THE CONNECTED COMPONENTS OF AFFINE DELIGNE–LUSZTIG VARIETIES.

IAN GLEASON, DONG GYU LIM, YUJIE XU

Abstract. We compute the connected components of arbitrary parahoric level affine Deligne–Lusztig varieties for quasisplit reductive groups, by relating them to the connected components of infinite level moduli spaces of p-adic shtukas, where we use v-sheaf-theoretic techniques such as the specialization map of *kimberlites*.

As an application, we deduce new CM lifting results on integral models of Shimura varieties at arbitrary parahoric levels. We also prove various results beyond the quasi-split case, by studying the cohomological dimensions of Newton strata inside the B^+_{dR}–affine Grassmannian.

Contents

1. Introduction

1.1. **Background.** In [Rap05], Rapoport introduced certain geometric objects called affine Deligne–Lusztig varieties (ADLVs), to study mod p reduction of Shimura varieties. Since then, ADLVs have played a prominent role in the geometric study of: Shimura varieties, Rapoport–Zink spaces, local Shimura varieties and moduli spaces of local shtukas. Moreover, results on connected components of affine Deligne–Lusztig varieties have found remarkable applications to Kottwitz' conjecture and Langlands–Rapoport conjecture, which describe mod p points of Shimura varieties in relation to L-functions, as part of the Langlands program (for more background on this, see for example [Kis17]). Historically, ADLVs were studied merely as sets, and their combinatoric structures played a predominant role. Recently,

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breakthroughs on perfect algebraic geometry have made it meaningful to talk about their geometric structures (see [Zhu17, BS17]).

Although there have been many successful approaches [Vie08, CKV15, Nie18, HZ20, Ham20, Nie21] to computing connected components of ADLVs in the past decade, as far as the authors know, the current article is the first one that substantially uses p-adic analytic geometry *à la Scholze*, to study affine Deligne–Lusztig varieties and compute their connected components for general quasi-split groups and arbitrary parahorics (with substantial progress towards computing connected components of ADLVs even for non-quasisplit groups). More precisely, we use a combination of Scholze's theory of diamonds [Sch17], the theory of *kimberlites* due to the first author [Gle22b], and the recent proof of the Scholze–Weinstein conjecture [AGLR22, GL22] to compute the connected components of ADLVs. Just as diamonds are generalizations of rigid analytic spaces, kimberlites and prekimberlites are v-sheaf analogues of formal schemes. Roughly speaking, they are diamondifications of formal schemes.

As is well-known to experts, affine Deligne–Lusztig varieties parametrize (at-p) isogeny classes on integral models of Shimura varieties. As an application of our main theorems, we deduce the isogeny lifting property for integral models for Shimura varieties at parahoric levels constructed in [KP18]. Moreover, we give a new CM lifting result on integral models for Shimura varieties–which is a generalization of the classical Honda-Tate theory–for quasi-split groups at p at arbitrary parahoric levels. This improves on previous CM lifting results, which were proved either assuming (1) $G_{\mathbb{Q}_p}$ residually split, or assuming (2) G unramified, or assuming that (3) the parahoric level is very special.

1.2. **Notations.** To not overload the introduction, we use common terms whose rigorous definitions we postpone till later (§2).

We denote by φ the lift of arithmetic Frobenius to $\breve{\mathbb{Q}}_p$. Let $\mathcal{I}$ and $\mathcal{K}_p$ be $\mathbb{Z}_p$-parahoric group schemes with common generic fiber a reductive group G. We let $K_p = \mathcal{K}_p(\mathbb{Z}_p)$, $\breve{\mathcal{I}} = \mathcal{I}(\breve{\mathbb{Z}}_p)$ and $\breve{K}_p := \mathcal{K}_p(\breve{\mathbb{Z}}_p)$. We require that $\mathcal{I}(\mathbb{Z}_p) \subseteq K_p$ and that $\mathcal{I}$ is an Iwahori subgroup of G.

Fix $S \subseteq G$, a $\mathbb{Q}_p$-torus that is maximally split over $\breve{\mathbb{Q}}_p$. Let $T = Z_G(S)$ be the centralizer of S, by Steinberg's theorem it is a maximal $\mathbb{Q}_p$-torus. Let $B \subseteq G_{\breve{\mathbb{Q}}_p}$ be a Borel containing $T_{\breve{\mathbb{Q}}_p}$, which may be defined only over $\breve{\mathbb{Q}}_p$. Let $\mu \in X_*^+(T)$ be a B-dominant cocharacter, and let $b \in G(\breve{\mathbb{Q}}_p)$. Let $\widetilde{W}$ be the Iwahori–Weyl group of G over $\breve{\mathbb{Q}}_p$. Let $\mathrm{Adm}(\mu) \subseteq \widetilde{W} = \breve{\mathcal{I}} \backslash G(\breve{\mathbb{Q}}_p)/\breve{\mathcal{I}}$ denote the μ-admissible set of Kottwitz–Rapoport [KR00].

The (closed) affine Deligne–Lusztig variety associated to (G, b, μ), denoted as $X_\mu(b)$, is a locally perfect finitely presented $\bar{\mathbb{F}}_p$-scheme (see [Zhu17]), with $\bar{\mathbb{F}}_p$-valued points given by:

$$X_\mu(b) = \{g\breve{\mathcal{I}} \mid g^{-1}b\varphi(g) \in \breve{\mathcal{I}}\,\mathrm{Adm}(\mu)\breve{\mathcal{I}}\}. \tag{1.1}$$

By definition, $X_\mu(b)$ embeds into the Witt vector affine flag variety $\mathcal{F}\ell_{\breve{\mathcal{I}}}$, whose $\bar{\mathbb{F}}_p$-valued points are the cosets $G(\breve{\mathbb{Q}}_p)/\breve{\mathcal{I}}$. We also consider the $\mathcal{K}_p$-version $X_\mu^{\mathcal{K}_p}(b)$ with $\bar{\mathbb{F}}_p$-points:

$$X_\mu^{\mathcal{K}_p}(b) = \{g\breve{K}_p \mid g^{-1}b\varphi(g) \in \breve{K}_p \operatorname{Adm}(\mu)\breve{K}_p\}. \tag{1.2}$$

Let $\mathbf{b} \in B(G)$ be the φ-conjugacy class of b, and let $\boldsymbol{\mu}$ be the conjugacy class of μ. Assume $\mathbf{b}$ lies in the Kottwitz set $B(G, \boldsymbol{\mu})$. Let $\mu^\diamond \in X_*(T)_{\mathbb{Q}}^+$ denote the "twisted Galois average" of μ (see (2.7)), and let $\boldsymbol{\nu}_{\mathbf{b}} \in X_*(T)_{\mathbb{Q}}^+$ denote the dominant Newton point. Recall that $\mathbf{b} \in B(G, \boldsymbol{\mu})$ implies that $\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}}$ is a non-negative sum of simple positive coroots with rational coefficients. We say that a pair $(\mathbf{b}, \boldsymbol{\mu})$ with $\mathbf{b} \in B(G, \boldsymbol{\mu})$ is *Hodge–Newton irreducible* (HN-irreducible) if all simple positive coroots have non-zero coefficient in $\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}}$.

Let Γ and I denote the Galois groups of $\mathbb{Q}_p$ and $\breve{\mathbb{Q}}_p$ respectively. Recall that the Kottwitz map [Kot97, 7.4]

$$\kappa_G : G(\breve{\mathbb{Q}}_p) \to \pi_1(G)_I \tag{1.3}$$

induces bijections $\pi_0(\mathcal{F}\ell_{\breve{\mathcal{I}}}) \cong \pi_0(\mathcal{F}\ell_{\breve{K}_p}) \cong \pi_1(G)_I$. Moreover, it is known that the induced map

$$\kappa_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to \pi_1(G)_I \tag{1.4}$$

factors surjectively onto $c_{b,\mu}\pi_1(G)_I^\varphi \subseteq \pi_1(G)_I$ for a unique coset element $c_{b,\mu} \in \pi_1(G)_I/\pi_1(G)_I^\varphi$ (see for example [HZ20, Lemma 6.1]).

1.3. **Main Results.**

The following conjecture describes $\pi_0(X_\mu^{\mathcal{K}_p}(b))$.

Conjecture 1.1. *If $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, the following map is bijective*

$$\kappa_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$$

Our main theorem is the following.

Theorem 1.2. *Let the notation be as in Conjecture 1.1.*

1. *If G is quasisplit, then Conjecture 1.1 holds.*
2. *If $\mathbf{b}$ is basic, then Conjecture 1.1 holds.*
3. *If Conjecture 1.20 holds, then Conjecture 1.1 holds.*

Remark 1.3. Our main theorem 1.2 (3) has no assumption on the tuple $(G, b, \mu, \mathcal{K}_p)$ other than HN-irreducibility of $(\mathbf{b}, \boldsymbol{\mu})$. In particular, we do not require μ to be minuscule.

To state the applications to the geometry of Shimura varieties, we shall also need the following notations. Suppose $\mathbf{G}$ splits over a tamely ramified extension. Let $(\mathbf{G}, X)$ be a Shimura datum of Hodge type. We shall always assume $p > 2$ and $p \nmid |\pi_1(\mathbf{G}^{\mathrm{der}})|$.[1] Let $K_p \subseteq \mathbf{G}(\mathbb{Q}_p)$ be an arbitrary parahoric

[1] although we certainly expect the same results to hold for $p = 2$, using similar ideas from [KMP16], which only addressed the hyperspecial level integral models.

subgroup. By [KP18][2], there is a normal integral model $\mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)$, for the Shimura variety $\mathrm{Sh}_{\mathcal{K}_p}(\mathbf{G}, X)$.

The following Corollary 1.4(1) (resp. Corollary 1.4(2)) is the parahoric analogue of [Kis17, Proposition 1.4.4] (resp. [Kis17, Theorem 2.2.3]) and can be obtained by combining our Theorem 1.2 with [Zho20, Proposition 6.5] (resp. [Zho20, Theorem 9.4]). These are generalizations of existing results in literature to $\mathbf{G}_{\mathbb{Q}_p}$ quasi-split and $\mathcal{K}_p$ arbitrary parahoric. Recall that to any $x \in \mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)(\bar{\mathbb{F}}_p)$, one can associate a $b \in G(\breve{\mathbb{Q}}_p)$ as in [Kis17, Lemma 1.1.12].

Corollary 1.4. *Let* $\mathbf{G}$ *be quasi-split at* p*. Let* $\mathcal{K}_p$ *be an arbitrary parahoric.*

(1) *For any* $x \in \mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)(\bar{\mathbb{F}}_p)$*, there exists a map of perfect schemes*
$$\iota_x : X_\mu^{\mathcal{K}_p}(b)(\bar{\mathbb{F}}_p) \to \mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)(\bar{\mathbb{F}}_p) \tag{1.5}$$
preserving crystalline tensors and equivariant with respect to the geometric r*-Frobenius.*

(2) *The isogeny class* $\iota_x(X_\mu^{\mathcal{K}_p}(b)(\bar{\mathbb{F}}_p)) \times \mathbf{G}(\mathbb{A}_f^p))$ *contains a point which lifts to a special point on* $\mathscr{S}_{\mathcal{K}_p}(G, X)$*.*

The following Corollary 1.5 is a parahoric analogue to [Kis17, Propositions 2.1.3, 2.1.5] when $\mathbf{G}_{\mathbb{Q}_p}$ is quasi-split, and can be obtained by combining our Theorem 1.2 with [Zho20, Proposition 9.1]. Notations as *loc.cit.*

Corollary 1.5. *Let* $\mathbf{G}$ *be quasisplit at* p*. Let* $\mathcal{G} := \mathcal{K}_p$ *be an arbitrary parahoric. Let* $k \subseteq \bar{\mathbb{F}}_p$ *be a finite field extension of* $\mathbb{F}_p$*. The following hold:*

(1) *The map* ι_x *in* (1.5) *induces an injective map*
$$\iota_x : I_x(\mathbb{Q})\backslash X_\mu^{\mathcal{K}_p}(b)(\bar{\mathbb{F}}_p) \times G(\mathbb{A}_f^p) \to \mathscr{S}_{\mathcal{K}_p}(G, X)(\bar{\mathbb{F}}_p), \tag{1.6}$$
where I_x *is a subgroup of the automorphism group of the abelian variety (base changed to* $\bar{\mathbb{F}}_p$*) associated to* x *fixing the Hodge tensors*[3]*.*

(2) *Let* $H^p = \prod_{\ell \neq p} I_{\ell/k}(\mathbb{Q}_\ell) \cap K^p$ *and* $H_p = I_{p/k}(\mathbb{Q}_p) \cap \mathcal{G}(W(k))$*. Then the map* (1.6) *induces an injective map*
$$I_{/k}(\mathbb{Q})\backslash \prod_\ell I_{\ell/k}(\mathbb{Q}_\ell)/H_p \times H^p \to \mathscr{S}_K(G, X)(k), \tag{1.7}$$
where $I_{/k}$ *is the analogue of* I_x *for the abelian variety over* k*.*
In particular, the left hand side of (1.7) *is finite.*

The following corollary verifies the He–Rapoport axioms [HR17] for integral models of Shimura varieties, and follows from combining our main theorem 1.2 with [Zho20, Theorem 8.1(2)].

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[2] In [KP18], the authors construct parahoric integral models assuming that $\mathbf{G}_{\mathbb{Q}_p}$ splits over a tamely ramified extension. We expect that this technical condition can be relaxed using [KZ21].

[3] As is standard in the theory of Shimura varieties, a Shimura variety of Hodge type carries a collection of Hodge tensors that "cut out" the reductive group $\mathbf{G}$

Corollary 1.6.

(1) *Let* $\mathbf{G}$ *be quasisplit at* p. *Let* $\mathcal{K}_p$ *be an arbitrary parahoric. The He–Rapoport axioms hold for the integral models* $\mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)$. *In particular, we obtain the non-emptyness of Newton strata.*

(2) *If Conjecture 1.20 holds, then the He–Rapoport axioms (thus the non-emptyness of Newton strata) hold even for non-quasisplit* $\mathbf{G}_{\mathbb{Q}_p}$.

Moreover, we also obtain the following corollary by combining [HK19, Theorem 2] with our Corollary 1.4 and Corollary 1.6, which allow us to verify Axiom A *loc.cit.*. We refer the reader *loc.cit.* for the precise formulation of the almost product structure of Newton strata.

Corollary 1.7. *Let* $\mathbf{G}$ *be quasisplit at* p. *Let* $\mathcal{K}_p$ *be an arbitrary parahoric. The "almost product structure" of Newton strata holds.*

Finally, as a corollary, we remove a certain technical assumption from the following theorem originally due to the third author [Xu21, Main Theorem][4].

Corollary 1.8. *Let* $\mathbf{G}$ *be quasisplit at* p. *Let* $\mathcal{K}_p$ *be an arbitrary parahoric. The normalization step in the construction of* $\mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)$ *is unnecessary, and the closure model* $\mathscr{S}^{-}_{\mathcal{K}_p}(\mathbf{G}, X)$ *is already normal. Therefore we obtain closed embeddings* $\mathscr{S}_{\mathcal{K}_p\mathcal{K}^p}(\mathbf{G}, X) \hookrightarrow \mathscr{S}_{\mathcal{K}'_p\mathcal{K}'^p}(\mathrm{GSp}, S^{\pm})$.
As a further consequence, the analogous statement holds for toroidal compactifications of integral models.

Remark 1.9. Our main theorem 1.2 results are independent of the integral models of Shimura varieties that one works with. For this reason, we expect our main theorems to have similar applications as the above corollaries to the recent integral models of Pappas–Rapoport [PR21, Theorem 4.8.6].

1.4. **Rough Sketch of the argument.** Many cases of Conjecture 1.1 have been proved in literature under various additional assumptions, see for example [Vie08, Theorem 2], [CKV15, Theorem 1.1], [Nie18, Theorem 1.1], [HZ20, Theorem 0.1], [Ham20, Theorem 1.1(3)], [Nie21, Theorem 0.2].

Previous attempts in literature used characteristic p perfect geometry and combinatorial arguments to construct enough "curves" connecting the components of the ADLV. In our approach, we use the theory of *kimberlites* and their specialization maps developed by the first author in [Gle22b], and the general kimberlite-theoretic unibranchness result for the Scholze–Weinstein local models (see [SW20]) recently established in [GL22, Theorem 1], to turn the problem of computing $\pi_0(X^{\mathcal{K}_p}_{\mu}(b))$ into the characteristic-zero question of computing $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)})$. We remark that when μ is non-minuscule, diamond-theoretic considerations are necessary, since the

[4]The original version of this theorem is stated assuming $\mathbf{G}_{\mathbb{Q}_p}$ residually split for integral models at parahoric levels; at hyperspecial levels, this assumption is not necessary. We are now able to relax "$\mathbf{G}_{\mathbb{Q}_p}$ residually split" to "$\mathbf{G}_{\mathbb{Q}_p}$ quasi-split" thanks to our main theorem 1.2.

spaces $\mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)}$ are not rigid-analytic spaces. Moreover, even when μ is minuscule, the theory of kimberlites is necessary here because: although $\mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)}$ is representable by a rigid-analytic space, its canonical integral model is not known to be representable by a formal scheme.

Once in characteristic zero, we are now able to exploit Fontaine's classical p-adic Hodge theory. In our approach, the role of "connecting curves" is played by "*generic* crystalline representations", inspired by the ideas in [Che14] (see §2.9). Intuitively speaking, the action of Galois groups can be interpreted as "analytic paths" in the moduli spaces of p-adic shtukas.

More precisely, the infinite level moduli space $\mathrm{Sht}_{G,b,\mu,\infty}$ of p-adic shtukas can be realized as the moduli space of trivializations of the universal crystalline $G(\mathbb{Q}_p)$-local system[5] over the b-admissible locus Gr_μ^b of the affine Grassmannian [SW20]. Then rational points of Gr_μ^b give rise to loops in Gr_μ^b, which correspond to "connecting paths" inside any covering space over Gr_μ^b (in particular the covering space $\mathrm{Sht}_{G,b,\mu,\infty}$). Thus it suffices to prove that the universal crystalline representation has enough monodromy to "connect" $\mathrm{Sht}_{G,b,\mu,\infty}$. We can then deduce our main theorem 1.2 at finite level $\mathrm{Sht}_{G,b,\mu,\mathcal{K}_p}$ from the analogous result at infinite level.

1.5. **More on the arguments.** We now dig in a bit deeper into the strategy for our main theorem 1.2, and sketch a few more results that led to our main theorem.

To each $(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))$, one can associate a diamond $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}$, which is the moduli space of p-adic shtukas at level $\mathcal{I}(\mathbb{Z}_p)$ defined in [SW20]. In [Gle22a], the first author constructed a specialization map

$$\mathrm{sp} : |\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}| \to |X_\mu(b)|. \tag{1.8}$$

By the unibranchness result of the first author joint with Lourenço [GL22, Theorem 1.3], and the v-sheaf local model diagram due to the first author [Gle22a, Theorem 3], the specialization map induces an isomorphism of sets

$$\mathrm{sp} : \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \cong \pi_0(X_\mu(b)). \tag{1.9}$$

Therefore we have now turned the question on $\pi_0(X_\mu(b))$ into a characteristic zero question on the connected components $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))})$ of the diamond $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}$, which we now compute.

For this purpose, we make use of the infinite level moduli space $\mathrm{Sht}_{(G,b,\mu,\infty)}$ of p-adic shtukas. Since $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} = \mathrm{Sht}_{(G,b,\mu,\infty)}/\mathcal{I}(\mathbb{Z}_p)$, we have

$$\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}) = \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)})/\mathcal{I}(\mathbb{Z}_p). \tag{1.10}$$

Our main theorem 1.2 follows from our Theorem 1.10 combined with Theorem 1.11, and Theorem 1.2 (3) follows from Theorem 1.10 combined with Theorem 1.14. Let $G^\circ := G(\mathbb{Q}_p)/\operatorname{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$ denote the maximal abelian quotient of $G(\mathbb{Q}_p)$.

[5]For local Shimura varieties coming from Rapoport–Zink spaces, this local system corresponds to the p-adic Tate module of the universal p-divisible group.

Theorem 1.10. *Let* $(\mathbf{b}, \boldsymbol{\mu})$ *be HN-irreducible. Suppose that* G^{ad} *does not have anisotropic factors. The following statements are equivalent:*

1. *The map* $\kappa_G : \pi_0(X_\mu(b)) \to c_{b,\mu}\pi_1(G)_I^{\varphi}$ *is bijective.*
2. *The map* $\kappa_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^{\varphi}$ *is bijective.*
3. *The action of* $G(\mathbb{Q}_p)$ *on* $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ *is transitive.*
4. *The action of* $G(\mathbb{Q}_p)$ *on* $\mathrm{Sht}_{(G,b,\mu,\infty)}$ *makes* $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ *into a* G°*-torsor.*

Theorem 1.11. *Keep the assumptions in Theorem 1.10. Suppose moreover that* G *is quasisplit. Then all four statements in Theorem 1.10 hold.*

Remark 1.12. Theorem 1.11 confirms many new cases (e.g. all quasisplit cases) of a conjecture of Rapoport–Viehmann [RV14, Conjecture 4.26]. Moreover, we generalize the statement to moduli spaces of p-adic shtukas, instead of only for local Shimura varieties as *loc.cit.*

Remark 1.13. Theorem 1.11 is a more general version of the main theorem of [Gle22a], where the first author proved the statement for unramified G, and computed the Weil group and $J_b(\mathbb{Q}_p)$-actions on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$. One should be able to combine the methods of our current paper with those *loc.cit.* to compute the Weil group and $J_b(\mathbb{Q}_p)$-actions in the more general setup of Theorem 1.11.

Theorem 1.14. *Keep the assumptions in Theorem 1.10 and assume moreover Conjecture 1.20. Then all four statements in Theorem 1.10 hold for arbitrary (not necessarily quasisplit)* G*.*

Remark 1.15. We expect that the condition on G^{ad} can be removed from Theorem 1.10. Indeed, one can reduce the proof of Theorem 1.10 to the case where G^{ad} is $\mathbb{Q}_p$ simple and anisotropic. In this case, G^{ad} is the adjoint quotient of the group of units of a division algebra over $\mathbb{Q}_p$. One should be able to calculate this case explicitly.

1.5.1. *Loop of the argument for Theorem 1.10.* We now discuss the proof of Theorem 1.10. Using ad-isomorphisms and z-extensions (see §2.8), we reduce all statements of Theorem 1.10 to the case where G^{der}–the derived subgroup of G–is simply connected (see Proposition 4.5). In this case, $G^\circ = G^{\mathrm{ab}}(\mathbb{Q}_p)$ and, using the HN-irreducibility condition, we prove a circle of implications

$$(1) \implies (2) \implies (3) \implies (4) \implies (1).$$

Now, assuming Conjecture 1.20, one can directly show that (3) holds true even without the assumption that $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible; whereas (1), (2) and (4) crucially rely on this HN-irreducibility assumption. We now explain the chain of implications.

Now, (1) $\implies$ (2) follows from [He16, Theorem 1.1], which says that the map $X_\mu(b) \to X_\mu^{\mathcal{K}_p}(b)$ is surjective. We give a new and simple proof of this result in Theorem 4.6, by observing that $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \to \mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)}$ is automatically surjective. This again exemplifies the advantage of working on the generic fiber (of the v-sheaf $\mathrm{Sht}_{\mu}^{\mathcal{K}_p}(b)$). For more details, see § 2.6.

By the key bijection in (1.9) (see also Theorem 2.18), (2) $\implies$ (3) follows from elementary group-theoretic manipulations exploiting the fact that $\mathrm{Sht}_{(G,b,\mu,\infty)} \to \mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}$ is a $\underline{\mathcal{I}(\mathbb{Z}_p)}$-torsor (for the v-sheaf $\underline{\mathcal{I}(\mathbb{Z}_p)}$ attached to $\mathcal{I}(\mathbb{Z}_p)$). For more details, see §4.5.

Since $\pi_0(X_\mu(b)) = \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}) = \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)})/\mathcal{I}(\mathbb{Z}_p)$, we can descend to finite level. Then the map $\det : G \to G^{\mathrm{ab}}$, combined with Lang's theorem, reduces (4) $\implies$ (1) to the tori case (see §4.1). For more details, see §4.4.

1.5.2. *Proof for* (3) $\implies$ (4) *in Theorem 1.10.* The core of the argument lies in (3) $\implies$ (4). For simplicity, we only discuss the case where G is semisimple and simply connected in the introduction (see §4.6 for the general argument). In this (simplified) case, G° is trivial, thus it suffices to show that $\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p$ is connected.

Let G_x denote the stabilizer of $x \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$. Since $G(\mathbb{Q}_p)$ acts transitively on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ by hypothesis (3) in Theorem 1.10, it suffices to prove $G_x = G(\mathbb{Q}_p)$.

For this, it suffices to show that: (i) G_x is open (see Lemma 4.10); (ii) the normalizer of G_x is of finite index in $G(\mathbb{Q}_p)$ (see Lemma 4.11). Indeed, since we assumed that G is semisimple and simply connected with only isotropic factors, a standard fact from [Mar91, Chapter II, Theorem 5.1] shows that $G(\mathbb{Q}_p)$ does not have finite index subgroups. Thus (ii) allows us to conclude that G_x is normal in $G(\mathbb{Q}_p)$. Moreover, the same standard fact *loc.cit.* shows that $G(\mathbb{Q}_p)$ does not have non-trivial open normal subgroups, therefore (i) implies $G_x = G(\mathbb{Q}_p)$.

We remark that the proof for $G_x \subseteq G(\mathbb{Q}_p)$ being open is the only part of the argument that uses HN-irreducibility of $(\mathbf{b}, \boldsymbol{\mu})$. This argument is close to the one used in [Gle22a, Theorem 1], except that the argument *loc.cit.* relies on [Che14, Théorème 5.0.6], which is only available when G is unramified. In §2.9, we push the methods *loc.cit.* and generalize the result to arbitrary reductive groups G (see also §1.5.3 in this introduction). To prove that the normalizer of G_x is finite we exploit that the actions of $J_b(\mathbb{Q}_p)$ and $G(\mathbb{Q}_p)$ commute, and the general finitness results of [HV20].

1.5.3. *The Mumford-Tate group of "generic crystalline representations".* Recall that in the proof sketch from §1.5.2, we used the fact that the stabilizer G_x is open in $G(\mathbb{Q}_p)$. Fix a finite extension $K/\breve{\mathbb{Q}}_p$ with Galois group $\Gamma_K := \mathrm{Gal}(\overline{K}/K)$, and let $\xi : \Gamma_K \to G(\mathbb{Q}_p)$ denote a conjugacy class of p-adic Hodge–Tate representations.

Definition 1.16. Let MT_ξ denote the connected component of the Zariski closure of the image of ξ in $G(\mathbb{Q}_p)$. This is the p-adic Mumford–Tate group attached to ξ which is well-defined up to conjugation.

It follows from results of Serre [Ser79, Théorème 1] and Sen [Sen73, §4, Théorème 1] (see also [Che14, Proposition 3.2.1]) that $\xi(\Gamma_K) \cap \mathrm{MT}_\xi(\mathbb{Q}_p)$ is open in $\mathrm{MT}_\xi(\mathbb{Q}_p)$.

Let $\mu^\eta : \mathbb{G}_m \to G_K$ be a cocharacter conjugate to μ. Suppose that (b, μ^η) defines an admissible pair in the sense of [RZ96, Definition 1.18]. Since $\mathbf{b} \in B(G, \boldsymbol{\mu})$, it induces a conjugacy class of crystalline representations $\xi_{(b,\mu^\eta)} : \Gamma_K \to G(\mathbb{Q}_p)$, and a p-adic Mumford–Tate group $\mathrm{MT}_{(b,\mu^\eta)}$ attached to $\xi_{(b,\mu^\eta)}$ (See Definition 2.25). Let $\mathrm{Fl}_\mu := G/P_\mu$ denote the generalized flag variety. We say that μ^η is *generic* if the map $\mathrm{Spec}\,(K) \to \mathrm{Fl}_\mu$ induced by μ^η lies over the generic point, this makes sense since $\breve{\mathbb{Q}}_p$ has infinite transcendence degree over $\mathbb{Q}_p$. Our third main theorem is the following generalization of [Che14, Théorème 5.0.6] to arbitrary reductive groups.

Theorem 1.17. *Let G be a reductive group over $\mathbb{Q}_p$. Let $b \in G(\breve{\mathbb{Q}}_p)$ and $\mu^\eta : \mathbb{G}_m \to G_K$ as above. Suppose that b is decent, that μ^η is generic and that $\mathbf{b} \in B(G, \boldsymbol{\mu})$. The following hold:*

1. *(b, μ^η) is admissible.*
2. *If $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, then $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}.*

We provide a partial converse to Theorem 1.17, assuming Conjecture 1.20. This gives an alternative characterization of Hodge-Newton irreducibility.

Proposition 1.18 (Proposition 5.11)**.** *Assume Conjecture 1.20 and that G^{ad} has only isotropic factors. If $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}, then $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*

Remark 1.19. Our Proposition 1.18 confirms the expectation that, at least when G has only isotropic factors, HN-irreducibility is equivalent to having full monodromy [Che14, Remarque 5.0.5].

1.5.4. *Proof of Theorem 1.2.* Consider the b-admissible locus Gr^b_μ in the B^+_{dR}-Grassmannian (see § 2.7). We expect the following:

Conjecture 1.20. *The space $\mathrm{Gr}^b_\mu \times \mathrm{Spd}\,\mathbb{C}_p$ is connected.*

We prove that Conjecture 1.20 implies Theorem 1.10 (3) in the general non-quasisplit case (see Lemma 5.7), which proves Theorem 1.2 (3). To prove that Theorem 1.2 (1) and (2) hold, we verify that Conjecture 1.20 holds either when G is quasi-split or when $\mathbf{b}$ is basic.

1.6. **Organization.** Finally, let us describe the organization of the paper.

§2 is a preliminary section. We start by collecting general notation and standard definitions that we omitted in this introduction. We recall the group-theoretic setup of [GHN19] necessary to discuss the Hodge-Newton decomposition for general reductive groups, and its relation to the connected components of affine Deligne–Lusztig varieties. Then we give a brief intuitive account of the theory of kimberlites. We also review the geometry of affine Deligne–Lusztig varieties and their relation to moduli spaces of p-adic shtukas. Moreover, we discuss ad-isomorphisms, z-extensions and compatibility with products (which will be used in §4 to reduce the proofs of Theorem 1.2 and Theorem 1.10 to the key cases).

In §3, we discuss Mumford–Tate groups. We review the results from [Che14] and discuss the modifications needed to prove Theorem 1.17.

In §4, we give a new proof of [He18, Theorem 7.1] (see Theorem 4.6). We complete proofs of our main results such as Theorem 1.2 and Theorem 1.10.

In §5, we discuss partial progress towards the general non-quasisplit case, and give a characterization of HN-irreducibility.

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2. Preliminaries and background

2.1. **The group-theoretic setup.** Given a group G, let G^{der} denote its derived subgroup, G^{sc} the simply connected cover of G^{der}, and $G^{\mathrm{ab}} := G/G^{\mathrm{der}}$. Since G^{ab} is a torus, it admits a unique parahoric model denoted by $\mathcal{G}^{\mathrm{ab}}$.

We continue the notation from §1.2. Recall that S is a maximal split $\mathbb{Q}_p$-torus of G. Let $\mathcal{N} = N_G(S)$ be the normalizer of S in G. Let $W_0 := \mathcal{N}(\breve{\mathbb{Q}}_p)/T(\breve{\mathbb{Q}}_p)$ be the relative Weyl group. Recall that $T = Z_G(S)$ is the centralizer of S. Let $\mathcal{T}$ denote its unique parahoric model[6]. Denote by $\widetilde{W}$ the Iwahori-Weyl group $\mathcal{N}(\breve{\mathbb{Q}}_p)/\mathcal{T}(\breve{\mathbb{Z}}_p)$. There is a φ-equivariant exact sequence ([HR08]):

$$0 \to X_*(T)_I \to \widetilde{W} \to W_0 \to 1 \tag{2.1}$$

Let $\mathcal{A}$ denote the apartment in the Bruhat–Tits building of $G_{\breve{\mathbb{Q}}_p}$ corresponding to S. Let $\mathbf{a} \subseteq \mathcal{A}$ denote the φ-invariant alcove determined by $\mathcal{I}(\mathbb{Z}_p)$. We choose a special vertex $\mathbf{o} \in \mathbf{a}$, and identify $\mathcal{A}$ with $X_*(T)^I \otimes \mathbb{R} = X_*(T)_I \otimes \mathbb{R}$ by sending the origin to $\mathbf{o}$. Let B be the Borel subgroup attached to $\mathbf{a}$ under

[6] This is the identity component of the locally of finite type Néron model of T.

this identification. Observe that the natural linear action of φ on $X_*(T)^I$ is the gradient of the affine action of φ on $\mathcal{A}$. Let $\Delta \subseteq \Phi^+ \subseteq \Phi \subseteq X^*(T)$ denote the set of simple positive roots, positive roots and roots attached to B, respectively.

The choice of $\mathbf{o}$ defines a splitting $W_0 \to \widetilde{W}$, which may not be φ-equivariant. For every element $\lambda \in X_*(T)_I$, let t_λ be its image in $\widetilde{W}$ under (2.1). Let $\mathbb{S}$ be the set of reflections along the walls of $\mathbf{a}$. Let W^{a} be the affine Weyl group generated by $\mathbb{S}$. It is a Coxeter group. There is a φ-equivariant exact sequence ([HR08, Lemma 14]):

$$1 \to W^{\mathrm{a}} \to \widetilde{W} \to \pi_1(G)_I \to 0 \tag{2.2}$$

This sequence splits and we can write $\widetilde{W} = W^{\mathrm{a}} \rtimes \pi_1(G)_I$. We can extend the Bruhat order $\preceq$ given on W^{a} to the one on $\widetilde{W}$ as follows: for elements $(w_i, \tau_i) \in \widetilde{W}$ with $i = 1, 2$, where $w_i \in W^{\mathrm{a}}$ and $\tau_i \in \pi_1(G)_I$, we say

$$(w_1, \tau_1) \preceq (w_2, \tau_2) \tag{2.3}$$

if $w_1 \preceq w_2$ in W^{a} and $\tau_1 = \tau_2 \in \pi_1(G)_I$. By [Hai18, Theorem 4.2], we can define the Kottwitz–Rapoport admissible set as

$$\mathrm{Adm}(\mu) = \{\tilde{w} \in \widetilde{W} \mid \tilde{w} \preceq t_\lambda \,\text{with}\, t_\lambda = t_{w(\mu)} \,\text{for}\, w \in W_0\}. \tag{2.4}$$

1. Let $\widetilde{W}^{\mathrm{ad}}$ denote the Iwahori–Weyl group of G^{ad}. By [HR08, Lemma 15][7], there exists an element $w^{\mathrm{ad}} \in \widetilde{W}^{\mathrm{ad}}$ such that $w^{\mathrm{ad}} \cdot \varphi(\mathbf{o}) = \mathbf{o}$ and $w^{\mathrm{ad}} \cdot \varphi(\mathbf{a}) = \mathbf{a}$. Conjugation by a lift of w^{ad} to $G^{\mathrm{ad}}(\breve{\mathbb{Q}}_p)$ gives the quasisplit inner form of G, which we denote by G^*. This defines a second action φ_0 on $G(\breve{\mathbb{Q}}_p)$ whose fixed points are $G^*(\mathbb{Q}_p)$ and that satisfies $\varphi_0(\mathcal{A}) = \mathcal{A}$, $\varphi_0(\mathbf{o}) = \mathbf{o}$, $\varphi_0(B) = B$.

2. Let $\mu \in X_*(T)^+$ be a dominant cocharacter. Denote by $\mu^\sharp \in \pi_1(G)_\Gamma$ the image of μ under the natural projection $X_*(T) \to \pi_1(G)_\Gamma$. As in [Kot97, (6.1.1)], we define

$$\mu^\diamond := \frac{1}{[\Gamma : \Gamma_\mu]} \sum_{\gamma \in \Gamma/\Gamma_\mu} \gamma(\mu) \in X_*(T)^+_{\mathbb{Q}}, \tag{2.5}$$

where the Galois action on $X_*(T)$ is the one coming from G^*. Via the isomorphism $X_*(T)_I \otimes \mathbb{Q} \simeq (X_*(T) \otimes \mathbb{Q})^I$ given by $[\mu] \mapsto \frac{1}{[I:I_\mu]} \sum_{\gamma \in I/I_\mu} \gamma(\mu)$, we may write $\mu^\diamond$ as follows (see [HN18, A.4]):

$$\underline{\mu} := \frac{1}{[I : I_\mu]} \sum_{\gamma \in I/I_\mu} \gamma(\mu) \tag{2.6}$$

$$\mu^\diamond = \frac{1}{N} \sum_{i=0}^{N-1} \varphi_0^i(\underline{\mu}) \tag{2.7}$$

[7]More precisely, $P^\vee$ *loc.cit.* acts transitively on the set of special vertices and σ sends a special vertex $\mathbf{o}$ to a special vertex. Thus $P^\vee$ and W_0 together make it possible to find this element w^{ad}.

Here N is any integer such that $\varphi_0^N(\underline{\mu}) = \underline{\mu}$, and I_μ is the stabilizer of μ associated to the action by the inertia group. Alternatively,

$$\mu^\diamond = \frac{1}{N}\sum_{i=0}^{N-1} \varphi^i(\mu)^{\mathrm{dom}}. \tag{2.8}$$

Here λ^{dom} denotes the unique B-dominant conjugate of λ for $\lambda \in X_*(T)\otimes\mathbb{Q}$.

3. Recall that attached to b, there is a slope decomposition map

$$\nu_b : \mathbb{D} \to G_{\breve{\mathbb{Q}}_p}, \tag{2.9}$$

where $\mathbb{D}$ is the pro-torus with $X^*(\mathbb{D}) = \mathbb{Q}$. We let the *Newton point*, denoted as $\boldsymbol{\nu}_{\mathbf{b}}$, be the unique conjugate in $X_*(T)^+_{\mathbb{Q}}$ of (2.9). Recall that there is a Kottwitz map $\kappa_G : B(G) \to \pi_1(G)_\Gamma$ [Kot85, Kot97].

Definition 2.1. Let $\mathbf{b} \in B(G)$.

(1) We write $\mathbf{b} \in B(G, \boldsymbol{\mu})$ if $\mu^\sharp = \kappa_G(\mathbf{b})$ and $\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} = \sum_{\alpha\in\Delta} c_\alpha \alpha^\vee$ with $c_\alpha \in \mathbb{Q}$ and $c_\alpha \geq 0$.

(2) We say $(\mathbf{b}, \boldsymbol{\mu})$ is *HN-irreducible (Hodge–Newton irreducible)* if $\mathbf{b} \in B(G, \boldsymbol{\mu})$ and $c_\alpha \neq 0$ for all $\alpha \in \Delta$.

Definition 2.2. [RZ96, Definition 1.8] Let $s \in \mathbb{N}$. We say that $b \in G(\breve{\mathbb{Q}}_p)$ is *s-decent* if $s\cdot\nu_b$ factors through a map $\mathbb{G}_m \to G_{\breve{\mathbb{Q}}_p}$, and the decency equation $(b\varphi)^s = s\cdot\nu_b(p)\varphi^s$ is satisfied in $G(\breve{\mathbb{Q}}_p) \rtimes \langle\varphi\rangle$. If the context is clear, we say that b is *decent* if it is s-decent for some s.

4. If b is s-decent, then $b \in G(\mathbb{Q}_{p^s})$ and ν_b is also defined over $\mathbb{Q}_{p^s}$, where $\mathbb{Q}_{p^s}$ is the degree s unramified extension of $\mathbb{Q}_p$. Moreover, for all $\mathbf{b} \in B(G)$, there exists an $s \in \mathbb{N}$ and an s-decent representative $b \in G(\mathbb{Q}_{p^s})$ of $\mathbf{b}$, such that $\nu_b = \boldsymbol{\nu}_{\mathbf{b}}$. Indeed, by [RZ96, 1.11], every $\mathbf{b}$ has a decent representative. Moreover, we can choose s large enough such that G is quasisplit over $\mathbb{Q}_{p^s}$, and then take an arbitrary s-decent element. Now, replacing b by a φ-conjugate in $G(\mathbb{Q}_{p^s})$ preserves decency and conjugates the map ν_b, thus we can assume without loss of generality that b is dominant.

Remark 2.3. One can define affine Deligne–Lusztig varieties over any local field F, and the statement of Theorem 1.2 is conjectured to hold in this generality. Our Theorem 1.2 holds when F is a finite extension of $\mathbb{Q}_p$, via a standard restriction of scalars argument (see for example [DOR10, §5 and §8]). It would be interesting to see if our methods also go through in the equal characteristic case.

2.2. **The Hodge–Newton decomposition.** We can classify elements in $B(G, \boldsymbol{\mu})$ into two kinds: Hodge-Newton decomposable or indecomposable.

Definition 2.4 (Hodge-Newton Decomposability)**.** Assume $\mathbf{b} \in B(G, \boldsymbol{\mu})$. We say $\mathbf{b}$ is *Hodge-Newton decomposable* (with respect to M) in $B(G, \boldsymbol{\mu})$ if there exists a φ_0-stable standard Levi subgroup M containing $M_{\boldsymbol{\nu}_{\mathbf{b}}}$, and

$$\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} \in \mathbb{Q}_{\geq 0}\Delta_M^\vee. \tag{2.10}$$

If no such M exists, $\mathbf{b}$ is said to be *Hodge-Newton indecomposable* in $B(G, \boldsymbol{\mu})$.

For a HN-decomposable $\mathbf{b}$ in $B(G, \boldsymbol{\mu})$, affine Deligne–Lusztig varieties admit a decomposition theorem (Theorem 2.5). More precisely, suppose $\mathbf{b}$ is HN-decomposable with respect to a Levi subgroup M. Let P be the standard parabolic subgroup containing M and B. As in [GHN19, 4.4], let $\mathfrak{P}^{\varphi}$ be the set of φ-stable parabolic subgroups containing the maximal torus T and conjugate to P. Given $P' \in \mathfrak{P}^{\varphi}$, let N' be the unipotent radical, and M' the Levi subgroup containing T such that $P' = M'N'$. Let W_K be the subgroup of W_0 generated by the set of simple reflections corresponding to $\mathcal{K}_p$. Let W_K^{φ} be the φ-invariant elements of W_K. We have the following.

Theorem 2.5 ([GHN19, Theorem A]). *Let $b \in B(G, \mu)$ be HN-decomposable with respect to $M \subset G$. Then there is an isomorphism*

$$X_{\mu}^{\mathcal{K}_p}(b) \simeq \bigsqcup_{P'=M'N'} X_{\mu_{P'}}^{\mathcal{K}_p^{M'}}(b_{P'}), \tag{2.11}$$

where P' ranges over the set $\mathfrak{P}^{\varphi}/W_K^{\varphi}$.

Note that the natural embedding

$$\phi_{P'} : X_{\mu_{P'}}^{\mathcal{K}_p^{M'}}(b_{P'}) \hookrightarrow X_{\mu}^{\mathcal{K}_p}(b) \tag{2.12}$$

is the composite of the closed immersion $\mathcal{F}\ell_{\mathcal{K}_p^{M'}} \hookrightarrow \mathcal{F}\ell_{\mathcal{K}_p}$ of affine flag varieties and the map $g\breve{\mathcal{K}}_p \mapsto h_{P'}g\breve{\mathcal{K}}_p$, where $h_{P'} \in G(\breve{\mathbb{Q}}_p)$ satisfies $b_{P'} = h_{P'}^{-1} b\sigma(h_{P'})$ ([GHN19, 4.5]).

By the following lemma, we may assume, without loss of generality, in the proof of Proposition 5.9 that each $(b_{P'}, \mu_{P'})$ is HN-indecomposable.

Lemma 2.6 ([Zho20, Lemma 5.7]). *There exists a unique φ_0-stable $M \subset G$ such that, for each P' appearing in decomposition (2.11), $b_{P'}$ is HN-indecomposable in $B(M', \mu_{P'})$.*

Example 2.7. A basic element $\mathbf{b}$ is always HN-indecomposable in $B(G, \boldsymbol{\mu})$ since $M_{\boldsymbol{\nu}_{\mathbf{b}}} = G$.

Example 2.8. When G^{ad} is simple , if b is basic and μ is not central, then b is Hodge-Newton irreducible (Definition 2.1) in $B(G, \mu)$ because a linear combination of coroots is dominant if and only if all the coefficients are positive ([LT92, 5.]).

Example 2.8 shows that, except for the "central cocharacter" case, HN-indecomposability is the same as HN-irreducibility in the b basic case. In fact, this is true even when b is not basic.

Proposition 2.9 (cf. [Zho20, Lemma 5.3]). *Suppose that $G = G^{\mathrm{ad}}$ and that G is $\mathbb{Q}_p$-simple. Let $b \in G(\breve{\mathbb{Q}}_p)$ and μ a dominant cocharacter, such that $\mathbf{b} \in B(G, \boldsymbol{\mu})$. Suppose $(\mathbf{b}, \boldsymbol{\mu})$ is HN-indecomposable. Then either $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible or b is φ-conjugate to some $p^{\underline{\mu}}$ with $\underline{\mu}$ central.*

Proposition 2.9 asserts that the gap between HN-indecomposable and HN-irreducible elements consists only of central elements. Moreover, when b is φ-conjugate to $p^{\underline{\mu}}$ for a central μ, the connected components of affine Deligne–Lusztig varieties have been computed in Proposition 2.10 below. Note that if μ is central, there is a unique $\mathbf{b} \in B(G, \boldsymbol{\mu})$. Moreover, this $\mathbf{b}$ is basic and represented by $p^{\underline{\mu}}$. We can then apply the following result.

Proposition 2.10 ([HZ20, Theorem 0.1 (1)])**.** *Suppose that G^{ad} is $\mathbb{Q}_p$-simple. Let $b \in G(\breve{\mathbb{Q}}_p)$ be a representative for a basic element $\mathbf{b} \in B(G)$. If μ is central and $\mathbf{b} \in B(G, \boldsymbol{\mu})$, then $X_\mu^{\mathcal{K}_p}(b)$ is discrete and*

$$X_\mu^{\mathcal{K}_p}(b) \simeq G(\mathbb{Q}_p)/\mathcal{K}_p(\mathbb{Z}_p). \tag{2.13}$$

5. Next we show that HN-irreducibility is preserved under ad-isomorphisms and taking projection onto direct factors. Let $f : G \to H$ be an ad-isomorphism. Let $b_H := f(b)$ and $\mu_H = \mu \circ f$. Let T_H denote a maximal torus containing $f(T)$. By functoriality, we have commutative diagrams

$$\begin{tikzcd} X_*(T) \arrow[r, "f_*"] \arrow[d] & X_*(T_H) \arrow[d] \\ \pi_1(G)_\Gamma \arrow[r] & \pi_1(H)_\Gamma \end{tikzcd} \tag{2.14}$$

and

$$\begin{tikzcd} B(G) \arrow[r] \arrow[d] & B(H) \arrow[d] \\ \pi_1(G)_\Gamma \arrow[r] & \pi_1(H)_\Gamma \end{tikzcd} \qquad \begin{tikzcd} B(G) \arrow[r] \arrow[d] & B(H) \arrow[d] \\ X_*(T)^+_{\mathbb{Q}} \arrow[r] & X_*(T)^+_{\mathbb{Q}} \end{tikzcd} \tag{2.15}$$

We have the following.

Proposition 2.11. *Let $\mathbf{b} \in B(G, \boldsymbol{\mu})$ and f an ad-isomorphism. Then $(\mathbf{b}_H, \boldsymbol{\mu}_H)$ is HN-irreducible if and only if $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*

Proof. Since $\mathbf{b} \in B(G, \boldsymbol{\mu})$, we have $\kappa_G(\mathbf{b}) = \mu^\sharp$ (see 2). By (2.14) and (2.15), we have $\kappa_H(\mathbf{b}_H) = \mu_H^\sharp$. Moreover, we can write

$$\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} = \sum_{\alpha \in \Phi^+} c_\alpha \alpha^\vee, \tag{2.16}$$

where $c_\alpha \geq 0$. On the other hand, note that $f_*(\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}}) = \mu_H^\diamond - \boldsymbol{\nu}_{\mathbf{b}_H}$. Since f is an ad-isomorphism, $f_*(\alpha^\vee) = \alpha^\vee$. Thus we have $\mu_H^\diamond - \boldsymbol{\nu}_{\mathbf{b}_H} = \sum_{\alpha \in \Phi^+} c_\alpha \alpha^\vee$, and hence $\mathbf{b}_H \in B(H, \boldsymbol{\mu}_H)$. Now, each $(\mathbf{b}_H, \boldsymbol{\mu}_H)$ is HN-irreducible if and only if $(\mathbf{b}, \boldsymbol{\mu})$ is, since this is in turn equivalent to $c_\alpha > 0$ for all $\alpha \in \Phi_G^+$. □

6. Let $G = G_1 \times G_2$, then $T = T_1 \times T_2$, $B(G) = B(G_1) \times B(G_2)$, $\pi_1(G)_\Gamma = \pi_1(G_1)_\Gamma \times \pi_1(G_2)_\Gamma$ and $X_*(T) = X_*(T_1) \times X_*(T_2)$. In this case, the Kottwitz and Newton maps 3 can be computed coordinatewise.

Proposition 2.12. *The following hold:*

1. $\mathbf{b} \in B(G, \boldsymbol{\mu}) \Leftrightarrow$ *each* $\mathbf{b}_i \in B(G_i, \boldsymbol{\mu}_i)$ *for* $i \in \{1, 2\}$.
2. $(\mathbf{b}, \boldsymbol{\mu})$ *is HN-irreducible* $\Leftrightarrow$ *each* $(\mathbf{b}_i, \mu_i)$ *is HN-irreducible for* $i \in \{1, 2\}$.

Proof. The condition $\kappa_G(\mathbf{b}) = \mu^\sharp$ can be checked component-wise. Moreover, since $\mu^\diamond - \boldsymbol{\nu}_\mathbf{b} = (\mu_1^\diamond - \boldsymbol{\nu}_{\mathbf{b}1}, \mu_2^\diamond - \boldsymbol{\nu}_{\mathbf{b}2})$, verifying whether it is a non-negative (resp. positive) sum of positive coroots (see Definition 2.1) can also be done component-wise.

□

2.3. **v-sheaf-theoretic setup.** We work within Scholze's framework of diamonds and v-sheaves [Sch17]. More precisely, we consider geometric objects that are functors

$$\mathcal{F} : \mathrm{Perf}_{\mathbb{F}_p} \to \mathrm{Sets}, \tag{2.17}$$

where $\mathrm{Perf}_{\mathbb{F}_p}$ is the site of affinoid perfectoid spaces in characteristic p, endowed with the v-topology (see [Sch17, Definition 8.1]). Recall that given a topological space T, we can define a v-sheaf $\underline{T}$ whose value on (R, R^+)-points is the set of continuous maps $|\operatorname{Spa}(R, R^+)| \to T$. We will mostly use this notation $\underline{T}$ for topological groups T.

Example 2.13. $\underline{\mathcal{I}(\mathbb{Z}_p)}$ and $\underline{G(\mathbb{Q}_p)}$ are the v-sheaf group objects attached to the topological groups $\mathcal{I}(\mathbb{Z}_p)$ and $G(\mathbb{Q}_p)$.

Conversely, to any diamond or v-sheaf $\mathcal{F}$, by [Sch17, Proposition 12.7], one can attach an underlying topological space that we denote by $|\mathcal{F}|$.

7. Recall that in the more classical setup of Rapoport–Zink spaces [RZ96], affine Deligne–Lusztig varieties arise, via Dieudonne theory, as the perfection of special fibers of Rapoport-Zink spaces. Moreover, the rigid generic fiber of such a Rapoport-Zink space is a special case of the so called *local Shimura varieties* [RV14]. In this way, Rapoport-Zink spaces (formal schemes) interpolate between local Shimura varieties and their corresponding affine Deligne–Lusztig varieties. Or in other words, Rapoport-Zink spaces serve as *integral models* of local Shimura varieties whose perfected special fibers are ADLVs. Moreover, by [SW20], the diamondification functor

$$\begin{aligned} \diamondsuit : \{\text{Adic Spaces}/\operatorname{Spa}\mathbb{Z}_p\} &\longrightarrow \{\text{v-sheaves}/\operatorname{Spd}\mathbb{Z}_p\} \\ X &\longmapsto X^\diamondsuit \end{aligned}$$

applied to a local Shimura variety is a locally spatial diamond that can be identified with a moduli space of p-adic shtukas (see §2.6).

Alternatively, one could consider the diamondification functor applied to the entire formal schemes (such as Rapoport-Zink spaces), rather than only their rigid generic fibres. The diamondification functor naturally takes values in v-sheaves, but contrary to the rigid-analytic case, these v-sheaves are no longer diamonds. Nevertheless, the v-sheaf associated to a formal scheme still has a lot of structure. Indeed, they are what the first author calls *kimberlites* [Gle22b], i.e. we have a commutative diagram

$$\begin{tikzcd} \{\text{Adic Spaces}/\operatorname{Spa}\mathbb{Z}_p\} \arrow[r, "\diamond"] & \{\text{v-sheaves}/\operatorname{Spd}\mathbb{Z}_p\} \\ \{\text{Formal Schemes}/\operatorname{Spf}\mathbb{Z}_p\} \arrow[r, "\diamond"] \arrow[u, hook] & \{\text{Kimberlites}/\operatorname{Spd}\mathbb{Z}_p\} \arrow[u, hook] \end{tikzcd} \tag{2.18}$$

Kimberlites share with formal schemes many pleasant properties that general v-sheaves do not. Let us list the main ones. Let $\mathfrak{X}$ be a kimberlite.

(1) Each Kimberlite has an open analytic locus $\mathfrak{X}^{\mathrm{an}}$ (which is a locally spatial diamond by definition), and a reduced locus $\mathfrak{X}^{\mathrm{red}}$ (which is by definition a perfect scheme).
(2) Each Kimberlite has a continuous "specialization map" whose source is $|\mathfrak{X}^{\mathrm{an}}|$ and whose target is $|\mathfrak{X}^{\mathrm{red}}|$ (see 8 for details).
(3) Kimberlites have a formal étale site and a formal nearby-cycles functor $R\Psi^{\mathrm{for}} : D_{\text{ét}}(\mathfrak{X}^{\mathrm{an}}, \Lambda) \to D_{\text{ét}}(\mathfrak{X}^{\mathrm{red}}, \Lambda)$.

Although we expect that every local Shimura variety admits a formal scheme "integral model" (see [PR22] for the strongest result on this direction), this is not known in full generality. Nevertheless, as the first author proved, every local Shimura variety (even the more general moduli spaces of p-adic shtukas) is modeled by a *prekimberlite*[8] whose perfected special fiber is the corresponding ADLV (see Theorem 2.17). We shall return to this discussion in §2.6.

8. Recall that given a formal scheme $\mathcal{X}$, one can attach a specialization triple $(\mathcal{X}_\eta, \mathcal{X}^{\mathrm{red}}, \mathrm{sp})$, where $\mathcal{X}_\eta$ is a rigid analytic space (the Raynaud generic fiber), $\mathcal{X}^{\mathrm{red}}$ is a reduced scheme (the reduced spacial fiber) and

$$\mathrm{sp} : |\mathcal{X}_\eta| \to |\mathcal{X}^{\mathrm{red}}| \tag{2.19}$$

is a continuous map.

Analogously, to a prekimberlite $\mathfrak{X}$ [Gle22b, Definition 4.15] over $\operatorname{Spd}(\mathbb{Z}_p)$, one can attach a specialization triple $(\mathfrak{X}_\eta, \mathfrak{X}^{\mathrm{red}}, \mathrm{sp})$ where

- $\mathfrak{X}_\eta$ is the generic fiber (which is an open subset of the analytic locus $\mathfrak{X}^{\mathrm{an}}$ [Gle22b, Definition 4.15] of $\mathfrak{X}$).
- $\mathfrak{X}^{\mathrm{red}}$ is a perfect scheme over $\mathbb{F}_p$ (obtained via the reduction functor [Gle22b, §3.2]) and
- sp is a continuous map [Gle22b, Proposition 4.14] analogous to (2.19).

For example, if $\mathfrak{X} = \mathcal{X}^\diamond$ for a formal scheme $\mathcal{X}$, then $\mathfrak{X}$ is a kimberlite, and we have $\mathfrak{X}_\eta = \mathcal{X}_\eta^\diamond$, $\mathfrak{X}^{\mathrm{red}} = (\mathcal{X}^{\mathrm{red}})^{\mathrm{perf}}$ and the specialization maps attached

[8] In fact, we expect moduli spaces of p-adic shtukas to be modeled by kimberlites, but for our purposes this difference is minor, as the specialization map is defined for both kimberlites and prekimberlites.

to $\mathcal{X}$ and $\mathfrak{X}$ agree, i.e. we have the following commutative diagram:

$$\begin{tikzcd} \mid \mathcal{X}_\eta \mid \arrow[r, "\cong"] \arrow[d, "\mathrm{sp}"'] & \mid \mathfrak{X}_\eta \mid \arrow[d, "\mathrm{sp}"] \\ \mid \mathcal{X}^{\mathrm{red}} \mid \arrow[r, "\cong"] & \mid \mathfrak{X}^{\mathrm{red}} \mid \end{tikzcd} \tag{2.20}$$

9. A *smelted kimberlite* is a pair $(\mathfrak{X}, X)$ where $\mathfrak{X}$ is a prekimberlite and $X \subseteq \mathfrak{X}^{\mathrm{an}}$ is an open subsheaf of the analytic locus, subject to some technical conditions. This is mainly used when $X = \mathfrak{X}^{\mathrm{an}}$ or when X is the generic fiber of a map to $\mathrm{Spd}\,\mathbb{Z}_p$ that is not p-adic.

Given a smelted kimberlite $(\mathfrak{X}, X)$ and a closed point $x \in |\mathfrak{X}^{\mathrm{red}}|$, one can define the tubular neighborhood $X_x^{\circledcirc}$ ([Gle22b, Definition 4.38]). It is an open subsehaf of X which, roughly speaking, is given as the locus in X of points that specialize to x.

2.4. B_{dR}^+**-Grassmannians and local models.**

Let Gr_G be the B_{dR}^+-Grassmannian attached to G [SW20, §19, 20]. This is an ind-diamond over $\mathrm{Spd}\,\breve{\mathbb{Q}}_p$. We omit G from the notation from now on, and denote by Gr_μ the Schubert variety [SW20, Definition 20.1.3] attached to G and μ. This is a spatial diamond over $\mathrm{Spd}\,\breve{E}$ where $\breve{E} = E \cdot \breve{\mathbb{Q}}_p$ and E is the field of definition of $\boldsymbol{\mu}$. Now, Gr_μ contains the Schubert cell attached to μ, which we denote by Gr_μ°. This is an open dense subdiamond of Gr_μ.

Recall the Bialynicki-Birula map (see [SW20, Proposition 19.4.2]) from the Schubert cell Gr_μ° to the generalized flag variety $\mathrm{Fl}_\mu := G/P_\mu$

$$\mathrm{BB} : \mathrm{Gr}_\mu^\circ \to \mathrm{Fl}_\mu\,. \tag{2.21}$$

In general, the map (2.21) is not an isomorphism (it is an isomorphism only when μ is minuscule), but it always induces a bijection on classical points, i.e. finite extensions of $\breve{E}$ (see for example [Gle21, Proposition 2.12] and [Vie21, Theorem 5.2]).

Let $\mathrm{Gr}_{\mathcal{K}_p}$ be the Beilinson–Drinfeld Grassmannian attached to $\mathcal{K}_p$. This is a v-sheaf that is ind-representable in diamonds over $\mathrm{Spd}\,\breve{\mathbb{Z}}_p$, whose generic fiber is Gr_G, and whose reduced special fiber is $\mathcal{F}\ell_{\breve{\mathcal{K}}_p}$. Let $\mathcal{M}_{\mathcal{K}_p,\mu}$ be the local models first introduced in [SW20, Definition 25.1.1] for minuscule μ and later extended to non-minuscule μ in [AGLR22, Definition 4.11].

A priori, these local models are defined only as v-sheaves over $\mathrm{Spd}\,O_{\breve{E}}$, but when μ is minuscule, $\mathcal{M}_{\mathcal{K}_p,\mu}$ is representable by a normal scheme flat over $\mathrm{Spec}\,O_{\breve{E}}$ by [AGLR22, Theorem 1.1][9] and [GL22, Corollary 1.4][10]. Moreover, in the general case, i.e. μ not necessarily minuscule, $\mathcal{M}_{\mathcal{K}_p,\mu}$ is a kimberlite by [AGLR22, Proposition 4.14], and it is unibranch by [GL22, Theorem

[9] Representability is proved in full generality *loc. cit.* and normality is proven when $p \geq 5$.

[10] We prove normality for all p, including when $p < 5$, and recover the normality part of [AGLR22, Theorem 1.1].

1.2]. Let $\mathcal{A}_{\mathcal{K}_p,\mu}$ denote the μ-admissible locus inside $\mathcal{F}\ell_{\breve{\mathcal{K}}_p}$ (see for example [AGLR22, Definition 3.11]). This is a perfect scheme whose $\bar{\mathbb{F}}_p$-valued points agree with $\breve{\mathcal{K}}_p \operatorname{Adm}(\mu)\breve{\mathcal{K}}_p/\breve{\mathcal{K}}_p$. The generic fiber of $\mathcal{M}_{\mathcal{K}_p,\mu}$ is Gr_μ and the reduced special fiber is $\mathcal{A}_{\mathcal{K}_p,\mu}$ by [AGLR22, Theorem 1.5].

2.5. **Functoriality of affine Deligne–Lusztig varieties.** The formation of affine Deligne–Lusztig varieties is functorial with respect to morphisms of tuples $(G_1, b_1, \mu_1, \mathcal{K}_{1,p}) \to (G_2, b_2, \mu_2, \mathcal{K}_{2,p})$. More precisely, we have the following lemma.

Lemma 2.14. *Let $f : G_1 \to G_2$ be a group homomorphism such that $b_2 = f(b_1)$, $\mu_2 = f \circ \mu_1$ and $f(\mathcal{K}_{1,p}) \subseteq \mathcal{K}_{2,p}$. Then we have a map $X^{\mathcal{K}_{1,p}}_{\mu_1}(b_1) \to X^{\mathcal{K}_{2,p}}_{\mu_2}(b_2)$ that fits in the following commutative diagram:*

$$\begin{array}{ccc} X^{\mathcal{K}_{1,p}}_{\mu_1}(b_1) & \longrightarrow & X^{\mathcal{K}_{2,p}}_{\mu_2}(b_2) \\ \downarrow & & \downarrow \\ \mathcal{F}\ell_{\breve{K}_{1,p}} & \longrightarrow & \mathcal{F}\ell_{\breve{K}_{2,p}} \end{array} \tag{2.22}$$

Proof. This follows directly from the definitions and from Lemma 2.15. $\square$

Lemma 2.15. $f(\breve{\mathcal{K}}_{1,p} \operatorname{Adm}(\mu_1)\breve{\mathcal{K}}_{1,p}) \subseteq \breve{\mathcal{K}}_{2,p} \operatorname{Adm}(\mu_2)\breve{\mathcal{K}}_{2,p}$.

Proof. We give a geometric argument. Let $\mathcal{M}_{\mathcal{K}_{1,p},\mu_1}$ and $\mathcal{M}_{\mathcal{K}_{2,p},\mu_2}$ denote the v-sheaf local models in [AGLR22, Definition 4.11]. Since $f(\mathcal{K}_{1,p}) \subseteq \mathcal{K}_{2,p}$, we have a morphism of parahoric group schemes $\mathcal{K}_{1,p} \to \mathcal{K}_{2,p}$. By the functoriality result of v-sheaf local models [AGLR22, Proposition 4.16], we obtain a morphism $\mathcal{M}_{\mathcal{K}_{1,p},\mu_1} \to \mathcal{M}_{\mathcal{K}_{2,p},\mu_2}$ of v-sheaves. Moreover, by [AGLR22, Theorem 6.16], we know that $\mathcal{M}_{\mathcal{K}_{i,p},\mu_i,\bar{\mathbb{F}}_p} \subseteq \mathcal{F}\ell_{\breve{K}_{i,p}}$ consists of Schubert cells parametrized by $\operatorname{Adm}(\mu_i)$. More precisely, $\mathcal{M}_{\mathcal{K}_{i,p},\mu_i,\bar{\mathbb{F}}_p}(\bar{\mathbb{F}}_p) = \breve{\mathcal{K}}_{i,p} \operatorname{Adm}(\mu_i)\breve{\mathcal{K}}_{i,p}/\breve{\mathcal{K}}_{i,p}$. Therefore, the existence of the map of perfect schemes $\mathcal{M}_{\mathcal{K}_{1,p},\mu_1,\bar{\mathbb{F}}_p} \to \mathcal{M}_{\mathcal{K}_{2,p},\mu_2,\bar{\mathbb{F}}_p}$ immediately implies that $f(\breve{\mathcal{K}}_{1,p} \operatorname{Adm}(\mu_1)\breve{\mathcal{K}}_{1,p}) \subseteq \breve{\mathcal{K}}_{2,p} \operatorname{Adm}(\mu_2)\breve{\mathcal{K}}_{2,p}$. $\square$

Lemma 2.15 is most relevant in the following situations:

(1) When $G_1 = G_2$, $f = \mathrm{id}$, and $\mathcal{K}_{1,p} \subseteq \mathcal{K}_{2,p}$.
(2) When $G_2 = G_1^{\mathrm{ab}}$ and $\mathcal{K}_{2,p}$ is the only parahoric of the torus G_1^{ab}.
(3) When $G_2 = G_1/Z$, where Z a central subgroup of G_1 and $\mathcal{K}_{2,p} = f(\mathcal{K}_{1,p})$.

To simplify certain proofs, we will also need the following statement.

Lemma 2.16. *Suppose $G = G_1 \times G_2$, $b = (b_1, b_2)$, $\mu = (\mu_1, \mu_2)$ and $\mathcal{K}_p = \mathcal{K}^1_p \times \mathcal{K}^2_p$. Then $X^{\mathcal{K}_p}_\mu(b) = X^{\mathcal{K}^1_p}_{\mu_1}(b_1) \times X^{\mathcal{K}^2_p}_{\mu_2}(b_2)$.*

Proof. This follows directly from the definition. $\square$

2.6. **Moduli spaces of p-adic shtukas.**

10. Recall from [SW20, §23] that to each (G, b, μ) and a closed subgroup $K \subseteq G(\mathbb{Q}_p)$, one can attach a locally spatial diamond $\mathrm{Sht}_{(G,b,\mu,K)}$ over $\mathrm{Spd}\,\breve{E}$, where $\breve{E} = \breve{\mathbb{Q}}_p \cdot E$ and E is the reflex field of $\boldsymbol{\mu}$, i.e. $\mathrm{Sht}_{(G,b,\mu,K)}$ is the moduli space of p-adic shtukas with level K.

This association is functorial in the tuple (G, b, μ, K), i.e. if $f : G \to H$ is a morphism of groups, we let $b_H := f(b)$, $\mu_H := f \circ \mu$ and we assume $f(K) \subseteq K_H$, then we have a morphism of diamonds

$$\mathrm{Sht}_{(G,b,\mu,K)} \to \mathrm{Sht}_{(H,b_H,\mu_H,K_H)}. \tag{2.23}$$

(1) When $H = G^{\mathrm{ab}}$, $f = \det : G \to G^{\mathrm{ab}}$ is the natural quotient map, and $K_H = \det(K)$, we let $b^{\mathrm{ab}} := \det(b)$, $\mu^{\mathrm{ab}} := \det \circ \mu$, $K^{\mathrm{ab}} := \det(K)$, and the morphism (2.23) in this case is called the "determinant map"

$$\det : \mathrm{Sht}_{(G,b,\mu,K)} \to \mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},K^{\mathrm{ab}})}. \tag{2.24}$$

(2) When $H = G$, $f = \mathrm{id}$, and the inclusion $K_1 \subseteq K_2$ is proper, we have a change-of-level-structures map:

$$\mathrm{Sht}_{(G,b,\mu,K_1)} \to \mathrm{Sht}_{(G,b,\mu,K_2)} \tag{2.25}$$

11. For parahoric levels K_p, $\mathrm{Sht}_{(G,b,\mu,K_p)}$ is the generic fiber of a canonical[11] integral model, which is a v-sheaf $\mathrm{Sht}_{\mu}^{\mathcal{K}_p}(b)$ over $\mathrm{Spd}\,\mathcal{O}_{\breve{E}}$ defined in [SW20, §25]. In [Gle22a, Theorem 2], the first author proved that $\mathrm{Sht}_{\mu}^{\mathcal{K}_p}(b)$ is a prekimberlite[12] (see [Gle22b, Definition 4.15]). Moreover, by [Gle22a, Proposition 2.30], its reduction (or its reduced special fiber in the sense of [Gle22b, §3.2]) can be identified with $X_{\mu}^{\mathcal{K}_p}(b)$. Furthermore, the formalism of kimberlites developed in [Gle22b] gives a continuous specialization map which turns out to be surjective (on the underlying topological spaces).

Theorem 2.17. [Gle22a, Theorem 2] *The pair* $(\mathrm{Sht}_{\mu}^{\mathcal{K}_p}(b), \mathrm{Sht}_{(G,b,\mu,K_p)})$ *is a rich smelted kimberlite*[13]*. Moreover,* $\mathrm{Sht}_{\mu}^{\mathcal{K}_p}(b)^{\mathrm{red}} = X_{\mu}^{\mathcal{K}_p}(b)$*. In particular, we have a surjective and continuous specialization map.*

$$\mathrm{sp} : |\mathrm{Sht}_{(G,b,\mu,K_p)}| \to |X_{\mu}^{\mathcal{K}_p}(b)|. \tag{2.26}$$

[11] canonical in the sense that $\mathrm{Sht}_{\mu}^{\mathcal{K}_p}(b)$ represents a moduli problem.

[12] Kimberlites [Gle22a, Definition 4.35] are analogues of formal schemes in the theory of v-sheaves. A prekimberlite is a weakening of this notion that still admit a specialization map [Gle22a, Proposition 4.14].

[13] The term "rich" refers to some technical finiteness assumption that ensures that the specialization map can be controlled by understanding the preimage of the closed points in the reduced special fiber.

12. Now we recall the v-sheaf-theoretic local model diagram of [Gle22a, Theorem 3]. It has the form

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "f"] & \\ (\mathrm{Sht}_\mu^{\mathcal{K}_p}(b) \times \mathrm{Spd}\,\mathcal{O}_F)^{\circledcirc}_x & & (\mathcal{M}_{\mathcal{K}_p,\mu} \times \mathrm{Spd}\,\mathcal{O}_F)^{\circledcirc}_y \end{tikzcd} \tag{2.27}$$

where $x \in X_\mu^{\mathcal{K}_p}(b)(k_F)$, $y \in \mathcal{A}_{\mathcal{K}_p,\mu}(k_F)$, and the maps f and g are $\widehat{L_W^+ G}$-torsors for a certain infinite-dimensional connected group v-sheaf $\widehat{L_W^+ G}$.

Using (2.27), one can show that the specialization map (2.26) induces a map $\pi_0(\mathrm{sp})$ on connected components. The following theorem is due to Lourenço and the first author.

Theorem 2.18 ([GL22])**.** *For any parahoric $K_p \subseteq G(\mathbb{Q}_p)$ and any field extension $\breve{E} \subseteq F \subseteq \mathbb{C}_p$, the map*

$$\pi_0(\mathrm{sp}) : \pi_0(\mathrm{Sht}_{(G,b,\mu,K_p)} \times \mathrm{Spd}\,F) \xrightarrow{\sim} \pi_0(X_\mu^{\mathcal{K}_p}(b)) \tag{2.28}$$

is bijective.

Proof. Recall that by [Gle22b, Lemma 4.55], whenever $(\mathfrak{X}, X)$ is a rich smelted kimberlite, to prove that

$$\pi_0(\mathrm{sp}) : \pi_0(X) \to \pi_0(\mathfrak{X}^{\mathrm{red}}) \tag{2.29}$$

is bijective, it suffices to prove that $(\mathfrak{X}, X)$ is unibranch[14] (in the sense of [Gle22b, Definition 4.52]), i.e. tubular neighborhoods are connected). By Theorem 2.17, $(\mathrm{Sht}_\mu^{\mathcal{K}_p}(b), \mathrm{Sht}_{(G,b,\mu,K_p)})$ is a rich smelted kimberlite, and thus it suffices to prove that $(\mathrm{Sht}_\mu^{\mathcal{K}_p}(b) \times \mathrm{Spd}\,\mathcal{O}_F, \mathrm{Sht}_{(G,b,\mu,K_p)} \times \mathrm{Spd}\,F)$ is unibranch, i.e. their tubular neighborhoods are connected.

By (2.27), it suffices to prove that the tubular neighborhoods of $(\mathcal{M}_{\mathcal{K}_p,\mu} \times \mathrm{Spd}\,\mathcal{O}_F, \mathrm{Gr}_\mu \times \mathrm{Spd}\,F)$ are connected, which is the content of [GL22, Theorem 1.3]. ☐

With a similar argument as in Lemma 2.15, one can prove that the formation of $\mathrm{Sht}_\mu^{\mathcal{K}_p}(b)$ is also functorial in tuples $(G, b, \mu, \mathcal{K}_p)$.

Lemma 2.19. *Let $f : G_1 \to G_2$ be a group homomorphism such that $b_2 = f(b_1)$, $\mu_2 = f \circ \mu_1$ and $f(\mathcal{K}_{1,p}) \subseteq \mathcal{K}_{2,p}$. Then we have a map*

$$\mathrm{Sht}_{\mu_1}^{\mathcal{K}_{1,p}}(b_1) \to \mathrm{Sht}_{\mu_2}^{\mathcal{K}_{2,p}}(b_2) \tag{2.30}$$

of v-sheaves. Moreover, taking the reduction functor [Gle22b, §3.2] *of map* (2.30) *induces the map $X_{\mu_1}^{\mathcal{K}_{1,p}}(b_1) \to X_{\mu_2}^{\mathcal{K}_{2,p}}(b_2)$ of Lemma 2.14.*

[14] The definition of unibranchness, or alternatively *topological normality*, for smelted kimberlites is inspired by a useful criterion for the unibranchness of a scheme (see [AGLR22, Proposition 2.38]).

Proof. The first statement follows from the definition of $\mathrm{Sht}_{\mu_1}^{\mathcal{K}_{1,p}}(b_1)$ (see for example [Gle22a, Definition 2.26]) and from Lemma 2.15. By [Gle22a, Proposition 2.30], we have the identity $X_{\mu_i}^{\mathcal{K}_{i,p}}(b_i) = \mathrm{Sht}_{\mu_i}^{\mathcal{K}_{i,p}}(b_i)^{\mathrm{red}}$, with $i \in \{1, 2\}$, where the right-hand side is the reduced special fiber (more precisely, it is the image under the reduction functor defined *loc.cit.*). $\square$

As a special case, if we fix a datum (G, b, μ) and two parahorics $K_1 \subseteq K_2$ of $G(\mathbb{Q}_p)$, we have a map

$$\mathrm{Sht}_{\mu}^{\mathcal{K}_1}(b) \to \mathrm{Sht}_{\mu}^{\mathcal{K}_2}(b) \tag{2.31}$$

of v-sheaves. On the generic fiber, the map (2.31) gives the change-of-level-structures map of (2.25). After applying the reduction functor to the map (2.31), we recover the map $X_{\mu}^{\mathcal{K}_1}(b) \to X_{\mu}^{\mathcal{K}_2}(b)$ from Lemma 2.14 applied to scenario (1).

2.7. **The Grothendieck–Messing period map.** Recall that given a triple (G, b, μ), there is a quasi-pro-étale *Grothendieck–Messing period morphism* (see for example [SW20, §23]):

$$\pi_{\mathrm{GM}} : \mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, \mathbb{C}_p \to \mathrm{Gr}_{\mu} \times \mathrm{Spd}\, \mathbb{C}_p. \tag{2.32}$$

Now, the b-admissible locus $\mathrm{Gr}_{\mu}^{b} \subseteq \mathrm{Gr}_{\mu}$, can be defined as the image of π_{GM}. Note that $\mathrm{Gr}_{\mu}^{b} \times \mathrm{Spd}\, \mathbb{C}_p \subseteq \mathrm{Gr}_{\mu} \times \mathrm{Spd}\, \mathbb{C}_p$ is a dense open subset. Moreover, there is a (universal) $\underline{G(\mathbb{Q}_p)}$-local system $\mathbb{L}_b$ over Gr_{μ}^{b}, such that for each finite extension K over $\breve{E}$ and $x \in \mathrm{Gr}_{\mu}^{b}(K)$, $x^*\mathbb{L}_b$ is a crystalline representation associated to the isocrystal with G-structure defined by b (for more details see for example [Gle21, §2.2-2.4]). The map in (2.32) can be then constructed as the geometric $\underline{G(\mathbb{Q}_p)}$-torsor attached to $\mathbb{L}_b$, i.e. $\mathrm{Sht}_{(G,b,\mu,\infty)}$ is the moduli space of trivializations of $\mathbb{L}_b$.

The following Conjecture 2.20 is generally expected to hold, and discussed carefully in a later section §5.1.

Conjecture 2.20. *Let (G, b, μ) be a p-adic shtuka datum with $\mathbf{b} \in B(G, \boldsymbol{\mu})$.*[15] *The b-admissible locus $\mathrm{Gr}_{\mu}^{b} \times \mathrm{Spd}\, \mathbb{C}_p$ is connected.*

Let $\mathrm{Fl}_{\mu}^{\mathrm{ad}} \subseteq \mathrm{Fl}_{\mu}$ denote the weakly admissible (or equivalently, semistable) locus inside the flag variety [DOR10, §5], and let $\mathrm{Gr}_{\mu}^{\circ,b} := \mathrm{Gr}_{\mu}^{\circ} \cap \mathrm{Gr}_{\mu}^{b}$. By (2.21), we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Gr}_{\mu}^{\circ,b} \arrow[r] \arrow[d, "\mathrm{BB}"'] & \mathrm{Gr}_{\mu}^{\circ} \arrow[d, "\mathrm{BB}"] \\ \mathrm{Fl}_{\mu}^{\mathrm{ad}} \arrow[r] & \mathrm{Fl}_{\mu}. \end{tikzcd} \tag{2.33}$$

[15] without requiring $(\mathbf{b}, \boldsymbol{\mu})$ to be HN-irreducible

Moreover, by [CF00], we have a bijection $\mathrm{Gr}_\mu^{\circ,b}(F) \cong \mathrm{Fl}_\mu^{\mathrm{ad}}(F)$ for all finite extensions F of $\breve{E}$ (see [Vie21, Theorem 5.2]).

We will also make use of the Beauville–Laszlo uniformization map obtained by modifying the G-bundle $\mathcal{E}_b$:

$$\mathcal{BL}_b : \mathrm{Gr}_\mu^\circ \to \mathrm{Bun}_G . \tag{2.34}$$

Note that $\mathrm{Gr}_\mu^{\circ,b} = \mathcal{BL}_b^{-1}(\mathrm{Bun}_G^1)$.

2.8. Ad-isomorphisms and z-extensions.

Definition 2.21. [Kot85, §4.8] A morphism $f : G \to H$ is called an *ad-isomorphism* if f sends the center of G to the center of H and induces an isomorphism of adjoint groups.

An important example of ad-isomorphisms are z-extensions.

Definition 2.22. [Kot82, §1] A map of connected reductive groups $f : G' \to G$ is a *z-extension* if: f is surjective, $Z = \mathrm{Ker}(f)$ is central in G', Z is isomorphic to a product of tori of the form $\mathrm{Res}_{F_i/\mathbb{Q}_p} \mathbb{G}_m$ for some finite extensions $F_i \subseteq \overline{\mathbb{Q}}_p$, and G' has simply connected derived subgroup.

Lemma 2.23. *Let $f : \tilde{G} \to G$ be a z-extension and $\mathbf{b} \in B(G, \boldsymbol{\mu})$.*
(1) There exist a conjugacy class of cocharacters $\tilde{\boldsymbol{\mu}}$ and an element $\tilde{\mathbf{b}} \in B(\tilde{G}, \tilde{\boldsymbol{\mu}})$ which, under the map $B(\tilde{G}, \tilde{\boldsymbol{\mu}}) \to B(G, \boldsymbol{\mu})$, map to $\boldsymbol{\mu}$ and $\mathbf{b}$, respectively.
(2) $c_{\tilde{b},\tilde{\mu}}\pi_1(\tilde{G})_I^\varphi \to c_{b,\mu}\pi_1(G)_I^\varphi$ is surjective.

Proof. (1) Let $T \subseteq G$ be a maximal torus and $\tilde{T} \subseteq \tilde{G}$ its preimage under f. Let $Z = \mathrm{Ker}(f)$. We have an exact sequence

$$0 \to Z \to \tilde{T} \to T \to 0 \tag{2.35}$$

Since Z is a torus, we have an exact sequence:

$$0 \to X_*(Z) \to X_*(\tilde{T}) \to X_*(T) \to 0 \tag{2.36}$$

In particular, we can lift $\boldsymbol{\mu}$ to an arbitrary $\tilde{\boldsymbol{\mu}} \in X_*(\tilde{T})$. To lift $\tilde{\mathbf{b}}$ compatibly, it suffices to recall from [Kot97, (6.5.1)] that

$$B(G, \boldsymbol{\mu}) \cong B(G^{\mathrm{ad}}, \boldsymbol{\mu}^{\mathrm{ad}}) \cong B(\tilde{G}, \tilde{\boldsymbol{\mu}}). \tag{2.37}$$

(2) Recall that the map $G(\mathbb{Q}_p) \to \pi_1(G)_I^\varphi$ is surjective (see for example [Zho20, Lemma 5.18]). Indeed, this follows from the exact sequence

$$0 \to \mathcal{T}(\breve{\mathbb{Z}}_p) \to T(\breve{\mathbb{Q}}_p) \to \pi_1(G)_I \to 0 \tag{2.38}$$

and the group cohomology vanishing $H^1(\mathbb{Z}, \mathcal{T}(\breve{\mathbb{Z}}_p)) = 0$, where $\mathcal{T}$ is the unique parahoric of T and the $\mathbb{Z}$-action on $\mathcal{T}(\breve{\mathbb{Z}}_p)$ is given by the Frobenius

φ. Consider the following commutative diagram:

$$\begin{tikzcd} \tilde{G}(\mathbb{Q}_p) \arrow[r] \arrow[d] & \pi_1(\tilde{G})_I^{\varphi} \arrow[d] \\ G(\mathbb{Q}_p) \arrow[r] & \pi_1(G)_I^{\varphi} \end{tikzcd} \tag{2.39}$$

The horizontal arrows in (2.39) are surjective. Since Z is an induced torus, $H^1_{\text{ét}}(\operatorname{Spec}\mathbb{Q}_p, Z) = 0$. Thus by the long exact sequence associated to

$$0 \to Z \to \tilde{G} \to G \to 0, \tag{2.40}$$

the map $\tilde{G}(\mathbb{Q}_p) \to G(\mathbb{Q}_p)$ is surjective. Therefore $\pi_1(\tilde{G})_I^{\varphi} \to \pi_1(G)_I^{\varphi}$ is surjective. Finally, since $\tilde{\mathbf{b}}$ and $\tilde{\boldsymbol{\mu}}$ map to $\mathbf{b}$ and $\boldsymbol{\mu}$, the coset $c_{\tilde{b},\tilde{\mu}}\pi_1(\tilde{G})_I^{\varphi}$ also maps to the coset $c_{b,\mu}\pi_1(\tilde{G})_I^{\varphi}$. □

Assume that f is an ad-isomorphism for the rest of this subsection. Let $b_H := f(b)$ and $\mu_H := f \circ \mu$. Let $\mathcal{K}_p^H$ denote the unique parahoric of H that corresponds to the same point in the Bruhat–Tits building as $\mathcal{K}_p$.

Proposition 2.24. *The following diagram is Cartesian:*

$$\begin{tikzcd} \pi_0(X_{\mu}^{\mathcal{K}_p}(b)) \arrow[r] \arrow[d] & c_{b,\mu}\pi_1(G)_I^{\varphi} \arrow[d] \\ \pi_0(X_{\mu}^{\mathcal{K}_p^H}(b)) \arrow[r] & c_{b_H,\mu_H}\pi_1(H)_I^{\varphi} \end{tikzcd} \tag{2.41}$$

Proof. This is a consequence of [PR22, Lemma 5.4.2], which is a generalization of [CKV15, Corollary 2.4.2] for arbitrary parahorics. □

2.9. **Mumford–Tate groups of crystalline representations.** We will use the theory of *crystalline representations with G-structures* (see for example [DOR10]). Let Rep_G be the category of algebraic representations of G in $\mathbb{Q}_p$-vector spaces. Let Isoc be the category of isocrystals.

Fix a finite extension K of $\breve{\mathbb{Q}}_p$. Let $\mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K}$ be the category of crystalline representations of Γ_K on finite-dimensional $\mathbb{Q}_p$-vector spaces. Let $\omega : \mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K} \to \mathrm{Vec}_{\mathbb{Q}_p}$ be the forgetful fibre functor. Let $\mathrm{IsocFil}_{K/\breve{\mathbb{Q}}_p}$ be the category of filtered isocrystals whose objects are pairs of an isocrystal N and a decreasing filtration of $N \otimes K$. Furthermore, let $\mathrm{IsocFil}^{\mathrm{ad}}_{K/\breve{\mathbb{Q}}_p}$ be Fontaine's subcategory of weakly admissible filtered isocrystals [Fon94]. This is a $\mathbb{Q}_p$-linear Tannakian category, which is equivalent to $\mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K}$ through Fontaine's functor V_{cris} [CF00].

13. Fix a pair (b, μ^{η}) with $b \in G(\breve{\mathbb{Q}}_p)$ and $\mu^{\eta} : \mathbb{G}_m \to G_K$ a group homomorphism over K. This defines a $\otimes$-functor

$$\mathcal{G}_{(b,\mu^{\eta})} : \mathrm{Rep}_G \to \mathrm{IsocFil}_{K/\breve{\mathbb{Q}}_p} \tag{2.42}$$

sending $\rho : G \to \mathrm{GL}(V)$ to the filtered isocrystal $(V \otimes \breve{\mathbb{Q}}_p, \rho(b)\sigma, \mathrm{Fil}^{\bullet}_{\mu^\eta} V \otimes K)$, where the filtration on $V \otimes K$ is the one induced by μ^η. The pair is called *admissible* [RZ96, Definition 1.18], if the image of $\mathcal{G}_{(b,\mu^\eta)}$ lies in $\mathrm{IsocFil}^{\mathrm{ad}}_{K/\breve{\mathbb{Q}}_p}$. Moreover, when $\mathbf{b} \in B(G, \boldsymbol{\mu})$, $V_{\mathrm{cris}} \circ \mathcal{G}_{(b,\mu^\eta)}$ defines a conjugacy class of crystalline representations $\xi_{(b,\mu^\eta)} : \Gamma_K \to G(\mathbb{Q}_p)$ [DOR10, Proposition 11.4.3].

Definition 2.25. With notation as above, let $\mathrm{MT}_{(b,\mu^\eta)}$ denote the identity component of the Zariski closure of $\xi_{(b,\mu^\eta)}(\Gamma_K)$ in $G(\mathbb{Q}_p)$. This is the *Mumford–Tate group* attached to (b, μ^η).

Lemma 2.26. *(*[Ser79, Théorème 1]*,*[Sen73, §4, Théorème 1]*,* [Che14, Proposition 3.2.1]*) The image of $\xi_{(b,\mu^\eta)}$ contains an open subgroup of $\mathrm{MT}_{(b,\mu^\eta)}$.*

14. As in [Che14, §3], we let $\mathscr{T}^{\mathrm{cris}}_{(b,\mu^\eta)} := \mathcal{G}_{(b,\mu^\eta)}(\mathrm{Rep}_G)$ and $\mathscr{T}_{(b,\mu^\eta)} := V_{\mathrm{cris}} \circ \mathcal{G}_{(b,\mu^\eta)}(\mathrm{Rep}_G)$ be the images of Rep_G. Then $\mathrm{MT}_{(b,\mu^\eta)}$ is the Tannakian group for the fiber functor $\omega : \mathscr{T}_{(b,\mu^\eta)} \to \mathrm{Vec}_{\mathbb{Q}_p}$ by [Che14, Proposition 3.2.3].

In [Che14, §3], there is a fiber functor $\omega_s : \mathscr{T}^{\mathrm{cris}}_{(b,\mu^\eta)} \to \mathrm{Vec}_{\mathbb{Q}_{p^s}}$ for s sufficiently large[16], with Tannakian group $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)} := \mathrm{Aut}^{\otimes}\omega_s$ as in [Che14, Définition 3.3.1]. When b is s-decent (see Definition 2.2), there is a canonical embedding $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)} \subseteq G_{\mathbb{Q}_{p^s}}$ [Che14, Lemme 3.3.2]. Moreover, $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$ and $\mathrm{MT}_{(b,\mu^\eta)} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^s}$ are pure inner forms of each other [Che14, Proposition 3.3.3]. Both claims follow immediately using Tannakian formalism. In particular, to prove that $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}, it suffices to prove $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$ contains $G^{\mathrm{der}}_{\mathbb{Q}_{p^s}}$ (since G^{der} is normal).

15. In fact, there is a more concrete description of $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$ given as follows.

Let $(V, \rho) \in \mathrm{Rep}_G$. The μ^η-filtration of V_K induces a degree function

$$\deg_{\mu^\eta} : V \setminus \{0\} \to \mathbb{Z}, \tag{2.43}$$

where $\deg_{\mu^\eta}(v) = i$ if $v \in \mathrm{Fil}^i V \setminus \mathrm{Fil}^{i+1} V$. We shall consider a subset $V^{s,k}_{(b,\mu^\eta)} \subseteq V \otimes \mathbb{Q}_{p^s}$ of elements that satisfy a certain "Newton equation" (2.44) and a certain "Hodge equation" (2.45) with respect to k.

Let $T^{s\cdot\nu_b}_\rho : V \otimes \mathbb{Q}_{p^s} \to V \otimes \mathbb{Q}_{p^s}$ be the operator with formula

$$T^{s\cdot\nu_b}_\rho := \rho \circ [s \cdot \nu_b](p). \tag{2.44}$$

Consider also the function $d^s_{\rho,\mu^\eta} : V \otimes \mathbb{Q}_{p^s} \setminus \{0\} \to \mathbb{Z}$ where

$$d^s_{\rho,\mu^\eta}(v) = \sum_{i=0}^{s-1} \deg_{\mu^\eta}([\rho(b)\varphi]^i(v)). \tag{2.45}$$

We consider the following subset of $V \otimes \mathbb{Q}_{p^s}$ given by

[16] Note that our notation ω_s differs from the notations *loc.cit.*, where the notation $\omega^{\mathrm{cris},s}_{b,\mu}$ is used instead.

$$V^{s,k}_{(b,\mu^\eta)} := \{v \in V \otimes \mathbb{Q}_{p^s} \mid T^{s\cdot\nu_b}_\rho(v) = p^k v,\ d^s_{\rho,\mu^\eta}(v) = k\} \tag{2.46}$$

By [Che14, Proposition 3.3.6], $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$ consists of those elements $g \in G_{\mathbb{Q}_{p^s}}$ such that: for any $(V,\rho) \in \mathrm{Rep}_G$ and any $k \in \mathbb{Z}$, all of the elements $v \in V^{s,k}_{(b,\mu^\eta)}$ are eigenvectors of $\rho(g)$. In particular, to prove $G^{\mathrm{der}}_{\mathbb{Q}_{p^s}} \subseteq \mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$, it suffices to prove that $G^{\mathrm{der}}_{\mathbb{Q}_{p^s}}$ acts trivially on $V^{s,k}_{(b,\mu^\eta)}$ for all V and k.

2.10. **Generic filtrations.**

16. As in [Che14, §4], we give a representation-theoretic formula for d^s_{ρ,μ^η} when μ^η is generic. In our case, G is not assumed to be neither unramified nor quasisplit.

We first recall some generalities, which we will apply later to $G_{\mathbb{Q}_{p^s}}$ for s-sufficiently large such that $G_{\mathbb{Q}_{p^s}}$ is quasisplit. Until further notice, K will denote an arbitrary field of characteristic 0, G a quasisplit reductive group over K, and $\boldsymbol{\mu}$ a conjugacy class of group homomorphisms $\mu : \mathbb{G}_m \to G_{\bar{K}}$. Let E/K be the reflex field of $\boldsymbol{\mu}$. Since G is quasisplit, we can choose a representative $\mu \in \boldsymbol{\mu}$ defined over E such that it is dominant for a choice of K-rational Borel $B \subseteq G$. To this data, we can associate a flag variety $\mathrm{Fl}_\mu := G_E/P_\mu$ over $\mathrm{Spec}\,(E)$ as in (2.21). It parametrizes filtrations of Rep_G of type $\boldsymbol{\mu}$. Given a field extension K'/K, $x \in \mathrm{Fl}_\mu(K')$ and $(V,\rho) \in \mathrm{Rep}_G$, we obtain a filtration $\mathrm{Fil}^\bullet_x V_{K'}$ as in [Che14, Définition 4.1.1].

Definition 2.27 ([Che14, Définition 4.2.1])**.** With the setup as above, let

$$\overline{\mathrm{Fil}}^\bullet_\mu V_E := \left(\bigcap_{x\in\mathrm{Fl}_\mu(E)} \mathrm{Fil}^\bullet_x V_E\right) \tag{2.47}$$

$$\overline{\mathrm{Fil}}^\bullet_{\boldsymbol{\mu}} V := V \cap \left(\bigcap_{x\in\mathrm{Fl}_\mu((E)} \mathrm{Fil}^\bullet_x V_E\right). \tag{2.48}$$

We refer to (2.47) (resp. (2.48)) as the *generic filtration* of V_E (resp. V) attached to μ (resp. $\boldsymbol{\mu}$).

17. Each step of $\overline{\mathrm{Fil}}^i_{\boldsymbol{\mu}} V$ is a subrepresentation of V. Moreover,

$$\overline{\mathrm{Fil}}^\bullet_{\boldsymbol{\mu}} V = V \cap \left(\bigcap_{g\in G(E)} \rho(g)\,\mathrm{Fil}^\bullet_\mu V_E\right). \tag{2.49}$$

This filtration $\overline{\mathrm{Fil}}^i_{\boldsymbol{\mu}} V$ gives rise to a degree function $\overline{\deg}_{\boldsymbol{\mu}} : V \setminus \{0\} \to \mathbb{Z}$ which can be computed as:

$$\overline{\deg}_{\boldsymbol{\mu}}(v) = \inf_{g\in G(E)} \deg_\mu(\rho(g)\cdot v). \tag{2.50}$$

Let K' be an arbitrary extension of K.

Definition 2.28. We say that a map $\operatorname{Spec}(K') \to \operatorname{Fl}_\mu$ is *generic* if, at the level of topological spaces $|\operatorname{Spec}(K')| \to |\operatorname{Fl}_\mu|$, the image of the unique point on the left is the generic point of Fl_μ.

The following statement relates $\overline{\operatorname{Fil}}^\bullet_{\boldsymbol{\mu}} V$ (see (2.48) or (2.49)) to the generic points of Fl_μ in the sense of Definition 2.28.

Proposition 2.29 ([Che14, Lemme 4.2.2])**.** *Let $\mu^\eta : \operatorname{Spec}(K') \to \operatorname{Fl}_\mu$ be generic (in the sense of Definition 2.28). Then for all $i \in \mathbb{Z}$, we have*

$$\overline{\operatorname{Fil}}^i_{\boldsymbol{\mu}} V = V \cap \operatorname{Fil}^i_{\mu^\eta} V_{K'}, \tag{2.51}$$

where the inclusion $V \subseteq V \otimes_K E \subseteq V \otimes_K K'$ is the natural one.

Proof. The following proof is in [Che14, 4.2.2]. We recall the argument for the convenience of the reader. Note that we do not assume G split over K, which is the running assumptions in *loc.cit.*

Let $\widetilde{\mathcal{Y}}_\mu$ be the the universal P_μ-bundle over $\operatorname{Fl}_\mu = G_E/P_\mu$ coming from the natural map to $[*/P_\mu]$. Consider the vector bundle $\mathcal{E} := \widetilde{\mathcal{Y}}_\mu \times_{P_{\mu,\rho}} V$, with a filtration

$$\cdots \supseteq \operatorname{Fil}^0 \mathcal{E} \supseteq \operatorname{Fil}^1 \mathcal{E} \supseteq \cdots \supseteq \operatorname{Fil}^n \mathcal{E} \supseteq \dots$$

of locally free locally direct factors of the form $\widetilde{\mathcal{Y}}_\mu \times_{P_{\mu,\rho}} \operatorname{Fil}^\bullet V$, where $\operatorname{Fil}^\bullet V$ is the natural filtration of V by subrepresentations of P_μ.

We may regard elements $v \in V$ as global sections of $\mathcal{E}$, and we have that

$$v \in \operatorname{Fil}^i_x V \Leftrightarrow v \in \ker\left(\Gamma(\operatorname{Fl}_\mu, \mathcal{E}/\operatorname{Fil}^i \mathcal{E}) \to \Gamma(\operatorname{Spec} \kappa(x), \mathcal{E}/\operatorname{Fil}^i \mathcal{E})\right).$$

The vanishing locus of such an element is a Zariski closed subset, and it contains the generic point if and only if it contains all the E-rational points. Thus $\overline{\operatorname{Fil}}^i_\mu V = V \cap \operatorname{Fil}^i_{\mu^\eta} V_{K'}$. □

18. We need a more easily computable description of $\overline{\operatorname{Fil}}^\bullet_{\boldsymbol{\mu}} V$. In [Che14, Proposition 4.3.2], there is such a description assuming that G is split over K. We now prove a generalization in the quasisplit case.

Let Γ_K denote the Galois group of K. We fix K-rational tori $S \subseteq T \subseteq B \subseteq G$ where S is maximally split and T is the centralizer of S. Recall that, by combining the theory of highest weights and Galois theory, one can classify all irreducible representations of a quasisplit group by the Galois orbits $\mathcal{O} \subseteq X_*(T)^+$ of dominant weights. Given $\lambda \in X_*(T)^+$, let $\mathcal{O}_\lambda := \Gamma_K \cdot \lambda$ denote its Galois orbit. We also consider $\mathcal{O}^E_\lambda := \Gamma_E \cdot \lambda$. Given a Γ_K-Galois orbit (resp. Γ_E-Galois orbit) $\mathcal{O} \subseteq X_*(T)^+$ (resp. $\mathcal{O}^E \subseteq X_*(T)^+$), let $V_{\mathcal{O}}$ (resp. $V_{\mathcal{O}^E}$) denote the $\mathcal{O}$-isotypic (resp. $\mathcal{O}^E$-isotypic) direct summand of V (resp. V_E). We have

$$V_{\mathcal{O}} \otimes_K \bar{K} = \bigoplus_{\lambda \in \mathcal{O}} V^\lambda_{\bar{K}}. \tag{2.52}$$

$$V_{\mathcal{O}^E} \otimes_E \bar{K} = \bigoplus_{\lambda \in \mathcal{O}^E} V^\lambda_{\bar{K}}. \tag{2.53}$$

Where $V_{\bar{K}}^{\lambda}$ is the λ-isotypic part of $V_{\bar{K}}$. Let $\underline{\mathcal{O}} \in (X_*(T)_{\mathbb{Q}}^{+})^{\Gamma_K}$ (resp. $(X_*(T)_{\mathbb{Q}}^{+})^{\Gamma_E}$) be given by $\underline{\mathcal{O}} := \frac{1}{|\mathcal{O}|}\sum_{\lambda\in\mathcal{O}} \lambda$. When $\mathcal{O}_\lambda = \Gamma_K \cdot \lambda$, we have

$$\underline{\mathcal{O}_\lambda} = \frac{1}{[\Gamma_K : \Gamma_\lambda]} \sum_{\gamma\in\Gamma_K/\Gamma_\lambda} \gamma(\lambda) \tag{2.54}$$

Analogously, we have $\underline{\mathcal{O}_\lambda^E} = \frac{1}{[\Gamma_E:\Gamma_\lambda]} \sum_{\gamma\in\Gamma_E/\Gamma_\lambda} \gamma(\lambda))$. Let $\mathcal{W}$ denote the absolute Weyl group of G. Let $w_0 \in \mathcal{W}$ be the longest element, which is Γ_K-invariant.

Proposition 2.30. *Let the setup be as above. For any $(V,\rho) \in \mathrm{Rep}_G$, the generic filtration attached to μ is given by the formula:*

$$\overline{\mathrm{Fil}}_{\boldsymbol{\mu}}^{k} V = \bigoplus_{\substack{\lambda\in X_*(T)^+ \\ \langle \underline{\mathcal{O}_{\tau(\lambda)}^E}, w_0.\mu\rangle \geq k \\ \tau\in\mathrm{Gal}(E/K)}} V_{\mathcal{O}_\lambda} \tag{2.55}$$

Proof. Since $\overline{\mathrm{Fil}}_{\boldsymbol{\mu}}^{k} V$ consists of subrepresentations, it suffices to show that

$$V_{\mathcal{O}_\lambda} \subseteq \overline{\mathrm{Fil}}_{\boldsymbol{\mu}}^{k} V \iff k \leq \langle \underline{\mathcal{O}_{\tau(\lambda)}^E}, w_0.\mu\rangle \quad \forall \tau \in \mathrm{Gal}(E/K). \tag{2.56}$$

Let us first prove "$\Longrightarrow$". Let $V = \bigoplus_{\sigma\in\mathrm{Irrep}(T)} V_\sigma$ be the decomposition of $\rho|_T$. Over an algebraic closure, each V_σ decomposes as $V_\sigma = \bigoplus_{\chi'\in\mathcal{O}_\chi} V_{\chi'}$ for some $\chi \in X^*(T)$.

Observe that $\overline{\mathrm{Fil}}_{\boldsymbol{\mu}}^{k} V \subseteq \mathrm{Fil}_{\tau(\mu)}^{k} V_E$ for all $\tau \in \mathrm{Gal}(E/K)$, and by definition we have

$$\mathrm{Fil}_{\tau(\mu)}^{k} V_{\bar{K}} = \bigoplus_{\substack{\langle\chi,\tau(\mu)\rangle\geq k, \\ \chi\in X^*(T)}} V_{\bar{K},\chi}. \tag{2.57}$$

In particular, the anti-dominant weights appearing in $V_{\mathcal{O}_\lambda}$ pair with $\tau(\mu)$ to a number greater than or equal to k. In other words, $k \leq \langle w_0.\xi, \tau(\mu)\rangle$ for $w_0.\xi \in \mathcal{O}_{w_0.\lambda}$, but then pairing $\tau(\mu)$ with their Γ_E-average $w_0.\underline{\mathcal{O}_\lambda^E}$ will still be greater than or equal to k, i.e. $k \leq \langle w_0.\underline{\mathcal{O}_\lambda^E}, \tau(\mu)\rangle = \langle \underline{\mathcal{O}_{\tau(\lambda)}^E}, w_0.\mu\rangle$.

Thus

$$\overline{\mathrm{Fil}}_{\boldsymbol{\mu}}^{k} V \subset \bigoplus_{\substack{\lambda\in X^*(T)^+ \\ \langle \underline{\mathcal{O}_{\tau(\lambda)}^E}, w_0.\mu\rangle \geq k \\ \tau\in\mathrm{Gal}(E/K)}} V_{\mathcal{O}_\lambda}. \tag{2.58}$$

Let us now prove "$\Longleftarrow$". Suppose $k \leq \langle w_0.\underline{\mathcal{O}_\lambda^E}, \tau(\mu)\rangle$ for all $\tau \in \mathrm{Gal}(E/K)$, this implies that for at least one $w_0.\xi \in \mathcal{O}_{w_0.\lambda}^E$, we have $k \leq \langle w_0.\xi, \tau(\mu)\rangle$. Since $\langle\cdot,\cdot\rangle$ is Γ_E-equivariant, $k \leq \langle w_0.\xi, \tau(\mu)\rangle$ for all $w_0.\xi \in \mathcal{O}_{w_0.\lambda}^E$. We can view $w_0.\xi$ as a cocharacter of T and $V_{w_0.\xi} \subseteq \mathrm{Fil}^k_{\tau(\mu)} V$. Consider $W_\xi := V^\xi_{\bar{K}}$ the isotypic part of $V_{\bar{K}}$ associated to the highest weight representation of ξ on an algebraic closure of K. If χ is a weight appearing in W_ξ, then $k \leq \langle w_0.\xi, \tau(\mu)\rangle \leq \langle \chi, \tau(\mu)\rangle$, and thus $W_\xi \subseteq \mathrm{Fil}^k_{\tau(\mu)} V_{\bar{K}}$. In particular,

$$W_E := \Big(\bigoplus_{w_0.\xi \in \mathcal{O}^E_{w_0\lambda}} W_\xi\Big)^{\Gamma_E} \tag{2.59}$$

is a subrepresentation of $\mathrm{Fil}^k_{\tau(\mu)} V_E$ defined over E. Thus $W_E \subseteq \overline{\mathrm{Fil}}^k_{\tau(\mu)} V_E$ for all $\tau \in \mathrm{Gal}(E/K)$. Then $W := \bigoplus_{\tau \in \mathrm{Gal}(E/K)} \tau(W_E)$ is contained in

$$\bigcap_{\tau \in \mathrm{Gal}(E/K)} \overline{\mathrm{Fil}}^k_{\tau(\mu)} V_E, \tag{2.60}$$

and the $\mathrm{Gal}(E/K)$-fixed points of W are contained in

$$\overline{\mathrm{Fil}}^k_\mu V = V \cap \bigcap_{\tau \in \mathrm{Gal}(E/K)} \overline{\mathrm{Fil}}^k_{\tau(\mu)} V_E.$$

But $W^{\mathrm{Gal}(E/K)} = V_{\mathcal{O}_\lambda}$, proving the claim.

□

3. Generic Mumford–Tate groups

3.1. **Mumford–Tate group computations.** The goal of this section is to prove Theorem 3.1 (or Theorem 1.17 in the introduction).

Let G be a reductive group over $\mathbb{Q}_p$. Let K be a finite extension of $\breve{\mathbb{Q}}_p$.

Let $b \in G(\breve{\mathbb{Q}}_p)$ be decent (Definition 2.2) and $\mu^\eta : \mathbb{G}_m \to G_K$ be generic (Definition 2.28) with $\mu^\eta \in \boldsymbol{\mu}$. As before, let $\mu \in X_*(T)^+$ be the unique B-dominant cocharacter of $\boldsymbol{\mu}$.

Theorem 3.1. *Suppose that b is decent, μ^η is generic and $\mathbf{b} \in B(G, \boldsymbol{\mu})$. The following hold:*

1. *(b, μ^η) is admissible.*
2. *If $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, then $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}.*

Proof. We fix s large enough so that b is s-decent, G is quasisplit over $\mathbb{Q}_{p^s}$ and splits over a totally ramified extension of $\mathbb{Q}_{p^s}$ that we denote by L. Recall that replacing b by $g^{-1}b\varphi(g)$ and μ^η by $g^{-1}\mu^\eta g$ gives isomorphic fiber functors $\mathcal{G}_{(b,\mu^\eta)}$ (see (2.42)). Moreover, via this kind of replacement, we can arrange that $\nu_b = \boldsymbol{\nu}_{\mathbf{b}}$ as in 4. Note that this replacement preserves genericity of μ^η.

(1) The argument in [Che14, Théorème 5.0.6.(1)] goes through in our setting. Indeed, the only part in the proof *loc.cit.* using that G is unramified is to justify that $\mathrm{Fl}^{\mathrm{ad}}_\mu \neq \emptyset$ whenever $\mathbf{b} \in B(G, \boldsymbol{\mu})$, but this is true by [DOR10, Theorem 9.5.10] in full generality.

(2) Let $(V,\rho)\in \mathrm{Rep}_G$ and let $v\in V^{s,k}_{(b,\mu^\eta)}$ as in (2.46). By 15, it suffices to show that $\rho(g)v=v$ for all $g\in G^{\mathrm{der}}_{\mathbb{Q}_{p^s}}$. Over L, we can write $v=\sum_{\lambda\in\Lambda_v} v_\lambda$ where $\Lambda_v\subseteq X^*(T)^+$, $v_\lambda\in V^\lambda$ and $v_\lambda\neq 0$. Since v is defined over $\mathbb{Q}_{p^s}$, we have $\gamma(v_\lambda)=v_{\gamma(\lambda)}$ for $\gamma\in\Gamma_{\mathbb{Q}_{p^s}}$.

Given $\mathcal{O}\in \mathrm{Irrep}_{G_{\mathbb{Q}_{p^s}}}$, let $v_{\mathcal{O}}=\sum_{\lambda\in\mathcal{O}\subseteq\Lambda_v} v_\lambda$. We have $v_{\mathcal{O}}\in V_{\mathbb{Q}_{p^s}}$. By Proposition 2.30 and Proposition 2.29, we can write

$$
\begin{aligned}
\deg_{\mu^\eta}(v) &= \overline{\deg}_{\boldsymbol{\mu}}(v) && (3.1)\\
&= \inf\nolimits_{\lambda\in\Lambda_v}\overline{\deg}_{\boldsymbol{\mu}}(v_{\mathcal{O}_\lambda}) && (3.2)\\
&\leq \overline{\deg}_{\boldsymbol{\mu}}(v_{\mathcal{O}_\lambda}) && (3.3)\\
&= \inf\nolimits_{\tau\in\mathrm{Gal}(E/K)}\langle \underline{\mathcal{O}^E_{\tau(\lambda)}}, w_0\cdot\mu\rangle && (3.4)\\
&\leq \langle \underline{\mathcal{O}_\lambda}, w_0\cdot\mu\rangle && (3.5)\\
&= \langle w_0\cdot\lambda, \underline{\mu}\rangle, && (3.6)
\end{aligned}
$$

Here (3.1) follows from Proposition 2.29. Since each step of $\overline{\mathrm{Fil}}^\bullet_{\boldsymbol{\mu}}V$ is a subrepresentation of V, in order for $v\in\overline{\mathrm{Fil}}^k_{\boldsymbol{\mu}}V$, each $v_{\mathcal{O}_\lambda}$ has to be in $\overline{\mathrm{Fil}}^\bullet_{\boldsymbol{\mu}}V$, and hence (3.2). Inequality (3.3) follows from the definition of infimum. (3.4) follows from Proposition 2.30, and the fact that

$$v_{\mathcal{O}_\lambda}=\sum_{\tau\in\mathrm{Gal}(E/K)} v_{\mathcal{O}^E_{\tau(\lambda)}}. \tag{3.7}$$

Since the average is smaller than the infimum, (3.5) follows. Finally, (3.6) follows from equivariance of the pairing $\langle\cdot,\cdot\rangle$ with respect to the Γ_K-action, and invariance of the pairing under the w_0-action.

Write $v^i=(\rho(b)\varphi)^i v$. Therefore, we have the following formula

$$
\begin{aligned}
d^s_{\rho,\mu^\eta}(v) &= \sum_{i=0}^{s-1} \deg_{\mu^\eta}((\rho(b)\varphi)^i v) && (3.8)\\
&= \sum_{i=0}^{s-1} \inf_{\lambda\in\Lambda_{v^i}} \overline{\deg}_{\boldsymbol{\mu}}(v^i_{\mathcal{O}_\lambda}) && (3.9)\\
&\leq \sum_{i=0}^{s-1} \langle \underline{\mathcal{O}_{\varphi^i(\lambda)^{\mathrm{dom}}}}, w_0\cdot\mu\rangle && (3.10)\\
&= \sum_{i=0}^{s-1} \langle w_0\cdot\varphi^i(\lambda)^{\mathrm{dom}}, \underline{\mu}\rangle && (3.11)\\
&= \sum_{i=0}^{s-1} \langle w_0\cdot\varphi_0^i(\lambda), \underline{\mu}\rangle && (3.12)\\
&= \sum_{i=0}^{s-1} \langle w_0\cdot\lambda, \varphi_0^i(\underline{\mu})\rangle && (3.13)\\
&= s\cdot\langle w_0\cdot\lambda, \mu^\diamond\rangle && (3.14)
\end{aligned}
$$

(3.8) follows from the definition in (2.45). Equality (3.9) follows from the inequalites (3.1) through (3.6) above. Since $\lambda\in\Lambda_v$, we have $\varphi^i(\lambda)^{\mathrm{dom}}\in\Lambda_{v^i}$. Thus by Proposition 2.30, we obtain (3.10). Equality (3.11) follows from equivariance of $\langle,\rangle$ under the Galois action and w_0-action. Equality (3.12) follows from the definition of φ_0 in (1). Since T is φ_0-stable, (3.13) follows from equivariance of $\langle,\rangle$ under the φ_0-action. Equality (3.14) follows from the definition of $\mu^\diamond$ (see (2.7)).

Since $v\in V^{s,k}_{(b,\mu^\eta)}$, by (3.8) through (3.14), we have $\frac{k}{s}\leq\langle w_0\cdot\lambda,\mu^\diamond\rangle$ for all $\lambda\in\Lambda_v$. On the other hand, over L, we have a decomposition $v=\sum_{\chi\in X^*(T)} v_\chi$. Since we have arranged that $\nu_b=\boldsymbol{\nu}_{\mathbf{b}}$, by (2.44) and (2.46) we have

$$
T_\rho^{s\cdot\nu_b}(v) = \sum_{\chi\in X^*(T)} T_\rho^{s\cdot\nu_b}(v_\chi) = \sum_{\chi\in X^*(T)} p^{\langle\chi, s\cdot\boldsymbol{\nu}_{\mathbf{b}}\rangle} v_\chi. \tag{3.15}
$$

The assumption $v\in V^{s,k}_{(b,\mu^\eta)}$ forces χ to satisfy $\langle\chi, s\cdot\boldsymbol{\nu}_{\mathbf{b}}\rangle=k$ for all χ where $v_\chi\neq 0$. In particular, since $w_0\cdot\lambda\leq\chi$ when $V_L^\chi\subseteq V_L^\lambda$, we have $\langle w_0\cdot\lambda,\boldsymbol{\nu}_{\mathbf{b}}\rangle\leq\frac{k}{s}$ for all $\lambda\in\Lambda_v$. Therefore $\langle w_0\cdot\lambda,\mu^\diamond-\boldsymbol{\nu}_{\mathbf{b}}\rangle\leq 0$. Since $(\mathbf{b},\boldsymbol{\mu})$ is HN-irreducible, we have $\langle w_0\cdot\lambda,\alpha^\vee\rangle=0$ for all $\alpha\in\Delta$. Therefore, the action of G_L^{der} on V^λ is trivial for all $\lambda\in\Lambda_v$. Thus we are done with the proof of (2) in Theorem 1.17. □

Corollary 3.2. *Let (G,b,μ) be a local shtuka datum over $\mathbb{Q}_p$ with $(\mathbf{b},\boldsymbol{\mu})$ HN-irreducible. There exists a finite extension K over $\breve{\mathbb{Q}}_p$ containing the reflex field of $\boldsymbol{\mu}$, and a point $x\in\mathrm{Gr}^b_\mu(K)$ whose induced (conjugacy class* *of) crystalline representation(s)*

$$\rho_x : \Gamma_K \to G(\mathbb{Q}_p)$$

satisfies that $\rho_x(\Gamma_K) \cap G^{\mathrm{der}}(\mathbb{Q}_p)$ *is open in* $G^{\mathrm{der}}(\mathbb{Q}_p)$.

Proof. Recall from [Gle22a, Proposition 2.12] (see also [Vie21, Theorem 5.2]) that the Bialynicki-Birula map BB in (2.21) induces a bijection of classical points. Therefore it suffices to construct the image $\mathrm{BB}(x) \in \mathrm{Fl}_\mu$, which corresponds to constructing a weakly admissible filtered isocrystal with G-structure.

By Lemma 3.3, we can take $\mathrm{BB}(x) = \mu^\eta$ to be generic (Definition 2.28). By Theorem 3.1(2), $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}. By Lemma 2.26, the image of the generic crystalline representation $\xi_{(b,\mu^\eta)}$ contains an open subgroup of $\mathrm{MT}_{(b,\mu^\eta)}$, thus containing an open subgroup of G^{der}. ∎

Lemma 3.3. *There exist a finite extension* K *over* $\breve{\mathbb{Q}}_p$ *and a map* μ^η : $\mathrm{Spec}(K) \to \mathrm{Fl}_\mu$ *such that* $|\mu^\eta| : \{*\} \to |\,\mathrm{Fl}_\mu\,|$ *maps to the generic point.*

Proof. Recall from [Che14, Proposition 2.0.3] that the transcendence degree of $\breve{\mathbb{Q}}_p$ over $\mathbb{Q}_p$ is infinite. By the structure theorem of smooth morphisms [Sta18, Tag 054L], one can find an open neighborhood $U \to \mathrm{Fl}_\mu$ that is étale over $\mathbb{A}^n_{\mathbb{Q}_p}$. On the other hand, one can always find a map $\mathrm{Spec}\,(\breve{\mathbb{Q}}_p) \to \mathbb{A}^n_{\mathbb{Q}_p}$ mapping to the generic point by choosing n trascendentally independent elements of $\breve{\mathbb{Q}}_p$. Its pullback to U is an étale neighborhood of $\mathrm{Spec}\,(\breve{\mathbb{Q}}_p)$ that consists of a finite disjoint union of finite extensions K of $\breve{\mathbb{Q}}_p$. Any of these components will give a map to the generic point of Fl_μ. ∎

The following is a partial converse to Theorem 1.17.

Proposition 3.4 (Proposition 5.11)**.** *Assume Conjecture 2.20 and that* G^{ad} *has only isotropic factors. If* $\mathrm{MT}_{(b,\mu^\eta)}$ *contains* G^{der}*, then* $(\mathbf{b}, \boldsymbol{\mu})$ *is HN-irreducible.*

We defer its proof till Proposition 5.11.

4. Proof of main theorems

The first goal in this section is to prove the following main theorem:

Theorem 4.1. *Let* $(\mathbf{b}, \boldsymbol{\mu})$ *be HN-irreducible. Suppose that* G^{ad} *does not have anisotropic factors. The following statements are equivalent:*

1. *The map* $\kappa_G : \pi_0(X_\mu(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ *is bijective.*
2. *The map* $\kappa_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ *is bijective.*
3. *The action of* $G(\mathbb{Q}_p)$ *on* $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ *is transitive.*
4. *The action of* $G(\mathbb{Q}_p)$ *on* $\mathrm{Sht}_{(G,b,\mu,\infty)}$ *makes* $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ *into a* G°*-torsor.*

The second goal in this section is to prove the following corollary of Theorem 4.1.

Theorem 4.2. *If G is quasisplit and $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, then all four statements of Theorem 4.1 hold. In particular, $\kappa_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ is bijective.*

The proof of the above main theorems proceeds as follows and will occupy the rest of section 4. We first prove the two theorems directly in the case of tori (see §4.1, specifically Lemma 4.3). We then use z-extensions and ad-isomorphisms to reduce the proof of Theorem 4.1 to the case where $G^{\mathrm{der}} = G^{\mathrm{sc}}$ (see Proposition 4.5). In this case, we use the determinant morphism (2.24) and finally deduce Theorem 4.2 in the general case using Theorem 4.1 and a result of Hamacher [Ham20, Theorem 1.1.(3)].

4.1. **The tori case.** When G^{ab} is a torus, there is only one parahoric model and we denote it by $\mathcal{G}^{\mathrm{ab}}$. In this case, both $X_\mu^{\mathcal{G}^{\mathrm{ab}}}(b)$ and $\mathrm{Sht}_{(G^{\mathrm{ab}},b,\mu,\mathcal{G}^{\mathrm{ab}})} \times \mathrm{Spd}\,\mathbb{C}_p$ are zero-dimensional. Since we are working over algebraically closed fields, they are of the form $\coprod_J \mathrm{Spec}\,\bar{\mathbb{F}}_p$ and $\coprod_I \mathrm{Spd}\,\mathbb{C}_p$ for some index sets I and J, respectively. Moreover, by Theorem 2.18, the specialization map (1.8) induces a bijection $\pi_0(\mathrm{sp}) : I \cong J$.

When G^{ab} is a torus, $G^\circ = G^{\mathrm{ab}}(\mathbb{Q}_p)$ and $\mathrm{Sht}_{(G^{\mathrm{ab}},b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p$ is a $\underline{G^{\mathrm{ab}}(\mathbb{Q}_p)}$-torsor over $\mathrm{Spd}\,\mathbb{C}_p$ (see for example [Gle22a, Theorem 1.24]). In particular, $\pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ is a $G^{\mathrm{ab}}(\mathbb{Q}_p)$-torsor and Theorem 4.1 (4) holds.

We remark that Theorem 4.1(3) is obvious, and the content of Theorem 4.1 (1) (equivalently (2)) becomes the following lemma.

Lemma 4.3. *Let G^{ab} be a torus. We have a $G^{\mathrm{ab}}(\mathbb{Q}_p)$-equivariant commutative diagram, where the horizontal arrows are isomorphisms:*

$$\begin{tikzcd}
\pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b,\mu,\mathcal{G}^{\mathrm{ab}})} \times \mathrm{Spd}\,\mathbb{C}_p) \arrow[r, "\cong"] & \pi_0(X_\mu^{\mathcal{G}^{\mathrm{ab}}}(b)) \arrow[r, "\cong"] \arrow[d] & c_{b,\mu}\pi_1(G^{\mathrm{ab}})_I^\varphi \arrow[d] \\
 & \pi_0(\mathcal{F}\ell_{\mathcal{G}^{\mathrm{ab}}}) \arrow[r, "\cong"] & \pi_1(G^{\mathrm{ab}})_I
\end{tikzcd}$$

Proof. Upon fixing an element of $\mathrm{Sht}_{(G^{\mathrm{ab}},b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p$, we can identify $\pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \cong G^{\mathrm{ab}}(\mathbb{Q}_p)$ (see for example [Gle22a, Theorem 1.24]), which then gives an identification $\mathrm{Sht}_{(G^{\mathrm{ab}},b,\mu,\mathcal{G}^{\mathrm{ab}})} \times \mathrm{Spd}\,\mathbb{C}_p \cong G^{\mathrm{ab}}(\mathbb{Q}_p)/\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \cong G^{\mathrm{ab}}(\breve{\mathbb{Q}}_p)^{\varphi=\mathrm{id}}/\mathcal{G}^{\mathrm{ab}}(\breve{\mathbb{Z}}_p)^{\varphi=\mathrm{id}}$. Since $H^1_{\acute{e}t}(\mathrm{Spec}\,\mathbb{Z}_p, \mathcal{G}^{\mathrm{ab}})$ vanishes, we can write

$$G^{\mathrm{ab}}(\breve{\mathbb{Q}}_p)^{\varphi=\mathrm{id}}/\mathcal{G}^{\mathrm{ab}}(\breve{\mathbb{Z}}_p)^{\varphi=\mathrm{id}} \cong (G^{\mathrm{ab}}(\breve{\mathbb{Q}}_p)/\mathcal{G}^{\mathrm{ab}}(\breve{\mathbb{Z}}_p))^{\varphi=\mathrm{id}}, \tag{4.1}$$

where the right-hand side is $X_*(G^{\mathrm{ab}})_I^{\varphi} = \pi_1(G^{\mathrm{ab}})_I^{\varphi}$. Therefore the $G^{\mathrm{ab}}(\mathbb{Q}_p)$-action makes $\pi_0(X_\mu^{\mathcal{G}^{\mathrm{ab}}}(b))$ and $\pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b,\mu,\mathcal{G}^{\mathrm{ab}})} \times \mathrm{Spd}\,\mathbb{C}_p)$ into $\pi_1(G^{\mathrm{ab}})_I^{\varphi}$-torsors (via the specialization map (1.8)). Thus by equivariance of $\pi_1(G^{\mathrm{ab}})_I^{\varphi}$-action, $\pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b,\mu,\mathcal{G}^{\mathrm{ab}})} \times \mathrm{Spd}\,\mathbb{C}_p)$ and $\pi_0(X_\mu^{\mathcal{G}^{\mathrm{ab}}}(b))$ can be identified with a unique coset $c_{b,\mu}\pi_1(G^{\mathrm{ab}})_I^{\varphi} \subseteq \pi_1(G^{\mathrm{ab}})_I$ (by the definition of $c_{b,\mu}$). ☐

4.2. **Reduction to the $G^{\mathrm{der}} = G^{\mathrm{sc}}$ case.** For the rest of this subsection, assume that f is an ad-isomorphism. Let $b_H := f(b)$ and $\mu_H := f \circ \mu$. Let $\mathcal{K}_p^H$ denote the unique parahoric of H that corresponds to the same point in the Bruhat–Tits building as $\mathcal{K}_p$.

Proposition 4.4. *(1) We have a canonical identification of diamonds*

$$\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \cong \mathrm{Sht}_{(G,b,\mu,\infty)} \times^{\underline{G(\mathbb{Q}_p)}} \underline{H(\mathbb{Q}_p)}. \tag{4.2}$$

(2) In particular, if $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ *is a* G°*-torsor, then*

$$\pi_0(\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$$

is a H°*-torsor.*

Proof. (1) A version of (4.2) was proven in [Gle21, Proposition 4.15], where the result is phrased in terms of the local system $\mathbb{L}_b$ from §2.7. [17] We sketch the proof for the reader's convenience:

Step 1. $\mathrm{Gr}_\mu = \mathrm{Gr}_{\mu_H}$: there is an obvious proper map $\mathrm{Gr}_\mu \to \mathrm{Gr}_{\mu_H}$ of spatial diamonds. Therefore, to prove that it is an isomorphism, it suffices to prove bijectivity on points, which can be done as in the classical Grassmannian case (see [AGLR22, Proposition 4.16] for a stronger statement).

Step 2. $\mathrm{Gr}_\mu^b = \mathrm{Gr}_{\mu_H}^{b_H}$: the b-admissible and b_H-admissible loci are open subsets of $\mathrm{Gr}_\mu = \mathrm{Gr}_{\mu_H}$. To prove that they agree, we can prove it on geometric points. This ultimately boils down to the fact that an element $e \in B(G)$ is basic if and only if $f(e) \in B(H)$ is basic, which holds because centrality of the Newton point ν_e can be checked after applying an ad-isomorphism.

Step 3. $\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \cong \mathrm{Sht}_{(G,b,\mu,\infty)} \times^{\underline{G(\mathbb{Q}_p)}} \underline{H(\mathbb{Q}_p)}$: recall from §2.7 that the Grothendieck–Messing period map (2.32) realizes $\mathrm{Sht}_{(G,b,\mu,\infty)}$ (respectively $\mathrm{Sht}_{(H,b_H,\mu_H,\infty)}$) as a $\underline{G(\mathbb{Q}_p)}$-torsor (respectively an $\underline{H(\mathbb{Q}_p)}$-torsor) over $\mathrm{Gr}_\mu^b = \mathrm{Gr}_{\mu_H}^{b_H}$. Since the $\underline{G(\mathbb{Q}_p)}$-equivariant map $\mathrm{Sht}_{(G,b,\mu,\infty)} \to \mathrm{Sht}_{(H,b_H,\mu_H,\infty)}$ extends to a map of $\underline{H(\mathbb{Q}_p)}$-torsors

$$\mathrm{Sht}_{(G,b,\mu,\infty)} \times^{\underline{G(\mathbb{Q}_p)}} \underline{H(\mathbb{Q}_p)} \to \mathrm{Sht}_{(H,b_H,\mu_H,\infty)}, \tag{4.3}$$

and any map of torsors is an isomorphism, the conclusion follows.

[17] Although [Gle21, Proposition 4.15] only considers unramified groups G (since this was the ongoing assumption in *loc.cit.*), the proof goes through without this assumption. A more detailed proof of Proposition 4.4 (1) can also be found in [PR22, Proposition 5.2.1], which was obtained independently as *loc.cit.*

(2) Recall that since $G \to H$ is an ad-isomorphism, we have an isomorphism $G^{\mathrm{sc}} \to H^{\mathrm{sc}}$. Recall $G^\circ := G(\mathbb{Q}_p)/\operatorname{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$ and $H^\circ := H(\mathbb{Q}_p)/\operatorname{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$. By (4.2), we have a canonical isomorphism

$$\pi_0(\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \times \operatorname{Spd}\mathbb{C}_p) \cong \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p)\times^{G(\mathbb{Q}_p)} H(\mathbb{Q}_p). \tag{4.4}$$

The right-hand side of (4.4) is by definition

$$\left(\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p)\times H(\mathbb{Q}_p)\right)/G(\mathbb{Q}_p), \tag{4.5}$$

where the quotient is via the diagonal action. Since $G^{\mathrm{sc}}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p)$, quotienting (4.5) by $G^{\mathrm{sc}}(\mathbb{Q}_p)$ first gives

$$\begin{aligned} &\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p)\times^{G(\mathbb{Q}_p)} H(\mathbb{Q}_p) && (4.6)\\ &\cong \left(\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p)\times(H(\mathbb{Q}_p)/\operatorname{Im} G^{\mathrm{sc}}(\mathbb{Q}_p))\right)/G^\circ, && (4.7)\end{aligned}$$

which simplifies, via (4.4) and since $G^{\mathrm{sc}}(\mathbb{Q}_p) = H^{\mathrm{sc}}(\mathbb{Q}_p)$, to

$$\pi_0(\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \times \operatorname{Spd}\mathbb{C}_p) = \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p) \times^{G^\circ} H^\circ. \tag{4.8}$$

The right-hand side of (4.8) is clearly an H°-torsor. □

Proposition 4.5. *If Theorem 4.1 holds for $G^{\mathrm{der}} = G^{\mathrm{sc}}$, then it holds in general as well.*

Proof. For each item $i \in \{1, \dots, 4\}$, we show that if (i) holds for $G^{\mathrm{der}} = G^{\mathrm{sc}}$, then (i) also holds for general G. For (1) and (2) of Theorem 4.1, we apply Proposition 2.24 (taking $G = G$ and $H = G^{\mathrm{ad}}$) to reduce to the case where $G = G^{\mathrm{ad}}$. Consider an arbitrary z-extension $\tilde{G} \to G$ (see Definition 2.22). By Lemma 2.23(1), there exist a conjugacy class of cocharacters $\tilde{\boldsymbol{\mu}}$ and an element $\tilde{\mathbf{b}} \in B(\tilde{G}, \tilde{\boldsymbol{\mu}})$ that map to $\boldsymbol{\mu}$ and $\mathbf{b}$, respectively, under the map $B(\tilde{G}, \tilde{\boldsymbol{\mu}}) \to B(G, \boldsymbol{\mu})$. By definition of z-extensions, $\tilde{G}^{\mathrm{der}} = \tilde{G}^{\mathrm{sc}}$. Moreover, by Lemma 2.23(2), $c_{\tilde{b},\tilde{\mu}}\pi_1(\tilde{G})_I^{\varphi} \to c_{b,\mu}\pi_1(G)_I^{\varphi}$ is surjective. We apply Proposition 2.24 again to the ad-isomorphism $\tilde{G} \to G$. Since the top horizontal arrow in (2.41) is a bijection (of sets), the bottom horizontal arrow in (2.41) is also a bijection of sets, as it is the pullback of the top horizontal arrow under a surjective map.

For (3) and (4) of Theorem 4.1, we take an arbitrary z-extension $\tilde{G} \to G$ and use Proposition 4.4 (since $\tilde{G}$ satisfies $\tilde{G}^{\mathrm{der}} = \tilde{G}^{\mathrm{sc}}$). □

4.3. Argument for (1) $\implies$ (2).

We can now give a new proof to [He18, Theorem 7.1].

Theorem 4.6. *The map $X_\mu^{\mathcal{I}}(b) \to X_\mu^{\mathcal{K}_p}(b)$ is surjective.*

Proof. By functoriality of the specialization map [Gle22b, Proposition 4.14] applied to $\mathrm{Sht}_\mu^{\mathcal{I}}(b) \to \mathrm{Sht}_\mu^{\mathcal{K}}(b)$ from (2.31), we get a commutative diagram:

$$\begin{tikzcd} \mid \mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \mid \arrow[r] \arrow[d, "\mathrm{sp}"'] & \mid \mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)} \mid \arrow[d, "\mathrm{sp}"] \\ \mid X^{\mathcal{I}}_\mu(b) \mid \arrow[r] & \mid X^{\mathcal{K}_p}_\mu(b) \mid \end{tikzcd}$$

The top arrow is given by (2.25). By [SW20, Proposition 23.3.1], it is a $\mathcal{K}_p/\mathcal{I}(\mathbb{Z}_p)$-torsor and thus surjective. It then suffices to prove that the specialization map is surjective, which follows directly from [Gle22a, Theorem 2 b)]. ☐

Now, Theorem 4.6 implies the (1) $\implies$ (2) part of Theorem 4.1: by Lemma 2.14, we have the following commutative diagram:

$$\begin{tikzcd} \pi_0(X^{\mathcal{I}}_\mu(b)) \arrow[r] \arrow[d] & \pi_0(X^{\mathcal{K}_p}_\mu(b)) \arrow[d] \\ \pi_0(\mathcal{F}\ell_{\breve{\mathcal{I}}}) \arrow[r, "\cong"] & \pi_0(\mathcal{F}\ell_{\breve{\mathcal{K}}_p}) \end{tikzcd} \tag{4.9}$$

For the bijection of the lower horizontal arrow, see for example [AGLR22, Lemma 4.17]. The left downward arrow is injective by assumption (1), and the top arrow is surjective by Theorem 4.6. Thus the right downward arrow is also injective.

4.4. **Argument for** (4) $\implies$ (1).

Proposition 4.7. (4) $\implies$ (1) *in Theorem 4.1.*

Proof. Consider the map $\det : G \to G^{\mathrm{ab}}$ where $G^{\mathrm{ab}} = G/G^{\mathrm{der}}$. Let $\mathcal{I}^{\mathrm{der}}$ denote the Iwahori subgroup of G^{der} attached to our alcove $\mathbf{a}$ (see §2). Let $\mathcal{G}^{\mathrm{ab}}$ be the unique parahoric group scheme of G^{ab}. We have an exact sequence:

$$e \to \mathcal{I}^{\mathrm{der}} \to \mathcal{I} \to \mathcal{G}^{\mathrm{ab}} \to e, \tag{4.10}$$

which induces maps $\mathrm{Sht}^{\mathcal{I}}_\mu(b) \to \mathrm{Sht}^{\mathcal{G}^{\mathrm{ab}}}_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}})$ and $X_\mu(b) \to X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}})$ by (2.30) and Lemma 2.14, respectively. To prove (1) of Theorem 4.1, it reduces to proving that $\pi_0(\det) : \pi_0(X_\mu(b)) \to \pi_0(X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}})) \cong X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}})$ is bijective.

Recall by Proposition 4.5, it suffices to assume $G^{\mathrm{der}} = G^{\mathrm{sc}}$. Indeed, when $G^{\mathrm{der}} = G^{\mathrm{sc}}$, we automatically have $G^\circ = G^{\mathrm{ab}}(\mathbb{Q}_p)$ and $\pi_1(G) = X_*(G^{\mathrm{ab}})$, which induces an isomorphism $\pi_1(G)_I = X_*(G^{\mathrm{ab}})_I$. In this case, by functoriality of the Kottwitz map $\widetilde{\kappa}$, we have the following commutative diagram

$$\begin{tikzcd} \pi_0(X_\mu(b)) \arrow[r, "\widetilde{\kappa}"] \arrow[d, "\pi_0(\det)"'] & \pi_1(G)_I \arrow[d, "\cong"] \\ X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}}) \arrow[r, "\widetilde{\kappa}"] & X_*(G^{\mathrm{ab}})_I. \end{tikzcd} \tag{4.11}$$

Recall that $c_{b,\mu}\pi_1(G)^\varphi_I$ can be defined as the unique coset of $\pi_1(G)_I$ (in the image of the affine Deligne–Lusztig variety) through which the Kottwitz map

$\widetilde{\kappa}$ factors. One defines $c_{b^{\mathrm{ab}},\mu^{\mathrm{ab}}}X_*(G^{\mathrm{ab}})_I^{\varphi}$ analogously. Clearly, the image of $\pi_0(X_\mu(b))$ (via commutativity of diagram (4.11)) lies in $c_{b^{\mathrm{ab}},\mu^{\mathrm{ab}}}X_*(G^{\mathrm{ab}})_I^{\varphi}$, through which the Kottwitz map $\widetilde{\kappa} : X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}}) \to X_*(G^{\mathrm{ab}})_I$ factors. Thus diagram (4.11) factors into the following commutative diagram:

$$\begin{tikzcd} \pi_0(X_\mu(b)) \arrow[r, "\kappa"] \arrow[d, "\pi_0(\det)"'] & c_{b,\mu}\pi_1(G)_I^{\varphi} \arrow[r, hook] \arrow[d] & \pi_1(G)_I \arrow[d, "\cong"] \\ X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}}) \arrow[r, "\kappa", "\cong"'] & c_{b^{\mathrm{ab}},\mu^{\mathrm{ab}}}X_*(G^{\mathrm{ab}})_I^{\varphi} \arrow[r, hook] & X_*(G^{\mathrm{ab}})_I. \end{tikzcd} \tag{4.12}$$

In particular, the middle downward arrow in (4.12) gives a bijection

$$c_{b,\mu}\pi_1(G)_I^{\varphi} \cong c_{b^{\mathrm{ab}},\mu^{\mathrm{ab}}}X_*(G^{\mathrm{ab}})_I^{\varphi}. \tag{4.13}$$

By functoriality of the specialization map [Gle22b, Proposition 4.14] and Theorem 2.18, we have

$$\begin{tikzcd} \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \arrow[r, "\pi_0(\mathrm{sp})", "\cong"'] \arrow[d, "\pi_0(\det)"'] & \pi_0(X_\mu(b)) \arrow[d, "\pi_0(\det)"] \\ \mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}})} \times \mathrm{Spd}\,\mathbb{C}_p \arrow[r, "\pi_0(\mathrm{sp})", "\cong"'] & X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}}) \end{tikzcd} \tag{4.14}$$

Note that we have the following identification

$$\begin{aligned} \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) &= \pi_0\left(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p/\underline{\mathcal{I}(\mathbb{Z}_p)}\right) \\ &= \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)/\mathcal{I}(\mathbb{Z}_p) \end{aligned}$$

Since $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ is a G^{ab}-torsor (i.e. assumption (4) of Theorem 4.1), up to choosing an $x \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$, we have compatible identifications $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \cong G^{\mathrm{ab}}(\mathbb{Q}_p)$ and

$$\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \cong G^{\mathrm{ab}}(\mathbb{Q}_p)/\mathcal{I}(\mathbb{Z}_p) = G^{\mathrm{ab}}(\mathbb{Q}_p)/\det(\mathcal{I}(\mathbb{Z}_p)). \tag{4.15}$$

Analogously, taking $x^{\mathrm{ab}} \in \pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ as $x^{\mathrm{ab}} = \pi_0(\det(x))$, we obtain a compatible identification $\pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}})} \times \mathrm{Spd}\,\mathbb{C}_p) = G^{\mathrm{ab}}(\mathbb{Q}_p)/\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ by Lemma 4.3. Moreover, the map $\det : \pi_0(\mathrm{Sht}_{G,b,\mu,\infty} \times \mathrm{Spd}\,\mathbb{C}_p) \to \pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ is equivariant with respect to the $G(\mathbb{Q}_p)$-action on the left and the $G^{\mathrm{ab}}(\mathbb{Q}_p)$-action on the right. Thus we have the following commutative diagram:

$$\begin{tikzcd} G^{\mathrm{ab}}(\mathbb{Q}_p)/\det(\mathcal{I}(\mathbb{Z}_p)) \arrow[r, "\cong"] \arrow[d, "\det"'] & \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \arrow[r, "\cong", "\pi_0(\mathrm{sp})"'] \arrow[d, "\det"] & \pi_0(X_\mu(b)) \arrow[d, "\det"] \\ G^{\mathrm{ab}}(\mathbb{Q}_p)/\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \arrow[r, "\cong"] & \mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}})} \times \mathrm{Spd}\,\mathbb{C}_p \arrow[r, "\cong", "\pi_0(\mathrm{sp})"'] & X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}}) \end{tikzcd}$$

Thus in order to prove that the vertical arrow on the left-hand side is a bijection, it suffices to show that $\mathcal{I} \to \mathcal{G}^{\mathrm{ab}}$ is surjective on the level of $\mathbb{Z}_p$-points. But this follows from Lang's theorem.

□

4.5. **Argument for** (2) $\implies$ (3)**.**

Proposition 4.8. (2) $\implies$ (3) *in Theorem 4.1.*

Proof. Take arbitrary $x, y \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$, we now construct an element $g \in G(\mathbb{Q}_p)$ such that $g \cdot x = y$. Recall the notation $\mathcal{G}^{\mathrm{ab}}$ to be the connected Néron model of G^{ab} (see §4.1). Let $\det(x), \det(y)$ denote their images in $\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}})} \times \mathrm{Spd}\,\mathbb{C}_p$. Since G^{der} is simply connected, by Steinberg's theorem, the map $G(\mathbb{Q}_p) \to G^{\mathrm{ab}}(\mathbb{Q}_p)$ is surjective. Replacing x by $g \cdot x$ for some $g \in G(\mathbb{Q}_p)$, we may assume $\det(x) = \det(y)$ by Lemma 4.3. Combining the diagram (4.14) with the assumption (2) of Theorem 4.1, we see that $\det(x) = \det(y)$ implies that the images of x and y in $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p)$ coincide. Since $\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p$ is a $\underline{\mathcal{I}(\mathbb{Z}_p)}$-torsor over $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p$ (recall the notation $\underline{\mathcal{I}(\mathbb{Z}_p)}$ from §2.13), there exists an element $i \in \mathcal{I}(\mathbb{Z}_p)$ such that $i \cdot x = y$.

□

4.6. **Argument for** (3) $\implies$ (4)**.**

Proposition 4.9. (3) $\implies$ (4) *in Theorem 4.1.*

Proof. As seen earlier (for example in §4.4), the map

$$\det : \pi_0(\mathrm{Sht}_{G,b,\mu,\infty} \times \mathrm{Spd}\,\mathbb{C}_p) \to \pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \tag{4.16}$$

is equivariant with respect to the $G(\mathbb{Q}_p)$-action on the left and the $G^{\mathrm{ab}}(\mathbb{Q}_p)$-action on the right. By the assumption that $G^{\mathrm{der}} = G^{\mathrm{sc}}$ (in particular $G^{\circ} = G^{\mathrm{ab}}(\mathbb{Q}_p)$), it suffices to show that

$$\pi_0(\det) : \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \to \mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty)} \times \mathrm{Spd}\,\mathbb{C}_p \tag{4.17}$$

is bijective. Since the map $G(\mathbb{Q}_p) \to G^{\mathrm{ab}}(\mathbb{Q}_p)$ is surjective, by equivariance of the respective group actions, the map (4.17) is always surjective. By hypothesis (3) in Theorem 4.1, $G(\mathbb{Q}_p)$ acts transitively on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$, thus–up to picking an $x \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$–as a set we have $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \cong G(\mathbb{Q}_p)/H_x$ for some subgroup $H_x := \mathrm{Stab}(x)$. To prove (4), it suffices to show that $H_x = G^{\mathrm{der}}(\mathbb{Q}_p)$. Firstly, it is easy to see that $H_x \subseteq G^{\mathrm{der}}(\mathbb{Q}_p)$: take any $g \in H_x$, we have $g \cdot x = x$; thus $\deg(g) \cdot \det(x) = \det(g \cdot x) = \det(x)$; by the tori case (see §4.1), $\det(g)$ is trivial, thus $g \in G^{\mathrm{der}}(\mathbb{Q}_p)$.

We now prove the other inclusion, i.e. that $G^{\mathrm{der}}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$. We may argue over finite extensions of $\breve{\mathbb{Q}}_p$.

Indeed, recall from [Sch17, Lemma 12.17] that, the underlying topological space of a cofiltered inverse limit of locally spatial diamonds along qcqs[18] transitions maps is the limit of the underlying topological spaces. Thus it suffices to prove that $G^{\mathrm{der}}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, K)$ for all finite degree extensions K over $\breve{\mathbb{Q}}_p$. For any fixed $x \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, K)$, we denote by $G_x \subseteq G(\mathbb{Q}_p)$ the stabilizer of x. Let $G_x^{\mathrm{der}} := G_x \cap G^{\mathrm{der}}(\mathbb{Q}_p)$. It suffices to prove that $G_x^{\mathrm{der}} = G^{\mathrm{der}}(\mathbb{Q}_p)$, which is shown in Lemma 4.12. □

Lemma 4.10. *G_x^{der} is open in $G^{\mathrm{der}}(\mathbb{Q}_p)$.*

Proof. For any $y \in \mathrm{Gr}_\mu^b(K)$, let $\mathcal{T}_y := \mathrm{Sht}_{(G,b,\mu,\infty)} \times_{\mathrm{Gr}_\mu^b} \mathrm{Spd}\, K$ be the fiber of y under the Grothendieck–Messing period morphism. Take an arbitrary $w \in \pi_0(\mathcal{T}_y)$, by hypothesis (3) of Theorem 4.1, we assume without loss of generality that $w \mapsto x$ under the surjection $\pi_0(\mathcal{T}_y) \to \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, K)$. Consider $G_w^{\mathrm{der}} := G_w \cap G^{\mathrm{der}}(\mathbb{Q}_p)$, and we have an inclusion of groups $G_w^{\mathrm{der}} \subseteq G_x^{\mathrm{der}} \subseteq G^{\mathrm{der}}(\mathbb{Q}_p)$. Thus it suffices to find a $y \in \mathrm{Gr}_\mu^b(K)$, such that

$$(*) \quad \text{there exists a } w \in \pi_0(\mathcal{T}_y) \text{ with } G_w^{\mathrm{der}} \text{ open in } G^{\mathrm{der}}(\mathbb{Q}_p).$$

Recall the $\underline{G(\mathbb{Q}_p)}$-local system $\mathbb{L}_b$ over Gr_μ^b from § 2.7. Let $y^*\mathbb{L}_b$ be the corresponding local system over $\mathrm{Spd}\, K$, which induces a crystalline representation $\rho_y : \Gamma_K \to G(\mathbb{Q}_p)$, well-defined up to conjugacy. We claim that G_w is equal to $\rho_y(\Gamma_K)$ up to $G(\mathbb{Q}_p)$-conjugacy. We now justify the claim. Consider the pullback $\mathcal{T}_t$ of $\mathcal{T}_y$ under the geometric point $t : \mathrm{Spd}\, \mathbb{C}_p \to \mathrm{Spd}\, K$. Thus $\mathcal{T}_t$ is a trivial torsor that gives a section $s : \mathrm{Spd}\, \mathbb{C}_p \to \mathcal{T}_t$. The Galois action of Γ_K on $\mathcal{T}_t$ defines a representative of the crystalline representation ρ_y. The orbit $\Gamma_K \cdot s$ descends to a unique component $w_s \in \pi_0(\mathcal{T}_y)$. Therefore, for any $g \in G(\mathbb{Q}_p)$ such that $g \cdot s \in \Gamma_K \cdot s$, we have $g \cdot w_s = w_s$. This gives us the desired claim.

By Corollary 3.2 (which is a consequence of our Theorem 1.17), any generic y satisfies property $(*)$. □

Lemma 4.11. *Assuming hypothesis (3) in Theorem 4.1. Let N_x denote the normalizer of G_x in $G(\mathbb{Q}_p)$. Then N_x has finite index in $G(\mathbb{Q}_p)$. In particular, N_x contains $G^{\mathrm{der}}(\mathbb{Q}_p)$.*

Proof. Let S be the set of orbits of $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\, K)$ under the $J_b(\mathbb{Q}_p)$-action. By [HV20, Theorem 1.2], S is finite. For each $s \in S$, we choose a representative $x_s \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, K)$ that maps to s under the map $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, K) \to \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\, K) \to S$. We can always arrange that x is in this set of representatives for some s. By the transitivity assumption of hypothesis (3) in Theorem 4.1, we can find an element $h_s \in G(\mathbb{Q}_p)$ such that $x_s \cdot h_s = x$, for each $s \in S$.

We wish to construct a surjection of the form $\coprod_{s \in S} \mathcal{I}(\mathbb{Z}_p) \cdot h_s \twoheadrightarrow G(\mathbb{Q}_p)/N_x$.

[18] i.e. quasi-compact quasi-separated

We do this in two steps. The first step is to construct, for any $g \in G(\mathbb{Q}_p)$, a triple (i, j, s) where $i \in \mathcal{I}(\mathbb{Z}_p)$, $j \in J_b(\mathbb{Q}_p)$ and $s \in S$ such that

$$j \cdot (x \cdot g) \cdot i = x_s \tag{4.18}$$

(note that s is uniquely determined by x and g). We do this by choosing j so that $j \cdot (x \cdot g)$ and x_s map to the same element in $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\, K)$. Since $\mathcal{I}(\mathbb{Z}_p)$ acts transitively on the fibers of the map $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, K) \to \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\, K)$, there exists an i satisfying (4.18). Thus we have

$$j \cdot x \cdot (gih_s) = x \tag{4.19}$$

The second step is to eliminate j from (4.19). By hypothesis (3) in Theorem 4.1, there exists an $n \in G(\mathbb{Q}_p)$ such that $x \cdot n = j \cdot x$. We now show that $n \in N_x$. Indeed, $n^{-1} G_x n = G_{x \cdot n} = G_{j \cdot x} = G_x$ since the actions of $J_b(\mathbb{Q}_p)$ and $G(\mathbb{Q}_p)$ commute. Thus we have $(x \cdot n) \cdot (gih_s) = x$. Since $G_x \subseteq N_x$, in particular $n \cdot (gih_s) \in G_x \subseteq N_x$. Thus $g \cdot i \cdot h_s \in N_x$, and we have a surjection:

$$\coprod_{s \in S} \mathcal{I}(\mathbb{Z}_p) \cdot h_s \twoheadrightarrow G(\mathbb{Q}_p)/N_x. \tag{4.20}$$

The target of (4.20) is discrete, and the source is compact. Thus the index of N_x in $G(\mathbb{Q}_p)$ is finite.

Recall that $G^{\mathrm{der}} = G^{\mathrm{sc}}$. Since G^{der} only has $\mathbb{Q}_p$-simple isotropic factors, and $N_x \cap G^{\mathrm{der}}(\mathbb{Q}_p)$ has finite index in $G^{\mathrm{der}}(\mathbb{Q}_p)$, we have $N_x \cap G^{\mathrm{der}}(\mathbb{Q}_p) = G^{\mathrm{der}}(\mathbb{Q}_p)$. Indeed, it is a standard fact that $G^{\mathrm{der}}(\mathbb{Q}_p)$ has no open subgroups of finite index [Mar91, Chapter II, Theorem 5.1], thus we are done. □

Lemma 4.12. *$G_x^{\mathrm{der}} = G^{\mathrm{der}}(\mathbb{Q}_p)$.*

Proof. By Lemma 4.10 and Lemma 4.11, $G_x^{\mathrm{der}} \subseteq G^{\mathrm{der}}(\mathbb{Q}_p)$ is open and normal. This already implies $G_x^{\mathrm{der}} = G^{\mathrm{der}}(\mathbb{Q}_p)$, since $G^{\mathrm{der}}(\mathbb{Q}_p)$ does not have open normal subgroups (recall that $G^{\mathrm{der}} = G^{\mathrm{sc}}$). □

This finishes the argument for (3) $\implies$ (4).

Proof of Theorem 4.1. We have now justified the circle of implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (1), i.e. Theorem 4.1 holds in the case $G^{\mathrm{der}} = G^{\mathrm{sc}}$. But by Proposition 4.5 the general case follows. □

Proof of Theorem 4.2. Recall from [Ham20, Theorem 1.1.(3)] that the map $\pi_0(X_\mu^{\mathcal{K}_p^{\mathrm{sp}}}(b)) \to \pi_1(G)_I$ in (1.4) is injective when G is quasisplit and $\mathcal{K}_p^{\mathrm{sp}}$ is a very special parahoric. By [HZ20, Lemma 6.1], the map (1.4) is also surjective (without any assumption on G and $\mathcal{K}_p^{\mathrm{sp}}$). Therefore condition (2) in Theorem 4.1 holds when G is quasisplit and $\mathcal{K}_p^{\mathrm{sp}}$ is a very special parahoric. By Theorem 4.1, condition (4) holds–note that $\mathcal{K}_p^{\mathrm{sp}}$ does not appear in the formulation of statement (4). Using the implication (4) $\implies$ (1) of Theorem 4.1, condition (1) also holds, i.e. the Iwahori-level map

$$\pi_0(X_\mu^{\mathcal{I}}(b)) \to \pi_1(G)_I \tag{4.21}$$

is also bijective for any Iwahori $\mathcal{I}$ not necessarily contained in $\mathcal{K}_p^{\mathrm{sp}}$. Now, given a parahoric $\mathcal{K}_p$ containing $\mathcal{I}(\mathbb{Z}_p)$, we apply the (1) $\implies$ (2) of Theorem 4.1 and obtain the result for arbitrary parahoric $\mathcal{K}_p$. ☐

5. Beyond the quasisplit case.

5.1. **Cohomological dimensions of Newton strata.** Let Gr_μ^b be as in the setup of Conjecture 2.20. Let $\mathrm{Gr}_\mu^\circ \times \mathrm{Spd}\,\mathbb{C}_p \subseteq \mathrm{Gr}_\mu \times \mathrm{Spd}\,\mathbb{C}_p$ denote the open Schubert cell, which is an ℓ-cohomologically smooth locally spatial diamond over $\mathrm{Spd}\,\mathbb{C}_p$ [FS21, Proposition IV.1.18].

Let $U := (\mathrm{Gr}_\mu^\circ \times \mathrm{Spd}\,\mathbb{C}_p) \cap \mathrm{Gr}_\mu^b$. Let $j : U \to \mathrm{Gr}_\mu^\circ \times \mathrm{Spd}\,\mathbb{C}_p$ denote the open immersion and $i : \mathcal{Z} \to \mathrm{Gr}_\mu^\circ \times \mathrm{Spd}\,\mathbb{C}_p$ the complementary closed immersion. Let d be the dimension [FS21, Definition IV.1.17] of the ℓ-cohomologically smooth map

$$\mathrm{Gr}_\mu^\circ \times \mathrm{Spd}\,\mathbb{C}_p \to \mathrm{Spd}\,\mathbb{C}_p. \tag{5.1}$$

Conjecture 5.1. *The ℓ-cohomological dimension of $(\mathcal{Z})_{\text{ét}}$ is smaller than or equal to $2d-2$.*

In Conjecture 5.1, $\mathcal{Z}$ is the preimage of the non-semistable strata in the Newton stratification of Bun_G under the Beauville–Laszlo unifmormization map (2.34). Let $f : \mathcal{Z} \to \mathrm{Spd}\,\mathbb{C}_p$ denote the structure map. We make the following stronger conjecture.

Conjecture 5.2. $\dim.\mathrm{trg}(f) \leq d-1$.

Lemma 5.3. *Conjecture 5.2 implies Conjecture 5.1.*

Proof. Since $i : \mathcal{Z} \to \mathrm{Gr}_\mu^\circ \times \mathrm{Spd}\,\mathbb{C}_p$ is a closed immersion, $\mathcal{Z}$ is a locally spatial diamond. Thus it admits a pro-étale cover by a disjoint union of affinoid perfectoid spaces. By [Sch17, Lemma 21.6] (applied to the pro-étale cover), the Krull dimension of $|\mathcal{Z}|$ is bounded by $d-1$. By [Sch17, Proposition 21.16] and Conjecture 5.2, at each maximal point $y \in |\mathcal{Z}|$,

$$\mathrm{cd}_\ell\, y \leq \dim.\mathrm{trg}(f) \leq d-1 \tag{5.2}$$

Finally, by [Sch17, Proposition 21.11], the cohomological dimension of $(\mathcal{Z})_{\text{ét}}$ is bounded by $\dim(|\mathcal{Z}|) + \mathrm{cd}_\ell\, y \leq 2d-2$. ☐

By the 6-functor formalism [Sch17], there is a distinguished triangle in $D_{\text{ét}}(\mathrm{Gr}_\mu^\circ \times \mathrm{Spd}\,\mathbb{C}_p, \mathbb{F}_\ell)$:

$$i_* i^! \mathbb{F}_\ell \to \mathbb{F}_\ell \to Rj_* j^* \mathbb{F}_\ell \to \tag{5.3}$$

Lemma 5.4. *Under Conjecture 5.1, $i^!\mathbb{F}_\ell$ is concentrated in degrees ≥ 2.*

Proof. Since $i^!\mathbb{F}_\ell$ is concentrated in non-negative degrees, it suffices to prove that $H^0(V, j_V^* i^! \mathbb{F}_\ell)$ and $H^0(V, j_V^* i^! \mathbb{F}_\ell)$ vanish for all étale maps $j_V : V \to \mathcal{Z}$, i.e. it suffices to show that

$$\mathrm{Hom}(\mathbb{F}_\ell, j_V^* i^! \mathbb{F}_\ell) = 0 = \mathrm{Hom}(\mathbb{F}_\ell[-1], j_V^* i^! \mathbb{F}_\ell). \tag{5.4}$$

Now, since f is ℓ-cohomologically smooth, we have $i^!\mathbb{F}_\ell = i^! f^! \mathbb{F}_\ell[-2d]$. Thus by adjunction [Sch17], it suffices to prove

$$\mathrm{Hom}(f_! i_! j_{V,!}\mathbb{F}_\ell[2d], \mathbb{F}_\ell) = 0 = \mathrm{Hom}(f_! i_! j_{V,!}\mathbb{F}_\ell[2d-1], \mathbb{F}_\ell). \tag{5.5}$$

In other words, it suffices to show that $R\Gamma_c(V, \mathbb{F}_\ell)$ vanishes in degrees greater than $2d-2$, which follows immediately from Conjecture 5.1. □

Proposition 5.5. *Conjecture 5.1 implies Conjecture 2.20.*

Proof. Since Gr°_μ is dense in Gr_μ (see for example [AGLR22, Proposition 4.5]), it suffices to prove that $U := (\mathrm{Gr}^\circ_\mu \times \mathrm{Spd}\,\mathbb{C}_p) \cap \mathrm{Gr}^b_\mu$ is dense in $\mathrm{Gr}^\circ_\mu \times \mathrm{Spd}\,\mathbb{C}_p$ and connected. By Lemma 5.4 and the long exact sequence for (5.5), we have $R^0 j_* \mathbb{F}_\ell = \mathbb{F}_\ell$. Thus U is dense–indeed it suffices to check that the stalks of $R^0 j_* \mathbb{F}_\ell$ do not vanish. Moreover, $\pi_0(U) = \pi_0(\mathrm{Gr}^\circ_\mu \times \mathrm{Spd}\,\mathbb{C}_p) = \{*\}$ because $H^0(U, \mathbb{F}_\ell) = H^0(U, R^0 j_* \mathbb{F}_\ell) = H^0(\mathrm{Gr}^\circ_\mu \times \mathrm{Spd}\,\mathbb{C}_p, \mathbb{F}_\ell) = \mathbb{F}_\ell$. □

When b is basic, Conjecture 2.20 is an unpublished result of Hansen and Weinstein, using the argument in Proposition 5.5. We verify Conjecture 5.2 when b is basic. For dimension considerations, we take the intersection of the moduli space of p-adic shtukas with the Schubert cell Gr°_μ. Let

$$\mathrm{Sht}^\circ_{G,b,\mu,\infty} \times \mathrm{Spd}\,\mathbb{C}_p := \pi^{-1}_{\mathrm{GM}}(\mathrm{Gr}^\circ_\mu \times \mathrm{Spd}\,\mathbb{C}_p). \tag{5.6}$$

Proposition 5.6. *If b is basic, then* $\dim.\mathrm{trg}(f) \leq d-1$. *(i.e. Conjecture 5.2 holds, and thus Conjecture 5.1 and Conjecture 2.20 holds.)*

Proof. For each $\beta \in B(G)$, let $\mathcal{Z}_\beta := (\mathrm{Gr}^\circ_\mu \times \mathrm{Spd}\,\mathbb{C}_p) \cap \mathcal{BL}_b^{-1}(\mathrm{Bun}^\beta_G)$, where $\mathcal{BL}_b$ is the map from (2.34). Let $f_\beta : \mathcal{Z}_\beta \to \mathrm{Spd}\,\mathbb{C}_p$ denote the structure map. It suffices to prove $\dim.\mathrm{trg}.(f_\beta) \leq d-1$ for all β such that $\mathcal{Z}_\beta \subseteq \mathcal{Z}$.

Recall that since b is basic, J_b is a pure inner form of G, and we have an identification $B(J_b) \cong B(G)$. By [FS21, Proposition III.5.3], $\mathrm{Bun}^\beta_G = [*/\tilde{G}_\beta]$ where $\tilde{G}_\beta = \underline{\mathrm{Aut}}(\mathcal{E}_\beta)$ is the automorphism v-sheaf of the G-bundle $\mathcal{E}_\beta$ considered in [FS21, § III.5.1]. Viewing μ^{-1} as a conjugacy class of cocharacters in J_b, and β as an element of $B(J_b, \mu^{-1})$, we have

$$\mathcal{Z}_\beta = \mathrm{Sht}^\circ_{J_b,\beta,\mu^{-1},\infty} \times \mathrm{Spd}\,\mathbb{C}_p/\tilde{G}_\beta. \tag{5.7}$$

The dimension of $\mathrm{Sht}^\circ_{J_b,\beta,\mu^{-1},\infty} \times \mathrm{Spd}\,\mathbb{C}_p$ over $\mathrm{Spd}\,\mathbb{C}_p$ is d. Therefore the dimension of $\mathcal{Z}_\beta$ is $d - \dim(\tilde{G}_\beta)$. Since $\mathcal{Z}$ consists of non-semistable strata, $\dim(\tilde{G}_\beta) \geq 1$ for all β such that $\mathcal{Z}_\beta \subseteq \mathcal{Z}$. □

When b is not necessarily basic, although Conjecture 5.2 is intuitively true (because the semistable locus is both open and dense inside $\mathrm{Gr}^\circ_\mu \times \mathrm{Spd}\,\mathbb{C}_p$), it becomes technically more challenging to justify that density implies the bound on dimension. We intend to come back to this in a future work.

5.2. **Main theorems revisited.** We now show that Conjecture 2.20 implies that all four statements in Theorem 4.1 hold, even for non-quasisplit (not necessarily unramified) groups. In particular, Conjecture 1.1 holds for all reductive groups G and all parahoric models $\mathcal{K}_p$.

Lemma 5.7. *For any p-adic shtuka datum (G, b, μ) over $\mathbb{Q}_p$ (with $\mathbf{b} \in B(G, \boldsymbol{\mu})$ but not necessarily HN-irreducible),*

Conjecture 2.20 holds $\Longleftrightarrow$ (3) in Theorem 4.1 holds.

In particular, if Conjecture 2.20 holds and $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, then all 4 statements in Theorem 4.1 hold.

Proof. Recall from [SW20, §23] that we have a canonical identification

$$\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p \cong (\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)/\underline{G(\mathbb{Q}_p)}. \tag{5.8}$$

Since π_0 is a left adjoint functor to the inclusion of totally disconnected topological spaces into v-sheaves, it commutes with coequalizers, which gives

$$\pi_0(\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p) \cong \pi_0\left(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p\right)/G(\mathbb{Q}_p). \tag{5.9}$$

Transitivity of $G(\mathbb{Q}_p)$-action (i.e. condition (3)) is equivalent to

$$\pi_0\left(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p\right)/G(\mathbb{Q}_p) = \{*\}, \tag{5.10}$$

whereas Conjecture 2.20 is equivalent to

$$\pi_0(\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p) = \{*\}. \tag{5.11}$$

Therefore (3) in Theorem 4.1 and Conjecture 2.20 are equivalent.

□

Corollary 5.8. *Conjecture 2.20 implies Conjecture 1.1, for all reductive groups G and all parahoric models $\mathcal{K}_p$.*

Proof. (i.e. also Proof of Theorem 1.2(2)) Using z-extension ad-isomorphisms and decomposition into products (Proposition 2.11, Proposition 2.12, Proposition 2.24 and Lemma 2.16), we may assume without loss of generality that $G^{\mathrm{der}} = G^{\mathrm{sc}}$ and G^{ad} is $\mathbb{Q}_p$-simple.

We split our discussion into two cases: (1) when G^{ad} is isotropic, and (2) when G^{ad} is anisotropic. The first case holds by combining Theorem 4.1 (1) and (2) with Lemma 5.7.

We now consider the case where G^{ad} is anisotropic. Recall that

$$\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p = \mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p/G(\mathbb{Q}_p). \tag{5.12}$$

When G^{ad} is $\mathbb{Q}_p$-simple and anisotropic, we have that $\mathcal{I}(\mathbb{Z}_p)$ is normal in $G(\mathbb{Q}_p)$ and $G(\mathbb{Q}_p)/\mathcal{I}(\mathbb{Z}_p) = G^{\mathrm{ab}}(\mathbb{Q}_p)/\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) = \pi_1(G)_I^{\varphi}$ (see § 4.1 for the last identification). Since $\mathcal{I}(\mathbb{Z}_p)$ is normal in $G(\mathbb{Q}_p)$, $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p$ becomes a $\pi_1(G)_I^{\varphi}$-torsor over $\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p$. Moreover, the determinant map

$$\det : \mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p \to \mathrm{Sht}_{(G,b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p \tag{5.13}$$

is $\pi_1(G)_I^{\varphi}$-equivariant. Since $\mathrm{Sht}_{(G,b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p))}\times \mathrm{Spd}\,\mathbb{C}_p$ is a $\pi_1(G)_I^{\varphi}$-torsor over $\mathrm{Spd}\,\mathbb{C}_p$, $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}\times \mathrm{Spd}\,\mathbb{C}_p$ is the trivial $\pi_1(G)_I^{\varphi}$-torsor over $\mathrm{Gr}_\mu^b\times \mathrm{Spd}\,\mathbb{C}_p$. That is

$$\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}\times \mathrm{Spd}\,\mathbb{C}_p \cong (\mathrm{Gr}_\mu^b\times \mathrm{Spd}\,\mathbb{C}_p)\times \underline{\pi_1(G)_I^{\varphi}}. \tag{5.14}$$

Taking π_0 in (5.14), we have

$$\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}\times \mathrm{Spd}\,\mathbb{C}_p)\cong \pi_0(\mathrm{Gr}_\mu^b\times \mathrm{Spd}\,\mathbb{C}_p)\times \pi_1(G)_I^{\varphi}. \tag{5.15}$$

By Conjecture 2.20, $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}\times \mathrm{Spd}\,\mathbb{C}_p)\cong \pi_1(G)_I^{\varphi}$ and the map

$$\pi_0(\kappa_G\circ \mathrm{sp}):\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))})\times \mathrm{Spd}\,\mathbb{C}_p)\to c_{b,\mu}\pi_1(G)_I^{\varphi}$$

is an isomorphism as we needed to show. We can finish by recalling that the map of (1.9) is bijective. □

5.3. A p-adic Hodge-theoretic characterization of HN-irreducibility.

In this section, we characterize the notion of HN-irreducibility in terms of the generic Mumford–Tate group $\mathrm{MT}_{(b,\mu^\eta)}$ from 2.25. We start by proving a converse to Conjecture 1.1 (or Theorem 1.2).

Proposition 5.9. *Assume G^{ad} has only isotropic factors. If the Kottwitz map $\kappa:\pi_0(X_\mu(b))\to c_{b,\mu}\pi_1(G)_I^{\varphi}$ is a bijection, then $(\mathbf{b},\boldsymbol{\mu})$ is HN-irreducible.*

Proof. By Proposition 2.11, Proposition 2.12, Lemma 2.16 and Proposition 2.24, we may assume without loss of generality that G is adjoint and $\mathbb{Q}_p$-simple.

We prove by contradiction and assume that $(\mathbf{b},\boldsymbol{\mu})$ is not HN-irreducible.

(I) If $\mathbf{b}$ is HN-decomposable in $B(G,\boldsymbol{\mu})$, then by Theorem 2.5, we have

$$\pi_0(X_\mu(b))=\bigsqcup_{P'\in\mathfrak{P}^{\varphi}/W_K^{\varphi}}\pi_0(X^{M'}_{\mu_{P'}}(b_{P'})). \tag{5.16}$$

Thus by Lemma 2.6, we may assume that each $b_{P'}$ is HN-indecomposable in $B(M',\mu_{P'})$. Recall the embedding $\phi_{P'}:X^{M'}_{\mu_{P'}}(b_{P'})\hookrightarrow X_\mu(b)$ from (2.12) for each $P'\in\mathfrak{P}^{\varphi}/W_K^{\varphi}$, which induces a map

$$\pi_0(\phi_{P'}):\pi_0(X^{M'}_{\mu_{P'}}(b_{P'}))\hookrightarrow \pi_0(X_\mu(b)). \tag{5.17}$$

Taking the disjoint union over $P'\in\mathfrak{P}^{\varphi}/W_K^{\varphi}$ in (5.17) recovers the bijection (5.16).

Consider $\iota: M'(\breve{F})\to G(\breve{F})$. Let $\iota_I:\pi_1(M')_I\to\pi_1(G)_I$ be the induced map, which then induces a map $\iota_I^{\varphi}:\pi_1(M')_I^{\varphi}\to\pi_1(G)_I^{\varphi}$. By [Kot97, 7.4], the following diagram commutes:

$$\begin{array}{ccc} M'(\breve{\mathbb{Q}}_p) & \xrightarrow{\kappa_{M'}} & \pi_1(M')_I \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota_I} \\ G(\breve{\mathbb{Q}}_p) & \xrightarrow{\kappa_G} & \pi_1(G)_I \end{array} \tag{5.18}$$

Denote by $+_{h_{P'}}: \pi_1(G)_I \to \pi_1(G)_I$ the addition-by-$\kappa_G(h_{P'})$ map. Then (5.18) shows that $+_{h_{P'}} \circ \iota_I$ sends $c_{b_{P'},\mu_{P'}}\pi_1(M')_I^{\varphi}$ to $c_{b,\mu}\pi_1(G)_I^{\varphi}$. Moreover, we have the following commutative diagram

$$\begin{tikzcd} \pi_0(X^{M'}_{\mu_{P'}}(b_{P'})) \arrow[r, two heads, "\kappa_{M'}"] \arrow[d, hook, "\pi_0(\phi_{P'})"'] & c_{b_{P'},\mu_{P'}}\pi_1(M')_I^{\varphi} \arrow[d, "+_{h_{P'}}\circ\iota_I"] \\ \pi_0(X_\mu(b)) \arrow[r, "\kappa_G", "\cong"'] & c_{b,\mu}\pi_1(G)_I^{\varphi} \end{tikzcd} \tag{5.19}$$

Here the surjectivity of $\kappa_{M'}$ follows from [HZ20, Lemma 6.1]. Now, if the lower horizontal arrow κ_G is a bijection, then the upper horizontal arrow $\kappa_{M'}$ should also be a bijection. Moreover, this implies that $+_{h_{P'}} \circ \iota_I$ is injective, which then implies that $\iota_I^{\varphi}: \pi_1(M')_I^{\varphi} \to \pi_1(G)_I^{\varphi}$ is injective. This contradicts Lemma 5.10.

(II) If $\mathbf{b}$ is HN-indecomposable in $B(G, \boldsymbol{\mu})$, by Proposition 2.9, we may assume that μ is central and $b = p^{\mu}$. We now show that $\pi_0(X_\mu(b)) \to \pi_1(G)_I^{\varphi}$ is not bijective.

By Proposition 2.10, there is a bijection $\pi_0(X_\mu(b)) \simeq G(\mathbb{Q}_p)/\mathcal{K}_p(\mathbb{Z}_p)$. Since G is not anisotropic, there exists a non-trivial $\mathbb{Q}_p$-split torus S, and we can consider the composition of maps

$$S(\mathbb{Q}_p) \hookrightarrow G(\mathbb{Q}_p) \twoheadrightarrow G(\mathbb{Q}_p)/\mathcal{K}_p(\mathbb{Z}_p). \tag{5.20}$$

Since $S(\mathbb{Q}_p) \cap \mathcal{K}_p(\mathbb{Z}_p)$ is compact, we have $S(\mathbb{Q}_p) \cap \mathcal{K}_p(\mathbb{Z}_p) \subseteq S(\mathbb{Z}_p)$. Therefore, we obtain an injective homomorphism

$$X_*(S) \cong S(\mathbb{Q}_p)/S(\mathbb{Z}_p) \hookrightarrow G(\mathbb{Q}_p)/\mathcal{K}_p(\mathbb{Z}_p). \tag{5.21}$$

Since G is adjoint, $\pi_1(G)_I^{\varphi}$ is finite. However, $X_*(S)$ is infinite, thus the map $\kappa_G: \pi_0(X_\mu(b)) \to \pi_1(G)_I^{\varphi}$ cannot be bijective. We have a contradiction. $\square$

Now we finish the proof of Proposition 5.9 by proving the following lemma.

Lemma 5.10. *Let G be adjoint and $\mathbb{Q}_p$-simple. For a proper Levi subgroup M of G, the natural map $\iota_I^{\varphi}: \pi_1(M)_I^{\varphi} \to \pi_1(G)_I^{\varphi}$ cannot be injective.*

Proof. Recall that $(\pi_1(G)_I)_{\hat{\mathbb{Z}}} \simeq \pi_1(G)_\Gamma$. We prove by contradiction and assume that the natural map $\iota_I^{\varphi}: \pi_1(M)_I^{\varphi} \to \pi_1(G)_I^{\varphi}$ is injective. In particular,

$$\pi_1(M)_I^{\varphi} \otimes \mathbb{Q} \hookrightarrow \pi_1(G)_I^{\varphi} \otimes \mathbb{Q}.$$

Via the "average map" under φ-action, we have

$$\pi_1(-)_I^{\varphi} \otimes \mathbb{Q} \simeq (\pi_1(-)_I)_{\langle\varphi\rangle} \otimes \mathbb{Q} \simeq \pi_1(-)_\Gamma \otimes \mathbb{Q} \simeq \pi_1(-)^\Gamma \otimes \mathbb{Q}. \tag{5.22}$$

Hence the natural map

$$\pi_1(M)^\Gamma \otimes \mathbb{Q} \to \pi_1(G)^\Gamma \otimes \mathbb{Q} \tag{5.23}$$

is injective. Let $M \subseteq P \subseteq G$ be the corresponding parabolic subgroup. Let $\theta_P = \sum_{\alpha\in\Phi_P} \alpha^\vee \in X_*(T)$ denote the sum of coroots of P. Now, θ_P is Γ-stable since P is defined over $\mathbb{Q}_p$. Moreover, its image under the natural projection map $q_M : X_*(T) \to \pi_1(M)$ is Γ-stable. One can check that $q_M(\theta) \neq 0$ in $\pi_1(M)^\Gamma \otimes \mathbb{Q}$. Since $q_G(\theta_P) = 0$ in $\pi_1(G)$, the map in (5.23) is not injective. We have a contradiction.

□

As a corollary, we obtain the following partial converse to Theorem 1.17, which was first stated, without proof, in Proposition 3.4.

Proposition 5.11. *Assume Conjecture 2.20 and that G^{ad} has only isotropic factors. If $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}, then $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*

Proof. First note that, if we replace the assumption of HN-irreducibility with the assumption that $G^{\mathrm{der}} \subseteq \mathrm{MT}_{(b,\mu^\eta)}$, the arguments in Section 4.6 go through. In other words, assuming $G^{\mathrm{der}} \subseteq \mathrm{MT}_{(b,\mu^\eta)}$, we have the chain (3) $\implies$ (4) $\implies$ (1) in Theorem 4.1. Now, if we assume Conjecture 2.20 we can conclude that (3) holds by Lemma 5.7. However, by Hodge-Newton decomposition (see Proposition 5.9), this can only happen if $(\mathbf{b}, \boldsymbol{\mu})$ is already HN-irreducible. □

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Mathematisches Institut der Universität Bonn, Endenicher Allee 60, Bonn, Germany

Email address: igleason@uni-bonn.de

University of California, Evans Hall, Berkeley, CA, USA

Email address: limath@math.berkeley.edu

M.I.T., 77 Massachusetts Avenue, Cambridge, MA, USA

Email address: yujiex@math.harvard.edu