

Specialization maps for Scholze's category of diamonds

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Abstract

The purpose of this article is to define and study the specialization map in the context of Scholze's category of diamonds and to prove some basic results on its behavior. Our specialization map generalizes the classical specialization map that appears in the theory of formal schemes. Afterwards, as an example of interest, we study the specialization map for p -adic Beilinson-Drinfeld Grassmanians and moduli spaces of mixed-characteristic shtukas associated to reductive groups over \mathbb{Z}_p . Finally, as an application, we describe the geometric connected components of moduli spaces of mixed-characteristic shtukas at hyper-special level in terms of the connected components of affine Deligne Lusztig varieties. This generalizes a similar theorem of Chen, Kisin, and Viehmann that describes the connected components of unramified Rapoport-Zink spaces in terms of connected components of affine Deligne Lusztig varieties.

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Introduction

The purpose of this paper is twofold, and each goal corresponds to a chapter. In the first chapter we construct the specialization map in the context of Scholze's category of diamonds, and we study abstract properties and related constructions in a very theoretical framework. In the second chapter, we apply the theory developed in the first chapter to study the specialization map for the p -adic Beilinson-Drinfeld Grassmanians and moduli spaces of mixed-characteristic shtukas. These moduli spaces were introduced in the Berkeley notes ([30]) as some of the most important examples that motivated the development of the theory of diamonds.

To fix ideas let us recall the classical case. Let \mathcal{X} be a separated formal scheme topologically of finite type over \mathbb{Z}_p . One can associate to \mathcal{X} a rigid analytic space over \mathbb{Q}_p , that we will denote by $X_\eta := \mathcal{X} \times_{\mathbb{Z}_p} \mathbb{Q}_p$. We can also associate to \mathcal{X} a finite type reduced scheme over \mathbb{F}_p , that we denote by $\overline{X} := (\mathcal{X} \times_{\mathbb{Z}_p} \mathbb{F}_p)^{\text{red}}$. Now, Huber's theory of adic spaces allows us to consider X_η as an adic space and in particular assign to it a locally spectral topological space $|X_\eta|$. Moreover, one can construct a continuous and spectral map of locally spectral spaces $\text{sp}_{\mathcal{X}} : |X_\eta| \rightarrow |\overline{X}|$, where $|\overline{X}|$ is the usual Zariski space underlying \overline{X} (See [4] 7.4.12 or [21] 6.4). It is this specialization map that our work generalizes, we elaborate below.

In [28] Scholze sets foundations for the theory of diamonds which can be defined as certain sheaves on the category of characteristic p perfectoid spaces endowed with a Grothendieck topology called the v -topology. He associates to any pre-adic space X over \mathbb{Z}_p (not necessarily analytic) a v -sheaf X^\diamond , and whenever X is analytic he proves that X^\diamond is a (locally spatial) diamond. Moreover, Scholze assigns to any v -sheaf \mathcal{F} an underlying topological space $|\mathcal{F}|$ and whenever $\mathcal{F} = X^\diamond$ he constructs a functorial surjective and continuous map $|\mathcal{F}| \rightarrow |X|$. When X is analytic it is proven in [30] that this map is a homeomorphism, but as we will discuss below this fails for non-analytic pre-adic spaces.

In the first chapter, we take as input what we call below a *specializing v -sheaf* \mathcal{F} and we assign to it: a scheme-theoretic v -sheaf \mathcal{F}^{red} which is the analogue of the reduced special fiber of a formal scheme, and a continuous map of topological spaces $\text{sp}_{\mathcal{F}} : |\mathcal{F}| \rightarrow |\mathcal{F}^{\text{red}}|$ that we call the *specialization map* of \mathcal{F} . If \mathcal{X} is a separated formal scheme over \mathbb{Z}_p , we can prove that \mathcal{X}^\diamond is a specializing v -sheaf, and in this case we have natural identifications $|\mathcal{X}_\eta| = |\mathcal{X}^\diamond \times_{\mathbb{Z}_p^\diamond} \mathbb{Q}_p^\diamond|$ and $|\overline{X}| = |(\mathcal{X}^\diamond)^{\text{red}}|$ together with a commutative diagram:

$$\begin{array}{ccc} |\mathcal{X}^\diamond \times_{\mathbb{Z}_p^\diamond} \mathbb{Q}_p^\diamond| & \xrightarrow{\cong} & |\mathcal{X}_\eta| \\ \downarrow \text{sp}_{\mathcal{X}^\diamond} & & \downarrow \text{sp}_{\mathcal{X}} \\ |(\mathcal{X}^\diamond)^{\text{red}}| & \xrightarrow{\cong} & |\overline{X}| \end{array}$$

It is in this sense that our specialization map generalizes the classical one.

The advantage of working in this broader context is that the categories of diamonds and v -sheaves are much more flexible than those of formal schemes and rigid analytic spaces. This allows us to construct interesting spaces that do not come from applying the \diamond -functor to pre-adic spaces. Actually, the main reason the author found the specialization map for diamonds interesting is that it has applications to the study of moduli spaces of mixed-characteristic shtukas. Typically, these moduli spaces are locally spatial diamonds that do not come from a pre-adic space. In forthcoming work of the author, we use the tools developed here to describe the profinite set of geometric connected components of some moduli spaces

of p -adic shtukas at any chosen level (including infinite level). This work builds on and generalizes the work of Chen on the geometric connected components of unramified Rapoport-Zink spaces (See [6]).

To describe the main results of our second chapter, we fix some notation. Let \mathcal{G} be a reductive group over \mathbb{Z}_p , and denote by G the generic fiber of \mathcal{G} over \mathbb{Q}_p . Fix $T \subseteq B \subseteq G$ a maximal \mathbb{Q}_p -rationally defined torus and a Borel respectively, and let \mathfrak{f} be an algebraically closed field extension of $\overline{\mathbb{F}}_p$. We let X_*^+ denote the subset of dominant cocharacters in $X_*(T_{\overline{\mathbb{Q}}_p})$, fix a $\mu \in X_*^+$ and an element $b \in G(W(\mathfrak{f})[\frac{1}{p}])$. Let $E := E(\mu)$ be the reflex field of μ . Since \mathcal{G} is reductive over \mathbb{Z}_p this is an unramified extension of \mathbb{Q}_p . Let J_b denote the σ -centralizer of b appearing in Kottwitz' theory of isocrystals with G -structure [20]. Let F_1 denote a complete non-Archimedean field extension of E , with ring of integers of O_{F_1} and residue field k_{F_1} . Let F_2 be a complete non-Archimedean field extension of $W(\mathfrak{f})[\frac{1}{p}]$ with residue field k_{F_2} . To this data one can associate the following objects:

- a.- A spatial diamond $\mathrm{Gr}_{F_1^\diamond}^{G, \leq \mu}$ proper over F_1^\diamond , parametrizing μ -bounded B_{dR}^+ -lattices with G -structure. Here B_{dR}^+ is the de Rham period ring of Fontaine, and this moduli is the B_{dR} -Grassmanian of the Berkeley notes [30].
- b.- A perfect scheme $\mathrm{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}$ proper and perfectly finitely presented over $\mathrm{Spec}(k_{F_1})$ parametrizing μ -bounded Witt-vector lattices with \mathcal{G} -structure. This is Zhu's Witt-vector Grassmanian [32], [5].
- c.- A locally spatial diamond $\mathrm{Sht}_{(\mathcal{G}, b, \mu), F_2^\diamond}$ partially proper over F_2^\diamond , parametrizing mixed-characteristic shtukas with G -structure that have relative position bounded by μ , and with level structure $\mathcal{G}(\mathbb{Z}_p)$. This is the moduli space of mixed-characteristic shtukas at hyperspecial level that appears in the Berkeley notes [30]. It comes endowed with a continuous $J_b(\mathbb{Q}_p)$ -action.
- d.- A perfect scheme $X_{\leq \mu}^{\mathcal{G}}(b)$ locally perfectly finitely presented over k_{F_2} , which on geometric points evaluates to affine Deligne-Lusztig sets of Rapoport [24]. This space also comes equipped with a continuous $J_b(\mathbb{Q}_p)$ -action.

Fix an algebraically closed non-Archimedean field C over F_1 with ring of integers O_C and let k_C denote the residue field of O_C . In [1], Anschütz constructs a map going from $\mathrm{Gr}_{F_1^\diamond}^{G, \leq \mu}(C, O_C)$ to $\mathrm{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}(k_C)$ which for now we denote sp_{Ans} . Before this work, the map was only known as a map of sets. Building on the work of Anschütz we upgrade that specialization map to construct a specialization map of topological spaces $\mathrm{sp}_{\mathrm{Gr}_{O_{F_1}^\diamond}^{\mathcal{G}, \leq \mu}} : |\mathrm{Gr}_{F_1^\diamond}^{G, \leq \mu}| \rightarrow |\mathrm{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}|$ making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Gr}_{F_1^\diamond}^{G, \leq \mu}(C, O_C) & \xrightarrow{\iota} & |\mathrm{Gr}_{F_1^\diamond}^{G, \leq \mu}| \\ \downarrow \mathrm{sp}_{Ans} & & \downarrow \mathrm{sp}_{\mathrm{Gr}_{O_{F_1}^\diamond}^{\mathcal{G}, \leq \mu}} \\ \mathrm{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}(k_C) & \xrightarrow{\iota} & |\mathrm{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}| \end{array}$$

Here ι associates to a $\mathrm{Spa}(C, O_C)$ -valued point (k_C -valued point respectively) its underlying topological point. We prove the following properties about our specialization map.

Theorem 1. a) *The specialization map*

$$\mathrm{sp}_{\mathrm{Gr}_{O_{F_1}^\diamond}^{\mathcal{G}, \leq \mu}} : |\mathrm{Gr}_{F_1^\diamond}^{G, \leq \mu}| \rightarrow |\mathrm{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}|$$

is a closed and spectral map of spectral topological spaces.

- b) *Given a closed point $x \in |\mathrm{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}|$ let $T_x := \mathrm{sp}_{\mathrm{Gr}_{O_{F_1}^\diamond}^{\mathcal{G}, \leq \mu}}^{-1}(x)$, then the interior T_x° of T_x in $|\mathrm{Gr}_{F_1^\diamond}^{G, \leq \mu}|$ is a dense subset of T_x .*

c) T_x and T_x° are non-empty and connected.

Using a technique that we learned from reading [30] together with the work of Anschütz, we construct a second specialization map but now its source is a moduli space of mixed-characteristic shtukas at hyperspecial level and the target is the affine Deligne-Lusztig variety associated to (\mathcal{G}, b, μ) .

Theorem 2. a) *There is a continuous specialization map*

$$\mathrm{sp}_{\mathrm{Sht}_{O_{F_2}}^{\mathcal{G}_b, \leq \mu}} : |\mathrm{Sht}_{(\mathcal{G}, b, \mu), F_2^\diamond}| \rightarrow |X_{\leq \mu}^{\mathcal{G}}(b)|,$$

this map is a specializing and spectral map of locally spectral topological spaces. It is a quotient map and it is $J_b(\mathbb{Q}_p)$ -equivariant.

b) *Given a closed point $x \in |X_{\leq \mu}^{\mathcal{G}}(b)|$ let $S_x = \mathrm{sp}_{\mathrm{Sht}_{O_{F_2}}^{\mathcal{G}_b, \leq \mu}}^{-1}(x)$, then the interior S_x° of S_x as a subspace of $|\mathrm{Sht}_{(\mathcal{G}, b, \mu), F_2^\diamond}|$ is dense in S_x .*

c) S_x and S_x° are non-empty and connected.

d) *The specialization map induces a $J_b(\mathbb{Q}_p)$ -equivariant bijection of connected components*

$$\mathrm{sp}_{\mathrm{Sht}_{O_{F_2}}^{\mathcal{G}_b, \leq \mu}} : \pi_0(\mathrm{Sht}_{(\mathcal{G}, b, \mu), F_2^\diamond}) \rightarrow \pi_0(X_{\leq \mu}^{\mathcal{G}}(b))$$

The work of Scholze and Weinstein identifies the diamond associated to moduli spaces of p -divisible groups as special instances of moduli spaces of mixed-characteristic shtukas (See [30] 24.3.5). Under this light, the last part of theorem 2 is a generalization of Theorem 5.1.5.(i) of [7] that describes the connected components of unramified Rapoport-Zink spaces at hyperspecial level. The study of the set of connected components of affine Deligne-Lusztig varieties had a lot of progress in the past 10 years. In the case of unramified groups at hyperspecial level the problem is very well understood, and the reader can find in [22] theorem 1.1, [7] theorem 1.1, [12] theorem 0.1, 0.2 concrete descriptions of these sets.

Our last main result compares the preimages of the specialization map of Grassmanians to those of moduli spaces of shtukas. The statement is philosophically aligned with Grothendieck-Messing's deformation theory of p -divisible groups and it is a key input in the proof of theorem 2. A precise statement is:

Theorem 3. *If we let $F_1 = F_2$, then for a closed point $x \in |X_{\leq \mu}^{\mathcal{G}}(b)|$ there is a closed point $y \in |\mathrm{Gr}_{\mathcal{W}, k_{F_2}}^{\mathcal{G}, \leq \mu}|$ such that S_x° considered as an open subdiamond of $\mathrm{Sht}_{(\mathcal{G}, b, \mu), F_2^\diamond}$ is isomorphic to T_y° when considered as an open subdiamond of $\mathrm{Gr}_{F_2^\diamond}^{G, \leq \mu}$. Here S_x and T_y are as in theorems 2 and 1 respectively.*

Now that we have stated our main results we would like to give a short summary of the theory of specialization of the first chapter and provide sketches of the proves of the main results of the second chapter.

Given a Tate Huber pair (A, A^+) with pseudo-uniformizer $\varpi \in A^+$ the specialization map $\mathrm{sp}_A : \mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spec}(A^+/\varpi)$ assigns to $x \in \mathrm{Spa}(A, A^+)$ the prime ideal \mathfrak{p}_x of those elements $a \in A^+$ for which $|a|_x < 1$. This is a continuous and closed map of spectral topological spaces and the construction is functorial in the category of Tate Huber pairs. The central idea of our theory is that, regardless of the definition, the specialization map for more general objects should also be functorial and should agree with the case of Tate Huber pairs. This desideratum naturally leads to defining the specialization map as the only map (if such a thing exists) that could be functorial. One is then forced to change perspective and to look for hypotheses that would prove that a functorial map exists and for conditions that would

make this map unique.

The first question one needs to answer is what should the target and source of the specialization map be? The case of Tate Huber pairs may be a little bit misleading in that Tate Huber pairs come, by design, with a canonical “integral model”. Namely, the integral model for $\mathrm{Spa}(A, A^+)$ is simply given by $\mathrm{Spa}(A^+, A^+)$. For more general spaces there is not a canonical “integral model” and one is forced to attach specialization maps to models rather than to the objects that one makes models of. In the case of Tate Huber pairs the specialization map can be extended to a map $\mathrm{sp}_{A^+} : \mathrm{Spa}(A^+, A^+) \rightarrow \mathrm{Spec}(A^+/\varpi)$ with the same formula. In the general case, one has to find an integral model for the diamond that one wishes to study. The integral models we will consider will be a subcategory of v -sheaves that satisfy some axioms.

An important result of the Berkeley notes proves that the \diamond -functor is a fully-faithful embedding of the category of characteristic p perfect schemes to Scholze’s category of v -sheaves. Our observation is that, with the correct setup, this functor admits a right adjoint which we call suggestively the reduction functor. Moreover, we compute directly that if B is a topological ring over \mathbb{Z}_p endowed with the I -adic topology for some finitely generated ideal I , then $(\mathrm{Spd}(B, B))^{\mathrm{red}}$ is given by the perfection of $\mathrm{Spec}(B/I)$. In particular, if (A, A^+) is a uniform Tate Huber pair, then $(\mathrm{Spd}(A^+, A^+))^{\mathrm{red}}$ is given by the perfection of $\mathrm{Spec}(A^+/\varpi)$. Recall that the underlying topological space of a scheme remains the same after taking its perfection. This suggests that one can define the target of our specialization to be the result of applying the reduction functor to the integral models that we want to associate a specialization map to. In general, the objects obtained in this way will not be perfect schemes but they will be what we call below scheme theoretic v -sheaves. These scheme theoretic v -sheaves come equipped with an underlying topological space that agrees with the Zariski topology whenever the sheaf is represented by a perfect scheme.

Once we have established the source and target, the next step is to construct the map. The key aspect that makes the specialization map for Tate Huber pairs functorial is that every map of Tate Huber pairs $\mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spa}(B, B^+)$ automatically upgrades to a map of “integral models” $\mathrm{Spa}(A^+, A^+) \rightarrow \mathrm{Spa}(B^+, B^+)$. This motivates the following definition: given a v -sheaf \mathcal{F} , an affinoid perfectoid space $\mathrm{Spa}(A, A^+)$ and a map $\iota : \mathrm{Spa}(A, A^+) \rightarrow \mathcal{F}$, we say that \mathcal{F} *formalizes* ι whenever it factors through a map $f : \mathrm{Spd}(A^+, A^+) \rightarrow \mathcal{F}$. We say that \mathcal{F} is v -formalizing if for every ι as above there is a v -cover $g : \mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spa}(A, A^+)$ such that \mathcal{F} formalizes $\iota \circ g$. Given a v -formalizing sheaf \mathcal{F} one can try to define the specialization map $\mathrm{sp}_{\mathcal{F}} : |\mathcal{F}| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ so that for any “formalized” map $f : \mathrm{Spd}(A^+, A^+) \rightarrow \mathcal{F}$ the following diagram is commutative:

$$\begin{array}{ccccc} |\mathrm{Spa}(A, A^+)| & \longrightarrow & |\mathrm{Spd}(A^+, A^+)| & \xrightarrow{f} & |\mathcal{F}| \\ \downarrow \mathrm{sp}_A & & & & \downarrow \mathrm{sp}_{\mathcal{F}} \\ |\mathrm{Spec}(A^+/\varpi)| & \xrightarrow{f^{\mathrm{red}}} & & & |\mathcal{F}^{\mathrm{red}}| \end{array}$$

The recipe to compute the specialization map would then be the following: given $x \in |\mathcal{F}|$ for \mathcal{F} v -formalizing we find an algebraically closed perfectoid field C and an open and bounded valuation ring C^+ together with a map $\iota_x : \mathrm{Spa}(C, C^+) \rightarrow \mathcal{F}$ such that the closed point of $\mathrm{Spa}(C, C^+)$ maps to x under ι_x . After replacing $\mathrm{Spa}(C, C^+)$ by a v -cover, we find a formalization $f_x : \mathrm{Spd}(C^+, C^+) \rightarrow \mathcal{F}$ of ι_x . We apply the reduction functor to f_x and obtain a map $f_x^{\mathrm{red}} : \mathrm{Spec}(C^+/\varpi) \rightarrow \mathcal{F}^{\mathrm{red}}$. Finally, we look at the topological image of the unique closed point of $\mathrm{Spec}(C^+/\varpi)$ under f_x^{red} . We define $\mathrm{sp}_{\mathcal{F}}(x)$ to be this image.

The natural question becomes whether or not this construction is well defined. The problem being that the map $\iota_x : \mathrm{Spa}(C, C^+) \rightarrow \mathcal{F}$ might have more than one formalization. The naive guess would be that this doesn't happen when \mathcal{F} is separated as a v -sheaf. Unfortunately, this is false. At the heart of the problem is the following pathology: although $|\mathrm{Spa}(C, C^+)|$ is dense within $|\mathrm{Spa}(C^+, C^+)|$ it is not true that $|\mathrm{Spd}(C, C^+)|$ is dense within $|\mathrm{Spd}(C^+, C^+)|$ whenever the valuation ring C^+ has rank larger than 1.

Resolving this problem forces us to understand better the topological spaces of the form $|\mathrm{Spd}(A, A^+)|$. To do so, we introduce what we call below the *olivine spectrum* of a Huber pair. The careful reader will notice that our work on the olivine spectrum of Huber pairs is a technification of Scholze and Weinstein's original approach to study perfect schemes as a full subcategory of the category of v -sheaves. Our work allow us to improve Scholze and Weinstein's full faithfulness result to the following statement:

Theorem 4. *Let Y be a perfect non-analytic adic space over \mathbb{F}_p and let X be a pre-adic space over \mathbb{Z}_p . The natural map*

$$\mathrm{Hom}_{\mathrm{PreAd}}(Y, X) \rightarrow \mathrm{Hom}(Y^\diamond, X^\diamond)$$

is bijective. In particular, \diamond is fully faithful when restricted to the category of perfect non-analytic adic spaces over \mathbb{F}_p .

After this rather subtle and long topological detour, we manage to identify a stronger notion of separatedness that we call *formal separatedness*. The main feature of a formally separated v -sheaf \mathcal{F} is that a map $\iota : \mathrm{Spa}(A, A^+) \rightarrow \mathcal{F}$ has at most one formalization (if any).

Combining the two inputs we say that a v -sheaf is specializing if it is v -formalizing and formally separated. We prove that specializing v -sheaves have a unique map that satisfies the commutative diagrams as above for any formalizable map. We prove that this specialization map is functorial in the full subcategory of specializing v -sheaves and that these specialization maps are continuous.

Although specializing v -sheaves produce the specialization maps that we are interested in, they are too general for practical purposes. For this reason, we focus our attention on a more restrictive class of v -sheaves that will have better behaved specialization maps. The central objects of our study is what we call below *kimberlites* (and *smelted kimberlites*). These will be specific kinds of specializing v -sheaves that satisfy other pleasant properties. For example, kimberlites come equipped with a good notion of “analytic locus” that is, by definition, an open subsheaf and a locally spatial diamond. The main advantage of kimberlites over more general specializing v -sheaves is that the specialization map of a kimberlite (when we restrict to the analytic locus) is a spectral maps of locally spectral spaces (i.e. continuous for the constructible topology). The author thinks of kimberlites as a first step towards the goal of formulating the notion of an “integral model” for diamonds. This is in the sense that if we wish to regard a v -sheaf as a “good” integral model for some (locally spatial) diamond, then it should at least satisfy the axioms to be a kimberlite.

Let us move on and discuss the content of the second chapter.

The construction of the specialization maps for the moduli spaces that we study follows from the general formalism that we discuss in the first chapter. To apply the theory one has to find a specializing v -sheaf “interpolating” the source and target of the desired specialization map. The candidates are already provided in the Berkeley notes [30]. More precisely, with the setup as in the beginning,

Scholze and Weinstein describe what we call here the p -adic Beilinson-Drinfeld Grassmanians $\mathrm{Gr}_{O_{F_1}}^{\mathcal{G}, \leq \mu}$ as a v -sheaf over $O_{F_1}^\diamond$ whose generic fiber is $\mathrm{Gr}_{F_1^\diamond}^{G, \leq \mu}$. Also, they describe a v -sheaf $\mathrm{Sht}_{O_{F_2}}^{\mathcal{G}_b, \leq \mu}$ over $O_{F_2}^\diamond$ whose generic fiber is $\mathrm{Sht}_{(\mathcal{G}, b, \mu), F_2^\diamond}$. We still call this v -sheaf the moduli space of mixed characteristic shtukas at hyperspecial level.

The proof that these v -sheaves are specializing uses all the machinery of modern p -adic Hodge theory as it is discussed in the Berkeley notes. Some key technical inputs are Kedlaya’s work [18] and Anschütz’ work (theorem 1.2 of [1]) on extending vector bundles and torsors over the punctured spectrum of A_{inf} . Once we know these v -sheaves are specializing, our theory produces the specialization maps. With more work, we prove that these specializing v -sheaves are even nicer. Namely, we prove that p -adic Beilinson-Drinfeld Grassmanians are kimberlites, and that moduli spaces of shtukas at hyperspecial level are smelted kimberlites. We also prove the identities: $(\mathrm{Gr}_{O_{F_1}}^{\mathcal{G}, \leq \mu})^{\mathrm{red}} = \mathrm{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}$ and $(\mathrm{Sht}_{O_{F_2}}^{\mathcal{G}_b, \leq \mu})^{\mathrm{red}} = X_{\leq \mu}^{\mathcal{G}}(b)$ which tell us that the targets of the specialization maps that our formalism constructs are the desired ones.

After that work, the difficulty becomes to understand the preimages of the specialization map. To tackle this difficulty we introduce some theoretical tools. In the first chapter, to a kimberlite (or a smelted kimberlite) \mathcal{F} and a chosen closed point $x \in |\mathcal{F}^{\mathrm{red}}|$ we attach *the tubular neighborhood* of \mathcal{F} at x which we denote by $\widehat{\mathcal{F}}_x$. Intuitively speaking, these tubular neighborhoods are the subsheaves of points that specialize to x . In general it is true that $\widehat{\mathcal{F}}_x$ is a subsheaf of \mathcal{F} and that $|\widehat{\mathcal{F}}_x| \subseteq \mathrm{sp}_{\mathcal{F}}^{-1}(x)$, but the equality usually doesn’t hold. One has to explore carefully the relation between these two sets. We identify a class of kimberlites (respectively smelted kimberlites), which we call *rich* kimberlites (respectively rich smelted kimberlites), for which tubular neighborhoods behave as nicely as possible. We prove that p -adic Beilinson-Drinfeld Grassmanians and moduli spaces of shtukas at hyperspecial level are rich kimberlites and rich smelted kimberlites, respectively. Being rich implies that $|\widehat{\mathcal{F}}_x|$ is dense within $\mathrm{sp}_{\mathcal{F}}^{-1}(x)$, which proves the density part of theorems 1 and 2.

Once we know that these v -sheaves are rich kimberlites most of the work required to prove theorem 1 and theorem 2 is subsumed by our formalism. The last thing that remains to be proved is that the preimages of the specialization map are non-empty and connected. As we have briefly mentioned, one can apply theorem 3 to reduce the non-empty and connected part of theorem 2 to the similar claim of theorem 1.

To better understand the preimages of the specialization map in the case of theorem 1 we construct a “Demazure resolution” in the spirit of [30] §19.3. Contrary to the case of GL_n , for other reductive groups, the subset of dominant minuscule cocharacters doesn’t generate the monoid of dominant cocharacters. This failure turns out to be a rather subtle matter and forces us to discuss what we call below “parahoric loop groups” for Chevalley groups. These groups are attached to points in the (Bruhat-Tits) apartment of G associated to T . They are subsheaves of the usual loop group given by the condition that their value on geometric points $\mathrm{Spa}(C^\sharp, C^{\sharp+})$ is precisely the corresponding parahoric subgroup of $G(B_{\mathrm{dR}})$ that the Bruhat-Tits theory attaches to the same point in the apartment and the (discrete valuation) period ring B_{dR}^+ .

The construction that we discuss relies on a group-theoretic construction of Pappas and Zhu and on theorem 3.1 of [23]. The tradeoff to using parahoric loop groups is that now the “Schubert varieties” are indexed by elements of the Iwahori-Weyl group, and they can all be resolved using simple reflections in the affine Weyl group. Functoriality of the specialization map will allow us to reduce questions on the

target of the resolution to questions on the source of the resolution. The v -sheaves that serve as source of this resolution, which we call Demazure kimberlites, are easier to understand.

Finally, we explain the content of each section:

- In the first section of chapter 1 we give a short review of the theory of diamonds, the v -topology and some facts about spectral topological spaces. We also review Scholze's \diamond functor that takes as input a pre-adic space over \mathbb{Z}_p and returns as output a v -sheaf.
- In the second section of chapter 1 we introduce and study what we call the olivine spectrum of a Huber pair (B, B^+) which we denote by $\text{Spo}(B, B^+)$. As we have mentioned already, for a pre-adic space X over \mathbb{Z}_p Scholze and Weinstein construct a surjective map of topological spaces $|X^\diamond| \rightarrow |X|$. This map is a homeomorphism whenever X is analytic, but the map will not be injective whenever X is not analytic, and in pathological cases not even a quotient map. The olivine spectrum is a very concrete topological space that one can associate to any Huber pair without any mention to the theory of perfectoid spaces or diamonds. We can summarize the results as follows:
 1. If (B, B^+) is any complete Huber pair over \mathbb{Z}_p we construct a continuous and bijective map $|\text{Spd}(B, B^+)| \rightarrow \text{Spo}(B, B^+)$.
 2. Whenever this map is a homeomorphism we say that (B, B^+) is *olivine*.
 3. Being olivine can be verified locally in the usual topology of $\text{Spa}(B, B^+)$ and it is compatible with rational localization.
 4. Affinoid fields (i.e. (K, K^+) with K a field and K^+ an open bounded valuation subring) are olivine.
 5. If (B, B^+) is uniform (i.e. B° is bounded) and B is a finite type B^+ -algebra then it is olivine.

Using the olivine spectrum we prove theorem 4.

- In the third section of chapter 1 we introduce and study a reduction functor that takes as input a small v -sheaf in the category of characteristic p perfectoid spaces, and returns a small v -sheaf in the category of perfect schemes in characteristic p . This functor generalizes the construction that assigns to a formal scheme topologically of finite type over \mathbb{Z}_p the perfection of its reduced special fiber. To the author's knowledge, although this construction is simple, it had not been considered in the literature before. As we have mentioned, this reduction functor will construct the target of our specialization map.
- In the fourth section of chapter 1 we develop the theoretical framework to study the specialization map. We introduce specializing v -sheaves, kimberlites, and smelted kimberlites. We introduce tubular neighborhoods and relate them with preimages of the specialization map. We define rich kimberlites which incorporate some "finiteness" conditions that are tailored to control the behavior of the preimages of the specialization map.
- In the first section of chapter 2 we review the main geometric objects of modern p -adic Hodge theory. We review Kedlaya and Liu's theory of vector bundles on adic spaces, and the theorems of Kedlaya and Anschütz' on extending vector bundles and G -torsors on the punctured spectrum of A_{inf} . One may think of Anschütz' result as a statement over a point, and Scholze and Weinstein prove in [30] a small improvement to this theorem by considering what we call here a product of points. We review Scholze and Weinstein's proof with our application in mind.

- In the second section of chapter 2 we study the specialization map for p -adic Beilinson-Drinfeld Grassmanians. We construct parahoric loop groups and construct Demazure kimberlites, which are the source of our “Demazure resolution”. We prove that Demazure kimberlites are rich kimberlites which allows us to prove that p -adic Beilinson-Drinfeld Grassmanians are also rich kimberlites with non-empty connected tubular neighborhoods. We prove theorem 1.
- In the third section of chapter 2 we study the specialization map for moduli spaces of mixed-characteristic shtukas. We prove that moduli spaces of mixed characteristic shtukas at hyperspecial level are rich smelted kimberlites. We prove theorems 2 and 3.

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A note on the terminology

The author would like to use this paragraph to make a small comment on the terminology. Some of the terms introduced below come with a metaphor. The incorporation of these metaphors into the text is nothing but a playful manner in which the author decided to interact with the mineralogical history of the field¹. In particular, they shouldn’t be taken seriously for any scientific or mathematical purposes.

The first term is the “olivine spectrum of a Huber pair”. Olivine minerals are a series of mineralogical structures that can be found most commonly in mafic and ultramafic igneous rocks, they are characteristic by their green olive like color. During the formation of a diamond small minerals like olivine, garnet, and chromite among others get surrounded by a host diamond. When these minerals get included in diamonds their morphology changes to resemble the structure that is found in diamonds. Similarly, the olivine spectrum of a Huber pair is a very small variation of the usual adic spectrum that has a subtle diamond-like change.

The second term that deserves an explanation is “kimberlite”. In mineralogy, kimberlites are hybrid rocks that are known to contain diamonds. The formation of diamonds happens in the depths of Earth

¹In other words, a joke.

and through geological processes, kimberlite magma pipes bring the diamonds to the surface. The interest in mining kimberlites comes from the hope of finding diamonds within. Similarly, the author thinks of kimberlites as a natural category for finding integral models of diamonds.

Chapter 1

Theoretical aspects of the specialization map

Throughout this article we assume that the reader is familiar with the basic theory of perfectoid spaces as discussed in ([30] §7) or ([28] §3). In most of our proofs we ignore the set-theoretic subtleties that arise from the theory, but we inherit the usage of the term “small” that is used to address such issues. We provide some indications on how to proceed when set-theoretic carefullness is absolutely necessary.

1.1 The v -topology

1.1.1 Diamonds and small v -sheaves

We let Perfd denote the category of perfectoid spaces and Perf the subcategory of perfectoid spaces in characteristic p . The following definition is taken from ([28] 7.8).

Definition 1.1.1. *Given a map of perfectoid spaces $f : Y \rightarrow X$ we say:*

1. *f is affinoid pro-étale if $Y = \mathrm{Spa}(S, S^+)$, $X = \mathrm{Spa}(R, R^+)$ and the map f is a small cofiltered limit of maps $f_i : \mathrm{Spa}(S_i, S_i^+) \rightarrow \mathrm{Spa}(R, R^+)$ where each f_i is étale.*
2. *f is pro-étale if for every $y \in Y$, there is an open neighborhood $V \subseteq Y$ containing y and an open $U \subseteq X$ satisfying $f(V) \subseteq U$ and $f|_V : V \rightarrow U$ is affinoid pro-étale.*

We can endow Perfd with two Grothendieck topologies, called the pro-étale topology and v -topology respectively, as follows:

Definition 1.1.2. *(See ([28] 8.1))*

1. *A family $\{f_i : Y_i \rightarrow X\}_{i \in I}$ of maps in Perfd is a cover for the pro-étale topology if each f_i is pro-étale and for every quasi-compact open $U \subseteq X$ there is a finite subset $I_U \subseteq I$ and quasi-compact open subsets $V_i \subseteq Y_i$ for all $i \in I_U$, such that $U = \bigcup f_i(V_i)$*
2. *A family $\{f_i : Y_i \rightarrow X\}_{i \in I}$ of maps in Perfd is a cover for the v -topology if for every quasi-compact open $U \subseteq X$ there is a finite subset $I_U \subseteq I$ and quasi-compact open subsets $V_i \subseteq Y_i$ for all $i \in I_U$, such that $U = \bigcup f_i(V_i)$*

Remark 1.1.3. *To make the pro-étale and v topologies useful, it is important to add the quasi-compactness hypothesis. Indeed, since open embeddings are étale the inclusion of points are pro-étale, but we do not want to consider the collection of inclusions of points as a cover.*

The following example of a cover for the v -topology will be used repeatedly.

Example 1.1.4. Let $\mathrm{Spa}(A, A^+)$ be an affinoid perfectoid space $\varpi \in A^+$ and a choice of a pseudo-uniformizer, we consider the following construction. For every point $x \in |\mathrm{Spa}(A, A^+)|$ consider the inclusion of affinoid residue field $\iota_x : \mathrm{Spa}(k(x), k(x)^+) \rightarrow \mathrm{Spa}(A, A^+)$. Note that by ([26] 6.7) each $\mathrm{Spa}(k(x), k(x)^+)$ is perfectoid. Consider $R^+ := \prod_{x \in |\mathrm{Spa}(A, A^+)|} k(x)^+$ as a topological ring with the ϖ -adic topology and let $R = R^+[\frac{1}{\varpi}]$. We have that $\mathrm{Spa}(R, R^+)$ is perfectoid and that the natural map $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spa}(A, A^+)$ is a cover for the v -topology.

Definition 1.1.5. Given a set I and a collection of tuples $\{(C_i, C_i^+), \varpi_i\}_{i \in I}$ we construct an adic space $\mathrm{Spa}(R, R^+)$. Here each C_i is an algebraically closed non-Archimedean field, the C_i^+ are open and bounded valuation subrings of C_i , and ϖ_i is a choice of pseudo-uniformizer. We let $R^+ := \prod_{i \in I} C_i^+$, we let $\varpi = (\varpi_i)_{i \in I}$, we endow R^+ with the ϖ -adic topology and we let $R := R^+[\frac{1}{\varpi}]$. Any space constructed in this way will be called a product of points.

Remark 1.1.6. We point out that different choices of pseudo-uniformizers $(\varpi_i)_{i \in I}$ will give rise to different adic spaces. Indeed, in general $R \subseteq \prod_{i \in I} C_i$ but if I is infinite this is a proper inclusion and the image of this inclusion depends of the choice of pseudo-uniformizers.

Example 1.1.4 proves that products of points form a basis for the v -topology in the category of perfectoid spaces. Recall the notion of totally disconnected spaces.

Definition 1.1.7. (See [28] 7.1, 7.15, 7.5) An affinoid perfectoid space $\mathrm{Spa}(R, R^+)$ is totally disconnected if it splits every open cover. Moreover, it is strictly totally disconnected if it splits every étale cover.

We have the following useful criterion:

Proposition 1.1.8. (See [28] 7.3, 7.16, 11.27) Let Y be an affinoid perfectoid space. Y is represented by a strictly totally disconnected space if and only if every connected component of Y is represented by $\mathrm{Spa}(C, C^+)$ for C an algebraically closed field and C^+ an open and bounded valuation subring.

Proposition 1.1.9. Product of points as in definition 1.1.5 are strictly totally disconnected perfectoid space.

Proof. Take $R^+ := \prod_{i \in I} C_i^+$ and pseudo-uniformizers $\varpi = (\varpi_i)_{i \in I}$ as in definition 1.1.5. The closed-opens subsets of $\mathrm{Spa}(R, R^+)$ are given by idempotents in R^+ , which in turn are given by subsets of I . A connected component $\pi \in \pi_0(\mathrm{Spa}(R, R^+))$ is computed by intersecting $\bigcap_{U \in \mathcal{U}} U$ for some ultrafilter and it is a Zariski closed subsets cut out by the ideal, $I_\pi = \langle 1_V \rangle_{V \notin \mathcal{U}}$, where the idempotents are indexed by the sets that do not belong to the ultrafilter. To compute the structure sheaf of this connected component we have to consider the ϖ -completion of R^+/I_π . Let $V = R^+/I_\pi$ and V' be the completion of V with respect to ϖ .

To prove that $\mathrm{Spa}(R, R^+)$ is a strictly totally disconnected perfectoid space it is enough, by proposition 1.1.8, to prove that V' is a valuation ring with algebraically closed fraction field. In general, if W is a valuation ring with algebraically closed fraction field and if $a \in W$ is not a unit, then the (a) -adic completion of W is also a valuation ring with algebraically closed fraction field. Applying this reasoning to V and V' , we see that it is enough to prove V is a valuation ring with algebraically closed fraction field.

To prove that V is a domain take two elements in $v_1, v_2 \in R^+$ with $v_1 \cdot v_2 = 0$. If we let $I_j \subseteq I$ with $j \in \{1, 2\}$ be the subsets of $i \in I$ such that $v_j = 0$ in C_i^+ then $I_1 \cup I_2 = I$ and one of I_1 or $I_2 \setminus I_1$ is in the ultrafilter, this implies that one of v_1 or v_2 equals 0 in V . Take an element of $v \in \mathrm{Frac}(V)$, this element may be represented by an element of $\prod_{i \in I} C_i$. Since each entry of the product defining R^+ is a valuation

ring one of the sets $\{i \in I \mid v_i \in C_i^+\}$ or $\{i \in I \mid v_i^{-1} \in C_i^+\}$ is in the ultrafilter, this implies $v \in V$ or $v^{-1} \in V$ and that V is a valuation ring. One can prove in a similar way that $\text{Frac}(V) = \prod_{i \in I} C_i / I_\pi$. In particular, it is an ultraproduct of algebraically closed fields, so $\text{Frac}(V)$ is algebraically closed. \square

Scholze proves that the v -topology (and consequently the pro-étale topology) on Perfd is subcanonical ([28] 8.6). To simplify notation, we denote a perfectoid space and the sheaf it represents with the same letter. In case we need to make a distinction, whenever X is a perfectoid space the sheaf it represents will be denoted by h_X . From now on we will focus most of our attention to the site Perf endowed with either the pro-étale or the v -topology. Let us recall Scholze's category of diamonds.

Definition 1.1.10. (See [28] 11.1) *A pro-étale sheaf Y on Perf is a diamond if it can be written as X/R where X and R are representable by perfectoid spaces and $R \subseteq X \times X$ is an equivalence relation for which the two projections to X are pro-étale maps of perfectoid spaces.*

Given a diamond Y we can associate to it a topological space, denoted $|Y|$, as follows:

Definition 1.1.11. *We say that a map $p : \text{Spa}(K, K^+) \rightarrow Y$ is a point if K is a perfectoid field in characteristic p and K^+ is an open and bounded valuation subring of K . Two points $p_i : \text{Spa}(K_i, K_i^+) \rightarrow Y$, $i \in \{1, 2\}$, are equivalent if there is a third point $p_3 : \text{Spa}(K_3, K_3^+) \rightarrow Y$, and surjective maps $q_i : \text{Spa}(K_3, K_3^+) \rightarrow \text{Spa}(K_i, K_i^+)$ making the following commutative diagram:*

$$\begin{array}{ccccc}
 & & \text{Spa}(K_1, K_1^+) & & \\
 & \nearrow q_1 & & \searrow p_1 & \\
 \text{Spa}(K_3, K_3^+) & \xrightarrow{p_3} & & & Y \\
 & \searrow q_2 & & \nearrow p_2 & \\
 & & \text{Spa}(K_2, K_2^+) & &
 \end{array}$$

We let $|Y|$ denote the set of equivalence classes of points of Y .

Scholze proves that if Y has a presentation X/R , then there is canonical bijection of sets between $|Y|$ and $|X|/|R|$ (where $|X|$ and $|R|$ are the topological space corresponding to the perfectoid spaces X and R). This gives a surjective map $|X| \rightarrow |Y|$ and we give $|Y|$ the quotient topology for such a map.

Proposition 1.1.12. (See [28] 11.13) *Let Y be a diamond. The topology on $|Y|$ is independent of the presentation of Y as a quotient $Y = X/R$, with X and R perfectoid spaces.*

We remark that if X is a perfectoid space, then the sheaf h_X represented by X is a diamond and that $|h_X|$ is canonically homeomorphic to $|X|$.

We refer to sheaves on Perf for the v -topology as v -sheaves and we say that a v -sheaf X is small if it admits a surjective map from a representable sheaf. This is a set theoretic condition.

Proposition 1.1.13. (See [28] 11.9) *Every diamond is a small v -sheaf.*

Recall that any Grothendieck site has an intrinsic notion of quasi-compactness. Quasi-compact v -sheaves are other important examples of small v -sheaves.

We denote by $\widetilde{\text{Perf}}$ the category of small v -sheaves, it may be constructed as follows. Given a cut-off cardinal κ (see [28] §4 and §8 for details) we denote by Perf_κ the category of κ -small perfectoid spaces

in characteristic p and by $\widetilde{\text{Perf}}_\kappa$ the topos of sheaves for the v -topology on this category. Objects in this topos will be called κ -small v -sheaves. We have natural fully-faithful embeddings $\widetilde{\text{Perf}}_\kappa \subseteq \widetilde{\text{Perf}}_\lambda$ for $\kappa < \lambda$ and we define $\widetilde{\text{Perf}} = \bigcup_\kappa \widetilde{\text{Perf}}_\kappa$ as a big filtered colimit over all cut-off cardinals κ .

Scholze associates to any small v -sheaf a topological space. The definition is almost identical to 1.1.11, the key point being that if $X \rightarrow Y$ is a map of small v -sheaves with X a diamond then $R = X \times_Y X$ is also a diamond and $Y = X/R$ ([28] 12.3). Scholze then defines $|Y|$ as $|X|/|R|$ with the quotient topology.

Proposition 1.1.14. (See [28] 12.7) *Let Y be a small v -sheaf. The set of equivalence classes of points of Y is in canonical bijection with $|X|/|R|$ for any presentation $Y = X/R$ with X and R diamonds. Moreover, the topology induced this way is independent of the presentation.*

Given a topological space T we can consider a presheaf on Perf , denoted \underline{T} , defined as

$$\underline{T}(R, R^+) = \{f : |\text{Spa}(R, R^+)| \rightarrow T \mid f \text{ is continuous}\}$$

This forms a v -sheaf but we warn the reader that it might not be small. There is a natural transformation:

$$X \rightarrow \underline{|X|}$$

A morphism of small v -sheaves $j : U \rightarrow X$ is said to be open if it is relatively representable in perfectoid spaces and after basechange by a perfectoid space it becomes an open embedding of perfectoid spaces. The following proposition shows that this is a purely topological notion.

Proposition 1.1.15. (See [28] 11.15 and 12.9) *Let Y be a small v -sheaf and let $|V|' \subseteq |Y|$ be an open subset. Define V as the Cartesian product:*

$$\begin{array}{ccc} V & \longrightarrow & |V|' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & |Y| \end{array}$$

The following assertions hold:

1. The map $V \rightarrow Y$ is an open embedding of small v -sheaves.
2. The induced map $|V| \rightarrow |Y|$ is an open embedding of topological spaces and factors through a homeomorphism to $|V|'$.
3. Every open embedding of small v -sheaves is isomorphic to one constructed in this way.
4. If Y is a diamond then V is also a diamond.

The concept of closed immersion is a little more subtle. It is not a purely topological condition in the sense that closed subsheaves of \mathcal{F} are not in one to one bijection with closed subsets of $|\mathcal{F}|$.

Definition 1.1.16. (See [28] 10.7, 10.11, 5.6) *A map of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ is a closed immersion if for every $X = \text{Spa}(R, R^+)$ a strictly totally disconnected space and a map $X \rightarrow \mathcal{G}$ the pullback $X \times_{\mathcal{F}} \mathcal{G} \subseteq X$ is representable by a closed immersion.*

The following result of Johan Anschütz, João Lourenço and Timo Richarz, that will appear in [16], characterizes closed immersions.

Proposition 1.1.17. (See [16]) For a v -sheaf \mathcal{F} we say a subset $X \subseteq |\mathcal{F}|$ is weakly generalizing if for any geometric point $f : \mathrm{Spa}(C, C^+) \rightarrow \mathcal{F}$ we have that $f^{-1}(X) \subseteq |\mathrm{Spa}(C, C^+)|$ is stable under generization. For any v -sheaf \mathcal{F} the rule

$$X \mapsto \mathcal{F} \times_{|\mathcal{F}|} \underline{X} \subseteq \mathcal{F}$$

gives a bijection between weakly generalizing closed subsets of $|\mathcal{F}|$ and closed subsheaves of \mathcal{F} .

1.1.2 Spectral spaces and locally spatial diamonds

The category of diamonds is too general for some purposes and one can construct many “pathological” examples of diamonds that do not arise from an algebro-geometric context. To control this flexibility Scholze considers some restrictions on the underlying topological space of a diamond.

We begin by recalling the basic theory of spectral topological spaces. This material is taken from section §2 of [28] where most of the proofs can be found.

Definition 1.1.18. For topological spaces S, T and a continuous map $f : S \rightarrow T$ we say that:

1. T is spectral if it is quasi-compact, quasi-separated, and it has a basis of open neighborhoods stable under intersection that consists of quasi-compact and quasi-separated subsets.
2. T is locally spectral if it admits an open cover by spectral spaces.
3. f is a spectral map of spectral spaces if S and T are spectral and f is quasi-compact.
4. f is a spectral map of locally spectral spaces if S and T are locally spectral and for every quasi-compact open $U \subseteq S$ and quasi-compact open $V \subseteq T$ with $f(U) \subseteq V$ the function $f|_U : U \rightarrow V$ is spectral.

Theorem 1.1.19. (Hochstetter) For a topological space T the following conditions are equivalent:

1. T is spectral.
2. T is homeomorphic to the spectrum of a ring.
3. T is a projective limit of finite T_0 topological spaces.

Moreover, the category of spectral topological spaces with spectral maps is equivalent to the pro-category of finite T_0 topological spaces.

Given a spectral space T , we say that a subset S is constructible if it lies in the Boolean algebra formed by quasi-compact open subsets of T . For a locally spectral space T , a subset S is constructible if for every quasi-compact open subset $U \subseteq T$ the subset $S \cap U$ is constructible in U . The patch (or constructible) topology on T is the one in which constructible subsets form a basis for the topology. A spectral space is Hausdorff and profinite for its patch topology and a locally spectral space is Hausdorff and locally profinite for the patch topology.

Proposition 1.1.20. A continuous map of locally spectral spaces $f : S \rightarrow T$ is spectral if and only if it is continuous for the patch topology.

Definition 1.1.21. 1. A map of topological spaces $f : S \rightarrow T$ is generalizing if for elements $t_1, t_2 \in T$ and $s_1 \in S$ such that $f(s_1) = t_1$ and t_2 generalizes t_1 , there exists an element s_2 generalizing s_1 with $f(s_2) = t_2$.

2. A map of topological spaces $f : S \rightarrow T$ is specializing if for elements $t_1, t_2 \in T$ and $s_1 \in S$ such that $f(s_1) = t_1$ and t_2 specializes from t_1 , there exists an element s_2 specializing from s_1 with $f(s_2) = t_2$.

For a locally spectral space T we say that a subset is pro-constructible if it is closed for the patch topology, or equivalently if it is an arbitrary intersection of constructible subsets. The following will be really useful for our purposes.

Proposition 1.1.22. (See [28] 2.4) *Let T be a spectral space and $S \subseteq T$ a pro-constructible subset. The closure \overline{S} of S in T consists of the points that specialize from a point in S .*

Corollary 1.1.23. *Let $f : S \rightarrow T$ be a spectral map of spectral spaces. If f is specializing then it is also a closed map.*

We warn the reader that the analogue of 1.1.23 for locally spectral spaces does not hold.

Proposition 1.1.24. (See [28] 2.5) *Let $f : S \rightarrow T$ be a spectral map of spectral topological spaces. Assume f is surjective and generalizing, then it is a quotient map.*

One can think of spectral spaces as the topological spaces that arise from an algebro-geometric situation. For this reason we will restrict our attention to diamonds that have this behavior.

Definition 1.1.25. (See [28] 11.17) *Let X be a diamond. We say that X is a spatial diamond if it is quasi-compact, quasi-separated and $|X|$ has a basis of open neighborhoods of the form $|U|$ where $U \subseteq X$ is a quasi-compact open embedding. We say that X is locally spatial if it has an open cover by spatial diamonds.*

As promised, the topology of spatial diamonds is spectral. Nevertheless, we remark that a diamond that has a spectral underlying topological space might not necessarily be spatial since the quasi-compactness and quasi-separatedness conditions of definition 1.1.25 are imposed on the topos-theoretic sense.

Proposition 1.1.26. (See [28] 11.18, 11.19) *Let X and Y be locally spatial diamonds and $f : X \rightarrow Y$ a morphism of v -sheaves. The following assertions hold:*

1. $|X|$ is a locally spectral topological space.
2. Any open subfunctor $U \subseteq X$ is a locally spatial diamond.
3. $|X|$ is quasi-compact (respectively quasi-separated) as a topological space if and only if X is quasi-compact (respectively quasi-separated) as a v -sheaf.
4. The topological map $|f|$ is spectral and generalizing. In particular, if $|X|$ is quasi-compact and $|f|$ is surjective then by proposition 1.1.24 it is also a quotient map.

1.1.3 Pre-adic spaces as v -sheaves

The theory of diamonds is mainly of “analytic” nature. On the other hand, we will need to consider some spaces that have a scheme-theoretic and formal-scheme-theoretic flavor instead. The category of v -sheaves allows us to consider these three types of spaces at the same time. In what follows, we show (following the Berkeley notes) how to consider any pre-adic space over \mathbb{Z}_p as a v -sheaf. Interestingly, this functor is by construction far from being fully-faithful, but we will justify below why the new morphisms are mostly of analytic nature.

Let us give a quick recollection of the appendix to lecture 3 of [30]. Let Caff^{op} denote the opposite category to the category of complete Huber pairs. This category can be regarded as a site when we consider the topology generated by rational covers. Although the topology in this site is not subcanonical any Huber pair $(A, A^+) \in \text{Caff}^{op}$ defines a sheaf $\text{Spa}(A, A^+)^Y : \text{Caff}^{op} \rightarrow \text{Sets}$ by taking sheafification of the functor $(B, B^+) \mapsto \text{Hom}_{\text{Caff}}((A, A^+), (B, B^+))$. Scholze and Weinstein define the category of Yoneda-adic spaces as sheaves $\mathcal{F} : \text{Caff}^{op} \rightarrow \text{Sets}$ that are locally isomorphic to $\text{Spa}(A, A^+)^Y$ for a suitable notion of open immersion of sheaves. Recall the category $(V)^{ind}$ whose objects are triples $(X, \mathcal{O}_X^{ind}, (|\cdot(x)|)_{x \in X})$ where X is a topological space, \mathcal{O}_X^{ind} is a sheaf of ind-topological rings and $|\cdot(x)|$ is an equivalence class of valuations on $\mathcal{O}_{X,x}$. For a Huber pair (A, A^+) Scholze and Weinstein define $\text{Spa}^{ind}(A, A^+)$ to be the object in $(V)^{ind}$ with underlying topological space $\text{Spa}(A, A^+)$ and with \mathcal{O}_X^{ind} defined as the sheafification of the structure presheaf of $\text{Spa}(A, A^+)$ in the category of ind-topological rings. A pre-adic space is an object $X \in (V)^{ind}$ that is locally of the form $\text{Spa}^{ind}(A, A^+)$.

Proposition 1.1.27. (See [30] 3.5.3) *For a pre-adic space X , the functor*

$$h_X = ((A, A^+) \mapsto \text{Hom}_{(V)^{ind}}(\text{Spa}^{ind}(A, A^+), X))$$

is a Yoneda adic space. The functor $X \mapsto h_X$ is an equivalence of categories between pre-adic spaces and Yoneda-adic spaces. This equivalence preserves the notions of open immersions and the functor associated to $\text{Spa}^{ind}(A, A^+)$ gets identified with $\text{Spa}^Y(A, A^+)$.

Certain things are easier to think in one perspective than in the other. An important aspect is that every pre-adic space X has an underlying topological space, and we can define the open analytic locus $|X|^{an}$ and the non-analytic locus $|X|^{na}$ in the naive way. That is, a point $x \in |X|$ is analytic if for every open affinoid $x \in \text{Spa}^{ind}(A, A^+) \subseteq |X|$ (equivalently one affinoid) x is analytic in $\text{Spa}(A, A^+)$.

Proposition 1.1.28. *Given a pre-adic space X there is a reduced non-analytic adic space X^{na} and a map $X^{na} \rightarrow X$ which is final in the category of maps $Y \rightarrow X$ with Y a reduced non-analytic adic space. Moreover, the map $|X^{na}| \rightarrow |X|^{na}$ is a homeomorphism.*

Proof. In the affinoid case we have that $\text{Spa}^{ind}(A, A^+)^{na}$ is given by

$$\text{Spa}(A/A^{\circ\circ} \cdot A, A^+/A^{\circ\circ} \cdot A^+).$$

Observe that since $A/A^{\circ\circ} \cdot A$ is discrete it is sheafy. Moreover if (B, B^+) has the discrete topology then $\text{Hom}(\text{Spa}^{ind}(B, B^+), \text{Spa}^{ind}(A, A^+))$ are given by maps of Huber pairs $(A, A^+) \rightarrow (B, B^+)$ and all topological nilpotents must map to 0 in B proving the universal property. The claim of topological spaces is clear.

For general pre-adic space X we define X^{na} to have underlying topological space $|X|^{na}$ and if $V \subseteq |X^{na}|$ is of the form $U \cap |X|^{na}$ for $U \subseteq |X|$ open and of the form $U = \text{Spa}^{ind}(A, A^+)$ we let

$$\mathcal{O}_{X^{na}}^{ind}(V) := \mathcal{O}_X^{ind}(U)/A^{\circ\circ} \cdot \mathcal{O}_X^{ind}(U).$$

Since the construction $A \mapsto A/A^{\circ\circ}$ is compatible with rational localization $\mathcal{O}_{X^{na}}^{ind}(V)$ is well-defined and glues to a sheaf of ind-topological rings on X^{na} . Moreover, locally the ind-topological rings are constant because $A/A^{\circ\circ}$ is sheafy. This implies X^{na} is an adic space. \square

Definition 1.1.29. 1. *We define the presheaf \mathbb{Z}_p^\diamond on Perf as the moduli of untilts, more precisely:*

$$\mathbb{Z}_p^\diamond(Y) = \{(Y^\sharp, \iota)\} / \cong$$

Where Y^\sharp is a perfectoid space in Perfd and $\iota : (Y^\sharp)^\flat \rightarrow Y$ is an isomorphism of perfectoid spaces in characteristic p .

2. Given a pre-adic space $X/\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ we define the presheaf X^\diamond on Perf as:

$$X^\diamond(Y) = \{(Y^\sharp, \iota, f)\} / \cong$$

Where $Y^\sharp \in \mathrm{Perfd}$, $\iota : (Y^\sharp)^\flat \rightarrow Y$ is an isomorphism of perfectoid spaces in characteristic p , and $f : Y^\sharp \rightarrow X$ is a morphism of pre-adic spaces.

Notice that there is a canonical morphism $X^\diamond \rightarrow \mathbb{Z}_p^\diamond$ given by forgetting the last entry of data.

Proposition 1.1.30. (See [30] 18.1.1) For any pre-adic space X (not necessarily analytic) over \mathbb{Z}_p , the presheaf X^\diamond is a small v -sheaf.

From now on, given a Huber pair (A, A^+) we will denote the v -sheaf $(\mathrm{Spa}(A, A^+)^Y)^\diamond$ by $\mathrm{Spd}(A, A^+)$ and we will also drop the decoration $(\cdot)^Y$ when referring to affinoid pre-adic spaces. If R is a base ring whose underlying topology (and ring of integral elements) is understood from the context we will abbreviate $\mathrm{Spd}(R, R^+)$ by R^\diamond . For example, $\mathbb{F}_p^\diamond, \mathbb{Z}_p^\diamond, \mathbb{Q}_p^\diamond$, etc.

Proposition 1.1.31. Let us collect some facts about \diamond , that are either in the literature or are easy to prove:

1. For any perfectoid space X we have that $X^\diamond \cong h_{X^\flat}$ as v -sheaves. (See [28] 15.2).
2. For any analytic pre-adic space X over \mathbb{Z}_p , the functor X^\diamond is a locally spatial diamond and $|X^\diamond| \cong |X|$. (See [28] 15.6).
3. For any pre-adic space over \mathbb{Z}_p there is a surjective map of topological spaces $|X^\diamond| \rightarrow |X|$ (See [30] 18.2.2).
4. If $\mathrm{PreAd}_{\mathbb{Z}_p}$ denotes the category of pre-adic spaces over \mathbb{Z}_p then $\diamond : \mathrm{PreAd}_{\mathbb{Z}_p} \rightarrow \widetilde{\mathrm{Perf}}$ commutes with limits and colimits. More precisely, if X_i is a family of pre-adic spaces indexed by a small category I and the functor $\varinjlim_{i \in I} X_i$ (respectively $\varprojlim_{i \in I} X_i$) is represented by a pre-adic space X then $X^\diamond = \varinjlim_{i \in I} X_i^\diamond$ (respectively $X^\diamond = \varprojlim_{i \in I} X_i^\diamond$). Indeed, both computations are done in the category of sheaves of a Grothendieck site. The only difference in the computations is the topology that one has to use to sheafify. But if a colimit is represented by a pre-adic space by proposition 1.1.30 it is already a v -sheaf.
5. For any complete Huber pair (B, B^+) over \mathbb{Z}_p the v -sheaf $\mathrm{Spd}(B, B^+)$ is separated over \mathbb{Z}_p^\diamond . Indeed, the basechange of the diagonal map $\mathrm{Spd}(B, B^+) \rightarrow \mathrm{Spd}(B, B^+) \times_{\mathbb{Z}_p^\diamond} \mathrm{Spd}(B, B^+)$ by any map $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(B, B^+) \times_{\mathbb{Z}_p^\diamond} \mathrm{Spd}(B, B^+)$ is given by the Zariski closed immersion defined by the ideal $R^\sharp \cdot I_\Delta$ with I_Δ the image of $\ker(B \otimes_{\mathbb{Z}_p} B \rightarrow B)$ in R^\sharp .
6. For any pre-adic space X the map of v -sheaves $(X^{na})^\diamond \rightarrow X^\diamond$ is a closed immersion of v -sheaves and $X^\diamond \setminus (X^{na})^\diamond = (X^{an})^\diamond$. Indeed, this can be verified locally so we may assume $X = \mathrm{Spa}(A, A^+)$. If $Y = \mathrm{Spa}(R, R^+)$ is a strictly totally disconnected space then $Y \times_{X^\diamond} (X^{na})^\diamond$ is a Zariski closed immersion defined by the image ideal of $A^{\circ\circ}$ in R^\sharp .

1.2 The olive spectrum

As we will see below, given a pre-adic space X over \mathbb{Z}_p the map $|X^\diamond| \rightarrow |X|$ of remark 1.1.31 will usually not be injective when X has non-analytic points. Although the map is always surjective, it might not be a quotient map in pathological cases. To develop our theory of specialization map we need better understanding of the topological spaces of the form $|\mathrm{Spd}(A, A^+)|$ for I -adic rings A over \mathbb{Z}_p . To tackle this

difficulty we introduce below what we call the *olivine spectrum* of a Huber pair, which is a very small variation of Huber's adic spectrum. The interest in studying the olivine spectrum is that if (B, B^+) is a complete Huber pair over \mathbb{Z}_p subject to some mild "finiteness" conditions then the topological space $|\mathrm{Spd}(B, B^+)|$ is homeomorphic to the olivine spectrum of (B, B^+) .

For the following fix (B, B^+) a Huber pair (not necessarily over \mathbb{Z}_p and not necessarily complete).

1.2.1 Review, terminology and conventions

We assume the reader to be familiar with the construction of Huber's adic spectrum, $\mathrm{Spa}(B, B^+)$, but we review some key aspects and definitions. We also fix some terminology.

1. Given $x \in \mathrm{Spa}(B, B^+)$ we define the support $\mathrm{supp}(x) \subseteq B$ as the set of elements $b \in B$ for which $|b|_x = 0$. This is a prime ideal of B .
2. We say that a point $x \in \mathrm{Spa}(B, B^+)$ is non-analytic if $\mathrm{supp}(x)$ is an open ideal of B , we say it is analytic otherwise.
3. Given an equivalence class of valuations on B , say represented by $|\cdot|_x : B \rightarrow \Gamma_x \cup \{0\}$, and a convex subgroup $H \subseteq \Gamma_x$, we define a second equivalence class of valuations represented by $|\cdot|_y : B \rightarrow (\Gamma_x/H) \cup \{0\}$ with $|b|_y = |b|_x + H \in \Gamma_x/H$ when $|b|_x \neq 0$ and $|b|_y = 0$ when $|b|_x = 0$. Any equivalence class of valuations constructed in this way is called a *vertical generization* of x .
4. Given a complete Huber pair (B, B^+) and a point $x \in \mathrm{Spa}(B, B^+)$ there is a residue field map of complete Huber pairs $\iota_x^* : (B, B^+) \rightarrow (K_x, K_x^+)$. In this case K_x is either a discrete field or a complete non-Archimedean field. In both cases, K_x^+ is an open and bounded valuation subring of K_x . The induced map $\iota_x : \mathrm{Spa}(K_x, K_x^+) \rightarrow \mathrm{Spa}(B, B^+)$ is a homeomorphism onto the subspace of $\mathrm{Spa}(B, B^+)$ consisting of continuous vertical generizations of x . The map satisfies the following universal property: For any map of complete Huber pairs $f^* : (B, B^+) \rightarrow (A, A^+)$ such that $f(\mathrm{Spa}(A, A^+)) \subseteq \mathrm{Spa}(B, B^+)$ consists of vertical generizations of x , there is a unique factorization $f^* = g^* \circ \iota_x^*$.
5. Vertical generizations and residue field maps have the following compatibility. Fix $x \in \mathrm{Spa}(B, B^+)$ with residue field (K_x, K_x^+) , and consider K_x° the subring of power-bounded elements. Given y a continuous vertical generization of x we can associate a valuation subring K_y^+ by letting $K_y^+ = \{b \in K_x \mid |b|_y \leq 1\}$. This association gives a bijection between the set of continuous vertical generizations of x and valuation subrings of K_x° containing K_x^+ . Moreover, in this case the residue field at y is (K_x, K_y^+) .
6. We say that a valuation x is *trivial* if it is equivalent to some valuation for which $\Gamma_x = \{1\}$. The residue field of a trivial valuation is discrete.
7. We say that a valuation is *microbial* if it has a non-trivial rank 1 vertical generization.
8. For technical reasons that will become clear to the reader below, we take the convention of considering trivial valuations as rank 1 valuations.
9. Given a valuation $|\cdot|_x$ of B we define the *characteristic subgroup* of $|\cdot|_x$, denoted by $c\Gamma_x$, as the smallest convex subgroup of Γ_x containing all elements of the form $\gamma = |b|_x$ for $b \in B$ with $1 \leq \gamma$.
10. Given an equivalence class of valuations $|\cdot|_x$ and a convex subgroup $H \subseteq \Gamma_x$ containing $c\Gamma_x$, we define a second equivalence class of valuations $|\cdot|_y$ with $|\cdot|_y : B \rightarrow H \cup \{0\}$. We let $|b|_y = |b|_x$ if $|b|_x \in H$ and we let $|b|_y = 0$ otherwise. Any equivalence class of valuations constructed in this

way is continuous if $|\cdot|_x$ is continuous. Equivalence classes of valuations constructed in this way are called *horizontal specialization* of x .

11. Horizontal specializations and residue field maps have the following compatibility. Fix $x \in \text{Spa}(B, B^+)$ with residue field (K_x, K_x^+) . We let K_B be the smallest valuation subring of K_x containing K_x^+ and the image of B in K_x . We get a natural map of Huber pairs $(B, B^+) \rightarrow (K_B, K_B^+)$, we consider the induced map $f : \text{Spec}(K_B) \rightarrow \text{Spec}(B)$. Horizontal specializations of x are in bijection with prime ideals of B that are in the image of f . Given a convex subgroup H containing $c\Gamma_x$ we can describe the prime ideal \mathfrak{p}_y associated to y as the set of elements of B with $|b|_x < \gamma$ for all $\gamma \in H$. We will denote $|\cdot|_y$ as $|\cdot|_x/\mathfrak{p}_y$.
12. Given a topological space T we construct a partial order on elements of T by letting $t_1 \leq t_2$ whenever $t_1 \in \overline{\{t_2\}}$. We call this partial order the *generization pattern* of T .
13. Vertical generizations and horizontal specializations completely describe the generization pattern of $\text{Spa}(B, B^+)$. More precisely, if y is a vertical generization of x then $x \in \overline{\{y\}}$ and we let xRy . If z is a horizontal specialization of x then $z \in \overline{\{x\}}$ and we let zRx . The generization pattern of $\text{Spa}(B, B^+)$ is the transitive closure of the relation R .

1.2.2 Definitions and basic properties

Definition 1.2.1. 1. We let $\text{Spo}(B, B^+)$, denote the subset of $\text{Spa}(B, B^+) \times \text{Spa}(B, B^+)$ consisting of pairs $x = (|\cdot|_x^h, |\cdot|_x^a)$ such that $|\cdot|_x^a$ is a rank 1 valuation and a vertical generization of $|\cdot|_x^h$.

2. Given two elements $b_1, b_2 \in B$ we let $U_{b_1 \leq b_2 \neq 0}$ be the set

$$\{x \in \text{Spo}(B, B^+) \mid |b_1|_x^h \leq |b_2|_x^h \neq 0\},$$

we call such subsets *classical localizations*.

3. Given two elements $b_1, b_2 \in B$ we let $N_{b_1 < b_2}$ to be the set

$$\{x \in \text{Spo}(B, B^+) \mid |b_1|_x^a < |b_2|_x^a \neq 0\},$$

we call such subsets *analytic localizations*.

4. We give $\text{Spo}(B, B^+)$ the topology generated by classical and analytic localizations, and we call the resulting topological space the *olivine spectrum* of (B, B^+) .

We will denote by $h : \text{Spo}(B, B^+) \rightarrow \text{Spa}(B, B^+)$ the projection onto the first coordinate. This map is continuous and both $\text{Spo}(-, -^+)$ and h are functorial in the category of Huber pairs.

Definition 1.2.2. Let $x \in \text{Spo}(B, B^+)$.

1. We say that x is *non-analytic* if $|\cdot|_x^a$ is trivial. We say that a non-analytic point is *microbial* if $h(x)$ is microbial. We say that a non-analytic point is *algebraic* if $|\cdot|_x^h$ is trivial.
2. We say that x is *d-analytic* if $|\cdot|_x^a$ is non-trivial. Suppose that x is d-analytic, we say that it is *analytic* if $h(x)$ is analytic and we say it is *meromorphic* otherwise.
3. We say that x is *bounded* if $|B|_x^a \leq 1$.
4. We say that x is *formal* if it is bounded and d-analytic.

Notice that for any point $x \in \text{Spo}(B, B^+)$ the set $h^{-1}(h(x))$ has at most one d-analytic point and at most one non-analytic point. The cardinality of $h^{-1}(h(x))$ is either one or two. If x is d-analytic then $h^{-1}(h(x))$ has cardinality one only when x is analytic. We warn the reader that although the definitions are designed so that $x \in \text{Spo}(B, B^+)$ is analytic if and only if $h(x) \in \text{Spa}(B, B^+)$ is analytic this is not the case for non-analytic points. Indeed, if x is meromorphic it is d-analytic but $h(x)$ is non-analytic. Meromorphic points behave as analytic points but they are not fully detached from their algebraic nature.

Definition 1.2.3. We define the support ideal, $\text{supp}(x) \subseteq B$, as $\text{supp}(h(x))$. We define the specialization ideal, $\text{sp}(x) \subseteq B^+$ as the set of elements of B^+ with $|b|_x^h < 1$. Whenever x is bounded we define the deformation ideal, denoted $\text{def}(x)$, as the prime ideal of elements of B for which $|b|_x^a < 1$.

Notice that x is bounded if and only if $c\Gamma_{x^a} = \{1\}$. Moreover, this only happens if x is either non-analytic or formal. When x is bounded it is non-analytic when $\text{supp}(x) = \text{def}(x)$ and formal otherwise. We can define the bounded locus, denoted $\text{Spo}(B, B^+)^{\dagger}$, to be the subset of points that are bounded. This is a closed subset since it is the complement of $\cup_{b \in B} N_{1 < b}$.

Definition 1.2.4. Let x and y be two points in $\text{Spo}(B, B^+)$.

1. We say that y is a vertical generization of x (x is a vertical specialization respectively) if $|\cdot|_x^a = |\cdot|_y^a$ and $|\cdot|_y^h$ is a vertical generization of $|\cdot|_x^h$ in $\text{Spa}(B, B^+)$.
2. We say that y is a meromorphic generization of x (meromorphic specialization respectively) if y is meromorphic, x is non-analytic and $h(x) = h(y)$.
3. We say that y is a formal generization of x (formal specialization respectively) if y is formal, x is non-analytic $\text{def}(y) = \text{supp}(x)$ and $|\cdot|_x^h = |\cdot|_y^h / \text{def}(y)$.

Given $x \in \text{Spo}(B, B^+)$ let $\mathcal{I}^{\leq}(x)$ denote the set of generizations of x in $\text{Spo}(B, B^+)$ and let $\mathcal{I}_{\text{ver}}^{\leq}(x)$ denote the set of vertical generizations of x . If the context is clear, for a point $y \in \text{Spa}(B, B^+)$ we will also use $\mathcal{I}_{\text{ver}}^{\leq}(y)$ to denote the vertical generizations of y in $\text{Spa}(B, B^+)$. Let us make some easy observations and set some convenient notation:

1. If x is non-analytic it has a meromorphic generization (necessarily unique) if and only if x is a microbial. We denote this generization by x^{mer} .
2. If x is meromorphic it has a unique meromorphic specialization, we denote it by x_{mer} .
3. If x is formal it has a unique formal specialization, we denote it by x_{for} . If x is non-analytic, we let x^{For} denote the set of formal generizations of x .

Example 1.2.5. If $B = \mathbb{F}_p[[t]]$, the ring of formal power series over \mathbb{F}_p endowed with the discrete topology, then $\text{Spa}(B, B)$ consists of 3 points:

$$\left\{ \eta = |\cdot|_{\eta}, s = |\cdot|_s, t = |\cdot|_t \right\}$$

Here $|\cdot|_{\eta}$ is the trivial valuation with residue field $\mathbb{F}_p((t))$, $|\cdot|_s$ is the trivial valuation with residue field \mathbb{F}_p and $|\cdot|_t$ is the (t) -adic valuation on $\mathbb{F}_p[[t]]$ with residue affinoid field $(\mathbb{F}_p((t)), \mathbb{F}_p[[t]])$. All three valuations have rank 1. The only non-trivial vertical generization in $\text{Spa}(B, B)$ goes from $|\cdot|_t$ to $|\cdot|_{\eta}$.

On the other hand $\text{Spo}(B)$ has 4 points:

$$\left\{ \eta = (|\cdot|_{\eta}, |\cdot|_{\eta}), s = (|\cdot|_s, |\cdot|_s), t^a = (|\cdot|_t, |\cdot|_t), t^h = (|\cdot|_t, |\cdot|_{\eta}) \right\}$$

One can verify directly from the definition that $\{\eta\} = U_{1 \leq t \neq 0}$, $\{\eta, t^h, t^a\} = U_{0 \leq t \neq 0}$, $\{t^a\} = N_{t^2 < t}$ and $\{t^a, s\} = N_{t < 1}$, and that these are the only proper open subsets.

In this example s , η and t^h are non-analytic. Moreover, t^h is microbial, and t^a is both a meromorphic and formal d -analytic point. The generization pattern is as follows: η is a vertical generization of t^h , t^h is the meromorphic specialization of t^a , and s is the formal specialization of t^a . We have that $\text{Spo}(\mathbb{F}_p[[t]], \mathbb{F}_p[[t]])^\dagger = \text{Spo}(\mathbb{F}_p[[t]], \mathbb{F}_p[[t]])$.

The following proposition shows that vertical generizations, formal specializations and meromorphic specializations completely describe the generization pattern in $\text{Spo}(B, B^+)$.

Proposition 1.2.6. *Let $x \in \text{Spo}(B, B^+)$.*

1. *If x is d -analytic then $\mathcal{I}^\leq(x) = \mathcal{I}_{\text{ver}}^\leq(x)$.*
2. *If x is non-analytic then $\mathcal{I}^\leq(x) = \mathcal{I}_{\text{ver}}^\leq(x) \cup \mathcal{I}_{\text{ver}}^\leq(x^{\text{mer}}) \cup (\bigcup_{z \in \mathcal{I}_{\text{ver}}^\leq(x)} z^{\text{For}})$.*

Proof. We start by proving the right to left inclusion. Let $y \in \mathcal{I}_{\text{ver}}^\leq(x) \cup \mathcal{I}_{\text{ver}}^\leq(x^{\text{mer}}) \cup (\bigcup_{z \in \mathcal{I}_{\text{ver}}^\leq(x)} z^{\text{For}})$ if x is non-analytic and let $y \in \mathcal{I}_{\text{ver}}^\leq(x)$ otherwise. Since h is continuous and $h(y)$ is a generization of $h(x)$ in $\text{Spa}(B, B^+)$ we have that y is contained in every classical localization containing x , so it is enough to check on analytic localizations. Suppose that $x \in N_{b_1 < b_2}$, if y is a vertical generization we have that $|\cdot|_y^a = |\cdot|_x^a$ so $y \in N_{b_1 < b_2}$. If x is non-analytic then $|b_1|_x^a = 0$ and $|b_2|_x^a = 1$, this implies that $|b_1|_{x^{\text{mer}}}^a = 0$ and that $|b_2|_{x^{\text{mer}}}^a \neq 0$, so $x^{\text{mer}} \in N_{b_1 < b_2}$ whenever x^{mer} exists. Moreover, for $y \in x^{\text{For}}$, we have that $\text{def}(y) = \text{supp}(x)$ which gives $|b_1|_y^a < 1$, $|b_2|_y^a = 1$, and $x^{\text{For}} \in N_{b_1 < b_2}$.

Now we prove the left to right inclusion, for this take $y \in \mathcal{I}^\leq(x)$. Using classical localizations one can deduce that $\text{supp}(y) \subseteq \text{supp}(x)$, and if x is d -analytic we claim that $\text{supp}(y) = \text{supp}(x)$. Indeed, let $b \in B$ such that $|b|_x^a \notin \{0, 1\}$, and let $b_1 \in \text{supp}(x)$. If $|b|_x^a < 1$ then $|b|_y^a < 1$, which implies that y is d -analytic. Additionally, the inequalities $|b_1|_y^a < |b^n|_y^a$ must hold for all n since $x \in N_{b_1 < b^n}$ and $y \in \mathcal{I}^\leq(x)$. In a similar way, if $1 < |b|_x^a$ then $1 < |b|_y^a$ and we may look at the inequalities $|b_1 \cdot b^n|_y^a < |b|_y^a$ instead. In both cases the Archimedean property of rank 1 valuations imply that $b_1 \in \text{supp}(y)$. Since the only generizations of $h(x)$ in $\text{Spa}(B, B^+)$ that have the same support as $h(x)$ are vertical generizations we must have $h(y) \in \mathcal{I}_{\text{ver}}^\leq(h(x))$. Consequently, $y \in \mathcal{I}_{\text{ver}}^\leq(x)$ holds in the d -analytic case.

Suppose now that x is non-analytic, if $\text{supp}(x) = \text{supp}(y)$ then we can reason as above to conclude $y \in \mathcal{I}_{\text{ver}}^\leq(x) \cup \mathcal{I}_{\text{ver}}^\leq(x^{\text{mer}})$. Let us assume there is $b \in \text{supp}(x) \setminus \text{supp}(y)$. Since $x \in N_{b < 1}$ we have $0 < |b|_y^a < 1$ and that y is d -analytic. By similar reasoning for all $b_1 \in B$ we have $|b \cdot b_1^n|_y^a < 1$ which implies that y is formal, we also have that $\text{supp}(x) \subseteq \text{def}(y)$. Moreover, for elements $b_2 \notin \text{supp}(x)$ we have $x \in U_{b \leq b_2^n \neq 0}$ for all n , this implies that $|b_2|_y^a = 1$ so $\text{def}(y) = \text{supp}(x)$. If we let $z = y^{\text{For}}$ then $\text{supp}(z) = \text{supp}(x)$ and one can check from the construction of horizontal specializations that z is also a generization of x . As above, we may conclude that $h(z)$ is a vertical generization of $h(x)$, and since both z and x are non-analytic then z is a vertical generization of x . In other words, $z \in \mathcal{I}_{\text{ver}}^\leq(x)$ and $y \in z^{\text{For}}$. \square

The olivine spectrum is compatible with completion and rational localization.

Proposition 1.2.7. *Let (\hat{B}, \hat{B}^+) denote the completion of (B, B^+) , the map $\text{Spo}(\hat{B}, \hat{B}^+) \rightarrow \text{Spo}(B, B^+)$ is a homeomorphism.*

Proof. The map $\text{Spo}(\hat{B}, \hat{B}^+) \rightarrow \text{Spo}(B, B^+)$ is a bijection of sets since $\text{Spa}(\hat{B}, \hat{B}^+) \rightarrow \text{Spa}(B, B^+)$ is. Since $\text{Spa}(\hat{B}, \hat{B}^+) \cong \text{Spa}(B, B^+)$ classical localizations of $\text{Spa}(\hat{B}, \hat{B}^+)$ are open in $\text{Spa}(B, B^+)$. It is enough to prove that if $f, g \in \hat{B}$ then $N_{g < f}$ is open in $\text{Spo}(B, B^+)$. Let $x \in N_{g < f}$ and let $f_x \in B$ with $|f_x|_x^h = |f|_x^h$. We have that

$$U_{f_x \leq f \neq 0} \cap U_{f \leq f_x \neq 0} \cap N_{g < f} = U_{f_x \leq f \neq 0} \cap U_{f \leq f_x \neq 0} \cap N_{g < f_x},$$

so we can reduce to the case in which $f \in B$. Take a ring of definition $B_0 \subseteq B$ and an ideal of definition $I \subseteq B_0$ with $|i|_x^h \leq |f|_x^h$ for a finite set of generators $i \in I$. Let $g_x \in B$ such that $g - g_x \in I^2 \cdot \hat{B}_0$. Then

$$\left(\bigcap_i U_{i \leq f \neq 0}\right) \cap N_{g_x < f} = \left(\bigcap_i U_{i \leq f \neq 0}\right) \cap N_{g < f}$$

which proves that the left hand side is also open in $\text{Spo}(B, B^+)$. Indeed, we have the inequalities $|g|_x^a \leq \max(|g_x|_x^a, |g - g_x|_x^a)$, $|g_x|_x^a \leq \max(|g|_x^a, |g - g_x|_x^a)$ and $|g - g_x|_x^a < |f|_x^a$ by the construction of g_x . \square

Proposition 1.2.8. *Let $s, t_1, \dots, t_n \in B$ such that the t_i generate an open ideal in B and consider the map of Huber pairs $(B, B^+) \rightarrow (R, R^+)$ associated to the rational localization $U(\frac{t_1, \dots, t_n}{s}) \subseteq \text{Spa}(B, B^+)$. The induced map $\text{Spo}(R, R^+) \rightarrow \text{Spo}(B, B^+)$ is a homeomorphism onto $h^{-1}(U(\frac{t_1, \dots, t_n}{s}))$.*

Proof. The only thing to verify is that for every $r_1, r_2 \in R$ the analytic localization $N_{r_1 < r_2} \subseteq \text{Spo}(R, R^+)$ is also open in $\text{Spo}(B, B^+)$. By proposition 1.2.7 and the construction of R as a rational localization we may assume that $r_1, r_2 \in B[\frac{1}{s}] \subseteq R$ since R is defined as the completion of $B[\frac{1}{s}]$ under certain linear topology. Let $r_1 = \frac{b_1}{s^{n_1}}$ and $r_2 = \frac{b_2}{s^{n_2}}$ and let $m = n_1 - n_2$. Then $N_{r_1 < r_2} = N_{b_1 < b_2 \cdot s^m} \cap \text{Spo}(R, R^+)$ when m is non-negative and $N_{r_1 < r_2} = N_{b_1 \cdot s^m < b_2} \cap \text{Spo}(R, R^+)$ otherwise. \square

The following example generalizes example 1.2.5. We encourage the reader to understand this example carefully before moving on. This will also be used in the proof of lemma 1.2.33 and theorem 4.

Example 1.2.9. *Suppose that B and B^+ are valuation rings with the same fraction field and suppose that B has the discrete topology. To describe $\text{Spo}(B, B^+)$ it is enough to realize that $\text{Spo}(B, B^+) \subseteq \text{Spo}(B^+, B^+)$ and that it acquires the subspace topology. Indeed, it corresponds to the intersection of the classical localizations $\cap_{b \in B \setminus B^+} U_{0 \leq b \neq 0} \subseteq \text{Spo}(B, B^+)$ (often times only one of the terms in this intersection is needed). To simplify notation we only describe explicitly $\text{Spo}(B^+, B^+)$, the advantage of this case is that all points are bounded and the deformation ideal def makes sense.*

Consider the map $\text{Spo}(B^+, B^+) \rightarrow \text{Spec}(B^+)^3$ given by

$$q \mapsto (\text{supp}(q), \text{def}(q), \text{sp}(q)).$$

The map is injective and the image consists of the set of triples (q_1, q_2, q_3) such that $q_1 \subseteq q_2 \subseteq q_3$, and such that the closed interval $[q_1, q_2]$ (in the sense of ordered sets) has cardinality one or two. A triple $q = (q_1, q_2, q_3)$ is d -analytic (necessarily meromorphic) if and only if $q_1 \neq q_2$, in this case there exists an element $b \in B^+$ with $b \in q_2 \setminus q_1$. We have that $q \in \mathcal{I}_{\text{ver}}^{\leq}(r)$ if $q_1 = r_1$, $q_2 = r_2$ and $q_3 \subseteq r_3$. Also, $q = r^{\text{mer}}$ if $r_1 = q_1 = r_2$, $q_2 \neq q_1$ and $r_3 = q_3$. With this setup formal generizations are unique and r is the formal specialization of q (i.e. $r = q_{\text{for}}$) if $q_1 \neq q_2$, $r_1 = q_2 = r_2$ and $r_3 = q_3$.

To each element $0 \neq g \in B^+ \setminus (B^+)^{\times}$ we can associate two ideals q_g^+ and q_g^- where q_g^+ is the largest prime ideal not containing g and q_g^- is the smallest prime ideal containing g . We have that $q_g^+ \subseteq q_g^-$ and that the interval $[q_g^+, q_g^-]$ has cardinality two.

When $\frac{f}{g} \in B^+$ then $U_{f \leq g \neq 0} = U_{0 \leq g \neq 0}$ and consists of triples (q_1, q_2, q_3) with $q_1 \leq q_g^-$. On the other hand if $\frac{g}{f} \in B^+$ we can let $b = \frac{g}{f}$ then $U_{f \leq g \neq 0} = U_{1 \leq b \neq 0}$ and it consists of triples with $q_3 \leq q_b^-$. Notice that the two families of open sets $U_{1 \leq b \neq 0}$ and $U_{0 \leq g \neq 0}$ are nested (i.e. one of $U_{1 \leq b_2 \neq 0} \subseteq U_{1 \leq b_1 \neq 0}$ or $U_{1 \leq b_1 \neq 0} \subseteq U_{1 \leq b_2 \neq 0}$ holds). In particular, finite intersections of classical localizations are all of the form $U_{1 \leq b \neq 0} \cap U_{0 \leq g \neq 0}$ for some $b, g \in B^+$.

When $n = \frac{f}{g} \in B^+$ then $N_{g < f}$ is empty and $N_{f < g} = U_{0 \leq g \neq 0} \cap N_{n < 1}$. The set $N_{n < 1}$ consists of the triples $q = (q_1, q_2, q_3)$ such that $q_n^+ \leq q_2$. The family of sets $N_{n < 1}$ is again nested.

In summary, if $x \in U \subseteq \text{Spo}(B^+, B^+)$ for U an open subset there are elements $g, b, n \in B^+$ with $x \in U_{0 \leq g \neq 0} \cap U_{1 \leq b \neq 0} \cap N_{n < 1} \subseteq U$ and elements of $U_{0 \leq g \neq 0} \cap U_{1 \leq b \neq 0} \cap N_{n < 1} \subseteq U$ correspond to triples (q_1, q_2, q_3) satisfying: $q_n^+, q_1 \subseteq q_2 \subseteq q_3 \subseteq q_b^-$ and $q_1 \subseteq q_g^-$.

1.2.3 Olivine Huber pairs

For the rest of the section (B, B^+) will denote a complete Huber pair over \mathbb{Z}_p .

Proposition 1.2.10. *If R is a Tate Huber pair the projection map $h : \mathrm{Spo}(R, R^+) \rightarrow \mathrm{Spa}(R, R^+)$ is a homeomorphism.*

Proof. Since (R, R^+) is Tate there are no trivial continuous valuations in $\mathrm{Spa}(R, R^+)$. In particular every point in $\mathrm{Spo}(R, R^+)$ is d-analytic and h is injective. If x^a is the maximal generization of x in $\mathrm{Spa}(R, R^+)$ then $h^{-1}(x) = \{(x, x^a)\}$. It is enough to prove that $h(N_{r_1 < r_2})$ is open. If $\varpi \in R$ is a topologically nilpotent unit, then

$$h(N_{r_1 < r_2}) = \bigcup_{0 < n} \{z \in \mathrm{Spa}(R, R^+) \mid |r_1^n|_z \leq |r_2^n \varpi|_z \neq 0\}$$

Indeed, a point $x \in \mathrm{Spa}(R, R^+)$ is in $h(N_{r_1 < r_2})$ if $|r_1|_{x^a} < |r_2|_{x^a}$. By the Archimedean property of rank 1 valuations there is $n \in \mathbb{N}$ such that $(\frac{|r_1|_{x^a}}{|r_2|_{x^a}})^n \leq |\varpi|_{x^a}$ since ϖ is a unit and $|\varpi|_{x^a} > 0$. On the other hand, if $|r_1^n|_x \leq |r_2^n \cdot \varpi|_x$ we also have $|r_1^n|_{x^a} \leq |r_2^n \cdot \varpi|_{x^a} < |r_2^n|_{x^a}$ since ϖ is topologically nilpotent. \square

If $m : (B, B^+) \rightarrow (R, R^+)$ is a map of Huber pairs, we denote by $\mathrm{Spo}(m) : \mathrm{Spo}(R, R^+) \rightarrow \mathrm{Spo}(B, B^+)$ the corresponding map of olivine spectra. In case (R, R^+) is Tate we have a continuous map $\mathrm{Spo}(m) \circ h^{-1} : \mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spo}(B, B^+)$. When the context is clear, we also abbreviate $\mathrm{Spo}(m) \circ h^{-1}$ by $\mathrm{Spo}(m)$.

Remark 1.2.11. *The topological considerations in what follows can be done purely in the context of adic spaces without any reference to perfectoid spaces. To do this one substitutes $|\mathrm{Spd}(B, B^+)|$ by $\mathrm{Spo}(B, B^+)$ where this second space has $\mathrm{Spo}(B, B^+)$ as underlying set but has the strongest topology making $\mathrm{Spo}(m)$ continuous for maps $m : (B, B^+) \rightarrow (R, R^+)$ ranging over all Tate Huber pairs. We do not pursue this.*

We define a map $\pi : |\mathrm{Spd}(B, B^+)| \rightarrow \mathrm{Spo}(B, B^+)$ as follows. Let $[x] \in |\mathrm{Spd}(B, B^+)|$ be represented by a geometric point $x : \mathrm{Spa}(C_x, C_x^+) \rightarrow \mathrm{Spd}(B, B^+)$, and let $s \in \mathrm{Spa}(C_x, C_x^+)$ denote the unique closed point. Recall that the map is given by an untilt C_x^\sharp and a map of Huber pairs $f_x^* : (B, B^+) \rightarrow (C_x^\sharp, C_x^{\sharp+})$. We define $\pi([x])$ to be $\mathrm{Spo}(f_x)(\mathfrak{b}^{-1}(s))$ where $\mathfrak{b} : \mathrm{Spa}(C_x^\sharp, C_x^{\sharp+}) \rightarrow \mathrm{Spa}(C_x, C_x^+)$ is the tilting homeomorphism. This map doesn't depend of the representative picked.

Proposition 1.2.12. *The map $\pi : |\mathrm{Spd}(B, B^+)| \rightarrow \mathrm{Spo}(B, B^+)$ defined above is continuous and bijective.*

Proof. To prove π is continuous one has to show that for any map $f : \mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(B, B^+)$ with (R, R^+) perfectoid the composition $|\mathrm{Spa}(R, R^+)| \rightarrow |\mathrm{Spd}(B, B^+)| \rightarrow \mathrm{Spo}(B, B^+)$ is continuous. But if $f^\sharp : (B, B^+) \rightarrow (R^\sharp, R^{\sharp+})$ is the map of Huber pairs associated to f then $\pi \circ |f|$ is given by $\mathrm{Spo}(f^\sharp) \circ \mathfrak{b}^{-1}$. Let us prove injectivity, take two geometric points $y_i : \mathrm{Spa}(C_i, C_i^+) \rightarrow \mathrm{Spd}(B, B^+)$ and suppose that $\pi(y_1) = \pi(y_2)$. We need to show that $[y_1] = [y_2]$. Let x be the common image in $\mathrm{Spo}(B, B^+)$ and let $(K_{h(x)}, K_{h(x)}^+)$ be the affinoid residue field of $h(x)$ in $\mathrm{Spa}(B, B^+)$. The map $(B, B^+) \rightarrow (C_i^\sharp, C_i^{\sharp+})$ factor through $(B, B^+) \rightarrow (K_{h(x)}, K_{h(x)}^+)$. We split our analysis in three cases.

Case 1: Suppose that x is analytic. In this case the closed point s_i of $\mathrm{Spa}(C_i^\sharp, C_i^{\sharp+})$ maps to $h(\pi(x))$ which is in the analytic locus $\mathrm{Spa}(B, B^+)^{an}$. This case follows from the bijectivity of $|X^\diamond| \rightarrow |X|$ for analytic pre-adic spaces (See remark 1.1.31).

Case 2: Suppose that x is meromorphic, we have that $h(x)$ is non-analytic in $\mathrm{Spa}(B, B^+)$. Let $K_{h(x)}^\circ := \{k \in K_{h(x)} \mid |k|_x^a \leq 1\}$ since $|\cdot|_x^a$ is non-trivial $K_{h(x)}^\circ$ is a proper valuation subring of $K_{h(x)}$. Choose $b \in B$ such that either $0 < |b|_x^a < 1$ or $|b|_x^a > 1$, then the subspace topology on $(K_{h(x)}^\circ) \subseteq_{y_i^*} O_{C_i^\sharp}$ coincides with the (b) -adic topology or, respectively, the $(\frac{1}{b})$ -adic topology. Taking the completion with respect to this topology we get a commutative diagram:

$$\begin{array}{ccc}
& \text{Spa}(C_1, C_1^+) & \\
& \downarrow p'_1 & \searrow y_1 \\
\text{Spa}(C_2, C_2^+) & \xrightarrow{p'_2} \text{Spd}(\hat{K}_{h(x)}, \hat{K}_{h(x)}^+) & \xrightarrow{\iota_x} \text{Spd}(K_{h(x)}, K_{h(x)}^+) \\
& \searrow y_2 & \\
& & \text{Spd}(K_{h(x)}, K_{h(x)}^+)
\end{array}$$

The maps p'_i map the respective closed points to the same topological point of $\text{Spd}(\hat{K}_{h(x)}, \hat{K}_{h(x)}^+)$. Since $\text{Spa}(\hat{K}_{h(x)}, \hat{K}_{h(x)}^+)$ is analytic we can conclude as in the first case.

Case 3: Suppose that x is non-analytic, in this case $h(x)$ is non-analytic in $\text{Spa}(B, B^+)$. We have that $(K_{h(x)}, K_{h(x)}^+)$ is given the discrete topology. Since $|\cdot|_x^a$ is trivial we have that $y_i(K_{h(x)}) \subseteq O_{C_i}^{\sharp, \times}$. After choosing pseudo-uniformizers $\varpi_i \in O_{C_i}^{\sharp}$ we may extend the y_i to continuous adic maps of topological rings $p_i^{f*} : K_{h(x)}[[t]] \rightarrow O_{C_i}^{\sharp}$ where $K_{h(x)}[[t]]$ is given the (t) -adic topology. These induce the following commutative diagram:

$$\begin{array}{ccc}
& \text{Spa}(C_1, C_1^+) & \\
& \downarrow p'_1 & \searrow y_1 \\
\text{Spa}(C_2, C_2^+) & \xrightarrow{p'_2} \text{Spd}(K_{h(x)}((t)), K_{h(x)}^+ + t \cdot K_{h(x)}[[t]]) & \xrightarrow{\iota_x} \text{Spd}(K_{h(x)}, K_{h(x)}^+) \\
& \searrow y_2 & \\
& & \text{Spd}(K_{h(x)}, K_{h(x)}^+)
\end{array}$$

The maps p'_i again send the closed point to the same point in $\text{Spd}(K_{h(x)}((t)), K_{h(x)}^+ + t \cdot K_{h(x)}[[t]])$ and this space is again the diamond associated to an analytic space so we may conclude as above. This finishes the proof of injectivity. The argument given above also explains how to construct a geometric point of $p_x : \text{Spa}(C, C^+) \rightarrow \text{Spa}(B, B^+)$ with $\text{Spo}(p_x)(s) = x$. Indeed, we can take a completed algebraic closure of $K_{h(x)}$ ($\hat{K}_{h(x)}$, or $K_{h(x)}((t))$ respectively) when x is analytic (meromorphic or non-analytic respectively). \square

Definition 1.2.13. Whenever x is d -analytic we let (K_x, K_x^+) denote $(\hat{K}_{h(x)}, \hat{K}_{h(x)}^+)$, and if x is non-analytic we let (K_x, K_x^+) denote $(K_{h(x)}((t)), K_{h(x)}^+ + t \cdot K_{h(x)}[[t]])$ as in the proof of proposition 1.2.12. In both cases we call (K_x, K_x^+) the pseudo-residue field at x .

Remark 1.2.14. The pseudo-residue field map $\text{Spo}(K_x, K_x^+) \rightarrow \text{Spo}(B, B^+)$ is a homeomorphism onto its image. The functor $\text{Spd}(K_x, K_x^+) \rightarrow \text{Spd}(B, B^+)$ surjects onto the subsheaf of $\text{Spd}(B, B^+)$ consisting of maps that factor through $\mathcal{I}_{\text{ver}}^{\leq}(x)$, but when x is non-analytic the map $\text{Spd}(K_x, K_x^+) \rightarrow \text{Spd}(B, B^+)$ is not injective. Actually, when x is non-analytic and $|\cdot|_x^h$ is non-trivial the subsheaf of points that factor through $\mathcal{I}_{\text{ver}}^{\leq}(x)$ is not representable by an adic space.

Corollary 1.2.15. For any map of Huber pairs $m^* : (B_1, B_1^+) \rightarrow (B_2, B_2^+)$ the map $\text{Spo}(m)$ is compatible with vertical generization. More precisely, if $x \in \text{Spo}(B_2, B_2^+)$, $y = \text{Spo}(m)(x)$ and y' is a vertical generization of y then there exist x' , a vertical generization of x , with $\text{Spo}(m)(x') = y'$.

Proof. Given $x \in \text{Spo}(B_2, B_2^+)$ and $y \in \text{Spo}(B_1, B_1^+)$ as in the statement we may, after making some choices if necessary, construct the following commutative diagram of pseudo-residue fields:

$$\begin{array}{ccc}
\text{Spd}(K_x, K_x^+) & \longrightarrow & \text{Spd}(K_y, K_y^+) \\
\downarrow & & \downarrow \\
\text{Spd}(B_2, B_2^+) & \longrightarrow & \text{Spd}(B_1, B_1^+)
\end{array}$$

Since the map $\mathrm{Spd}(K_x, K_x^+) \rightarrow \mathrm{Spd}(K_y, K_y^+)$ is a map of locally spatial diamonds it is generalizing and consequently surjective. But $|\mathrm{Spd}(K_x, K_x^+)| = \mathcal{I}_{\mathrm{ver}}^{\leq}(x)$ and analogously for y . \square

Lemma 1.2.16. *The topological spaces $\mathrm{Spo}(B, B^+)$ and $|\mathrm{Spd}(B, B^+)|$ have the same generization pattern.*

Proof. Since the map $|\mathrm{Spd}(B, B^+)| \rightarrow \mathrm{Spo}(B, B^+)$ is continuous the generization pattern of $|\mathrm{Spd}(B, B^+)|$ is smaller than that of $\mathrm{Spo}(B, B^+)$, it is enough by proposition 1.2.6 to prove that formal, meromorphic and vertical specializations are specializations in $|\mathrm{Spd}(B, B^+)|$. For any $x \in \mathrm{Spo}(B, B^+)$ the pseudo-residue field map $\iota_x : \mathrm{Spd}(K_x, K_x^+) \rightarrow \mathrm{Spd}(B, B^+)$ in the proof of proposition 1.2.12 defines a bijection onto $\mathcal{I}_{\mathrm{ver}}^{\leq}(x)$ which proves that vertical specializations are specializations in $\mathrm{Spd}(B, B^+)$.

Let $x \in \mathrm{Spo}(B, B^+)$ be d-analytic and let b such that $|b|_x^a \notin \{0, 1\}$. Let $p : \mathrm{Spa}(C, C^+) \rightarrow \mathrm{Spa}(B, B^+)$ be a geometric point mapping to x and let $\varpi \in C^{\circ\circ}$ be either $p^*(b)$ or $\frac{1}{p^*(b)}$. To this choice we will associate two product of points as follows. Let $R^+ = \prod_{i=1}^{\infty} C^+$, let $\varpi_0 = (\varpi^{\frac{1}{n}})_{n=1}^{\infty}$ and $\varpi_{\infty} = (\varpi^n)_{n=1}^{\infty}$. Let R_0^+ (R_{∞}^+ respectively) be R^+ endowed with the ϖ_0 -topology (ϖ_{∞} -topology respectively), and let $R_0 = R_0^+[\frac{1}{\varpi_0}]$ ($R_{\infty} = R_{\infty}^+[\frac{1}{\varpi_{\infty}}]$ respectively). We have diagonal maps of rings $C^+ \rightarrow R_0^+$ and $C \rightarrow R_{\infty}$, but we warn the reader that this maps are not continuous. On the other hand, the map $C^+ \rightarrow R_0^+$ is continuous but ϖ is not invertible in R_0 so the map does not extend to a map $C \rightarrow R_0$. Intuitively speaking, the product of points $\mathrm{Spa}(R_{\infty}, R_{\infty}^+)$ “converges outside” of the locus in which ϖ is topologically nilpotent and the product of points $\mathrm{Spa}(R_0, R_0^+)$ “converges outside” of the locus in which ϖ is invertible.

Suppose that x is meromorphic, then the affinoid residue field $(K_{h(x)}, K_{h(x)}^+)$ is given the discrete topology. In particular, the diagonal map $f : B \rightarrow K_{h(x)} \rightarrow R_{\infty}$ is continuous and defines a map $\mathrm{Spa}(R_{\infty}, R_{\infty}^+) \rightarrow \mathrm{Spa}(B, B^+)$. The space of connected components $\pi_0(|\mathrm{Spa}(R_{\infty}, R_{\infty}^+)|)$ is the Stone-Ćech compactification of \mathbb{N} which has as underlying set the set of ultrafilters of \mathbb{N} . Principal ultrafilters $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ define inclusions $\iota_n : \mathrm{Spa}(C, C^+) \rightarrow \mathrm{Spa}(R_{\infty}, R_{\infty}^+)$ that correspond to the n th-projection in the coordinate rings. In particular, the closed point of a principal connected component maps to x under $\mathrm{Spo}(f)$. We claim that the closed point of a non-principal connected component maps to x_{mer} . It is enough to construct a commutative diagram as below:

$$\begin{array}{ccccc} \mathrm{Spa}(C_{\mathcal{U}}, C_{\mathcal{U}}^+) & \longrightarrow & \mathrm{Spa}(K_{x_{\mathrm{mer}}}, K_{x_{\mathrm{mer}}}^+) & & \\ \downarrow & & \downarrow & & \\ \mathrm{Spa}(R_{\infty}, R_{\infty}^+) & \longrightarrow & \mathrm{Spa}(K_{h(x)}, K_{h(x)}^+) & \longrightarrow & \mathrm{Spa}(B, B^+) \end{array}$$

We claim that the natural map $K_{h(x)} \rightarrow C_{\mathcal{U}}$ maps to $O_{C_{\mathcal{U}}}$. Indeed, it is enough to prove $\varpi_{\infty} \cdot K_{h(x)} \subseteq O_{C_{\mathcal{U}}}$ since then every element of $K_{h(x)}$ would be power bounded. Clearly $K_{h(x)}^+ \subseteq O_{C_{\mathcal{U}}}$ and since $K_{h(x)} = K_{h(x)}^+[b, \frac{1}{b}]$ it is enough to prove that $\frac{\varpi_{\infty}}{\varpi^n} \in O_{C_{\mathcal{U}}}$ for $n \in \mathbb{N}$. Clearly $\frac{\varpi_{\infty}}{\varpi^n} \in \prod_{i=n+1}^{\infty} O_C$ and since our ultrafilter is non-principal complements of finite sets are in \mathcal{U} , which finishes the proof of the claim.

By letting t map to ϖ_{∞} we get a map $K_{h(x)}((t)) \rightarrow C_{\mathcal{U}}$, the intersection of $K_{h(x)}[[t]]$ with $C_{\mathcal{U}}^+$ in $O_{C_{\mathcal{U}}}$ is $K_{h(x)}^+ + t \cdot K_{h(x)}[[t]] = K_{x_{\mathrm{mer}}}^+$ which gives our factorization. Since the set of points that are contained in a principal connected component and that are closed in $|\mathrm{Spa}(R, R^+)|$ is dense within the set of closed points of $|\mathrm{Spa}(R, R^+)|$, meromorphic specializations in $\mathrm{Spo}(B, B^+)$ are specializations in $|\mathrm{Spd}(B, B^+)|$.

Suppose now that x is formal, since $|B|_x^a \leq 1$ the map $(B, B^+) \rightarrow (C, C^+)$ factors through a map to (O_C, C^+) and we have $\mathrm{def}(x) = B \cap C^{\circ\circ}$. This allows us to define a map $\mathrm{Spa}(R_0, R_0^+) \rightarrow \mathrm{Spa}(B, B^+)$. As in the previous case the space of connected components of $\mathrm{Spa}(R_0, R_0^+)$ is Stone-Ćech compactification of \mathbb{N} , principal connected components of $\mathrm{Spa}(R_0, R_0^+)$ map to x in $\mathrm{Spo}(B, B^+)$ and we will show that the non-principal ones map to x_{for} .

Let $k = O_C/C^{\circ\circ}$ and $k^+ = C^+/C^{\circ\circ}$, it is enough to prove that the map $(O_C, C^+) \rightarrow (C_U, C_U^+)$ factors as:

$$(O_C, C^+) \rightarrow (k, k^+) \rightarrow (k((t)), k^+ + t \cdot k[[t]]) \rightarrow (C_U, C_U^+)$$

Now $\frac{\varpi}{\varpi_0^n} \in \prod_{i=n+1}^{\infty} O_C$ which implies that $|\varpi|_U \leq |\varpi_0^n|_U$. Since ϖ_0 is a pseudo-uniformizer in C_U this implies $|\varpi|_U = 0$. Clearly $k \subseteq O_{C_U}$ and we may send t to ϖ_0 to construct our factorization. We may conclude the proof that formal specializations are specializations in $|\mathrm{Spo}(B, B^+)|$ as in the previous case. \square

We say that a Huber pair is formal if it is of the form (B, B) where B is an I -adic ring with finitely generated ideal I . For the moment we restrict to studying the olivine spectrum of formal Huber pairs. The main technical advantage of restricting to this case is that the open unit ball over $\mathrm{Spd}(B, B)$ is easy to describe. Indeed, it is represented by $\mathrm{Spd}(B[[t]], B[[t]])$ when this ring is given the (I, t) -adic topology.

Proposition 1.2.17. *Let (B, B) be a formal Huber pair then the map $|\mathrm{Spd}(B, B)| \rightarrow \mathrm{Spo}(B, B)$ is a homeomorphism.*

Proof. By proposition 1.2.12 we the map is a continuous bijection. Let $Y = \mathrm{Spa}(B[[t]], B[[t]])^{t \neq 0}$ and recall that $|Y| = |Y^\diamond|$ since this is an analytic pre-adic space. Let U be an open subset of $|\mathrm{Spd}(B, B^+)|$, let $x \in U$ and let $y \in Y$ be a point mapping to x , such that $|t|_y \neq 0$. We will construct a neighborhood of x contained in U that is open in $\mathrm{Spo}(B, B)$.

Given a classical localization $U_{b_1 \leq b_2 \neq 0}$ or an analytic localization $N_{b_1 < b_2}$ containing x we choose quasi-compact neighborhoods of y in $\mathrm{Spa}(B[[t]], B[[t]])$, that we denote $U_{b_1, b_2, y}$ and $N_{b_1, b_2, y}$, whose image in $\mathrm{Spo}(B, B)$ are contained in $U_{b_1 \leq b_2 \neq 0}$ and $N_{b_1 < b_2}$ respectively. The construction is as follows, given the classical localization $U_{b_1 \leq b_2 \neq 0}$ we pick a finite set of elements $S \subseteq B$ and a positive integer n such that $|s|_y \leq |b_2|_y$ for $s \in S$, that $|t^n|_y \leq |b_2|_y$, and that the ideal generated by S is open in B . We let $U_{b_1, b_2, y}$ be the rational localization $U(\frac{S, t^n, b_1}{b_2}) \subseteq \mathrm{Spa}(B[[t]], B[[t]])$. Rational localizations of affinoid adic spaces are always quasi-compact open subsets and clearly $\mathrm{Spo}(f)(h^{-1}(U_{b_1, b_2, y})) \subseteq U_{b_1 \leq b_2 \neq 0}$.

Analogously, given $N_{b_1 < b_2}$ we pick a set S and two positive integers, n_1 and n_2 , such that $|b_1^{n_1}|_y \leq |b_2^{n_1} \cdot t|_y$, that $|s|_y \leq |b_2^{n_1} \cdot t|_y$ for $s \in S$, that $|t^{n_2}|_y \leq |t \cdot b_2^{n_1}|_y$ and that S generates an open ideal in B . We let $N_{b_1, b_2, y} = U(\frac{S, t^{n_2}, b_1^{n_1}}{b_2^{n_1} \cdot t})$. Since t is topologically nilpotent in $B[[t]]$, for any point $z \in \mathrm{Spa}(B[[t]], B[[t]])$ we must have $|t|_z < 1$, which proves $\mathrm{Spo}(f)(h^{-1}(N_{b_1, b_2, y})) \subseteq N_{b_1 < b_2}$. Notice that $N_{b_1, b_2, y} \subseteq \mathrm{Spa}(B[[t]], B[[t]])^{t \neq 0}$.

Let X denote the intersection of all neighborhoods of y of the form $N_{b_1, b_2, y}$ and $U_{b_1, b_2, y}$ that were chosen in this way, then $\mathrm{Spo}(f)(X)$ is contained in $\mathcal{I}^{\leq}(x)$. By lemma 1.2.16 $|\mathrm{Spd}(B, B^+)|$ and $\mathrm{Spo}(B, B)$ have the same generization pattern, so we also have that $\mathrm{Spo}(f)(X) \subseteq U$. We have that $\mathrm{Spo}(f)^{-1}(U)$ is open in $\mathrm{Spa}(B[[t]], B[[t]])^{t \neq 0}$ and that the two families, $U_{b_1, b_2, y} \cap N_{0, 1, y}$ and $N_{b_1, b_2, y}$, consist of quasi-compact open subsets of $\mathrm{Spa}(B[[t]], B[[t]])^{t \neq 0}$. The standard compactness argument in the patch topology of $\mathrm{Spa}(B[[t]], B[[t]])$ will prove that a finite intersection of these neighborhoods is contained in $\mathrm{Spo}(f)^{-1}(U)$. We prove below that the image under $\mathrm{Spo}(f)$ of such a finite intersection is open in $\mathrm{Spo}(B, B)$.

It is enough to show that if a set Z is a finite intersections of sets of the form

$$V_{b'_1, b'_2} := \{z \in \mathrm{Spa}(B[[t]], B[[t]])^{t \neq 0} \mid |b'_1|_z \leq |b'_2|_z \neq 0\}$$

where $b'_1 \in B \cup \{t^n\}_{n \in \mathbb{N}}$ and $b'_2 \in B \cup t \cdot B$, then $\mathrm{Spo}(f)(Z)$ is open in $\mathrm{Spo}(B, B)$. Observe that if $b'_1, b'_2 \in B$ then $V_{b'_1, b'_2} = \mathrm{Spo}(f)^{-1}(U_{b'_1 \leq b'_2 \neq 0})$ and that for any Z as above we have $\mathrm{Spo}(f)(Z \cap V_{b'_1, b'_2}) = \mathrm{Spo}(f)(Z) \cap U_{b'_1 \leq b'_2 \neq 0}$. This allow us to reduce to the case in which Z is an intersections of opens such that at least one of $b'_1 \in \{t^n\}_{n \in \mathbb{N}}$ or $b'_2 = b_2 \cdot t$ holds.

Let T_Z^n be the subset of B for which either $V_{t^n, b}$ or $V_{t^{n+1}, (b \cdot t)}$ appear in the expression of Z as an intersection, we let T_Z^{\leq} be the set of pairs $(b_1, b_2) \in B \times B$ such that $V_{b_1, (b_2 \cdot t)}$ appears in the expression

of Z as an intersection, and we let T_Z^- and T_Z^+ denote the image of $T_Z^{<<}$ under the projection onto the first and second factors respectively. We claim, and prove below, that $\text{Spo}(f)(Z)$ is the intersection of all the sets of the form $U_{b_1^n \leq b_2^n \cdot b_3 \neq 0}$ where $(b_1, b_2) \in T_Z^{<<}$ and $b_3 \in T_Z^n$ and all the sets of the form $N_{b_1 < b_2}$, with $(b_1, b_2) \in T_Z^{<<}$. This proves $\text{Spo}(f)(Z)$ is open.

Let $z \in Z$ with associated rank 1 point $z^a \in Z$, let $w = \text{Spo}(f)(z)$ and fix b_1, b_2 and b_3 as above. By raising to the n th-power we have that $|b_1|_z \leq |b_2|_z \cdot |t^n|_z$ and $|t^n|_z \leq |b_3|_z$ hold. In particular,

$$|b_1|_z = |b_1|_w^h \leq |b_2|_w^h \cdot |b_3|_w^h = |b_2|_z \cdot |b_3|_z$$

holds as well and we can conclude that $w \in U_{b_1^n \leq b_2^n \cdot b_3 \neq 0}$. Similarly, since t is topologically nilpotent we have $|t|_{z^a} < 1$ which implies that $|b_1|_{z^a} < |b_2|_{z^a}$ and consequently that $|b_1|_w^a < |b_2|_w^a$. This says that $w \in N_{b_1 < b_2}$.

To prove the converse containment given a point w in the intersection of those sets we need to construct a lift landing in Z . Pick a geometric point $q : \text{Spa}(C, C^+) \rightarrow \text{Spa}(B, B)$ mapping to w in $\text{Spo}(B, B)$, the choice of an element $\varpi \in C^{\circ\circ, \times}$ defines a lift of q to a map $\text{Spa}(C, C^+) \rightarrow \text{Spa}(B[[t]], B[[t]])^{t \neq 0}$ simply by letting t map to ϖ . If w is non-analytic then $|b_1|_w^a = 0$ for every $b_1 \in T_Z^-$ and $|b_2|_w^a = |b_3|_w^a = 1$ for every $b_2 \in T_Z^+$ and $b_3 \in T_Z^n$. In this case, any choice of ϖ defines a lift landing inside of Z .

If w is d-analytic ϖ must be chosen more carefully. Since C is algebraically closed we may choose n th-roots of (b_3) for all $b_3 \in T_Z^n$. For a lift of q to land in Z , ϖ must satisfy the following: $|\varpi|_q \leq |(b_3)^{\frac{1}{n}}|_q$ for all $b_3 \in T_Z^n$ and $\frac{|(b_1)|_q}{|(b_2)|_q} \leq |\varpi|_q$ for all $(b_1, b_2) \in T_Z^{<<}$. We let m be the smallest of the values in Γ_q of the form $|b_3^{\frac{1}{n}}|_q$ with $b_3 \in T_Z^n$ and we let M be the largest of the values of the form $|\frac{b_1}{b_2}|_q$ with $(b_1, b_2) \in T_Z^{<<}$. Since $w \in U_{b_1^n \leq b_2^n \cdot b_3 \neq 0}$ we have $M \leq m$. Since $w \in N_{b_1 < b_2}$ for all pairs $(b_1, b_2) \in T_Z^{<<}$ we also have $M < 1$. Any $\varpi \in C$ with $|\varpi|_q < 1$ and $M \leq |\varpi|_q \leq m$ defines a lift of q in Z . This finishes the proof of the first claim. \square

Definition 1.2.18. Let (B, B^+) be a complete Huber pair over \mathbb{Z}_p , we say that (B, B^+) is *olivine* if the map $|\text{Spd}(B, B^+)| \rightarrow \text{Spo}(B, B^+)$ is a homeomorphism.

Question 1.2.19. Is every complete Huber pair over \mathbb{Z}_p an olivine Huber pair?

We have some partial progress in answering this question. Although we do not know what to expect in full generality, for all of the Huber pairs that we consider this is true. Let us clarify. By remark 1.1.31 we know that Tate Huber pairs are olivine. By proposition 1.2.17 we know that formal Huber pairs are olivine. Proposition 1.2.8 allows us to conclude that if $(B, B^+) \rightarrow (R, R^+)$ induces an open immersion $\text{Spa}(R, R^+) \subseteq \text{Spa}(B, B^+)$ and $\text{Spa}(B, B^+)$ is olivine then $\text{Spa}(R, R^+)$ is olivine. Moreover, we can conclude that being olivine is a property that can be verified locally in the analytic topology of $\text{Spa}(B, B^+)$. The following proposition is the most general criterion we could come up with.

Proposition 1.2.20. Suppose that (B, B^+) is a complete Huber pair over \mathbb{Z}_p such that B^+ is a ring of definition with ideal of definition I and such that B is a finite type B^+ -algebra. Then (B, B^+) is an olivine Huber pair.

Proof. Write $B = B^+[b_1, \dots, b_n]$ we prove that (B, B^+) is olivine by induction on the size of n , the case $n = 0$ is the content of proposition 1.2.17. Let $\text{Spa}(R, R^+)$ be the rational localization corresponding to $\{x \in \text{Spa}(B, B^+) \mid |b_1|_x \leq |1|_x \neq 0\}$, we claim that (R, R^+) is olivine. By proposition 1.2.7 we may compute R without taking completions. Up to completion R^+ is the integral closure of $B^+[b_1]$ in B and the underlying rings of R and B coincide (although they are not homeomorphic). Clearly R is generated over R^+ by less than n elements so by induction it is olivine. We let $\text{Spa}(S, S^+)$ be the

rational localization corresponding to $\{x \in \mathrm{Spa}(B, B^+) \mid |1|_x \leq |b_1|_x \neq 0\}$. Up to completion $S = B[\frac{1}{b_1}]$ as rings and S^+ is the integral closure of $B^+[\frac{1}{b_1}]$ in $B[\frac{1}{b_1}]$. The Huber pair $(S^+[b_2, \dots, b_n], S^+)$ is olivine by induction. But $\mathrm{Spa}(S, S^+)$ is the locus in $\mathrm{Spa}(S^+[b_2, \dots, b_n], S^+)$ in which $\frac{1}{b_1} \neq 0$, so this one is also olivine. Since $\mathrm{Spa}(B, B^+) = \mathrm{Spa}(R, R^+) \cup \mathrm{Spa}(S, S^+)$ this finishes the proof that (B, B^+) is olivine. \square

Remark 1.2.21. For an arbitrary Huber pair for which B^+ serves as a ring of definition we can consider the commutative diagram

$$\begin{array}{ccc} |\mathrm{Spd}(B, B^+)| & \longrightarrow & \mathrm{Spo}(B, B^+) \\ \downarrow & & \downarrow \\ \varprojlim_i |\mathrm{Spd}(B_i, B^+)| & \longrightarrow & \varprojlim_i \mathrm{Spo}(B_i) \end{array}$$

where B_i ranges over all subrings of B that are of finite type. By proposition 1.2.20 the bottom horizontal arrow is a homeomorphism and one can verify directly that the right vertical arrow is also a homeomorphism. It is not clear to us if the left vertical arrow is a homeomorphism or not since taking limits of v -sheaf does not necessarily commute with taking underlying topological spaces. Adding to the complexity of the situation the transition maps $\mathrm{Spd}(B_i, B^+) \rightarrow \mathrm{Spd}(B_j, B^+)$ might not be quasi-compact. Any counterexample to question 1.2.19 should come from this failure.

Question 1.2.22. Let $B = \mathbb{F}_p[T_1, \dots, T_n, \dots]$ be the free algebra in infinitely many variables endowed with the discrete topology and let $B^+ = \mathbb{F}_p$. Is (B, B^+) olivine?

1.2.4 Some open and closed subsheaves of $\mathrm{Spd}(B, B^+)$

By [28] 12.9 open subsets of $\mathrm{Spo}(B, B^+)$ define open subsheaves of $\mathrm{Spd}(B, B^+)$, and when (B, B^+) is olivine this association is bijective. Since the formation of $\mathrm{Spd}(B, B^+)$ commutes with localization in $\mathrm{Spa}(B, B^+)$, one can compute the open subsheaf corresponding to classical localizations. The following lemma describes, in some cases, the open subsheaf associated to an analytic localizations.

Lemma 1.2.23. Let (B, B^+) be a complete Huber pair over \mathbb{Z}_p and suppose that B is adic with ideal of definition I . Let $b \in B$, let B_b be the completion of B with respect to the (b, I) -adic topology and let B_b^+ be the integral closure of $B^+ + (B_b)^{\circ\circ}$ in B_b . The open subsheaf associated to the analytic localization $N_{b < 1} \subseteq |\mathrm{Spd}(B, B^+)|$ is represented by $\mathrm{Spd}(B_b, B_b^+)$.

Proof. If a map $f : \mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(B, B^+)$ factors through $\mathrm{Spd}(B_b, B_b^+)$ then $f^*(b)$ is topologically nilpotent in R^\sharp . This implies that $\mathrm{Spo}(f)(\mathrm{Spa}(R, R^+)) \subseteq N_{b < 1}$ and since this happens for every test space $(R, R^+) \in \mathrm{Perf}$, the map of v -sheaves $\mathrm{Spd}(B_b, B_b^+) \rightarrow \mathrm{Spd}(B, B^+)$ must factor through the subsheaf associated to $N_{b < 1}$. Moreover, since $B \subseteq B_b$ is dense, the maps $f^* : B_b \rightarrow R^\sharp$ are determined by their restriction to B , this implies that $\mathrm{Spd}(B_b, B_b^+) \rightarrow \mathrm{Spd}(B, B^+)$ is an injective map.

We must prove that if $f : \mathrm{Spa}(R^\sharp, R^{\sharp+}) \rightarrow \mathrm{Spa}(B, B^+)$ is such that $\mathrm{Spo}(f)(\mathrm{Spa}(R^\sharp, R^{\sharp+})) \subseteq N_{b < 1}$, then it factors through a (unique) map $\mathrm{Spa}(R^\sharp, R^{\sharp+}) \rightarrow \mathrm{Spa}(B_b, B_b^+)$. Given a point $x \in \mathrm{Spa}(R^\sharp, R^{\sharp+})$ and a pseudo-uniformizer $\varpi \in R^{\sharp+}$ we let x^a denote the rank 1 generization of x , we have that $|f^*b^n|_x \leq |\varpi|_x$ for some n since by hypothesis $|f^*b|_{x^a} < 1$.

Let $U = U(\frac{f^*b^n}{\varpi}) \subseteq \mathrm{Spa}(R^\sharp, R^{\sharp+})$. If $R_1 = \widehat{(R^{\sharp+})[\frac{b^n}{\varpi}]}$ where we complete by the ϖ -adic topology, then $R' = H^0(U, \mathcal{O}_X) = R_1[\frac{1}{\varpi}]$ and $R'^+ = H^0(U, \mathcal{O}_X^+)$ is the integral closure of $R^{\sharp+} + R_1^{\circ\circ}$ in R' . Since $f^*b^n \in R'^{\circ\circ}$ the map $B \rightarrow R'$ is continuous when B is given the (I, b) -adic topology. Moreover, since R'° is complete we get a map $B_b \rightarrow R'^\circ$. This gives a factorization $\mathrm{Spa}(R', R'^+) \rightarrow \mathrm{Spa}(B_b, B_b^+)$, and a map $\mathrm{Spa}(R', R'^+)^b \rightarrow \mathrm{Spd}(B_b, B_b^+)$. We have proved that locally f factors through $\mathrm{Spd}(B_b, B_b^+)$, since $\mathrm{Spd}(B_b, B_b^+)$ is a v -sheaf and the factorization is unique we may glue this to a map $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(B_b, B_b^+)$. \square

In general the subsheaf $N_{b < 1}$ might not be of the form $\mathrm{Spd}(R, R^+)$ for a Huber pair (R, R^+) , but it can always be described as the locus in which b is topologically nilpotent.

Recall that we defined $\mathrm{Spo}(B, B^+)^\dagger \subseteq \mathrm{Spo}(B, B^+)$ as the closed subset of bounded points. The definition of boundedness implies that $\mathrm{Spo}(B, B^+)^\dagger$ is stable under vertical generization and by proposition 1.1.17 it defines a closed subsheaf of $\mathrm{Spd}(B, B^+)$. Here is a different description.

Proposition 1.2.24. *Define $\mathrm{Spd}(B, B^+)^\dagger : \mathrm{Perf} \rightarrow \mathrm{Sets}$ to parametrize triples (R^\sharp, ι, f) where (R^\sharp, ι) is an untilt of R and $f : \mathrm{Spa}(R^{\sharp, \circ}, R^{\sharp, +}) \rightarrow \mathrm{Spa}(B, B^+)$ is a morphism of pre-adic spaces. We get the following Cartesian diagram*

$$\begin{array}{ccc} \mathrm{Spd}(B, B^+)^\dagger & \longrightarrow & \mathrm{Spd}(B, B^+) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Spo}(B, B^+)^\dagger} & \longrightarrow & \underline{\mathrm{Spo}(B, B^+)} \end{array}$$

Proof. We first prove that $\mathrm{Spd}(B, B^+)^\dagger \rightarrow \mathrm{Spd}(B, B^+)$ is a closed immersion. Let $\mathbb{A}_{\mathbb{Z}_p}^{|B|}$ denote the functor sending $(R, R^+) \mapsto (R^\sharp, \iota, x)$ where (R^\sharp, ι) is an untilt and x is a tuple with values in R^\sharp indexed by elements of B . The similarly defined space $\mathbb{A}_{\mathbb{Z}_p}^{|B|, \dagger}$ parametrizing tuples in $R^{\sharp, \circ}$ sits inside $\mathbb{A}_{\mathbb{Z}_p}^{|B|}$, and we have a basechange identity $\mathrm{Spd}(B, B^+)^\dagger = \mathbb{A}_{\mathbb{Z}_p}^{|B|, \dagger} \times_{\mathbb{A}_{\mathbb{Z}_p}^{|B|}} \mathrm{Spd}(B, B^+)$. Since the limit of closed immersions is a closed immersion we can reduce to prove that $\mathbb{A}_{\mathbb{Z}_p}^{1, \dagger} \rightarrow \mathbb{A}_{\mathbb{Z}_p}^1$ is a closed immersion. Consider the basechange by maps with source an affinoid perfectoid, $f_r : \mathrm{Spa}(R, R^+) \rightarrow \mathbb{A}_{\mathbb{Z}_p}^1$ with $r \in R^\sharp$. Then $\mathbb{A}_{\mathbb{Z}_p}^{1, \dagger} \times_{\mathbb{A}_{\mathbb{Z}_p}^1} \mathrm{Spa}(R, R^+)$ is the complement in $\mathrm{Spa}(R^\sharp, R^{\sharp, +})$ of

$$\bigcup_{\varpi} \{x \in \mathrm{Spa}(R^\sharp, R^{\sharp, +}) \mid |1|_x \leq |r \cdot \varpi|_x \neq 0\}$$

where ϖ ranges over elements of $R^{\sharp, \circ}$. This is a closed subset stable under vertical generization and defines a closed immersion into $\mathrm{Spa}(R, R^+)$ as we wanted to show.

Once we know $\mathrm{Spd}(B, B^+)^\dagger$ and $\mathrm{Spd}(B, B^+) \times_{\mathrm{Spo}(B, B^+)} \mathrm{Spo}(B, B^+)^\dagger$ are closed immersions it suffices to verify on geometric points that they agree. Let $q : \mathrm{Spa}(C, C^+) \rightarrow \mathrm{Spd}(B, B^+)$ be a geometric point with rank 1 generization q^a . That q maps to a bounded point in $\mathrm{Spo}(B, B^+)$ means by definition that $|B|_{q^a} \leq 1$, which is precisely the condition that the map $B \rightarrow C^\sharp$ factors through O_{C^\sharp} . \square

Remark 1.2.25. *One should be careful with the notion of bounded points since this notion is not compatible with localization (and it shouldn't). If $\mathrm{Spa}(R, R^+) \subseteq \mathrm{Spa}(B, B^+)$ is a rational localization and $x \in \mathrm{Spo}(R, R^+)$ it might happen that $x \in \mathrm{Spo}(B, B^+)^\dagger \cap \mathrm{Spo}(R, R^+)$ but $x \notin \mathrm{Spo}(R, R^+)^\dagger$. For example, $(\mathbb{Z}_p^\diamond)^\dagger = \mathbb{Z}_p^\diamond$ but $(\mathbb{Q}_p^\diamond)^\dagger = \emptyset$.*

Lemma 1.2.26. *Suppose that (A, A^+) and (B, B^+) are complete Huber pairs over \mathbb{Z}_p and that we have an adic homomorphism $(B, B^+) \rightarrow (A, A^+)$. Then the induced map of v -sheaves $\mathrm{Spd}(A, A^+)^\dagger \rightarrow \mathrm{Spd}(B, B^+)^\dagger$ is representable in spatial diamonds and in particular qcqs.*

Proof. Since the map $(B, B^+) \rightarrow (A, A^+)$ is adic we can write (A, A^+) as a filtered colimit $\varinjlim_{i \in I} (A_i, A_i^+)$ where each (A_i, A_i^+) is topologically of finite type over (B, B^+) , and the transition maps realize $A_i \rightarrow A_j$ as a topological subrings for $i < j$. One can see directly that $\mathrm{Spd}(A, A^+)^\dagger = \varprojlim_i \mathrm{Spd}(A_i, A_i^+)^\dagger$ and by [28] 12.17 it is enough to prove that $\mathrm{Spd}(A_i, A_i^+)^\dagger \rightarrow \mathrm{Spd}(B, B^+)^\dagger$ is representable in spatial diamonds. By definition of being topologically of finite type there is a family of sets $M = \{M_i\}_{i=1}^n$ with $B \cdot M_i$ an open ideal and a strict surjection $B\langle T_1, \dots, T_n \rangle_{M_1, \dots, M_n} \rightarrow A_i$ compatible with rings of integral elements. One can verify directly that the induced map

$$\mathrm{Spd}(A_i, A_i^+) \rightarrow \mathrm{Spd}(B\langle T_1, \dots, T_n \rangle_{M_1, \dots, M_n}, B\langle T_1, \dots, T_n \rangle_{M_1, \dots, M_n}^+)$$

is a closed immersion and that $\mathrm{Spd}(A_i, A_i^+)^\dagger$ is the basechange of the corresponding bounded subsheaf. Since closed immersions are representable in spatial diamonds it is enough to verify the case in which $A_i = B\langle T_1 \rangle_{M_1}$. There is an open immersion $\mathrm{Spd}(B\langle T_1 \rangle_{M_1}, B\langle T_1 \rangle_{M_1}^+) \rightarrow \mathbb{A}_B^1$ and such that $\mathrm{Spd}(B\langle T_1 \rangle_{M_1}, B\langle T_1 \rangle_{M_1}^+) \cap \mathbb{A}_B^{1,\dagger} = \mathrm{Spd}(B\langle T_1 \rangle_{M_1}, B\langle T_1 \rangle_{M_1}^+)^\dagger$. To prove $\mathbb{A}^{1,\dagger} \rightarrow \mathrm{Spd}(B, B^+)^\dagger$ is representable in spatial diamonds it is enough to verify that basechanges by affinoid perfectoid are spatial diamonds. The basechange of a map $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(B, B^+)^\dagger$ is representable by $\mathrm{Spd}(R^\sharp\langle T \rangle, R')$ where R' is the minimal ring of integral elements containing $R^{\sharp,+}$. Since $R^\sharp\langle T \rangle$ is Tate this space is a spatial diamond. \square

The following statement says that at least the bounded locus of a Huber pair is always olivine.

Proposition 1.2.27. *Suppose that (B, B^+) is a complete Huber pair over \mathbb{Z}_p . The natural map*

$$|\mathrm{Spd}(B, B^+)^\dagger| \rightarrow \mathrm{Spo}(B, B^+)^\dagger$$

is a homeomorphism.

Proof. Let $B_0 \subseteq B^+$ be a ring of definition and express (B, B^+) as a filtered colimit of Huber pairs of the form $\varinjlim_{i \in J} (B_i, B_i^+)$ where both B_i and B_i^+ are of finite type over B_0 . We have that $\mathrm{Spd}(B, B^+)^\dagger = \varprojlim \mathrm{Spd}(B_i, B_i^+)^\dagger$. By proposition 1.2.20 each (B_i, B_i^+) is olivine and by lemma 1.2.26 the transition maps are representable in spatial diamonds. Let $\mathbb{D}_{B_i^\dagger}^\times$ denote the punctured open unit disc over $\mathrm{Spd}(B_i, B_i^+)^\dagger$, and let π_i be the projection map $\pi_i : \mathbb{D}_{B_i^\dagger}^\times \rightarrow \mathrm{Spd}(B_i, B_i^+)^\dagger$. Observe that π_i is universally open since it is ℓ -cohomologically smooth. We have that $\mathbb{D}_{B_0^\dagger}^\times$ is a locally spatial diamond represented by $(\mathrm{Spa}(B_0[[t]], B_0[[t]])^{t \neq 0})^\diamond$. In particular, $\mathbb{D}_{B_i^\dagger}^\times$ is also a locally spatial diamond and since the transition maps $\mathbb{D}_{B_i^\dagger}^\times \rightarrow \mathbb{D}_{B_j^\dagger}^\times$ are qcqs we see that by [28] 12.17 $|\mathbb{D}_{B^\dagger}^\times| = \varprojlim |\mathbb{D}_{B_i^\dagger}^\times|$. It is enough to prove that $\pi : |\mathbb{D}_{B^\dagger}^\times| \rightarrow \mathrm{Spo}(B, B^+)^\dagger$ is a quotient map. Let $S \subseteq \mathrm{Spo}(B, B^+)^\dagger$ with $\pi^{-1}(S)$ open. For every point $y \in \pi^{-1}(S)$ there is an index $j_y \in J$ and an open subset of $U_y \subseteq \mathbb{D}_{B_{j_y}^\dagger}^\times$ whose preimage in $\mathbb{D}_{B^\dagger}^\times$ is contained in $\pi^{-1}(S)$ and contains y . Now, $\pi_{j_y}(U_y)$ is open in $|\mathrm{Spd}(B_{j_y}, B_{j_y}^+)^\dagger|$ and since $(B_{j_y}, B_{j_y}^+)$ is olivine it is also open in $\mathrm{Spo}(B_{j_y}, B_{j_y}^+)^\dagger$. The preimage of $\pi_{j_y}(U_y)$ in $\mathrm{Spo}(B, B^+)^\dagger$ contains $\pi(y)$, is open and it is contained in $h^{-1}(S)$ which finishes the proof that S is open in $\mathrm{Spo}(B, B^+)^\dagger$. \square

1.2.5 Discrete Huber pairs in characteristic p

In the following section, when we discuss the reduction functor, we will need to understand the olivine spectrum of Huber pairs associated to perfect schemes. For this reason we discuss this case in detail. For the rest of the subsection A denotes a discrete perfect ring in characteristic p and $A^+ \subseteq A$ is integrally closed.

Proposition 1.2.28. *Let (A, A^+) be as above. The projection map $\mathrm{Spo}(A, A^+)^\dagger \rightarrow \mathrm{Spa}(A, A^+)$ is surjective. Moreover, if $\mathfrak{S} \subseteq \mathrm{Spa}(A, A^+)$ is stable under arbitrary generization and $h^{-1}(\mathfrak{S})$ is open in $\mathrm{Spo}(A, A^+)^\dagger$ then \mathfrak{S} is open in $\mathrm{Spa}(A, A^+)$.*

Proof. The complement of the bounded locus consists of d-analytic points. Since A has the discrete topology every d-analytic point is meromorphic. If $x \in \mathrm{Spo}(A, A^+)^\dagger$ is a meromorphic point it has a meromorphic specialization $y \in \mathrm{Spo}(A, A^+)^\dagger$ with $h(x) = h(y)$, y is non-analytic and in particular bounded. This gives $h(\mathrm{Spo}(A, A^+)^\dagger) = h(\mathrm{Spo}(A, A^+)^\dagger)$ which is $\mathrm{Spa}(A, A^+)$.

For the second claim observe that the map $\mathrm{Spd}(A((t)), A^+ + t \cdot A[[t]]) \rightarrow \mathrm{Spd}(A, A^+)$ surjects onto $\mathrm{Spd}(A, A^+)^\dagger$ and represents the punctured open unit ball over it. Let f denote the map of adic spaces

$f : \mathrm{Spa}(A((t)), A^+ + t \cdot A[[t]]) \rightarrow \mathrm{Spa}(A, A^+)$, it is enough to prove that if \mathfrak{S} is stable under generization and $f^{-1}(\mathfrak{S})$ is open then \mathfrak{S} is open. The rest of the argument is an easier version of the proof of proposition 1.2.17. In this case one exploits the constructible topology of $\mathrm{Spa}(A((t)), A^+ + t \cdot A[[t]])$.

Let $x \in \mathfrak{S}$, and let $y \in \mathrm{Spa}(A((t)), A^+ + t \cdot A[[t]])$ a lift of x . For every open $U_{x,a_1,a_2} = \{x \in \mathrm{Spa}(A, A^+) \mid |a_1|_x \leq |a_2|_x \neq 0\}$ we choose $n \in \mathbb{N}$ such that $|t^n|_y \leq |a_2|_y$ and define U_{y,a_1,a_2} as

$$\{z \in \mathrm{Spa}(A((t)), A^+ + t \cdot A[[t]]) \mid |a_1|_z, |t^n|_z \leq |a_2|_z \neq 0\}.$$

Observe that $y \in U_{y,a_1,a_2}$, that it is quasi-compact and that $f(U_{y,a_1,a_2}) = U_{x,a_1,a_2}$. Notice that since \mathfrak{S} is stable under generization in $\mathrm{Spa}(A, A^+)$ the intersection of all U_{x,a_1,a_2} that contain x is contained in \mathfrak{S} . This implies that the intersection of the U_{y,a_1,a_2} is contained in $f^{-1}(\mathfrak{S})$. Since $f^{-1}(\mathfrak{S})$ is open, by the usual compactness argument in the patch topology there is a finite subset with $\bigcap U_{y,a_1,a_2} \subseteq f^{-1}(\mathfrak{S})$. We denote by Z this intersection and we observe that $f(Z) = \bigcap U_{x,a_1,a_2}$ which is open in $\mathrm{Spa}(A, A^+)$ and contains x . \square

Proposition 1.2.29. *If A and A^+ are discrete perfect valuation rings with the same fraction field then (A, A^+) is olivine.*

Proof. If $\mathrm{Spo}(A, A^+)^\dagger = \mathrm{Spo}(A, A^+)$ then proposition 1.2.27 proves that (A, A^+) is olivine. Suppose $x \in \mathrm{Spo}(A, A^+) \setminus \mathrm{Spo}(A, A^+)^\dagger$, then x is meromorphic and there is $a \in A$ with $1 < |a|_x^a$. We must have $\frac{1}{a} \in A^+$ since A^+ is a valuation ring and $a \notin A^+$. Let $b = \frac{1}{a}$, we claim that $A = A^+[\frac{1}{b}]$. By the Archimedean property of $|\cdot|_x^a$ for every $a' \in A$ there is a big enough $n \in \mathbb{N}$ with $|b^n \cdot a'|_x^a < 1$. Since A^+ is a valuation ring either $a' \cdot b^n \in A^+$ or $\frac{1}{a' \cdot b^n} \in A^+$, but the second case contradicts that $|\cdot|_x^a \in \mathrm{Spa}(A, A^+)$.

By proposition 1.2.17 (A^+, A^+) is olivine and since $\mathrm{Spo}(A, A^+) \subseteq \mathrm{Spo}(A^+, A^+)$ is the open locus in which $b \neq 0$ we conclude by proposition 1.2.8 that (A, A^+) is also olivine. \square

In what follows we prove some lemmas to prepare the proof of theorem 4. As we have mentioned in the introduction the statements and techniques are derivative of Scholze and Weinstein's full faithfulness result [30] 18.3.1.

Lemma 1.2.30. *Let (A, A^+) be a non-analytic perfect Huber pair in characteristic p . Then there is a unique morphism of v -sheaves $\mathrm{Spd}(A, A^+) \rightarrow \mathbb{Z}_p^\diamond$. It is given by the composition $\mathrm{Spd}(A, A^+) \rightarrow \mathbb{F}_p^\diamond \rightarrow \mathbb{Z}_p^\diamond$.*

Proof. It is enough to prove that for every geometric point $\mathrm{Spa}(C, C^+) \rightarrow \mathrm{Spd}(A, A^+)$ the composition to \mathbb{Z}_p^\diamond factors through \mathbb{F}_p^\diamond . Consider the product of points $\mathrm{Spa}(R_\infty, R_\infty^+) \rightarrow \mathrm{Spd}(A, A^+)$ as in the proof of lemma 1.2.16, with $R_\infty^+ = \prod_{i=1}^\infty C^+$ and $\varpi_\infty = (\varpi^{p^i})$. The composition $\mathrm{Spa}(R_\infty, R_\infty^+) \rightarrow \mathrm{Spd}(A, A^+) \rightarrow \mathbb{Z}_p^\diamond$ defines an untilt of R_∞ given by an element $\xi = p + (\varpi_\infty)^{\frac{1}{p^k}} \cdot \alpha$ with $\alpha \in W(R^+)$. For any $i \in \mathbb{N}$ the projection map $\iota_i : R_\infty \rightarrow C$ defines an untilt of C and since the composition $(A, A^+) \rightarrow (R_\infty, R_\infty^+) \xrightarrow{\iota_i} (C, C^+)$ is independent of the projection map chosen, all of these untilts agree. This says that the ideal I_i generated by $\iota_i(\xi)$ in $W(C^+)$ agree, we call this ideal I . Since $\iota_i(\xi) = p - \varpi^{\frac{p^i}{p^k}} \iota_i(\alpha)$ the sequence $\iota_i(\xi)$ converges to p in the (p, ϖ) -adic topology. But the ideal associated to an untilt is closed, so $p \in I$ and $\mathrm{Spa}(C, C^+) \rightarrow \mathbb{Z}_p^\diamond$ factors through \mathbb{F}_p^\diamond . \square

Lemma 1.2.31. *Let (A, A^+) be as above and let (B, B^+) be a complete Huber pairs over \mathbb{Z}_p . Then every morphism of v -sheaves $\mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(B, B^+)$ comes from a unique morphism of Huber pairs $(B, B^+) \rightarrow (A, A^+)$.*

Proof. Given a map $\mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(B, B^+)$ we associate to it a map of pre-adic spaces $\mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spa}(B, B^+)$. Let $S = \mathrm{Spd}(A, A^+)$, $R = A((t^{\frac{1}{p^\infty}}))$, $R^+ = A^+ + (t^{\frac{1}{p^\infty}})A[[t^{\frac{1}{p^\infty}}]]$, $X = \mathrm{Spa}(R, R^+)$ and $X' = X \times_S X$. Notice that X is an affinoid perfectoid space and X' is a perfectoid space (which is

not affinoid). The natural map $X \rightarrow \mathrm{Spd}(A, A^+)$ surjects onto $\mathrm{Spd}(A, A^+)^{\dagger}$ and we get an equalizer diagram:

$$0 \rightarrow \mathrm{Hom}(\mathrm{Spd}(A, A^+)^{\dagger}, \mathrm{Spd}(B, B^+)) \rightarrow \mathrm{Hom}(X, \mathrm{Spd}(B, B^+)) \rightrightarrows \mathrm{Hom}(X', \mathrm{Spd}(B, B^+)).$$

Since X is affinoid perfectoid a homomorphism $f : X \rightarrow \mathrm{Spd}(B, B^+)$ is given by an untilt R^{\sharp} and a continuous ring map $f^* : B \rightarrow R^{\sharp}$ such that $f^*(B^+) \subseteq R^{+, \sharp}$. By lemma 1.2.30 the untilt must be R .

A necessary condition for such a morphism to glue, is that $f^*(B)$ must be invariant under any automorphism of R over A . In particular, we may replace t by any topological nilpotent unit in R without changing the image of $f^*(B)$. Take an element $b \in B$, we want to show $f^*(b) \in A$. Observe that $t^{p^n} \cdot f^*(b)$ is topologically nilpotent for big enough n . Replacing by $t \mapsto t^{\frac{1}{p^n}}$ we conclude that $t^{p^n} f^*(b)$ is topologically nilpotent for all $n \in \mathbb{Z}$. This proves $f^*(b)$ is power-bounded which gives $f^*(b) \in A[[t^{\frac{1}{p^\infty}}]]$. We can write $f^*(b)$ as $a_0 + t^{\frac{1}{p^n}} q$ with $a_0 \in A$ and $q \in A[[t^{\frac{1}{p^\infty}}]]$. Since the second term converges to 0 under the substitution $t \mapsto t^{p^n}$ we see that $f^*(b) = a_0$. This defines a map of rings $B \rightarrow A$. Since the subspace topology of A in R is the discrete topology this ring morphism is continuous if and only if the original one was. Finally, we see that B^+ maps to $R^+ \cap A$ which is easily seen to be A^+ . So far we have constructed a map $\mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spa}(B, B^+)$ with the property that the induced map $\mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(B, B^+)$ agrees with our original map in the locus $\mathrm{Spd}(B, B^+)^{\dagger}$. We wish to prove that the two maps agree.

Consider the map $\mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(B, B^+) \times \mathrm{Spd}(B, B^+)$ we will show that $\mathrm{Spd}(A, A^+)$ factors through the diagonal embedding $\Delta : \mathrm{Spd}(B, B^+) \rightarrow \mathrm{Spd}(B, B^+) \times \mathrm{Spd}(B, B^+)$. This can be verified on geometric points $\mathrm{Spd}(C, C^+) \rightarrow \mathrm{Spd}(A, A^+)$, and since the maps already agree on $\mathrm{Spd}(A, A^+)^{\dagger}$ it is enough to verify this on meromorphic points $x \in \mathrm{Spo}(A, A^+)$. Pick a pseudo-uniformizer $\varpi \in C$ and consider the product of points R_∞ as in the proof lemma 1.2.16 together with the map $\mathrm{Spa}(R_\infty, R_\infty^+) \rightarrow \mathrm{Spd}(A, A^+)$ given by the diagonal morphism $A \rightarrow \prod C$. Recall that for the product of points constructed in this way we have that

$$C \subseteq_{\Delta} R_\infty \subseteq \prod_{i=1}^{\infty} C,$$

and that although the diagonal embedding $C \subseteq_{\Delta} R_\infty$ is not continuous the composition $A \rightarrow R_\infty$ is continuous since A has the discrete topology.

The composition $\mathrm{Spa}(R_\infty, R_\infty^+) \rightarrow \mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(B, B^+) \times \mathrm{Spd}(B, B^+)$ gives two morphisms $f_1, f_2 : B \rightarrow R_\infty$, and one verifies that at the level of rings they both have to factor through the diagonal $C \subseteq_{\Delta} R_\infty$. The connected components of $\mathrm{Spa}(R_\infty, R_\infty^+)$ are in bijection with ultrafilters of \mathbb{N} . By the proof of lemma 1.2.16 the residue field at a non-principal ultrafilter \mathcal{U} maps to the meromorphic specialization x_{mer} . Since $\mathrm{Spa}(C_{\mathcal{U}}, C_{\mathcal{U}}^+) \rightarrow \mathrm{Spd}(B, B^+) \times \mathrm{Spd}(B, B^+)$ factors through $\mathrm{Spd}(A, A^+)^{\dagger}$ (being non-analytic in $\mathrm{Spo}(A, A^+)$), it also factors through the diagonal. These maps are given by the composition $f_i : B \rightarrow C \rightarrow R_\infty \rightarrow C_{\mathcal{U}}$. We can conclude $f_1 = f_2$ since the map $C \rightarrow C_{\mathcal{U}}$ is injective. \square

We are not ready to prove theorem 4, which is the global version of lemma 1.2.31. For the convenience of the reader we state it again.

Theorem 1.2.32. *Let Y be a perfect non-analytic adic space over \mathbb{F}_p and let X be a pre-adic space over \mathbb{Z}_p . The natural map*

$$\mathrm{Hom}_{\mathrm{PreAd}}(Y, X) \rightarrow \mathrm{Hom}(Y^{\diamond}, X^{\diamond})$$

is bijective. In particular, \diamond is fully faithful when restricted to the category of perfect non-analytic adic spaces over \mathbb{F}_p .

This theorem says, intuitively speaking, that (up to perfection) one does not get new morphisms of v -sheaves when the source is a non-analytic adic space.

Proof. It is not hard to prove injectivity. For surjectivity, the hard part is to prove that a morphism $g : X^\diamond \rightarrow Y^\diamond$ induces a unique map of topological spaces $f : |X| \rightarrow |Y|$ that makes the following diagram commutative.

$$\begin{array}{ccc} |Y^\diamond| & \xrightarrow{g} & |X^\diamond| \\ \downarrow & & \downarrow \\ |Y| & \xrightarrow{f} & |X| \end{array}$$

Assume for the moment that this is the case. Let $U = \coprod_{i \in I} \mathrm{Spa}(B_i, B_i^+)$ be an open cover for X and let $V = \coprod_{j \in J} \mathrm{Spa}(B'_j, B'^+_j)$ be an open cover of $U \times_X U$. Given a map $Y^\diamond \rightarrow X^\diamond$ we can pullback U and V through f to obtain an open cover of adic spaces

$$Y_V \rightrightarrows Y_U \rightarrow Y$$

satisfying $Y_U^\diamond = Y^\diamond \times_{X^\diamond} U^\diamond$ and $Y_V^\diamond = Y^\diamond \times_{X^\diamond} U^\diamond$. It is enough to prove that $Y_U^\diamond \rightarrow U^\diamond$ and $Y_V^\diamond \rightarrow V^\diamond$ come from morphisms of pre-adic spaces. This follows from lemma 1.2.31 by a standard glueing argument, since U and V are a disjoint union of affinoid pre-adic spaces.

Verifying that $g : |Y^\diamond| \rightarrow |X^\diamond|$ descends to a continuous map $f : |Y| \rightarrow |X|$ can be done locally on $|Y|$, we may assume $Y = \mathrm{Spa}(A, A^+)$. For $y \in |Y|$ and $z \in \mathrm{Spo}(A, A^+)$ with $h(z) = y$, we define $f(y) := h(g(z))$. We must verify that this doesn't depend of the choice of z and that it is continuous. The map f is well defined if and only if $h(g(z)) = h(g(z_{\mathrm{mer}}))$ when z is meromorphic, and by proposition 1.2.28 to prove continuity it is enough to prove that if $\mathfrak{S} \subseteq |X|$ is open then $f^{-1}(\mathfrak{S})$ is stable under arbitrary generization in $\mathrm{Spa}(A, A^+)$. Let $w \in \mathrm{Spa}(A, A^+)$ a horizontal generization of y . Let (k_y, k_y^+) and (k_w, k_w^+) denote the affinoid residue fields of w and y and let K_w denote the smallest ring containing k_w^+ and A . It is enough to prove that the induced maps $|\mathrm{Spd}(K_w, k_w^+)| \rightarrow |X^\diamond|$ and $|\mathrm{Spd}(k_y, k_y^+)| \rightarrow |X^\diamond|$ descend to continuous maps $|\mathrm{Spa}(K_w, k_w^+)| \rightarrow |X|$ and $|\mathrm{Spa}(k_y, k_y^+)| \rightarrow |X|$. In summary, we have reduced the initial claim to the case in which $Y = \mathrm{Spa}(A, A^+)$ where $A^+ \subseteq A$ and the two rings are perfect non-analytic valuation rings that have the same fraction field.

Lemma 1.2.33. *Let X be a pre-adic space over \mathbb{Z}_p as above and let $Y = \mathrm{Spa}(A, A^+)$ where A^+ and A are perfect non-analytic valuation rings that have the same fraction field, and let $g : \mathrm{Spd}(A, A^+) \rightarrow X^\diamond$ be a map. Let $h(c) \in \mathrm{Spa}(A, A^+)$ denote the unique closed point and let $c \in \mathrm{Spo}(A, A^+)$ denote the unique non-analytic point mapping to $h(c)$. If $h(g(c)) \in |X|$ lies in $\mathrm{Spa}(B_1, B_1^+) \subseteq X$ then g factors through a map $\mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(B_1, B_1^+) \subseteq X^\diamond$. In particular, g is coming from a map of pre-adic spaces $\mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spa}(B_1, B_1^+) \subseteq X$.*

Proof. Suppose to get a contradiction that there is an “exotic” g that does not satisfy this property. By proposition 1.2.29 (A, A^+) is olivine so we may treat $|\mathrm{Spd}(A, A^+)|$ and $\mathrm{Spo}(A, A^+)$ as the same object. Let $U_1 \subseteq \mathrm{Spo}(A, A^+)$ be the open subset associated to the pullback of $\mathrm{Spd}(B_1, B_1^+)$, this is by assumption a proper open subset. Let $Z = \mathrm{Spo}(A, A^+) \setminus U_1$, it is a quasi-compact topological space and we may use [30] 18.3.2 to find the largest prime $\mathfrak{p}_m \in \mathrm{Spec}(A)$ that is the support of an element in Z . Replacing A and A^+ by A/\mathfrak{p}_m and A^+/\mathfrak{p}_m we may assume that all elements of $z \in Z$ satisfy $\mathrm{supp}(z) = 0$, we let $K = \mathrm{Frac}(A)$. In this case $Z \subseteq \mathrm{Spo}(K, A^+)$. Since Z is a closed subset it contains the unique closed point q of $\mathrm{Spo}(K, A^+)$, this is the unique non-analytic point such that $h(q)$ is closed in $\mathrm{Spa}(K, A^+)$. We claim that U_1 contains every analytic localization of the form $N_{n < 1}$ with $n \neq 0$ and $n \in \mathrm{supp}(c)$. Indeed, if $z \in Z \cap N_{n < 1}$ then $|n|_z^a < 1$ and either z or its formal specialization would have non-trivial support contradicting the assumption that \mathfrak{p}_m was the largest.

The composition $\mathrm{Spd}(K, A^+) \rightarrow X^\diamond$ must also factor through some other open affine subsheaf $\mathrm{Spd}(B_2, B_2^+)$, since it has a unique closed point. We let U_2 be the open in $\mathrm{Spo}(A, A^+)$ associated

to the pullback of $\mathrm{Spd}(B_2, B_2^+)$. By example 1.2.9 there is an open neighborhood of q of the form $U_{0 \leq b \neq 0} \cap U_{1 \leq b' \neq 0} \cap N_{n < 1}$ and contained in U_2 . Moreover, in example 1.2.9 q corresponds to the triple of prime ideals $(0, 0, \mathfrak{m})$ where \mathfrak{m} denotes the maximal ideal of A^+ . With this description it is easy to see that $U_{1 \leq b' \neq 0} = \mathrm{Spo}(A, A^+)$ and that $N_{n < 1} = \mathrm{Spo}(A, A^+)$ since they contain q . In summary, there is a classical localization $U_{0 \leq b \neq 0}$ containing q and contained in U_2 .

We have found neighborhoods $N_{b < 1} \subseteq U_1$ and $U_{0 \leq b \neq 0} \subseteq U_2$ with $N_{b < 1} \cap U_{0 \leq b \neq 0} \subseteq U_1 \cap U_2$, observe that $\mathrm{Spo}(A, A^+) = N_{b < 1} \cup U_{0 \leq b \neq 0}$. Let A_b and A_b^+ denote the (b) -adic completion of A and A^+ respectively. Lemma 1.2.23 shows that $N_{b < 1}$ is represented by $\mathrm{Spd}(A_b, A_b^+)$. We also have that $U_{0 \leq b \neq 0}$ is represented by $\mathrm{Spd}(A[\frac{1}{b}], A^+)$ and that the intersection $N_{b < 1} \cap U_{0 \leq b \neq 0}$ is represented by $\mathrm{Spa}(A_b[\frac{1}{b}], A_b^+)$, notice that this last one is a perfectoid field. We let q_b denote the closed point of $N_{b < 1} \cap U_{0 \leq b \neq 0}$ which is meromorphic in $\mathrm{Spo}(A, A^+)$.

Since these morphisms glue, there is an affinoid open subspace $\mathrm{Spa}(B_3, B_3^+) \subseteq \mathrm{Spa}(B_1, B_1^+) \times_X \mathrm{Spa}(B_2, B_2^+)$ and a map $\mathrm{Spa}(A_b[\frac{1}{b}], A_b^+) \rightarrow \mathrm{Spd}(B_3, B_3^+)$ making the following diagram commutative:

$$\begin{array}{ccccc}
\mathrm{Spa}(A_b[\frac{1}{b}], A_b^+) & \xrightarrow{\hspace{2cm}} & \mathrm{Spd}(A[\frac{1}{b}], A^+) & & \\
\downarrow & \searrow & \downarrow & & \\
& & \mathrm{Spd}(B_3, B_3^+) & \longrightarrow & \mathrm{Spd}(B_2, B_2^+) \\
& & \downarrow & & \downarrow \\
\mathrm{Spd}(A_b, A_b^+) & \longrightarrow & \mathrm{Spd}(B_1, B_1^+) & \longrightarrow & X^\diamond
\end{array}$$

By lemma 1.2.31 the map $\mathrm{Spd}(A[\frac{1}{b}], A^+) \rightarrow \mathrm{Spd}(B_2, B_2^+)$ is given by a map of Huber pairs $(B_2, B_2^+) \rightarrow (A[\frac{1}{b}], A^+)$. Since $\mathrm{Spo}(B_3, B_3^+) \subseteq \mathrm{Spo}(B_2, B_2^+)$ has the form $h^{-1}(\mathrm{Spa}(B_3, B_3^+))$ the pullback of $\mathrm{Spo}(B_3, B_3^+)$ to $\mathrm{Spo}(A[\frac{1}{b}], A^+)$ has the form $h^{-1}(U_3)$ for some $U_3 \subseteq \mathrm{Spa}(A[\frac{1}{b}], A^+)$. Moreover, $h(q_b)$ is the closed point of $\mathrm{Spa}(A[\frac{1}{b}], A^+)$. This proves that $\mathrm{Spd}(A[\frac{1}{b}], A^+)$ factors through $\mathrm{Spd}(B_3, B_3^+)$ and consequently through $\mathrm{Spd}(B_1, B_1^+)$ contradicting our initial assumption. □

□

We now study perfect non-analytic Huber pairs of the form (A, A) . These are the type of Huber pairs that we will associate to perfect affine schemes to develop our theory of specialization.

Proposition 1.2.34. *Let A be a ring endowed with the discrete topology and $f^* : (B, B^+) \rightarrow (A, A)$ a map of Huber pairs, then the following hold:*

1. $\mathrm{Spo}(f)$ is saturated with respect to h , in other words

$$\mathrm{Spo}(f)(\mathrm{Spo}(A, A)) = h^{-1}(f(\mathrm{Spa}(A, A))).$$

2. $f(\mathrm{Spa}(A, A))$ is stable under horizontal specializations in $\mathrm{Spa}(B, B^+)$.

3. $f(\mathrm{Spa}(A, A))$ is stable under vertical generization in $\mathrm{Spa}(B, B^+)$.

Proof. To prove the first claim let $y \in \mathrm{Spo}(A, A)$ and let $x = \mathrm{Spo}(f)(y)$. If x is d-analytic, y must be meromorphic and y_{mer} maps to x_{mer} under $\mathrm{Spo}(f)$, that is $h^{-1}(h(x)) \subseteq \mathrm{Im}(\mathrm{Spo}(f))$.

Suppose now that x is non-analytic and that x^{mer} exists in $\mathrm{Spo}(B, B^+)$. In this case, y^{mer} might not exist if $|\cdot|_y^h$ is not microbial, and even when y^{mer} exists it may not be true that y^{mer} maps to x^{mer} under $\mathrm{Spo}(f)$. Indeed, it may happen that y^{mer} maps to x instead. For these reasons we use a different construction.

Consider $h(y) \in \text{Spa}(A, A)$ together with its residue field map $\iota_{h(y)} : \text{Spa}(K_{h(y)}, K_{h(y)}^+) \rightarrow \text{Spa}(A, A)$. Notice that $\iota_{h(y)}$ factors through a map $g : \text{Spa}(K_{h(y)}^+, K_{h(y)}^+) \rightarrow \text{Spa}(A, A)$, so it is enough to prove that x^{mer} is in the image of $\text{Spo}(f \circ g)$. Take an element $b \in B$ with $|b|_{x^{mer}}^a \notin \{0, 1\}$ and replace it by its inverse in $K_{h(y)}$, whenever it is necessary, so that $b \in K_{h(y)}^+$. Define K^+ as the (b) -adic completion of $K_{h(y)}^+$, and let $K = K^+[\frac{1}{b}]$. We get the following commutative diagram,

$$\begin{array}{ccc} \text{Spa}(K, K^+) & \xrightarrow{\quad} & \text{Spo}(K_{h(y)}^+, K_{h(y)}^+) \\ & \searrow & \swarrow \\ & \text{Spo}(B, B^+) & \end{array}$$

and one can verify that the image of the closed point of $\text{Spa}(K, K^+)$ in $\text{Spo}(B, B^+)$ is x^{mer} . This proves that $h^{-1}(h(x)) \subseteq \text{Im}(\text{Spo}(f))$.

The proof of the second claim also follows from observing that the residue field map $\iota_{h(y)}$ factors through g . Indeed, we get the following commutative diagram of adic spaces:

$$\begin{array}{ccccc} \text{Spa}(K_{h(y)}, K_{h(y)}^+) & \longrightarrow & \text{Spa}(K_{h(y)}^+, K_{h(y)}^+) & \xrightarrow{g} & \text{Spa}(A, A) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spa}(K_{h(x)}, K_{h(x)}^+) & \longrightarrow & \text{Spa}(K_{h(x)}^+, K_{h(x)}^+) & \xrightarrow{g'} & \text{Spa}(B, B^+) \end{array}$$

Where we use that $K_{h(x)}^+ = K_{h(y)}^+ \cap K_{h(x)}$ to define g' . Moreover, the vertical map on the left is surjective since $h(x) = f(h(y))$ and one can deduce that the vertical map in the middle column is also surjective because the map of valuation rings is local. A prime ideal of $J \subseteq K_{h(x)}^+$ determines a horizontal specializations of $|\cdot|_{h(x)}$, namely $|\cdot|_{h(x)}/J$, and every horizontal specialization of $h(x)$ can be constructed in this way. For J as above we let $K_J^+ = K_{h(x)}^+/J$ and $K_J = \text{Frac}(K_J^+)$, we get the following commutative diagram:

$$\begin{array}{ccccc} \text{Spa}(K_J, K_J^+) & \longrightarrow & \text{Spa}(K_J^+, K_J^+) & \xrightarrow{\quad} & \text{Spa}(K_{h(y)}^+, K_{h(y)}^+) \\ & \searrow & \downarrow & \swarrow & \\ & & \text{Spo}(B, B^+) & & \end{array}$$

The closed point of $\text{Spa}(K_J, K_J^+)$ maps to the horizontal specialization of $h(x)$ associated to the ideal J .

The third claim follows from corollary 1.2.15 and from the first claim. \square

Definition 1.2.35. We say that a subset of $\text{Spo}(B, B^+)$ is a *schematic subset* if it is a union of sets of the form $\text{Spo}(m)(\text{Spo}(A, A))$ where (A, A) is given the discrete topology and $m^* : (B, B^+) \rightarrow (A, A)$ is a map of Huber pairs.

Proposition 1.2.36. Suppose that $Z \subseteq \text{Spo}(B, B^+)$ is a schematic closed subset. Let $\sigma : \text{Spo}(B, B^+) \rightarrow \text{Spec}(B)$ denote the map $x \mapsto \text{supp}(x)$ attaching to every point of $\text{Spo}(B, B^+)$ its support ideal. Notice that $\sigma = \text{supp} \circ h$ where $\text{supp} : \text{Spa}(B, B^+) \rightarrow \text{Spec}(B)$ also attaches the support ideal. Then $Z = \sigma^{-1}(V(I))$ for some prime ideal $I \subseteq B$ open for the topology in B .

Proof. Any map $m^* : (B, B^+) \rightarrow (A, A)$ with A a discrete ring must factor through $(B/B^{\circ\circ}, B^+/B^{\circ\circ})$, by reducing to this case we may assume that B has the discrete topology. By proposition 1.2.34, $Z = h^{-1}(h(Z))$ and by corollary 1.2.15, Z is closed under vertical generization. Moreover, since Z is closed in $\text{Spo}(B, B^+)$ it is also stable under vertical specialization. This implies that $Z = \sigma^{-1}(\sigma(Z))$.

Indeed, any two points $x, y \in \mathrm{Spa}(B, B^+)$ with $\mathrm{supp}(x) = \mathrm{supp}(y) = \mathfrak{p}$ are vertical specializations of the trivial valuation on B with support \mathfrak{p} .

We only have left to prove that $\sigma(Z)$ is closed in the Zariski topology of $\mathrm{Spec}(B)$. Since B has the discrete topology, the support map admits a continuous section $\mathrm{triv} : \mathrm{Spec}(B) \rightarrow \mathrm{Spa}(B, B^+)$ that assigns to a prime ideal $\mathfrak{p} \subseteq B$ the trivial valuation with support \mathfrak{p} . We have $\sigma(Z) = \mathrm{triv}^{-1}(h(Z))$ so we may prove $h(Z)$ is closed instead. By proposition 1.2.34, $h(Z)$ is also closed under horizontal specialization, this gives that the complement of $h(Z)$ in $\mathrm{Spa}(B, B^+)$ is stable under (arbitrary) generization. $\mathrm{Spo}(B, B^+) \setminus Z = h^{-1}(\mathrm{Spa}(B, B^+) \setminus h(Z))$ by proposition 1.2.28 the set $\mathrm{Spa}(B, B^+) \setminus h(Z)$ is open as we needed to show. \square

1.3 The reduction functor

1.3.1 The v -topology for perfect schemes

This is the only section in which we will be forced to be set-theoretically careful, we advise the reader that does not wish to ignore the set-theoretic subtleties that arise in this section to review the definition and basic properties of cut-off cardinals that are given in [28] §4.

We will denote by $\mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$ the category of perfect affine schemes over \mathbb{F}_p . If κ is a cut-off cardinal we denote by $\mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}}$ the category of perfect affine schemes over \mathbb{F}_p whose underlying topological space and whose ring of global sections have cardinality bounded by κ . Given $S = \mathrm{Spec}(A) \in \mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$ we associate to it a v -sheaf in Perf given by:

$$S^\diamond((R, R^+)) = \{f : A \rightarrow R^+ \mid f \text{ is a morphism of rings}\}$$

Remark 1.3.1. Notice that $\mathrm{Spec}(A)^\diamond = \mathrm{Spd}(A, A)$ when A is given the discrete topology. Later on, we will work with v -sheaves of the form $\mathrm{Spd}(A, A)$ where A can be given either the discrete topology or a more interesting topology and we might consider both kind of sheaves at the same time. To avoid having to specify the topology given to A every time, we will use $\mathrm{Spec}(A)^\diamond$ whenever A is given the discrete topology and we will use $\mathrm{Spd}(A, A)$ when A is given a more interesting topology.

Proposition 1.3.2. If κ is a cut-off cardinal and $S \in \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}}$ then S^\diamond is a κ -small v -sheaf.

Proof. Let $S = \mathrm{Spec}(A)$ then $\mathrm{Spa}(A((t^{\frac{1}{p^\infty}})), A[[t^{\frac{1}{p^\infty}}]])$ is a κ -small perfectoid space and the map

$$\mathrm{Spa}(A((t^{\frac{1}{p^\infty}})), A[[t^{\frac{1}{p^\infty}}]]) \rightarrow S^\diamond$$

is surjective. \square

Proposition 1.3.2 gives rise to functors $\diamond_\kappa : \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}} \rightarrow \widetilde{\mathrm{Perf}}_\kappa$ that are compatible when we vary κ and give rise to a functor $\diamond : \mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}} \rightarrow \widetilde{\mathrm{Perf}}$

Proposition 1.3.3. The functors $\diamond : \mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}} \rightarrow \widetilde{\mathrm{Perf}}$ and $\diamond_\kappa : \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}} \rightarrow \widetilde{\mathrm{Perf}}_\kappa$ are fully-faithful and commute with finite limits.

Proof. This is a direct consequence of 1.2.32. \square

After embedding $\mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$ in $\widetilde{\mathrm{Perf}}$ one can define a Grothendieck topology on $\mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$ by considering a small family of maps of affine schemes, $(S_i \rightarrow T)_{i \in \mathcal{F}}$, to be a cover if the map $\coprod_{i \in \mathcal{F}} S_i^\diamond \rightarrow T^\diamond$ is a surjective map of v -sheaves. However, there is an intrinsic way of defining this topology which we now discuss.

Definition 1.3.4. (See [5] 2.1)

1. A morphisms of qcqs schemes $S \rightarrow T$, is said to be *universally subtrusive* (or a *v-cover*) if for any valuation ring V and a map $\text{Spec}(V) \rightarrow T$ there is an extension of valuation rings $V \subseteq W$ (see [31] 0ASG) and a map $\text{Spec}(W) \rightarrow S$ making the following diagram commutative:

$$\begin{array}{ccc} \text{Spec}(W) & \longrightarrow & S \\ \downarrow & & \downarrow \\ \text{Spec}(V) & \longrightarrow & T \end{array}$$

2. A small family of morphisms in $\text{PCAlg}_{\mathbb{F}_p}^{\text{op}}$, $(S_i \rightarrow T)_{i \in \mathcal{F}}$, is said to be *universally subtrusive* (or a *v-cover*) if there is a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ for which $\coprod_{i \in \mathcal{F}'} S_i \rightarrow T$ is universally subtrusive.

Lemma 1.3.5. (See [5] 2.2) A morphism $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ of affine schemes (not necessarily over \mathbb{F}_p) is universally subtrusive if and only if the map of topological spaces $|f^{ad}| : |\text{Spa}(B, B)| \rightarrow |\text{Spa}(A, A)|$ is surjective.

Proof. Let $T = \text{Spec}(A)$, $S = \text{Spec}(B)$, $T^{ad} = \text{Spa}(A, A)$ and $S^{ad} = \text{Spa}(B, B)$. Assume f to be universally subtrusive and take $x \in |T^{ad}|$. Taking a representative we can consider x as a valuation $|\cdot|_x : A \rightarrow \Gamma_x$, which gives a valuation subring V of $\text{Frac}(A/\text{supp}(|\cdot|_x))$ together with a map $\text{Spec}(V) \rightarrow \text{Spec}(A)$. Since f is universally subtrusive we can take an extension of valuation rings W/V and a map $\text{Spec}(W) \rightarrow \text{Spec}(B)$ making diagram 1 above commutative. The map $B \rightarrow W$ induces a valuation $|\cdot|_y : B \rightarrow \Gamma_y$ and consequently a point $y \in S^{ad}$. Moreover, the composition $|\cdot|_{f(y)} : A \rightarrow B \rightarrow \Gamma_y$ is equivalent to $|\cdot|_x$ which proves that $|S^{ad}| \rightarrow |T^{ad}|$ is surjective. For the converse, given a map $\text{Spec}(V) \rightarrow T$ we may consider the induced map $\text{Spa}(K, V) \rightarrow T^{ad}$ with $K = \text{Frac}(V)$. The closed point of $\text{Spa}(K, V)$ gives a point $x \in T^{ad}$ and by surjectivity of f^{ad} we may pick a point $y \in \text{Spa}(B, B)$ lifting x . Consider the affinoid residue fields (K_x, K_x^+) and (K_y, K_y^+) at x and y respectively. We get the following commutative diagram:

$$\begin{array}{ccccc} & & \text{Spec}(K_y^+) & \longrightarrow & S \\ & & \downarrow & & \downarrow \\ \text{Spec}(V) & \longrightarrow & \text{Spec}(K_x^+) & \longrightarrow & T \end{array}$$

Both K_y^+ and V are valuation extensions of K_x^+ , consequently there is a valuation ring W extending both K_y^+ and V making the diagram commute (See [15] 1.1.14-f). This proves that f is universally subtrusive. \square

Lemma 1.3.6. Let $f : S \rightarrow T$ be a morphism of perfect affine schemes over \mathbb{F}_p . The map $f^\diamond : S^\diamond \rightarrow T^\diamond$ is a quasi-compact map of *v-sheaves*.

Proof. By writing $B = A[t_i]_{i \in I}/JA[t_i]_{i \in I}$ for some variables t_i and an ideal J we can reduce to the cases where either f is a closed embedding or f is the base change of the structure map $g : \text{Spec}(\mathbb{F}_p[t_i]_{i \in I}) \rightarrow \text{Spec}(\mathbb{F}_p)$.

Let $X = \text{Spa}(R, R^+) \in \text{Perf}$ it is enough to prove that $X \times_{T^\diamond} S^\diamond$ is quasi-compact for any map $\text{Spa}(R, R^+) \rightarrow S^\diamond$. For the later case, the basechange gives the sheaf $X \times \text{Spec}(\mathbb{F}_p[t_i])^\diamond$. This functor is represented by

$$\text{Spa}(R\langle t_i^{\frac{1}{p^\infty}} \rangle_{i \in I}, R^+\langle t_i^{\frac{1}{p^\infty}} \rangle_{i \in I}),$$

which is affinoid perfectoid and consequently quasi-compact. For the former case let $B = A/J$, and let $Z = X \times_{T^\diamond} S^\diamond$. For a perfectoid Huber pair (L, L^+) we have:

$$Z(L, L^+) = \{r : (R, R^+) \rightarrow (L, L^+) \mid r(R \cdot J) = 0\}$$

This is the definition of a Zariski closed subset of X and by ([27] Lemma II.2.2) representable by an affinoid perfectoid space. In particular, Z is a quasi-compact v -sheaf. \square

Remark 1.3.7. *One can prove lemma 1.3.6 by observing that for a perfect discrete ring A we have the identity $\mathrm{Spd}(A, A)^\dagger = \mathrm{Spd}(A, A)$. Indeed, we can apply lemma 1.2.26.*

Proposition 1.3.8. *1. Let $f : S \rightarrow T$ be a morphism of perfect affine schemes over \mathbb{F}_p . The map f is universally subtrusive if and only if $f^\diamond : S^\diamond \rightarrow T^\diamond$ is a surjective map of v -sheaves.*

2. A family of morphisms $(S_i \rightarrow T)_{i \in \mathcal{F}}$ is universally subtrusive if and only if $(\coprod_{i \in \mathcal{F}} S_i) \rightarrow T$ is a surjective map of v -sheaves.

Proof. Let $T = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(B)$. Since the map of v -sheaves $f^\diamond : S^\diamond \rightarrow T^\diamond$ is quasi-compact, by ([28] 12.11) it is a surjective map of v -sheaves if and only if $|f^\diamond|$ is a surjective map of topological spaces. By proposition 1.2.12 and lemma 1.3.5, it suffices to prove that the map $\mathrm{Spo}(B, B) \rightarrow \mathrm{Spo}(A, A)$ is surjective if and only if the map $\mathrm{Spa}(B, B) \rightarrow \mathrm{Spa}(A, A)$ is. Functoriality and surjectivity of h proves one direction, and the converse direction is a direct consequence of 1.2.34.

For the second claim, it follows easily from above that a universally subtrusive family of maps $(S_i \rightarrow T)_{i \in \mathcal{F}}$ induces a surjective map of v -sheaves $(\coprod_{i \in \mathcal{F}} S_i) \rightarrow T$, actually a finite subfamily is already surjective. To prove the converse we have to take a family of maps $(S_i \rightarrow T)_{i \in \mathcal{F}}$ such that $(\coprod_{i \in \mathcal{F}} S_i^\diamond) \rightarrow T^\diamond$ is a surjective map of v -sheaves and prove there is a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ for which $\coprod_{i \in \mathcal{F}'} S_i^\diamond \rightarrow T^\diamond$ is still surjective. Let $S_i = \mathrm{Spec}(R_i)$ and $T = \mathrm{Spec}(P)$ and consider affinoid perfectoid spaces $Y = \mathrm{Spa}(P((t^{\frac{1}{p^\infty}})), P[[t^{\frac{1}{p^\infty}}]])$ and $X_i = \mathrm{Spa}(R_i((t^{\frac{1}{p^\infty}})), R_i[[t^{\frac{1}{p^\infty}}]])$. The map $(\coprod_{i \in \mathcal{F}} X_i) \rightarrow Y$ is surjective and since Y is quasi-compact there is a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ such that $(\coprod_{i \in \mathcal{F}'} X_i) \rightarrow Y$ is still surjective. An easy argument proves that for \mathcal{F}' chosen in this way $(\coprod_{i \in \mathcal{F}'} S_i^\diamond) \rightarrow T^\diamond$ is also surjective. \square

Remark 1.3.9. *In this context, one can discuss the analogue of example 1.1.4. Given an index set I and $\{V_i\}_{i \in I}$ a family of perfect valuation rings over \mathbb{F}_p , we construct the ring $R = \prod_{i \in I} V_i$. We call the affine schemes constructed in this way a scheme-theoretic product of points. They form a basis for the v -topology on $\mathrm{PCAlg}_{\mathbb{F}_p}^{op}$ (See [5] 6.2).*

Given a cut-off cardinal κ we let $\widetilde{\mathrm{SchPerf}}_\kappa$ be the topos associated to the site $\mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{op}$ with the v -topology, and we will refer to an object in this topos as a κ -small scheme-theoretic v -sheaf. For any pair of cut-off cardinals $\kappa < \lambda$ we have a continuous fully-faithful embedding of sites $\iota_{\kappa, \lambda}^* : \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{op} \rightarrow \mathrm{PCAlg}_{\mathbb{F}_p, \lambda}^{op}$, which induces a morphism of topoi $\iota_{\kappa, \lambda} : \widetilde{\mathrm{SchPerf}}_\lambda \rightarrow \widetilde{\mathrm{SchPerf}}_\kappa$.

Proposition 1.3.10. *The functor $\iota_{\kappa, \lambda}^* : \widetilde{\mathrm{SchPerf}}_\kappa \rightarrow \widetilde{\mathrm{SchPerf}}_\lambda$ is fully-faithful (See [28] 8.2).*

Proof. It is enough to prove that the adjunction $\mathcal{F} \rightarrow \iota_{\kappa, \lambda, *} \iota_{\kappa, \lambda}^* \mathcal{F}$ is an isomorphism. Let

$$\mathcal{G} : \mathrm{PCAlg}_{\mathbb{F}_p, \lambda}^{op} \rightarrow \mathrm{Sets}$$

be the presheaf with $S \mapsto \mathcal{G}(S)$ constructed as follows. Let \mathcal{C}_S^κ denote the category of maps of affine schemes $S \rightarrow T$ with $T \in \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{op}$. This category is cofiltered and there is a λ -small set of objects $I_S^\kappa \subseteq \mathcal{C}_S^\kappa$, that is cofinal in \mathcal{C}_S^κ . We let $\mathcal{G}(S) = \lim_{T \in I_S^\kappa} \mathcal{F}(T)$, for any choice of I_S^κ . Unraveling the definitions we see that $\iota_{\kappa, \lambda}^* \mathcal{F}$ is the sheafification of \mathcal{G} .

We claim that \mathcal{G} is already a sheaf. Indeed, since filtered colimits are exact it is enough to prove that any v -cover $S' \rightarrow S$ in $\text{PCAlg}_{\mathbb{F}_p, \lambda}^{op}$ can be expressed as a filtered colimit of v -covers in $\text{PCAlg}_{\mathbb{F}_p, \kappa}^{op}$. Let $S = \text{Spec}(A)$ and let $S' = \text{Spec}(B)$, write $A = \varinjlim_{i \in I_S^\kappa} A_i$ and $B = \varinjlim_{j \in I_{S'}^\kappa} B_j$ with A_i and B_j κ -small rings, we may assume that the transition maps are all injective. By lemma 1.3.11 below we may assume that all morphisms $\text{Spec}(A) \rightarrow \text{Spec}(A_i)$ are v -covers. Consequently, the composition $S' \rightarrow S \rightarrow \text{Spec}(A_i)$ is also a v -cover and whenever $S' \rightarrow \text{Spec}(A_i)$ factors through a map $\text{Spec}(B_j) \rightarrow \text{Spec}(A_i)$ this later one is also a v -cover. We can replace our index sets I_S^κ and $I_{S'}^\kappa$ by a common index set I and replace the rings B_j by the smallest subring of B containing B_j and A_i for some $i \in I_S^\kappa$ so that we get a family indexed by I for which $(\text{Spec}(B_i) \rightarrow \text{Spec}(A_i))_{i \in I}$ is always defined and is a v -cover. We get our desired expression

$$(S' \rightarrow S) = \varinjlim_{i \in I} (\text{Spec}(B_i) \rightarrow \text{Spec}(A_i))_{i \in I}.$$

Once we know $\iota_{\kappa, \lambda}^* \mathcal{F} = \mathcal{G}$, we compute $\iota_{\kappa, \lambda, *}\iota_{\kappa, \lambda}^* \mathcal{F}(S)$ for $S \in \text{PCAlg}_{\mathbb{F}_p, \kappa}^{op}$ to be $\varinjlim_{T \in I_S^\kappa} \mathcal{F}(T)$, but since S is κ -small the identity map is cofinal in \mathcal{C}_S^κ and $\varinjlim_{I_S^\kappa} \mathcal{F}(S) = \mathcal{F}(S)$ as we needed to show. \square

Lemma 1.3.11. *Let κ be a cut-off cardinal, $S \in \text{PCAlg}_{\mathbb{F}_p}^{op}$ and $T \in \text{PCAlg}_{\mathbb{F}_p, \kappa}^{op}$. Given a morphism $g : S \rightarrow T$, there is $T' \in \text{PCAlg}_{\mathbb{F}_p, \kappa}^{op}$ together with morphisms $f : S \rightarrow T'$ and $h : T' \rightarrow T$ such that f is a v -cover and $g = h \circ f$.*

Proof. This lemma is purely of set-theoretic nature and contentless otherwise. Indeed, if S was κ -small we could simply choose $T' = S$ and f to be the identity. Lets treat the general case, let $S = \text{Spec}(B)$ and $T = \text{Spec}(A)$. By replacing A by its image in B we may assume $g^* : A \rightarrow B$ to be injective. We construct a countable sequence of subrings

$$A = A_0 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots \subseteq B$$

with the property that each A_i is κ -small and that the image of the map $\text{Spa}(B, B) \rightarrow \text{Spa}(A_n, A_n)$ coincides with the image of $\text{Spa}(A_{n+1}, A_{n+1}) \rightarrow \text{Spa}(A_n, A_n)$. We do this inductively as follows: Assume A_n to be defined and let $Z_n \subseteq \text{Spa}(A_n, A_n)$ be the image of $\text{Spa}(B, B)$ in $\text{Spa}(A_n, A_n)$. If x is an element of $\text{Spa}(A_n, A_n) \setminus Z_n$ the valuation $|\cdot|_x : A_n \rightarrow \Gamma_x$ can't be extended to a valuation $|\cdot| : B \rightarrow \Gamma$. A compactness argument proves there are finitely many elements $\{a_1, \dots, a_m\}$ such that $|\cdot|_x$ does not extend to $A_n[a_1, \dots, a_m] \subseteq B$. Since $\text{Spa}(A_n, A_n) \setminus Z_n$ is κ -small, there is $\lambda < \kappa$ and a set $\{a_i\}_{i \in \lambda} \subseteq B$ such that $A_n[a_i]_{i \in \lambda}$ does not extend any $x \in \text{Spa}(A_n, A_n) \setminus Z_n$. We let $A_{n+1} = A_n[a_i^{\frac{1}{p^\infty}}]_{i \in \lambda}$, clearly A_{n+1} satisfies the desired properties.

Let $A_\infty = \varinjlim_{i \in \mathbb{N}} A_i$, we claim that the map $\text{Spec}(B) \rightarrow \text{Spec}(A_\infty)$ is a v -cover and that A_∞ is κ -small. Indeed, since each A_i is κ -small and since the cofinality of κ is larger than ω (See [28] 4.1) A_∞ is also κ -small. To prove it is a v -cover, we can use lemma 1.3.5 to prove instead that $\text{Spa}(B, B) \rightarrow \text{Spa}(A_\infty, A_\infty)$ is surjective. One verifies that $\text{Spa}(A_\infty, A_\infty) = \varprojlim_{i \in \mathbb{N}} \text{Spa}(A_i, A_i)$ as topological spaces. Given a compatible sequence $x_i \in \text{Spa}(A_i, A_i)$ we define M_i to be the preimage of x_i in $\text{Spa}(B, B)$. This gives a sequence of sets

$$\text{Spa}(B, B) \supseteq M_0 \supseteq M_1 \dots M_n \supseteq \dots$$

Since the maps $\text{Spa}(B, B) \rightarrow \text{Spa}(A_i, A_i)$ are spectral maps of spectral topological spaces, each of the M_i is closed and compact in the patch topology and their intersection is non-empty. Any element in this intersection will map to the element $x_\infty \in \text{Spa}(A_\infty, A_\infty)$ represented by the compatible sequence x_i . \square

We define $\widetilde{\text{SchPerf}}$ as the big colimit $\bigcup_\kappa \widetilde{\text{SchPerf}}_\kappa$ along all cut-off cardinals and the fully-faithful embeddings $\iota_{\kappa, \lambda}^*$. Objects in $\widetilde{\text{SchPerf}}$ are called small scheme-theoretic v -sheaves.

The general formalism of topoi, specifically ([3] IV 4.9.4), allows us to promote $\diamond_\kappa : \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{op} \rightarrow \widetilde{\mathrm{Perf}}_\kappa$ to a morphism of topoi $f_\kappa : \widetilde{\mathrm{Perf}}_\kappa \rightarrow \widetilde{\mathrm{SchPerf}}_\kappa$ for which $f_\kappa^*|_{\mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{op}} = \diamond_\kappa$. Indeed, proposition 1.3.3 shows that \diamond_κ is left-exact and proposition 1.3.8 gives us continuity of \diamond_κ .

Proposition 1.3.12. 1. *Given two cut-off cardinals $\kappa < \lambda$ we have a commutative diagram of morphism of topoi:*

$$\begin{array}{ccc} \widetilde{\mathrm{Perf}}_\lambda & \xrightarrow{f_\lambda} & \widetilde{\mathrm{SchPerf}}_\lambda \\ \downarrow \iota_{\kappa, \lambda} & & \downarrow \iota_{\kappa, \lambda} \\ \widetilde{\mathrm{Perf}}_\kappa & \xrightarrow{f_\kappa} & \widetilde{\mathrm{SchPerf}}_\kappa \end{array}$$

2. *We also have that the natural morphism $\iota_{\kappa, \lambda}^* \circ f_{\kappa, *} \rightarrow f_{\lambda, *} \circ \iota_{\kappa, \lambda}^*$ is an isomorphism.*

Proof. The commutativity of morphism of topoi follows formally from the similar commutativity of continuous functors:

$$\begin{array}{ccc} \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{op} & \xrightarrow{\diamond_\kappa} & \widetilde{\mathrm{Perf}}_\kappa \\ \downarrow \iota_{\kappa, \lambda}^* & & \downarrow \iota_{\kappa, \lambda}^* \\ \mathrm{PCAlg}_{\mathbb{F}_p, \lambda}^{op} & \xrightarrow{\diamond_\lambda} & \widetilde{\mathrm{Perf}}_\lambda \end{array}$$

For the second claim, given an element $S \in \mathrm{PCAlg}_{\mathbb{F}_p, \lambda}^{op}$ we let I_S^κ be an index set category as in the proof of 1.3.10. If $S = \mathrm{Spec}(A)$ we let $X = \mathrm{Spa}(A((t^{\frac{1}{p^\infty}})), A[[t^{\frac{1}{p^\infty}}]])$ and $Y = X \times_S X$. In a similar way, for $T \in I_S^\kappa$ with $T = \mathrm{Spec}(B)$ we let $X_T = \mathrm{Spa}(B((t^{\frac{1}{p^\infty}})), B[[t^{\frac{1}{p^\infty}}]])$ and $Y_T = X_T \times_T X_T$. The family of perfectoid spaces $(X_T)_{T \in I_S^\kappa}$ ($(Y_T)_{T \in I_S^\kappa}$ respectively) is cofinal in the category \mathcal{C}_X^κ of maps $X \rightarrow X'$ with X' a κ -small perfectoid space (\mathcal{C}_Y^κ respectively). We get the following chain of isomorphisms:

$$\iota_{\kappa, \lambda}^* f_{\kappa, *} \mathcal{F}(S) = \varinjlim_{T \in I_S^\kappa} \mathrm{Hom}_{\widetilde{\mathrm{SchPerf}}_\kappa}(h_T, f_{\kappa, *} \mathcal{F}) \quad (1.1)$$

$$= \varinjlim_{T \in I_S^\kappa} \mathrm{Hom}_{\widetilde{\mathrm{Perf}}_\kappa}(f_\kappa^* h_T, \mathcal{F}) \quad (1.2)$$

$$= \varinjlim_{T \in I_S^\kappa} \mathrm{Hom}_{\widetilde{\mathrm{Perf}}_\kappa}(T^{\diamond_\kappa}, \mathcal{F}) \quad (1.3)$$

$$= \varinjlim_{T \in I_S^\kappa} \mathrm{Eq}_{\widetilde{\mathrm{Perf}}_\kappa}(\mathrm{Hom}(X_T, \mathcal{F}) \rightrightarrows \mathrm{Hom}(Y_T, \mathcal{F})) \quad (1.4)$$

$$= \mathrm{Eq}_{\widetilde{\mathrm{Perf}}_\lambda}(\varinjlim_{T \in I_S^\kappa} \mathrm{Hom}(X_T, \mathcal{F}) \rightrightarrows \varinjlim_{T \in I_S^\kappa} \mathrm{Hom}(Y_T, \mathcal{F})) \quad (1.5)$$

$$= \mathrm{Eq}_{\widetilde{\mathrm{Perf}}_\lambda}(\mathrm{Hom}(X_S, \iota_{\kappa, \lambda}^* \mathcal{F}) \rightrightarrows \mathrm{Hom}(Y_S, \iota_{\kappa, \lambda}^* \mathcal{F})) \quad (1.6)$$

$$= \mathrm{Hom}_{\widetilde{\mathrm{Perf}}_\lambda}(S^{\diamond_\lambda}, \iota_{\kappa, \lambda}^* \mathcal{F}) \quad (1.7)$$

$$= \mathrm{Hom}_{\widetilde{\mathrm{SchPerf}}_\lambda}(h_S, f_{\lambda, *} \iota_{\kappa, \lambda}^* \mathcal{F}) \quad (1.8)$$

$$= f_{\lambda, *} \iota_{\kappa, \lambda}^* \mathcal{F}(S) \quad (1.9)$$

□

Recall that a morphism of topoi consists of a pair of adjoint functors (f^*, f_*) such that f^* commutes with finite limits. By proposition 1.3.12 above we can gather all of the morphisms of topoi $f_\kappa : \widetilde{\mathrm{Perf}}_\kappa \rightarrow \widetilde{\mathrm{SchPerf}}_\kappa$ into a pair of adjoint functors $(f^*, f_*) : \widetilde{\mathrm{Perf}} \rightarrow \widetilde{\mathrm{SchPerf}}$ such that f^* commutes with finite limits. This is not a morphism of topoi because $\widetilde{\mathrm{Perf}}$ and $\widetilde{\mathrm{SchPerf}}$ are not topoi, but they behave as such.

Definition 1.3.13. Let (f^*, f_*) the pair of adjoint functors described above, given $\mathcal{F} \in \widetilde{\text{SchPerf}}$ we will denote $f^*\mathcal{F}$ by \mathcal{F}^\diamond and given $\mathcal{G} \in \widetilde{\text{Perf}}$ we will denote $f_*\mathcal{G}$ by $(\mathcal{G})^{\text{red}}$. We refer to $(-)^{\text{red}}$ as the reduction functor.

Remark 1.3.14. The functor $(-)^{\text{red}}$ will be very important for our purposes. To make this functor explicit take a small v -sheaf $\mathcal{F} \in \widetilde{\text{Perf}}$ and $S \in \text{PCAlg}_{\mathbb{F}_p}^{\text{op}}$. Adjunction tells us that $\mathcal{F}^{\text{red}}(S) = \text{Hom}_{\widetilde{\text{Perf}}}(S^\diamond, \mathcal{F})$. We could have defined the functor in this way without invoking the formalism of topoi, but it will be useful to know that the “reduction” of small v -sheaf is a small scheme-theoretic v -sheaf.

We can endow any small scheme-theoretic v -sheaf with a topological space in a similar fashion to definition 1.1.11. Given $\mathfrak{S} \in \widetilde{\text{SchPerf}}$ we let $|\mathfrak{S}|$ denote the set of equivalence classes of maps $\text{Spec}(k) \rightarrow \mathfrak{S}$, where k is a perfect field over \mathbb{F}_p . Two maps p_1, p_2 are equivalent if we can complete a commutative diagram as below:

$$\begin{array}{ccccc}
 & & \text{Spec}(k_1) & & \\
 & \nearrow q_1 & & \searrow p_1 & \\
 \text{Spec}(k_3) & & & & \mathfrak{S} \\
 & \searrow q_2 & & \nearrow p_2 & \\
 & & \text{Spec}(k_2) & &
 \end{array}
 \quad \begin{array}{c} \xrightarrow{p_3} \end{array}$$

Proposition 1.3.15. Let $\mathfrak{S} \in \widetilde{\text{SchPerf}}$ the following hold:

1. There is a pair of cut-off cardinals $\kappa < \lambda$ and a λ -small family $\{S_i\}_{i \in I}$ of objects in $\text{PCAlg}_{\mathbb{F}_p, \kappa}^{\text{op}}$ together with a surjective map $X = (\coprod_{i \in I} S_i) \rightarrow \mathfrak{S}$.
2. The small scheme-theoretic v -sheaf $R = X \times_{\mathfrak{S}} X$ has a similar cover $Y = (\coprod_{j \in J} T_j) \rightarrow R$, there is a natural map $|X| \rightarrow |\mathfrak{S}|$ which induces a bijection $|\mathfrak{S}| \cong |X|/|Y|$. We endow $|\mathfrak{S}|$ with the quotient topology induced by this bijection.
3. The topology on $|\mathfrak{S}|$ does not depend on the choices of X or Y .
4. Any map of small v -sheaves $\mathfrak{S}_1 \rightarrow \mathfrak{S}_2$ induces a continuous map of topological spaces $|\mathfrak{S}_1| \rightarrow |\mathfrak{S}_2|$.

Proof. By definition $\mathfrak{S} \in \widetilde{\text{SchPerf}}_\kappa$ for some cut-off cardinal κ , the category $\text{PCAlg}_{\mathbb{F}_p, \kappa}^{\text{op}}$ is a small category. By cofinality of cut-off cardinals we may pick λ larger than $\sup_{T \in \text{PCAlg}_{\mathbb{F}_p, \kappa}^{\text{op}}} \mathfrak{S}(T)$. We let $X = (\coprod_{T \in \text{PCAlg}_{\mathbb{F}_p, \kappa}^{\text{op}}} \coprod_{x \in \mathfrak{S}(T)} T)$ with the evident projection map to \mathfrak{S} . We claim this map is surjective. This map is defined in $\widetilde{\text{SchPerf}}_\lambda$ and it is enough to prove surjectivity there. Given $S \in \text{PCAlg}_{\mathbb{F}_p, \lambda}^{\text{op}}$, we have $\mathfrak{S}(S) = \varinjlim_{T \in I_S^\kappa} \mathfrak{S}(T)$. Since this colimit is filtered, for a fixed map $g : S \rightarrow \mathfrak{S}$ we can find $(f : S \rightarrow T) \in I_S^\kappa$ and a map $h : T \rightarrow \mathfrak{S}$ with $g = h \circ f$. In particular, g factors through a map to X since X contains a copy of T mapping to \mathfrak{S} via h .

We move on to the second claim. Given $x \in |X|$ we take the residue field inclusion $\iota_x : \text{Spec}(k_x) \rightarrow X$. The composition $\text{Spec}(k_x) \rightarrow X \rightarrow \mathfrak{S}$ defines an element of $|\mathfrak{S}|$. Suppose now that $x_1, x_2 \in |X|$, we must show that $(x_1, x_2) \in |X| \times |X|$ is in the image of $|Y|$ if and only if x_1 and x_2 define the same element in $|\mathfrak{S}|$. If the maps ι_{x_1} and ι_{x_2} are equivalent we get a map $\iota_{x_3} : \text{Spec}(k_3) \rightarrow X \times_{\mathfrak{S}} X$. By replacing k_3 by a larger field if necessary, we may assume that ι_{x_3} lifts to Y and defines an element $y \in |Y|$. We see that y maps to (x_1, x_2) in $|X| \times |X|$. On the other hand, if there is $y \in Y$ mapping to (x_1, x_2) and $\iota_y : \text{Spec}(k_y) \rightarrow Y$ is the residue field map, the compositions $\text{Spec}(k_y) \rightarrow Y \xrightarrow{\pi_i} X$ factor through $\iota_{x_i} : \text{Spec}(k_{x_i}) \rightarrow X$. This proves that x_1 and x_2 map to the same point $|\mathfrak{S}|$.

For the third claim suppose we are given two covers $X_i \rightarrow \mathfrak{S}$ with $i \in \{1, 2\}$, we must show that the two quotient topologies coming from the surjections $|X_i| \rightarrow |\mathfrak{S}|$ agree. The small scheme theoretic v -sheaf $R = X_1 \times_{\mathfrak{S}} X_2$ admits a v -cover $X_3 \rightarrow R$ by the first claim. By replacing X_2 by X_3 we may assume that we have a commutative diagram of surjective maps:

$$\begin{array}{ccc} X_2 & & \\ \downarrow & \searrow & \\ & & \mathfrak{S} \\ \uparrow & \swarrow & \\ X_1 & & \end{array}$$

Since $X_2 \rightarrow X_1$ is a v -cover we get a quotient map of topological spaces $|X_2| \rightarrow |X_1|$. If we give $|\mathfrak{S}|$ the quotient topology coming from the surjection $|X_1| \rightarrow |\mathfrak{S}|$ the composition map $|X_2| \rightarrow |\mathfrak{S}|$ is also a quotient map. This implies that the two topologies agree.

For the last claim, we may find covers \mathfrak{S}_1 and \mathfrak{S}_2 by X_1 and X_2 respectively forming the following commutative diagram:

$$\begin{array}{ccc} |X_2| & \xrightarrow{q} & |\mathfrak{S}_2| \\ \downarrow & & \downarrow \\ |X_1| & \xrightarrow{q} & |\mathfrak{S}_1| \end{array}$$

Both horizontal maps are quotient maps and the leftmost vertical map is continuous since it is induced by a morphism of unions of affine schemes, this prove the required continuity. \square

1.3.2 Reduction functor and formal adicness

Definition 1.3.16. We say that a small scheme-theoretic v -sheaf \mathcal{F} is reduced if the adjunction morphism $\mathcal{F} \rightarrow (\mathcal{F}^\diamond)^{\text{red}}$ is an isomorphism in $\widetilde{\text{SchPerf}}$.

We have the following formal consequences of our definition.

Proposition 1.3.17. 1. If S is a perfect scheme in characteristic p then the Yoneda functor h_S is reduced. (See [30] 18.3.1)

2. The functor $\diamond : \widetilde{\text{SchPerf}} \rightarrow \widetilde{\text{Perf}}$ is fully-faithful when restricted to small reduced v -sheaves.

Proof. The first claim follows from theorem 1.2.32. The second claim follows from adjunction. Indeed, if \mathcal{F} is reduced and $\mathcal{G} \in \widetilde{\text{SchPerf}}$ then:

$$\text{Hom}_{\widetilde{\text{Perf}}}(\mathcal{G}^\diamond, \mathcal{F}^\diamond) = \text{Hom}_{\widetilde{\text{SchPerf}}}(\mathcal{G}, (\mathcal{F}^\diamond)^{\text{red}}) = \text{Hom}_{\widetilde{\text{SchPerf}}}(\mathcal{G}, \mathcal{F})$$

\square

Question 1.3.18. Are perfect algebraic spaces reduced? If this was the case much of the formalism of specialization goes through in this generality.

Intuitively speaking, the reduction functor kills all topological nilpotents and removes analytic points from our v -sheaf. Below, we try to justify why one can think of this reduction functor as an analogue of taking the underlying reduced subscheme of a formal scheme.

Lemma 1.3.19. The scheme-theoretic v -sheaf $(\mathbb{Z}_p^\diamond)^{\text{red}}$ is represented by \mathbb{F}_p .

Proof. This is a direct consequence of lemma 1.2.30. \square

For a Huber pair (A, A^+) over \mathbb{Z}_p , we let A_{red} denote the perfection of $A/(A \cdot A^{\circ\circ})$ where $A \cdot A^{\circ\circ}$ is the ideal generated by the set of topologically nilpotent elements. The following statement generalizes lemma 1.3.19

Proposition 1.3.20. *Let X be a pre-adic space over \mathbb{Z}_p and let X^{na} be the reduced adic space associated to the non-analytic locus of proposition 1.1.28. The following hold:*

1. *The map $(X^{na, \diamond})^{\text{red}} \rightarrow (X^\diamond)^{\text{red}}$ is an isomorphism.*
2. *If $X = \text{Spa}(A, A^+)$ for (A, A^+) a Huber pair over \mathbb{Z}_p , then $\text{Spd}(A, A^+)^{\text{red}}$ is represented by $\text{Spec}(A_{\text{red}})$.*

Proof. By theorem 1.2.32 if $S = \text{Spec}(R) \in \text{PCAlg}_{\mathbb{F}_p}^{\text{op}}$ then morphisms $S^\diamond \rightarrow X$ are given by maps of pre-adic spaces $f : \text{Spa}(R, R) \rightarrow X$, and they must factor through the non-analytic locus which proves the first claim.

The non-analytic locus of $\text{Spa}(A, A^+)$ is represented by the Huber pair $(A/A^{\circ\circ} \cdot A, A^{\circ\circ} \cdot A^+)$. Since R is perfect the map $f^* : A/A \cdot A^{\circ\circ} \rightarrow R$ factors uniquely through its perfection. \square

We now justify the intuition behind thinking of diamonds as purely analytic objects.

Proposition 1.3.21. *For a quasi-separated diamond Y the associated reduced functor Y^{red} is the empty-sheaf.*

Proof. We need to prove that for a perfect scheme S there are no morphisms $S^\diamond \rightarrow Y$. It is enough to prove this for $S = \text{Spec}(k)$ the spectrum of an algebraically closed field. Suppose there is such a map $f : S^\diamond \rightarrow Y$, and let $y \in |Y|$ be the unique point in the image of $|f|$. We consider Y_y the sub- v -sheaf of points of $\text{Spa}(R, R^+) \rightarrow Y$ for which $|\text{Spa}(R, R^+)| \rightarrow |Y|$ factors through y . The map f factors through Y_y and by ([28] 11.10) it is a quasi-separated diamond with $|Y_y|$ consisting of one point. We can use ([28] 21.9) to write $Y_y = \text{Spa}(C, O_C)/\underline{G}$ with C a non-Archimedean algebraically closed field in characteristic p and \underline{G} a profinite group acting continuously and faithfully on C .

Consider the v -cover $S' = \text{Spa}(K_1, O_{K_1}) \rightarrow \text{Spec}(k)^\diamond$ where K_1 is an algebraic closure of $k((t^{\frac{1}{p^\infty}}))$. Similarly, let $T = \text{Spa}(K_2, O_{K_2})$ where K_2 is an algebraically closed non-Archimedean field containing k discretely and whose value group $\Gamma_{K_2} \subseteq \mathbb{R}^{\geq 0}$ has at least two elements that are linearly independent when we treat $\Gamma_{K_2} \setminus \{0\}$ as vector space over \mathbb{Q} . By our hypothesis on K_2 , we can find two continuous embeddings $\iota_i^* : K_1 \rightarrow K_2$ such that $|\iota_1^*(K_1)|_{\Gamma_{K_2}} \cap |\iota_2^*(K_1)|_{\Gamma_{K_2}} = 1$ and in particular, such that $\iota_1^*(K_1) \cap \iota_2^*(K_1) = k$.

The composition of $f : S^\diamond \rightarrow Y_y$ with the natural projection $\text{Spa}(K_1, K_1^+) \rightarrow S^\diamond$ gives a map $[g] : \text{Spa}(K_1, K_1^+) \rightarrow Y_y$ such that $[g] \circ \iota_1 = [g] \circ \iota_2$. Since both $\text{Spa}(K_1, K_1^+)$ and $\text{Spa}(K_2, K_2^+)$ are algebraically closed fields the sets of maps to Y_y are given by G -orbits of maps to $\text{Spa}(C, O_C)$, that is $\text{Hom}(\text{Spa}(K_i, K_i^+), Y_y) = \text{Hom}(\text{Spa}(K_i, K_i^+), \text{Spa}(C, O_C))/G$. Let $g^* : (C, O_C) \rightarrow (K_1, O_{K_1})$ represent $[g]$ in $\text{Hom}(\text{Spa}(K_1, K_1^+), Y_y)$, we get maps $\iota_i^* \circ g^* : (C, O_C) \rightarrow (K_2, O_{K_2})$ and since $[g] \circ \iota_1 = [g] \circ \iota_2$ we have $\iota_1^* \circ g^*(C) = \iota_2^* \circ g^*(C) \subseteq k$. This contradicts that k has the discrete topology and that C is a non-Archimedean field, the contradiction shows that the map $f : S^\diamond \rightarrow Y_y$ does not exist. \square

Recall that a morphism of adic spaces $X \rightarrow Y$ is said to be adic if the image of an analytic point is again an analytic point. For v -sheaves we can define a related notion.

Definition 1.3.22. We say that a morphism of v -sheaves $\mathcal{F} \rightarrow \mathcal{G}$ is formally adic if the commutative diagram that one obtains from adjunction:

$$\begin{array}{ccc} (\mathcal{F}^{\text{red}})^{\diamond} & \longrightarrow & (\mathcal{G}^{\text{red}})^{\diamond} \\ \downarrow & & \downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{G} \end{array}$$

is a Cartesian diagram.

We warn the reader that although the notion of a morphism of adic spaces to be adic is related to the morphism of v -sheaves being formally adic neither of these notions implies the other.

Example 1.3.23. Take a perfect field k in characteristic p together with a rank 1 valuation subring $O_k \subseteq k$ with the discrete topology. The morphism of adic spaces $\text{Spa}(k, O_k) \rightarrow \text{Spa}(\mathbb{F}_p, \mathbb{F}_p)$ is adic. Nevertheless, the induced morphism $\text{Spd}(k, O_k) \rightarrow \text{Spd}(\mathbb{F}_p, \mathbb{F}_p)$ is not formally adic since $\text{Spd}(k, O_k)^{\text{red}}$ is represented by $\text{Spec}(k)$. Observe that $\text{Spd}(k, O_k)$ has a meromorphic point that is not bounded.

Example 1.3.24. Take a non-Archimedean perfect field K in characteristic p and consider the morphism $\text{id} : \text{Spa}(K_1, O_{K_1}) \rightarrow \text{Spa}(K_2, O_{K_2})$ where $K_2 = K$ given the discrete topology and $K_1 = K$ given the topology induced by the norm. This morphism is not adic, nevertheless the reduction diagram looks like this:

$$\begin{array}{ccc} \emptyset & \longrightarrow & \text{Spec}(K_2)^{\diamond} \\ \downarrow & & \downarrow \\ \text{Spd}(K_1, O_{K_1}) & \longrightarrow & \text{Spd}(K_2, O_{K_2}) \end{array}$$

Which is Cartesian.

Although the notion of formal adicness does not recover the notion of adicness in general, it will in some important situations:

Proposition 1.3.25. Let (A, A) and (B, B) be formal Huber pairs over \mathbb{Z}_p with ideals of definition I_A and I_B respectively. A morphism of adic spaces $\text{Spa}(A, A) \rightarrow \text{Spa}(B, B)$ is adic if and only if the corresponding morphism of v -sheaves $\text{Spd}(A, A) \rightarrow \text{Spd}(B, B)$ is formally adic.

Proof. The reduction diagram looks as follows:

$$\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ \text{Spec}(A_{\text{red}})^{\diamond} & & \xrightarrow{\quad} & & \text{Spec}(B_{\text{red}})^{\diamond} \\ & \searrow & & \nearrow & \\ & & (\text{Spec}(A/I_B)^{\text{perf}})^{\diamond} & \longrightarrow & \text{Spec}(B_{\text{red}})^{\diamond} \\ & & \downarrow & & \downarrow \\ & & \text{Spd}(A, A) & \longrightarrow & \text{Spd}(B, B) \end{array}$$

Continuity of the morphism $B \rightarrow A$ ensures that $I_B^n \subseteq I_A$ for some n . In this context, the morphism is adic if and only if $I_B \cdot A$ is an ideal of definition of A which happens if and only if $I_A^m \subseteq A \cdot I_B$ for some m . If the morphism is adic, then A/I_A and $(A/A \cdot I_B)$ become isomorphic after taking perfection which gives formal adicness. Conversely, if the morphism is formally adic, by hypothesis the rings $(A/I_B)^{\text{perf}}$, and A_{red} are isomorphic with the isomorphism being induced by the natural surjective ring map with source $(A/p)^{\text{perf}}$. This implies that the ideals I_A and I_B define the same Zariski closed subset in $\text{Spec}(A)$. In particular, the elements of I_A are nilpotent in A/I_B , and since I_A is finitely generated $I_A^m \subseteq I_B$ for some m . \square

Proposition 1.3.26. 1. If $\mathcal{F} \rightarrow \mathcal{H}$ and $\mathcal{H} \rightarrow \mathcal{G}$ are formally adic, the composition $\mathcal{F} \rightarrow \mathcal{G}$ is formally adic.

2. If $\mathcal{F} \rightarrow \mathcal{H}$ is formally adic, the basechange $\mathcal{G} \times_{\mathcal{H}} \mathcal{F} \rightarrow \mathcal{G}$ is formally adic.

Proof. The first claim is clear. For the second one we get a Cartesian diagram:

$$\begin{array}{ccccc} (\mathcal{G}^{\text{red}})^{\diamond} \times_{(\mathcal{H}^{\text{red}})^{\diamond}} (\mathcal{F}^{\text{red}})^{\diamond} & \longrightarrow & (\mathcal{F}^{\text{red}})^{\diamond} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow & & \downarrow \\ (\mathcal{G}^{\text{red}})^{\diamond} & \longrightarrow & (\mathcal{H}^{\text{red}})^{\diamond} & \longrightarrow & \mathcal{H} \end{array}$$

The functors $(-)^{\text{red}}$ and $(-)^{\diamond}$ commute with finite limits. This gives $(\mathcal{G}^{\text{red}})^{\diamond} \times_{(\mathcal{H}^{\text{red}})^{\diamond}} (\mathcal{F}^{\text{red}})^{\diamond} = ((\mathcal{G} \times_{\mathcal{H}} \mathcal{F})^{\text{red}})^{\diamond}$, and proves that

$$\begin{array}{ccccc} ((\mathcal{G} \times_{\mathcal{H}} \mathcal{F})^{\text{red}})^{\diamond} & \longrightarrow & \mathcal{G} \times_{\mathcal{H}} \mathcal{F} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow & & \downarrow \\ (\mathcal{G}^{\text{red}})^{\diamond} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} \end{array}$$

is also Cartesian. \square

Definition 1.3.27. We say that a v -sheaf \mathcal{F} over \mathbb{Z}_p^{\diamond} is formally p -adic if the morphism $\mathcal{F} \rightarrow \mathbb{Z}_p^{\diamond}$ is formally adic.

Over \mathbb{Z}_p the situation of example 1.3.24 does not happen.

Proposition 1.3.28. Suppose we have a Huber pair (A, A^+) and a map $f : \text{Spa}(A, A^+) \rightarrow \text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, if f^{\diamond} is formally adic then f is adic (as a morphism of adic spaces).

Proof. Let $U \subseteq \text{Spa}(A, A^+)$ the open subset of analytic points. It is easy to verify that this open embedding is formally adic because $\text{Spec}(A_{\text{red}})^{\diamond} \rightarrow \text{Spd}(A, A^+)$ factors through the complement of U^{\diamond} and because by proposition 1.3.21 $(U^{\diamond})^{\text{red}} = \emptyset$ holds. Since formal adicness is preserved by composition $U^{\diamond} \rightarrow \mathbb{Z}_p^{\diamond}$ is formally adic. By formal adicness the map $U^{\diamond} \rightarrow \mathbb{Z}_p^{\diamond}$ must factor through \mathbb{Q}_p^{\diamond} . This proves $f(U) \subseteq \text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which proves that f is adic. \square

Recall that a v -sheaves \mathcal{F} is said to be separated if the diagonal $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is a closed immersion (See [28] 10.7). We need the following related notion:

Definition 1.3.29. 1. We say that a map of v -sheaves $\mathcal{F} \rightarrow \mathcal{G}$ is formally closed if it is a formally adic closed immersion.

2. We say that a v -sheaf is formally separated if the diagonal map $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is formally closed.

Lemma 1.3.30. The v -sheaf \mathbb{Z}_p^{\diamond} is formally separated.

Proof. We need to prove that the diagonal $\mathbb{Z}_p^{\diamond} \rightarrow \mathbb{Z}_p^{\diamond} \times \mathbb{Z}_p^{\diamond}$ is a closed immersion of perfectoid spaces after any basechange by maps $\text{Spa}(R, R^+) \rightarrow \mathbb{Z}_p^{\diamond} \times \mathbb{Z}_p^{\diamond}$, with $\text{Spa}(R, R^+) \in \text{Perf}$. This amounts to proving that the locus on which two untilts agree is closed inside $|\text{Spa}(R, R^+)|$ and representable by a perfectoid space. Now, each untilt is individually cut out of $\text{Spa}(W(R^+), W(R^+)) \setminus \{V([\varpi])\}$ as a closed Cartier divisor (See [30] 11.3.1). We can take the intersection which will define a Zariski closed subset in each of the untilts, but Zariski closed subsets of a perfectoid space are representable by some other perfectoid space. The tilt of such a closed immersion represents this basechange.

To prove the diagonal is formally adic we compute directly $(\mathbb{Z}_p^{\diamond} \times_{\mathbb{F}_p^{\diamond}} \mathbb{Z}_p^{\diamond})^{\text{red}} = \mathbb{F}_p$ since $(-)^{\text{red}}$ commutes with limits. The basechange $\mathbb{F}_p^{\diamond} \times_{\mathbb{Z}_p^{\diamond} \times_{\mathbb{F}_p^{\diamond}} \mathbb{Z}_p^{\diamond}} \mathbb{Z}_p^{\diamond}$ agrees with \mathbb{F}_p^{\diamond} . \square

Proposition 1.3.31. *If a v -sheaf \mathcal{F} is formally p -adic, then the diagonal $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is formally adic.*

Proof. We have a formally adic map $\mathcal{F} \rightarrow \mathbb{Z}_p^\diamond$, and since formal adicness is preserved by basechange and composition we get a formally adic map $\mathcal{F} \times_{\mathbb{Z}_p^\diamond} \mathcal{F} \rightarrow \mathbb{Z}_p^\diamond$. By the two out of three property of Cartesian diagrams, the diagonal map $\mathcal{F} \rightarrow \mathcal{F} \times_{\mathbb{Z}_p^\diamond} \mathcal{F}$ is also formally adic. Now, $\mathcal{F} \times_{\mathbb{Z}_p^\diamond} \mathcal{F}$ is the basechange of the diagonal map $\mathbb{Z}_p^\diamond \rightarrow \mathbb{Z}_p^\diamond \times \mathbb{Z}_p^\diamond$ by the projection map $\mathcal{F} \times \mathcal{F} \rightarrow \mathbb{Z}_p^\diamond \times \mathbb{Z}_p^\diamond$. This gives us that $\mathcal{F} \times_{\mathbb{Z}_p^\diamond} \mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is also formally adic. Since formal adicness is preserved by composition, $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is also formally adic as we needed to show. \square

Lemma 1.3.32. *The diagonal $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is formally adic if and only if the adjunction morphism $(\mathcal{F}^{\text{red}})^\diamond \rightarrow \mathcal{F}$ is injective. In this case, if (A, A^+) is a perfectoid Huber pair, and $m \in \mathcal{F}(A, A^+)$ then $m \in (\mathcal{F}^{\text{red}})^\diamond(A, A^+)$ if and only if $\text{Spa}(A, A^+)$ admits a v -cover $\text{Spa}(R, R^+) \rightarrow \text{Spa}(A, A^+)$ and a morphism $\text{Spec}(R^+)^\diamond \rightarrow \mathcal{F}$ making the following diagram commutative:*

$$\begin{array}{ccc} \text{Spa}(R, R^+) & \longrightarrow & \text{Spec}(R^+)^\diamond \\ \downarrow & & \downarrow \\ \text{Spa}(A, A^+) & \xrightarrow{m} & \mathcal{F} \end{array}$$

Proof. In general, a map of sheaves $\mathcal{G} \rightarrow \mathcal{F}$ is injective if and only if $(\mathcal{G} \times \mathcal{G}) \times_{\mathcal{F} \times \mathcal{F}} \mathcal{F} = \mathcal{G}$. We can apply this reasoning to the map $(\mathcal{F}^{\text{red}})^\diamond \rightarrow \mathcal{F}$.

For the second claim let \mathcal{C}_R be the category of maps $\text{Spa}(R, R^+) \rightarrow S^\diamond$ with $S \in \text{PCAlg}_{\mathbb{F}_p}^{\text{op}}$, this category is cofiltered. Now, $(\mathcal{F}^{\text{red}})^\diamond$ is the sheafification of the functor that assigns to (R, R^+) :

$$\varinjlim_{S^\diamond \in \mathcal{C}_R} \text{Hom}(S^\diamond, \mathcal{F}).$$

But the evident map $\text{Spa}(R, R^+) \rightarrow \text{Spec}(R^+)^\diamond$ is cofinal in \mathcal{C}_R . That is, $(\mathcal{F}^{\text{red}})^\diamond$ is the sheafification of the presheaf that assigns $(R, R^+) \mapsto \text{Hom}(\text{Spec}(R^+)^\diamond, \mathcal{F})$. The description given in the statement above is what one gets from taking sheafification and assuming injectivity of $(\mathcal{F}^{\text{red}})^\diamond \rightarrow \mathcal{F}$. \square

The following lemma will be key for our theory of specialization, it roughly says that formally adic closed immersions behave as expected:

Lemma 1.3.33. *Let (A, A^+) be a perfectoid Huber pair and let $\mathcal{F} \rightarrow \text{Spd}(A^+, A^+)$ be formally adic closed immersion. Then $(\mathcal{F}^{\text{red}})^\diamond = \text{Spec}(A^+/J)^\diamond$ for some open ideal $J \subseteq A^+$.*

Proof. Since $\mathcal{F} \rightarrow \text{Spd}(A^+, A^+)$ is a closed immersion, $|\mathcal{F}| \subseteq \text{Spo}(A^+, A^+)$ is a closed subset and we have a Cartesian diagram,

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \text{Spd}(A^+, A^+) \\ \downarrow & & \downarrow \\ |\mathcal{F}| & \longrightarrow & \text{Spo}(A^+, A^+) \end{array}$$

By proposition 1.3.20, $(\text{Spd}(A^+, A^+)^{\text{red}})^\diamond = \text{Spec}(A_{\text{red}}^+)^\diamond$ which is also a closed subsheaf of $\text{Spd}(A^+, A^+)$. By formal adicness $(\mathcal{F}^{\text{red}})^\diamond$ is a closed subsheaf of $\text{Spd}(A^+, A^+)$ given again by the topological condition $|\mathcal{F}| \cap |\text{Spec}(A_{\text{red}}^+)|$. By lemma 1.3.32 a map $\text{Spa}(R, R^+) \rightarrow \mathcal{F}$ factors through $(\mathcal{F}^{\text{red}})^\diamond$ if after possibly replacing R by a v -cover it factors through $\text{Spec}(R^+)^\diamond \rightarrow \mathcal{F} \cap \text{Spec}(A_{\text{red}}^+)^\diamond$. This proves that $|(\mathcal{F}^{\text{red}})^\diamond|$ is a schematic closed subset of $\text{Spo}(A^+, A^+)$ as in definition 1.2.35. By proposition 1.2.36 it is a Zariski closed subset corresponding to an open ideal $J \subseteq A^+$, this proves the claim. \square

Example 1.3.34. Let Z denote the complement $\mathrm{Spd}(\mathbb{F}_p[t], \mathbb{F}_p[t]) \setminus N_{t < 1}$ of the analytic localization $N_{t < 1}$. Then $Z \rightarrow \mathrm{Spd}(\mathbb{F}_p[t], \mathbb{F}_p[t])$ is a closed immersion that is not formally closed. One may glue two copies of $\mathrm{Spd}(\mathbb{F}_p[t], \mathbb{F}_p[t])$ along Z to get a v -sheaf Y that is separated but not formally separated. Y^{red} is represented by the affine line with two origins but the adjunction map $(Y^{\mathrm{red}})^{\diamond} \rightarrow Y$ is not injective.

We will often use implicitly the following easy result.

Lemma 1.3.35. Let \mathcal{F} and \mathcal{G} be two small v -sheaves, and $f : \mathcal{F} \rightarrow \mathcal{G}$ a map between them. Suppose that $\mathcal{F} \times_{\mathcal{G}} (\mathcal{G}^{\mathrm{red}})^{\diamond}$ is representable by a scheme-theoretic v -sheaf, then f is formally adic.

Proof. Let $T \in \widetilde{\mathrm{SchPerf}}$ be such that $T^{\diamond} = \mathcal{F} \times_{\mathcal{G}} (\mathcal{G}^{\mathrm{red}})^{\diamond}$. Recall that for any pair of adjoint functors (L, R) the compositions $R \rightarrow R \circ L \circ R \rightarrow R$ and $L \rightarrow L \circ R \circ L \rightarrow L$ are the identity. We compute directly:

$$\begin{aligned} (T^{\diamond})^{\mathrm{red}} &= (\mathcal{F} \times_{\mathcal{G}} (\mathcal{G}^{\mathrm{red}})^{\diamond})^{\mathrm{red}} & (\mathcal{F}^{\mathrm{red}})^{\diamond} &= ((T^{\diamond})^{\mathrm{red}})^{\diamond} \\ &= \mathcal{F}^{\mathrm{red}} \times_{\mathcal{G}^{\mathrm{red}}} ((\mathcal{G}^{\mathrm{red}})^{\diamond})^{\mathrm{red}} & &= T^{\diamond} \\ &= \mathcal{F}^{\mathrm{red}} \times_{\mathcal{G}^{\mathrm{red}}} \mathcal{G}^{\mathrm{red}} \\ &= \mathcal{F}^{\mathrm{red}} \end{aligned}$$

□

1.4 Specialization

In this section we review the specialization map in the context of formal schemes and generalize it to the context of v -sheaves. We identify a class of v -sheaves, that we call kimberlites, whose specialization maps behave like those of formal schemes. We prove some abstract statement on the behavior of the specialization map in this context, which we will later use when we discuss the examples of interest.

1.4.1 Specialization for Tate Huber pairs

Definition 1.4.1. Given a Tate Huber pair (A, A^+) over \mathbb{Z}_p and a pseudo-uniformizer $\varpi \in A$, we define the specialization map $\mathrm{sp}_A : |\mathrm{Spa}(A, A^+)| \rightarrow |\mathrm{Spec}(A_{\mathrm{red}}^+)|$ by sending a valuation $|\cdot|_x \in |\mathrm{Spa}(A, A^+)|$ to the ideal $\mathfrak{p} \subseteq A^+$ given by $\mathfrak{p} = \{a \in A^+ \mid |a|_x < 1\}$

These maps of sets are functorial in the category of Tate Huber pairs.

Proposition 1.4.2. (See [4] 8.1.2) The specialization map $\mathrm{sp}_A : |\mathrm{Spa}(A, A^+)| \rightarrow |\mathrm{Spec}(A_{\mathrm{red}}^+)|$ is a continuous, surjective, spectral and closed map of spectral topological spaces.

Strictly totally disconnected spaces form a basis for the pro-étale topology on Perf . In particular, any small v -sheaf admits a surjective map from a union of totally disconnected spaces. Moreover, as the following proposition shows, the specialization map for these spaces is usefully nice.

Proposition 1.4.3. For a strictly totally disconnected space $\mathrm{Spa}(R, R^+)$, the specialization map sp_R is a homeomorphism.

Proof. By proposition 1.4.2 the map is surjective and a quotient map so it is enough to prove injectivity. Suppose $x, y \in |\mathrm{Spa}(R, R^+)|$ map to the same point in $|\mathrm{Spec}(R_{\mathrm{red}}^+)|$. We claim that x and y are in the same connected component of $|\mathrm{Spa}(R, R^+)|$. Indeed, let π_x and π_y be the connected components of x and y respectively. The closed-open subsets $U \subseteq \mathrm{Spa}(R, R^+)$ are Zariski closed subsets defined by an idempotent $1_U \in R^+$. The ones containing x are precisely those for which $|1_U|_x = 1$ or equivalently

for which $1_U \notin \mathrm{sp}_R(x) \subseteq R^+$. By assumption $\mathrm{sp}_R(x) = \mathrm{sp}_R(y)$ so x and y are contained in the same closed-opens, this gives $\pi_x = \pi_y$.

By proposition 1.1.8, π_x is representable by $\mathrm{Spa}(C, C^+)$ for some perfectoid field C and open valuation subring C^+ . By functoriality of the specialization map it is enough to prove that the maps sp_C and $|\mathrm{Spec}(C^+/\varpi)| \rightarrow |\mathrm{Spec}(R^+/\varpi)|$ are injective. The former is injective by lemma 1.4.4 below. To prove injectivity of the later map we argue as follows: $\pi_x = \bigcap U$ where U ranges over the closed-open subsets of $|\mathrm{Spa}(R, R^+)|$ containing x . Each closed-open $U \subseteq \mathrm{Spa}(R, R^+)$ is of the form $U = \mathrm{Spa}(R_U, R_U^+)$ and if U^c denotes the complement of U then $R^+ = R_U^+ \times R_{U^c}^+$ as topological rings. In particular, the map $R^+ \rightarrow R_U^+$ is surjective. We have that C^+ is the ϖ -adic completion of $\varinjlim_{x \in U} R_U^+$ which implies that the image of $R^+ \rightarrow C^+$ is dense. Consequently, $\mathrm{Spec}(C^+/\varpi) \rightarrow \mathrm{Spec}(R^+/\varpi)$ is a closed immersion and injective. \square

Lemma 1.4.4. 1. *Given a non-Archimedean field K there is an order preserving bijection between open and bounded valuation subrings K^+ of K , and valuation subrings of $O_K/K^{\circ\circ}$, given by $K^+ \mapsto K^+/K^{\circ\circ}$*

2. *Given K as above and an open and bounded valuation subring K^+ the specialization map sp_K is a homeomorphism.*

Proof. This is well known and the proof is left to the reader. \square

Remark 1.4.5. *Although the construction of V in the proof above does not depend of the choice of $\varpi = (\varpi_i)_{i \in I}$, the ring V' very much depends of this choice. This is in agreement with remark 1.1.6.*

1.4.2 Specializing v -sheaves

We now discuss the specialization map for v -sheaves. The idea is to descend the specialization map from the case of formal Huber pairs.

Definition 1.4.6. *We say that a small v -sheaf \mathcal{F} is v -locally formal if there is a set I , a family $(B_i, B_i)_{i \in I}$ of formal Huber pairs over \mathbb{Z}_p and a surjective map of v -sheaves*

$$\coprod_{i \in I} \mathrm{Spd}(B_i, B_i) \rightarrow \mathcal{F}.$$

Definition 1.4.7. *Let \mathcal{F} be a small v -sheaf, $\mathrm{Spa}(A, A^+)$ an affinoid perfectoid space in characteristic p and $f : \mathrm{Spa}(A, A^+) \rightarrow \mathcal{F}$ a map of v -sheaves.*

1. *We say that \mathcal{F} formalizes f (or that f is formalizable) if there exists a commutative diagram as follows:*

$$\begin{array}{ccc} \mathrm{Spa}(A, A^+) & \xrightarrow{\quad} & \mathcal{F} \\ \downarrow \iota & \nearrow & \\ \mathrm{Spd}(A^+, A^+) & & \end{array}$$

2. *We say that \mathcal{F} v -formalizes f if there is a v -cover $g : \mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spa}(A, A^+)$ of affinoid perfectoid spaces for which \mathcal{F} formalizes $f \circ g$.*

3. *We say that \mathcal{F} is formalizing if it formalizes any f as above.*

4. *We say that \mathcal{F} is v -formalizing if it v -formalizes any f as above.*

The previous technical definition will be used extensively, because it gives an abstract criterion to verify that a v -sheaf is v -locally formal.

Lemma 1.4.8. *The following statements hold:*

1. *The v -sheaf \mathbb{Z}_p^\diamond is formalizing.*
2. *$\mathrm{Spd}(B, B)$ is formalizing for any formal Huber pair over \mathbb{Z}_p .*
3. *A small v -sheaf \mathcal{F} is v -formalizing if and only if it is v -locally formal.*

Proof. Given an affinoid perfectoid $\mathrm{Spa}(R, R^+)$ in characteristic p and an untilt $\iota : (R^\sharp)^\flat \rightarrow R$ we need to produce a natural transformation $\mathrm{Spd}(R^+, R^+) \rightarrow \mathbb{Z}_p^\diamond$ for which the composition with the canonical map $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(R^+, R^+)$ gets mapped to R^\sharp . Let $\xi = p + [\varpi]\alpha$ be a generator of the kernel of $W(R^+) \rightarrow (R^\sharp)^+$, where $\varpi \in R^+$ denotes a pseudo-uniformizer and $\alpha \in W(R^+)$. Let $\mathrm{Spa}(A, A^+)$ be some other affinoid perfectoid space in characteristic p . Recall that, since R^+ is in characteristic p

$$\mathrm{Spd}(R^+, R^+)(A, A^+) = \{f : \mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spa}(R^+, R^+)\}$$

Consider the following construction, take the map of topological rings $f^* : R^+ \rightarrow A^+$ defined by f , apply the Witt vector functor to f^* to get $W(f^*) : W(R^+) \rightarrow W(A^+)$ and consider the element $W(f^*)(\xi) \in W(A^+)$. We claim that $W(f^*)(\xi)$ is primitive of degree 1 (See [30] 6.2.8) and defines an untilt of $\mathrm{Spa}(A, A^+)$ over $\mathrm{Spa}(R^\sharp, R^{\sharp+})$. Indeed $W(f^*)(\xi) = p + [f^*(\varpi)]f^*(\alpha)$ and it is enough to prove that there is a pseudo-uniformizer ϖ_A that divides $f^*(\varpi)$. This follows from the fact that $f^*(\varpi)$ is topologically nilpotent.

For the second claim, if we fix an untilt and a morphism $\mathrm{Spa}(R^\sharp, R^{\sharp+}) \rightarrow \mathrm{Spa}(B, B)$ we can promote this to a morphism $\mathrm{Spa}(R^{\sharp+}, R^{\sharp+}) \rightarrow \mathrm{Spa}(B, B)$ and induce a formalization $\mathrm{Spd}(R^{\sharp+}, R^{\sharp+}) \rightarrow \mathrm{Spd}(B, B)$.

For the third claim, assume that \mathcal{F} is v -formalizing. Since it is small there is a set I and a surjective map by a union of affinoid perfectoid spaces $\coprod_{i \in I} \mathrm{Spa}(R_i, R_i^+) \rightarrow \mathcal{F}$. After refining this cover we may assume that each of the maps $\mathrm{Spa}(R_i, R_i^+) \rightarrow \mathcal{F}$ formalizes to a map $\mathrm{Spd}(R_i^+, R_i^+) \rightarrow \mathcal{F}$, then $\coprod_{i \in I} \mathrm{Spd}(R_i^+, R_i^+) \rightarrow \mathcal{F}$ is also surjective proving that \mathcal{F} is v -locally formal. If \mathcal{F} is v -locally formal a map $\mathrm{Spa}(R, R^+) \rightarrow \mathcal{F}$ will v -locally factor through a map $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(B_i, B_i)$. By the second claim this map formalizes $\mathrm{Spd}(R^+, R^+) \rightarrow \mathrm{Spd}(B_i, B_i)$ and the composition to \mathcal{F} is a formalization of the original map. □

Since the notions of being v -locally formal and being v -formalizing are equivalent we will use them interchangeably, without mentioning it.

Proposition 1.4.9. *The following properties are easy to verify.*

1. *If $f : \mathcal{F} \rightarrow \mathcal{G}$ is a surjective map of small v -sheaves and \mathcal{F} is v -formalizing then \mathcal{G} is v -formalizing.*
2. *If $\mathrm{Spec}(R) \in \mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$ then $\mathrm{Spec}(R)^\diamond$ is formalizing.*
3. *If $X \in \widetilde{\mathrm{SchPerf}}$ then X^\diamond is v -formalizing by lemma 1.3.32.*
4. *Non-empty v -formalizing v -sheaves have non-empty reduction. Consequently, diamonds are not v -formalizing.*
5. *If \mathcal{F} formalizes $f : \mathrm{Spa}(A, A^+) \rightarrow \mathcal{F}$ then \mathcal{F} formalizes $f \circ g$ for any map $g : \mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spa}(A, A^+)$*

Proposition 1.4.10. *Let \mathcal{F} be a small v -sheaf, and $f : \mathrm{Spa}(R, R^+) \rightarrow \mathcal{F}$ a map with $\mathrm{Spa}(R, R^+)$ affinoid perfectoid in characteristic p . If \mathcal{F} is formally separated then f admits at most one formalization.*

Proof. Suppose we are given two formalizations $g_i : \mathrm{Spd}(R^+, R^+) \rightarrow \mathcal{F}$ that agree on $\mathrm{Spa}(R, R^+)$. We get a map $(g_1, g_2) : \mathrm{Spd}(R^+, R^+) \rightarrow \mathcal{F} \times \mathcal{F}$, and we can pullback along the diagonal $\Delta_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ to get $\mathcal{G} \subseteq \mathrm{Spd}(R^+, R^+)$ a formally closed subsheaf. We want to prove that $\mathcal{G} = \mathrm{Spd}(R^+, R^+)$, and to do so it is enough to prove the equality at the level of topological spaces, $|\mathcal{G}| = |\mathrm{Spd}(R^+, R^+)|$. Moreover, since $|\mathrm{Spa}(R, R^+)| \subseteq |\mathcal{G}|$ and $|\mathrm{Spo}(R^+, R^+)| = |\mathrm{Spa}(R, R^+)| \cup |\mathrm{Spec}(R_{\mathrm{red}}^+)^{\diamond}|$ it is enough to prove $|(\mathcal{G}^{\mathrm{red}})^{\diamond}| = |\mathrm{Spec}(R_{\mathrm{red}}^+)^{\diamond}|$. We warn the reader that we can't use a direct density argument because although $\mathrm{Spa}(R, R^+)$ is dense in $\mathrm{Spa}(R^+, R^+)$, it is no longer dense in $\mathrm{Spo}(R^+, R^+)$.

We first deal with the case in which C is a non-Archimedean field and $C^+ \subseteq C$ is an open and bounded valuation subring. Let $k^+ = C_{\mathrm{red}}^+$ and $k = \mathrm{Frac}(k^+)$, we have that $\mathrm{Spec}(k^+) = \mathrm{Spd}(C^+, C^+)^{\mathrm{red}}$ and by lemma 1.3.33 $(\mathcal{G}^{\mathrm{red}})^{\diamond} = \mathrm{Spec}(k^+/I)^{\diamond}$ for some ideal I . On the other hand, since $\mathrm{Spa}(C, C^+) \subseteq \mathcal{G}$ and $|\mathcal{G}|$ is closed in $\mathrm{Spo}(C^+, C^+)$, $|\mathcal{G}|$ contains the formal specialization of $\mathrm{Spa}(C, C^+)$ in $\mathrm{Spo}(C^+, C^+)$, this corresponds to the image of $\mathrm{Spec}(k)^{\diamond}$. By formal adicness $|(\mathcal{G}^{\mathrm{red}})^{\diamond}| = |\mathcal{G}| \cap |\mathrm{Spec}(k^+)^{\diamond}|$ and we can conclude that $\mathrm{Spec}(k)^{\diamond} \subseteq (\mathcal{G}^{\mathrm{red}})^{\diamond}$. This proves that $I = \{0\}$ and that $(\mathcal{G}^{\mathrm{red}})^{\diamond} = \mathrm{Spec}(k^+)^{\diamond}$ as we needed to show.

For the general case, we get that for every map $\mathrm{Spa}(C, C^+) \rightarrow \mathrm{Spa}(R, R^+)$ the canonical formalization $\mathrm{Spd}(C^+, C^+) \rightarrow \mathrm{Spd}(R^+, R^+)$ factors through \mathcal{G} . In particular, after taking reduction, the map $\mathrm{Spec}(k^+) \rightarrow \mathrm{Spec}(R_{\mathrm{red}}^+)$ factors through $\mathcal{G}^{\mathrm{red}}$. This says that $|\mathcal{G}^{\mathrm{red}}|$ contains every point of $|\mathrm{Spec}(R_{\mathrm{red}}^+)|$ in the image of the specialization map. By lemma 1.3.33 $\mathcal{G}^{\mathrm{red}} \rightarrow \mathrm{Spec}(R_{\mathrm{red}}^+)$ is a closed immersion and by proposition 1.4.2 the specialization map is surjective, these two imply that $\mathcal{G}^{\mathrm{red}} = \mathrm{Spec}(R_{\mathrm{red}}^+)$. This also shows that $|(\mathcal{G}^{\mathrm{red}})^{\diamond}| = |\mathrm{Spec}(R_{\mathrm{red}}^+)^{\diamond}|$ and concludes the proof. \square

Proposition 1.4.11. *The following statements hold:*

1. *Given two maps of v -sheaves $\mathcal{F} \rightarrow \mathcal{H}$, $\mathcal{G} \rightarrow \mathcal{H}$ if \mathcal{F} and \mathcal{G} are v -formalizing and \mathcal{H} is formally separated then $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is v -formalizing.*
2. *The subcategory of v -sheaves that are v -formalizing and formally separated is stable under fiber product and contains \mathbb{Z}_p^{\diamond} .*

Proof. Given a map $\mathrm{Spa}(A, A^+) \rightarrow \mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ we can find a cover $\mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spa}(A, A^+)$ for which the compositions with the projections to \mathcal{F} and \mathcal{G} are both formalizable. By formal separatedness any pair of choices of formalizations $\mathrm{Spd}(B^+, B^+) \rightarrow \mathcal{G}$ and to $\mathrm{Spd}(B^+, B^+) \rightarrow \mathcal{F}$ define the same formalization to \mathcal{H} and a map to $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$. The second claim follows from the stability of separatedness by basechange and composition, from lemma 1.3.30 and from lemma 1.3.32. Indeed, we need to prove that $(\mathcal{F}^{\mathrm{red}})^{\diamond} \times_{(\mathcal{H}^{\mathrm{red}})^{\diamond}} (\mathcal{G}^{\mathrm{red}})^{\diamond}$ is a subsheaf of $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$, but this follows from knowing that $\mathcal{F}^{\mathrm{red}}$ (respectively \mathcal{H}, \mathcal{G}) is a subsheaf of \mathcal{F} (respectively \mathcal{H}, \mathcal{G}). \square

Definition 1.4.12. *Let \mathcal{F} be a small v -sheaf, we say that it is specializing if it is formally separated and v -locally formal.*

Definition 1.4.13. *Let \mathcal{F} be a specializing v -sheaf and let $f : \coprod_{i \in I} \mathrm{Spd}(B_i, B_i) \rightarrow \mathcal{F}$ be a surjective map. The specialization map for \mathcal{F} , denoted $\mathrm{sp}_{\mathcal{F}}$, is the unique map $\mathrm{sp}_{\mathcal{F}} : |\mathcal{F}| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ making the following diagram commutative:*

$$\begin{array}{ccc} \coprod_{i \in I} |\mathrm{Spd}(B_i, B_i)| & \xrightarrow{f} & |\mathcal{F}| \\ \downarrow \mathrm{sp}_{B_i} & & \downarrow \\ \coprod_{i \in I} |\mathrm{Spec}((B_i)_{\mathrm{red}})| & \xrightarrow{|f^{\mathrm{red}}|} & |\mathcal{F}^{\mathrm{red}}| \end{array}$$

Remark 1.4.14. We use proposition 1.4.10 to prove that this map of sets is well defined and does not depend on the choices taken. Indeed, given a point $[x] \in |\mathcal{F}|$ we may take a formalizable representative $x : \mathrm{Spa}(K_x, K_x^+) \rightarrow \mathcal{F}$. We take its unique formalization $\mathrm{Spd}(K_x^+, K_x^+) \rightarrow \mathcal{F}$ and we apply the reduction functor to this map. We obtain a map $\mathrm{Spec}((K_x^+)_{\mathrm{red}}) \rightarrow \mathcal{F}^{\mathrm{red}}$, and the maximal ideal of $(K_x^+)_{\mathrm{red}}$ maps to a point in $|\mathcal{F}^{\mathrm{red}}|$, $\mathrm{sp}_{\mathcal{F}}([x])$ is this point. Suppose $y : \mathrm{Spa}(K_y, K_y^+) \rightarrow \mathcal{F}$ is another formalizable map with $[x] = [y]$ after replacing y by a cover we may assume that the map factors as $\mathrm{Spa}(K_y, K_y^+) \rightarrow \mathrm{Spa}(K_x, K_x^+) \rightarrow \mathcal{F}$. Since formalizations are unique we get a map $\mathrm{Spd}(K_y^+, K_y^+) \rightarrow \mathrm{Spd}(K_x^+, K_x^+) \rightarrow \mathcal{F}$ and the maximal ideal of $(K_y^+)_{\mathrm{red}}$ maps to the maximal ideal of $(K_x^+)_{\mathrm{red}}$.

Proposition 1.4.15. For any specializing v -sheaf \mathcal{F} the specialization map $\mathrm{sp}_{\mathcal{F}} : |\mathcal{F}| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ is continuous. Moreover, this construction is functorial in the category of specializing v -sheaves.

Proof. We prove functoriality first, take a map of v -sheaves as above $g : \mathcal{F} \rightarrow \mathcal{G}$. Given a point $x : \mathrm{Spa}(K, K^+) \rightarrow \mathcal{F}$ the image in $|\mathcal{G}|$ is given by composition. A formalization for x gives a formalization for $g(x)$ and we get maps $\mathrm{Spec}(K_{\mathrm{red}}^+) \rightarrow \mathcal{F}^{\mathrm{red}} \rightarrow \mathcal{G}^{\mathrm{red}}$, we get that $g^{\mathrm{red}}(\mathrm{sp}_{\mathcal{F}}(x)) = \mathrm{sp}_{\mathcal{G}}(g(x))$.

Let us prove continuity, take a cover $f : \coprod_{i \in I} \mathrm{Spd}(B_i, B_i) \rightarrow \mathcal{F}$ with each (B_i, B_i) a formal Huber pair. By definition we get the following commutative diagram:

$$\begin{array}{ccc} |\coprod_{i \in I} \mathrm{Spd}(B_i, B_i)| & \xrightarrow{f} & |\mathcal{F}| \\ \downarrow \mathrm{sp}_{B_i} & & \downarrow \mathrm{sp}_{\mathcal{F}} \\ |\mathrm{Spec}((B_i)_{\mathrm{red}})| & \xrightarrow{f^{\mathrm{red}}} & |\mathcal{F}^{\mathrm{red}}| \end{array}$$

The map f^{red} is continuous by proposition 1.3.15, the map f is continuous and a quotient map, and the maps sp_{B_i} are continuous by proposition 1.4.2. Since the diagram is commutative, the map $\mathrm{sp}_{\mathcal{F}}$ is also continuous. \square

1.4.3 Kimberlites, formal schemes and tubular neighborhoods

To prove pleasant properties of the specialization map we need to restrict our discussion to certain types of specializing v -sheaves.

Definition 1.4.16. 1. A pre-kimberlite is a small v -sheaf \mathcal{F} such that:

- a) \mathcal{F} is specializing.
- b) $\mathcal{F}^{\mathrm{red}}$ is represented by a scheme.
- c) The map $(\mathcal{F}^{\mathrm{red}})^{\diamond} \rightarrow \mathcal{F}$ coming from adjunction is a closed immersion.

- 2. For a pre-kimberlite \mathcal{F} , we define the analytic locus $\mathcal{F}^{\mathrm{an}}$ as the open subsheaf $\mathcal{F} \setminus \mathcal{F}^{\mathrm{red}}$. If $\mathcal{F}^{\mathrm{an}}$ is a locally spatial diamond we say that \mathcal{F} is a kimberlite.

There are situations in which we will be interested in studying the specialization map when restricted to a proper subset of the analytic locus, for this reason we consider the following slightly more general concept.

Definition 1.4.17. A smelted kimberlite is a pair $(\mathcal{F}, \mathcal{D})$ where \mathcal{F} is a pre-kimberlite and $\mathcal{D} \subseteq \mathcal{F}^{\mathrm{an}}$ is an open subsheaf such that \mathcal{D} is a locally spatial diamond. A morphism of smelted kimberlites $f : (\mathcal{F}_1, \mathcal{D}_1) \rightarrow (\mathcal{F}_2, \mathcal{D}_2)$ is a morphism of v -sheaves $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ such that $f(\mathcal{D}_1) \subseteq \mathcal{D}_2$.

If we are given a kimberlite \mathcal{F} over \mathbb{Z}_p^{\diamond} there are two different smelted kimberlites that one can naturally associate to \mathcal{F} . That is, $(\mathcal{F}, \mathcal{F}^{\mathrm{an}})$ and $(\mathcal{F}, \mathcal{F} \times_{\mathbb{Z}_p^{\diamond}} \mathbb{Q}_p^{\diamond})$. These two will only coincide if $\mathcal{F} \rightarrow \mathbb{Z}_p^{\diamond}$ is formally adic. We will use the following definition to abbreviate some sentences.

Definition 1.4.18. A kimberlite (respectively prekimberlite) \mathcal{F} together with a formally adic map $\mathcal{F} \rightarrow \mathbb{Z}_p^\diamond$ is said to be a p -adic kimberlite (respectively p -adic prekimberlite). Given a map of sheaves $\mathcal{F} \rightarrow \mathbb{Z}_p^\diamond$ we denote by \mathcal{F}_η the basechange $\mathcal{F} \times_{\mathbb{Z}_p^\diamond} \mathbb{Q}_p^\diamond$. A p -smelted kimberlite is a prekimberlite \mathcal{F} over \mathbb{Z}_p^\diamond for which $(\mathcal{F}, \mathcal{F}_\eta)$ is a smelted kimberlite.

Definition 1.4.19. Given a smelted kimberlite $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ we define the map of topological spaces $\mathrm{sp}_{\mathcal{K}} : |\mathcal{D}| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ as the composition of $|\mathcal{D}| \rightarrow |\mathcal{F}| \xrightarrow{\mathrm{sp}_{\mathcal{F}}} |\mathcal{F}^{\mathrm{red}}|$. For a kimberlite \mathcal{F} we abbreviate $\mathrm{sp}_{(\mathcal{F}, \mathcal{F}^{\mathrm{an}})}$ by $\mathrm{sp}_{\mathcal{F}^{\mathrm{an}}}$.

Proposition 1.4.20. Let $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ be a smelted kimberlite, then $\mathrm{sp}_{\mathcal{K}}$ is a spectral map of locally spectral spaces. The construction of $\mathrm{sp}_{\mathcal{K}}$ is functorial in the category of smelted kimberlites.

Proof. Continuity and functoriality follows directly from proposition 1.4.15. We need to prove that this map is also continuous for the constructible topology. Since it is enough to prove continuity on an open cover of $|\mathcal{D}|$, we may assume that \mathcal{D} is a spatial diamond. We cover \mathcal{D} by an affinoid perfectoid space $X = \mathrm{Spa}(A, A^+)$. Consider the diagram,

$$\begin{array}{ccc} |\mathrm{Spa}(A, A^+)|^{\mathrm{cons}} & \xrightarrow{g} & |\mathcal{D}|^{\mathrm{cons}} \\ \downarrow \mathrm{sp}_X & & \downarrow \mathrm{sp}_{\mathcal{K}} \\ |\mathrm{Spec}(A_{\mathrm{red}}^+)^{\mathrm{red}}|^{\mathrm{cons}} & \xrightarrow{g^{\mathrm{red}}} & |\mathcal{F}^{\mathrm{red}}|^{\mathrm{cons}} \end{array}$$

where the spaces are given the constructible topology. Since $\mathcal{F}^{\mathrm{red}}$ is represented by a scheme proposition 1.3.17 implies that g^{red} is continuous for the constructible topology. Indeed, morphisms of schemes induce spectral maps. Similarly, the map sp_X is continuous and since X is a spatial diamond proposition 1.1.26 shows that the map g is also continuous. Moreover, g gives a surjective map of compact spaces and is consequently a quotient map. Since the diagram commutes, $\mathrm{sp}_{\mathcal{K}}$ is continuous for the patch topology. \square

The author thinks of kimberlites as a natural category in which one can consider “integral models” for diamonds. In what follows we will prove that v -sheaf associated to a separated formal scheme is a kimberlite.

For this one has to choose conventions carefully of what it means to be a “formal scheme”. We take the convention given [29] section 2.2.

Convention 1. Denote by $\mathrm{Nilp}_{\mathbb{Z}_p}$ the category of algebras in which p is nilpotent, and endow $\mathrm{Nilp}_{\mathbb{Z}_p}^{\mathrm{op}}$ with the structure of a site by giving it the Zariski topology. By a formal scheme \mathfrak{X} over \mathbb{Z}_p we mean a Zariski sheaf on $\mathrm{Nilp}_{\mathbb{Z}_p}^{\mathrm{op}}$ which is Zariski locally of the form $\mathrm{Spf}(A)$. Here A is a topological ring given the I -adic topology for a finitely generated ideal of A containing p , and $\mathrm{Spf}(A)$ denotes the functor $\mathrm{Spec}(B) \mapsto \varinjlim_n \mathrm{Hom}(A/I^n, B)$.

For a formal scheme \mathfrak{X} over \mathbb{Z}_p we let $\mathfrak{X}_{\mathrm{red}}$ denote its reduction in the sense of formal schemes (See [31] Tag 0AIN). Recall that this is a sheaf in $\mathrm{Nilp}_{\mathbb{Z}_p}^{\mathrm{op}}$ which is representable by a scheme. Moreover, the map $\mathfrak{X}_{\mathrm{red}} \rightarrow \mathfrak{X}$ is relatively representable in schemes, it is a closed immersion and for any open $\mathrm{Spf}(A) \subseteq \mathfrak{X}$ the pullback to $\mathfrak{X}_{\mathrm{red}}$ is given by the reduced subscheme of $\mathrm{Spec}(A/I)$ (for an ideal of definition $I \subseteq A$).

We say that \mathfrak{X} is separated if $\mathfrak{X}_{\mathrm{red}}$ is a separated scheme (See [31] Tag 0AJ7).

Recall the following result of Scholze and Weinstein

Proposition 1.4.21. (See [29] 2.2.1) The functor $\mathrm{Spf}(A) \mapsto \mathrm{Spa}(A, A)$ extends to a fully faithful functor $\mathfrak{X} \mapsto \mathfrak{X}^{\mathrm{ad}}$ from formal schemes over \mathbb{Z}_p as in convention 1 to the category of pre-adic spaces.

Remark 1.4.22. We warn that what is called adic spaces in [29] is what we call pre-adic spaces here and in [30].

Proposition 1.4.23. If \mathfrak{X} is a separated formal scheme over \mathbb{Z}_p as in convention 1 then $(\mathfrak{X}^{ad})^\diamond$ is a kimberlite.

Proof. Let $X = \mathfrak{X}^{ad}$ and let $W = X^{na}$, we have that $W = (\mathfrak{X}_{red})^{ad}$. Clearly X^\diamond is v -locally formal since it is open locally of the form $\mathrm{Spd}(B, B)$. By proposition 1.3.20 we have $(W^\diamond)^{red} = (X^\diamond)^{red}$ which is the perfection of \mathfrak{X}_{red} . The adjunction morphism agrees with the map $W^\diamond \rightarrow X^\diamond$ which by proposition 1.1.31 is a closed immersion. Moreover, this says $X^\diamond \setminus ((X^\diamond)^{red})^\diamond = (X^{an})^\diamond$ so by proposition 1.1.31 this is a locally spatial diamond.

The only thing left to prove is that $X^\diamond \rightarrow \mathbb{Z}_p^\diamond$ is separated, we first prove that X^\diamond is quasi-separated. Let $Z = \mathrm{Spa}(R, R^+)$ be a strictly totally disconnected space and take a map $f : Z \rightarrow X^\diamond \times_{\mathbb{Z}_p^\diamond} X^\diamond$. Since Z splits any open cover we may assume that f factors through an open neighborhood of the form $\mathrm{Spd}(B_1, B_1) \times_{\mathbb{Z}_p^\diamond} \mathrm{Spd}(B_2, B_2)$ for an open subset $\mathrm{Spf}(B_1) \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(B_2) \subseteq \mathfrak{X} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathfrak{X}$. Consider the following basechange diagrams, where $Y = \mathfrak{Y}^{ad}$

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & \mathrm{Spf}(B_1) \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(B_2) \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{X} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathfrak{X} \end{array} \quad \begin{array}{ccc} Y & \longrightarrow & \mathrm{Spa}(B_1, B_1) \times_{\mathbb{Z}_p} \mathrm{Spa}(B_2, B_2) \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times_{\mathbb{Z}_p} X \end{array}$$

Since \mathfrak{X} is separated \mathfrak{Y} is quasi-compact. This implies that Y admits a finite open cover of the form $\coprod_{i=1}^n \mathrm{Spa}(A_i, A_i) \rightarrow Y$. Moreover, the diagonal map $X \rightarrow X \times_{\mathbb{Z}_p} X$ is adic (sends analytic points to analytic points). Indeed, composing the diagonal map with one of the projections gives the identity. Since open immersions preserve adicness the maps of pre-adic spaces $\mathrm{Spa}(A_i, A_i) \rightarrow \mathrm{Spa}(B_1, B_1) \times_{\mathbb{Z}_p} \mathrm{Spa}(B_2, B_2)$ are adic. By lemma 1.2.26 the maps $\mathrm{Spd}(A_i, A_i) \rightarrow \mathrm{Spd}(B_1, B_1) \times_{\mathbb{Z}_p} \mathrm{Spd}(B_2, B_2)$ are quasi-compact, which proves that $Y^\diamond \rightarrow \mathrm{Spd}(B_1, B_1) \times_{\mathbb{Z}_p} \mathrm{Spd}(B_2, B_2)$ and any basechange of it is also quasi-compact. After proving that the map $X^\diamond \rightarrow \mathbb{Z}_p^\diamond$ is quasi-separated we may use the valuative criterion of separatedness (See [28] 10.9). We must prove that for a perfectoid field and a map $\mathrm{Spa}(K, O_K) \rightarrow X^\diamond$ there is at most one extension to $\mathrm{Spa}(K, K^+) \rightarrow X^\diamond$ where $K^+ \subseteq O_K$ is an open and bounded valuation subring. Maps $\mathrm{Spa}(K, K^+) \rightarrow X^\diamond$ are in bijection with maps $\mathrm{Spf}(K^+) \rightarrow \mathfrak{X}$. On the other hand, maps $g : \mathrm{Spf}(K^+) \rightarrow \mathfrak{X}$ are in bijection with pairs (g_η, g_s) where $g_\eta : \mathrm{Spf}(O_K) \rightarrow \mathfrak{X}$, $g_s : \mathrm{Spec}(K^+/K^{\circ\circ}) \rightarrow \mathfrak{X}_{red}$ and such that $g_\eta = g_s$ when we restrict the maps to $\mathrm{Spec}(O_K/K^{\circ\circ})$. At this point we may use the valuative criterion of separatedness of \mathfrak{X}_{red} . □

The following concept is central to our purposes.

Definition 1.4.24. Let \mathcal{F} be a prekimberlite and let $S \subseteq \mathcal{F}^{red}$ be a locally closed immersion of schemes.

1. We define the tubular neighborhood of S on \mathcal{F} , denoted $\widehat{\mathcal{F}}_{/S}$, as the sub- v -sheaf of \mathcal{F} defined by the following Cartesian diagram:

$$\begin{array}{ccc} \widehat{\mathcal{F}}_{/S} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ |S| & \longrightarrow & |\mathcal{F}| \\ & & \downarrow \mathrm{sp}_{\mathcal{F}} \\ |S| & \longrightarrow & |\mathcal{F}^{red}| \end{array}$$

2. If $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ is a smelted kimberlite we let $\widehat{\mathcal{D}}_{/S}$ be $\widehat{\mathcal{F}}_{/S} \cap \mathcal{D}$ and we refer to this sheaf as the smelted tubular neighborhood.
3. If \mathcal{F} comes equipped with a map to \mathbb{Z}_p^\diamond (not necessarily formally adic) we define the p -adic tubular neighborhood of \mathcal{F} , denoted $(\widehat{\mathcal{F}}_{/S})_\eta$, as the basechange $\widehat{\mathcal{F}}_{/S} \times_{\mathbb{Z}_p^\diamond} \mathbb{Q}_p^\diamond$.

Intuitively speaking, $\widehat{\mathcal{F}}_{/S}$ is the subsheaf of points whose specialization map factors through S . This notion generalizes completions along a closed subscheme in formal geometry:

Proposition 1.4.25. *Suppose (A, A) is a formal Huber pair over \mathbb{Z}_p with ideal of definition I . Let $J \subseteq A$ be a finitely generated ideal containing I and B the completion of A with respect to J . The closed immersion of schemes $S = \mathrm{Spec}(B_{\mathrm{red}}) \rightarrow \mathrm{Spec}(A_{\mathrm{red}})$, induces an identification $\mathrm{Spd}(\widehat{A, A})_{/S} = \mathrm{Spd}(B, B)$.*

Proof. Let $S = \mathrm{Spec}(B_{\mathrm{red}})$ and $T = \mathrm{Spec}(A_{\mathrm{red}})$. The reduction of the map $\mathrm{Spd}(B, B) \rightarrow \mathrm{Spd}(A, A)$ induces the map $S \rightarrow T$. Since specialization is functorial any point coming from $\mathrm{Spd}(B, B)$ has to specialize to S . Consequently the map factors as $\mathrm{Spd}(B, B) \rightarrow \mathrm{Spd}(\widehat{A, A})_{/S} \rightarrow \mathrm{Spd}(A, A)$. Since A is dense in B , it is easy to see that this map is an injection. To prove surjectivity onto $\mathrm{Spd}(\widehat{A, A})_{/S}$, suppose we have a map $f : A \rightarrow R^+$ for which the induced map $f : \mathrm{Spec}(R_{\mathrm{red}}^+) \rightarrow \mathrm{Spec}(A_{\mathrm{red}})$ factors through $|S|$. Then for every $a \in J$ the element $f(a)$ is nilpotent in $\mathrm{Spec}(R^+/\varpi^n)$. Since J is finitely generated there is an m for which $J^m \subseteq (\varpi^n)$ in R^+ . This proves that the map $f : A \rightarrow R^+$ is continuous for the J -adic topology on A . Since R^+ is complete the map $f : A \rightarrow R^+$ factors through B , which proves that any map $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(\widehat{A, A})_{/S}$ factors through a map to $\mathrm{Spd}(B, B)$. \square

Proposition 1.4.26. *Let $f : \mathcal{G} \rightarrow \mathcal{F}$ be a morphism prekimberlites and let $S \subseteq |\mathcal{F}^{\mathrm{red}}|$ a locally closed subscheme. If we define $T = S \times_{\mathcal{F}^{\mathrm{red}}} \mathcal{G}^{\mathrm{red}}$, then $\widehat{\mathcal{F}}_{/S} \times_{\mathcal{F}} \mathcal{G} = \widehat{\mathcal{G}}_{/T}$. In particular, a map of prekimberlites $\mathcal{G} \rightarrow \mathcal{F}$ factors through $\widehat{\mathcal{F}}_{/S}$ if and only if $\mathcal{G}^{\mathrm{red}} \rightarrow \mathcal{F}^{\mathrm{red}}$ factors through S .*

Proof. Since S is a locally closed immersion we have $|T| = |S| \times_{|\mathcal{F}^{\mathrm{red}}|} |\mathcal{G}^{\mathrm{red}}|$. We can look at the following commutative diagram:

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ |G^{\mathrm{red}}| & \longrightarrow & |\mathcal{F}^{\mathrm{red}}| \end{array}$$

The first claim follows by basechanging this diagram by the map $|S| \rightarrow |\mathcal{F}^{\mathrm{red}}|$. For the second claim, observe that the map $\mathcal{G} \rightarrow \mathcal{F}$ factoring through $\widehat{\mathcal{F}}_{/S}$ is equivalent to $\mathcal{G} \times_{\mathcal{F}} \widehat{\mathcal{F}}_{/S} = \mathcal{G}$. By the first claim this is equivalent to $\widehat{\mathcal{G}}_{/T} = \mathcal{G}$, which happens if and only if $T = \mathcal{G}^{\mathrm{red}}$ and $\mathcal{G}^{\mathrm{red}} \rightarrow \mathcal{F}^{\mathrm{red}}$ factors through S . \square

Remark 1.4.27. *Let \mathcal{F} be a prekimberlite and $S \subseteq \mathcal{F}^{\mathrm{red}}$ locally closed subset. One can prove that the v -sheaf $\widehat{\mathcal{F}}_{/S}$ is a small v -sheaf but this is not automatic. The problem is that the v -sheaf \underline{T} is not small whenever the topological space T does not satisfy the separation axiom $T1$.*

Proposition 1.4.28. *Let \mathcal{F} be a prekimberlite and let $S \subseteq |\mathcal{F}^{\mathrm{red}}|$ a locally closed subset, then $\widehat{\mathcal{F}}_{/S}$ is a prekimberlite and $(\widehat{\mathcal{F}}_{/S})^{\mathrm{red}} = S$.*

Proof. The formula $(\widehat{\mathcal{F}}_{/S})^{\mathrm{red}} = S$ follows easily from observing that by proposition 1.4.26 a map $\mathrm{Spec}(A)^\diamond \rightarrow \mathcal{F}$ factors through $\widehat{\mathcal{F}}_{/S}$ if and only if the map obtained by adjunction $\mathrm{Spec}(A) \rightarrow \mathcal{F}^{\mathrm{red}}$ factors through S . Indeed, S and $\widehat{\mathcal{F}}_{/S}^{\mathrm{red}}$ represent the same functor in this case.

For the first claim, since $\widehat{\mathcal{F}}_S$ is a subsheaf of a formally separated v -sheaf it is formally separated as well. To prove it is v -formalizing take a map $\mathrm{Spa}(R, R^+) \rightarrow \widehat{\mathcal{F}}_S \subseteq \mathcal{F}$. After replacing $\mathrm{Spa}(R, R^+)$ by a v -cover if necessary we get a formalization $\mathrm{Spd}(R^+, R^+) \rightarrow \mathcal{F}$. By proposition 1.4.26 this formalization factors through $\widehat{\mathcal{F}}_S$ if and only if $\mathrm{Spec}(R^+_{\mathrm{red}}) \rightarrow \mathcal{F}^{\mathrm{red}}$ factors through S . But this later condition holds since $\mathrm{Spa}(R, R^+) \rightarrow \mathcal{F}$ factors through $\widehat{\mathcal{F}}_S$. To finish the proof we need to show that $S^\diamond \rightarrow \widehat{\mathcal{F}}_S$ is a closed immersion. Consider the base change $\widehat{\mathcal{F}}_S \times_{\mathcal{F}} (\mathcal{F}^{\mathrm{red}})^\diamond$. On one hand the projection to $\widehat{\mathcal{F}}_S$ is a closed immersion, and on the other hand by proposition 1.4.26 this identifies with $((\widehat{\mathcal{F}^{\mathrm{red}}})^\diamond)_{/S}$. In case S is a closed subscheme of $\mathcal{F}^{\mathrm{red}}$ we have that the map of v -sheaves $S^\diamond \rightarrow (\mathcal{F}^{\mathrm{red}})^\diamond$ is proper so that $S^\diamond \rightarrow ((\widehat{\mathcal{F}^{\mathrm{red}}})^\diamond)_{/S}$ is also a closed immersion. In case S is an open subscheme of $\mathcal{F}^{\mathrm{red}}$, we can verify $((\widehat{\mathcal{F}^{\mathrm{red}}})^\diamond)_{/S} = S^\diamond$. The general case follows from these two cases. \square

Whenever S is a constructible subset we can say more:

Proposition 1.4.29. *Let \mathcal{F} be a prekimberlite, $S \subseteq |\mathcal{F}^{\mathrm{red}}|$ a locally closed constructible subset then:*

1. *The map $\widehat{\mathcal{F}}_S \rightarrow \mathcal{F}$ is an open immersion.*
2. *If $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ is a smelted kimberlite, then $\widehat{\mathcal{F}}_S \cap \mathcal{D}$ is the open subsheaf corresponding to the interior of $\mathrm{sp}_{\mathcal{K}}^{-1}(S)$ in $|\mathcal{D}|$.*

Proof. For the first claim we begin by observing that the question is Zariski local in $\mathcal{F}^{\mathrm{red}}$. Indeed, an open cover $\coprod_{i \in I} U_i \rightarrow \mathcal{F}^{\mathrm{red}}$ induces an open cover $\coprod_{i \in I} \widehat{\mathcal{F}}_{/U_i} \rightarrow \mathcal{F}$. After localizing, we may assume that $\mathcal{F}^{\mathrm{red}} = \mathrm{Spec}(A)$ and that S is closed subset of $\mathrm{Spec}(A)$ that is open for the constructible topology. Write $S = \mathrm{Spec}(A/I)$ for $I \subseteq A$ an ideal, since we are only interested in S as a topological space, a compactness argument allows us to assume that I is finitely generated. Let (i_1, \dots, i_n) be a list of generators for I , let (R, R^+) be a perfectoid Huber pair and $\mathrm{Spd}(R^+, R^+) \rightarrow \mathcal{F}$ a map. We can describe the basechange $X := \mathrm{Spd}(R^+, R^+) \times_{\mathcal{F}} \widehat{\mathcal{F}}_S$ as follows. Let $\varpi \in R^+$ be a pseudo-uniformizer and (j_1, \dots, j_n) a list of lifts of (i_1, \dots, i_n) in R^+_{red} . We claim X is the open subsheaf of $\mathrm{Spd}(R^+, R^+)$ defined by $\bigcap_{k=1}^n N_{j_k} < 1$. Indeed, by proposition 1.4.26 X is given by $\mathrm{Spd}(\widehat{R^+, R^+})_{/V(I)}$ and by proposition 1.4.25 if we let B^+ be the completion of R^+ by the (I, ϖ) -adic topology then $X = \mathrm{Spd}(B^+, B^+)$. That $\mathrm{Spd}(B^+, B^+) = \bigcap_{k=1}^n N_{j_k} < 1$ is a direct consequence of lemma 1.2.23. Since \mathcal{F} is v -formalizing every map $\mathrm{Spa}(R, R^+) \rightarrow \mathcal{F}$ factors through $\mathrm{Spd}(R^+, R^+)$ after replacing $\mathrm{Spa}(R, R^+)$ by a v -cover. In particular, the basechanges $\mathrm{Spa}(R, R^+) \times_{\mathcal{F}} \widehat{\mathcal{F}}_S$ are open after taking a v -cover. By [28] 10.11 $\widehat{\mathcal{F}}_S \rightarrow \mathcal{F}$ is open.

For the second claim let $T \subseteq \mathrm{sp}_{\mathcal{K}}^{-1}(S)$ be the largest subset stable under generization. We prove that $T \subseteq \widehat{\mathcal{F}}_S \cap \mathcal{D}$ since we already have a chain of inclusions:

$$\widehat{\mathcal{F}}_S \cap \mathcal{D} \subseteq (\mathrm{sp}_{\mathcal{K}}^{-1}(S))^{\mathrm{int}} \subseteq T.$$

Take $x \in T$ and a formalizable geometric point $\iota_x : \mathrm{Spa}(C_x, C_x^+) \rightarrow \mathcal{F}$ over x . Since every generization of x is in $\mathrm{sp}_{\mathcal{K}}^{-1}(S)$ the map $\mathrm{Spec}((C_x^+)_{\mathrm{red}}) \rightarrow \mathcal{F}^{\mathrm{red}}$ factors through S so that the composition $|\mathrm{Spa}(C_x, C_x^+)| \rightarrow |\mathcal{D}| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ factors through $|S|$, giving that ι_x factors through $\widehat{\mathcal{F}}_S$. \square

Proposition 1.4.30. *Let $f : \mathcal{G} \rightarrow \mathcal{F}$ be a formally closed immersion of small v -sheaves. The following hold:*

1. *If \mathcal{F} is a specializing v -sheaf, then \mathcal{G} is a specializing v -sheaf.*
2. *If \mathcal{F} is a prekimberlite, then \mathcal{G} is a prekimberlite.*

3. If \mathcal{F} is a kimberlite, then \mathcal{G} is a kimberlite.

4. If $(\mathcal{F}, \mathcal{D})$ forms a smelted kimberlite then $(\mathcal{G}, \mathcal{G} \cap \mathcal{D})$ forms a smelted kimberlite.

Proof. Suppose \mathcal{F} is specializing, since \mathcal{G} is a subsheaf of \mathcal{F} it is formally separated. Observe that for a perfectoid Huber pair (R, R^+) and a map $\mathrm{Spd}(R^+, R^+) \rightarrow \mathcal{F}$ the basechange $X := \mathcal{G} \times_{\mathcal{F}} \mathrm{Spd}(R^+, R^+)$ is a formally closed subsheaf of $\mathrm{Spd}(R^+, R^+)$. We may reason as in the proof of proposition 1.4.10 to conclude $X = \mathrm{Spd}(R^+, R^+)$ whenever $\mathrm{Spa}(R, R^+) \rightarrow \mathcal{F}$ factors through \mathcal{G} . This proves that \mathcal{G} is also v -formalizing and a specializing sheaf. Suppose \mathcal{F} is a prekimberlite, we have a commutative diagram:

$$\begin{array}{ccc} (\mathcal{G}^{\mathrm{red}})^{\diamond} & \longrightarrow & (\mathcal{F}^{\mathrm{red}})^{\diamond} \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{F} \end{array}$$

By formal adicness this diagram is Cartesian which gives that $(\mathcal{G}^{\mathrm{red}})^{\diamond} \rightarrow \mathcal{G}$ is a closed immersion. Since the map $(\mathcal{G}^{\mathrm{red}})^{\diamond} \rightarrow (\mathcal{F}^{\mathrm{red}})^{\diamond}$ is also a formally closed immersion by lemma 1.3.33 $\mathcal{G}^{\mathrm{red}}$ is represented by a closed subscheme of \mathcal{F} , finishing the proof that \mathcal{G} is a prekimberlite. Suppose now that \mathcal{F} is a kimberlite, then $\mathcal{G}^{\mathrm{an}} = \mathcal{F}^{\mathrm{an}} \times_{\mathcal{F}} \mathcal{G}$ and by ([28] 11.20) it is a locally spatial diamond, so \mathcal{G} is a kimberlite. The same applies for $\mathcal{G} \times_{\mathcal{F}} \mathcal{D}$ in the smelted kimberlite case. \square

1.4.4 cJ-diamonds and rich kimberlites

Suppose we have a formal scheme \mathcal{X} topologically of finite type over \mathbb{Z}_p , suppose we let X_{η} denote the generic fiber of \mathcal{X} considered as an adic space over \mathbb{Q}_p and suppose we let X^{red} denote the reduced special fiber of \mathcal{X} considered as a scheme over \mathbb{F}_p . In this classical situation we have a specialization map $\mathrm{sp}_{X_{\eta}} : |X_{\eta}| \rightarrow |X^{\mathrm{red}}|$, and for a fixed closed point $x \in |X^{\mathrm{red}}|$ we have the following chain of inclusions $|(\hat{\mathcal{X}}/x)_{\eta}| \subseteq \mathrm{sp}_{X_{\eta}}^{-1}(x) \subseteq |X_{\eta}|$. These inclusions satisfy that:

1. $\mathrm{sp}_{X_{\eta}}^{-1}(x)$ is a closed subset.
2. $|(\hat{\mathcal{X}}/x)_{\eta}|$ is the interior of $\mathrm{sp}_{X_{\eta}}^{-1}(x)$ in $|X_{\eta}|$.
3. $|(\hat{\mathcal{X}}/x)_{\eta}|$ is dense in $\mathrm{sp}_{X_{\eta}}^{-1}(x)$

The first two conditions generalize, by proposition 1.4.29, to the case of kimberlites for which closed points are constructible. In this section we give sufficient conditions that make a kimberlite have the third property as well. Before discussing these condition we give an example showing that some sort of finiteness hypothesis need to be imposed for the third property to hold.

Example 1.4.31. Let C be a p -adic non-Archimedean field and C^+ an open and bounded valuation subring whose rank is strictly larger than 1. We have that sp_C is a homeomorphism between $\mathrm{Spa}(C, C^+)$ and $\mathrm{Spec}(C^+/C^{\circ\circ})$. In particular, if x denotes the closed point of $\mathrm{Spec}(C^+/C^{\circ\circ})$ then $\mathrm{sp}_C^{-1}(x)$ is the closed point of $y \in \mathrm{Spa}(C, C^+)$. The interior of $\{y\}$ is empty, therefore it is not a dense subset of $\{y\}$.

Definition 1.4.32. We say that a locally spatial diamond X is constructibly Jacobson if the subset of rank 1 points are dense for the constructible topology of $|X|$. Locally spatial diamonds with this property will be called *cJ-diamonds*.

Proposition 1.4.33. Suppose that $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ is a smelted kimberlite with \mathcal{D} a cJ-diamond, let $S \subseteq |\mathcal{F}|$ a constructible subset. Then $|\mathcal{D} \cap \hat{\mathcal{F}}_S|$ is dense in $\mathrm{sp}_{\mathcal{K}}^{-1}(S)$.

Proof. By the proof of proposition 1.4.29, we know that $|\mathcal{D} \cap \widehat{\mathcal{F}}_S|$ is the largest subset of $\mathrm{sp}_K^{-1}(S)$ stable under generization. Since S is constructible and sp_K is a spectral map, the set $\mathrm{sp}_K^{-1}(S)$ is open in the constructible topology of $|\mathcal{D}|$ and rank 1 points contained in this set are dense in it. Since rank 1 points are stable under generization, they belong to $|\mathcal{D} \cap \widehat{\mathcal{F}}_S|$. This proves that $|\mathcal{D} \cap \widehat{\mathcal{F}}_S|$ is dense in $\mathrm{sp}_K^{-1}(S)$ for the constructible topology, but the usual topology is coarser so it is dense for the usual topology as well. \square

We discuss some properties of this concept.

Proposition 1.4.34. *Let $f : X \rightarrow Y$ be a morphism of locally spatial diamonds the following hold:*

1. *Suppose that $|f|$ is a surjective map of topological spaces and that X is a cJ-diamond, then Y is a cJ-diamond.*
2. *Suppose that f is an open immersion and that Y is a cJ-diamond, then X is a cJ-diamond.*
3. *Suppose that f realizes X as a quasi-pro-étale \underline{J} -torsor over Y for some profinite group J and that X is a cJ-diamond, then Y is a cJ-diamond.*
4. *Suppose that f is étale and that Y is a cJ-diamond, then X is a cJ-diamond.*

Proof. Maps of locally spatial diamonds induce continuous spectral maps of locally spectral spaces. Surjective maps send dense subsets to dense subsets. Moreover, maps of locally spatial diamonds are generalizing which implies that rank 1 points can only map to rank 1 points. This proves the first claim.

Suppose now that Y is a cJ-diamond. If f is an open immersion, any open in the patch topology of X is also open in the patch topology of Y and contains a rank 1 point, this proves the second claim. Moreover this allow us to localize in the analytic topology, so we can assume for the rest of the argument X and Y are spatial.

If f is étale, by ([28] 11.31) locally for the analytic topology we can write f as the composition of an open immersion and a finite étale map. The category of finite étale morphisms over a fixed spatial diamond is a Galois category and using the first claim we may reduce to the case in which f is Galois with finite Galois group G . In this way, the fourth claim follows from the third.

In the setup of the third claim, we claim (and prove below) that f is an open mapping for the patch topology, this would finish the proof. Indeed, if a point y maps to x under a quasi-pro-étale map and x is rank 1 then y is also rank 1.

Let $J = \varprojlim_i J_i$ with J_i a cofiltered family of finite groups and denote by $f_i : X_i \rightarrow Y$ the induced J_i -torsors. We get action maps $J_i \times |X_i| \rightarrow |X_i|$ that are continuous for the discrete topology on J_i and the constructible topology on $|X_i|$. Moreover, for any set $S \subseteq |X_i|$ we have that $f_i^{-1}(f_i(S)) = J_i \cdot S$. Now, the formation of the patch topology on a spectral space commutes with limits along spectral maps. This gives an action map $J \times |X| \rightarrow |X|$ that is continuous when $|X|$ is given the patch topology and J is given its profinite topology. Let $U \subseteq X$ be open in the constructible topology, then $f^{-1}(f(U)) = J \cdot U$ which is also open. The map $|f|^{cons} : |X|^{cons} \rightarrow |Y|^{cons}$ is a surjective continuous map of compact spaces, so it is a quotient map. Since $J \cdot U$ is open and saturated $f(J \cdot U) = f(U)$ is open as we wanted to show. \square

Let's recall the following theorem of Huber:

Theorem 1.4.35. (See [13] Theorem 4.1) *Let k be a complete field with respect to a rank 1 valuation, and let A be a k -algebra of topologically finite type over k . Then the subset $\mathrm{Max}(A) \subseteq \mathrm{Spa}(A, A^\circ)$ is dense for the constructible topology.*

Huber's statement says something a bit stronger, but this weaker form of the statement is easier to state and the one we will use in applications.

Corollary 1.4.36. *If X is an adic space topologically of finite type over $\mathrm{Spa}(k, k^\circ)$, where $\mathrm{Spa}(k, k^\circ)$ is a non-Archimedean field over \mathbb{Z}_p . Then X^\diamond is a cJ-diamond.*

Proof. The claim is local on X so we can assume $X = \mathrm{Spa}(A, A^\circ)$ for a Tate algebra, A/k . In this case every point $\mathrm{Max}(A)$, when considered as a valuation, is a rank 1 valuation. \square

Example 1.4.37. *The perfectoid unit ball $\mathbb{B}_n = \mathrm{Spa}(C\langle T_1^{\frac{1}{p^\infty}} \dots T_n^{\frac{1}{p^\infty}} \rangle, O_C\langle T_1^{\frac{1}{p^\infty}} \dots T_n^{\frac{1}{p^\infty}} \rangle)$ over a perfectoid field C of characteristic p , is a cJ-diamond. Indeed, we have the equality of diamonds*

$$\mathrm{Spa}(C\langle T_1 \dots T_n \rangle, O_C\langle T_1 \dots T_n \rangle)^\diamond = \mathbb{B}_n,$$

and we may conclude by theorem 1.4.35.

Definition 1.4.38. *Let C be a perfectoid field in characteristic p and X a locally spatial diamond over $\mathrm{Spa}(C, O_C)$. We say that X has “enough facets” over C if it admits a v -cover of the form $\coprod_{i \in I} \mathrm{Spd}(A_i, A_i^\circ) \rightarrow X$ where each A_i is an algebra topologically of finite type over C .*

Proposition 1.4.39. *Let X and Y be two locally spatial diamonds with enough facets over C , and let C^\sharp denote an untilt of C . The following hold:*

1. *For any morphism of perfectoid fields $\mathrm{Spa}(C', O_{C'}) \rightarrow \mathrm{Spa}(C, O_C)$ the base change $X \times_{\mathrm{Spa}(C, O_C)} \mathrm{Spa}(C', O_{C'})$ has enough facets over C' .*
2. *The fiber product $X \times_{\mathrm{Spa}(C, O_C)} Y$ has enough facets over C .*
3. *X is a cJ-diamond.*
4. *If $X = \mathrm{Spd}(A, A^\circ)$ for a smooth and topologically of finite type C^\sharp -algebra A , then X has enough facets.*

Proof. Since the property of being topologically of finite type is stable under products and change of the base ground field one can prove easily the first two claims. The third claim follows from corollary 1.4.36 and proposition 1.4.34.

For the last claim, let $\mathbb{T}_{C^\sharp}^n$ denote $\mathrm{Spa}(C^\sharp\langle T_1^\pm, \dots, T_n^\pm \rangle, O_{C^\sharp}\langle T_1^\pm, \dots, T_n^\pm \rangle)$, and let $\tilde{\mathbb{T}}_{C^\sharp}^n = \varprojlim_{T_i \mapsto T_i^p} \mathbb{T}_{C^\sharp}^n$ analogously for \mathbb{T}_C^n and $\tilde{\mathbb{T}}_C^n$. For any point $x \in \mathrm{Spa}(A, A^\circ)$ we may find an open neighborhood U of x together with an étale map $\eta : U \rightarrow \mathbb{T}_{C^\sharp}^n$. Let \tilde{U} denote the pullback of η along $\tilde{\mathbb{T}}_{C^\sharp}^n \rightarrow \mathbb{T}_{C^\sharp}^n$, we get an étale map $\tilde{U}^\flat \rightarrow \tilde{\mathbb{T}}_C^n$. By the invariance of the étale site under perfection (see [28] lemma 15.6) $\tilde{U}^\flat = U'^\diamond$ for an adic space U' that is étale over \mathbb{T}_C^n . Now, U' admits an open cover of the form $\coprod_{i \in I} \mathrm{Spa}(A_i, A_i^\circ) \rightarrow U'$ with each A_i topologically of finite type over C . This gives a cover,

$$\coprod_{i \in I} \mathrm{Spd}(A_i, A_i^\circ) \rightarrow \tilde{U}^\flat \rightarrow U'^\diamond.$$

\square

We now define rich kimberlites, which are some of the kimberlites that will satisfy the third condition we discussed above.

Definition 1.4.40. *Let \mathcal{F} be a prekimberlite and $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ a smelted kimberlite.*

1. *We say that \mathcal{K} is rich if the following conditions hold:*

- \mathcal{D} is a cJ -diamond.
- $|\mathcal{F}^{\text{red}}|$ is a locally Noetherian topological space.
- The specialization map $\text{sp}_{\mathcal{K}} : |\mathcal{D}| \rightarrow |\mathcal{F}^{\text{red}}|$ is specializing and a quotient map.

2. If \mathcal{F} is a kimberlite we say it is rich if $(\mathcal{F}, \mathcal{F}^{\text{an}})$ is rich. If \mathcal{F} is a p -smelted kimberlite we say it is rich if $(\mathcal{F}, \mathcal{F}_{\eta})$ is rich.

Remark 1.4.41. To the author's knowledge, the theory of diamonds and v -sheaves doesn't have a good notion of what it means to be of "finite type". Being rich, is an ad hoc condition that is good enough for the applications that we have in mind.

The following fact about rich smelted kimberlites is a crucial property that we use later on in our applications.

Proposition 1.4.42. Let $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ be a rich smelted kimberlite and suppose that for any closed point $x \in |\mathcal{F}^{\text{red}}|$ the smelted tubular neighborhood $\widehat{\mathcal{D}}_{/x}$ is connected, then $\pi_0(\text{sp}_{\mathcal{D}}) : \pi_0(|\mathcal{D}|) \rightarrow \pi_0(|\mathcal{F}^{\text{red}}|)$ is a bijection between sets of connected components.

Proof. Let $U, V \subseteq |\mathcal{D}|$ be two non-empty closed-open subsets with $V \cup U = |\mathcal{D}|$. Since $\text{sp}_{\mathcal{D}}$ is a quotient map the map of connected components is surjective. Suppose now that $\emptyset \neq \text{sp}_{\mathcal{F}}(U) \cap \text{sp}_{\mathcal{F}}(V)$ we want to show that $U \cap V \neq \emptyset$ which implies that $|\mathcal{D}|$ and $|\mathcal{F}^{\text{red}}|$ have the same families of closed-open subsets. Since $\text{sp}_{\mathcal{D}}$ is specializing we can assume there is a closed point $x \in \text{sp}_{\mathcal{D}}(U) \cap \text{sp}_{\mathcal{D}}(V)$. Since $|\mathcal{F}^{\text{red}}|$ is locally Noetherian the closed points are open in the constructible topology. By proposition 1.4.33, $\widehat{\mathcal{D}}_{/x}$ is dense in $\text{sp}_{\mathcal{D}}^{-1}(x)$, this implies that $\text{sp}_{\mathcal{D}}^{-1}(x)$ is connected. Connectedness gives that $(\text{sp}_{\mathcal{D}}^{-1}(x) \cap U) \cap (\text{sp}_{\mathcal{D}}^{-1}(x) \cap V) \neq \emptyset$ and in particular $U \cap V \neq \emptyset$ which is what we wanted to show. \square

The following three technical lemmas will be used later on to prove that certain kimberlites are rich kimberlites.

Lemma 1.4.43. Suppose \mathcal{F} is a p -smelted kimberlite and that \mathcal{F}_{η} is partially proper over \mathbb{Q}_p^{\diamond} , then:

1. $\text{sp}_{\mathcal{F}_{\eta}} : |\mathcal{F}_{\eta}| \rightarrow |\mathcal{F}^{\text{red}}|$ is specializing.
2. If $\text{sp}_{\mathcal{F}_{\eta}}$ is surjective and $|\mathcal{F}^{\text{red}}|$ is a locally Noetherian topological space then it is also a quotient map.

Proof. Take a point $r \in |\mathcal{F}_{\eta}|$ mapping to $x \in |\mathcal{F}^{\text{red}}|$ and take $y \in |\mathcal{F}^{\text{red}}|$ specializing from x . We need to find q specializing from r that maps to y . Suppose r is represented by a map $f_r : \text{Spa}(C, C^+) \rightarrow \mathcal{F}$ and suppose that \mathcal{F} formalizes f_r . Let $K = O_C/C^{\circ\circ}$ and $K^+ = C^+/C^{\circ\circ}$, then x is the image of the maximal ideal of K^+ under the map $f_x : \text{Spec}(K^+) \rightarrow \mathcal{F}^{\text{red}}$. Consider the local ring R , constructed from \mathcal{F}^{red} by taking the reduced subscheme whose underlying topological spaces is the intersection of the closure of x and the localization at y . We let $k = K^+/\mathfrak{m}_{K^+}$, and so we have $R \subseteq k$. By ([31] Tag 00IA), we have a valuation subring $R \subseteq V \subseteq k$ such that $\text{Frac}(V) = k$ and V dominates R . This induces a valuation subring $K'^+ \subseteq K^+$ and a map $f_y : \text{Spec}(K'^+) \rightarrow \mathcal{F}^{\text{red}}$ whose closed point maps to y . In turn, this induces a valuation subring $C'^+ \subseteq C^+$ with $C'^+/C^{\circ\circ} = K'^+$ by lemma 1.4.4. Since \mathcal{F}_{η} is partially proper, we get a map $f_q : \text{Spa}(C, C'^+) \rightarrow \mathcal{F}_{\eta}$ extending f_r . Separatedness of \mathcal{F}^{red} can be used to prove that the point $q = [f_q] \in |\mathcal{F}_{\eta}|$ maps to y .

For the second claim, we first prove the case in which $|\mathcal{F}^{\text{red}}|$ is irreducible. Let g be the generic point of $|\mathcal{F}^{\text{red}}|$, and take a rank 1 point in $r \in |\mathcal{F}_{\eta}|$ mapping to g . Take a map $f_r : \text{Spa}(C, O_C) \rightarrow \mathcal{F}_{\eta}$ representing r , and let C^{min} be the minimal integrally closed subring of C containing \mathbb{Z}_p and $C^{\circ\circ}$, this is the minimal ring of integral elements for C . By partial properness we get a map $\text{Spa}(C, C^{\text{min}}) \rightarrow \mathcal{F}_{\eta}$

whose image consists of the set of specializations of x in $|\mathcal{F}_\eta|$. The composition of the map $f^{\min} : |\mathrm{Spa}(C, C^{\min})| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ is specializing, surjective and a spectral map of spectral spaces (surjectivity of this map proves that $|\mathcal{F}^{\mathrm{red}}|$ is also spectral instead of just locally spectral). By corollary 1.1.23 f^{\min} is a closed map and consequently a quotient map of topological spaces.

The case in which $|\mathcal{F}^{\mathrm{red}}|$ has a finite number of irreducible components is analogous. For the general case, it is enough to prove locally on $|\mathcal{F}^{\mathrm{red}}|$ (for the Zariski topology) that $\mathrm{sp}_{\mathcal{F}_\eta}$ is a quotient map. By assumption around each point $x \in |\mathcal{F}^{\mathrm{red}}|$ there is an open neighborhood $U_x \subseteq |\mathcal{F}^{\mathrm{red}}|$ for which $|U_x|$ is a Noetherian topological space. In particular, U_x has a finite number of irreducible components and the closure $\overline{U}_x \subseteq |\mathcal{F}^{\mathrm{red}}|$ also has a finite number of irreducible components. Let $T = \mathrm{sp}_{\mathcal{F}_\eta}^{-1}(\overline{U}_x)$, this set is closed and consequently stable under specialization. If we lift the generic points of \overline{U}_x to T we can argue as above to prove that the map $\mathrm{sp}_{\mathcal{F}_\eta} : T \rightarrow \overline{U}_x$ is a quotient map. This finishes the proof. \square

Lemma 1.4.44. *Let C be a characteristic zero non-Archimedean algebraically closed field, and let $k = O_C/\mathfrak{m}_C$. Let \mathcal{F} be a p -smelted kimberlite over $\mathrm{Spd}(O_C, O_C)$. Suppose that for every algebraically closed non-Archimedean field extension C'/C the basechange $\mathcal{F}_{O_{C'}} = \mathcal{F} \times_{\mathrm{Spd}(O_C, O_C)} \mathrm{Spd}(O_{C'}, O_{C'})$ satisfies that for closed point $x \in |\mathcal{F}_{O_{C'}}^{\mathrm{red}}|$ the p -adic tubular neighborhood $(\widehat{\mathcal{F}_{O_{C'}}}_{/x})_\eta$ is non-empty. Then $\mathrm{sp}_{\mathcal{F}_\eta}$ is a surjection.*

Proof. Given a point $x \in |\mathcal{F}^{\mathrm{red}}|$ we can find a field extension of perfect fields K/k for which $\mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K)$ has a section $y : \mathrm{Spec}(K) \rightarrow \mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K)$ mapping to x under $\mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K) \rightarrow \mathcal{F}^{\mathrm{red}}$. Since \mathcal{F} is formally separated, $\mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K)$ is also separated and sections to the structure map define closed points. We can construct a non-Archimedean field C' with $C \subseteq C'$ and $W(k)[\frac{1}{p}] \subseteq W(K)[\frac{1}{p}] \subseteq C'$. We get a map of p -smelted kimberlites $\mathcal{F}_{O_{C'}} \rightarrow \mathcal{F}$, and in $|\mathcal{F}_{O_{C'}}^{\mathrm{red}}|$ there is a closed point y mapping to x . Any point $r \in |\mathcal{F}_{C'}|$ with $\mathrm{sp}_{\mathcal{F}_{C'}}(r) = y$ maps to a point whose image under the specialization map is x . This proves surjectivity. \square

Lemma 1.4.45. *Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a map of p -adic kimberlites over \mathbb{Z}_p^\diamond . Suppose that f is surjective, that \mathcal{F} is a rich kimberlite, that $|\mathcal{G}^{\mathrm{red}}|$ is locally Noetherian and that $|f^{\mathrm{red}}|$ is a specializing map of topological spaces. Then the following hold:*

1. \mathcal{G} is rich.
2. If \mathcal{F} has connected p -adic tubular neighborhoods and f^{red} has connected geometric fibers then \mathcal{G} has connected p -adic tubular neighborhoods.

Proof. Since the map $\mathcal{F}_\eta \rightarrow \mathcal{G}_\eta$ is surjective we have that, by proposition 1.4.34, \mathcal{G}_η is a cJ-diamond. Since we assumed the kimberlites to be p -adic the map $\mathcal{F}^{\mathrm{red}} \rightarrow \mathcal{G}^{\mathrm{red}}$ is surjective, $|\mathcal{F}^{\mathrm{red}}| \rightarrow |\mathcal{G}^{\mathrm{red}}|$ is a quotient map by proposition 1.3.15 and a specializing map by hypothesis. Since we assumed that $|\mathcal{G}^{\mathrm{red}}|$ is locally Noetherian we only need to prove that $\mathrm{sp}_{\mathcal{G}}$ is specializing and a quotient map. Observe that the composition $|f^{\mathrm{red}}| \circ \mathrm{sp}_{\mathcal{F}}$ is specializing and quotient map, from which we can conclude.

For the claim on p -adic tubular neighborhoods pick a closed point $x \in |\mathcal{G}^{\mathrm{red}}|$, by proposition 1.4.26 $(\widehat{\mathcal{G}}_{/x})_\eta \times_{\mathcal{G}} \mathcal{F} = (\widehat{\mathcal{F}}_{/S})_\eta$ with $S = |f^{\mathrm{red}}|^{-1}(x)$. One can easily deduce from the hypothesis on geometric fibers that S is connected which implies by propositions 1.4.42 and 1.4.29 that $(\widehat{\mathcal{F}}_{/S})_\eta$ is also connected. Since f is surjective $(\widehat{\mathcal{G}}_{/x})_\eta$ is also connected. \square

Chapter 2

Specialization maps for some moduli problems

2.1 G -torsors, lattices and shtukas

In this section we recall the theory of vector bundles over the Fargues-Fontaine curve, and point to the technical statements that allow us to discuss the specialization map for the p -adic Beilinson-Drinfeld Grassmanians and moduli spaces of mixed-characteristic shtukas. Nothing in this section is essentially new and it is all written in some form in ([30], [19], [9], [1]). Nevertheless, we need specific formulations for some of these results that are not explicit in the literature. For the convenience of the reader, we justify how our formulations follow from other (harder) statements that can be explicitly found in the literature.

For the rest of this Chapter we let k be an auxiliary perfect field in characteristic p and we let \mathcal{G} be a parahoric group scheme over $\mathrm{Spec}(W(k))$ with reductive generic fiber G . Depending on the context, we will introduce more notation and add restrictive hypothesis on what \mathcal{G} and k are allowed to be. We will often times abbreviate $\mathrm{Spd}(W(k), W(k))$ by $W(k)^\diamond$, and $\mathrm{Spec}(k)^\diamond$ by k^\diamond .

2.1.1 Vector bundles, torsors and meromorphicity

We give a quick review of the theory of vector bundles for adic and perfectoid spaces. Given an analytic Huber pair (A, A^+) , and an A module M we can define \tilde{M} as a presheaf on the open sets of $\mathrm{Spa}(A, A^+)$ defined as $\tilde{M}(U) = \varprojlim_{\mathrm{Spa}(B, B^+) \subseteq U} M \otimes_A B$ running over all the rational subsets $\mathrm{Spa}(B, B^+) \subseteq \mathrm{Spa}(A, A^+)$, and where $M \otimes_A B$ refers to the usual tensor product of A -modules ignoring the topology (See [17] 1.3.2). Kedlaya and Liu prove that whenever (A, A^+) is sheafy and M is a finite projective A -module \tilde{M} is an acyclic sheaf.

Definition 2.1.1. *Given an adic space X , a vector bundle of rank n over X is a sheaf of \mathcal{O}_X -modules which is locally isomorphic to $\tilde{M}(U_i)$ for some affinoid open cover $U_i = \mathrm{Spa}(A_i, A_i^+)$ and rank n projective modules $M(U_i)$ over A_i .*

In what follows we will need to work with categories of vector bundles over adic spaces and over schemes at the same time. The following important result of Kedlaya and Liu makes the bridge between these categories:

Theorem 2.1.2. (See [17] 1.4.2) *Let $X = \mathrm{Spa}(A, A^+)$ be an analytic affinoid adic space, suppose that*

A is sheafy. The functor

$$H^0(\mathrm{Spa}(A, A^+), -) : \mathrm{Vec}_{\mathrm{Spa}(A, A^+)} \rightarrow \mathrm{Vec}_{\mathrm{Spec}(A)}$$

from the category of vector bundles over $\mathrm{Spa}(A, A^+)$ to the category of finite projective A -modules is an exact equivalence of exact categories.

Remark 2.1.3. The acyclicity of \tilde{M} proves that $H^0(\mathrm{Spa}(A, A^+), -)$ is exact. The quasi-inverse $(\tilde{-})$ is also exact since $\mathrm{Tor}_i(M, B) = 0$ for M finite projective A -module and $i > 0$.

As in the theory of analytic functions on a complex variable one can introduce the notion of poles and meromorphic functions between vector bundles. We discuss how to do this:

Definition 2.1.4. (See [30] 5.3.1, 5.3.2, 5.3.7) Given a uniform analytic adic space X , and an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, we say that \mathcal{I} defines a Cartier divisor if \mathcal{I} is a line bundle over X . Let $Z \subseteq X$ denote the support of $\mathcal{O}_X/\mathcal{I}$. We say that \mathcal{I} is a closed Cartier divisor if the topologically ringed topological space equipped with valuations $(Z, \mathcal{O}_X/\mathcal{I}, |\cdot|_{x \in Z})$ is an adic space.

The data of a Cartier divisor allows us to define the notion of meromorphicity.

Proposition 2.1.5. (See [30] 5.3.4) Let X be a uniform analytic adic space and $\mathcal{I} \subseteq \mathcal{O}_X$ a Cartier divisor. Let $U = X \setminus V(\mathcal{I})$ be the complement of the Cartier divisor and denote $j : U \subseteq X$ the natural inclusion, we then have inclusions of \mathcal{O}_X -modules:

$$\mathcal{O}_X \subseteq \varinjlim \mathcal{I}^{\otimes(-n)} \subseteq j_*(\mathcal{O}_U)$$

Definition 2.1.6. Let X be a uniform analytic adic space, let \mathcal{V}_1 and \mathcal{V}_2 be two vector bundles over X and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a Cartier divisor. Let us denote by U the complement of the support of \mathcal{I} . We say that a map in $\mathrm{Hom}_U(\mathcal{V}_1, \mathcal{V}_2)$ is meromorphic along \mathcal{I} if it is in

$$H^0(X, \underline{\mathrm{Hom}}(\mathcal{V}_1, \mathcal{V}_2) \otimes (\varinjlim \mathcal{I}^{\otimes(-n)}))$$

where $\underline{\mathrm{Hom}}(\mathcal{V}_1, \mathcal{V}_2)$ denotes the internal Hom vector bundle.

Definition 2.1.7. Let X be a uniform analytic adic space over $\mathrm{Spa}(W(k), W(k))$. We define a \mathcal{G} -torsor over X to be a \otimes -exact functor from the category of algebraic representations over finite free $W(k)$ -modules, $\mathrm{Rep}(\mathcal{G})$, to the category of vector bundles over X , Vec_X .

We can then generalize the notion of meromorphicity to that of \mathcal{G} -torsors.

Definition 2.1.8. With X as in definition 2.1.6 and definition 2.1.7, we will say that a morphism, $f : \mathcal{T}_1|_U \rightarrow \mathcal{T}_2|_U$, of \mathcal{G} -torsors over X , is meromorphic along \mathcal{I} if for all representation $\pi \in \mathrm{Rep}(\mathcal{G})$ the corresponding map $f(\pi) : \mathcal{T}_1(\pi)|_U \rightarrow \mathcal{T}_2(\pi)|_U$ is meromorphic along \mathcal{I} .

We will often use the following fact.

Theorem 2.1.9. (See [30] 17.1.8) The category of vector bundles fibered over Perfd forms a stack for the v -topology.

2.1.2 Vector bundles on \mathcal{Y}

We defined \mathbb{Z}_p^\diamond as the v -sheaf parametrizing untilts. Although \mathbb{Z}_p^\diamond is not itself represented by an analytic adic space, the product $\mathbb{Z}_p^\diamond \times S$ for any $S \in \mathrm{Perf}$ can be represented by an analytic adic space. Let us recall this construction.

Definition 2.1.10. Given a perfectoid Huber pair (R, R^+) and a pseudo-uniformizer $\varpi \in R^+$, we define $\mathcal{Y}_{[0,\infty)}^{R^+}$ as $\mathrm{Spa}(W(R^+), W(R^+)) \setminus V([\varpi])$. Where $[\varpi]$ denotes a Teichmüller lift of ϖ , and where $W(R^+)$ is given the $(p, [\varpi])$ -adic topology. We also define \mathcal{Y}_{R^+} as $\mathrm{Spa}(W(R^+), W(R^+)) \setminus V(p, [\varpi])$.

Proposition 2.1.11. (See [18] 3.6, [30] 11.2.1) For any perfectoid Huber pair (R, R^+) the space \mathcal{Y}_{R^+} has a cover by sheafy Huber pairs. Consequently, \mathcal{Y}_{R^+} and $\mathcal{Y}_{[0,\infty)}^{R^+}$ are adic spaces. Moreover, $(\mathcal{Y}_{[0,\infty)}^{R^+})^\diamond = \mathbb{Z}_p^\diamond \times \mathrm{Spa}(R, R^+)$.

Let us review the geometry of \mathcal{Y}_{R^+} , for this fix a pseudo-uniformizer $\varpi \in R^+$. One defines a continuous map $\kappa_\varpi : |\mathcal{Y}_{R^+}| \rightarrow [0, \infty]$ characterized by the property that $\kappa(y) = r$ if and only if for any positive rational number $r \leq \frac{m}{n}$ the inequality $|p|_y^m \leq |[\varpi]|_y^n$ holds and for any positive rational number $\frac{m}{n} \leq r$ the inequality $|[\varpi]|_y^n \leq |p|_y^m$ holds.

Given an interval $I \subseteq [0, \infty]$ we denote by $\mathcal{Y}_I^{R^+}$ the open subset corresponding to the interior of $\kappa_\varpi^{-1}(I)$. For example, $\mathcal{Y}_{(0,\infty)}^{R^+}$ corresponds to the locus in \mathcal{Y}_{R^+} where $|p| \neq 0$ and $\mathcal{Y}_{[0,\infty)}^{R^+}$ corresponds to the locus where $|[\varpi]| \neq 0$. For intervals of the form $[0, \frac{h}{d}]$ where h and d are integers the space $\mathcal{Y}_{[0,\frac{h}{d}]}^{R^+}$ is represented by $\mathrm{Spa}(R', R'^+)$ corresponding to the rational localization,

$$\{x \in \mathrm{Spa}(W(R^+), W(R^+)) \mid |p^h|_x \leq |[\varpi]^d|_x \neq 0\}.$$

In this case, we can compute R'^+ explicitly as the $[\varpi]$ -adic completion of $W(R^+)[\frac{p^h}{[\varpi]^d}]$ and R' as $R'^+[\frac{1}{[\varpi]}]$. A direct computation shows that R' does not depend of R^+ . In particular, the exact category of vector bundles over $\mathcal{Y}_{[0,\infty)}^{R^+}$ does not depend of the choice of R^+ either.

We will also need to work with an algebraic version of \mathcal{Y}_{R^+} , which we will denote Y_{R^+} . This is defined as the scheme $\mathrm{Spec}(W(R^+)) \setminus V(p, [\varpi])$. Since $W(R^+) \subseteq \mathcal{O}_{\mathcal{Y}_{R^+}}$ and since $p, [\varpi]$, do not vanish simultaneously on \mathcal{Y}_{R^+} we get a map of locally ringed spaces $f : \mathcal{Y}_{R^+} \rightarrow Y_{R^+} \subseteq \mathrm{Spec}(W(R^+))$.

Recall that given an untilt R^\sharp of R there is a canonical surjection $W(R^+) \rightarrow R^{\sharp+}$ whose kernel is generated by an element $\xi \in W(R^+)$ primitive of degree 1 (See [30] 6.2.8). The element ξ defines a closed Cartier divisor over \mathcal{Y}_{R^+} and also defines a Cartier divisor on the scheme Y_{R^+} . In what follows, we compare the categories of vector bundles over $\mathrm{Spec}(W(R^+))$, Y_{R^+} and \mathcal{Y}_{R^+} with morphisms being functions that are meromorphic along the ideal (ξ) .

Recall GAGA-type theorem of Kedlaya and Liu:

Theorem 2.1.12. (See [18] 3.8) Suppose (R, R^+) is a perfectoid Huber pair in characteristic p . The natural morphisms of locally ringed spaces $f : \mathcal{Y}_{R^+} \rightarrow Y_{R^+}$ gives, via the pullback functor $f^* : \mathrm{Vec}_{Y_{R^+}} \rightarrow \mathrm{Vec}_{\mathcal{Y}_{R^+}}$, an exact equivalence of exact categories.

Remark 2.1.13. Although the reference does not explicitly claim that this equivalence is exact, one can simply follow the proof loc. cit. exchanging the word “equivalence” by “exact equivalence” since every arrow involved in the proof is an exact functor.

Corollary 2.1.14. With the notation as above, the pullback f^* induces an equivalence

$$f^* : (\mathrm{Vec}_{Y_{R^+}^{\xi \neq 0}})^{\mathrm{mer}} \rightarrow (\mathrm{Vec}_{\mathcal{Y}_{R^+}^{\xi \neq 0}})^{\mathrm{mer}}$$

between the category whose objects are vector bundles over \mathcal{Y}_{R^+} (respectively vector bundles over Y_{R^+}) and morphisms are functions meromorphic along the ideal (ξ) (respectively functions over $Y_{R^+} \setminus V(\xi)$).

Proof. By theorem 2.1.12 it is enough to prove that f^* is fully-faithful. Using internal $\underline{\mathrm{Hom}}$ we can reduce to proving $H^0(\mathcal{Y}_{R^+}, f^*\mathcal{V})_{\xi \neq 0}^{\mathrm{mer}} = H^0(Y_{R^+}^{\xi \neq 0}, \mathcal{V})$. For quasi-compact, quasi-separated schemes the global sections of a quasi-coherent sheaf after localizing by a global section of the structure sheaf is given simply by localization. That is $H^0(Y_{R^+}^{\xi \neq 0}, \mathcal{V}) = H^0(Y_{R^+}, \mathcal{V})[\frac{1}{\xi}]$. On the other hand, by definition

$H^0(\mathcal{Y}_{R^+}, f^*\mathcal{V})_{\xi \neq 0}^{mer} = H^0(\mathcal{Y}_{R^+}, f^*\mathcal{V} \otimes \varinjlim (\xi)^{\otimes(-n)})$. Now the ideal sheaf (ξ) is isomorphic to $\mathcal{O}_{\mathcal{Y}_{R^+}}$ since it is a principal Cartier divisor so we can view $f^*\mathcal{V} \otimes \varinjlim (\xi)$ as:

$$f^*\mathcal{V} \xrightarrow{\xi} f^*\mathcal{V} \xrightarrow{\xi} \dots$$

And since H^0 commutes with filtered colimits we get precisely $H^0(\mathcal{Y}_{R^+}, f^*\mathcal{V})[\frac{1}{\xi}]$. \square

Since we defined \mathcal{G} -torsors Tannakianly these statements immediately generalize to those for \mathcal{G} -torsors. Kedlaya proves another important statement.

Theorem 2.1.15. (See [18] 2.3, 2.7, 3.11) *With notation as above, and letting j be the open embedding, $j : Y_{R^+} \rightarrow \text{Spec}(W(R^+))$ the following statements hold:*

1. *The pullback functor $j^* : \text{Vec}_{\text{Spec}(W(R^+))} \rightarrow \text{Vec}_{Y_{R^+}}$ is fully-faithful.*
2. *If R^+ is a valuation ring then j^* is an equivalence.*
3. *Taking categories of quasi-coherent sheaves the adjunction morphism $j^*j_*\mathcal{V} \rightarrow \mathcal{V}$ is an isomorphism.*

Remark 2.1.16. *One may think that the third statement together with the first statement of theorem 2.1.15 above would give an equivalence of categories of vector bundles for any ring R^+ . This is not the case because even if \mathcal{V} is a vector bundle, $j_*\mathcal{V}$ might not be a vector bundle over $\text{Spec}(W(R^+))$.*

We will need a small modification of theorem 2.1.15.

For this we recall a few facts about topological modules on a Tate ring, this material is taken from ([30] 14.2.3). Let A be a complete Tate ring, f a topological nilpotent unit, A_0 a ring of definition, and N a projective module over A . One can endow N with its canonical topology (See [17] 1.1.11).

1. An A_0 -submodule $M \subseteq N$ is open if and only if $M[\frac{1}{f}] = N$.
2. An A_0 -submodule $M \subseteq N$ is bounded if and only if M is contained in a finitely generated A_0 -submodule.
3. If $A \subseteq B$ with the subspace topology, B is Tate and complete for its topology, and $S \subseteq N \otimes_A B$ is bounded, then $S \cap N \subseteq N$ is also bounded.

The following statement is implicitly used and proved in ([30] 25.1.2).

Proposition 2.1.17. *Let $\text{Spa}(R, R^+)$ be the product of points constructed from the family $\{(C_i, C_i^+), \varpi_i\}_{i \in I}$ as in definition 1.1.5. The pullback functor $j^* : \text{Vec}_{\text{Spec}(W(R^+))} \rightarrow Y_{R^+}$ gives an equivalence of categories of vector bundles with fixed rank.*

Proof. We already have a fully-faithful embedding by theorem 2.1.15, so it is enough to prove it is essentially surjective. Let \mathcal{V} be a vector bundle over Y_{R^+} of constant rank n , we let $M' = H^0(Y_{R^+}, \mathcal{V})$ which is a $W(R^+)$ -module whose pullback to Y_{R^+} identifies with \mathcal{V} by theorem 2.1.15, we want to prove that M' is a projective module. Let $N = M' \otimes_{W(R^+)} W(R^+)[\frac{1}{p}]$, this module is projective since N is the pullback of \mathcal{V} to $\text{Spec}(W(R^+)[\frac{1}{p}])$.

Define M_i as $H^0(Y_{C_i^+}, \iota_i^*\mathcal{V})$ where $\iota_i : Y_{C_i^+} \rightarrow Y_{R^+}$ is the closed embedding produced by the idempotent $1_i \in W(R^+) = \prod_{i \in I} W(C_i^+)$. For each i , this is a free $W(C_i^+)$ -module by theorem 2.1.15 and because $W(C_i^+)$ is a local ring. We define $M = \prod_{i \in I} M_i$ which is a free $W(R^+)$ -module of constant rank n . Since we have maps of $W(R^+)$ modules

$$M' = H^0(Y_{R^+}, \mathcal{V}) \rightarrow H^0(Y_{C_i^+}, \iota_i^*\mathcal{V}) = M_i$$

We get a map $M' \rightarrow M$. This map is injective since the family $Y_{C_i^+}$ is dense in the Zariski topology of Y_{R^+} . We want to prove that this map is an isomorphism.

As a first step we prove that the map induces an isomorphism $M'[\frac{1}{p}] \rightarrow M[\frac{1}{p}]$. For this we will consider $W(R^+)[\frac{1}{p}]$ as a Tate ring with its p -adic topology, and $W(R^+)$ as a ring of definition. In this context $M' \subseteq N$ is an open subset when N is given its canonical topology. This follows from property 1, since by construction $N = M'[\frac{1}{p}]$. The map of schemes $\text{Spec}(W(R)) \rightarrow \text{Spec}(W(R^+))$ factors through Y_{R^+} . This implies that $M' \otimes_{W(R^+)} W(R)$ is a finite projective module over $W(R)$. The usual map realizes $W(R^+)[\frac{1}{p}]$ as a topological subring of $W(R)[\frac{1}{p}]$. Moreover, $M' \otimes_{W(R^+)} W(R)$ is a bounded subset of $N \otimes_{W(R^+)[\frac{1}{p}]} W(R)[\frac{1}{p}]$ by property 2. On the other hand, property 3 readily implies M' is a bounded subset of N .

We construct an injection $M \subseteq N$ as $W(R^+)$ -modules. Consider $N_i = 1_i N$ as a $W(C_i^+)[\frac{1}{p}]$ -module but also as a subset of N . We have an injection $N \subseteq \prod_{i \in I} N_i$ and an element $(n_i)_{i \in I}$ is in the image of N if and only if the set $S = \{n_i\}_{i \in I} \subseteq N$ is bounded in N . There is a clear injection $M = \prod_{i \in I} M_i \rightarrow \prod_{i \in I} N_i$ and we claim that it factors through N . To prove the claim observe that if 1_i^c denotes the complementary idempotent of 1_i then $1_i \cdot M' = M'[\frac{1}{1_i^c}]$. Since taking global sections commutes with localization on qcqs schemes, we have that $M_i = 1_i \cdot M'$. Then the image of any element $m \in M$ in $\prod_{i \in I} N_i$ has the form $(m_i)_{i \in I}$ with $m_i \in 1_i \cdot M'$. Since M' is bounded in N , the set $\prod_{i \in I} M_i$ is bounded and the map $M \rightarrow \prod_{i \in I} N_i$ defines an embedding into N . We have $M' \subseteq M \subseteq N$ and in particular $M'[\frac{1}{p}] = M[\frac{1}{p}]$, which finishes the first step.

We define \mathcal{V}_2 to be j^*M , which is a vector bundle over Y_{R^+} . The situation is as follows, we have a morphism of vector bundles $\mathcal{V}_1 \rightarrow \mathcal{V}_2$ over Y_{R^+} with \mathcal{V}_2 a trivial vector bundle, that becomes an isomorphism over $Y_{C_i^+}$ for every $i \in I$ and also becomes an isomorphism over $\text{Spec}(W(R^+)[\frac{1}{p}]) \subseteq Y_{R^+}$. We prove that it is already an isomorphism over Y_{R^+} . After taking determinant bundles and fixing a trivialization we get a map $\wedge \mathcal{V}_1 \rightarrow \mathcal{O}_X$, and it is enough to prove this one is an isomorphism.

Upon applying Beauville-Laszlo lemma (See [30] 5.2.9) to $p \in W(R^+)[\frac{1}{\varpi}]$ the morphism $\wedge \mathcal{V}_1 \rightarrow \mathcal{O}_X$ produces for us a family of lattices over $W(R) = \widehat{(W(R^+)[\frac{1}{\varpi}])}_p$ parametrized by $\text{Spec}(R)$. This is the same as a morphism of schemes $\text{Spec}(R) \rightarrow \text{Gr}_{\mathcal{W}}^{\mathbb{G}_m}$ to the 1-dimensional Witt-vector Grassmanian (See [5] 8.1). Pullback of the map $\mathcal{V}_1 \rightarrow \mathcal{O}_X$ to $W(C_i)$ gives a lattice corresponding to the composition $\text{Spec}(C_i) \rightarrow \text{Spec}(R) \rightarrow \text{Gr}_{\mathcal{W}}^{\mathbb{G}_m}$, but for each $i \in I$ the restriction of the morphism $\mathcal{V}_1 \rightarrow \mathcal{O}_X$ to $W(C_i)$ is an isomorphism. In particular, we get the following commutative diagram,

$$\begin{array}{ccc} \prod_{i \in I} \text{Spec}(C_i) & \longrightarrow & \text{Spec}(R) \\ \downarrow & & \downarrow \\ \text{Spec}(\mathbb{F}_p) & \xrightarrow{e} & \text{Gr}_{\mathcal{W}}^{\mathbb{G}_m} \end{array}$$

where the map $e : \text{Spec}(\mathbb{F}_p) \rightarrow \text{Gr}_{\mathcal{W}}^{\mathbb{G}_m}$ is the one associated to the identity of \mathcal{O}_X . The image of $|\prod_{i \in I} \text{Spec}(C_i)|$ in $|\text{Spec}(R)|$ is dense since the map of rings $R \rightarrow \prod_{i \in I} C_i$ is injective. But $\text{Gr}_{\mathcal{W}}^{\mathbb{G}_m}$ is representable by a discrete disjoint union of points of the form $\text{Spec}(\mathbb{F}_p)$. So the map $\text{Spec}(R) \rightarrow \text{Gr}_{\mathcal{W}}^{\mathbb{G}_m}$ factors through the identity section which finishes the proof. \square

Given $\xi \in W(R^+)$ primitive of degree 1 as before, observe that since both $\text{Spec}(W(R^+))$ and Y_{R^+} are qcqs schemes the equivalence of vector bundles of proposition 2.1.17 generalizes to the categories where the objects are the same, but morphism are allowed to have poles along ξ on both categories.

Interestingly, extending \mathcal{G} -torsors from Y_{R^+} to $\text{Spec}(W(R^+))$ adds yet another layer of complexity. Indeed, the equivalences of theorem 2.1.15 and proposition 2.1.17 are not exact equivalences, so Tan-

nakian formalism can't be used directly. As a matter of fact, only the pullback functor j^* is exact. J. Anschütz gives a detailed study of the problem of extending \mathcal{G} -torsors along j in [1]. We emphasize that, as we discussed in the introduction, the methods of [1] allow Anschütz to construct a point-wise specialization map for the p -adic Beilinson-Drinfeld Grassmanians attached to any group \mathcal{G} with parahoric reduction. Proposition 2.1.19 below, which is nothing but a small improvement to theorem 2.1.18 of Anschütz, is the main technical input that we will need to upgrade Anschütz map to a map of topological spaces. In the case that \mathcal{G} is reductive we will be able to say more about the specialization map.

Theorem 2.1.18. (See [1] 7.2, 7.3, 7.9, 6.5, 7.6 and [2] 11.6) *Let C be an algebraically closed non-Archimedean field over k , let $C^+ \subseteq C$ an open and bounded valuation subring with $k \subseteq C^+$, and let \mathcal{G} be a parahoric group scheme over $W(k)$. Then every \mathcal{G} -torsor \mathcal{T} over Y_{C^+} extends to $\text{Spec}(W(C^+))$.*

We now state an analogue of proposition 2.1.17 for \mathcal{G} -torsors.

Proposition 2.1.19. *Keep the notation as in theorem 2.1.18, and let $\text{Spa}(R, R^+)$ be a product of points over k . Every \mathcal{G} -torsor \mathcal{T} over Y_{R^+} extends along $j : Y_{R^+} \rightarrow \text{Spec}(W(R^+))$ to a torsor \mathcal{G} torsor over $\text{Spec}(W(R^+))$.*

Proof. We need to prove that the functor $j_*\mathcal{T} : \text{Rep}(\mathcal{G}) \rightarrow \text{Vec}_{\text{Spec}(W(R^+))}$ is exact, and since the functor j_* is always left-exact we only have to prove right-exactness of $j_*\mathcal{T}$. Suppose we have a morphism of free modules $f : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ over $\text{Spec}(W(R^+))$ and we have that the basechange to $\text{Spec}(W(C_i^+))$ is surjective for every $i \in I$, we need to prove that the morphism is surjective. Taking determinant bundles we can reduce to the case that \mathcal{V}_2 is free of rank 1. After taking trivializations we have n sections $f_1, \dots, f_n \in W(R^+)$ and we need to prove that they generate the unit ideal. Consider the family of subsets $\{I_m\}_{1 \leq m \leq n}$ defined by

$$I_m = \{i \in I \mid f_m \in W(C_i^+)^\times\}$$

By construction 1_{I_m} is in the ideal generated by the f_i . Since each $W(C_i^+)$ is a local ring and the $\{f_m\}_{1 \leq m \leq n}$ generate the unit ideal in $W(C_i^+)$ the union $\bigcup I_m$ has to be I . This finishes the proof. \square

We need the following descent result which is similar to theorem 2.1.9.

Proposition 2.1.20. (See [30] 19.5.3) *Let S be a perfectoid space over k and let $U \subseteq \mathcal{Y}_{[0, \infty)}^S$ be an open subset. For map of perfectoid spaces $f : S' \rightarrow S$, let $\mathcal{C}_{S'}$ denote the category of \mathcal{G} -torsors over $\mathcal{Y}_{[0, \infty)}^{S'} \times_{\mathcal{Y}_{[0, \infty)}^S} U$. Then the assignment $S' \mapsto \mathcal{C}_{S'}$, as a fibered category over Perfs , is a v -stack.*

2.1.3 Lattices and shtukas

For this section fix $\text{Spa}(R, R^+)$ an affinoid perfectoid space over k , $\varpi \in R^+$ a choice of pseudo-uniformizer, R^\sharp an untilt of R and ξ_{R^\sharp} a generator for the kernel of the map $W(R^+) \rightarrow R^{\sharp, +}$.

Definition 2.1.21. *We define the category of $B_{dR}^+(R^\sharp)$ -lattices with \mathcal{G} -structure to have as objects pairs (\mathcal{T}, ψ) where \mathcal{T} is a \mathcal{G} -torsor over $\mathcal{Y}_{[0, \infty)}^{R^+}$ and $\psi : \mathcal{T} \rightarrow \mathcal{G}$ is an isomorphism over $\mathcal{Y}_{[0, \infty)}^{R^+} \setminus V(\xi_{R^\sharp})$ that is meromorphic along (ξ_{R^\sharp}) . Morphisms are the evident isomorphisms of pairs.*

Given data (\mathcal{T}, ψ) as above we can choose a big enough number $r_\varpi \in \mathbb{R}$ for which $\mathcal{Y}_{[r_\varpi, \infty)}^{R^+}$ is disjoint from $V(\xi_{R^\sharp})$. Over this locus we can glue along ψ to extend \mathcal{T} canonically to a \mathcal{G} -torsor over \mathcal{Y}_{R^+} . Using Corollary 2.1.14 and Beauville-Laszlo on the scheme $\text{Spec}(W_R^+)[\frac{1}{\varpi}]$ we get an equivalence of categories with the category of pairs (Ξ, ψ) where Ξ is \mathcal{G} -torsor over $\text{Spec}(B_{dR}^+(R^\sharp))$ and $\psi : \Xi \rightarrow \mathcal{G}$ is a trivialization over $\text{Spec}(B_{dR}(R^\sharp))$, where $B_{dR}^+(R^\sharp)$ denotes the completion of $W(R^+)[\frac{1}{\varpi}]$ along ξ_{R^\sharp} , and $B_{dR}(R^\sharp) = B_{dR}^+(R^\sharp)[\frac{1}{\xi_{R^\sharp}}]$.

Recall that for an algebraically closed non-Archimedean field C the ring $B_{dR}(C^\sharp)$ is a complete discrete valuation field so that the set of isomorphism classes of \mathcal{G} -torsors over $\mathrm{Spec}(B_{dR}^+(C^\sharp))$ is in canonical bijection with $\mathcal{G}(B_{dR}(C^\sharp))/\mathcal{G}(B_{dR}^+(C^\sharp))$. In case $\frac{1}{p} \in C^\sharp$ we also have that $\frac{1}{p} \in B_{dR}^+(C^\sharp)$, and we will find that $\mathcal{G}_{B_{dR}^+(C^\sharp)} = G_{B_{dR}^+(C^\sharp)}$ is split reductive. After fixing auxiliary groups $T \subseteq B \subseteq G_{B_{dR}^+(C^\sharp)}$, a maximal torus and a Borel respectively, the Cartan decomposition gives an identification:

$$\mathcal{G}(B_{dR}^+(C^\sharp)) \backslash \mathcal{G}(B_{dR}(C^\sharp)) / \mathcal{G}(B_{dR}^+(C^\sharp)) = G(B_{dR}^+(C^\sharp)) \backslash G(B_{dR}(C^\sharp)) / G(B_{dR}^+(C^\sharp)) = X_*^+(T) \quad (2.1)$$

Suppose that B and T are fixed and understood from the context, and let $\mu \in X_*^+(T)$. We say that a $B_{dR}^+(C^\sharp)$ -lattice (Ξ, ψ) is of type μ if the isomorphism class of (Ξ, ψ) maps to μ under the identification above.

We now consider mixed-characteristic shtukas. Recall that the spaces $\mathrm{Spec}(W(R^+))$, $\mathcal{Y}_{[0,\infty)}^{R^+}$, Y_{R^+} and \mathcal{Y}_{R^+} come equipped with a Frobenious action. All of these actions are coming from the usual Frobenious action on $W(R^+)$ given by:

$$\phi^* \left(\sum_{i=0}^{\infty} [\alpha_i] p^i \right) = \sum_{i=0}^{\infty} [\alpha_i^p] p^i$$

A computation shows that $\phi(\mathcal{Y}_{[a,b]}^{R^+}) = \mathcal{Y}_{[pa,pb]}^{R^+}$ which proves that all of the loci considered above are preserved by Frobenious.

Definition 2.1.22. We define the category of shtukas with one paw over $\mathrm{Spa}(R^\sharp, R^{\sharp+})$ and \mathcal{G} -structure. For this we require that $k = \mathbb{F}_p$ and that \mathcal{G} is defined over \mathbb{Z}_p . This category has as objects pairs (\mathcal{T}, Φ) where \mathcal{T} is a \mathcal{G} -torsor over $\mathcal{Y}_{[0,\infty)}^{R^+}$ and $\Phi : \phi^* \mathcal{T} \rightarrow \mathcal{T}$ is an isomorphism over $\mathcal{Y}_{[0,\infty)}^{R^+} \setminus V(\xi_{R^\sharp})$ meromorphic along (ξ_{R^\sharp}) . Morphisms are the evident isomorphisms of pairs.

Definition 2.1.23. Given a ϕ -module with \mathcal{G} -structure $(\mathcal{E}, \Phi_{\mathcal{E}})$ over $\mathcal{Y}_{(0,\infty)}^{R^+}$ and a shtuka $(\mathcal{T}, \Phi_{\mathcal{T}})$ we say that $(\mathcal{T}, \Phi_{\mathcal{T}})$ is isogenous to $(\mathcal{E}, \Phi_{\mathcal{E}})$ if there is a number $r \in \mathbb{R}$ (that depends of the choice of ϖ) and a ϕ -equivariant isomorphism $f : (\mathcal{T}, \Phi_{\mathcal{T}}) \rightarrow (\mathcal{E}, \Phi_{\mathcal{E}})$ defined over $\mathcal{Y}_{[r,\infty)}^{R^+}$. We call such a pair (r, f) an isogeny. Two isogenies (r_1, f_1) and (r_2, f_2) are equivalent if there is a third isogeny (r_3, f_3) with $r_3 > r_1, r_2$ and $f_1 = f_3 = f_2$ when restricted to $\mathcal{Y}_{[r_3,\infty)}^{R^+}$. We also refer by isogenies to the elements of the set of equivalence classes of pairs (r, f) .

After the work of Scholze and Weinstein one may think of mixed-characteristic shtukas as a generalization of p -divisible groups (See [30] 14.11, [29] Theorem B). We do not make this precise, but isogenies as defined above are closely related with isogenies of p -divisible groups. In what follows, we prove three technical lemmas that intuitively speaking allow us to “deform” lattices and shtukas with \mathcal{G} -structure. Later on it will become clear why we think of these lemmas as “deformation” statements.

For any $r \in [0, \infty)$ let $B_{[r,\infty)}^{R^+} = H^0(\mathcal{Y}_{[r,\infty)}^{R^+}, \mathcal{O}_{\mathcal{Y}_{[r,\infty)}^{R^+}})$, and consider the ring $R_{\mathrm{red}}^+ = (R^+/\varpi)^{\mathrm{perf}}$. We observe that the universal property of $\mathcal{Y}_{[r,\infty)}^{R^+}$ as a rational subset of $\mathrm{Spa}(W(R^+), W(R^+))$ induces compatible maps of rings $B_{[r,\infty)}^{R^+} \rightarrow W(R_{\mathrm{red}}^+)[\frac{1}{p}]$ for varying r . We denote this family of reduction maps by $(-_{\mathrm{red}})$.

Lemma 2.1.24. Let $s \in B_{[r,\infty)}^{R^+}$ and suppose that the reduction s_{red} , originally defined over $W(R_{\mathrm{red}}^+)[\frac{1}{p}]$, lies in $W(R_{\mathrm{red}}^+)$, then there are: a number r' with $r \leq r'$, elements $a \in W(R^+)$, $b \in B_{[r',\infty)}^{R^+}$ and a pseudo-uniformizer $\varpi_s \in R^+$ such that $s = a + b$ and $b \in [\varpi_s] \cdot B_{[r',\infty)}^{R^+}$.

Proof. By enlarging r if necessary we can assume $\mathcal{Y}_{[r,\infty)}^{R^+}$ is of the form:

$$\{x \in \mathrm{Spa}(W(R^+), W(R^+)) \mid ||\varpi||_x \leq |p^m|_x \neq 0\}$$

for some m , we compute $B_{[r,\infty]}^{R^+}$ explicitly. If S^+ denotes the p -adic completion of $W(R^+)[\frac{[\varpi]}{p^m}]$, then $B_{[r,\infty]}^{R^+} = S^+[\frac{1}{p}]$. Any element $s \in B_{[r,\infty]}^{R^+}$ is of the form $s = \frac{1}{p^n} \cdot \sum_{i=0}^{\infty} [a_i] x^{m(i)} p^i$ where $a_i \in R^+$, $x = \frac{[\varpi]}{p^m}$, and $m(i)$ denotes a non-negative integer. We can decompose $p^n \cdot s$ as

$$x \cdot \left(\sum_{i=0, m(i)>0}^{\infty} [a_i] x^{m(i)-1} p^i \right) + \sum_{i=0, m(i)=0}^{\infty} [a_i] p^i.$$

Since $x = \frac{[\varpi]}{p^m}$, we have that $[\varpi]$ divides in $B_{[r,\infty]}^{R^+}$ the first term of this decomposition. As long as we pick a ϖ_s that divides ϖ , we may and do reduce to the case $s = \sum_{i=0}^{\infty} [a_i] p^{i-n}$. In this case, $s_{\text{red}} = \sum_{i=0}^{\infty} [(a_i)_{\text{red}}] p^{i-n}$ and by hypothesis we have that for $i < n$ $(a_i)_{\text{red}} = 0$ in R_{red}^+ . We can choose a pseudo-uniformizer ϖ_s for which all of a_i , for $i < n$, are zero in R^+/ϖ_s . We can take $a = \sum_{i=n}^{\infty} [a_i] p^{i-n}$ and $b = \sum_{i=0}^{n-1} [a_i] p^{i-n}$. These clearly satisfy the properties. \square

Lemma 2.1.25. *Suppose that \mathcal{T}_1 and \mathcal{T}_2 are trivial \mathcal{G} -torsor over $\text{Spec}(W(R^+))$ and that $\lambda : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ is an isomorphism defined over $\mathcal{Y}_{[r,\infty]}^{R^+}$ which upon reduction to $\text{Spec}(W(R_{\text{red}}^+)[\frac{1}{p}])$ extends to an isomorphism over $\text{Spec}(W(R_{\text{red}}^+))$. Then, there is an isomorphism $\tilde{\lambda} : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ defined over $\text{Spec}(W(R^+))$, a pseudo-uniformizer $\varpi_\lambda \in R^+$ and a number $r' \in \mathbb{R}$ with $r \leq r'$ such that $\lambda = \tilde{\lambda}$ in:*

$$\text{Hom}_{\text{Spec}(B_{[r',\infty]}^{R^+}/[\varpi_\lambda])}(\mathcal{T}_1, \mathcal{T}_2)$$

Proof. Fix for the rest of the proof trivializations $\iota_i : \mathcal{T}_i \rightarrow \mathcal{G}$, and consider $\iota_2 \circ \lambda \circ \iota_1^{-1}$ as an element $g \in H^0(\mathcal{Y}_{[r,\infty]}^{R^+}, \mathcal{G}) \subseteq H^0(\mathcal{Y}_{[r,\infty]}^{R^+}, GL_n)$ for some n and some embedding $\mathcal{G} \rightarrow GL_n$ defined over $W(k)$. By lemma 2.1.24 we can find ϖ_λ such that one can write g as $M_1 + [\varpi_\lambda]M_2$ where $M_1 \in GL_n(W(R^+))$ and $M_2 \in M_{n \times n}(B_{[r',\infty]}^{R^+})$. With this setup the reduction of M_1 to $GL_n(B_{[r',\infty]}^{R^+}/[\varpi_\lambda])$ lies in $\mathcal{G}(W(R^+)/[\varpi_\lambda])$. Moreover, since \mathcal{G} is a smooth group and $W(R^+)$ is $[\varpi_\lambda]$ -complete, we can lift this to an element $g' \in \mathcal{G}(W(R^+))$ with $g' = M_1$ in $GL_n(W(R^+)/[\varpi_\lambda])$. Consequently $g' = g$ in $\mathcal{G}(B_{[r',\infty]}^{R^+}/[\varpi_\lambda])$, and by letting $\tilde{\lambda} = \iota_2^{-1} \circ g' \circ \iota_1$ we get the desired isomorphism. \square

Remark 2.1.26. *In lemmas 2.1.25 and 2.1.24 above one can take $r = r'$ but that would extend the arguments and we will not need this.*

The proof of the following lemma is inspired by the computations that appear in [11] Theorem 5.6, and it is a key input in the proof of theorem 2.3.14.

Lemma 2.1.27 (Unique liftability of isogenies). *Let \mathcal{T} be a trivial \mathcal{G} -torsor over $\text{Spec}(W(R^+))$ and let \mathcal{G}_b denote the trivial \mathcal{G} -torsor endowed with the ϕ -module structure over $\mathcal{Y}_{(0,\infty]}^{R^+}$ given by an element $b \in \mathcal{G}(\mathcal{Y}_{(0,\infty]}^{R^+})$. Let $\Phi : \phi^* \mathcal{T} \rightarrow \mathcal{T}$ be an isomorphism defined over $\text{Spec}(W(R^+)[\frac{1}{\xi}])$ and $\lambda : \mathcal{T} \rightarrow \mathcal{G}_b$ a ϕ -equivariant isomorphism defined over $B_{[r,\infty]}^{R^+}/[\varpi]$ for some r big enough so that ξ_{R^+} becomes a unit. Then, there is a unique ϕ -equivariant isomorphism $\tilde{\lambda} : \mathcal{T} \rightarrow \mathcal{G}_b$ defined over $\mathcal{Y}_{[r,\infty]}^{R^+}$ such that $\tilde{\lambda} = \lambda$ in $B_{[r,\infty]}^{R^+}/[\varpi]$.*

Proof. After fixing a trivialization $\iota : \mathcal{T} \rightarrow \mathcal{G}$ we may assume, by transport of structure, that $\mathcal{G} = \mathcal{T}$, that Φ is given by an element $\mathcal{G}(W(R^+)[\frac{1}{\xi}])$, and that λ is given by an element $\mathcal{G}(B_{[r,\infty]}^{R^+}/[\varpi])$. We need to find an element $\tilde{\lambda} \in \mathcal{G}(B_{[r,\infty]}^{R^+})$ reducing to λ and satisfying $\Phi = \tilde{\lambda}^{-1} \circ b \circ \phi^*(\tilde{\lambda})$. Choose an arbitrary lift $\lambda_0 \in \mathcal{G}(B_{[r,\infty]}^{R^+})$ of λ , and let $\eta_0 = \lambda_0^{-1} \circ b \circ \phi^*(\lambda_0) \circ \Phi^{-1}$. We construct a pair of sequences of maps, $\lambda_i : \mathcal{G} \rightarrow \mathcal{G}_b$ and $\eta_i : \mathcal{G} \rightarrow \mathcal{G}$ defined recursively as follows:

$$\lambda_{n+1} = \lambda_n \circ \eta_n$$

$$\eta_n = \lambda_n^{-1} \circ b \circ \phi^*(\lambda_n) \circ \Phi^{-1}$$

We make the observation that $\eta_0 = Id$ in $\mathcal{G}(B_{[r,\infty]}^{R^+}/[\varpi])$ and we prove inductively that $\eta_n = Id$ in $\mathcal{G}(B_{[r,\infty]}^{R^+}/[\varpi^{p^n}])$. If $g \in \mathcal{G}(B_{[r,\infty]}^{R^+})$ is such that $g = Id$ in $\mathcal{G}(B_{[r,\infty]}^{R^+}/[\varpi^{p^n}])$, then $\phi^*(g) = Id$ in

$$\mathcal{G}(B_{[\frac{r}{p},\infty]}^{R^+}/[\varpi^{p^{n+1}}]) \subseteq \mathcal{G}(B_{[r,\infty]}^{R^+}/[\varpi^{p^{n+1}}]).$$

The induction then follows from the computation:

$$\eta_{n+1} = \lambda_{n+1}^{-1} \circ b \circ \phi^*(\lambda_{n+1}) \circ \Phi^{-1} \quad (2.2)$$

$$= \eta_n^{-1} \circ \lambda_n^{-1} \circ b \circ \phi^*(\lambda_{n+1}) \circ \Phi^{-1} \quad (2.3)$$

$$= \Phi \circ \phi^*(\lambda_n)^{-1} \circ b^{-1} \circ \lambda_n \circ \lambda_n^{-1} \circ b \circ \phi^*(\lambda_{n+1}) \circ \Phi^{-1} \quad (2.4)$$

$$= \Phi \circ \phi^*(\lambda_n)^{-1} \circ \phi^*(\lambda_{n+1}) \circ \Phi^{-1} \quad (2.5)$$

$$= \Phi \circ \phi^*(\eta_n) \circ \Phi^{-1} \quad (2.6)$$

Since $\phi^*(\eta_n) = Id$ in $\mathcal{G}(B_{[r,\infty]}^{R^+}/[\varpi^{p^{n+1}}])$ we also have that $\eta_{n+1} = Id$ in $\mathcal{G}(B_{[r,\infty]}^{R^+}/[\varpi^{p^{n+1}}])$.

This let us conclude that η_i converges to Id in $\mathcal{G}(B_{[r,\infty]}^{R^+})$. We define $\tilde{\lambda} \in \mathcal{G}(B_{[r,\infty]}^{R^+})$ as the limit of the λ_i . Taking limits we deduce the identities $Id = \eta_\infty = \tilde{\lambda} \circ b \circ \phi^*(\tilde{\lambda}) \circ \Phi^{-1}$ and $\tilde{\lambda} = \lambda_0 = \lambda$ in $\mathcal{G}(B_{[r,\infty]}^{R^+}/[\varpi])$ as we needed to show.

Suppose that there are two lifts $\tilde{\lambda}_i$ of λ with the required properties. We get a ϕ -equivariant automorphism $\tilde{\lambda}_1 \circ \tilde{\lambda}_2^{-1}$ of \mathcal{G}_b which we may think of as an element of $g \in \mathcal{G}(B_{[r,\infty]}^{R^+})$ that reduces to the identity in $B_{[r,\infty]}^{R^+}/[\varpi]$. Now, ϕ -equivariance gives $b = g^{-1} \circ b \circ \phi^*(g)$, and since $g = Id$ in $\mathcal{G}(B_{[r,\infty]}^{R^+}/[\varpi])$ then $\phi^*(g) = Id$ in $\mathcal{G}(B_{[r,\infty]}^{R^+}/[\varpi^p])$ and we get the identity $b = g^{-1} \circ b \circ Id$ in $\mathcal{G}(B_{[r,\infty]}^{R^+}/[\varpi^p])$. We may proceed inductively to prove that $g = Id$ in $\mathcal{G}(B_{[r,\infty]}^{R^+}/[\varpi^{p^n}])$ for every n . Since $B_{[r,\infty]}^{R^+}$ is complete and separated for the $[\varpi]$ -adic topology we conclude that $g = Id$ in $\mathcal{G}(B_{[r,\infty]}^{R^+})$. \square

2.2 The specialization map for p -adic Beilinson-Drinfeld Grassmanians

2.2.1 Grassmanians as kimberlites

In the Berkeley notes, Scholze and Weinstein define a p -adic analogue of the Beilinson-Drinfeld Grassmanian where the parameter “curve” is given by \mathbb{Z}_p^\diamond , or in our case $W(k)^\diamond = \text{Spec}(k)^\diamond \times \mathbb{Z}_p^\diamond$. We will adopt the definition that is the most convenient for studying the specialization map for this object.

Definition 2.2.1. (See [30] 20.3.1) We let $\text{Gr}_{W(k)^\diamond}^{\mathcal{G}}$ denote the presheaf that assigns to an affinoid perfectoid pair (R, R^+) the set:

$$\text{Gr}_{W(k)^\diamond}^{\mathcal{G}}(R, R^+) = \{(R^\sharp, \iota, f, \mathcal{T}, \psi)\} / \cong$$

Where (R^\sharp, ι, f) is an untilt over $W(k)$ and (\mathcal{T}, ψ) is a lattice with \mathcal{G} -structure as in definition 2.1.21.

Whenever \mathcal{G} is reductive over $W(k)$ with quasi-split fibers we fix $\mathfrak{T} \subseteq \mathfrak{B} \subseteq \mathcal{G}$, integrally defined maximal torus and Borel subgroups respectively.

Definition 2.2.2. Suppose that \mathcal{G} is reductive and $\mu \in X_*^+(\mathfrak{T})$ is a dominant cocharacter with reflex field E . We define $\text{Gr}_{O_E}^{\mathcal{G}, \leq \mu}$ as the subsheaf of $\text{Gr}_{O_E}^{\mathcal{G}} := \text{Gr}_{W(k)^\diamond}^{\mathcal{G}} \times_{W(k)^\diamond} O_E^\diamond$ that on geometric points evaluates to \mathcal{G} -lattices (Ξ, ψ) whose type is bounded by μ in the Bruhat order. We use equation 2.1 to compare μ_Ξ with μ .

Recall the following theorem of the Berkeley notes.

Theorem 2.2.3. (See [30] 20.3.2, 21.2.1) For any parahoric group scheme \mathcal{G} over $W(k)$, the presheaf $\mathrm{Gr}_{W(k)^\diamond}^\mathcal{G}$ is a small v -sheaf and ind-proper over $W(k)^\diamond$. Moreover, if \mathcal{G} is reductive and $\mu \in X_*(\mathfrak{T})$ the functor $\mathrm{Gr}_{O_E}^{\mathcal{G}, \leq \mu}$ is proper and representable in spatial diamonds over O_E^\diamond . The inclusion of sheaves $\mathrm{Gr}_{O_E}^{\mathcal{G}, \leq \mu} \rightarrow \mathrm{Gr}_{O_E}^\mathcal{G}$ is a closed embedding.

Proposition 2.2.4. The v -sheaf $\mathrm{Gr}_{W(k)^\diamond}^\mathcal{G}$ formalizes products of points. In particular, it is v -formalizing.

Proof. Let $\mathrm{Spa}(R, R^+)$ be a product of points and $f : \mathrm{Spa}(R, R^+) \rightarrow \mathrm{Gr}_{W(k)^\diamond}^\mathcal{G}$ a map. By definition, associated to this map we have an untilt (R^\sharp, ι, m) over $W(k)$ and a \mathcal{G} -torsor \mathcal{T} over $\mathcal{Y}_{[0, \infty)}^{R^+}$ together with a trivialization $\psi : \mathcal{T} \rightarrow \mathcal{G}$ over $\mathcal{Y}_{[0, \infty)}^{R^+} \setminus V(\xi_{R^\sharp})$ meromorphic along ξ_{R^\sharp} . We can use ψ to glue \mathcal{T} and \mathcal{G} along $\mathcal{Y}_{[r, \infty)}^{R^+}$ (for big enough r) and get a \mathcal{G} -torsor defined over \mathcal{Y}_{R^+} , together with a meromorphic isomorphism over $\mathcal{Y}_{R^+} \setminus V(\xi_{R^\sharp})$ which restricts to the original data. Using corollary 2.1.14, proposition 2.1.19 and the fact that by construction \mathcal{T} is trivial on $Y_{R^+} \setminus V(\xi)$ we can extend \mathcal{T} to a \mathcal{G} -torsor over $\mathrm{Spec}(W(R^+))$ together with a trivialization over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi_{R^\sharp}}])$. We claim that this is enough to define a map $\mathrm{Spd}(R^+, R^+) \rightarrow \mathrm{Gr}_{W(k)^\diamond}^\mathcal{G}$. Indeed, take a second affinoid perfectoid $\mathrm{Spa}(S, S^+)$ and a map $g : \mathrm{Spa}(S, S^+) \rightarrow \mathrm{Spd}(R^+, R^+)$, we want to produce a map $\mathrm{Spa}(S, S^+) \rightarrow \mathrm{Gr}_{W(k)^\diamond}^\mathcal{G}$ in a functorial way. We may construct an untilt (S^\sharp, ι, m) as in lemma 1.4.8. The map g gives a map $g' : W(R^+) \rightarrow W(S^+)$ with $g'(\xi_{R^\sharp}) = \xi_{S^\sharp}$. Basechange along g' gives a \mathcal{G} -torsor over $\mathrm{Spec}(W(S^+))$ together with a trivialization over $\mathrm{Spec}(W(S^+)[\frac{1}{g'(\xi)}])$. This restricts to a \mathcal{G} -torsor over $\mathcal{Y}_{[0, \infty)}^{S^+}$ and a trivialization over $\mathcal{Y}_{[0, \infty)}^{S^+} \setminus V(g'(\xi))$ that is meromorphic along $g'(\xi)$. This gives our desired natural transformation $\mathrm{Spd}(R^+, R^+) \rightarrow \mathrm{Gr}_{W(k)^\diamond}^\mathcal{G}$. Clearly the composition $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(R^+, R^+) \rightarrow \mathrm{Gr}_{W(k)^\diamond}^\mathcal{G}$ agrees with f , so this map is a formalization. \square

Proposition 2.2.5. (See [30] Section 20.3) The v -sheaf $\mathrm{Gr}_{W(k)^\diamond}^\mathcal{G}$ is specializing and formally p -adic, and $(\mathrm{Gr}_{W(k)^\diamond}^\mathcal{G})^{\mathrm{red}}$ is represented by the Witt-vector Grassmanian, $\mathrm{Gr}_{\mathcal{W}, k}^\mathcal{G}$. Moreover, if \mathcal{G} is reductive and k_E denotes the residue field of O_E , then $\mathrm{Gr}_{O_E}^{\mathcal{G}, \leq \mu}$ is also formally p -adic and $(\mathrm{Gr}_{O_E}^{\mathcal{G}, \leq \mu})^{\mathrm{red}} = \mathrm{Gr}_{\mathcal{W}, k_E}^{\mathcal{G}, \leq \mu}$.

Proof. That $\mathrm{Gr}_{W(k)^\diamond}^\mathcal{G}$ is specializing would follow from proposition 2.2.4, theorem 2.2.3 and proposition 1.3.31 once we establish that it is formally p -adic. We begin by constructing a map $\mathrm{Gr}_{\mathcal{W}, k}^\mathcal{G} \rightarrow (\mathrm{Gr}_{W(k)^\diamond}^\mathcal{G})^{\mathrm{red}}$. Given a map $\mathrm{Spec}(R) \rightarrow \mathrm{Gr}_{\mathcal{W}, k}^\mathcal{G}$ we need to produce a map $\mathrm{Spec}(R)^\diamond \rightarrow \mathrm{Gr}_{W(k)^\diamond}^\mathcal{G}$ in a functorial way. The map to $\mathrm{Gr}_{\mathcal{W}, k}^\mathcal{G}$ is given by a \mathcal{G} -torsor \mathcal{T} over $\mathrm{Spec}(W(R))$ together with a trivialization $\psi : \mathcal{T} \rightarrow \mathcal{G}$ over $\mathrm{Spec}(W(R)[\frac{1}{p}])$. Given an affinoid perfectoid $\mathrm{Spa}(S, S^+)$ and a map $f : \mathrm{Spa}(S, S^+) \rightarrow \mathrm{Spec}(R)^\diamond$ we need to produce a map $\mathrm{Spa}(S, S^+) \rightarrow \mathrm{Gr}_{W(k)^\diamond}^\mathcal{G}$. The morphism f induces the ring map $f' : W(R) \rightarrow W(S^+)$. We can assign to f the characteristic p untilt and assign the \mathcal{G} -bundle $f'^*\mathcal{T}$ over $\mathcal{Y}_{[0, \infty)}^{S^+}$ with trivialization $f'^*\psi$, using corollary 2.1.14 we see that it is meromorphic along p . This construction is clearly functorial and gives the desired map.

Now, by Beauville-Laszlo theorem, we may also think of $(f'^*\mathcal{T}, f'^*\psi)$ as a pair (Ξ_S, ψ_S) with Ξ_S a \mathcal{G} -torsor over $\mathrm{Spec}(B_{dR}^+(S))$ and $\psi_S : \Xi_S \rightarrow \mathcal{G}$ a trivialization over $\mathrm{Spec}(B_{dR}(S))$. Since S is the characteristic p untilt, we have $B_{dR}^+(S) = W(S)$ and $B_{dR}(S) = W(S)[\frac{1}{p}]$. In this case, (Ξ_S, ψ_S) is simply the pullback of (\mathcal{T}, ψ) along $W(R) \rightarrow W(S)$. In particular, if \mathcal{G} is reductive and the type of (\mathcal{T}, ψ) is pointwise bounded by $\mu \in X_*^+(\mathfrak{T})$, then the type of (Ξ_S, ψ_S) is also pointwise bounded by μ . This last observation gives us a commutative diagram which we will use later in the proof:

$$\begin{array}{ccc} (\mathrm{Gr}_{\mathcal{W}, k_E}^{\mathcal{G}, \leq \mu})^\diamond & \longrightarrow & (\mathrm{Gr}_{\mathcal{W}, k_E}^\mathcal{G})^\diamond \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{O_E}^{\mathcal{G}, \leq \mu} & \longrightarrow & \mathrm{Gr}_{O_E}^\mathcal{G} \end{array} \quad (2.7)$$

For the moment, let us move on and prove explicitly that for any (R, R^+) we have bijection of sets:

$$(\mathrm{Gr}_{\mathcal{W},k}^{\mathcal{G}})^{\diamond}(R, R^+) \rightarrow \mathrm{Gr}_{W(k)^{\diamond}}^{\mathcal{G}} \times_{W(k)^{\diamond}} \mathrm{Spec}(k)^{\diamond}(R, R^+).$$

By lemma 1.3.35, this would give that $\mathrm{Gr}_{W(k)^{\diamond}}^{\mathcal{G}} \rightarrow W(k)^{\diamond}$ is formally adic and would prove $(\mathrm{Gr}_{W(k)^{\diamond}}^{\mathcal{G}})^{\mathrm{red}} = \mathrm{Gr}_{\mathcal{W},k}^{\mathcal{G}}$. To prove injectivity, suppose we are given two maps $g_i : \mathrm{Spa}(R, R^+) \rightarrow (\mathrm{Gr}_{\mathcal{W},k}^{\mathcal{G}})^{\diamond}$ in characteristic p whose composition agree. It is enough to prove that $g_1 = g_2$ after taking a v -cover of $\mathrm{Spa}(R, R^+)$. Locally for the v -topology we can assume that both maps factor through morphisms $g'_i : \mathrm{Spec}(R^+) \rightarrow \mathrm{Gr}_{\mathcal{W},k}^{\mathcal{G}}$ given by pairs (\mathcal{T}_i, ψ_i) . Since the compositions agree, these pairs become isomorphic over $\mathcal{Y}_{[0,\infty)}^{R^+}$. Since both \mathcal{T}_i are defined over $\mathrm{Spec}(W(R^+))$ and the pullback functor $j^* : \mathrm{Vec}_{\mathrm{Spec}(W(R^+))} \rightarrow \mathrm{Vec}_{Y_{R^+}}$ of theorem 2.1.15 is fully faithful (even when it is not an equivalence), we can conclude that $g'_1 = g'_2$.

To prove surjectivity take a map $f : \mathrm{Spa}(R, R^+) \rightarrow \mathrm{Gr}_{W(k)^{\diamond}}^{\mathcal{G}} \times_{W(k)^{\diamond}} \mathrm{Spec}(k)^{\diamond}$. Since surjectivity can be checked v -locally we can assume that $\mathrm{Spa}(R, R^+)$ is a product of points. By the proof of proposition 2.2.4 we get a \mathcal{G} -torsor \mathcal{T} over $\mathrm{Spec}(W(R^+))$ and a trivialization over $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$ which gives a map $\mathrm{Spec}(R^+) \rightarrow \mathrm{Gr}_{\mathcal{W},k}^{\mathcal{G}}$ and consequently the required lift to our original map $\mathrm{Spa}(R, R^+) \rightarrow (\mathrm{Gr}_{\mathcal{W},k}^{\mathcal{G}})^{\diamond}$.

The second claim will follow from proving that the commutative diagram 2.7 is Cartesian. Indeed, that would prove that the closed immersion $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu} \rightarrow \mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G}}$ is formally adic, and that $(\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu})^{\mathrm{red}} = \mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G},\leq\mu}$. All of the morphisms in diagram 2.7 are closed immersions, so it is enough to check that the diagram is Cartesian on (C, O_C) -points. Suppose we have a map $m : \mathrm{Spa}(C, O_C) \rightarrow \mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu} \cap (\mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G}})^{\diamond}$, this factors through a map $m' : \mathrm{Spec}(O_C)^{\diamond} \rightarrow (\mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G}})^{\diamond}$. The map is given by a lattice $(\mathcal{T}_{O_C}, \psi_{O_C})$ over $W(O_C)$ whose basechange to $W(C)$ is bounded by μ . Indeed, using the Beauville-Laszlo theorem, this is the translation of what it means for the composition $\mathrm{Spa}(C, O_C) \subseteq \mathrm{Spec}^{\diamond}(O_C) \rightarrow (\mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G}})^{\diamond}$ to factor through $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu}$. By proposition 1.3.17 m' is coming from a map $\mathrm{Spec}(O_C) \rightarrow \mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G}}$ for which the composition $\mathrm{Spec}(C) \rightarrow \mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G}}$ factors through $\mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G},\leq\mu}$. Since $\mathrm{Spec}(C)$ is Zariski dense in $\mathrm{Spec}(O_C)$, m' factors through $\mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G},\leq\mu}$ and m factors through $(\mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G},\leq\mu})^{\diamond}$. \square

Remark 2.2.6. We want to remark that although the v -sheaf $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu}$ is formally p -adic, the similarly defined moduli $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\mu}$ of B_{dR}^+ -lattices of type exactly μ is not formally p -adic if μ is not minuscule. Indeed, in that case there are points $p : \mathrm{Spa}(C, O_C) \rightarrow (\mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G}})^{\diamond}$ with formalization $m : \mathrm{Spd}(O_C, O_C) \rightarrow (\mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G}})^{\diamond}$ such that p factors through $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\mu}$ but m doesn't. This implies that $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\mu} \cap (\mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G}})^{\diamond}$ is not v -formalizing and consequently not represented by a scheme-theoretic v -sheaf. Similarly, this proves that if μ is not minuscule $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\mu} \cap (\mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G}})^{\diamond}$ properly contains $(\mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G},\mu})^{\diamond}$ as open subsheaves of $(\mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G},\leq\mu})^{\diamond}$.

Corollary 2.2.7. If \mathcal{G} is reductive over $W(k)$ and $\mu \in X_*^+(\mathfrak{T})$, then the v -sheaf $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu}$ is a p -adic kimberlite.

Proof. We know $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu}$ is separated by theorem 2.2.3. Since $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu}$ is formally p -adic, by proposition 1.3.31 it is also formally separated. The map $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu} \rightarrow \mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G}}$ is a formally closed immersion and $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G}}$ is specializing so by proposition 1.4.30 $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu}$ is also specializing. The morphism $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu} \rightarrow O_E^{\diamond}$ is formally adic which implies that the adjunction morphism $((\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu})^{\mathrm{red}})^{\diamond} \rightarrow \mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu}$ is a closed embedding. By proposition 2.2.5 $(\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu})^{\mathrm{red}} = \mathrm{Gr}_{\mathcal{W},k_E}^{\mathcal{G},\leq\mu}$, which is represented by a scheme (See [5] Theorem 8.3). We also have $(\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu})^{\mathrm{an}} = \mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu} \times_{O_E^{\diamond}} E^{\diamond}$, which is represented by a spatial diamond by theorem 2.2.3. \square

The purpose of the rest of this section is to upgrade corollary 2.2.7 and prove that $\mathrm{Gr}_{O_E^{\diamond}}^{\mathcal{G},\leq\mu}$ is a rich p -adic kimberlite and that its p -adic tubular neighborhoods are connected.

Remark 2.2.8. One could try to generalize corollary 2.2.7 to the parahoric case. This is more subtle to deal with because it is not possible to define boundedness conditions through a cocharacter. What one can

do is to define the boundedness condition on the generic fiber and take the closure in the sense of v -sheaves to obtain a closed subsheaf. To prove that this closed subsheaf is a kimberlite the only real difficulty one can run into is that it is not a priori clear whether the special fiber of this subsheaf is represented by a scheme or not. It is the author's understanding that these subtleties will be tackled in [16], and that their methods can prove that the special fiber will be represented by the "expected" perfect scheme. With this result at hand one can prove that the bounded parahoric Beilinson-Drinfeld Grassmanians are also rich p -adic kimberlites.

2.2.2 Twisted loop sheaves

We begin by discussing two constructions that are related to twisted loop sheaves and that we will use below. Given an affine scheme $X = \mathrm{Spec}(A)$ of finite type over $W(k)$ with structure morphism $\pi : X \rightarrow \mathrm{Spec}(W(k))$, we can associate to it two v -sheaves over $W(k)^\diamond$ which we will denote by $X(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+})$. Here $X(\mathcal{O}^\sharp) : \mathrm{Perf}_k \rightarrow \mathrm{Sets}$ is defined to be the presheaf that assigns to $\mathrm{Spa}(R, R^+)$ the set of triples (R^\sharp, ι, f) where (R^\sharp, ι) is an untilt and $f \in \mathrm{Hom}_{W(k)}(A, R^\sharp)$ is a $W(k)$ -algebra homomorphism. On the other hand, $X(\mathcal{O}^{\sharp,+})$ assigns triples (R^\sharp, ι, f) with $f \in \mathrm{Hom}_{W(k)}(A, R^{\sharp,+})$. Notice that we have an open inclusion of v -sheaves $X(\mathcal{O}^{\sharp,+}) \subseteq X(\mathcal{O}^\sharp)$. Both of these functors glue to give a construction that is now defined for every scheme X locally of finite type over $\mathrm{Spec}(W(k))$. Visibly, these two constructions are very related to the functor $\diamond : \mathrm{PreAd}_{W(k)} \rightarrow \widehat{\mathrm{Perf}}$, we make this explicit below.

We still assume $X = \mathrm{Spec}(A)$, and we let X_p denote the p -adic completion of X . Now, X_p is a p -adic Noetherian formal scheme that we may regard as an affinoid adic space $\mathrm{Spa}(A_p, A_p)$. Since for any untilt of R the ring $R^{\sharp,+}$ is p -adically complete, we have an identification $X_p^\diamond = X(\mathcal{O}^{\sharp,+})$. Also, if $Y \rightarrow X$ is an open cover of the form $Y = \coprod_{i=1}^n \mathrm{Spec}(A[\frac{1}{f_i}])$ with $f_i \in A$, then $Y_p \rightarrow X_p$ is also an open cover of adic spaces. Indeed, $\mathrm{Spec}(A[\frac{1}{f_i}])_p$ corresponds to the open subset of X_p where $1 \leq |f_i|$.

The construction of $X(\mathcal{O}^\sharp)$ is a little more elaborate. Given an adic space S (thought of as a triple $(|S|, \mathcal{O}_S, \{v_s : s \in |S|\})$ in Huber's category \mathcal{V} see [14]), we let S^H denote the topologically ringed space $(|S|, \mathcal{O}_S)$ that is obtained from S by forgetting the last entry of data. Suppose we are given a morphism of schemes $f : X \rightarrow Y$ that is locally of finite type and a morphism $g : S^H \rightarrow Y$ of locally ringed spaces where S is an adic space for which every point $s \in S$ has an affinoid open neighborhood with Noetherian ring of definition. In [14] (proposition 3.8) Huber constructs an adic space " $S \times_Y X$ " together with a map of adic spaces $p_1 : "S \times_Y X" \rightarrow S$ and a map of locally ringed spaces $p_2 : ("S \times_Y X")^H \rightarrow X$ with the following universal property. If T is an adic space, $\pi_1 : T \rightarrow S$ is a map of adic spaces and $\pi_2 : T^H \rightarrow X$ is a map of locally ringed spaces such that $f \circ \pi_1 = g \circ \pi_2^H$, then there is a unique map $\pi : T \rightarrow "S \times_Y X"$ such that $p_1 \circ \pi = \pi_1$ and $p_2 \circ \pi^H = \pi_2$.

With this adic space at hand we can let $Y = \mathrm{Spec}(W(k))$, $S = \mathrm{Spa}(W(k), W(k))$ and X the finite type scheme over Y that we started with and define X^{ad} as $("S \times_Y X")$. With this definition we have $X(\mathcal{O}^\sharp) = (X^{ad})^\diamond$. Moreover, if $X = \mathrm{Spec}(A)$ and $X_f = \mathrm{Spec}(A[\frac{1}{f}])$ for $f \in A$ we can see from the universal property that X_f^{ad} is the open locus of X^{ad} where $f \neq 0$.

The advantage of these two-step constructions is that it makes it clear, by proposition 1.1.30, that $X(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+})$ are small v -sheaves and it also clarifies the glueing process for $X(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+})$ when X is an arbitrary finite type scheme. These two constructions already appear in [28] §27.

We will later on use the following facts about these constructions:

Proposition 2.2.9. *If $X \rightarrow \mathrm{Spec}(W(k))$ is a proper map of schemes, then the natural map $X(\mathcal{O}^{\sharp,+}) \rightarrow X(\mathcal{O}^\sharp)$ is an isomorphism.*

Proof. By [14] remark 4.6.(iv).d we have an isomorphism of adic spaces $X_p \rightarrow X^{ad}$ where X_p and X^{ad} are as above. Since $X(\mathcal{O}^\sharp) = (X^{ad})^\diamond$ and $X(\mathcal{O}^{\sharp,+}) = X_p^\diamond$ the conclusion holds. \square

Proposition 2.2.10. *Suppose that X and Y are qcqs finite type schemes over $\mathrm{Spec}(W(k))$ and that $X \rightarrow Y$ is universally subtrusive as in definition 1.3.4, then $X(\mathcal{O}^\sharp) \rightarrow Y(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+}) \rightarrow Y(\mathcal{O}^{\sharp,+})$ are surjective maps of v -sheaves.*

Proof. Replacing Y by an open cover we may assume that $Y = \mathrm{Spec}(A)$ for a ring A of finite type over $W(k)$. By [25] theorem 3.12 we may assume that $X \rightarrow Y$ factors as $X \rightarrow Y' \rightarrow Y$ with $Y' \rightarrow Y$ proper and surjective and $X \rightarrow Y'$ a quasi-compact open covering. Since open covers of adic spaces induce surjective maps of v -sheaves we only need to deal with the proper case. Moreover, by Chow's lemma ([31] Tag 0200) we may assume $Y' \rightarrow Y$ is projective. We claim that both maps of v -sheaves $Y'(\mathcal{O}^\sharp) \rightarrow Y(\mathcal{O}^\sharp)$ and $Y'(\mathcal{O}^{\sharp,+}) \rightarrow Y(\mathcal{O}^{\sharp,+})$ are quasi-compact. Indeed, they are both the composition of a closed immersion and the first projection map of $(\mathbb{P}_{W(k)}^n)^\diamond \times_{W(k)^\diamond} Y(\mathcal{O}^\sharp)$ and $(\mathbb{P}_{W(k)}^n)^\diamond \times_{W(k)^\diamond} Y(\mathcal{O}^{\sharp,+})$ respectively. By ([28] 12.11) we may check surjectivity at a topological level. Take an algebraically closed non-Archimedean field C with open and bounded valuation subring $C^+ \subseteq C$, and consider ring maps $r^* : A \rightarrow C$ and $s^* : A \rightarrow C^+$ representing (C, C^+) -valued points in $r \in Y(\mathcal{O}^\sharp)$ and $s \in Y(\mathcal{O}^{\sharp,+})$ respectively. Since $Y' \rightarrow Y$ is proper and surjective the map of schemes $\mathrm{Spec}(C) \times_Y Y' \rightarrow \mathrm{Spec}(C)$ admits a section which induces a lift of r to $Y'(\mathcal{O}^\sharp)$. Analogously, the map of schemes $\mathrm{Spec}(C^+) \times_Y Y' \rightarrow \mathrm{Spec}(C^+)$ admits a section (by the valuative criterion of properness). This defines an element of $Y'(\mathcal{O}^{\sharp,+})$ lifting s . \square

Perhaps unsurprisingly, for a scheme X over $\mathrm{Spec}(W(k))$ the reduction functor applied to $X(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+})$ give the same scheme-theoretic v -sheaf.

Proposition 2.2.11. *Given X a scheme locally of finite type over $\mathrm{Spec}(W(k))$ we have identifications in $\mathrm{SchPerf}$:*

$$X(\mathcal{O}^\sharp)^{\mathrm{red}} \cong X \times_{W(k)} \mathrm{Spec}(k) \cong X(\mathcal{O}^{\sharp,+})^{\mathrm{red}}$$

Proof. Both identifications follow from proposition 1.3.20. By the construction of X_p as a p -adic completion in the case of $X(\mathcal{O}^{\sharp,+})$, and by the universal property of X^{ad} in the category of adic spaces in the case of $X(\mathcal{O}^\sharp)$. \square

We move on to discuss twisted loop sheaves. For the rest of this section we let C be an algebraically closed non-Archimedean field over k with ring of integers O_C and residue field k_C . We fix a characteristic 0 untilt C^\sharp and we pick $\xi \in W(O_C)$ a generator for the kernel of $W(O_C) \rightarrow O_{C^\sharp}$. The choice of untilt determines a unique map $O_C^\diamond \rightarrow \mathbb{Z}_p^\diamond$ that we also fix throughout this section.

- Definition 2.2.12.**
1. We let $W^+(\mathcal{O}) : \mathrm{Perf}_{O_C^\diamond} \rightarrow \mathrm{Sets}$ denote the presheaf that assigns to $\mathrm{Spa}(R, R^+) \rightarrow O_C^\diamond$ the ring $W(R^+)$.
 2. We let $B_{dR}^+(\mathcal{O}^\sharp) : \mathrm{Perf}_{O_C^\diamond} \rightarrow \mathrm{Sets}$ denote the presheaf that assigns to $\mathrm{Spa}(R, R^+) \rightarrow O_C^\diamond$ the ring $B_{dR}^+(R^\sharp)$ where R^\sharp is the untilt associated with our fixed choice of $\xi \in W(O_C)$.
 3. We let $W(\mathcal{O}) : \mathrm{Perf}_{O_C^\diamond} \rightarrow \mathrm{Sets}$ denote the presheaf that assigns to $\mathrm{Spa}(R, R^+) \rightarrow O_C^\diamond$ the ring $W(R^+)[\frac{1}{\xi}]$.
 4. We let $B_{dR}(\mathcal{O}^\sharp) : \mathrm{Perf}_{O_C^\diamond} \rightarrow \mathrm{Sets}$ denote the presheaf that assigns to $\mathrm{Spa}(R, R^+) \rightarrow O_C^\diamond$ the ring $B_{dR}(R^\sharp) := B_{dR}^+(R^\sharp)[\frac{1}{\xi}]$.

Proposition 2.2.13. *The presheaves $W^+(\mathcal{O})$, $B_{dR}^+(\mathcal{O}^\sharp)$, $W(\mathcal{O})$, and $B_{dR}(\mathcal{O}^\sharp)$ are small v -sheaves.*

Proof. Ignoring the ring structure, we see that the Teichmüller expansion of $W(R^+)$ gives a bijection to $(R^+)^{\mathbb{N}}$ which is a small v -sheaf. We can prove inductively that $B_{dR}^+(\mathcal{O}^\sharp)/\xi^n$ is a small v -sheaf. Indeed, it sits in the exact sequence of presheaves:

$$0 \rightarrow B_{dR}^+(\mathcal{O}^\sharp)/\xi^{n-1} \xrightarrow{\cdot\xi} B_{dR}^+(\mathcal{O}^\sharp)/\xi^n \rightarrow \mathcal{O}^\sharp \rightarrow 0$$

By induction the leftmost term is a small v -sheaf and we already know that the rightmost term is a small v -sheaf. A diagram chase gives that the middle one is also a small v -sheaf. Since $B_{dR}^+(\mathcal{O}^\sharp) = \varprojlim_n B_{dR}^+(\mathcal{O}^\sharp)/\xi^n$ this other one is also a small v -sheaf.

Now $W(\mathcal{O}) = \varinjlim (W^+(\mathcal{O}) \xrightarrow{\cdot\xi} W^+(\mathcal{O}) \xrightarrow{\cdot\xi} \dots)$ and $B_{dR}(\mathcal{O}^\sharp) = \varinjlim (B_{dR}^+(\mathcal{O}^\sharp) \xrightarrow{\cdot\xi} B_{dR}^+(\mathcal{O}^\sharp) \xrightarrow{\cdot\xi} \dots)$. Since these are filtered colimit of sheaves they define small v -sheaves as well. \square

Notice that $W^+(\mathcal{O})$ and $B_{dR}^+(\mathcal{O}^\sharp)$ come equipped with reduction maps

$$red : W^+(\mathcal{O}) \rightarrow W^+(\mathcal{O})/\xi = \mathcal{O}^{\sharp,+}$$

and

$$red : B_{dR}^+(\mathcal{O}^\sharp) \rightarrow B_{dR}^+(\mathcal{O}^\sharp)/\xi = \mathcal{O}^\sharp.$$

Definition 2.2.14. Let H be a finite type affine scheme over $\text{Spec}(W(k)[t, t^{-1}])$, and let (\mathcal{H}, ρ) be a finite type affine scheme over $\text{Spec}(W(k)[t])$ together with an isomorphism $\rho : \mathcal{H} \times_{\mathbb{A}_{W(k)}^1} \mathbb{G}_m \rightarrow H$. To this setup we associate the following presheaves over O_C^\diamond :

1. $W^+\mathcal{H}$ assigns to $\text{Spa}(R, R^+) \rightarrow O_C^\diamond$ the set of sections $\text{Spec}(W(R^+)) \rightarrow \mathcal{H} \times_{\mathbb{A}_{W(k)}^1} \text{Spec}(W(R^+))$.
2. WH assigns to $\text{Spa}(R, R^+) \rightarrow O_C^\diamond$ the set of sections $\text{Spec}(W(R^+)[\frac{1}{\xi}]) \rightarrow H \times_{\mathbb{G}_m, W(k)} \text{Spec}(W(R^+)[\frac{1}{\xi}])$.
3. $L^+\mathcal{H}$ assigns to $\text{Spa}(R, R^+) \rightarrow O_C^\diamond$ the set of sections $\text{Spec}(B_{dR}^+(R^\sharp)) \rightarrow \mathcal{H} \times_{\mathbb{A}_{W(k)}^1} \text{Spec}(B_{dR}^+(R^\sharp))$.
4. LH assigns to $\text{Spa}(R, R^+) \rightarrow O_C^\diamond$ the set of sections $\text{Spec}(B_{dR}(R^\sharp)) \rightarrow H \times_{\mathbb{G}_m, W(k)} \text{Spec}(B_{dR}(R^\sharp))$.

where the base change in all cases is given by the usual map on $W(k)$ deduced from the composition $k \rightarrow O_C \rightarrow R^+$ and given by $t \mapsto \xi$.

Proposition 2.2.15. With the notation as above $W^+\mathcal{H}$, WH , $L^+\mathcal{H}$ and LH are small v -sheaves.

Proof. Let $\mathcal{R} \in \{W^+(\mathcal{O}), W(\mathcal{O}), B_{dR}^+(\mathcal{O}^\sharp), B_{dR}(\mathcal{O}^\sharp)\}$ denote one of the sheaves of rings of definition 2.2.12. Suppose that $\mathcal{H} = \text{Spec}(W(k)[t][x_1, \dots, x_n]/(f_1(t, \bar{x}), \dots, f_m(t, \bar{x}))$. Notice there is a map of v -sheaves $O_C^\diamond \xrightarrow{0} \mathcal{R}$ corresponding to the constant 0 section. Consider the following basechange diagram:

$$\begin{array}{ccc} X & \longrightarrow & O_C^\diamond \\ \downarrow & & \downarrow 0 \\ \mathcal{R}^n & \xrightarrow{F} & \mathcal{R}^m \end{array}$$

Where $F(\bar{r}) = (f_1(\xi, \bar{r}), \dots, f_m(\xi, \bar{r}))$. Whenever \mathcal{R} is $W^+(\mathcal{O})$, $W(\mathcal{O})$, $B_{dR}^+(\mathcal{O}^\sharp)$, or $B_{dR}(\mathcal{O}^\sharp)$ then X is isomorphic as presheaves to $W^+\mathcal{H}$, WH , $L^+\mathcal{H}$, or LH respectively. From this diagram, it is clear that X is a small v -sheaf. \square

In our setup, ρ will induce maps of v -sheaves $L^+\mathcal{H} \xrightarrow{\rho} LH$ and $W^+\mathcal{H} \xrightarrow{\rho} WH$. We get the following diagrams of inclusions:

$$\begin{array}{ccc}
& W(\mathcal{O}) & \\
\swarrow & & \searrow \\
W^+(\mathcal{O}) & & B_{dR}(\mathcal{O}^\sharp) \\
\searrow & & \swarrow \\
& B_{dR}^+(\mathcal{O}^\sharp) &
\end{array}
\quad
\begin{array}{ccc}
& WH & \\
\swarrow \rho & & \searrow \\
W^+\mathcal{H} & & LH \\
\searrow & & \swarrow \rho \\
& L^+\mathcal{H} &
\end{array}$$

Moreover, if we let $\overline{\mathcal{H}}$ denote the basechange $\mathcal{H} \times_{\mathbb{A}_{W(k)}^1} \mathrm{Spec}(W(k))$ at $t = 0$ we get reduction morphisms $W^+\mathcal{H} \rightarrow \overline{\mathcal{H}}(\mathcal{O}^\sharp, +)_{O_C^\diamond}$ and $L^+\mathcal{H} \rightarrow \overline{\mathcal{H}}(\mathcal{O}^\sharp)_{O_C^\diamond}$.

Proposition 2.2.16. *If \mathcal{H} is smooth over $\mathrm{Spec}(W(k)[t])$ then the reduction maps $W^+\mathcal{H} \rightarrow \overline{\mathcal{H}}(\mathcal{O}^\sharp, +)_{O_C^\diamond}$ and $L^+\mathcal{H} \rightarrow \overline{\mathcal{H}}(\mathcal{O}^\sharp)_{O_C^\diamond}$ are surjective maps of v -sheaves.*

Proof. We claim that the map is surjective even at the level of presheaves. The (R, R^+) -valued points of $\overline{\mathcal{H}}(\mathcal{O}^\sharp)$ and $\overline{\mathcal{H}}(\mathcal{O}^\sharp, +)$ can be seen as maps $\mathrm{Spec}(R^\sharp) \rightarrow \mathcal{H}_{B_{dR}(R^\sharp)}$ and $\mathrm{Spec}(R^{\sharp, +}) \rightarrow \mathcal{H}_{W(R^+)}$ whose composition with the projections to $\mathrm{Spec}(B_{dR}(R^\sharp))$ and $\mathrm{Spec}(W(R^+))$ are the usual closed embeddings. By smoothness of \mathcal{H} , for any $n \in \mathbb{N}$ the maps can be lifted to maps $\mathrm{Spec}(B_{dR}(R^\sharp)/\xi^n) \rightarrow \mathcal{H}_{B_{dR}(R^\sharp)}$ and $\mathrm{Spec}(W(R^+)/\xi^n) \rightarrow \mathcal{H}_{W(R^+)}$ respectively. Since \mathcal{H} is an affine scheme and since both $B_{dR}(R^\sharp)$ and $W(R^+)$ are (ξ) -adically complete we may pass to the inverse limit by choosing compatible lifts. \square

Definition 2.2.17. 1. *With the setup as above, consider the ring k_C with the discrete topology, we let $\mathcal{W}_{\mathrm{red}}^+\mathcal{H} \in \widehat{\mathrm{SchPerf}}_{k_C}$ be the scheme-theoretic v -sheaf that assigns to $\mathrm{Spec}(R) \in \mathrm{PCAlg}_{k_C}^{\mathrm{op}}$ sections $\mathrm{Spec}(W(R)) \rightarrow \mathcal{H} \times_{\mathbb{A}_{W(k)}^1} \mathrm{Spec}(W(R))$ where the basechange is given by $t \mapsto p$.*

2. *We let $\mathcal{W}_{\mathrm{red}}^+(\mathcal{O}) : \mathrm{PCAlg}_{k_C}^{\mathrm{op}} \rightarrow \mathrm{Sets}$ denote sheaf that sends $\mathrm{Spec}(R)$ to $W(R)$. This sheaf can also be expressed as $\mathcal{W}_{\mathrm{red}}^+ \mathbb{A}_{W(k)[t]}^1$.*

Remark 2.2.18. *The scheme-theoretic v -sheaves $\mathcal{W}_{\mathrm{red}}^+\mathcal{H}$ of definition 2.2.17 are the v -sheaves that Zhu calls p -adic jet spaces in [32] 1.1.1. These sheaves are represented by perfect affine schemes.*

Proposition 2.2.19. *With the notation as above the v -sheaf $W^+\mathcal{H}$ is a p -adic kimberlite and $(W^+\mathcal{H})^{\mathrm{red}} = (\mathcal{W}_{\mathrm{red}}^+\mathcal{H})$.*

Proof. Observe that $W^+(\mathcal{O})$ is represented by $\mathrm{Spd}(O_C \langle T_n \rangle_{n \in \mathbb{N}}, O_C \langle T_n \rangle_{n \in \mathbb{N}})$, by proposition 1.4.23 and proposition 1.3.25 $W^+(\mathcal{O})$ is a p -adic kimberlite. Moreover, by proposition 1.3.20 $W^+(\mathcal{O})^{\mathrm{red}}$ is represented by $\mathrm{Spec}(k_C[T_n]_{n \in \mathbb{N}})$ which is $\mathcal{W}_{\mathrm{red}}^+(\mathcal{O})$. Lets move on to the general case, recall from the proof of proposition 2.2.15 that if $\mathcal{H} = \mathrm{Spec}(A)$ is presented as $A = W(k)[t][\overline{x}]/I$ with $I = (f_1(t, \overline{x}), \dots, f_m(t, \overline{x}))$. Then $W^+\mathcal{H}$ fits in the commutative diagram with Cartesian square:

$$\begin{array}{ccc}
W^+\mathcal{H} & \longrightarrow & O_C^\diamond \\
\downarrow & & \downarrow 0 \\
W^+(\mathcal{O})^n & \xrightarrow{F} & W^+(\mathcal{O})^m \longrightarrow O_C^\diamond
\end{array}
\quad \begin{array}{c} \nearrow id \end{array}$$

We claim that all of these maps are formally adic, and in particular $W^+\mathcal{H}$ is formally p -adic. This follows from the fact that formal adicness is stable under basechange, that it has the 2-out-of-3 property and that, as we proved above, $W^+(\mathcal{O}) \rightarrow O_C^\diamond$ is formally adic. Since $W^+(\mathcal{O})$ is separated over O_C^\diamond the section $O_C^\diamond \xrightarrow{0} W^+(\mathcal{O})^m$ is a formally adic closed immersion. We can conclude that $W^+\mathcal{H}$ is a p -adic kimberlite by using lemma 1.4.30. Finally, since we can basechange by $\mathrm{Spec}(k_C)^\diamond \rightarrow \mathrm{Spd}(O_C, O_C)$ to compute reductions we get the following Cartesian diagram:

$$\begin{array}{ccc}
W^+ \mathcal{H}^{\text{red}} & \longrightarrow & \text{Spec}(k_C) \\
\downarrow & & \downarrow 0 \\
\mathcal{W}_{\text{red}}^+(\mathcal{O})^n & \xrightarrow{F} & \mathcal{W}_{\text{red}}^+(\mathcal{O})^m
\end{array}$$

which gives the isomorphism $W^+ \mathcal{H}^{\text{red}} = \mathcal{W}_{\text{red}}^+ \mathcal{H}$. □

2.2.3 Demazure kimberlites

In this subsection we use twisted loop sheaves to construct a family of kimberlites that will allow us to understand how the specialization map for the p -adic Beilinson-Drinfeld Grassmanians behave. We change the setup a little bit and fix some notation first:

1. Let H be a split reductive group over $W(k)$, let $T \subseteq B \subseteq H$ a choice of maximal split torus and a Borel respectively.
2. Let $(X^*, \Phi, X_*, \Phi^\vee)$ be the root datum associated to (H, T) .
3. We let $\langle \cdot, \cdot \rangle : X^* \times X_* \rightarrow \mathbb{Z}$ denote perfect pairing between roots and coroots.
4. Let Φ^+ be the set of positive roots associated to B .
5. Let N be the normalizer of T in H .
6. Let $W = N/T$ be the Weyl group of H .
7. We let $\mathcal{A} = \mathcal{A}(H, T)$ denote $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
8. We let $\Psi = \{\alpha + n \mid \alpha \in \Phi, n \in \mathbb{Z}\}$ denote the set of affine functionals on \mathcal{A} coming from the natural perfect pairing $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \rightarrow \mathbb{Z}$. We call them affine roots.
9. Given a point $q \in \mathcal{A}$ we let $\Phi_q = \{\alpha \in \Phi \mid \alpha(q) \in \mathbb{Z}\}$ this is clearly a closed sub-root system. We let M_q be the generalized Levi subgroup of H containing T with root datum given by $(X^*, \Phi_q, X_*, \Phi_q^\vee)$.
10. Ψ defines a hyperplane structure on \mathcal{A} , and for any point $q \in \mathcal{A}$ we can associate a polysimplicial closed region of \mathcal{A} that we will denote by F_q . In case H is semisimple this region is bounded and forms a polytope. We let o denote the vertex associated to the origin in \mathcal{A} and \mathcal{C} the unique alcove containing o and contained in the Bruhat chamber associated to B .
11. We denote by \mathbb{S} the set of reflections along the walls of \mathcal{C} , we let W^{aff} the affine Weyl group generated by \mathbb{S} . Given any facet $\mathcal{F} \subseteq \mathcal{C}$ we let $\mathbb{S}_{\mathcal{F}}$ be the subset of elements of \mathbb{S} fixing \mathcal{F} and we let $W_{\mathcal{F}}$ be the subgroup of W^{aff} generated by $\mathbb{S}_{\mathcal{F}}$.
12. We let \tilde{W} denote the Iwahori-Weyl group of H . Recall that $W^{\text{aff}} \subseteq \tilde{W}$ and that if we let $\Omega_H = \pi_1(H^{\text{der}})$ we have a decomposition $\tilde{W} = W^{\text{aff}} \rtimes \Omega_H$.

Fix a point $q \in \mathcal{A}$. In ([23] §3) Pappas and Zhu use Bruhat-Tits theory and dilatation techniques to construct smooth affine algebraic groups \mathcal{H}_q over $\text{Spec}(W(k)[t])$ together with an isomorphism ρ from $\mathcal{H}_q \times_{W(k)[t]} \text{Spec}(W(k)[t, t^{-1}])$ to $H \times_{W(k)} \text{Spec}(W(k)[t, t^{-1}])$ with the following list of properties:

- a) For any discrete valuation ring V and a map $W(k)[t] \rightarrow V$ given by $t \mapsto \pi$ with $\pi \in V$ a uniformizer, the basechange $\mathcal{H}_q \times_{\mathbb{A}_{W(k)}^1} \text{Spec}(V)$ is the parahoric group scheme associated to $q \in \mathcal{A}(H, V[\frac{1}{\pi}])$ by Bruhat-Tits theory under the identification $\mathcal{A}(H, V[\frac{1}{\pi}]) = X^*(T_{V[\frac{1}{\pi}]}) \otimes \mathbb{R} = X^*(T) \otimes \mathbb{R}$.

- b) For any root $\alpha \in \Phi$ there are smooth connected closed subgroups $\mathcal{U}_\alpha^q \subseteq \mathcal{H}_q$ (respectively $\mathcal{T} \subseteq \mathcal{H}_q$) extending the usual root subgroup $U_\alpha \subseteq H$ (respectively extending the torus $T \subseteq H$). Over $W(k)[t]$, the groups \mathcal{U}_α^q are isomorphic to \mathbb{G}_a and \mathcal{T} is isomorphic to \mathbb{G}_m^n for some n .
- c) There is an open cell decomposition:

$$\mathcal{V}_q := \prod_{\alpha \in \Phi^-} \mathcal{U}_\alpha^q \times \mathcal{T} \times \prod_{\alpha \in \Phi^+} \mathcal{U}_\alpha^q \rightarrow \mathcal{H}_q.$$

This map forms an open embedding onto a fiberwise Zariski-dense neighborhood of the identity section.

- d) The group multiplication map $\mathcal{V}_q \times \mathcal{V}_q \xrightarrow{\mu} \mathcal{H}_q$ is smooth and surjective.
- e) The basechange $\overline{\mathcal{H}}_q := \mathcal{H}_q \times_{\mathbb{A}_{W(k)}^1} \text{Spec}(W(k))$ along $t = 0$ supports a split reductive quotient $\overline{\mathcal{H}}_q^{Red}$ over $W(k)$ with root datum canonically identified with $(X^*, \Phi_q, X_*, \Phi_q^\vee)$. In particular we can identify M_q with $\overline{\mathcal{H}}_q^{Red}$.
- f) If $\alpha \in \Phi_q$ the composition $\overline{\mathcal{U}}_\alpha^q \rightarrow \overline{\mathcal{H}}_q^{Red}$ at $t = 0$ defines an isomorphism onto the root group of $\overline{\mathcal{H}}_q^{Red}$ corresponding to α . On the other hand, if $\alpha \in \Phi \setminus \Phi_q$ then the composition $\overline{\mathcal{U}}_\alpha^q \rightarrow \overline{\mathcal{H}}_q^{Red}$ factors through the identity section.
- g) We have a commutative diagram of open cell decomposition:

$$\begin{array}{ccccc} \prod_{\alpha \in \Phi \setminus \Phi_q} \overline{\mathcal{U}}_\alpha & \longrightarrow & \prod_{\alpha \in \Phi^-} \overline{\mathcal{U}}_\alpha \times \overline{\mathcal{T}} \times \prod_{\alpha \in \Phi^+} \overline{\mathcal{U}}_\alpha & \xrightarrow{\pi} & \prod_{\alpha \in \Phi_q^-} \overline{\mathcal{U}}_\alpha \times \overline{\mathcal{T}} \times \prod_{\alpha \in \Phi_q^+} \overline{\mathcal{U}}_\alpha \\ \downarrow \cong & & \downarrow \mu & & \downarrow \mu \\ \text{Ker}(m) & \longrightarrow & \overline{\mathcal{H}}_q & \xrightarrow{m} & \overline{\mathcal{H}}_q^{Red} \end{array}$$

where π is the projection map and the left vertical arrow is an isomorphism.

If we are given two points $q_1, q_2 \in \mathcal{A}$ such that $F_{q_1} \subseteq F_{q_2}$ we also get a map of algebraic groups $f : \mathcal{H}_{q_2} \rightarrow \mathcal{H}_{q_1}$. This map has the following properties:

- a) $\rho_1 \circ f = \rho_2$ over $W(k)[t, t^{-1}]$.
- b) The composition $\overline{\mathcal{H}}_{q_2} \rightarrow \overline{\mathcal{H}}_{q_1}^{Red}$ surjects onto the parabolic subgroup of $\overline{\mathcal{H}}_{q_1}^{Red}$ associated to the closed sub-root system given by $\Phi_{q_1, q_2} := \{\alpha \in \Phi_{q_1} \mid \lfloor \alpha(q_2) \rfloor = \alpha(q_1)\}$. Moreover, the kernel of this map is fiberwise a vector group.

We are now prepared to define “parahoric” versions of the positive loop groups which we will use to define Demazure kimberlites.

Definition 2.2.20. 1. We define the loop group LH to be as in definition 2.2.14 when we consider H as a scheme over $W(k)[t, t^{-1}]$ by taking the appropriate basechange.

2. Given a point $q \in \mathcal{A}$ define the parahoric loop group to be $L^+ \mathcal{H}_q$ as in definition 2.2.14.

3. Associated to the same point we also define the formal parahoric loop group to be $W^+ \mathcal{H}_q$.

Notice that we have injective maps of v -sheaves $W^+ \mathcal{H}_q \subseteq L^+ \mathcal{H}_q \xrightarrow{\rho} LH$.

Proposition 2.2.21. *With the notation as above, for any point $q \in \mathcal{A}$ we have surjective morphisms of v -sheaves in groups:*

$$\begin{aligned} L^+ \mathcal{H}_q &\rightarrow \overline{\mathcal{H}}_q^{\text{Red}}(\mathcal{O}_X^\#) = M_q(\mathcal{O}_X^\#) \\ W^+ \mathcal{H}_q &\rightarrow \overline{\mathcal{H}}_q^{\text{Red}}(\mathcal{O}_X^{\#,+}) = M_q(\mathcal{O}_X^{\#,+}) \end{aligned}$$

Proof. This is a direct consequence of proposition 2.2.16 and proposition 2.2.10 since the map $\overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}^{\text{Red}}$ is smooth surjective and consequently universally subtrusive. \square

We let $L^u \mathcal{H}_q$ and $W^u \mathcal{H}_q$ denote respectively the kernels of the morphisms of proposition 2.2.21 above.

Proposition 2.2.22. *If $q_1, q_2 \in \mathcal{A}$ are such that $F_{q_1} \subseteq F_{q_2}$, then we get inclusions of v -sheaves in groups:*

$$\begin{aligned} L^u \mathcal{H}_{q_1} &\subseteq L^u \mathcal{H}_{q_2} \subseteq L^+ \mathcal{H}_{q_2} \subseteq L^+ \mathcal{H}_{q_1} \subseteq LH \\ W^u \mathcal{H}_{q_1} &\subseteq W^u \mathcal{H}_{q_2} \subseteq W^+ \mathcal{H}_{q_2} \subseteq W^+ \mathcal{H}_{q_1} \subseteq WH \end{aligned}$$

Moreover, the map from $L^+ \mathcal{H}_{q_2}$ to $M_{q_1}(\mathcal{O}^\#)$ surjects onto $P_{\Phi_{q_1, q_2}}(\mathcal{O}^\#) \subseteq M_{q_1}(\mathcal{O}^\#)$. Analogously, $W^+ \mathcal{H}_{q_2}$ surjects onto $P_{\Phi_{q_1, q_2}}(\mathcal{O}^{\#,+}) \subseteq M_{q_1}(\mathcal{O}^{\#,+})$.

Proof. We will deal only with the case of parahoric loop groups since the other case is completely analogous. Recall that we have a morphism of algebraic groups over $W(k)[t]$, $f : \mathcal{H}_{q_2} \rightarrow \mathcal{H}_{q_1}$, such that $\rho_1 \circ f = \rho_2$. Functoriality of L^+ , gives us maps $L^+ \mathcal{H}_{q_2} \rightarrow L^+ \mathcal{H}_{q_1} \rightarrow LH$, since $L^+ \mathcal{H}_{q_2} \rightarrow LH$ is an injection, then $L^+ \mathcal{H}_{q_2} \rightarrow L^+ \mathcal{H}_{q_1}$ is also injective.

Now, since the map of affine schemes $\overline{\mathcal{H}}_{q_2} \rightarrow P_{\Phi_{q_1, q_2}}$ is faithfully flat of finite presentation it is universally subtrusive. This implies, by proposition 2.2.16 and proposition 2.2.10, that the composition of $L^+ \mathcal{H}_{q_2} \rightarrow \overline{\mathcal{H}}_{q_2}(\mathcal{O}^\#)$ with $\overline{\mathcal{H}}_{q_2}(\mathcal{O}^\#) \rightarrow P_{\Phi_{q_1, q_2}}(\mathcal{O}^\#)$ is surjective.

Finally, we claim that any map $g : \text{Spec}(B_{dR}^+(R^\#)) \rightarrow \mathcal{H}_{q_1, B_{dR}^+(R^\#)}$ whose reduction $\text{Spec}(R^\#) \rightarrow M_q$ factors through the identity section lifts to a map $\text{Spec}(B_{dR}^+(R^\#)) \rightarrow \mathcal{H}_{q_2, B_{dR}^+(R^\#)}$. Indeed, observe that $\text{Spec}(R^\#) \rightarrow \overline{\mathcal{H}}_{q_1}$ factors through the open cell $\overline{\mathcal{V}}_{q_1}$, which implies that g is of the form

$$g = \left(\prod_{\alpha \in \Phi_{q_1}^-} u_\alpha(g) \right) \cdot t(g) \cdot \left(\prod_{\alpha \in \Phi_{q_1}^+} u_\alpha(g) \right)$$

with $t(g)$ and $\{u_\alpha(g)\}_{\alpha \in \Phi \setminus \Phi_{q_1}}$ reducing to the identity.

We can verify directly from the construction of the map $\mathcal{H}_{q_2} \rightarrow \mathcal{H}_{q_1}$ that each of this elements lifts uniquely to an element in \mathcal{V}_{q_2} . Indeed, on the torus \mathcal{T} and on \mathcal{U}_α with $\alpha \notin \Phi_{q_1}$ f induces an isomorphism because $F_{q_1} \subseteq F_{q_2}$. For $\alpha \in \Phi_{q_1} \setminus \Phi_{q_1, q_2}$ we may, after making some choices, write $\mathcal{U}_\alpha^{q_1}$ as $\text{Spec}(W(k)[t, u])$ and $\mathcal{U}_\alpha^{q_2}$ as $\text{Spec}(W(k)[t, \frac{u}{t}])$. In this case f restricted to $\mathcal{U}_\alpha^{q_2}$ is given by the natural inclusion of rings. The map of rings $u_\alpha(g)^* : W(k)[t, u] \rightarrow B_{dR}^+(R^\#)$ with $t \mapsto \xi$ extends to a map $W(k)[t, \frac{u}{t}] \rightarrow B_{dR}^+(R^\#)$ whenever ξ divides the image of u , but this happens whenever $u_\alpha(g)$ reduces to identity. \square

Proposition 2.2.23. *Let $q_1, q_2 \in \mathcal{A}$ such that $F_{q_1} \subseteq F_{q_2}$, then we have isomorphisms of quotient v -sheaves $L^+ \mathcal{H}_{q_1} / L^+ \mathcal{H}_{q_2} = W^+ \mathcal{H}_{q_1} / W^+ \mathcal{H}_{q_2}$. Moreover, we can identify both of these quotients with $(Fl_{q_1, q_2, \mathcal{O}_C})^\diamond$, where Fl_{q_1, q_2} denotes the flag variety $M_{q_1} / P_{\Phi_{q_1, q_2}}$ when thought of as a p -adic formal scheme.*

Proof. We have sequence of equalities:

$$\begin{aligned} L^+ \mathcal{H}_{p_1} / L^+ \mathcal{H}_{p_2} &= (L^+ \mathcal{H}_{p_1} / L^u \mathcal{H}_{p_1}) / (L^+ \mathcal{H}_{p_2} / L^u \mathcal{H}_{p_1}) \\ &= M_{p_1}(\mathcal{O}^\#) / P_{\Phi_{p_1, p_2}}(\mathcal{O}^\#) \\ &= Fl_{p_1, p_2}(\mathcal{O}^\#) \\ &= Fl_{p_1, p_2}^\diamond \end{aligned}$$

The last equality follows from proposition 2.2.9.

Analogously:

$$\begin{aligned} W^+ \mathcal{H}_{p_1} / W^+ \mathcal{H}_{p_2} &= (W^+ \mathcal{H}_{p_1} / W^u \mathcal{H}_{p_1}) / (W^+ \mathcal{H}_{p_2} / W^u \mathcal{H}_{p_2}) \\ &= M_{p_1}(\mathcal{O}^\sharp, +) / P_{\Phi_{p_1, p_2}}(\mathcal{O}^\sharp, +) \\ &= Fl_{p_1, p_2}^\diamond \end{aligned}$$

□

Lemma 2.2.24. 1. Let \mathcal{F} be a locally spatial diamond with a map $\mathcal{F} \rightarrow O_C^\diamond$ and fix two points $q_1, q_2 \in \mathcal{A}$ with $F_{q_1} \subseteq F_{q_2}$. The natural map $L^+ \mathcal{H}_{q_1} \times_{O_C^\diamond} \mathcal{F} \rightarrow Fl_{q_1, q_2}^\diamond \times_{O_C^\diamond} \mathcal{F}$ admits pro-étale locally a section.

2. If $\mathrm{Spa}(R, R^+)$ is affinoid perfectoid and we are given a map $\mathrm{Spa}(R, R^+) \rightarrow Fl_{q_1, q_2}^\diamond$ then the pullback $L^+ \mathcal{H}_{q_1} \times_{Fl_{q_1, q_2}^\diamond} \mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spa}(R, R^+)$ admits a section locally on the analytic topology of $\mathrm{Spa}(R, R^+)$.

Proof. We may reduce the first claim to the second one by [28] 11.24. Indeed, by proposition 2.2.23 the map in question forms a $L^+ \mathcal{H}_{q_2}$ -torsor so it is enough to prove it is pro-étale locally the trivial torsor. Since the map $Fl_{q_1, q_2}^\diamond \rightarrow O_C^\diamond$ is representable in spatial diamonds we can find a pro-étale cover $\mathrm{Spa}(R, R^+) \rightarrow Fl_{q_1, q_2}^\diamond \times_{O_C^\diamond} \mathcal{F}$ with $\mathrm{Spa}(R, R^+)$ affinoid perfectoid.

Let us prove the second claim. The obstruction to the triviality of the $L^+ \mathcal{H}_{q_2}$ -torsor over $\mathrm{Spa}(R, R^+)$ is an element \mathbf{obs} in $H_v^1(\mathrm{Spa}(R, R^+), L^+ \mathcal{H}_{q_2})$. We prove that this obstruction vanishes after a localization in the analytic topology, recall the following two sequences of maps:

$$\begin{aligned} (L^+ \mathcal{H}_{q_1}) &\rightarrow M_{q_1}(\mathcal{O}_X^\sharp) \rightarrow Fl_{q_1, q_2}^\diamond \\ e &\rightarrow L^u \mathcal{H}_{q_1} \rightarrow L^+ \mathcal{H}_{q_2} \rightarrow P_{\Phi_{q_1, q_2}}(\mathcal{O}_X^\sharp) \rightarrow e \end{aligned}$$

The map $M_{q_1}(\mathcal{O}_X^\sharp) \rightarrow Fl_{q_1, q_2}^\diamond$ is a $P_{\Phi_{q_1, q_2}}(\mathcal{O}_X^\sharp)$ -torsor with obstruction to triviality lying in

$$H_v^1(Fl_{q_1, q_2}^\diamond, P_{\Phi_{q_1, q_2}}(\mathcal{O}_X^\sharp)).$$

Since the map of schemes $M_{q_1} \rightarrow Fl_{q_1, q_2}$ admits Zariski locally a section we may replace $\mathrm{Spa}(R, R^+)$ by an analytic cover for which \mathbf{obs} in $H_v^1(\mathrm{Spa}(R, R^+), P_{\Phi_{q_1, q_2}}(\mathcal{O}_X^\sharp))$ is trivial. Observe that \mathbf{obs} comes from an element of $H_v^1(\mathrm{Spa}(R, R^+), L^u \mathcal{H}_{q_1})$, we prove that in this case it is already trivial. Consider the exact sequence

$$e \rightarrow \mathrm{Ker}(L^+ \mathcal{H}_{q_1} \rightarrow \overline{\mathcal{H}}_{q_1}(\mathcal{O}^\sharp)) \rightarrow L^u \mathcal{H}_{q_1} \rightarrow \mathrm{Ker}(\overline{\mathcal{H}}_{q_1}(\mathcal{O}^\sharp) \rightarrow \overline{\mathcal{H}}_{q_1}^{\mathrm{Red}}(\mathcal{O}^\sharp)) \rightarrow e,$$

we prove that after applying $H^1(\mathrm{Spa}(R, R^+), -)$ to the two groups in the extremes we obtain the trivial pointed set.

For the first group we construct a family $\{L^{u, n}\}_{n=1}^\infty$ of groups filtering $L^{u, 1} := \mathrm{Ker}(L^+ \mathcal{H}_{q_1} \rightarrow \overline{\mathcal{H}}_{q_1}(\mathcal{O}^\sharp))$. We define them as:

$$L^{u, n}(R, R^+) := \mathrm{Ker}(\mathcal{H}_{q_1, B_{dR}^+(R^\sharp)}(B_{dR}^+(R^\sharp)) \rightarrow \mathcal{H}_{q_1, B_{dR}^+(R^\sharp)}(B_{dR}^+(R^\sharp)/\xi^n))$$

Successive quotients $L^{u, n}/L^{u, n+1}$ get identified with the sheaves assigning to $\mathrm{Spa}(R, R^+)$ the groups:

$$\mathrm{Ker}(\mathcal{H}_{q_1, B_{dR}^+(R^\sharp)}(B_{dR}^+(R^\sharp)/\xi^{n+1}) \rightarrow \mathcal{H}_{q_1, B_{dR}^+(R^\sharp)}(B_{dR}^+(R^\sharp)/\xi^n))$$

Since $\mathrm{Spec}(B_{dR}^+(R^\sharp)/\xi^n) \rightarrow \mathrm{Spec}(B_{dR}^+(R^\sharp)/\xi^{n+1})$ is a first order nilpotent thickening, deformation theory gives:

$$L^{u,n}/L^{u,n+1} = \mathrm{Hom}(e^*\Omega_{\mathcal{H}_{q_1}}^1 \otimes_{W(k)[t]} B_{dR}^+(R^\sharp), (\xi^n \cdot B_{dR}^+(R^\sharp)/\xi^{n+1})) = \mathrm{Hom}(e^*\Omega_{\mathcal{H}_{q_1}}^1 \otimes R^\sharp, R^\sharp)$$

Since \mathcal{H}_{q_1} has an open cell decomposition we can see explicitly that $e^*\Omega_{\mathcal{H}_{q_1}/W(k)[t]}^1$ is a finite free module over $W(k)[t]$ (a priori it is only projective), and after fixing a basis we get an identification $L^{u,n}/L^{u,n+1} = (\mathcal{O}^\sharp)^n$. By ([28] 8.8) the cohomology group $H_v^1(\mathrm{Spa}(R, R^+), \mathcal{O}^\sharp) = 0$. Let $I^{u,n}$ be the image of $H_v^1(\mathrm{Spa}(R, R^+), L^{u,n})$ in $H_v^1(\mathrm{Spa}(R, R^+), L^{u,1})$. The argument above shows that $\mathbf{obs} \in \bigcap_{n \in \mathbb{N}} I^{u,n}$. On the other hand,

$$L^{u,1} = \varprojlim L^{u,1}/L^{u,n}$$

with transition maps that are surjective at the level of presheaves. One can use Čech cohomology to prove that $\bigcap_{n \in \mathbb{N}} I^{u,n} = \{e\}$. So far we have proved $H_v^1(\mathrm{Spa}(R, R^+), L^{u,1}) = \{e\}$, but

$$H_v^1(\mathrm{Spa}(R, R^+), \mathrm{Ker}(\overline{\mathcal{H}}_{q_1}(\mathcal{O}^\sharp) \rightarrow \overline{\mathcal{H}}_{q_1}^{Red}(\mathcal{O}^\sharp)))$$

is also trivial since $\mathrm{Ker}(\overline{\mathcal{H}}_{q_1} \rightarrow \overline{\mathcal{H}}_{q_1}^{Red})$ is a vector group over $W(k)$ and we may use [28] 8.8 again. This proves that $H_v^1(\mathrm{Spa}(R, R^+), L^u \mathcal{H}_{q_1}) = \{e\}$ and finishes the proof. \square

Definition 2.2.25. Let $\sigma_r := \{r_i\}_{1 \leq i \leq n}$ and $\sigma_q := \{q_i\}_{1 \leq i \leq n}$ denote a pair of sequences of points in \mathcal{A} such that $F_{r_i}, F_{r_{i+1}} \subseteq F_{q_i}$, and let σ denote the tuple (σ_r, σ_q) . To each σ of this form we associate a v -sheaf that we call the Demazure kimberlite of σ . We define them as the contracted group product:

$$D(\sigma) = L^+ \mathcal{H}_{r_1} \times_{O_C^\diamond}^{L^+ \mathcal{H}_{q_1}} L^+ \mathcal{H}_{r_2} \times_{O_C^\diamond}^{L^+ \mathcal{H}_{q_2}} \dots \times_{O_C^\diamond}^{L^+ \mathcal{H}_{q_{n-1}}} L^+ \mathcal{H}_{r_n} / L^+ \mathcal{H}_{q_n}$$

In what follows we will prove that for any σ as above the $D(\sigma)$ are rich p -adic kimberlites that are proper and smooth over O_C^\diamond .

Proposition 2.2.26. The map of v -sheaves $D(\sigma) \rightarrow O_C^\diamond$ is representable in spatial diamonds, proper and ℓ -cohomologically smooth for any $\ell \neq p$.

Proof. Let σ be as above and let $\sigma' = (\{r_i\}_{1 \leq i \leq n-1}, \{q_i\}_{1 \leq i \leq n-1})$ be the subsequence of the first $n-1$ points of σ . We have a projection morphism of v -sheaves $f : D(\sigma) \rightarrow D(\sigma')$ given by forgetting the last entry corresponding to r_n . It is enough to inductively show that this map satisfies all of the properties in the hypothesis. Since the definition of $D(\sigma)(R, R^+)$ is independent of R^+ to prove the map is proper it is enough to prove it is quasi-compact and separated over $D(\sigma')$. By ([28] 23.15, 10.11, 13.4) separatedness, quasi-compactness and almost all of the requirements that a map needs to satisfy to be ℓ -cohomologically smooth ([28] 23.8) can be checked v -locally. The following diagram is Cartesian with surjective horizontal arrows:

$$\begin{array}{ccc} L^+ \mathcal{H}_{r_1} \times_{O_C^\diamond} \dots \times_{O_C^\diamond} L^+ \mathcal{H}_{r_{n-1}} \times_{O_C^\diamond} (L^+ \mathcal{H}_{r_n} / L^+ \mathcal{H}_{q_n}) & \longrightarrow & D(\sigma) \\ \downarrow & & \downarrow \\ L^+ \mathcal{H}_{r_1} \times_{O_C^\diamond} \dots \times_{O_C^\diamond} L^+ \mathcal{H}_{r_{n-1}} & \longrightarrow & D(\sigma') \end{array}$$

By proposition 2.2.23 we have that $L^+ \mathcal{H}_{r_n} / L^+ \mathcal{H}_{q_n} = \mathrm{Fl}_{r_n, q_n, O_C}^\diamond$ which is proper, representable in spatial diamonds and ℓ -cohomologically smooth over O_C^\diamond for any $\ell \neq p$. This proves the map in the left column of the diagram is representable in spatial diamonds, ℓ -cohomologically smooth and proper. It also proves that the map in the right column of the diagram satisfy the properties that can be checked v -locally.

We now verify the finer properties that cannot be checked v -locally. Namely, we need to check that $f : D(\sigma) \rightarrow D(\sigma')$ is representable in spatial diamonds and that f has bounded topological transcendence

degree. By ([28] 13.4) the first property can be checked pro-étale locally. Given a map from a spatial diamond $\mathcal{F} \rightarrow D(\sigma')$ we let $X = \mathcal{F} \times_{D(\sigma')} D(\sigma)$. Applying lemma 2.2.24 repeatedly to the quotients $L^+\mathcal{H}_{r_k}/L^+\mathcal{H}_{q_k}$ we get that pro-étale locally on \mathcal{F} , X is of the form $\mathcal{F} \times_{O_C^\diamond} Fl_{r_n, q_n}^\diamond$ which is a spatial diamond. Moreover, if $\mathcal{F} = \text{Spa}(C', C'^+)$ with C' algebraically closed non-Archimedean field and C'^+ an open and bounded valuation subring then $X = Fl_{r_n, q_n, C'}^\diamond$ and $\dim.trg.(f) = \dim(Fl_{r_n, q_n}) < \infty$ (See [28] 21.7).

□

Proposition 2.2.27. *Fix σ as above. The projection map $\pi : W^+\mathcal{H}_{r_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} W^+\mathcal{H}_{r_n} \rightarrow D(\sigma)$ induced from the family of injections $W^+\mathcal{H}_{r_i} \subseteq L^+\mathcal{H}_{r_i}$ is a surjective map of v -sheaves. It induces an identification:*

$$\iota : D(\sigma) \cong W^+\mathcal{H}_{r_1} \times_{O_C^\diamond}^{W^+\mathcal{H}_{q_1}} \cdots \times_{O_C^\diamond}^{W^+\mathcal{H}_{q_{n-1}}} W^+\mathcal{H}_{r_n}/W^+\mathcal{H}_{q_n}$$

Consequently, $D(\sigma)$ is v -formalizing.

Proof. Consider the following basechange diagram:

$$\begin{array}{ccccc} W^+\mathcal{H}_{r_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} (L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n}) & \longrightarrow & L^+\mathcal{H}_{r_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} (L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n}) & \longrightarrow & D(\sigma) \\ \downarrow & & \downarrow & & \downarrow \\ W^+\mathcal{H}_{r_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} W^+\mathcal{H}_{r_{n-1}} & \longrightarrow & L^+\mathcal{H}_{r_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} L^+\mathcal{H}_{r_{n-1}} & \longrightarrow & D(\sigma') \end{array} \quad (2.8)$$

Proposition 2.2.23 gives the equality $W^+\mathcal{H}_{r_n}/W^+\mathcal{H}_{q_n} = L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n}$ and allow us to conclude surjectivity by induction. Assume that we have an identification:

$$\iota' : D(\sigma') \cong W^+\mathcal{H}_{r_1} \times_{O_C^\diamond}^{W^+\mathcal{H}_{q_1}} \cdots \times_{O_C^\diamond}^{W^+\mathcal{H}_{q_{n-2}}} W^+\mathcal{H}_{r_{n-1}}/W^+\mathcal{H}_{q_{n-1}}$$

Since $W^+\mathcal{H}_{q_k} \subseteq L^+\mathcal{H}_{q_k}$ the map ι is defined and surjective, we need to prove that ι is also injective. Let $[g_1]$ and $[g_2]$ be two maps

$$[g_1], [g_2] : \text{Spa}(R, R^+) \rightarrow W^+\mathcal{H}_{r_1} \times_{O_C^\diamond}^{W^+\mathcal{H}_{q_1}} \cdots \times_{O_C^\diamond}^{W^+\mathcal{H}_{q_{n-1}}} W^+\mathcal{H}_{r_n}/W^+\mathcal{H}_{q_n}$$

and suppose that they get identified after mapping to $D(\sigma)(R, R^+)$. By our inductive hypothesis on $D(\sigma')$ we may locally for the v -topology find representatives g_1 and g_2 of $[g_1]$ and $[g_2]$ whose projection to the first $n-1$ coordinates is the same. That is g_i are of the form (g_i^1, \dots, g_i^n) in $W^+\mathcal{H}_{r_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} W^+\mathcal{H}_{r_n}$ with $g_1^j = g_2^j$ for $j \in \{1, \dots, n-1\}$. Since $[g_1]$ and $[g_2]$ get identified in $D(\sigma)$ we must have that (v -locally) g_1 and g_2 are on the same $L^+\mathcal{H}_{q_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} L^+\mathcal{H}_{q_n}$ -orbit. Since g_1 and g_2 share all of their entries except possibly the last we have that g_1^n and g_2^n are in the same $L^+\mathcal{H}_{q_n}$ -orbit. Since $g_1^n, g_2^n \in W^+\mathcal{H}_{r_n}$ and $W^+\mathcal{H}_{q_n} = W^+\mathcal{H}_{r_n} \cap L^+\mathcal{H}_{q_n}$ they are in the same $W^+\mathcal{H}_{q_n}$ -orbit which proves $[g_1] = [g_2]$.

Finally, by proposition 2.2.19 each $W^+\mathcal{H}_{r_i}$ is formalizing, proposition 1.4.11 implies the same for the product, and since $D(\sigma)$ is the quotient of a v -formalizing sheaf it is also v -formalizing. □

Lemma 2.2.28. *Given two points $r_1, r_2 \in \mathcal{A}$ with $F_{r_1} \subseteq F_{r_2}$ the projection map of perfect schemes $\mathcal{W}_{\text{red}}^+\mathcal{H}_{r_1} \rightarrow \mathcal{W}_{\text{red}}^+\mathcal{H}_{r_1}/\mathcal{W}_{\text{red}}^+\mathcal{H}_{r_2} = Fl_{r_1, r_2}^{\text{perf}}$ admits Zariski locally a section.*

Proof. Let $\text{obs} \in H_{v\text{-Sch}}^1(Fl_{r_1, r_2}^{\text{perf}}, \mathcal{W}_{\text{red}}^+\mathcal{H}_{r_2})$ be the obstruction to finding a section. Consider the reduction morphism exact sequence:

$$e \rightarrow \mathcal{W}\mathcal{H}_{r_1}^u \rightarrow \mathcal{W}_{\text{red}}^+\mathcal{H}_{r_2} \rightarrow P_{\Phi_{r_1, r_2}}^{\text{perf}} \rightarrow e$$

We also have $Fl_{r_1, r_2}^{perf} = M_{r_1}^{perf} / P_{\Phi_{r_1, r_2}}^{perf}$ and the cohomology class in $H_{v-Sch}^1(Fl_{r_1, r_2}^{perf}, P_{\Phi_{r_1, r_2}}^{perf})$ associated to the $P_{\Phi_{r_1, r_2}}^{perf}$ -torsor $M_{r_1}^{perf} \rightarrow Fl_{r_1, r_2}^{perf}$ is the image of \mathbf{obs} . Zariski locally on Fl_{r_1, r_2}^{perf} the $P_{\Phi_{r_1, r_2}}^{perf}$ -torsor is trivial. This is known for the classical flag variety Fl_{r_1, r_2} over $\mathrm{Spec}(k_C)$ and the result will follow from taking perfection. Indeed, consider a commutative diagram trivializing the $P_{\Phi_{r_1, r_2}}$ -torsor:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\iota} & Fl_{p_1, p_2} \\ & \searrow s & \uparrow \\ & & M_{p_1} \end{array}$$

Any such diagram having ι as Zariski open cover will produce a similar diagram after we take perfections and will trivialize the $P_{\Phi_{r_1, r_2}}^{perf}$ -torsor.

Fix an affine cover $\mathrm{Spec}(R) \rightarrow Fl_{r_1, r_2}^{perf}$ trivializing $\mathrm{red}(\mathbf{obs})$. We claim that $\mathrm{Spec}(R)$ has a section to $\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{r_1} \times_{Fl_{r_1, r_2}^{perf}} \mathrm{Spec}(R)$. By construction, we know that $\mathbf{obs} \in H_{v-Sch}^1(\mathrm{Spec}(R), \mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{r_2})$ is in the image of $H_{v-Sch}^1(\mathrm{Spec}(R), \mathcal{W}^u \mathcal{H}_{r_1})$, but this pointed set is trivial. Indeed, we have that $H_{v-Sch}^1(\mathrm{Spec}(R), \mathbb{G}_a) = \{e\}$ which is a particular case of theorem 4.1 in [5]. One can finish the proof by using the argument given in lemma 2.2.24. \square

Proposition 2.2.29. *The map $D(\sigma) \rightarrow O_C^\diamond$ is formally adic. Moreover, $D(\sigma)^{\mathrm{red}}$ is represented by a qcqs scheme that is perfectly finitely presented and proper over $\mathrm{Spec}(k_C)$ (See [5] 3.11 and 3.14 for definitions).*

Proof. Let $\sigma = (\{r_i\}_{1 \leq i \leq n}, \{q_i\}_{1 \leq i \leq n})$ and let $\sigma' = (\{r_i\}_{1 \leq i \leq n-1}, \{q_i\}_{1 \leq i \leq n-1})$. In any Grothendieck topos pullback commutes with finite limits and colimits, so by proposition 2.2.19 we have:

$$D(\sigma) \times_{O_C^\diamond} \mathrm{Spec}(k_C)^\diamond = (\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{r_1})^\diamond \times_{k_C^\diamond}^{(\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_1})^\diamond} \cdots \times_{k_C^\diamond}^{(\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_{n-1}})^\diamond} (\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{r_n})^\diamond / (\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_n})^\diamond$$

Since the functor $(\cdot)^\diamond$ is a left adjoint it commutes with colimits, so we get:

$$D(\sigma) \times_{O_C^\diamond} \mathrm{Spec}(k_C)^\diamond = \left(\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{p_1} \times_{k_C}^{(\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_1})^\diamond} \cdots \times_{k_C}^{(\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_{n-1}})^\diamond} \mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{p_n} / \mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_n} \right)^\diamond$$

Lemma 1.3.35 proves that $D(\sigma) \rightarrow O_C^\diamond$ is formally adic and that $D(\sigma)^{\mathrm{red}} = D(\sigma) \times_{O_C^\diamond} \mathrm{Spec}(k_C)^\diamond$. In particular $\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{r_k} / \mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_k} = Fl_{r_k, q_k}$.

We prove inductively that $D(\sigma)^{\mathrm{red}}$ is represented by a qcqs scheme perfectly finitely presented and proper over $\mathrm{Spec}(k_C)$. Iterating lemma 2.2.28 we see that the map $D(\sigma)^{\mathrm{red}} \rightarrow D(\sigma')^{\mathrm{red}}$ is Zariski locally a trivial Fl_{r_k, q_k}^{perf} -bundle. Now, quasi-compactness, quasi-separatedness, separatedness, representability in schemes, being perfectly of finite presentation and being proper can all be checked Zariski locally on the target and are stable under basechange and composition (See [31] Tag 02YJ). By induction, $D(\sigma')$ enjoys all of these properties over $\mathrm{Spec}(k)$ and $Fl_{r_n, q_n} \times_{k_C} \mathrm{Spec}(A)$ enjoys them over $\mathrm{Spec}(A)$ for any affine open $\mathrm{Spec}(A) \subseteq D(\sigma')^{\mathrm{red}}$. This proves that $D(\sigma)$ also enjoys them over $\mathrm{Spec}(k_C)$, which finishes the proof. \square

Proposition 2.2.30. *For any σ and any geometric point $\mathrm{Spa}(C', O_{C'}) \rightarrow O_C^\diamond$ the base change $D(\sigma) \times_{O_C^\diamond} \mathrm{Spa}(C', O_{C'})$ is a cJ-diamond.*

Proof. We prove by induction that $D(\sigma)$ has enough facets, let $\sigma = (\{r_i\}_{1 \leq i \leq n}, \{q_i\}_{1 \leq i \leq n})$ and let $\sigma' = (\{r_i\}_{1 \leq i \leq n-1}, \{q_i\}_{1 \leq i \leq n-1})$. Suppose that $D(\sigma')_{C'}$ has enough facets, let $S := \coprod_{i \in I} \mathrm{Spd}(B_i, B_i^\circ)$ with each B_i a topologically of finite type C' -algebra and let $f : S \rightarrow D(\sigma')_{C'}$ be a surjective map. Let $\mathcal{F} = D(\sigma)_{C'} \times_{D(\sigma')_{C'}} S$, we prove that \mathcal{F} has a enough facets. Analytically locally on S the projection map $\mathcal{F} \rightarrow S$ is a trivial $(Fl_{r_n, q_n, C'^\#})^\diamond$ -fibration. The proof of this claim is an iteration of the argument given

on lemma 2.2.24 together with the observation that S is already a disjoint union of affinoid perfectoid spaces. We may replace S by an analytic cover S' so that we get the expression:

$$\mathcal{F}' = D(\sigma)_{C'} \times_{D(\sigma')_{C'}} S' = \coprod \mathrm{Spd}(B'_i, (B'_i)^\circ) \times_{\mathrm{Spa}(C', O_{C'})} (Fl_{r_n, q_n, C'^\#})^\diamond$$

By proposition 1.4.39 having enough facets is stable under products so \mathcal{F}' has enough facets and consequently $D(\sigma)$ as well. \square

We can summarize this subsection with the following theorem:

Theorem 2.2.31. *For any σ as in definition 2.2.25 the Demazure kimberlite $D(\sigma)$ is a rich p -adic kimberlite. The p -adic tubular neighborhoods are non-empty and connected, and the structure morphism $D(\sigma) \rightarrow O_C^\diamond$ is proper and ℓ -cohomologically smooth.*

Proof. Separatedness and the properties of the structure morphism were proven on proposition 2.2.26. That it is v -formalizing is proven in proposition 2.2.27. We have

$$(D(\sigma)^{\mathrm{red}})^\diamond = D(\sigma) \times_{O_C^\diamond} \mathrm{Spec}(k_C)^\diamond$$

this implies that the adjunction morphism $(D(\sigma)^{\mathrm{red}})^\diamond \rightarrow D(\sigma)$ is a closed embedding. By proposition 1.3.31 $D(\sigma)$ is formally separated and specializing. By proposition 2.2.29 $D(\sigma)^{\mathrm{red}}$ is represented by a scheme. At this point we have proved that $D(\sigma)$ is a p -adic pre-kimberlite.

Since $D(\sigma)^{\mathrm{red}}$ is represented by a proper perfectly finitely presented scheme over k_C then $|D(\sigma)^{\mathrm{red}}|$ is a Noetherian and spectral topological space. The analytic locus $D(\sigma)^{\mathrm{an}}$ coincides with the generic fiber $D(\sigma) \times_{O_C^\diamond} \mathrm{Spa}(C, O_C)$ and by proposition 2.2.30 this is a cJ -diamond. One can easily prove inductively over the map $D(\sigma) \rightarrow D(\sigma')$ that the specialization map is surjective on closed points. By lemmas 1.4.43 and 1.4.44 it is a quotient map. This finishes the proof that $D(\sigma)$ is a rich kimberlite.

The connectedness of tubular neighborhoods will follow from lemma 2.2.32 below. Indeed, we have already verified that all but conditions 4 and 5 of this lemma hold. Condition 5 holds by induction over the maps $D(\sigma) \rightarrow D(\sigma')$ and condition 4 follows from the diagram 2.8 since each of the $W^+ \mathcal{H}_{r_i}$ is formalizing and basechanges along maps that factor through $W^+ \mathcal{H}_{r_1} \times_{O_C^\diamond} \cdots \times_{O_C^\diamond} W^+ \mathcal{H}_{r_{n-1}}$ will give a trivial bundle. \square

Lemma 2.2.32. *Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a map of kimberlites over O_C^\diamond , let $X \rightarrow \mathrm{Spec}(O_{C^\#})$ be a smooth projective scheme. Suppose the following properties hold:*

1. *f is ℓ -cohomologically smooth for some $\ell \neq p$.*
2. *f is proper.*
3. *$f : \mathcal{F} \rightarrow \mathcal{G}$ is a $X(\mathcal{O}^{\#,+})$ -bundle locally trivial for the v -topology.*
4. *For any non-Archimedean field C' in characteristic 0 and any map $t : \mathrm{Spa}(C', O_{C'}) \rightarrow \mathcal{G}$ there is a v -cover $r : \mathrm{Spa}(C'', O_{C''}) \rightarrow \mathrm{Spa}(C', O_{C'})$ such that \mathcal{G} formalizes $t \circ r$ and the base change $\mathcal{F} \times_{\mathcal{G}} O_{C''}^\diamond$ is isomorphic to $X(\mathcal{O}^{\#,+}) \times_{O_C^\diamond} O_{C''}^\diamond$.*
5. *For any closed point $x \in |\mathcal{G}^{\mathrm{red}}|$ the p -adic tubular neighborhood $(\widehat{\mathcal{G}}_{/x})_\eta$ is connected.*
6. *$\mathcal{G}^{\mathrm{red}}$ and $\mathcal{F}^{\mathrm{red}}$ are perfectly finitely presented (See [5] 3.10) over $\mathrm{Spec}(k_C)$.*

Then, for any closed point $y \in |\mathcal{F}^{\mathrm{red}}|$ the p -adic tubular neighborhood $(\widehat{\mathcal{F}}_{/y})_\eta$ is also connected.

Proof. We begin by making the observation that f is open and closed since it is ℓ -cohomologically smooth and proper (See [28] 23.11). Take a closed point $y \in |\mathcal{F}^{\text{red}}|$ with $x = f(y)$ and consider the map $f : (\widehat{\mathcal{F}}_y)_\eta \rightarrow (\widehat{\mathcal{G}}_x)_\eta$. Assume for the moment that given C' an arbitrary algebraically closed non-Archimedean field and a map of the form $\text{Spa}(C', O_{C'}) \rightarrow (\widehat{\mathcal{G}}_x)_\eta$ the base change $(\widehat{\mathcal{F}}_y)_\eta \times_{(\widehat{\mathcal{G}}_x)_\eta} \text{Spa}(C', O_{C'})$ is always non-empty and connected, we finish the proof under this assumption. Observe that the map of topological spaces $|(\widehat{\mathcal{F}}_y)_\eta| \rightarrow |(\widehat{\mathcal{G}}_x)_\eta|$ is specializing, and by assumption surjective on rank 1 points. Take two non-empty open and closed subsets U and V with $U \cup V = |(\widehat{\mathcal{F}}_y)_\eta|$. Then $f(U) \cup f(V) = |(\widehat{\mathcal{G}}_x)_\eta|$ and consequently $f(U) \cap f(V) \neq \emptyset$. Since f is an open map $f(U)$ and $f(V)$ must meet in a rank 1 point, this implies that U and V also meet which finishes the proof under our assumption.

Let us prove our assumption holds. Take a map $t : \text{Spa}(C', O_{C'}) \rightarrow (\widehat{\mathcal{G}}_x)_\eta$ and after replacing $\text{Spa}(C', O_{C'})$ by a v -cover we can assume \mathcal{G} formalizes the composition $\text{Spa}(C', O_{C'}) \rightarrow \mathcal{G}$ and has the base change property of condition 4 with respect to the unique formalization $O_{C'}^\diamond \rightarrow \mathcal{G}$. We get a Cartesian diagram:

$$\begin{array}{ccccc} \widehat{\mathcal{F}}_y \times_{\mathcal{G}} O_{C'}^\diamond & \longrightarrow & X(\mathcal{O}^{\sharp,+}) \times_{O_{\mathcal{G}}^\diamond} O_{C'}^\diamond & \longrightarrow & O_{C'}^\diamond \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathcal{F}}_y & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} \end{array}$$

After taking reduction functor of this diagram we get the following Cartesian diagram:

$$\begin{array}{ccccc} Z & \longrightarrow & X \times \text{Spec}(k') & \longrightarrow & \text{Spec}(k') \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(k(y)) & \xrightarrow{y} & \mathcal{F}^{\text{red}} & \longrightarrow & \mathcal{G}^{\text{red}} \end{array}$$

Since $\mathcal{F}^{\text{red}} \rightarrow \mathcal{G}^{\text{red}}$ is perfectly finitely presented and k is algebraically closed we have that $k = k(y) = k(x)$ and the composition $y : \text{Spec}(k) \rightarrow \mathcal{G}^{\text{red}}$ is the closed immersion corresponding to the point x . Consequently $Z \rightarrow \text{Spec}(k')$ is a closed immersion, therefore an isomorphism. We have that

$$\widehat{\mathcal{F}}_y \times_{\mathcal{G}} O_{C'}^\diamond = X(\mathcal{O}^{\sharp,+})_{O_{C'}} \times_{\mathcal{F}} \widehat{\mathcal{F}}_y = (X(\widehat{\mathcal{O}^{\sharp,+}})_{O_{C'}})_{/Z}$$

by proposition 1.4.26. But $Z \rightarrow X \times \text{Spec}(k')$ is a closed point, so $(X(\widehat{\mathcal{O}^{\sharp,+}})_{O_{C'}})_{/Z}$ is isomorphic to an open unit ball $\mathbb{B}_n^{<1}$ over C'^\sharp , where $n = \dim(X)$ and C'^\sharp is the untilt determined by $O_{C'}^\sharp$. We have proved that the fibers are non-empty and connected. \square

2.2.4 Resolution of p -adic Beilinson-Drinfeld Grassmanians

In this subsection we discuss an analogue of the Demazure resolution for split reductive groups in the context of v -sheaves (also known as Bott-Samelson resolution). We keep the notation from the beginning of the previous subsection and we restrict our attention to parahoric loop groups associated to points contained in our chosen alcove \mathcal{C} . Given $s_j \in \mathbb{S}$ we denote by $L^+ \mathcal{H}_{s_j}$ the parahoric loop group associated to the wall F_{s_j} in \mathcal{C} corresponding to the reflection s_j . For a point $r \in \mathcal{C}$ we let $J_r \subseteq \mathbb{S}$ denote the set $\{s_j \mid r \in F_{s_j}\}$. We will denote by $L^+ B$ the parahoric loop group associated \mathcal{C} .

By functoriality of $L(-)$ we can define a loop group version of the Weyl group by the formula $LW := LN/LT$. We can also define the Iwahori-Weyl group as $L\tilde{W} := LN/L^+ \mathcal{T}$. There is an exact sequence of v -sheaves in groups:

$$e \rightarrow LT/L^+ \mathcal{T} \rightarrow L\tilde{W} \rightarrow LW \rightarrow e$$

One can prove by a direct computation that $LW = L(N/T) = \underline{W} \times O_C^\diamond$ and that $LT/L^+T = \underline{X}_*(T) \times O_C^\diamond$ by using the Cartan decomposition. These two imply that $L\tilde{W} = \tilde{W} \times O_C^\diamond$.

Since H is a split reductive group over $W(k)[t, t^{-1}]$, for any element $w \in W$ we can find a section $n_w : \text{Spec}(W(k)[t, t^{-1}]) \rightarrow N$ whose projection to W is w (See [8] 5.1.11). This allow us to define a similar section $n_w : O_C^\diamond \rightarrow LN \subseteq LH$. Also for any $\mu \in X_*(T)$ and any $\text{Spa}(R, R^+) \rightarrow O_C^\diamond$ we can consider the element $\xi^\mu \in T(B_{dR}(R^\sharp))$. This is functorial and defines a section $O_C^\diamond \rightarrow LT$ mapping to $\mu \in \underline{X}_*(T) \times O_C^\diamond$. In particular, for any element $\tilde{w} \in \tilde{W}$ there is a section $n_{\tilde{w}} : O_C^\diamond \rightarrow LN$ projecting to \tilde{w} in $L\tilde{W}$. We can use $n_{\tilde{w}}$ to construct an automorphism $n_{\tilde{w}} : \text{Gr}_{O_C^\diamond}^H \rightarrow \text{Gr}_{O_C^\diamond}^H$ with

$$n_{\tilde{w}}(x \cdot L^+H) := n_{\tilde{w}} \cdot x \cdot L^+H.$$

We will use this discussion in the proof of theorem 2.2.34.

Proposition 2.2.33. *Let $\sigma = (\sigma_r, \sigma_q)$ with σ as in the previous subsection except that we require $\sigma_r, \sigma_q \subseteq \mathcal{C}$. Suppose that $L^+\mathcal{H}_{q_n} = L^+\mathcal{H}_{r_n} = L^+H$ then the multiplication map $\mu : D(\sigma) \rightarrow \text{Gr}_{O_C}^H = LH/L^+H$ has geometrically connected fibers.*

Proof. This proof follows the classical one. The key inputs are as follows, the basechange of $D(\sigma) \rightarrow O_C^\diamond$ by geometric points are proper spatial diamonds, rank 1 points are dense for any spatial diamond and the group of rank 1 geometric points of a parahoric loop group coincide with the “parabolic subgroups” of a Tits-systems (or BN -pair). These two observations together with ([28] 12.11) reduces the proof to the classical combinatorial case. Indeed, properness (which includes quasi-compactness) will allow us to prove all surjectivity claims at the level of rank 1 geometric points. We provide further details below for the convenience of the reader.

Fix a geometric point $\text{Spa}(C', O_{C'}) \rightarrow O_C^\diamond$, all of the objects considered in our argument below are considered over $\text{Spa}(C', O_{C'})$ but we omit the basechange from the notation. Let us start by making some reductions, observe that since we are assuming that $\sigma_q \subseteq \mathcal{C}$ we have $L^+B \subseteq L^+\mathcal{H}_{q_i}$ so we get a surjective map:

$$D(\tau) := L^+\mathcal{H}_{r_1} \overset{L^+B}{\times} \dots \overset{L^+B}{\times} L^+\mathcal{H}_{r_n}/L^+B \rightarrow D(\sigma)$$

Surjectivity allows us to replace $D(\sigma)$ for $D(\tau)$ so we may assume $L^+\mathcal{H}_{q_i} = L^+B$ for all $i \leq n$. Now the flag varieties $L^+\mathcal{H}_{r_i}/L^+B$ admit a surjective map from a finite contracted product of the form:

$$L^+\mathcal{H}_{s_{j_1}}, \overset{L^+B}{\times} \dots \overset{L^+B}{\times} L^+\mathcal{H}_{s_{j_m}}/L^+B \rightarrow L^+\mathcal{H}_{r_1}/L^+B$$

Where $s_{j_k} \in J_{r_i}$ and the product $s_{j_1} \cdots s_{j_m}$ is a reduced expression for the longest word in the finite Coxeter group generated by J_{r_i} . This lets us reduce to the case in which for all $i \leq n$, $L^+\mathcal{H}_{r_i} = L^+\mathcal{H}_{s_j}$ for some j . Moreover, in this case the map $D(\tau) \rightarrow \text{Gr}_{C'}^H$ factors through $LH/L^+B \rightarrow \text{Gr}_{C'}^H$, which is a L^+H/L^+B -bundle.

We prove inductively that $D(\tau) \rightarrow LH/L^+B$ has connected geometric fibers. Write $S(\tau)$ for the image of $D(\tau)$ in LH/L^+B . The multiplication map factors as:

$$D(\tau) = L^+\mathcal{H}_{s_{j_1}}, \overset{L^+B}{\times} D(\tau') \rightarrow L^+\mathcal{H}_{s_{j_1}}, \overset{L^+B}{\times} S(\tau') \rightarrow S(\tau)$$

If we assume inductively that the map $D(\tau') \rightarrow S(\tau')$ has connected geometric fibers, then it suffices to prove that $L^+\mathcal{H}_{s_{j_1}}, \overset{L^+B}{\times} S(\tau') \rightarrow S(\tau)$ also has connected geometric fibers.

Notice that by construction $S(\tau') \subseteq LH/L^+B$ is a closed subsheaf that is stable under the action of L^+B . As in the classical case the L^+B -orbits of geometric points in LH/L^+B are indexed \tilde{W} . Given an element $w \in \tilde{W}$ we can consider $C(w)$ the locally-closed subsheaf of LH/L^+B associated to this

L^+B -orbit and we can let $S(w) = \bigcup_{w' \leq w} C(w')$ where ' \leq ' denotes the Bruhat order of the quasi-Coxeter system $\mathbb{S} \subseteq W^{aff} \subseteq \tilde{W}$. We also assume in our inductive hypothesis that $S(\tau') = S(w)$ for $w \in W^{aff}$ of the form $w = s_{j_{k_1}} \dots s_{j_{k_l}}$ where s_{j_k} is a subsequence of elements in \mathbb{S} of the sequence appearing in the definition of $D(\tau')$.

For the induction step we have two cases, either $s_{j_1} \cdot w < w$ or $s_{j_1} \cdot w > w$. In the first case we will have that the action of $L^+\mathcal{H}_{s_{j_1}}$ on LH/L^+B stabilizes $S(w)$ so that the multiplication map

$$L^+\mathcal{H}_{s_{j_1}} \overset{L^+B}{\times} S(w) \rightarrow S(w)$$

decomposes as the composition of an isomorphism $L^+\mathcal{H}_{s_{j_1}} \overset{L^+B}{\times} S(w) \rightarrow L^+\mathcal{H}_{s_{j_1}}/L^+B \times S(w)$ followed by the second projection. In this case geometric fibers are isomorphic to $L^+\mathcal{H}_{s_{j_1}, C''}/L^+B_{C''} \cong (\mathbb{P}_{C''}^1)^\diamond$, and we have that $S(\tau) = S(\tau') = S(w)$ which is of the form assumed in our inductive hypothesis.

On the other hand, if $s_{j_1} \cdot w > w$ we can consider the collection T of $w' < w$ for which $s \cdot w' < w'$ we have that the multiplication map

$$L^+\mathcal{H}_{s_{j_1}} \overset{L^+B}{\times} S(w) \setminus \bigcup_{w' \in T} S(w') \rightarrow S(s_{j_1} \cdot w) \setminus \bigcup_{w' \in T} S(w')$$

is an isomorphism while the map

$$L^+\mathcal{H}_{s_{j_1}} \overset{L^+B}{\times} \bigcup_{w' \in T} S(w') \rightarrow \bigcup_{w' \in T} S(w')$$

has geometric fibers as in the previous case since this set is also $L^+\mathcal{H}_{s_{j_1}}$ -stable. Moreover, we have $S(\tau) = S(s_{j_1} \cdot w)$ which is again of the form assumed in our induction hypothesis, this finishes inductive step and the proof. \square

Theorem 2.2.34. *Let \mathcal{G} be a quasi-split reductive group over $W(k)$, $\mathfrak{T} \subseteq \mathfrak{B} \subseteq \mathcal{G}$ a Borel and a maximal torus in \mathcal{G} defined over $W(k)$ and take a cocharacter $\mu \in X_*(\mathfrak{T})$ defined over an algebraic closure of $W(k)[\frac{1}{p}]$. Let F be a non-Archimedean field extension of $W(k)[\frac{1}{p}]$ containing $E(\mu)$ the reflex field of μ . We let O_F the ring of integers of F and the residue field k_F , assume that F is complete for the p -adic topology and that k_F is perfect. Then $\mathrm{Gr}_{O_F^\diamond}^{\mathcal{G}, \leq \mu}$ is an rich p -adic kimberlite over O_F^\diamond . Moreover, the p -adic tubular neighborhoods of $\mathrm{Gr}_{O_F^\diamond}^{\mathcal{G}, \leq \mu}$ at closed points are non-empty and connected.*

Proof. In corollary 2.2.7 we show that $\mathrm{Gr}_{O_F^\diamond}^{\mathcal{G}, \leq \mu}$ is a p -adic kimberlite so the only thing left to prove are the statements related to the structure of p -adic tubular neighborhoods. We first prove the case in which F is a complete algebraically closed extension of $W(k)[\frac{1}{p}]$ which we will denote instead by C . In this case $\mathcal{G} \times_{W(k)} W(k_C)$ is isomorphic to a split reductive group, and since the functor $\mathrm{Gr}_{O_C^\diamond}^{\mathcal{G}, \leq \mu}$ only depends on the isomorphism class of $\mathcal{G}_{W(k_C)}$, we may assume $\mathcal{G} = H$ with H split reductive. Furthermore, we discuss first the case in which H is semisimple and simply connected, in this case $\tilde{W} = W^{aff}$.

Recall that we have inclusions $X_*^+(\mathfrak{T}) \subseteq X_*(\mathfrak{T}) \subseteq \tilde{W}$ so we may think of μ as an element of the Iwahori-Weyl group. By definition, $\mathrm{Gr}_{O_C^\diamond}^{H, \leq \mu}(R, R^+)$ consists of those elements in $\mathrm{Gr}_{O_C^\diamond}^H(R, R^+)$ satisfying that for any geometric point $q : \mathrm{Spa}(C', C'^+) \rightarrow \mathrm{Spa}(R, R^+)$ the type of q , μ_q , is in the double coset

$$H(B_{dR}^+(C'^\sharp)) \backslash H(B_{dR}(C'^\sharp)) / H(B_{dR}^+(C'^\sharp)) = X_*^+(T) = W_o \backslash W^{aff} / W_o$$

satisfies that $\mu_q \leq \mu$ in the Bruhat order. Now given any element $w \in \tilde{W}$ we may consider the subsheaf $\mathrm{Gr}_{O_C^\diamond}^{G, \leq w} \subseteq \mathrm{Gr}_{O_C^\diamond}^G$ given instead by the property that on a geometric point $q : \mathrm{Spa}(C', C'^+) \rightarrow \mathrm{Spa}(R, R^+)$ the type of q , $[w_q]$, in the double coset

$$\mathfrak{B}(B_{dR}^+(C'^\sharp)) \backslash H(B_{dR}(C'^\sharp)) / H(B_{dR}^+(C'^\sharp)) = W^{aff} / W_o$$

satisfies that $[w_q] \leq [w]$ in the Bruhat order. The projection map $\pi : W^{aff}/W_o \rightarrow W_o \backslash W^{aff}/W_o$ is order preserving and $\pi^{-1}(\mu)$ has a unique element $[w_\mu]$ of largest length, it has the property that $v \leq w_\mu$ if and only if $\pi(v) \leq \mu$. In particular, we have equalities of sheaves $\text{Gr}_{O_C^\diamond}^{H, \leq w_\mu} = \text{Gr}_{O_C^\diamond}^{H, \leq \mu}$. We prove that for any word $w \in W^{aff}$ the v -sheaf $\text{Gr}_{O_C^\diamond}^{H, \leq w}$ satisfies the conclusions of the theorem. If we find a reduced expression for $w = s_{j_1} \dots s_{j_n}$ we can use the theory of BN-pairs to construct a Demazure kimberlite

$$D(w) := L^+ H_{s_{j_1}} \overset{L^+ \mathfrak{B}}{\times}_{O_C^\diamond} \dots L^+ H_{s_{j_n}} / L^+ H$$

for which the multiplication map $m : D(w) \rightarrow \text{Gr}_{O_C^\diamond}^H$ factors through $\text{Gr}_{O_C^\diamond}^{H, \leq w}$ and surjects onto it at the level of rank 1 geometric points. But m is a proper map so that by ([28] 12.11) it is actually a surjection of v -sheaves. Moreover, this also proves that $\text{Gr}_{O_C^\diamond}^{H, \leq w}$ is a closed subsheaf of $\text{Gr}_{O_C^\diamond}^H$. Theorem 2.2.31 and proposition 2.2.33 combined with lemma 1.4.45 allow us to conclude in this case.

Suppose now that H is an arbitrary split reductive group. In this case, $\mu \in \tilde{W}$ can be expressed as $\mu = (w, \omega)$ with $w \in W^{aff}$ and $\omega \in \Omega_H$ for the decomposition $\tilde{W} = W^{aff} \rtimes \Omega_H$. We may find a section $n_\omega : O_C^\diamond \rightarrow L\tilde{W}$ projecting to (e, ω) in $L\tilde{W}$, this section induces an isomorphism between $\text{Gr}_{O_C^\diamond}^{H, \leq \mu}$ and $\text{Gr}_{O_C^\diamond}^{H, \leq (w, e)}$. But for any $w \in W^{aff}$ the v -sheaf $\text{Gr}_{O_C^\diamond}^{H, \leq (w, e)}$ admits a surjective map by a Demazure kimberlite as in the previous case.

Finally, let us deal with the general case in which F is not assumed to be algebraically closed. Let C be the completion of an algebraic closure of F and F' the completion of the maximal unramified subextension of F inside C . We have surjective maps of v -sheaves:

$$\text{Gr}_{O_C^\diamond}^{\mathcal{G}, \leq \mu} \rightarrow \text{Gr}_{O_{F'}^\diamond}^{\mathcal{G}, \leq \mu} \rightarrow \text{Gr}_{O_F^\diamond}^{\mathcal{G}, \leq \mu}$$

Lemma 1.4.45 implies that $\text{Gr}_{O_F^\diamond}^{\mathcal{G}, \leq \mu}$ and $\text{Gr}_{O_{F'}^\diamond}^{\mathcal{G}, \leq \mu}$ are rich kimberlites. Moreover, we can infer by proposition 1.4.26 that $\text{Gr}_{O_{F'}^\diamond}^{\mathcal{G}, \leq \mu}$ has connected p -adic tubular neighborhoods since we have an identification $(\text{Gr}_{O_C^\diamond}^{\mathcal{G}, \leq \mu})^{\text{red}} = (\text{Gr}_{O_{F'}^\diamond}^{\mathcal{G}, \leq \mu})^{\text{red}}$. On the other hand, the map $\text{Gr}_{O_{F'}^\diamond}^{\mathcal{G}, \leq \mu} \rightarrow \text{Gr}_{O_F^\diamond}^{\mathcal{G}, \leq \mu}$ is a $\pi_1^{f\acute{e}t}(\text{Spec}(O_F))$ -torsor and for any closed point $x \in |\text{Gr}_{O_F^\diamond}^{\mathcal{G}, \leq \mu}{}^{\text{red}}|$ the action of $\pi_1^{f\acute{e}t}(\text{Spec}(O_F))$ will permute transitively the closed points $y \in |\text{Gr}_{O_{F'}^\diamond}^{\mathcal{G}, \leq \mu}{}^{\text{red}}|$ over x . In particular, the action permutes transitively the p -adic tubular neighborhoods associated to such y . This proves that the tubular neighborhood over x is also connected. \square

We finish this section with the proof of theorem 1 which is just a rephrasing of theorem 2.2.34 in less technical language. For the convenience of the reader we write the statement again.

Theorem 2.2.35. *With notation as in the introduction the following holds:*

a) *The specialization map*

$$\text{sp}_{\text{Gr}_{O_{F_1}^\diamond}^{\mathcal{G}, \leq \mu}} : |\text{Gr}_{F_1^\diamond}^{G, \leq \mu}| \rightarrow |\text{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}|$$

is a closed and spectral map of spectral topological spaces.

b) *Given a closed point $x \in |\text{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}|$ let $T_x := \text{sp}_{\text{Gr}_{O_{F_1}^\diamond}^{\mathcal{G}, \leq \mu}}^{-1}(x)$, then the interior T_x° of T_x in $|\text{Gr}_{F_1^\diamond}^{G, \leq \mu}|$ is a dense subset of T_x .*

c) *T_x and T_x° are connected.*

Proof of theorem 1. We may apply theorem 2.2.34 and proposition 2.2.5 to the case in which $k = \mathbb{F}_p$ to conclude that $\mathrm{Gr}_{O_{F_1}^\diamond}^{\mathcal{G}, \leq \mu}$ is a rich p -adic kimberlite with generic fiber $\mathrm{Gr}_{F_1^\diamond}^{G, \leq \mu}$ and with reduction $\mathrm{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}$. Since $\mathrm{Gr}_{O_{F_1}^\diamond}^{\mathcal{G}, \leq \mu}$ is a kimberlite by proposition 1.4.20 the specialization map

$$\mathrm{sp}_{\mathrm{Gr}_{O_{F_1}^\diamond}^{\mathcal{G}, \leq \mu}} : |\mathrm{Gr}_{F_1^\diamond}^{G, \leq \mu}| \rightarrow |\mathrm{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}|$$

is a spectral map of locally spectral spaces. Since $\mathrm{Gr}_{O_{F_1}^\diamond}^{\mathcal{G}, \leq \mu}$ is rich, the map is surjective and specializing, and since $\mathrm{Gr}_{F_1^\diamond}^{G, \leq \mu}$ is quasi-compact proposition 1.1.22 gives that the specialization map is closed, this finishes the proof of the first claim. For the second claim let $x \in |\mathrm{Gr}_{\mathcal{W}, k_{F_1}}^{\mathcal{G}, \leq \mu}|$, we can use proposition 1.4.29 to identify T_x° with $|\widehat{(\mathrm{Gr}_{O_{F_1}^\diamond}^{\mathcal{G}, \leq \mu})_\eta}|$. Since $\mathrm{Gr}_{O_{F_1}^\diamond}^{\mathcal{G}, \leq \mu}$ is rich we can apply proposition 1.4.33 to prove that T_x° is dense in T_x giving the second claim. By 2.2.34 T_x° is connected and since it is dense in T_x this later one is also connected. \square

2.3 Specialization for moduli of mixed characteristic shtukas

For the rest of this section we will assume that $k = \mathbb{F}_p$ and that \mathcal{G} is a reductive group over \mathbb{Z}_p . We fix a torus and a Borel $\mathfrak{T} \subseteq \mathfrak{B} \subseteq \mathcal{G}$. We fix \mathfrak{f} an algebraically closed field extension of $\overline{\mathbb{F}}_p$ and we let $K_0 = W(\mathfrak{f})[\frac{1}{p}]$, we fix an element $b \in \mathcal{G}(K_0)$ and we let $\mathcal{G}_b : \mathrm{Rep}_{\mathbb{Z}_p}^{\mathcal{G}} \rightarrow \mathrm{IsoCrys}_{K_0}$ denote the \otimes -exact functor from the category of algebraic representations of \mathcal{G} to the category of isocrystals over K_0 associated to b .

Definition 2.3.1. We define the moduli space of mixed characteristic shtukas associated to \mathcal{G}_b , which we denote by $\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$, as the functor $\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b} : \mathrm{Perf} \rightarrow \mathrm{Sets}$:

$$\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}(R, R^+) = \{(R^\sharp, \iota, f), \mathcal{T}, \Phi, \lambda\} / \cong$$

Where (R^\sharp, ι, f) denotes an untilt of R over $\mathrm{Spa}(W(\mathfrak{f}), W(\mathfrak{f}))$, the pair (\mathcal{T}, Φ) is a shtuka as in definition 2.1.22 and $\lambda : \mathcal{T} \rightarrow \mathcal{G}_b|_{\mathcal{Y}_{[r, \infty)}^{R^+}}$ is an equivalence class of isogenies as in definition 2.1.23. Here $\mathcal{G}_b|_{\mathcal{Y}_{[r, \infty)}^{R^+}}$ denotes the pullback along the natural map of locally ringed spaces $\mathcal{Y}_{[r, \infty)}^{R^+} \rightarrow \mathrm{Spec}(K_0)$ induced by f .

As with p -adic Beilinson-Drinfeld Grassmanians, moduli spaces of shtukas admit bounded versions. Given a geometric point of our moduli $\mathrm{Spa}(C, C^+) \rightarrow \mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ the torsor \mathcal{T} can be glued with \mathcal{G}_b along $\mathcal{Y}_{[0, \infty)}^{C^+}$ to extend it to \mathcal{Y}_{C^+} . This gives a \mathcal{G} -torsor over Y_{C^+} by theorem 2.1.12. One can basechange this torsor to $B_{dR}^+(C^\sharp)$ where we can choose a trivialization of $\tau : \mathcal{T} \rightarrow \mathcal{G}$. The morphism $\tau \circ \Phi : \phi^* \mathcal{T} \rightarrow \mathcal{G}$ defines an element of $\mathcal{G}(B_{dR}(C^\sharp)) / \mathcal{G}(B_{dR}^+(C^\sharp))$ whose image, $\mu_{(\mathcal{T}, \Phi)}$, in the double coset

$$\mathcal{G}(B_{dR}^+(C^\sharp)) \backslash \mathcal{G}(B_{dR}(C^\sharp)) / \mathcal{G}(B_{dR}^+(C^\sharp)) = X_*^+(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$$

does not depend on the choice of τ . We call $\mu_{(\mathcal{T}, \Phi)}$ the relative position of the shtuka at that geometric point.

Definition 2.3.2. Let $\mu \in X_*^+(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$. We define the moduli space of mixed characteristic shtukas associated to \mathcal{G}_b and bounded by μ , which we denote by $\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$, as the functor $\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu} : \mathrm{Perf} \rightarrow \mathrm{Sets}$:

$$\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}(R, R^+) = \{(R^\sharp, \iota, f), \mathcal{T}, \Phi, \lambda\} / \cong$$

Where (R^\sharp, ι, f) denotes an untilt of R over $\mathrm{Spa}(W(\mathfrak{f}), W(\mathfrak{f}))$, the pair (\mathcal{T}, Φ) is a shtuka whose relative position is point-wise bounded by μ in the Bruhat order and $\lambda : \mathcal{T} \rightarrow \mathcal{G}_b|_{\mathcal{Y}_{[r, \infty)}^{R^+}}$ is an (equivalence class of) isogenies.

Remark 2.3.3. In definition 2.3.2, let $E(\mu)$ denote the reflex field of μ . Since \mathcal{G} is reductive over \mathbb{Z}_p , $E(\mu)$ is an unramified extension of \mathbb{Q}_p . Moreover, since \mathfrak{f} comes equipped with an inclusion $\overline{\mathbb{F}}_p \rightarrow \mathfrak{f}$ we get an inclusion $E(\mu) \rightarrow W(\mathfrak{f})[\frac{1}{p}]$. We are implicitly using this inclusion of fields to compare the relative positions.

The purpose of this section is to prove that the $\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$ are rich p -smelted kimberlites that have connected p -adic tubular neighborhoods.

2.3.1 Moduli spaces of shtukas are kimberlites

In this subsection we verify that the map $\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu} \rightarrow W(\mathfrak{f})^\diamond$ forms a p -smelted kimberlite. We will need to define auxiliary spaces to simplify some of the arguments below:

Definition 2.3.4. We let $\text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ denote the functor $\text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b} : \text{Perf} \rightarrow \text{Sets}$:

$$\text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b}(R, R^+) = \{(R^\sharp, \iota, f), M, \lambda\}$$

Where the triple (R^\sharp, ι, f) denotes an untilt over $\text{Spa}(W(\mathfrak{f}), W(\mathfrak{f}))$, $M \in \mathcal{G}(W(R^+)[\frac{1}{\xi_{R^\sharp}}])$ and $\lambda : \mathcal{G}_M \rightarrow \mathcal{G}_b$ is an equivalence class of isogenies defined over $\mathcal{Y}_{[r, \infty]}^{R^+}$ for some r . Here \mathcal{G}_M denotes the tuple (\mathcal{G}, Φ_M) with $\Phi_M : \phi^*\mathcal{G} \rightarrow \mathcal{G}$ an isomorphism given by M and defined over $\text{Spec}(W(R^+)[\frac{1}{\xi_{R^\sharp}}])$.

Notice that there is a natural map $\text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ given by restriction and assigning $(M, \lambda) \mapsto (\mathcal{G}, \Phi_M, \lambda)$. We denote by $\mathbb{W}^+\mathcal{G}$ the sheaf in groups $\mathbb{W}^+\mathcal{G}(R, R^+) = \mathcal{G}(W(R^+))$, notice that $(\mathbb{W}^+\mathcal{G})_{O_C} = W^+(\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[t])$ as in definition 2.2.14.

Proposition 2.3.5. 1. The functors $\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ and $\text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ are small v -sheaves.

2. The map $\text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ is a $\mathbb{W}^+\mathcal{G}$ -torsor for the v -topology.

3. $\text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ is formalizing and $\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ is v -formalizing.

Proof. To prove that it is a v -sheaf one has to prove that each of the entries descend. A standard argument using 2.1.20 repeatedly proves this. Given $N \in \mathbb{W}^+\mathcal{G}(R, R^+)$ and $(M, \lambda) \in \text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b}(R, R^+)$ we let $N \cdot (M, \lambda) = (N^{-1}M\phi(N), \lambda \circ N)$. This specifies an action of $\mathbb{W}^+\mathcal{G}$ on $\text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ that makes the map $\text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ equivariant when the target is endowed with the trivial action. It is enough to prove that the basechange of $\text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ along product of points gives a trivial $\mathbb{W}^+\mathcal{G}$ -torsor.

Let $\text{Spa}(R, R^+)$ be a product of points, and take $(\mathcal{T}, \Phi, \lambda) \in \text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}(R, R^+)$. We can glue \mathcal{T} along λ over $\mathcal{Y}_{[r, \infty]}^{R^+}$ to extend to a \mathcal{G} -bundle over \mathcal{Y}_{R^+} , a meromorphic isomorphism Φ over $\mathcal{Y}_{R^+} \setminus V(\xi_{R^\sharp})$ and an isogeny λ over $\mathcal{Y}_{[r, \infty]}^{R^+}$. We can use theorem 2.1.12 and theorem 2.1.18 to get a \mathcal{G} -bundle over $\text{Spec}(W(R^+))$ with a meromorphic Φ that restrict to the previous ones. Since R^+ is a product of valuation rings with algebraically closed fraction field any \mathcal{G} -bundle on $\text{Spec}(W(R^+))$ is trivial. This is the case because $\text{Spec}(W(R^+))$ splits every étale cover. The choice of a trivialization specifies a section $(M, \lambda) \in \text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b}(R, R^+)$ and after chasing definitions one can see that the natural action of $\mathbb{W}^+\mathcal{G}$ on the set of trivialization acts compatibly with the action specified above.

We prove that $\text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ is formalizing, this already implies that $\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ is v -formalizing since the map $\text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ is surjective. Let $\text{Spa}(S, S^+) \in \text{Perf}$, fix $\varpi_S \in S^+$ a pseudo-uniformizer and take

$$((S^\sharp, \iota, f), M, \lambda) \in \text{LSht}_{W(\mathfrak{f})}^{\mathcal{G}_b}(S, S^+)$$

Given a map $f : \text{Spa}(L, L^+) \rightarrow \text{Spd}(S^+, S^+)$ we get a map of rings $f : W(S^+)[\frac{1}{\xi_{S^\sharp}}] \rightarrow W(L^+)[\frac{1}{\xi_{L^\sharp}}]$, and we can let M_L be $f(M)$. Moreover, fix a pseudo-uniformizer $\varpi_L \in L^+$, we claim that for any such choice and for any $r \in \mathbb{R}$ there is a large enough $r' \in \mathbb{R}$ for which the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{Y}_{[r', \infty]}^{L^+} & \longrightarrow & \mathcal{Y}_{[r, \infty]}^{R^+} \\
\downarrow & & \downarrow \\
\mathrm{Spa}(W(L^+), W(L^+)) & \longrightarrow & \mathrm{Spa}(W(S^+), W(S^+))
\end{array}$$

This map allows us to pullback the isogeny λ to $\mathrm{Spa}(L, L^+)$. The equivalence class of isogenies constructed this way does not depend of the choices of ϖ_S , ϖ_L , r or r' and the construction is functorial, so it defines a map $\mathrm{Spd}(S^+, S^+) \rightarrow \mathrm{LSht}_{W(f)}^{\mathcal{G}_b}$. \square

Moduli spaces of shtukas satisfy the valuative criterion for partial properness over $W(f)^\diamond$ since the definition of all of the data involved (via Tannakian formalism) takes place in the exact category of vector bundles over $\mathcal{Y}_{[0, \infty]}^{R^+}$ which is equivalent (by an exact equivalence) to the category of vector bundles over $\mathcal{Y}_{[0, \infty]}^{R^\circ}$.

Lemma 2.3.6. *Let $\mathcal{G}_1 \rightarrow \mathcal{G}_2$ be a closed embeddings of reductive groups over \mathbb{Z}_p and \mathcal{G}_b an isocrystal with \mathcal{G}_1 structure. Let $\mathcal{G}'_b = \mathcal{G}_b \times_{\mathcal{G}_2} \mathcal{G}_1$, the induced map $\mathrm{LSht}_{W(f)}^{\mathcal{G}_b} \rightarrow \mathrm{LSht}_{W(f)}^{\mathcal{G}'_b}$ is a closed immersion.*

Proof. It is enough to prove that the basechange by any totally disconnected perfectoid space is a closed immersion. Let $\mathrm{Spa}(S, S^+)$ in Perf be totally disconnected, and let $(M, \lambda) \in \mathrm{LSht}_{W(f)}^{\mathcal{G}'_b}(S, S^+)$. Abusing notation we let λ denote a choice of representative of the equivalence class of isogenies and we let $r \in \mathbb{R}$ such that λ is defined over $\mathcal{Y}_{[r, \infty]}^{S^+}$. By unraveling the definitions we can think of M as a ring map $\mathcal{O}_{\mathcal{G}_2} \rightarrow W(S^+)[\frac{1}{\xi_{S^+}}]$ and we think of λ as a ring map $\mathcal{O}_{\mathcal{G}_2} \rightarrow B_{S^+}^{[r, \infty]}$ (with the notation as in lemma 2.1.24). Since $\mathcal{G}_1 \rightarrow \mathcal{G}_2$ is a closed embedding of affine algebraic groups we have that $\mathcal{O}_{\mathcal{G}_1} = \mathcal{O}_{\mathcal{G}_2}/I$ for some finitely generated ideal $I \subseteq \mathcal{O}_{\mathcal{G}_2}$. The basechange

$$\mathrm{Spa}(S, S^+) \times_{\mathrm{LSht}_{W(f)}^{\mathcal{G}'_b}} \mathrm{LSht}_{W(f)}^{\mathcal{G}_b}$$

is representing the moduli of maps $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spa}(S, S^+)$ for which the compositions:

$$\begin{aligned}
M : \mathcal{O}_{\mathcal{G}_2} &\rightarrow W(S^+)[\frac{1}{\xi_{S^+}}] \rightarrow W(R^+)[\frac{1}{\xi_{R^+}}] \\
\lambda : \mathcal{O}_{\mathcal{G}_2} &\rightarrow B_{[r, \infty]}^{S^+} \rightarrow B_{[r, \infty]}^{R^+}
\end{aligned}$$

map elements of I to 0.

Let us prove that for any element $t \in W(S^+)[\frac{1}{\xi_{S^+}}]$ (or $t \in B_{S^+}^{[r, \infty]}$) the moduli of points in $\mathrm{Spa}(S, S^+)$ where t is identically 0 forms a closed immersion. Fix $t \in W(S^+)[\frac{1}{\xi_{S^+}}]$, since ξ_{S^+} is not a zero-divisor the moduli of points where t is 0 is the same as that of $\xi^n \cdot t$ so we may assume $t \in W(S^+)$. Using the Teichmüller expansion we have $t \in (S^+)^\mathbb{N}$ and t is 0 if and only if each entry is 0. This defines a Zariski closed subset of $\mathrm{Spa}(S, S^+)$. We prove the other case, fix $t \in B_{S^+}^{[r, \infty]} \subseteq B_{S^+}^{[r, \infty)}$ and let $Z \subseteq |\mathcal{Y}_{[r, \infty]}^{R^+}|$ be the set of valuations with $|t|_z = 0$. We have a projection map of diamonds $\pi : (\mathcal{Y}_{[r, \infty]}^{S^+})^\diamond \rightarrow \mathrm{Spd}(S, S^+)$ which is ℓ -cohomologically smooth and consequently universally open (See [28] 24.5). The moduli of points we are considering is given by maps to $\mathrm{Spa}(S, S^+)$ that factor through $Z' = |\mathrm{Spa}(S, S^+)| \setminus \pi(|\mathcal{Y}_{[r, \infty]}^{R^+}| \setminus Z)$ which is a closed subset. Since the subset of interest is closed and generalizing, it defines a closed immersion of $\mathrm{Spa}(S, S^+)$ (See [28] 7.6). \square

Proposition 2.3.7. *With notation as in lemma 2.3.6 the map $\mathrm{Sht}_{W(f)}^{\mathcal{G}_b} \rightarrow \mathrm{Sht}_{W(f)}^{\mathcal{G}'_b}$ is a closed immersion. In particular, if we let $\mathcal{G}_2 = \mathcal{G}_1 \times_{\mathbb{Z}_p} \mathcal{G}_1$ and we apply the result to the diagonal embedding $\Delta : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ we deduce that $\mathrm{Sht}_{W(f)}^{\mathcal{G}_b}$ is separated over $W(f)^\diamond$.*

Proof. We begin by proving that the map is injective. For this consider two sets of triples

$$t_i = (\mathcal{T}_i, \Phi_i, \lambda_i) \in \text{Sht}_{W(f)}^{\mathcal{G}_b}(R, R^+) \text{ with } i \in \{1, 2\}$$

and suppose that the $t_i \times^{\mathcal{G}_1} \mathcal{G}_2 := (\mathcal{T}_i \times^{\mathcal{G}_1} \mathcal{G}_2, \Phi_i, \lambda_i)$ become isomorphic, we need to prove $t_1 \cong t_2$. Since products of points form a basis for the v -topology we can assume $\text{Spa}(R, R^+)$ to be a product of points. For a product of points any map $\text{Spa}(R, R^+) \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b}$ factors through $\text{LSht}_{W(f)}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b}$. Let $T_i \in \text{LSht}_{W(f)}^{\mathcal{G}_b}(R, R^+)$ factoring t_i with $T_i := (M_i, \lambda_i)$. The set of choices for T_i mapping to t_i forms a $\mathbb{W}^+ \mathcal{G}_1(R, R^+)$ -torsors. Since $t_1 \times^{\mathcal{G}_1} \mathcal{G}_2 \cong t_2 \times^{\mathcal{G}_1} \mathcal{G}_2$ we have that $T_1 \times^{\mathcal{G}_1} \mathcal{G}_2$ and $T_2 \times^{\mathcal{G}_1} \mathcal{G}_2$ are in the same $\mathbb{W}^+ \mathcal{G}_2(R, R^+)$ -orbit. But $\lambda_i \in \mathcal{G}_1(B_{R^+}^{[r, \infty)})$ so that $\lambda_1 \circ \lambda_2^{-1} \in \mathcal{G}_1(B_{R^+}^{[r, \infty)}) \cap \mathcal{G}_2(W(R^+))$, since $W(R^+) \rightarrow B_{R^+}^{[r, \infty)}$ is injective this intersection is $\mathcal{G}_1(W(R^+))$. This, together with the injectivity of lemma 2.3.6, proves that T_1 and T_2 are in the same $\mathbb{W}^+ \mathcal{G}_1$ -orbit, which proves $t_1 = t_2$.

Once we know $\text{Sht}_{W(f)}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}'_b}$ is injective it is enough prove that the map is proper for it to be a closed immersion. Injectivity implies the map is a separated map of v -sheaves and since each of them satisfies the valuative criterion of partial properness over \mathbb{F}_p^\diamond , the map between them is a partially proper map. We only have left to prove that the map $\text{Sht}_{W(f)}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}'_b}$ is quasi-compact. Consider the following commutative diagram:

$$\begin{array}{ccc} \text{LSht}_{W(f)}^{\mathcal{G}_b} & \longrightarrow & \text{LSht}_{W(f)}^{\mathcal{G}'_b} \\ \downarrow & & \downarrow \\ \text{Sht}_{W(f)}^{\mathcal{G}_b} & \longrightarrow & \text{Sht}_{W(f)}^{\mathcal{G}'_b} \end{array}$$

The composition $\text{LSht}_{W(f)}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}'_b}$ is a quasi-compact map, and the map $\text{LSht}_{W(f)}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b}$ is surjective which formally implies that the map $\text{Sht}_{W(f)}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}'_b}$ is quasi-compact. \square

Proposition 2.3.8. *For any $\mu \in X_*^+(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$ we have that $\text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu} \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b}$ is a closed immersion. Moreover, $\text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$ is v -formalizing.*

Proof. Let $\text{LSht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$ denote the basechange of $\text{LSht}_{W(f)}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b}$ by $\text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$. Given an element $(M, \lambda) \in \text{LSht}_{W(f)}^{\mathcal{G}_b}(R, R^+)$, M naturally defines an (R, R^+) -valued point of $\text{Gr}_{W(f)}^{\mathcal{G}}$ when we think of M as an element of $\mathcal{G}(B_{dR}(R^\sharp))$.

We have the following pair of Cartesian diagrams:

$$\begin{array}{ccc} \text{LSht}_{W(f)}^{\mathcal{G}_b, \leq \mu} & \longrightarrow & \text{LSht}_{W(f)}^{\mathcal{G}_b} \\ \downarrow & & \downarrow \\ \text{Gr}_{W(f)}^{\mathcal{G}, \leq \mu} & \longrightarrow & \text{Gr}_{W(f)}^{\mathcal{G}} \end{array} \quad \begin{array}{ccc} \text{LSht}_{W(f)}^{\mathcal{G}_b, \leq \mu} & \longrightarrow & \text{LSht}_{W(f)}^{\mathcal{G}_b} \\ \downarrow & & \downarrow \\ \text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu} & \longrightarrow & \text{Sht}_{W(f)}^{\mathcal{G}_b} \end{array}$$

Since being a closed immersion can be checked v -locally on the target (See [28] 10.11), and since $\text{LSht}_{W(f)}^{\mathcal{G}_b} \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b}$ is surjective $\text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu} \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b}$ is a closed immersion. Moreover, by 2.2.5 the map $\text{Gr}_{W(f)}^{\mathcal{G}, \leq \mu} \rightarrow \text{Gr}_{W(f)}^{\mathcal{G}}$ is formally adic which implies that $\text{LSht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$ is formalizing and consequently that $\text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$ is v -formalizing. \square

In what follows we will prove that the functors $(\text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu})^{\text{red}}$ are represented by affine Deligne Lusztig varieties. We warn the reader that the definition that we take of affine Deligne Lusztig varieties is not the standard one. Nevertheless, it is well known and easy to establish that the definition we take defines the same objects as the standard definition.

Definition 2.3.9. Let \mathcal{G}_b be an isocrystal with \mathcal{G} -structure and $\mu : \mathbb{G}_{m, \overline{\mathbb{Q}}} \rightarrow \mathfrak{T}_{\overline{\mathbb{Q}}_p}$ a cocharacter. We define the v -sheaf $X_{\leq \mu}^{\mathcal{G}_b} : \text{PCAlg}_{/f}^{\text{op}} \rightarrow \text{Sets}$ as:

$$X_{\leq \mu}^{\mathcal{G}_b}(R) = \{(\mathcal{T}, \Phi, \lambda)\} / \cong$$

Where \mathcal{T} is a \mathcal{G} -torsor over $\text{Spec}(W(R))$, $\Phi : \phi^* \mathcal{T} \rightarrow \mathcal{T}$ is an isomorphism over $\text{Spec}(W(R)[\frac{1}{p}])$ of relative position bounded by μ and $\lambda : \mathcal{T} \rightarrow \mathcal{G}_b$ is a ϕ -equivariant isomorphism over $\text{Spec}(W(R)[\frac{1}{p}])$.

Proposition 2.3.10. We have an identification $X_{\leq \mu}^{\mathcal{G}_b} = (\text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu})^{\text{red}}$. Moreover, the map $(X_{\leq \mu}^{\mathcal{G}_b})^\diamond \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$ is injective and $\text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$ is a specializing v -sheaf.

Proof. Given a map $\text{Spec}(R) \rightarrow X_{\leq \mu}^{\mathcal{G}_b}$ we construct functorially a map $\text{Spec}(R)^\diamond \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$ in what follows. The untill is always the characteristic p untill. For any perfectoid space $f : \text{Spa}(S, S^+) \rightarrow \text{Spec}(R)^\diamond$ we get a triple $(f^* \mathcal{T}, f^* \Phi, f^* \lambda)$ coming from the map of rings $f : W(R) \rightarrow W(S^+)$ and by restriction to the appropriate loci $\mathcal{Y}_{[0, \infty)}^{S^+}$, $\mathcal{Y}_{[0, \infty)}^{S^+} \setminus V(p)$ and $\mathcal{Y}_{[r, \infty)}^{S^+}$ respectively. This data defines functorially a map $\text{Spa}(S, S^+) \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$, consequently a map $\text{Spec}(R)^\diamond \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$. The construction of $\text{Spec}(R)^\diamond \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$ is clearly functorial in $\text{PCAlg}_{/k}^{\text{op}}$. By adjunction, this gives a map $X_{\leq \mu}^{\mathcal{G}_b} \rightarrow (\text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu})^{\text{red}}$, we claim this map is an isomorphism.

We begin by proving it is injective, since $X_{\leq \mu}^{\mathcal{G}_b}$ is represented by a perfect scheme $((X_{\leq \mu}^{\mathcal{G}_b})^\diamond)^{\text{red}} = X_{\leq \mu}^{\mathcal{G}_b}$ and we may prove the map $(X_{\leq \mu}^{\mathcal{G}_b})^\diamond \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$ is injective instead. Take two arbitrary maps $g_i : \text{Spa}(R, R^+) \rightarrow (X_{\leq \mu}^{\mathcal{G}_b})^\diamond$, it is enough to prove injectivity v -locally so we may assume the maps factor through maps of the form $g'_i : \text{Spec}(R^+)^\diamond \rightarrow (X_{\leq \mu}^{\mathcal{G}_b})^\diamond$. The g'_i are given by data $(\mathcal{T}_i, \Phi_i, \lambda_i)$ over $\text{Spec}(W(R^+))$, $\text{Spec}(W(R^+)[\frac{1}{p}])$ and $\text{Spec}(W(R^+)[\frac{1}{p}])$ respectively and the g_i are given by restricting these data to $\mathcal{Y}_{[0, \infty)}^{R^+}$, $\mathcal{Y}_{[0, \infty)}^{R^+} \setminus V(p)$ and $\mathcal{Y}_{[r, \infty)}^{R^+}$ respectively. Nevertheless, we can recover g'_i from g_i since we can use λ_i to glue back the restricted data as in the proof of proposition 2.3.5.

Let us now prove surjectivity. Let $f : \text{Spec}(A)^\diamond \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$ and $g : \text{Spa}(R, R^+) \rightarrow \text{Spec}(A)^\diamond$ be a map with $\text{Spa}(R, R^+)$ a product of points and $A \in \text{PCAlg}_{/f}^{\text{op}}$. We will show below how to construct the following commutative diagram:

$$\begin{array}{ccc} \text{Spa}(R, R^+) & \longrightarrow & \text{Spec}(A)^\diamond \\ \downarrow & & \downarrow \\ (X_{\leq \mu}^{\mathcal{G}_b})^\diamond & \longrightarrow & \text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu} \end{array}$$

Since products of points are a basis for the topology, and since $(X_{\leq \mu}^{\mathcal{G}_b})^\diamond \rightarrow \text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$ is injective this defines a map $\text{Spec}(A)^\diamond \rightarrow (X_{\leq \mu}^{\mathcal{G}_b})^\diamond$ factoring our original map to $\text{Sht}_{W(f)}^{\mathcal{G}_b, \leq \mu}$ and proves the desired surjectivity.

Fix a pseudo-uniformizer $\varpi \in R^+$, we let $\text{Spa}(R_\infty, R_\infty^+)$ be a second product of points defined as follows: $R_\infty^+ = \prod_{i=1}^\infty R^+$ with pseudo-uniformizer now given by $\varpi_{R_\infty} = (\varpi^i)_{i=1}^\infty$. The product of points $\text{Spa}(R_\infty, R_\infty^+)$ comes equipped with a family of closed embeddings $\iota_i : \text{Spa}(R, R^+) \rightarrow \text{Spa}(R_\infty, R_\infty^+)$ given in coordinates by the projection onto the i th-factor. The diagonal ring map $\Delta_g : A \rightarrow \prod_{i=1}^\infty R^+$ induces a map $\Delta_g : \text{Spa}(R_\infty, R_\infty^+) \rightarrow \text{Spec}(A)^\diamond$ with the property that $\Delta_g \circ \iota_i = g$ for every i . Since $\text{Spa}(R_\infty, R_\infty^+)$ is a product of points, by proposition 2.1.19, the map $f \circ \Delta_g$ can be represented by a triple $(\mathcal{G}_{R_\infty}, \Phi_{R_\infty}, \lambda_{R_\infty})$ with \mathcal{G}_{R_∞} trivial. After choosing a trivialization for \mathcal{G}_{R_∞} we can think of λ_{R_∞} as a ring map $\mathcal{O}_{\mathcal{G}} \rightarrow B_{[r, \infty)}^{R_\infty^+}$. Moreover, since $f \circ \Delta_g \circ \iota_i = f \circ \Delta_g \circ \iota_j$ we have that for all i the composition

$$\lambda_i : \mathcal{O}_{\mathcal{G}} \rightarrow B_{[r, \infty)}^{R_\infty^+} \rightarrow B_{[r_i, \infty)}^{R_i^+} = B_{[r_i, \infty)}^{R^+}$$

lies in the same $\mathcal{G}(W(R^+))$ -orbit. Clearly $\mathcal{G}(W(R_\infty^+)) = \prod_{i=1}^\infty \mathcal{G}(W(R^+))$ so after a change of trivialization we may assume that $r_i = r_j$ and that $\lambda_i = \lambda_j =: \lambda_R$ for all $1 \leq i, j < \infty$. We claim that

λ_R factors through the inclusion of rings $W(R^+)[\frac{1}{p}] \subseteq B_{[r,\infty]}^{R^+}$. Take an element $t \in \mathcal{O}_{\mathcal{G}}$ and consider $s = \lambda_{R_\infty}(t) \in B_{[r,\infty]}^{R_\infty+}$, after replacing r by a larger number if necessary we may assume $r = n \in \mathbb{N}$. In particular, $p^k \cdot s$ lies in the p -adic completion of $W(R_\infty^+)[\frac{[\varpi R_\infty]}{p^n}]$ for some large enough $k \in \mathbb{N}$. Let us write $p^k \cdot s$ as $\sum_{j=0}^\infty x^{n(j)}[\alpha_j]p^j$ where x denotes $\frac{[\varpi R_\infty]}{p^n}$, $0 \leq n(j)$ is a multiplicity, and $\alpha_j \in R_\infty^+$. We have that $\iota_i(p^k \cdot s) = \sum_{j=0}^\infty (\frac{[\varpi]}{p^n})^{n(j)}[\iota_i(\alpha_j)]p^j$ with $\iota_i(\alpha_j) \in R^+$. In particular,

$$p^k \cdot \lambda_R(t) \in \bigcap_{i \in \mathbb{N}} (H^0(\mathcal{Y}_{[\frac{n}{i}, \infty]}^{R^+}, \mathcal{O}^+)),$$

but this intersection is $W(R^+)$ proving the claim.

Since the elements of the triple $(\mathcal{G}_{R_\infty}, \Phi_{R_\infty}, \lambda_{R_\infty})$ are defined over $\text{Spec}(W(R^+))$ and $\text{Spec}(W(R^+)[\frac{1}{p}])$ they define a map to $\text{Spec}(R^+) \rightarrow X_{\leq \mu}^{\mathcal{G}_b}$. The composition $\text{Spa}(R, R^+) \rightarrow \text{Spec}(R^+)^\diamond \rightarrow (X_{\leq \mu}^{\mathcal{G}_b})^\diamond$ gives the factorization we were looking for.

That $\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$ is formally separated follows from lemma 1.3.32 and proposition 2.3.7, that $\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$ is v -formalizing follows from proposition 2.3.8. \square

Lemma 2.3.11. *The adjunction map $(X_{\leq \mu}^{\mathcal{G}_b})^\diamond \rightarrow \text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$ arising from the identification of proposition 2.3.10 is a closed immersion.*

Proof. We will use that $X_{\leq \mu}^{\mathcal{G}_b}$ admits a closed immersion into the Witt vector Grassmanian $Gr_{\mathcal{W}, \mathfrak{f}}^{\mathcal{G}}$. We have that

$$(X_{\leq \mu}^{\mathcal{G}_b})^\diamond = \bigcup_{\nu \in X_*(\mathfrak{T}_{\overline{\mathbb{Q}}_p})} (X_{\leq \mu}^{\mathcal{G}_b} \cap Gr_{\mathcal{W}, \mathfrak{f}}^{\mathcal{G}, \leq \nu})^\diamond$$

and since each of this subsheaves are coming from a perfectly finitely presented proper scheme over \mathfrak{f} , they are proper as v -sheaves over \mathfrak{f}^\diamond . Consequently, the map $(X_{\leq \mu}^{\mathcal{G}_b} \cap Gr_{\mathcal{W}, \mathfrak{f}}^{\mathcal{G}, \leq \nu})^\diamond \rightarrow \text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$ is proper and since it is injective a closed immersion.

Now, $X_{\leq \mu}^{\mathcal{G}_b}$ is a scheme which is locally perfectly of finite type (See [10] theorem 1.1), and in particular each point admits an open neighborhood that is spectral and Noetherian as a topological space. Using a compactness argument in the patch topology, to every point $x \in |X_{\leq \mu}^{\mathcal{G}_b}|$ we may associate an open neighborhood $U_x \subseteq X_{\leq \mu}^{\mathcal{G}_b}$ and finite number of $\nu_i \in X_*(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$ for which $U = U \cap (\bigcup_{i \in I_x} Gr_{\mathcal{W}, \mathfrak{f}}^{\mathcal{G}, \leq \nu_i})$. Indeed, if U_x is Noetherian every closed subset is open in the constructible topology.

In proposition 2.3.10, we proved that $\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$ is a specializing v -sheaf, so by proposition 1.4.15 we get a specialization map $\text{sp}_{\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}} : |\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}| \rightarrow |X_{\leq \mu}^{\mathcal{G}_b}|$. We let $V_x = (\text{sp}_{\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}})^{-1}(U_x)$ for $x \in |X_{\leq \mu}^{\mathcal{G}_b}|$ and U_x as above, this forms an open cover of $\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$. Since being a closed immersion is v -local on the target and $V_x \rightarrow \text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$ is a formally adic open immersion it is enough to verify that $(V_x^{\text{red}})^\diamond \rightarrow V_x$ is a closed immersion. But the adjunction map $(U_x)^\diamond \rightarrow V_x$ fits in the following Cartesian diagram:

$$\begin{array}{ccccc} U_x^\diamond & \xrightarrow{Id} & U_x^\diamond & \longrightarrow & V_x \\ \downarrow & & \downarrow & & \downarrow \\ (\bigcup_{i \in I_x} Gr_{\mathcal{W}, \mathfrak{f}}^{\mathcal{G}, \leq \nu_i} \cap X_{\leq \mu}^{\mathcal{G}_b})^\diamond & \longrightarrow & (X_{\leq \mu}^{\mathcal{G}_b})^\diamond & \longrightarrow & \text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu} \end{array}$$

Since the union of a finite number of closed immersions still defines a closed immersion we can conclude by basechange that $U_x^\diamond \rightarrow V_x$ is also a closed immersion. \square

Proposition 2.3.12. *With the notation as above the map $\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu} \rightarrow W(\mathfrak{f})^\diamond$ is a p -smelted kimberlite as in definition 1.4.18.*

Proof. Proposition 2.3.10 proves that $\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$ is a specializing v -sheaf, in proposition 2.3.10 we proved that $(\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu})^{\mathrm{red}}$ is represented by a scheme and by lemma 2.3.11 the adjunction map $((\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu})^{\mathrm{red}})^{\diamond} \rightarrow \mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$ is a closed immersion which finished the proof that $\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$ is prekimberlite. Theorem 23.1.4 of [30] proves that $(\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu})_{\eta}$ is a locally spatial diamond. \square

2.3.2 Comparison of tubular neighborhoods

Recall that in this section \mathcal{G} is a reductive group over $\mathrm{Spec}(\mathbb{Z}_p)$, let $\mathcal{D} = (\mathcal{D}, \Phi_{\mathcal{D}})$ be a \mathcal{G} -torsor over $\mathrm{Spec}(W(\mathfrak{f}))$ together with an isomorphism $\Phi_{\mathcal{D}} : \phi^* \mathcal{D} \rightarrow \mathcal{D}$ defined over $\mathrm{Spec}(W(\mathfrak{f})[\frac{1}{p}])$, and fix $\mu \in X_*^+(\mathfrak{T}_{\overline{\mathbb{Q}_p}})$. We can define some objects associated to this data, which are nothing but “coordinate-free” versions of the moduli we defined in the previous sections:

Definition 2.3.13. 1. We denote the functor $\mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{D}} : \mathrm{Perf}_{\mathfrak{f}} \rightarrow \mathrm{Sets}$ with

$$\mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{D}}(R, R^+) = \{((R^{\sharp}, \iota, f), \mathcal{T}, \psi)\} / \cong$$

Where (R^{\sharp}, ι, f) is an untilt over $W(\mathfrak{f})$, \mathcal{T} is a \mathcal{G} -torsor over \mathcal{Y}_{R^+} and $\psi : \mathcal{T} \rightarrow \mathcal{D}$ is an isomorphism defined over $\mathcal{Y}_{R^+} \setminus V(\xi_{R^{\sharp}})$ that is meromorphic along $\xi_{R^{\sharp}}$.

2. We denote the functor $\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{D}} : \mathrm{Perf}_{\mathfrak{f}} \rightarrow \mathrm{Sets}$ with

$$\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{D}}(R, R^+) = \{((R^{\sharp}, \iota, f), \mathcal{T}, \Phi, \lambda)\} / \cong$$

Where (R^{\sharp}, ι, f) is an untilt over $W(\mathfrak{f})$, (\mathcal{T}, Φ) is a shtuka with \mathcal{G} -structure, and $\lambda : \mathcal{T} \rightarrow \mathcal{D}$ is an isogeny.

The functors $\mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{D}}$ and $\mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{D}}$ come with a canonical section $\mathrm{can}_{\mathcal{D}} : \mathrm{Spec}(\mathfrak{f})^{\diamond} \rightarrow \mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{D}}$ given by the data $(\phi^* \mathcal{D}, \Phi_{\mathcal{D}})$ and $\mathrm{can}_{\mathcal{D}} : \mathrm{Spec}(\mathfrak{f})^{\diamond} \rightarrow \mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{D}}$ given by $(\mathcal{D}, \Phi_{\mathcal{D}}, Id)$ respectively. We point out that if we fix an isomorphism $\tau : \mathcal{D} \rightarrow \mathcal{G}$ we get isomorphisms $\tau : \mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{D}} \rightarrow \mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{G}}$, and $\tau : \mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{D}, \leq \mu} \rightarrow \mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{G}, \leq \mu}$. Moreover, if we are given a section $\sigma : \mathrm{Spec}(\mathfrak{f})^{\diamond} \rightarrow \mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{G}}$ we can construct a pair $(\mathcal{D}, \Phi_{\mathcal{D}})$ and an isomorphism $\tau : \mathcal{D} \rightarrow \mathcal{G}$ such that the following diagram is commutative:

$$\begin{array}{ccc} & & \mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{D}} \\ & \nearrow \mathrm{can}_{\mathcal{D}} & \downarrow \tau \\ \mathrm{Spec}(\mathfrak{f})^{\diamond} & & \mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{G}} \\ & \searrow \sigma & \end{array}$$

Analogously, if we find a ϕ -equivariant isomorphism $\tau : \mathcal{D} \rightarrow \mathcal{G}_b$ over $\mathrm{Spec}(W(\mathfrak{f})[\frac{1}{p}])$ we get an isomorphism $\tau : \mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{D}, \leq \mu} \rightarrow \mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu}$, and given a section $\sigma : \mathrm{Spec}(\mathfrak{f})^{\diamond} \rightarrow \mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b}$ we can construct $(\mathcal{D}, \Phi_{\mathcal{D}})$ and τ making the following diagram commutative:

$$\begin{array}{ccc} & & \mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{D}} \\ & \nearrow \mathrm{can}_{\mathcal{D}} & \downarrow \tau \\ \mathrm{Spec}(\mathfrak{f})^{\diamond} & & \mathrm{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b} \\ & \searrow \sigma & \end{array}$$

Since \mathfrak{f} is algebraically closed every tubular neighborhood of Grassmanians and moduli of mixed characteristic shtukas at closed points are coming from the canonical one associated to some pair $(\mathcal{D}, \Phi_{\mathcal{D}})$. Indeed, every closed point of $X_{\leq \mu}^{\mathcal{G}_b}$ and $\mathrm{Gr}_{W, \mathfrak{f}}^{\mathcal{G}}$ is the image of a section since the bounded version of these ind-schemes are locally perfectly of finite presentation over $\mathrm{Spec}(\mathfrak{f})$.

Theorem 2.3.14. *Given $(\mathcal{D}, \Phi_{\mathcal{D}})$ and $\mu \in X_*(\mathfrak{T}_{\mathbb{Q}_p})$ as above, we have canonical identifications of tubular neighborhoods $\widehat{\mathrm{Sht}}_{W(\mathfrak{f})/\mathrm{can}_{\mathcal{D}}}^{\mathcal{D}} \cong \widehat{\mathrm{Gr}}_{W(\mathfrak{f})/\mathrm{can}_{\mathcal{D}}}^{\mathcal{D}}$ and $\widehat{\mathrm{Sht}}_{W(\mathfrak{f})/\mathrm{can}_{\mathcal{D}}}^{\mathcal{D}, \leq \mu} \cong \widehat{\mathrm{Gr}}_{W(\mathfrak{f})/\mathrm{can}_{\mathcal{D}}}^{\mathcal{D}, \leq \mu}$.*

Before proving the theorem and describing the map identifying these tubular neighborhoods we will need some preparation.

Definition 2.3.15. 1. We let $\widehat{LGr}_{\mathcal{D}}$ denote the sheaf of groups over $W(\mathfrak{f})^{\diamond}$ given by $\widehat{LGr}_{\mathcal{D}}(R, R^+) = \{((R^{\sharp}, \iota, f), g)\}$. Where (R^{\sharp}, ι, f) is an untilt over $W(\mathfrak{f})$ and $g : \mathcal{D} \rightarrow \mathcal{D}$ is an automorphism of \mathcal{G} -torsors defined over $\mathrm{Spec}(W(R^+))$ for which there is a pseudo-uniformizer $\varpi_g \in R^+$, depending of g , such that g restricts to the identity over $\mathrm{Spec}(W(R^+)/[\varpi_g])$. We define $\widehat{LGr}_{\phi^*\mathcal{D}}$ in a similar way exchanging the role of \mathcal{D} for that of $\phi^*\mathcal{D}$.

2. We let $\widehat{LGr}_{\mathcal{D}}$ be the v -sheaf over $W(\mathfrak{f})^{\diamond}$ assigning

$$\widehat{LGr}_{\mathcal{D}}(R, R^+) = \{(R^{\sharp}, \iota, f), \mathcal{T}, \psi, \sigma\} / \cong$$

Where (R^{\sharp}, ι, f) is an untilt over $W(\mathfrak{f})$, \mathcal{T} is a \mathcal{G} -torsor over $\mathrm{Spec}(W(R^+))$, $\psi : \mathcal{T} \rightarrow \mathcal{D}$ is an isomorphism over $\mathrm{Spec}(W(R^+)/[\frac{1}{\xi}])$ and $\sigma : \mathcal{T} \rightarrow \phi^*\mathcal{D}$ is an isomorphism of \mathcal{G} -torsors over $\mathrm{Spec}(W(R^+))$ such that there is a pseudo-uniformizer $\varpi \in R^+$ depending on the data for which $\Phi_{\mathcal{D}} \circ \sigma = \psi$ when restricted to $\mathrm{Spec}(W(R^+)/[\varpi])$.

3. We let $\widehat{LSht}_{\mathcal{D}}$ be the v -sheaf over $W(\mathfrak{f})^{\diamond}$ assigning

$$\widehat{LSht}_{\mathcal{D}}(R, R^+) = \{(R^{\sharp}, \iota, f), \mathcal{T}, \Phi, \lambda, \sigma\} / \cong$$

Where (R^{\sharp}, ι, f) is an untilt over $W(\mathfrak{f})$, \mathcal{T} is a \mathcal{G} -torsor over $\mathrm{Spec}(W(R^+))$, $\Phi : \phi^*\mathcal{T} \rightarrow \mathcal{T}$ is an isomorphism over $\mathrm{Spec}(W(R^+)/[\frac{1}{\xi}])$, $\lambda : \mathcal{T} \rightarrow \mathcal{D}$ is an isogeny over $\mathcal{Y}_{[r, \infty]}^{R^+}$ and $\sigma : \mathcal{T} \rightarrow \mathcal{D}$ is an isomorphism of \mathcal{G} -torsors over $\mathrm{Spec}(W(R^+))$ such that there is a pseudo-uniformizer $\varpi \in R^+$ depending on the data for which $\sigma = \lambda$ when restricted to $\mathrm{Spec}(B_{[r, \infty]}^{R^+}/[\varpi])$.

Standard arguments using proposition 2.1.20 will prove that the objects in definition 2.3.15 are v -sheaves. Notice though, that the category of vector bundles over $\mathrm{Spec}(W(R^+))$ fibered over Perf does not form a stack for the v -topology. Nevertheless, the category fibered over Perf that assigns to $\mathrm{Spa}(R, R^+)$ the category of pairs (\mathcal{T}, σ) where \mathcal{T} is a \mathcal{G} -torsor over $\mathrm{Spec}(W(R^+))$ and $\sigma : \mathcal{T} \rightarrow \mathcal{G}$ is a trivialization does form a stack for the v -topology on Perf .

There is a natural map $\widehat{LGr}_{\mathcal{D}} \rightarrow \mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{D}}$ that takes a triple $(\mathcal{T}, \psi, \sigma)$ and assigns the pair (\mathcal{T}, ψ) restricted to $\mathcal{Y}_{[0, \infty]}^{R^+}$ and $\mathcal{Y}_{[0, \infty]}^{R^+} \setminus V(\xi)$ respectively. This map is $\widehat{LGr}_{\phi^*\mathcal{D}}$ -equivariant when we consider the left action $\widehat{LGr}_{\phi^*\mathcal{D}} \times \widehat{LGr}_{\mathcal{D}} \rightarrow \widehat{LGr}_{\mathcal{D}}$ sending an element $(g, (\mathcal{T}, \psi, \sigma))$ to $(\mathcal{T}, \psi, g \circ \sigma)$ and $\mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{D}}$ is given the trivial $\widehat{LGr}_{\phi^*\mathcal{D}}$ -action.

Lemma 2.3.16. *The natural map $\widehat{LGr}_{\mathcal{D}} \rightarrow \mathrm{Gr}_{W(\mathfrak{f})}^{\mathcal{D}}$ factors through $\widehat{\mathrm{Gr}}_{W(\mathfrak{f})/\mathrm{can}_{\mathcal{D}}}^{\mathcal{D}}$. Moreover, the map $\widehat{LGr}_{\mathcal{D}} \rightarrow \widehat{\mathrm{Gr}}_{W(\mathfrak{f})/\mathrm{can}_{\mathcal{D}}}^{\mathcal{D}}$ is a $\widehat{LGr}_{\phi^*\mathcal{D}}$ -torsor.*

Proof. We begin by proving that $\widehat{LGr}_{\mathcal{D}}$ formalizes any map coming from an affinoid perfectoid space $\mathrm{Spa}(A, A^+)$. Indeed, take a map $\mathrm{Spa}(A, A^+) \rightarrow \widehat{LGr}_{\mathcal{D}}$ given by an untilt and a triple $(\mathcal{T}, \psi, \sigma)$ and

take a map $f : \mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spd}(A^+, A^+)$, we have to define functorially a map $\mathrm{Spa}(B, B^+) \rightarrow \widehat{LGr}_{\mathcal{D}}$. We can construct an untilt for B functorially as in lemma 1.4.8. We have a map of affine schemes $f : \mathrm{Spec}(W(B^+)) \rightarrow \mathrm{Spec}(W(A^+))$ along which we can pullback to get the triple $(f^*\mathcal{T}, f^*\psi, f^*\sigma)$ where in this case $f^*\mathcal{T}$ is a \mathcal{G} -torsor over $\mathrm{Spec}(W(B^+))$, $f^*\psi : f^*\mathcal{T} \rightarrow \mathcal{D}$ is an isomorphism over $\mathrm{Spec}(W(B^+)[\frac{1}{f(\xi)}])$ and $f^*\sigma : f^*\mathcal{T} \rightarrow \phi^*\mathcal{D}$ is an isomorphism over $\mathrm{Spec}(W(B^+))$. We need to verify that this triple satisfies the constraints. Take a pseudo-uniformizer $\varpi_A \in A^+$ for which $\Phi_{\mathcal{D}} \circ \sigma = \psi$ in $\mathrm{Spec}(W(A^+)/[\varpi_A])$. By continuity, $f(\varpi_A)$ is topologically nilpotent and there is a pseudo-uniformizer $\varpi_B \in B^+$ with $f(\varpi_A) = \varpi_B \cdot t$ for some $t \in B^+$. We have $\Phi_{\mathcal{D}} \circ f^*\sigma = f^*\psi$ over $\mathrm{Spec}(W(B^+)/[\varpi_B])$ proving the constraint holds.

To prove that $\widehat{LGr}_{\mathcal{D}} \rightarrow \mathrm{Gr}_{W(f)}^{\mathcal{D}}$ factors through $\widehat{\mathrm{Gr}_{W(f)}^{\mathcal{D}}/can_{\mathcal{D}}}$ it is enough to prove that for any map $\mathrm{Spd}(R^+, R^+) \rightarrow \widehat{LGr}_{\mathcal{D}}$ the map of reductions $(\mathrm{Spd}(R^+, R^+))^{\mathrm{red}} = \mathrm{Spec}(R_{\mathrm{red}}^+)^{\diamond} \rightarrow (\mathrm{Gr}_{W(f)}^{\mathcal{D}})^{\mathrm{red}}$ factors through the canonical map $can_{\mathcal{D}} : \mathrm{Spec}(f)^{\diamond} \rightarrow (\mathrm{Gr}_{W(f)}^{\mathcal{D}})^{\mathrm{red}}$. After restricting the data $(\mathcal{T}, \psi, \sigma)$ to $\mathrm{Spec}(W(R_{\mathrm{red}}^+))$ we get the identity $\Phi_{\mathcal{D}} \circ \sigma = \psi$. After pullback, the map $\mathrm{Spec}(R_{\mathrm{red}}^+)^{\diamond} \rightarrow \mathrm{Gr}_{W(f)}^{\mathcal{D}}$ is given by the tuple (\mathcal{T}, ψ) . Since this data is isomorphic via σ to $(\phi^*\mathcal{D}, \Phi_{\mathcal{D}})$, the map factors through $can_{\mathcal{D}} : \mathrm{Spec}(f)^{\diamond} \rightarrow \mathrm{Gr}_{W(f)}^{\mathcal{D}}$.

We now prove that $\widehat{LGr}_{\mathcal{D}} \rightarrow \widehat{\mathrm{Gr}_{W(f)}^{\mathcal{D}}/can_{\mathcal{D}}}$ is surjective. It is enough to prove this for a product of points which we denote $\mathrm{Spa}(R, R^+)$, with pseudo-uniformizer $\varpi \in R^+$. In this case, by proposition 2.1.19, a (R, R^+) -valued point is given by (\mathcal{T}, ψ) with \mathcal{T} defined over $\mathrm{Spec}(W(R^+))$ and $\psi : \mathcal{T} \rightarrow \mathcal{D}$ defined over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$, with the additional condition that (\mathcal{T}, ψ) is isomorphic to $(\phi^*\mathcal{D}, \Phi_{\mathcal{D}})$ when restricted to $W(R_{\mathrm{red}}^+)$ and $W(R_{\mathrm{red}}^+[\frac{1}{p}])$. Such an isomorphism $\sigma_{\mathrm{red}} : (\mathcal{T}, \psi) \rightarrow (\phi^*\mathcal{D}, \Phi_{\mathcal{D}})$ is unique and has to be given by $\sigma_{\mathrm{red}} = \Phi_{\mathcal{D}}^{-1} \circ \psi_{\mathrm{red}}$, since it has to satisfy the commutative diagram:

$$\begin{array}{ccc} \mathcal{T} & & \\ \downarrow \sigma_{\mathrm{red}} & \searrow \psi & \\ \phi^*\mathcal{D} & & \mathcal{D} \end{array} \quad \begin{array}{c} \Phi_{\mathcal{D}} \nearrow \end{array}$$

The morphism $\tilde{\sigma} = \Phi_{\mathcal{D}}^{-1} \circ \psi : \mathcal{T} \rightarrow \phi^*\mathcal{D}$ is defined over $\mathcal{Y}_{[r, \infty]}^{R^+}$ for r sufficiently big (so that it avoids $V(\xi)$) and it restricts to σ_{red} . We can use lemma 2.1.25 to construct an isomorphism $\sigma : \mathcal{T} \rightarrow \phi^*\mathcal{D}$ such that $\sigma = \tilde{\sigma}$ when restricted to $\mathrm{Spec}(B_{[r, \infty]}^{R^+}/[\varpi'])$ for some pseudo-uniformizer $\varpi' \in R^+$. In particular $\Phi_{\mathcal{D}} \circ \sigma = \psi$ over $\mathrm{Spec}(W(R^+)/[\varpi'])$. The data $(\mathcal{T}, \psi, \sigma)$ constructs a map $\mathrm{Spa}(R, R^+) \rightarrow \widehat{LGr}_{\mathcal{D}}$ which evidently composes to the original map $\mathrm{Spa}(R, R^+) \rightarrow \widehat{\mathrm{Gr}_{W(f)}^{\mathcal{D}}/can_{\mathcal{D}}}$.

Finally, we need to prove $\widehat{LGr}_{\mathcal{D}} \times_{\mathrm{Gr}_{W(f)}^{\mathcal{D}}} \widehat{LGr}_{\mathcal{D}} \cong \widehat{LG}_{\phi^*\mathcal{D}} \times_{W(f)^{\diamond}} \widehat{LGr}_{\mathcal{D}}$. Take two sets of data $(\mathcal{T}_i, \psi_i, \sigma_i)$ over $\mathrm{Spa}(A, A^+)$ and suppose that $(\mathcal{T}_1|_{\mathcal{Y}_{A^+}}, \psi_1|_{\mathcal{Y}_{A^+} \setminus V(\xi)}) \cong (\mathcal{T}_2|_{\mathcal{Y}_{A^+}}, \psi_2|_{\mathcal{Y}_{A^+} \setminus V(\xi)})$. The isomorphism must be given by $\psi_1^{-1} \circ \psi_2 : \mathcal{T}_2 \rightarrow \mathcal{T}_1$ and by the fully-faithfulness part of theorem 2.1.15 the isomorphism extends over $\mathrm{Spec}(W(A^+))$. Moreover, we can define $g = \sigma_1 \circ \psi_1^{-1} \circ \psi_2 \circ \sigma_2^{-1} : \phi^*\mathcal{D} \rightarrow \phi^*\mathcal{D}$. By hypothesis, $\sigma_1 \circ \psi_1^{-1} = \Phi_{\mathcal{D}}^{-1}$ and $\psi_2 \circ \sigma_2^{-1} = \Phi_{\mathcal{D}}$ on $\mathrm{Spec}(W(A^+)/[\varpi_A])$ for some suitable choice of pseudo-uniformizer $\varpi_A \in A^+$. Consequently we can associate to the original data the tuple $(g, \mathcal{T}_2, \psi_2, \sigma_2) \in \widehat{LG}_{\phi^*\mathcal{D}} \times_{W(f)^{\diamond}} \widehat{LGr}_{\mathcal{D}}(A, A^+)$. On the other hand, to a tuple $(g, \mathcal{T}, \psi, \sigma)$ the action of $\widehat{LG}_{\phi^*\mathcal{D}}$ associates the pair of tuples $(\mathcal{T}, \psi, g \circ \sigma)$ and $(\mathcal{T}, \psi, \sigma)$. Since these two constructions are functorial and compose to the identity they define isomorphisms. \square

For moduli spaces of shtukas we have a very similar story. We have a projection map $\pi : \widehat{\mathrm{LSht}}_{\mathcal{D}} \rightarrow \mathrm{Sht}_{W(f)}^{\mathcal{D}}$, which we can construct by assigning to a tuple $(\mathcal{T}, \Phi, \lambda, \sigma)$ the tuple $(\mathcal{T}|_{\mathcal{Y}_{[0, \infty]}^{R^+}}, \Phi|_{\mathcal{Y}_{[0, \infty]}^{R^+} \setminus V(\xi)}, \lambda)$. Moreover, this projection is $\widehat{LG}_{\mathcal{D}}$ -equivariant when we endow $\widehat{\mathrm{LSht}}_{\mathcal{D}}$ with the left action $\widehat{LG}_{\mathcal{D}} \times \widehat{\mathrm{LSht}}_{\mathcal{D}} \rightarrow$

$\widehat{\text{LSht}}_{\mathcal{D}}$ sending the tuple $(g, (\mathcal{T}, \Phi, \lambda, \sigma))$ to the tuple $(\mathcal{T}, \Phi, \lambda, g \circ \sigma)$ and when $\text{Sht}_{W(f)}^{\mathcal{D}}$ is given the trivial action.

Lemma 2.3.17. *The natural map $\widehat{\text{LSht}}_{\mathcal{D}} \rightarrow \text{Sht}_{W(f)}^{\mathcal{D}}$ factors through $\widehat{\text{Sht}}_{W(f)/\text{can}_{\mathcal{D}}}^{\mathcal{D}}$. Moreover, the map $\widehat{\text{LSht}}_{\mathcal{D}} \rightarrow \widehat{\text{Sht}}_{W(f)/\text{can}_{\mathcal{D}}}^{\mathcal{D}}$ is a $\widehat{LG}_{\mathcal{D}}$ -torsor.*

Proof. The proves that $\widehat{\text{LSht}}_{\mathcal{D}}$ formalizes any map $\text{Spa}(A, A^+) \rightarrow \widehat{\text{LSht}}_{\mathcal{D}}$ with $\text{Spa}(A, A^+) \in \text{Perf}$, that the map $\widehat{\text{LSht}}_{\mathcal{D}} \rightarrow \text{Sht}_{W(f)}^{\mathcal{D}}$ factors through $\widehat{\text{Sht}}_{W(f)/\text{can}_{\mathcal{D}}}^{\mathcal{D}}$ and that this later map is surjective in that locus follow very similar arguments to those given in the proof of lemma 2.3.16. We omit the details.

Let us prove that $\widehat{\text{LSht}}_{\mathcal{D}} \times_{\text{Sht}_{W(f)}^{\mathcal{D}}} \widehat{\text{LSht}}_{\mathcal{D}} \cong \widehat{LG}_{\mathcal{D}} \times_{W(f) \diamond} \widehat{\text{LSht}}_{\mathcal{D}}$. Take two sets of data $(\mathcal{T}_i, \Phi_i, \lambda_i, \sigma_i)$ over $\text{Spa}(A, A^+)$ and suppose that $(\mathcal{T}_1|_{\mathcal{Y}_{[0, \infty)}^{A^+}}, \Phi_1|_{\mathcal{Y}_{[0, \infty)}^{A^+} \setminus V(\xi)}, \lambda_1) \cong (\mathcal{T}_2|_{\mathcal{Y}_{[0, \infty)}^{A^+}}, \Phi_2|_{\mathcal{Y}_{[0, \infty)}^{A^+} \setminus V(\xi)}, \lambda_2)$. The isomorphism must be the unique lift of $\lambda_1^{-1} \circ \lambda_2 : \mathcal{T}_2 \rightarrow \mathcal{T}_1$ to $\mathcal{Y}_{[0, \infty)}^{A^+}$. Glueing along the λ_i we can also define a lift to \mathcal{Y}_{A^+} . Since the \mathcal{T}_i are defined over $\text{Spec}(W(A^+))$ and by the fully-faithfulness part of theorem 2.1.15 the isomorphism extends to $\text{Spec}(W(A^+))$. Moreover, we can define $g = \sigma_1 \circ \lambda_1^{-1} \circ \lambda_2 \circ \sigma_2^{-1} : \mathcal{D} \rightarrow \mathcal{D}$. By hypothesis, $\sigma_1 \circ \lambda_1^{-1} = \text{Id} = \lambda_2 \circ \sigma_2^{-1}$ over $\text{Spec}(B_{[r, \infty]}^{A^+}/[\varpi_A])$ for some suitable choice of pseudo-uniformizer $\varpi_A \in A^+$. Consequently we can associate to the original data the tuple $(g, \mathcal{T}_2, \Phi_2, \lambda_2, \sigma_2) \in \widehat{LG}_{\mathcal{D}} \times_{W(f) \diamond} \widehat{\text{LSht}}_{\mathcal{D}}(A, A^+)$. The action map gives back isomorphic data. \square

We can now prove the theorem.

Proof. (of theorem 2.3.14). For this proof we define φ to be the inverse of Frobenious, $\varphi = \phi^{-1}$. We begin by observing there is an isomorphism $\theta : \widehat{LG}_{\mathcal{D}} \rightarrow \widehat{LG}_{\phi^* \mathcal{D}}$ given by sending $g \in \widehat{LG}_{\mathcal{D}}(R, R^+)$ with $g : \mathcal{D} \rightarrow \mathcal{D}$ to $\phi^* g : \phi^* \mathcal{D} \rightarrow \phi^* \mathcal{D}$. By definition of $\widehat{LG}_{\mathcal{D}}$, there is a pseudo-uniformizer $\varpi_g \in R^+$ for which $g = \text{Id}$ in $\text{Spec}(W(R^+)/[\varpi_g])$. One can verify that $\phi^* g = \text{Id}$ in $\text{Spec}(W(R^+)/[\varpi^p])$ so that $\phi^* g \in \widehat{LG}_{\phi^* \mathcal{D}}$ the inverse of this group homomorphism is of course given by sending $h \in \widehat{LG}_{\phi^* \mathcal{D}}(R, R^+)$ to $\varphi^* h$. Using θ one can then endow $\widehat{LGr}_{\mathcal{D}}$ with a $\widehat{LG}_{\mathcal{D}}$ action for which the projection $\pi : \widehat{LGr}_{\mathcal{D}} \rightarrow \text{Gr}_{W(f)}^{\mathcal{D}}$ of lemma 2.3.16 is a $\widehat{LG}_{\mathcal{D}}$ -torsor.

In what follows we construct a $\widehat{LG}_{\mathcal{D}}$ -equivariant isomorphism between $\widehat{LGr}_{\mathcal{D}} \rightarrow \widehat{\text{LSht}}_{\mathcal{D}}$. Take a perfectoid Huber pair (A, A^+) and tuple $(\mathcal{T}, \psi, \sigma) \in \widehat{LGr}_{\mathcal{D}}(A, A^+)$. Consider the \mathcal{G} -torsor $\varphi^* \mathcal{T}$, and consider the map $\Phi : \mathcal{T} \rightarrow \varphi^* \mathcal{T}$ defined by $\Phi = (\varphi^* \sigma)^{-1} \circ \psi$. We now construct a ϕ -equivariant map $\lambda : \varphi^* \mathcal{T} \rightarrow \mathcal{D}$. Consider the following (non-commutative!!!) diagram:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\sigma} & \phi^* \mathcal{D} \\ \downarrow \Phi & & \downarrow \Phi_{\mathcal{D}} \\ \varphi^* \mathcal{T} & \xrightarrow{\varphi^* \sigma} & \mathcal{D} \end{array}$$

Each of the arrows of the diagram is defined over $\mathcal{Y}_{[r, \infty]}^{A^+}$ for big enough r avoiding $V(\xi)$. Moreover, by hypothesis there is a pseudo-uniformizer $\varpi \in A^+$ for which $\psi = \Phi_{\mathcal{D}} \circ \sigma$ over $\text{Spec}(W(R^+)/[\varpi])$. We can see that $\varphi^* \sigma \circ \Phi = \Phi_{\mathcal{D}} \circ \sigma$ over $\text{Spec}(B_{[r, \infty]}^{A^+}/[\varpi])$ and in particular the morphism $\varphi^* \sigma : \varphi^* \mathcal{T} \rightarrow \mathcal{D}$ is ϕ -equivariant over this locus. By lemma 2.1.27 there is a unique isogeny over $\mathcal{Y}_{[r, \infty]}^{A^+}$ denoted $\lambda : \varphi^* \mathcal{T} \rightarrow \mathcal{D}$ such that $\lambda = \varphi^* \sigma$ when restricted to $\text{Spec}(B_{[r, \infty]}^{A^+}/[\varpi])$. We can associate to our original data:

$$(\mathcal{T}, \psi, \sigma) \mapsto (\varphi^* \mathcal{T}, \Phi, \lambda, \varphi^* \sigma)$$

This construction is functorial when we let (A, A^+) vary by the uniqueness of λ . This gives a map $\Theta : \widehat{LGr}_{\mathcal{D}} \rightarrow \widehat{\text{LSht}}_{\mathcal{D}}$. Moreover, we have $g \cdot (\varphi^* \mathcal{T}, \Phi, \tau, \varphi^* \sigma) = (\varphi^* \mathcal{T}, \Phi, \tau, g \circ \varphi^* \sigma)$ and $g \cdot (\mathcal{T}, \lambda, \sigma) = (\mathcal{T}, \lambda, \phi^* g \circ \sigma)$ so the map Θ is $\widehat{LG}_{\mathcal{D}}$ -equivariant.

We construct explicitly the inverse Θ^{-1} . Given a tuple $(\mathcal{T}, \Phi, \lambda, \sigma) \in \widehat{\text{LSht}}_{\mathcal{D}}(A, A^+)$ we can assign:

$$(\mathcal{T}, \Phi, \lambda, \sigma) \mapsto (\phi^* \mathcal{T}, \sigma \circ \Phi, \phi^* \sigma)$$

this construction is clearly functorial in (A, A^+) , and if $\varpi_A \in A^+$ is such that $\lambda = \sigma$ over $B_{[r, \infty]}^{A^+}/[\varpi_A]$ then $\Phi_{\mathcal{D}} \circ \phi^* \sigma = \sigma \circ \Phi$ over $\text{Spec}(W(A^+)/[\varpi_A])$ since λ is ϕ -equivariant. This gives a map $\Omega : \widehat{\text{LSht}}_{\mathcal{D}} \rightarrow \widehat{\text{LGr}}_{\mathcal{D}}$ the composition $\Omega \circ \Theta$ is clearly the identity. One can verify directly that $\Theta \circ \Omega(\mathcal{T}, \Phi, \lambda, \sigma) = (\mathcal{T}, \Phi, \lambda', \sigma)$ for some λ' nevertheless $\lambda' = \sigma = \lambda$ over $B_{[r, \infty]}^{A^+}/[\varpi]$ as ϕ -equivariant maps for some $\varpi \in A^+$. By the uniqueness part of lemma 2.1.27 we have $\lambda = \lambda'$.

Since Θ gives a $\widehat{\text{LGr}}_{\mathcal{D}}$ -equivariant isomorphism we also get isomorphisms

$$\Theta : \widehat{\text{Gr}}_{W(\mathfrak{f})/can_{\mathcal{D}}}^{\mathcal{D}} = \widehat{\text{LGr}}_{\mathcal{D}}/\widehat{\text{LGr}}_{\mathcal{D}} \cong \widehat{\text{LSht}}_{\mathcal{D}}/\widehat{\text{LGr}}_{\mathcal{D}} = \widehat{\text{Sht}}_{W(\mathfrak{f})/can_{\mathcal{D}}}^{\mathcal{D}}$$

One can also verify directly by the construction of Θ that it preserves the boundedness condition so that $\Theta : \widehat{\text{Gr}}_{W(\mathfrak{f})/can_{\mathcal{D}}}^{\mathcal{D}, \leq \mu} \rightarrow \widehat{\text{Sht}}_{W(\mathfrak{f})/can_{\mathcal{D}}}^{\mathcal{D}, \leq \mu}$ is also an isomorphism. \square

Let us prove that moduli spaces of mixed characteristic shtukas are rich p -smelted kimberlites.

Theorem 2.3.18. *With the notation as in the beginning of this section we have that the map $\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu} \rightarrow W(\mathfrak{f})^{\diamond}$ forms a rich p -smelted kimberlite with connected p -adic tubular neighborhoods.*

Proof. Proposition 2.3.12 proves this map forms a p -smelted kimberlite. In [30] 23.3.3 it is proven that the period morphism $\text{Sht}_{W(\mathfrak{f})}^{\mathcal{G}_b, \leq \mu} \rightarrow \text{Gr}_{W(\mathfrak{f})}^{\mathcal{G}, \leq \mu}$ is étale. By proposition 1.4.34 and theorem 2.2.34 we know that $\text{Sht}_{W(\mathfrak{f})[\frac{1}{p}]}^{\mathcal{G}_b, \leq \mu}$ is a cJ-diamond.

By theorem 1.1 of [10] we know that $X_{\leq \mu}^{\mathcal{G}_b}$ is locally Noetherian. By lemmas 1.4.43 and 1.4.44 to prove that the specialization map is a quotient and specializing map we only need to prove that for any non-Archimedean field extension $C/W(\mathfrak{f})[\frac{1}{p}]$ with C algebraically closed the specialization map of the base change $\text{Sht}_{O_C}^{\mathcal{G}_b, \leq \mu}$ is surjective on closed points. It is then enough to prove that for any such C the p -adic tubular neighborhoods of $\text{Sht}_{O_C}^{\mathcal{G}_b, \leq \mu}$ are non-empty and connected.

Take a closed point $x \in |(\text{Sht}_{O_C}^{\mathcal{G}_b, \leq \mu})^{\text{red}}|$ we claim that we have an isomorphism $(\widehat{\text{Sht}}_{O_C}^{\mathcal{G}_b, \leq \mu})_x \cong (\widehat{\text{Gr}}_{O_C}^{\mathcal{G}, \leq \mu})_y$ for some $y \in |(\text{Gr}_{O_C}^{\mathcal{G}, \leq \mu})^{\text{red}}|$. Indeed, if \mathfrak{f}_C denotes the residue field of O_C we may apply theorem 2.3.14 to compare $(\widehat{\text{Sht}}_{W(\mathfrak{f}_C)}^{\mathcal{G}_b, \leq \mu})_x$ with $(\widehat{\text{Gr}}_{W(\mathfrak{f}_C)}^{\mathcal{G}, \leq \mu})_y$ for some y . Since $C/W(\mathfrak{f}_C)[\frac{1}{p}]$ is purely ramified we have identifications $|(\text{Sht}_{O_C}^{\mathcal{G}_b, \leq \mu})^{\text{red}}| = |(\text{Sht}_{W(\mathfrak{f}_C)}^{\mathcal{G}_b, \leq \mu})^{\text{red}}|$ and $|(\text{Gr}_{O_C}^{\mathcal{G}, \leq \mu})^{\text{red}}| = |(\text{Gr}_{W(\mathfrak{f}_C)}^{\mathcal{G}, \leq \mu})^{\text{red}}|$. For any $x \in |(\text{Sht}_{O_C}^{\mathcal{G}_b, \leq \mu})^{\text{red}}|$ and any $y \in |(\text{Gr}_{O_C}^{\mathcal{G}, \leq \mu})^{\text{red}}|$ we have the identities $(\widehat{\text{Sht}}_{O_C}^{\mathcal{G}_b, \leq \mu})_x = (\widehat{\text{Sht}}_{W(\mathfrak{f}_C)}^{\mathcal{G}_b, \leq \mu})_x \times_{W(\mathfrak{f})^{\diamond}} O_C^{\diamond}$, and $(\widehat{\text{Gr}}_{O_C}^{\mathcal{G}, \leq \mu})_y = (\widehat{\text{Gr}}_{W(\mathfrak{f}_C)}^{\mathcal{G}, \leq \mu})_y \times_{W(\mathfrak{f})^{\diamond}} O_C^{\diamond}$ which finishes the proof of the claim. Finally, by theorem 2.2.34 $(\widehat{\text{Gr}}_{O_C}^{\mathcal{G}, \leq \mu})_y$ is non-empty and connected as we wanted to show. \square

We finish this section with the proof of theorem 2 which is a rephrasing of 2.3.18 in less technical language. For the convenience of the reader we write the statement again.

Theorem 2.3.19. *With notation as in the introduction the following holds:*

a) *There is a continuous specialization map*

$$\text{sp}_{\text{Sht}_{O_{F_2}}^{\mathcal{G}_b, \leq \mu}} : |\text{Sht}_{(\mathcal{G}, b, \mu), F_2^{\diamond}}| \rightarrow |X_{\leq \mu}^{\mathcal{G}}(b)|,$$

this map is a specializing and spectral map of locally spectral topological spaces. It is a quotient map and $J_b(\mathbb{Q}_p)$ -equivariant.

- b) Given a closed point $x \in |X_{\leq \mu}^{\mathcal{G}}(b)|$ let $S_x = \text{sp}_{\text{Sht}_{O_{F_2}}^{\mathcal{G}_b, \leq \mu}}^{-1}(x)$, then the interior S_x° of S_x as a subspace of $|\text{Sht}_{(\mathcal{G}, b, \mu), F_2^\diamond}|$ is dense in S_x .
- c) S_x and S_x° are non-empty and connected.
- d) The specialization map induces a $J_b(\mathbb{Q}_p)$ -equivariant bijection of connected components

$$\text{sp}_{\text{Sht}_{O_{F_2}}^{\mathcal{G}_b, \leq \mu}} : \pi_0(\text{Sht}_{(\mathcal{G}, b, \mu), F_2^\diamond}) \rightarrow \pi_0(X_{\leq \mu}^{\mathcal{G}}(b))$$

Proof of theorem 1. We may apply theorem 2.3.18 and proposition 2.3.10 to conclude that the pair $(\text{Sht}_{O_{F_2}^\diamond}^{\mathcal{G}_b, \leq \mu}, \text{Sht}_{(\mathcal{G}, b, \mu), F_2^\diamond})$ is a rich smelted kimberlite with reduction $X_{\leq \mu}^{\mathcal{G}}(b)$. This implies by proposition 1.4.20 that the specialization map $\text{sp}_{\text{Sht}_{O_{F_2}}^{\mathcal{G}_b, \leq \mu}} : |\text{Sht}_{(\mathcal{G}, b, \mu), F_2^\diamond}| \rightarrow |X_{\leq \mu}^{\mathcal{G}}(b)|$ is a spectral map of locally spectral spaces. $\text{Sht}_{O_{F_2}^\diamond}^{\mathcal{G}_b, \leq \mu}$ is rich, by definition the specialization map is specializing and a quotient map. Moreover, $J_b(\mathbb{Q}_p)$ acts on $\text{Sht}_{O_{F_2}}^{\mathcal{G}_b, \leq \mu}$ by ϕ -equivariant automorphisms of \mathcal{G}_b , since the construction of the specialization map is functorial in the category of smelted kimberlites the map is equivariant, this finishes the proof of the first claim. Let $x \in |X_{\leq \mu}^{\mathcal{G}}(b)|$, we can use proposition 1.4.29 to identify S_x° with $|\widehat{(\text{Sht}_{O_{F_2}^\diamond}^{\mathcal{G}_b, \leq \mu})}_x|_\eta$. Since $\text{Sht}_{O_{F_2}}^{\mathcal{G}_b, \leq \mu}$ is rich we can apply proposition 1.4.33 to prove that S_x° is dense in S_x giving the second claim. By theorem 2.3.18 S_x° is connected and since it is dense in S_x this later one is also connected giving the third claim. For the last claim we may apply proposition 1.4.42. \square

Bibliography

- [1] Johannes Anschütz, *Extending torsors on the punctured $\mathrm{Spec}(A_{\mathrm{inf}})$* , 2018.
- [2] ———, *Extending torsors on the punctured $\mathrm{Spec}(A_{\mathrm{inf}})$* , 2020.
- [3] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier, *Theorie de topos et cohomologie étale des schémas I, II, III*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer, 1971.
- [4] Bhargav Bhatt, *Lecture notes for a class on perfectoid spaces*, <http://www-personal.umich.edu/~bhattb/teaching/mat679w17/lectures.pdf>, 2017.
- [5] Bhargav Bhatt and Peter Scholze, *Projectivity of the Witt vector affine Grassmannian*, Invent. Math. **209** (2017), no. 2, 329–423. MR 3674218
- [6] Miaofen Chen, *Composantes connexes géométriques de la tour des espaces de modules de groupes p -divisibles*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 4, 723–764. MR 3250062
- [7] Miaofen Chen, Mark Kisin, and Eva Viehmann, *Connected components of affine Deligne-Lusztig varieties in mixed characteristic*, Compos. Math. **151** (2015), no. 9, 1697–1762. MR 3406443
- [8] Brian Conrad, *Reductive group schemes*, Autour des schémas en groupes. Vol. I, Panor. Synthèses, vol. 42/43, Soc. Math. France, Paris, 2014, pp. 93–444. MR 3362641
- [9] Laurent Fargues and Jean-Marc Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p -adique*, Astérisque (2018), no. 406, xiii+382, With a preface by Pierre Colmez. MR 3917141
- [10] Paul Hamacher and Eva Viehmann, *Finiteness properties of affine deligne-lusztig varieties*, 2019.
- [11] Urs Hartl and Eva Viehmann, *The Newton stratification on deformations of local G -shtukas*, J. Reine Angew. Math. **656** (2011), 87–129. MR 2818857
- [12] X. He and R. Zhou, *On the connected components of affine deligne-lusztig varieties*, arXiv: Algebraic Geometry (2016).
- [13] R. Huber, *Continuous valuations*, Math. Z. **212** (1993), no. 3, 455–477. MR 1207303
- [14] ———, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513–551. MR 1306024
- [15] Roland Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996. MR 1734903
- [16] J. N. Pereira Lourenço J. Anschütz and T. Richarz, *The B_{dR}^+ -affine grassmannian and local models*, In preparation.

- [17] Kiran Kedlaya, *Sheaves, stacks and shtukas; aws lecture notes*, <http://swc.math.arizona.edu/aws/2017/2017KedlayaNotes.pdf>, 2017.
- [18] Kiran S. Kedlaya, *Some ring-theoretic properties of A_{inf}* , 2016.
- [19] Kiran S. Kedlaya and Ruochuan Liu, *Relative p -adic Hodge theory: foundations*, Astérisque (2015), no. 371, 239. MR 3379653
- [20] Robert E. Kottwitz, *Isocrystals with additional structure*, Compositio Math. **56** (1985), no. 2, 201–220. MR 809866
- [21] João N. P. Lourenço, *The riemannian hebbbarkeitssätze for pseudorigid spaces*, 2017.
- [22] Sian Nie, *Connected components of closed affine Deligne-Lusztig varieties in affine Grassmannians*, Amer. J. Math. **140** (2018), no. 5, 1357–1397. MR 3862068
- [23] G. Pappas and X. Zhu, *Local models of Shimura varieties and a conjecture of Kottwitz*, Invent. Math. **194** (2013), no. 1, 147–254. MR 3103258
- [24] Michael Rapoport, *A guide to the reduction modulo p of Shimura varieties*, no. 298, 2005, Automorphic forms. I, pp. 271–318. MR 2141705
- [25] David Rydh, *Submersions and effective descent of étale morphisms*, Bull. Soc. Math. France **138** (2010), no. 2, 181–230. MR 2679038
- [26] Peter Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313. MR 3090258
- [27] ———, *On torsion in the cohomology of locally symmetric varieties*, Ann. of Math. (2) **182** (2015), no. 3, 945–1066. MR 3418533
- [28] Peter Scholze, *Etale cohomology of diamonds*, 2017.
- [29] Peter Scholze and Jared Weinstein, *Moduli of p -divisible groups*, Camb. J. Math. **1** (2013), no. 2, 145–237. MR 3272049
- [30] Peter Scholze and Jared Weinstein, *Berkeley lectures on p -adic geometry*, <http://www.math.uni-bonn.de/people/scholze/Berkeley.pdf>, 2019.
- [31] The Stacks Project Authors, *Stacks Project*, <https://stacks.math.columbia.edu>, 2018.
- [32] Xinwen Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, Ann. of Math. (2) **185** (2017), no. 2, 403–492. MR 3612002