Comp 350 Assignment 4

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1. (a) We start by considering the taylor series of $f(c)$ around c_n and $f[c_n + f(c_n)]$ around c_n:

$$f(c) = f(c_n) + f'(c_n)(c - c_n) + \frac{f''(z)}{2}(c - c_n)^2,$$

$$f[c_n + f(c_n)] = f(c_n) + f'(c_n)f(c_n) + \frac{f''(X_n)}{2}f(c_n)^2$$

, where X_n is a point between c_n and $c_n + f(c_n)$. Moving the latter around we get the following:

$$\frac{f[c_n - f(c_n)]}{f(c_n)^2} = \frac{f'(c_n) + \frac{f''(X_n)f(c_n)}{2}}{f(c_n)}$$

Taking the inverse and then forming the difference $c - c_{n+1} = c - c_n + \frac{f(c_n)}{f'(c_n)+\frac{f''(X_n)}{2}f(c_n)} = \frac{f(c_n)+f'(c_n)(c-c_n)+\frac{f''(X_n)}{2}f(c_n)(c-c_n)}{f'(c_n)+\frac{f''(X_n)}{2}f(c_n)}$. Taking the expansion of $f(c)$ around c_n, we can shorten it to

$$f(c) = f(c_n) + f'(z*)(c - c_n)$$

where $z*$ does not necessarily equal z or X_n. Combining these and simplifying we get the following:

$$c - c_{n+1} = -\frac{\frac{f''(z)}{2}(c - c_n)^2 + \frac{f''(X_n)}{2}f'(z*)(x - x_n)^2}{f'(c_n) + \frac{f''(X_n)}{2}f(c_n)}$$

Using the fact the $f'(x)$ and $f''(x)$ are continuous; taking the limit of this we get:

$$\lim_{n\to\infty} \frac{|c - c_{n+1}|}{|c - c_n|^2} = \lim_{n\to\infty} \left| \frac{\frac{f''(z)}{2} + \frac{f''(X_n)}{2}f'(z*)}{f'(c_n) + \frac{f''(X_n)}{2}f(c_n)} \right| = \frac{1}{2}\left|\frac{f''(c)}{f'(c)}\right| |1 + f'(c)|$$

Since $f''(c)$ and $f'(c)$ are both continuous, this is equal to a constant. So this shows that the convergence is indeed quadratic unless $\frac{1}{2}\left|\frac{f''(c)}{f'(c)}\right| |1 + f'(c)| = 0$ when the order is higher.

(b) The output of all my programs are attached at the bottom of this document. All three of the functions (newtons, secant and steffensen) found the correct answer of 2.1746 but what is interesting is how many steps it took each and how costly they are and why.
Newton's method was the fastest with 7 iterations for $x_0 = 3$ and 6 with $x_0 = 2.5$ which is expected because it computes df separately, though this is inconvenient, quite costly sometimes and may not always be possible. Steffensen method, is a derivative of Newton's method because it replaces the derivative calculation with a forward difference approximation to it and can therefore be used to find roots of any function. It is more costly though because since calculating $f(x)$ and $f(x+h)$ are required. Steffensen's takes more steps than newton's method because the forward approximation uses a small interval at each iteration which is correlated to $f(x)$, quadratically as proved above.
The secant method is slower than newton's because it does not need to compute fd. It is faster than the steffenson method because it uses super-linear convergence.

2. From Newton's method, we know that $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ and $y_{i+1} = y_i - \frac{g(y_i)}{g'(y_i)}$.

   Expanding both $f(x, y)$ and $g(x, y)$ around (x_0, y_0)we get

$$f(x, y) = f(x_0, y_0) + (x - x_0)f_x + (y - y_0)f_y$$

   and

$$g(x, y) = g(x_0, y_0) + (x - x_0)g_x + (y - y_0)g_y$$

   This can be generalized with the following function:

$$f_{i+1} = f_i + (x_{i+1} - x_i)f_x + (y_{i+1} - y_i)f_y$$

   This can also be done for $g(x, y)$.
   Equating this to zero (since they converge to 0) and rearranging, we get

$$f_x x_{i+1} + f_y y_{i+1} = -f_i + x_i f_x + f_y y_i,$$

$$g_x x_{i+1} + g_y y_{i+1} = -g_i + x_i g_x + y_i g_y$$

   After solving for $x_{i+1} - x_i$ in both equations, we get

$$x_{i+1} - x_i = \frac{-f_i - (y_{i+1} - y_i)f_y}{f_x} = \frac{-g_i - (y_{i+1} - y_i)g_y}{g_x}$$

$$\Rightarrow -f_x g_i - f_x(y_{i+1} - y_i)g_y = -g_x f_i - g_x(y_{i+1} - y_i)f_y$$

$$\Rightarrow (y_{i+1} - y_i)(f_y g_x - f_x g_y) = -g_x f_i + f_x g_i \Rightarrow y_{i+1} - y_i = \frac{f_x g_i - g_x f_i}{f_y g_x - f_x g_y}$$

   and similarly,

$$x_{i+1} - x_i = \frac{f_i g_y - g_i f_y}{f_y g_x - f_x g_y}$$

   which are equivalent to the formulas in question.

```
>> run main
--------------------------------------------------
Newton with x0=3.0
   n            x                    f(x)
------------------------------------
   0   3.000000000000000e+00   9.000000000000000e+00

   1   2.437500000000000e+00   2.036865234375000e+00
   2   2.213032716315110e+00   2.563633850614186e-01
   3   2.175554938721488e+00   6.463361488812325e-03
   4   2.174560100666446e+00   4.479068050233792e-06
   5   2.174559410293313e+00   2.158273559871304e-12
   6   2.174559410292980e+00  -4.440892098500626e-16
   7   2.174559410292980e+00  -4.440892098500626e-16

Nroot1 =

    2.1746

Newton with x0=2.5
   n            x                    f(x)
------------------------------------
   0   2.500000000000000e+00   2.625000000000000e+00

   1   2.230769230769231e+00   3.791533909877129e-01
   2   2.176650207900208e+00   1.358464532314763e-02
   3   2.174562452216795e+00   1.973570361801791e-05
   4   2.174559410299432e+00   4.185984892046690e-11
   5   2.174559410292980e+00  -4.440892098500626e-16
   6   2.174559410292980e+00  -4.440892098500626e-16

Nroot2 =

    2.1746

--------------------------------------------------
Secant with x0=3, x1=2.5
   n           x            f(x)
-------------------------------------
   0   2.500000000000000e+00   2.625000000000000e+00
   1   2.294117647058823e+00   8.420516995725604e-01
   2   2.196883428228948e+00   1.471012873173900e-01
   3   2.176301702527455e+00   1.131753459442031e-02
   4   2.174586221934466e+00   1.739541855689808e-04
   5   2.174559442836684e+00   2.111399166615513e-07
   6   2.174559410293588e+00   3.945732629517806e-12
   7   2.174559410292980e+00   1.776356839400250e-15
   8   2.174559410292980e+00  -4.440892098500626e-16

SecantRoot1 =

    2.1746

--------------------------------------------------
Steffensen with x0=3
   n           x            f(x)
-------------------------------------
   0   3.000000000000000e+00   9.000000000000000e+00
   1   2.943750000000000e+00   8.121970458984373e+00
```

```
   2   2.884323096080784e+00   7.241289670443109e+00
   3   2.821373434940594e+00   6.359627589116213e+00
   4   2.754534966186990e+00   5.479541013752657e+00
   5   2.683456373334138e+00   4.604983416949556e+00
   6   2.607886435699827e+00   3.742165646952511e+00
   7   2.527871896333360e+00   2.901045275655515e+00
   8   2.444203924423074e+00   2.097937169941939e+00
   9   2.359385350870704e+00   1.359975508238714e+00
  10   2.279472615676871e+00   7.316110427307083e-01
  11   2.216103280265766e+00   2.774110711407629e-01
  12   2.182417991833528e+00   5.126545429903606e-02
  13   2.174873316004761e+00   2.037030957350972e-03
  14   2.174559924191816e+00   3.334119409892367e-06
  15   2.174559410294359e+00   8.945733043219661e-12
  16   2.174559410292980e+00  -4.440892098500626e-16
  17   2.174559410292980e+00  -4.440892098500626e-16

StefRoot1 =

    2.1746

Steffensen with x0=2.5
   n            x             f(x)
-------------------------------------
   0   2.500000000000000e+00   2.625000000000000e+00
   1   2.415535444947210e+00   1.840106431880669e+00
   2   2.331445502952307e+00   1.133063317889633e+00
   3   2.255522585795603e+00   5.554636238698674e-01
   4   2.201045344174577e+00   1.750297406198946e-01
   5   2.177911443493598e+00   2.179848308591747e-02
   6   2.174617403524390e+00   3.762688186799146e-04
   7   2.174559427848583e+00   1.138987881788012e-07
   8   2.174559410292982e+00   1.199040866595169e-14
   9   2.174559410292980e+00  -4.440892098500626e-16

StefRoot2 =

    2.1746

>>
```