mathrm{Sht}_{(G,b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p \tag{6.23}$$

is $\pi_1(G)_I^\varphi$-equivariant. Since $\mathrm{Sht}_{(G,b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p$ is a $\pi_1(G)_I^\varphi$-torsor over $\mathrm{Spd}\,\mathbb{C}_p$, $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p$ is the trivial $\pi_1(G)_I^\varphi$-torsor over $\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p$. That is

$$\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p \cong (\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p) \times \underline{\pi_1(G)_I^\varphi}. \tag{6.24}$$

Taking π_0 in (6.24), we have

$$\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \cong \pi_0(\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p) \times \pi_1(G)_I^\varphi. \tag{6.25}$$

By Theorem 3.9, $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \cong \pi_1(G)_I^\varphi$ and the map

$$\omega_G \circ \pi_0(\mathrm{sp}) : \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}) \times \mathrm{Spd}\,\mathbb{C}_p) \to c_{b,\mu}\pi_1(G)_I^\varphi$$

is an isomorphism as we needed to show. We can finish by recalling that the map of (1.9) is bijective. □

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Mathematisches Institut der Universität Bonn, Endenicher Allee 60, Bonn, Germany
Email address: igleason@uni-bonn.de

University of California, Evans Hall, Berkeley, CA, USA
Email address: limath@math.berkeley.edu

M.I.T., 77 Massachusetts Avenue, Cambridge, MA, USA
Email address: yujiexu@mit.edu