

PERFECTOID NULLSTELLENSATZ: RESULTS AND COUNTEREXAMPLES.

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ABSTRACT. We give necessary conditions and we give sufficient conditions for perfectoid Nullstellensatz to hold. As a consequence, we prove that perfectoid Nullstellensatz does not hold for \mathbb{C}_p and other natural p -adic fields.

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1. INTRODUCTION

A weak version of Hilbert's Nullstellensatz says that every finite collection

$$S := \{p_1, \dots, p_m\} \subseteq \mathbb{C}[x_1, \dots, x_n]$$

of polynomial functions in n -variables over the complex numbers fall into two cases: either all elements of S have a common solution in \mathbb{C}^n , or one can find a set of polynomials q_1, \dots, q_m such that

$$1 = \sum_{i=1}^m q_i p_i.$$

Recall that for a tuple $\bar{y} \in \mathbb{C}^n$ we have an evaluation map

$$\text{ev}_{\bar{y}} : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C} \tag{1.1}$$

$$\text{ev}_{\bar{y}}(f) = f(\bar{y}). \tag{1.2}$$

Let $I_{\bar{y}} := \ker(\text{ev}_{\bar{y}})$, we call ideals obtained in this way *evaluation ideals*. One can reformulate the statement to say that every maximal ideal of $\mathbb{C}[x_1, \dots, x_n]$ is an evaluation ideal. With this reformulation in mind we make the following definitions. Fix a field C and a C -algebra R .

Definition 1.1. Let $I \subseteq R$ be an ideal. We say that I is an *evaluation ideal* if the composition map $C \rightarrow R/I$ is an isomorphism.

Definition 1.2. We say that (C, R) satisfies the Nullstellensatz theorem, or that (C, R) is a *Nullstellensatz pair* if every maximal ideal of R is an evaluation ideal.

The following is another reformulation, see [Har77, Example 1.4.4].

Theorem 1.3. *The pair $(C, C[x_1, \dots, x_n])$ is a Nullstellensatz pair if and only if C is algebraically closed.*

Now suppose that C is a complete non-Archimedean analytic field. We denote by $O_C \subseteq C$ its ring of integers, by $|\cdot| : C \rightarrow \mathbb{R}^+ \cup \{0\}$ its norm, by $\varpi \in O_C$ a choice of uniformizer, by $\mathfrak{m} \subseteq O_C$ the unique maximal ideal and by $k := O_C/\mathfrak{m}$ the residue field of O_C . Let $T_n = C\langle x_1, \dots, x_n \rangle$ denote the Tate algebra of convergent power series in n -variables. The following is the version of the Nullstellensatz in the context of rigid geometry, see [Bos14, Theorem 4, Corollary 6 §3.2].

Theorem 1.4. *The pair (C, T_n) is a Nullstellensatz pair if and only if C is algebraically closed.*

Now suppose that C is a perfectoid field [Sch17, Definition 3.6]. Let $R_n = C\langle x_1^{1/p^\infty}, \dots, x_n^{1/p^\infty} \rangle$ denote the perfectoid Tate algebra in n -variables. We are concerned with the following question:

Question 1.5. *What conditions are sufficient and what conditions are necessary for (C, R_n) to be a Nullstellensatz pair?*

Evidently for (C, R_n) to be a Nullstellensatz pair it is necessary that C be algebraically closed, and one may be tempted by Theorem 1.3 and Theorem 1.4 to believe that this condition is also sufficient. Unfortunately, this is not the case.

Theorem 1.6. *Suppose that $n \geq 2$ and that $|C^\times| \subseteq \mathbb{R}^+$ is a proper subset. Then (C, R_n) is **not** a Nullstellensatz pair.*

Theorem 1.6 gives, outside of the case $n = 1$, an additional necessary condition for perfectoid Nullstellensatz to hold. Namely, that the value group of its norm map must be as large as possible. The following example shows that perfectoid Nullstellensatz fails for the fields that show up in nature.

Example 1.7. Let \mathbb{C}_p denote the p -adic completion of an algebraic closure of \mathbb{Q}_p . The pair $(\mathbb{C}_p, \mathbb{C}_p\langle x_1^{1/p^\infty}, \dots, x_n^{1/p^\infty} \rangle)$ is a Nullstellensatz pair if and only if $n = 1$. Indeed, $|\mathbb{C}_p^\times| = p^\mathbb{Q} \subseteq \mathbb{R}^+$. One can argue in an analogous way for the completions of the algebraic closures of fields that are topologically of finite transcendence degree over \mathbb{Q}_p .

We now give positive results, the first one says that perfectoid Nullstellensatz holds in dimension 1.

Proposition 1.8. *The pair $(C, C\langle x^{1/p^\infty} \rangle)$ is a Nullstellensatz pair if and only if C is algebraically closed.*

Our second result says that perfectoid Nullstellensatz holds as long as one works with fields that are large enough and suitably complete. This might be useful in a situation where one is allowed to work v -locally.

Theorem 1.9. *Suppose that all of the following conditions hold:*

- (1) C is algebraically closed.
- (2) $|C^\times| = \mathbb{R}^+$.
- (3) C is spherically complete.
- (4) k is uncountable.

then (C, R_n) is a Nullstellensatz pair.

Remark 1.10. As explained in Theorem 1.6 conditions 1 and 2 are necessary as long as $n \geq 2$. We suspect that condition 3 is also necessary, but that condition 4 is not. Nevertheless, we do not know this.

Let us describe briefly our methods. One can first reduce the proofs of Theorem 1.6, Proposition 1.8 and Theorem 1.9 to the case in which C is a characteristic p field by using the tilting equivalence. In characteristic p , the algebra R_n represents the unit ball \mathbb{B}_C^n in the category of diamonds, and to every maximal ideal of R_n one can associate a unique point in \mathbb{B}_C^n . We use the Berkovich classification of points in \mathbb{B}_C^1 for our arguments.

For proving Theorem 1.9 we use the assumptions on C to conclude that the Berkovich unit ball only has Type I points and Type II points. Type I points correspond to evaluation ideals and when $x \in \mathbb{B}_C^n$ projects to a Type II point in \mathbb{B}_C^1 then $k(x) = \mathcal{O}_{C(x)}/\mathfrak{m}$ is a transcendental extension of k . We exploit the fact that $k(x)$ has uncountable dimension as a k -vector space whereas “the elements coming from R_n ” are countably dimensional.

For proving Theorem 1.6, we consider a Type III point in $y \in \mathbb{B}_C^1$ corresponding to an element $r \in \mathbb{R}^+ \setminus |C^\times|$. This point induces a field $C(y)$ which is a proper extension of C , and we construct explicitly a surjection $R_3 \rightarrow C(y)$ exhibiting the failure of perfectoid Nullstellensatz, when $n = 3$. This construction resembles a division algorithm. The cases $n = 2$ and $n \geq 4$ can be reduced to the case $n = 3$.

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2. PREPARATIONS

2.1. Zariski closed subsets, tilting and points in the unit ball. Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space.

Definition 2.1. ([Sch17, Definition 5.7]) A closed subset $Z \subseteq X$ is said to be a *Zariski closed* set if there is an ideal $I \subseteq R$ such that Z is the vanishing locus of I .

By [Sch15, Lemma II.2.2], a Zariski closed subsets of X is always represented by an affinoid perfectoid space $Z = \mathrm{Spa}(S, S^+)$. Moreover, by [Sch17, Theorem 5.8] the natural map $R \rightarrow S$ is always surjective.

Recall that tilting induces an equivalence between perfectoid spaces over X and perfectoid spaces over its tilt X^\flat [Sch17, Corollary 3.20]. By [Sch15, Lemma II.2.7] and [Sch17, Theorem 5.8] a map of affinoid perfectoid spaces $Z \rightarrow X$ is a Zariski closed immersion if and only if the map of tilts $Z^\flat \rightarrow X^\flat$ is.

Corollary 2.2. (C, R_n) is a *Nullstellensatz pair* if and only if (C^\flat, R_n^\flat) is a *Nullstellensatz pair*.

By Corollary 2.2 we may and do assume from now on that C is a characteristic p field. In characteristic p , the algebra R_n represents \mathbb{B}_C^n , the unit ball in n -variables. Note that $\mathrm{Spa} C\langle x_1, \dots, x_n \rangle \cong \mathrm{Spa} C\langle x_1^{1/p^\infty}, \dots, x_n^{1/p^\infty} \rangle$ as topological spaces. For this reason, the notion of closed subset agrees for both adic spaces, but the notion of Zariski closed subset in one and the other are very different. Question 1.5 is asking which closed points of \mathbb{B}_C^1 can arise as Zariski closed subsets of the perfectoid algebra.

2.2. Spherically complete fields. Recall that a field extension of non-archimedean valued fields F over C is said to be an *immediate extension* [BCS18, Definition 6.9] if $|F^\times| = |C^\times|$ and the map of residue fields $O_C/\mathfrak{m} \rightarrow O_F/\mathfrak{m}_F$ is an isomorphism. A non-archimedean valued field is said to be *maximally complete* if it has no proper immediate extensions. By [BCS18, Theorem 6.12] maximally complete fields coincide with spherically complete fields, and by [BCS18, Theorem 6.13] any non-archimedean valued field admits a maximally complete immediate extension, which we will refer to as a *spherical completion*.

In general, a spherical completion of a field C might be hard to describe and two different spherical completions might not even be isomorphic [BCS18, Remark 6.21]. Nevertheless, every non-archimedean field admits a valuation preserving injection into a field of a very simple form.

Let $\Gamma \subseteq \mathbb{R}$ be an ordered abelian subgroup, and let k be field. We let $k((\Gamma))$ denote the set of pairs $\{(\mathcal{P}, g)\}$ where $\mathcal{P} \subseteq \Gamma$ is a well-ordered subset and $g : \mathcal{P} \rightarrow k^\times$ is a function. Alternatively, one can think of $k((\Gamma))$ as the set of functions $g : \mathbb{R} \rightarrow k$ whose support is well-ordered and contained in Γ . The addition rule is given point-wise addition $[g_1 + g_2](\gamma) = g_1(\gamma) + g_2(\gamma)$ and multiplication is given by convolution $[g_1 \cdot g_2](\gamma) = \sum_{\gamma_1 \in \Gamma} g_1(\gamma_1)g_2(\gamma - \gamma_1)$.

We can endow this ring with a valuation by letting $|g|_{k((\Gamma))} = e^{-\gamma_g} \in \mathbb{R}$ where γ_g is the smallest element in the support of g and $|g|_{k((\Gamma))} = 0$ when $g = 0$.

Theorem 2.3. ([BCS18, Theorem 3.16]) *For all k and $\Gamma \subseteq \mathbb{R}$, $k((\Gamma))$ is a complete non-archimedean valued field with residue field isomorphic to k and value group Γ .*

Definition 2.4. Any field of the form $k((\Gamma))$ is called a *Hahn field*.

Theorem 2.5. ([BCS18, Theorem 7.3]) *Suppose that C is spherically complete and that k is algebraically closed. Let $\Gamma = \log(|C^\times|) \subseteq \mathbb{R}$ and suppose that Γ is divisible. Then C is isomorphic to $k((\Gamma))$.*

A consequence of Theorem 2.5 is that every field C may be embedded into a Hahn field \mathfrak{C} in such a way that the residue field of \mathfrak{C} is an algebraic extension of k .

3. RESULTS

In this section we give the proof of Theorem 1.9.

Theorem 3.1. *Suppose that all of the following conditions hold:*

- (1) C is algebraically closed.
- (2) $|C^\times| = \mathbb{R}^+$.
- (3) C is spherically complete.
- (4) k is uncountable.

then (C, R_n) is a Nullstellensatz pair.

Proof. By Corollary 2.2 we may assume that C has characteristic p . Let $x \in \mathbb{B}_C^n$ denote the point associated to a maximal ideal $I \subseteq R_n$, let $C(x) = R_n/I$ be the residue field at x , and assume for the sake of contradiction that $C(x) \neq C$. By induction, we may also choose n to be minimal for which such an ideal I exists. Let x_i be the image of x under the projections $\pi_i : \mathbb{B}_C^n \rightarrow \mathbb{B}_C^1$. Recall the Berkovich classification of points in the unit ball [Ber90, §1.4.4]. Since $|C^\times| = \mathbb{R}^+$ and C is spherically complete, \mathbb{B}_C^1 does not have Type III or Type IV points. By minimality of n , x_i is also not a Type I point, so x_i must be a Type II point. This implies that the residue field $k(x)$ of $C(x)$ is a transcendental extension of k .

Let $R'_n = f^{-1}(O_{C(x)})$, and let $A_n \subseteq k(x)$ the image of R'_n in $k(x)$. We claim that A_n is a proper subring of $k(x)$, which implies that $f : R_n \rightarrow C(x)$ is not surjective. This gives the contradiction.

By Theorem 2.5 we may embed $C(x)$ into a Hahn field $\mathfrak{C}(x)$. We may interpret $x \in \mathbb{B}_C^n$ as a map $f : R_n \rightarrow \mathfrak{C}(x)$ with $f(R_n) = C(x)$. One can describe $\mathfrak{C}(x)$ explicitly, it is of the form $\overline{k(x)}((\mathbb{R}))$, and maps $R_n \rightarrow \mathfrak{C}(x)$ are determined by the choice of a tuple $(c_1, \dots, c_n) \in O_{\mathfrak{C}(x)}^n$. We may interpret each element c_i as a pair $(\mathcal{P}_{c_i}, g_{c_i})$ where $\mathcal{P}_{c_i} \subseteq \mathbb{R}$ is well-ordered subset and $g_{c_i} : \mathcal{P}_{c_i} \rightarrow \overline{k(x)}^\times$ is a function. Similarly, for $q \in R_n$ we may interpret $f(q)$ as a tuple (\mathcal{P}_q, g_q) , and $q \in R'_n$ if and only if the smallest element of \mathcal{P}_q is larger or equal to 0. Moreover, if $q \in R'_n$ its image in A_n is $g_q(0)$.

Let $S \subseteq \overline{k(x)}$ denote the set of elements of the form $s = g_{c_i}(r_i)$ where $r_i \in \mathcal{P}_{c_i}$ for some $i \in \{1, \dots, n\}$. It is not hard to see that

$$A_n \subseteq k[S^{1/p^\infty}] \cap k(x) \subseteq \overline{k(x)}.$$

Furthermore, note that S is countable. Indeed, for $i \in \{1, \dots, n\}$ the set $\mathcal{P}_i \subseteq \mathbb{R}$ is countable since it is a well-ordered subset of the real numbers. This readily implies that $k[S^{1/p^\infty}]$ has a countable basis as a k -vector space, which implies the same of A_n .

We finish by observing that $k(x)$ has uncountable dimension over k . Indeed, if $t \in k(x)$ is a transcendental element, then the set $\{\frac{1}{t-c} \mid c \in k\}$ is linearly independent over k . By our assumption 4 this set is uncountable. \square

In the case of one variable, the perfectoid Nullstellensatz holds more generally.

Proposition 3.2. *The pair (C, R_1) is a Nullstellensatz pair if and only if C is algebraically closed.*

Proof. Necessity is evident. Suppose C is algebraically closed and of characteristic p . We use the classifications of points in the unit ball \mathbb{B}_C^1 . Take $I \subseteq C\langle x^{1/p^\infty} \rangle$ a maximal ideal and let $f \in I$ be a non-zero element. Let Z_f denote the zero locus of f , we claim that $Z_f = S \times \mathrm{Spa}(C)$ for a profinite set S . Equivalently, we claim that $Z_f \rightarrow \mathrm{Spa}(C)$ is quasi-pro-étale. By [Sch17, Proposition 10.11.(v)], it suffices to prove this after a v-cover of $\mathrm{Spa}(C)$, so we may enlarge C to assume that $|C^\times| = \mathbb{R}^+$ and that C is spherically complete.

Let $x \in \mathbb{B}_C^1$, we prove that $x \notin Z_f$ whenever x is of Type II. By changing coordinates we may always assume that x corresponds to the ball of radius $r \in \mathbb{R}$ centered at the origin. Recall that $C\langle x^{1/p^\infty} \rangle$ is the completed colimit of the system of rings:

$$C\langle x \rangle \xrightarrow{x \mapsto x^p} C\langle x \rangle \xrightarrow{x \mapsto x^p} \dots \quad (3.1)$$

Write f in the form

$$f = \sum_{m \in \mathbb{Z}[\frac{1}{p}]} a_m x^m \quad (3.2)$$

with $a_m \in C$ such that for all $s \in \mathbb{R}^+$ only finitely many terms in the sum have value $|a_m| > s$. Choose a monomial of the form $a_M x^M$ appearing in f , and let $\epsilon = |a_M| \cdot r^M$. We can approximate f by the finite sum of terms $f_\epsilon = \sum_{j=1}^k a_{m(j)} x^{m(j)}$ with $m(j) \in \mathbb{Z}[\frac{1}{p}]$ of those monomials appearing in the expression of f for which $|a_{m(j)}| \geq \epsilon$. By construction $|f - f_\epsilon|_x < \epsilon$. Moreover, since $a_M x^M$ shows up as a monomial of f_ϵ we also know that $|f_\epsilon|_x \geq \epsilon$ from which it follows that $|f|_x \geq \epsilon$.

Now that we know Z_f consists only of Type I points, we claim that the natural map $Z_f \rightarrow \pi_0(Z_f) \times \mathrm{Spa}(C)$, is an isomorphism. Since both spaces are proper over $\mathrm{Spa}(C)$, by [Sch17, Lemma 11.11] it suffices to prove that

every component of Z_f consists of one point. But if two Type I points lie in the same connected component of Z_f , the whole Berkovich path between them in \mathbb{B}_C^1 would also lie in Z_f . This contradicts the fact that Z_f does not contain Type II points. From the above it follows easily that (C, R_n) is a Nullstellensatz pair. Indeed, the map $\mathrm{Spa} R_n/I \rightarrow Z_f \rightarrow \mathrm{Spa} C$ must factor through a connected component of $y \in \pi_0(Z_f)$, which has residue field C . \square

4. THE COUNTEREXAMPLE

We now discuss the proof of Theorem 1.6. In other words, we give a counterexample to perfectoid Nullstellensatz for every field C such that $|C^\times| \neq \mathbb{R}^+$.

Theorem 4.1. *Suppose that $n \geq 2$ and that $|C^\times| \subseteq \mathbb{R}^+$ is a proper subset. Then (C, R_n) is **not** a Nullstellensatz pair.*

Proof. Assume that $n \geq 3$, and that $|C^\times| \neq \mathbb{R}^+$. With this setup, we construct a proper field extension $C(y)$ over C and a surjective map $f_y : R_3 \rightarrow C(y)$. The kernel of f_y is a maximal ideal that is not an evaluation ideal. Since for every map $n \geq 3$ we have a surjective map $R_n \rightarrow R_3$, this proves that (C, R_n) is not a Nullstellensatz pair whenever $n \geq 3$. We reduce the case $n = 2$ to the case $n = 3$ on Proposition 4.5 below.

Let $r \in \mathbb{R}^+ \setminus |C^\times|$ with $r < 1$. We can consider the unit ball of radius r centered at the origin and this gives rise to a Type III point in $y \in \mathbb{B}_C^1$. We let $C(y)$ denote the (perfected) residue field at this point. We can describe this field explicitly. It consists of power series expressions

$$\alpha = \sum_{q \in \mathbb{Z}[\frac{1}{p}]} a_q x^q \quad (4.1)$$

where $q = \frac{i}{p^m}$ for some $i \in \mathbb{Z}$ relatively prime to p , where $a_q \in C$ and subject to the constraint that for any $\epsilon > 0$ the set of values $|a_q| r^q > \epsilon$ is finite. Addition and multiplications are given by the evident formulas and the value $|\alpha|_y$ is given by $\sup_{q \in \mathbb{Z}[\frac{1}{p}]} |a_q| r^q$. Moreover, the supremum is always a

maximum and it is uniquely attained, we call this term the *leading monomial*.

We divide the proof on two steps. The first step, corresponds to Lemma 4.3, it gives a criterion for when a map $R_3 \rightarrow C(y)$ is surjective. The second step, corresponding to Proposition 4.4, gives an explicit construction that satisfies the criterion of Lemma 4.3.

Definition 4.2. Let $0 < s < 1$ and let $q \in \mathbb{Z}[\frac{1}{p}]$ be of the form $q = \frac{i}{p^m}$ for some integer $i \in \mathbb{Z}$ relatively prime to p . We say that an element $\alpha \in C(y)$ is (q, s) -adapted if the following conditions hold:

- (1) $s < |\alpha|_y \leq 1$
- (2) If $\alpha = \sum_{j \in \mathbb{Z}[\frac{1}{p}]} a_j x^j$ then $|a_q| r^q = |\alpha|_y$. In other words $a_q x^q$ is the leading monomial.

$$(3) \quad |\alpha - a_q x^q| \leq s \cdot |\varpi|.$$

Lemma 4.3. *Let $f : R_3 \rightarrow C(y)$ be a continuous map. Suppose there exists $0 < s < 1$ such that for any $q \in \mathbb{Z}[\frac{1}{p}]$, there is an element $a \in O_C\langle x_1^{1/p^\infty}, x_2^{1/p^\infty}, x_3^{1/p^\infty} \rangle$ such that $f(a)$ is (q, s) -adapted. Then f is surjective.*

Proof. Given $\beta \in C(y)$ we construct an element $a_\infty \in C\langle x_1^{1/p^\infty}, x_2^{1/p^\infty}, x_3^{1/p^\infty} \rangle$ mapping to β . Replacing β by $\varpi^k \beta$ for some k , we may assume $|\beta|_y \leq s$. We can write

$$\beta = \sum_{q \in \mathbb{Z}[\frac{1}{p}]} b_q x^q. \quad (4.2)$$

We construct recursively elements $a_m \in R_3$ and $\beta_m \in C(y)$ for which we prove inductively that $|\beta_m|_y \leq |\varpi^m| \cdot s$, that $|a_{m+1} - a_m| \leq |\varpi^m|$, and that a_m maps to $\beta - \beta_m$. In particular, the sequence a_m is a Cauchy sequence in $C\langle x_1^{1/p^\infty}, x_2^{1/p^\infty}, x_3^{1/p^\infty} \rangle$ and letting a_∞ be the limit of the a_m exhibits an element with $f(a_\infty) = \beta - \beta_\infty = \beta$. One should think of this recursion as a division algorithm and it is done as follows:

We let $\beta_0 = \beta$ and $a_0 = 0$, and by our hypothesis above $|\beta_0| \leq |\varpi^0| \cdot s$. If $\beta_m = \sum_{q \in \mathbb{Z}[\frac{1}{p}]} b_{q,m} x^q$, we let $\{\beta_m\} = \sum_{q \in S_m \subseteq \mathbb{Z}[\frac{1}{p}]} b_{q,m} x^q$ denote the sum of those terms for which $|\varpi^{m+1}| \cdot s \leq |b_{q,m} x^q|_y \leq |\varpi^m| \cdot s$. Since $\beta_m \in C(y)$, this consists of finitely many terms. For each term $b_{q,m} x^q$ appearing in $\{\beta_m\}$ we choose an element $e_{q,m} \in O_C\langle x_1^{1/p^\infty}, x_2^{1/p^\infty}, x_3^{1/p^\infty} \rangle$ mapping to an element $f(e_{q,m})$ that is (q, s) -adapted. Moreover, we let $c_{q,m} x^q$ denote the leading monomial of $f(e_{q,m})$.

Now, by construction $|\varpi^{m+1}| \cdot s \leq |b_{q,m} x^q|_y \leq |\varpi^m| \cdot s$ and since $f(e_{q,m})$ is (q, s) -adapted $|\frac{b_{q,m} x^q}{\varpi^m}|_y < |c_{q,m} x^q|_y$. We may find an element $d_{q,m} \in O_C$ with $|d_{q,m}| < 1$ and such that $b_{q,m} = c_{q,m} d_{q,m} \varpi^m$. We let $e_m = \sum d_{q,m} \varpi^m e_{q,m}$.

Notice that by construction, and since $f(e_{q,m})$ is (q, s) -adapted, $|f(e_m) - \{\beta_m\}|_y \leq s \cdot |\varpi^{m+1}|$. We let $a_{m+1} = a_m + e_m$ and we let $\beta_{m+1} = \beta_m - f(e_m)$. Note that $|\beta_{m+1}|_y$ is bounded by the maximum of $|\beta_m - \{\beta_m\}|_y$ and $|\{\beta_m\} - f(e_m)|_y$. In particular, $|\beta_{m+1}|_y \leq |\varpi^{m+1}| \cdot s$. It is clear from the definition of a_{m+1} that $|a_{m+1} - a_m| \leq |\varpi^m|$ since $|e_m| \leq \varpi^m$ holds by construction. Now, $f(a_{m+1}) = f(a_m) + f(e_m) = \beta - \beta_m + f(e_m) = \beta - \beta_{m+1}$, by the definition of β_{m+1} . \square

We now pass to the second part of the proof. Namely, we construct a map $R_3 \rightarrow C(y)$ satisfying the hypothesis of Lemma 4.3.

Proposition 4.4. *For any $0 < s < 1$, there is a continuous map $f : R_3 \rightarrow C(y)$ satisfying that for any $q \in \mathbb{Z}[\frac{1}{p}]$ there is an element $a_q \in O_C\langle x_1^{1/p^\infty}, x_2^{1/p^\infty}, x_3^{1/p^\infty} \rangle$ such that $f(a_q)$ is (q, s) -adapted.*

Proof. We fix $0 < s < 1$ and construct the map. Since we are working in characteristic p it suffices to specify the images of x_1, x_2 and x_3 . We let $x_1 \mapsto$

x and we let $x_2 \mapsto cx^{-1}$ for some element $c \in O_C$ so that $s < |cx^{-1}|_y < 1$. The construction of the image of x_3 is more elaborate.

Choose a bijection $\omega : \mathbb{N} \rightarrow \mathbb{Z}[\frac{1}{p}]$, we use ω to well-order the elements of $\mathbb{Z}[\frac{1}{p}]$. We let $x_3 \mapsto \alpha$ with $\alpha \in C(y)$ constructed as a sum $\sum_{m=1}^{\infty} \alpha_m$ given by:

$$\alpha = \sum_{m=1}^{\infty} \alpha_m = \sum_{m=1}^{\infty} [e_m x^{\omega(m)}]^{p^{b_m}} \quad (4.3)$$

Here $\alpha_m = [e_m x^{\omega(m)}]^{p^{b_m}}$, where we fix $e_m \in C$ so that $s < |e_m x^{\omega(m)}|_y < 1$ and then we construct $b_m \in \mathbb{N}$ recursively.

We let $b_1 = 0$ and for $m > 1$ we choose b_m recursively depending on $\{b_1, \dots, b_{m-1}\}$ and sufficiently large so that b_m satisfies the following constraints:

- (1) We require that b_m be large enough so that $|\alpha_m| < |\varpi|^m$.
- (2) Choose $\epsilon_m \in O_C \setminus \{0\}$, with the property that for all $1 \leq j < m$,

$$|\epsilon_m \alpha_j| \leq |x^{\omega(j) \cdot p^{b_j}}| \quad \text{when} \quad 0 < \omega(j) \quad (4.4)$$

$$|\epsilon_m \alpha_j| \leq |(cx^{-1})^{-\omega(j)p^{b_j}}| \quad \text{when} \quad \omega(j) < 0. \quad (4.5)$$

With ϵ_m chosen in this way we require that b_m be large enough so that $s < |\epsilon_m^{p^{-b_m}} e_m x^{\omega(m)}| < 1$.

- (3) For all $1 \leq j < m$, we require $b_m - b_j$ to be large enough so that $|(e_m x^{\omega(m)})^{p^{b_m - b_j}}| < |\varpi| \cdot s$.

The first condition ensures that α is a well defined element in $C(y)$. We will exploit the second and third condition to construct for all $q \in \mathbb{Z}[\frac{1}{p}]$ a (q, s) -adapted element in the image of $O_C \langle x_1^{1/p^\infty}, x_2^{1/p^\infty}, x_3^{1/p^\infty} \rangle$.

Let $W_{\omega(j)} = x_1^{\omega(j)p^{b_j}}$ when $\omega(j) \geq 0$ and $W_{\omega(j)} = x_2^{-\omega(j)p^{b_j}}$ if $\omega(j) < 0$. By the requirement in 2, for $1 \leq j \leq m-1$ the j th-term of $\epsilon_m \cdot f(x_3)$ is divisible by $f(W_{\omega(j)})$, let $d_{g(j)} \in O_C$ denote $\frac{\epsilon_m \cdot \alpha_j}{f(W_{\omega(j)})}$. Let

$$a_{\omega(m)}^{p^{b_m}} = \epsilon_m x_3 - \sum_{i=1}^{m-1} d_{g(i)} W_{\omega(i)}, \quad (4.6)$$

we claim that $a_{\omega(m)} \in O_C \langle x_1^{1/p^\infty}, x_2^{1/p^\infty}, x_3^{1/p^\infty} \rangle$ satisfies that $f(a_{\omega(m)})$ is $(\omega(m), s)$ -adapted. Indeed, $f(a_{\omega(m)})^{p^{b_m}} = \epsilon_m \cdot \sum_{j=m}^{\infty} \alpha_j$, so that $f(a_{\omega(m)})$

has leading term $\epsilon_m^{p^{-b_m}} \cdot e_m x^{\omega(m)}$, with value $s < |\epsilon_m^{p^{-b_m}} \cdot e_m x^{\omega(m)}| < 1$ by 2. The next terms have the form $\epsilon_m^{p^{-b_m}} \cdot (e_k x^{\omega(k)})^{p^{b_k - b_m}}$ with value $|\epsilon_m^{p^{-b_m}} \cdot (e_k x^{\omega(k)})^{p^{b_k - b_m}}| < |\varpi| \cdot s$ by 3. This finishes the proof that $f(a_{\omega(m)})$ is $(\omega(m), s)$ -adapted. \square

\square

Proposition 4.5. *If $|C^\times| \neq \mathbb{R}^+$, then (C, R_2) is not a Nullstellensatz pair.*

Proof. Let $I_c \subseteq R_3$ denote the ideal generated by $x_1 \cdot x_2 - c$, and let $S_c = R_3/I_c$. By Lemma 4.3 and the explicit construction of Proposition 4.4 we know that for c chosen so that $s < |c \cdot x^{-1}|_y < 1$ the pair (C, S_c) is not a Nullstellensatz pair since there is a surjection $S_c \rightarrow C(y)$. We observe that $\mathrm{Spa} S_c = U \times \mathbb{B}_C^1$ for $U \subseteq \mathbb{G}_{m,C}$ an open subset of the multiplicative group. Indeed, the variable x_3 does not interact with the other variables and the variety cut by the equation $x_1 x_2 = c$ can be parametrized by the map $t \mapsto (t, c \cdot t^{-1})$, U corresponds to the locus where $|t| \leq 1$ and $|c \cdot t^{-1}| \leq 1$.

We can now regard $\mathrm{Spa} S_c = U \times \mathbb{B}^1$ as an open subset of \mathbb{B}_C^2 and by [Zav21, Corollary D.5] we may find a finite étale morphism $\mathrm{Spa} S_c \rightarrow \mathbb{B}_C^2$. This gives a map $R_2 \rightarrow C(y)$, and we let K denote the image of this map. Since $S_c \rightarrow C(y)$ is surjective $C(y)$ is a finite K -module. This implies that K is a field. We have constructed a surjective map $R_2 \rightarrow K$ and $K \neq C$ since $C(y)$ is a finite extension of K but not of C . \square

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