SOLUTION OF BOUNDARY LAYER EQUATIONS

Prabal Talukdar

Associate Professor

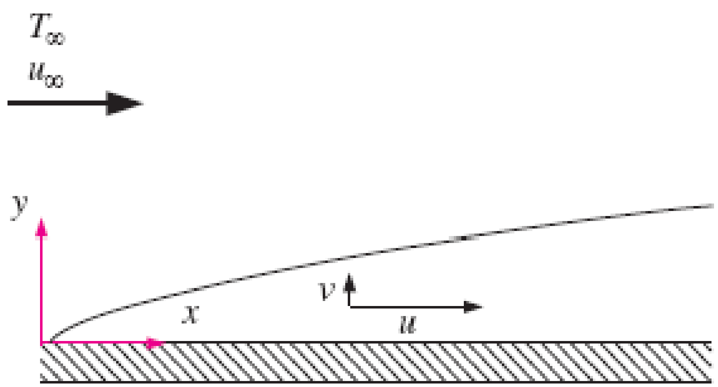
Department of Mechanical Engineering

IIT Delhi

E-mail: prabal@mech.iitd.ac.in

Boundary layer Approximation

X momentum: $$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}-\frac{\partial P}{\partial x}$$



1) Velocity components:

$v \ll u$

2) Velocity grandients:

$$\frac{\partial v}{\partial x}\approx 0,\ \frac{\partial v}{\partial y}\approx 0$$

$$\frac{\partial u}{\partial x}\ll\frac{\partial u}{\partial y}$$

3) Temperature gradients:

$$\frac{\partial T}{\partial x}\ll\frac{\partial T}{\partial y}$$

Applying Newton's 2nd law in the y-direction, we get y-momentum equation

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\mu\frac{\partial^2 v}{\partial x^2}-\frac{\partial P}{\partial y}$$

$$\frac{\partial P}{\partial y}=0$$

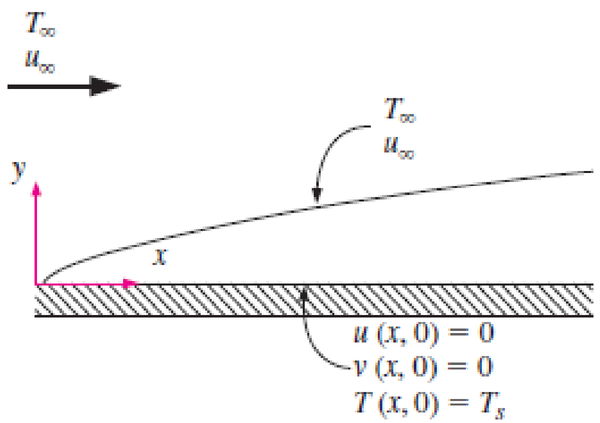
Thus, $P = P(x)$ Hence, $\dfrac{\partial P}{\partial x}=\dfrac{dP}{dx}$

For a flate plate, since $u = U_\infty$ = constant and $v = 0$ outside the boundary layer, X-momentum equation gives

$$\frac{\partial P}{\partial x}=0$$

Therefore, for flow over a flat plate, the pressure remains constant over the entire plate (both inside and outside the boundary layer).

Boundary layer over a flat plate



Continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

Energy: $$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

with the boundary conditions

At $x = 0$:	$u(0, y) = u_\infty$,	$T(0, y) = T_\infty$
At $y = 0$:	$u(x, 0) = 0$,	$v(x, 0) = 0$, $T(x, 0) = T_s$
As $y \to \infty$:	$u(x, \infty) = u_\infty$,	$T(x, \infty) = T_\infty$

Paul Richard Heinrich Blasius (1883 – 1970) was a German Fluid Dynamic Engineer. He was one of the first students of Prandtl.

Born	9 August 1883 Berlin, Germany
Died	24 April 1970 (aged 86) Hamburg, West Germany
Fields	Fluid mechanics and mechanical engineering
Alma mater	University of Göttingen
Doctoral advisor	Ludwig Prandtl

The continuity and momentum equations were first solved in 1908 by the German engineer H. Blasius, a student of L. Prandtl.

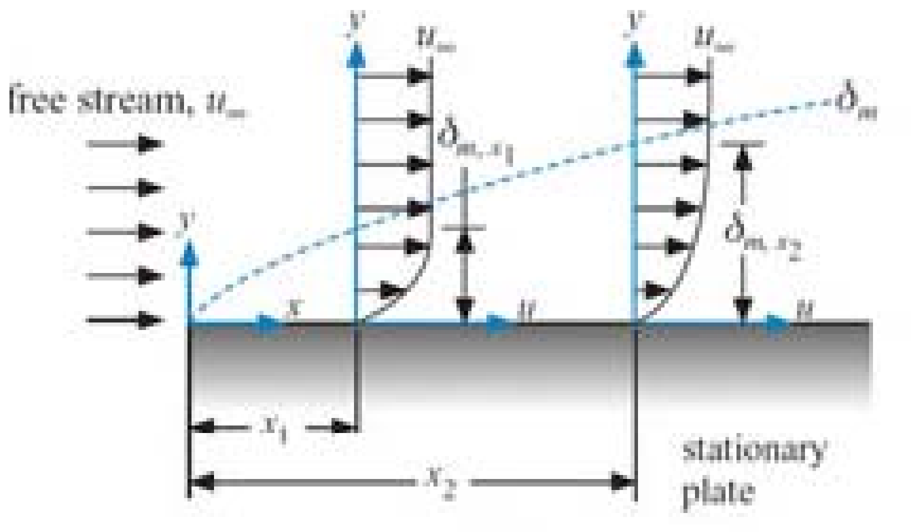
This was done by transforming the two partial differential equations into a single ordinary differential equation by introducing a new independent variable, called the similarity variable.

The finding of such a variable, assuming it exists, is more of an art than science, and it requires to have a good insight of the problem.

The shape of the velocity profile remains the same along the plate.

Blasius reasoned that the nondimensional velocity profile u/u_∞ should remain unchanged when plotted against the nondimensional distance y/δ, where δ is the thickness of the local velocity boundary layer at a given x.

That is, although both δ and u at a given y vary with x, the velocity u at a fixed y/δ remains constant



Blasius was also aware from the work of Stokes that δ is proportional to

$$\sqrt{\frac{\nu x}{u_\infty}}$$

Scale Analysis

$$u \sim u_\infty$$

$$y \sim \delta$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{u_\infty}{x} + \frac{v}{\delta} \approx 0$$

$$v \sim \frac{u_\infty \delta}{x}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$u_\infty\frac{u_\infty}{x} + \frac{u_\infty \delta}{x}\frac{u_\infty}{\delta} \approx \nu\frac{u_\infty}{\delta^2}$$

$$\delta^2 \sim \frac{\nu x}{u_\infty}$$

$$\delta \sim \sqrt{\frac{\nu x}{u_\infty}}$$

Dividing by x to express the result in dimensionless form gives

$$\frac{\delta}{x} \sim \sqrt{\frac{\nu}{u_\infty x}} = \frac{1}{\sqrt{\mathrm{Re}_x}}$$

Similarity Variable

The significant variable is y/δ, and we assume that the velocity may be expressed as a function of this variable. We then have

$$\frac{u}{u_\infty} = g\left(\frac{y}{\delta}\right)$$

$$\frac{u}{u_\infty} = g(\eta)$$

$$\delta \sim \sqrt{\frac{\nu x}{u_\infty}}$$

We define $\eta = y \Big/ \left(\sqrt{\dfrac{\nu x}{u_\infty}}\right)$

This makes, $\eta \approx y / \delta$

Here, η is called the similarity variable, and g(η) is the function we seek as a solution

Variable Transformation

A stream function was defined such that:

$$u = \frac{\partial \Psi}{\partial y} \qquad v = -\frac{\partial \Psi}{\partial x}$$ to get rid of continuity equation

$$d\Psi = u\,dy$$

$$\eta = y\sqrt{\frac{u_\infty}{\nu x}}$$

$$\psi = \int u_\infty g(\eta)\, dy = \int u_\infty \sqrt{\frac{\nu x}{u_\infty}} g(\eta)\, d\eta$$

$$d\eta = dy\sqrt{\frac{u_\infty}{\nu x}}$$

$$\Psi = u_\infty \sqrt{\nu x / u_\infty}\, f(\eta) \quad \text{where} \quad f(\eta) = \int g(\eta) d\eta$$

$$f(\eta) = \frac{\Psi}{u_\infty \sqrt{\nu x / u_\infty}} \Rightarrow \Psi = f(\eta) u_\infty \sqrt{\nu x / u_\infty}$$

$$u = \frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\infty \sqrt{\frac{\nu x}{u_\infty}} \frac{df}{d\eta} \sqrt{\frac{u_\infty}{\nu x}} = u_\infty \frac{df}{d\eta}$$

$$v = -\frac{\partial \Psi}{\partial x} = -u_\infty \sqrt{\frac{\nu x}{u_\infty}} \frac{df}{d\eta}\frac{\partial \eta}{\partial x} - \frac{u_\infty}{2}\sqrt{\frac{\nu}{u_\infty x}} f = \frac{1}{2}\sqrt{\frac{u_\infty \nu}{x}}\left(\eta \frac{df}{d\eta} - f\right)$$

Differentiating the previous equation with respect to x and y

$$\frac{\partial u}{\partial x} = -\frac{u_\infty}{2x}\eta\frac{d^2 f}{d\eta^2},$$

$$\frac{\partial u}{\partial y} = u_\infty\sqrt{\frac{u_\infty}{\nu x}}\frac{d^2 f}{d\eta^2}, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{u_\infty{}^2}{\nu x}\frac{d^3 f}{d\eta^3}$$

Substituting these relations into the momentum equation and simplifying, we obtain

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0$$

This is a third-order nonlinear differential equation.

This way the system of two PDEs is converted to one ODE.

Blasius Solution

Similarity function f and its derivatives for laminar boundary layer along a flat plate.

η	f	$\frac{df}{d\eta} = \frac{u}{u_\infty}$	$\frac{d^2f}{d\eta^2}$
0	0	0	0.332
0.5	0.042	0.166	0.331
1.0	0.166	0.330	0.323
1.5	0.370	0.487	0.303
2.0	0.650	0.630	0.267
2.5	0.996	0.751	0.217
3.0	1.397	0.846	0.161
3.5	1.838	0.913	0.108
4.0	2.306	0.956	0.064
4.5	2.790	0.980	0.034
5.0	3.283	0.992	0.016
5.5	3.781	0.997	0.007
6.0	4.280	0.999	0.002
∞	∞	1	0

$$2\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} = 0$$

Physical coordinates		Similarity coordinates	
$u = 0$	at $y = 0$	$\frac{df}{d\eta} = 0$	at $\eta = 0$
$v = 0$	at $y = 0$	$f = 0$	at $\eta = 0$
$\frac{\partial u}{\partial y} = 0$	at $y \to \infty$	$\frac{df}{d\eta} = 1.0$	at $\eta \to \infty$

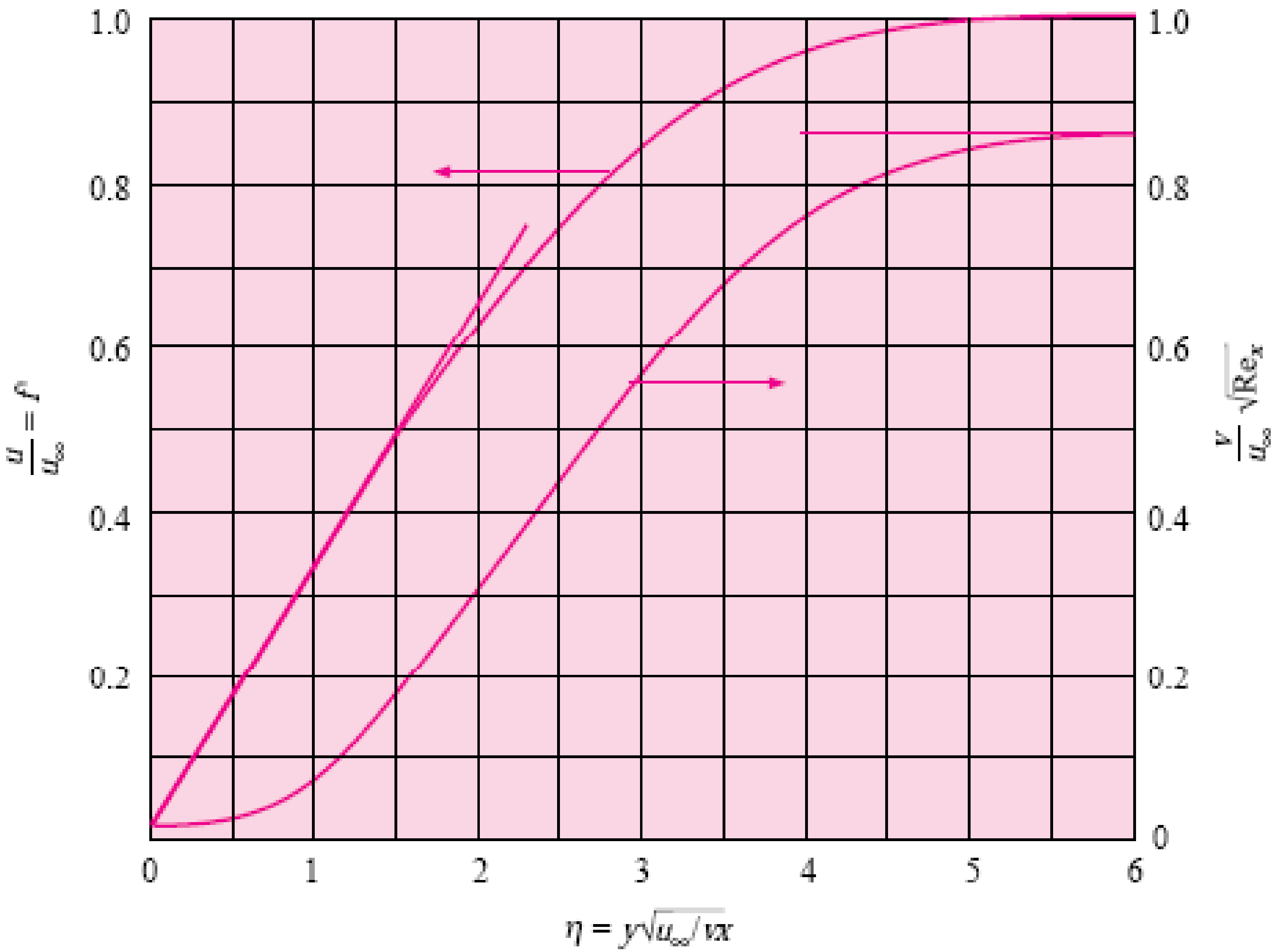
The value of η corresponding to $u/u_\infty = 0.992$ is 5.0

$$\eta = y\sqrt{\frac{u_\infty}{\nu x}} \Rightarrow 5 = \delta\sqrt{\frac{u_\infty}{\nu x}}$$

$$\delta = \frac{5.0}{\sqrt{u_\infty/\nu x}} = \frac{5.0x}{\sqrt{Re_x}}$$

Significance of u_∞, ν, x

Figure B-1 | Velocity profiles in laminar boundary layer. Slope $du/d\eta = 0.332$ at $\eta = 0$.



P.

Similarity function f and its derivatives for laminar boundary layer along a flat plate.

η	f	$\frac{df}{d\eta} = \frac{u}{u_\infty}$	$\frac{d^2f}{d\eta^2}$
0	0	0	0.332
0.5	0.042	0.166	0.331
1.0	0.166	0.330	0.323
1.5	0.370	0.487	0.303
2.0	0.650	0.630	0.267
2.5	0.996	0.751	0.217
3.0	1.397	0.846	0.161
3.5	1.838	0.913	0.108
4.0	2.306	0.956	0.064
4.5	2.790	0.980	0.034
5.0	3.283	0.992	0.016
5.5	3.781	0.997	0.007
6.0	4.280	0.999	0.002
∞	∞	1	0

The shear stress at the wall can be determined from:

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu u_\infty \sqrt{\frac{u_\infty}{\nu x}} \left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0}$$

$$\tau_w = 0.332\, u_\infty \sqrt{\frac{\rho\mu u_\infty}{x}} = \frac{0.332\, \rho u_\infty^2}{\sqrt{Re_x}}$$

Local skin friction coefficient becomes

$$C_{f,x} = \frac{\tau_w}{\rho V^2/2} = \frac{\tau_w}{\rho u_\infty^2/2} = 0.664 Re_x^{-1/2}$$

Note that unlike the boundary layer thickness, wall shear stress and the skin friction coefficient decrease along the plate as $x^{-1/2}$.

Energy Equation

Introduce a non-dimensional temperature $$\theta(x,y)=\frac{T(x,y)-T_s}{T_\infty - T_s}$$

Substitution gives an energy equation of the form: $$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}=\alpha\frac{\partial^2\theta}{\partial y^2}$$

Temperature profiles for flow over an isothermal flat plate are similar like the velocity profiles.

Thus, we expect a similarity solution for temperature to exist.

Further, the thickness of the thermal boundary layer is proportional to $\sqrt{\nu x / u_\infty}$

Using the chain rule and substituting the u and v expressions into the energy equation gives

$$u_\infty \frac{df}{d\eta}\frac{d\theta}{d\eta}\frac{\partial\eta}{\partial x}+\frac{1}{2}\sqrt{\frac{u_\infty \nu}{x}}\left(\eta\frac{df}{d\eta}-f\right)\frac{d\theta}{d\eta}\frac{\partial\eta}{\partial y}=\alpha\frac{d^2\theta}{d\eta^2}\left(\frac{\partial\eta}{\partial y}\right)^2$$

Using the chain rule and substituting the u and v expressions into the energy equation gives

$$u_\infty \frac{df}{d\eta}\frac{d\theta}{d\eta}\frac{\partial\eta}{\partial x}+\frac{1}{2}\sqrt{\frac{u_\infty \nu}{x}}\left(\eta\frac{df}{d\eta}-f\right)\frac{d\theta}{d\eta}\frac{\partial\eta}{\partial y}=\alpha\frac{d^2\theta}{d\eta^2}\left(\frac{\partial\eta}{\partial y}\right)^2$$

df/dη is replaced by θ

$$2\frac{d^2\theta}{d\eta^2}+\Pr f\frac{d\theta}{d\eta}=0$$

$$\theta(0)=0 \text{ and } \theta(\infty)=1$$

Compare
For Pr = 1

$$2\frac{d^3 f}{d\eta^3}+f\frac{d^2 f}{d\eta^2}=0$$

$$\left.\frac{df}{d\eta}\right|_{\eta=0}=0 \quad \text{and} \quad \left.\frac{df}{d\eta}\right|_{\eta=\infty}=1$$

Thus we conclude that the velocity and thermal boundary layers coincide, and the nondimensional velocity and temperature profiles (u/u_∞ and θ) are identical for steady, incompressible, laminar flow of a fluid with constant properties and Pr = 1 over an isothermal flat plate

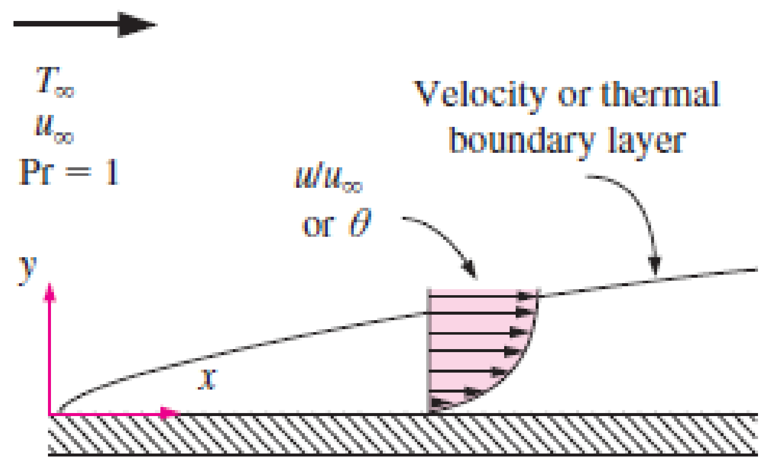
The value of the temperature gradient at the surface (Pr =1) ??

$$\frac{d\theta}{d\eta}=\frac{d^2 f}{d\eta^2}=0.332$$

$$2\frac{d^2\theta}{d\eta^2} + Pr\ f\ \frac{d\theta}{d\eta} = 0$$

This eq. is solved for numerous values of Prandtl numbers. For Pr > 0.6, the nondimensional temperature gradient at the surface is found to be proportional to $Pr^{1/3}$

Pr > 0.6 $$\left.\frac{d\theta}{d\eta}\right|_{\eta=0} = 0.332\ Pr^{1/3}$$



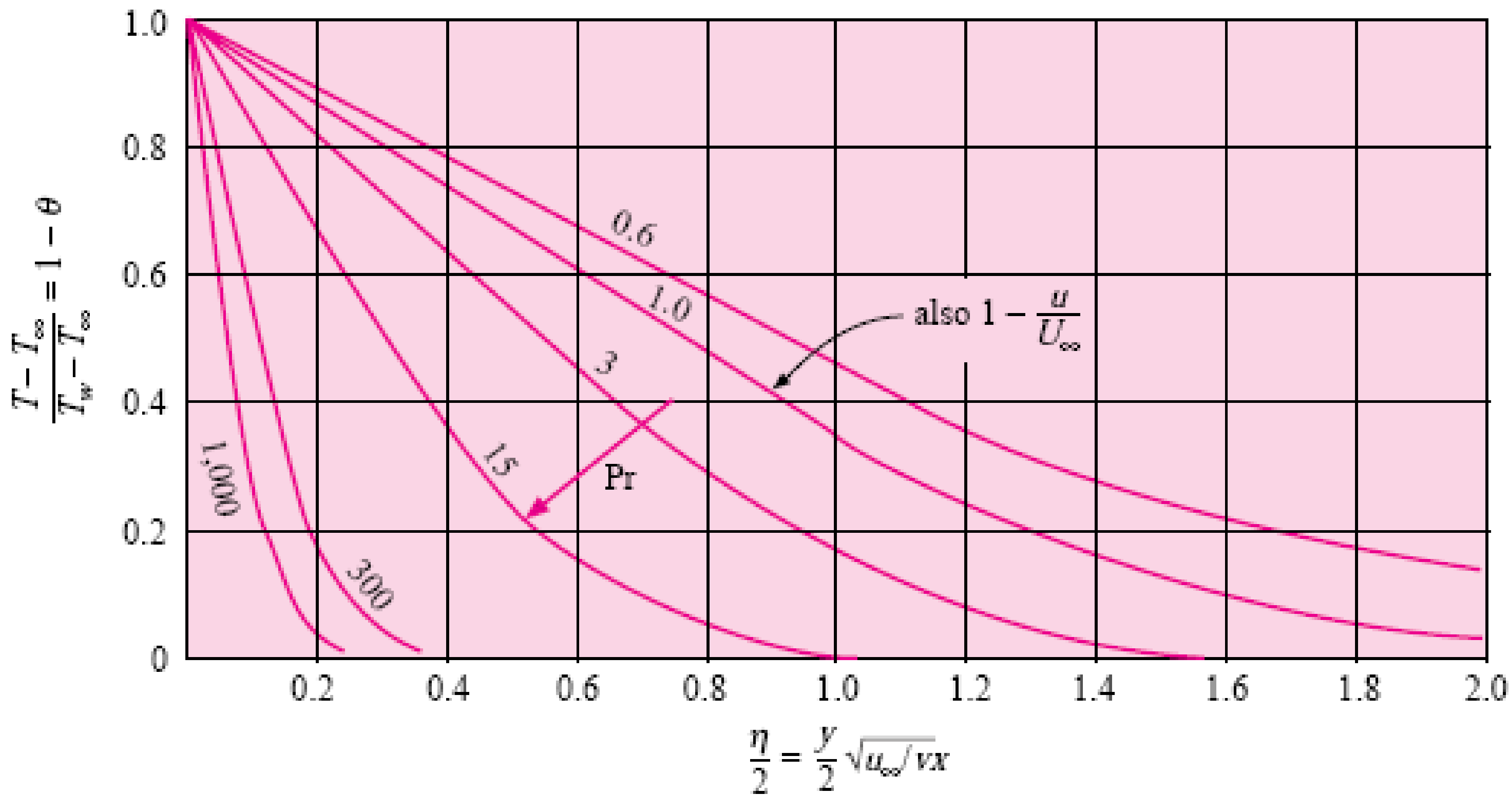
$$\theta(x,y) = \frac{T(x,y) - T_s}{T_\infty - T_s} \qquad \eta = \frac{y}{\delta} = y\sqrt{\frac{u_\infty}{\nu x}}$$

The temperature gradient at the surface is

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = (T_\infty - T_s)\left.\frac{\partial \theta}{\partial y}\right|_{y=0} =$$

$$=$$

Figure B-2 | Temperature profiles in laminar boundary layer with isothermal wall.



This solution is given by Pohlhausen

The local convection coefficient can be expressed as:

$$h_x = \frac{q_s}{(T_s - T_\infty)} = \frac{-k\left.\frac{\partial T}{\partial y}\right|_{y=0}}{(T_s - T_\infty)} = 0.332\ Pr^{1/3}\ k\sqrt{\frac{u_\infty}{\nu x}}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0.332\ Pr^{1/3}(T_\infty - T_s)\sqrt{\frac{u_\infty}{\nu x}}$$

And the local Nusselt number becomes

$$Nu_x = \frac{h_x x}{k} = 0.332\ Pr^{1/3}\ Re_x^{1/2} \qquad Pr > 0.6$$

Solving the thermal boundary layer equation numerically for the temperature profile for different Prandtl numbers, and using the definition of the thermal boundary layer, it is determined that

$$\frac{\delta}{\delta_t} \cong Pr^{1/3}$$

$$\delta_t = \frac{\delta}{Pr^{1/3}} = \frac{5.0x}{Pr^{1/3}\sqrt{Re_x}}$$

Non-dimensionalization

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$x^* = \frac{x}{L}, y^* = \frac{y}{L}, u^* = \frac{u}{V}, v^* = \frac{v}{V}, p^* = \frac{p}{\rho V^2} \text{ and } T^* = \frac{T - T_s}{T_\infty - T_s}$$

Continuity:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

Momentum:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L} \frac{\partial^2 u^*}{\partial y^{*2}} - \frac{dp^*}{dx^*}$$

Energy:

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L \Pr} \frac{\partial^2 T^*}{\partial y^{*2}}$$

With the boundary conditions:

$$u^*(0, y^*) = 1,\ u^*(x^*, 0) = 0,\ u^*(x^*, \infty) = 1,\ v^*(x^*, 0) = 0$$

$$T^*(0, y^*) = 1,\ T^*(x^*, 0) = 0,\ T^*(x^*, \infty) = 1$$

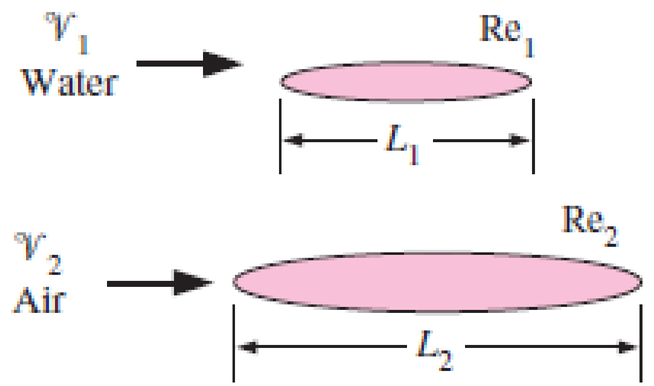
Geometrically Similar

Parameters before nondimensionalizing

$L, \mathcal{V}, T_\infty, T_s, \nu, \alpha$

Parameters after nondimensionalizing:

Re, Pr



If $\mathrm{Re}_1 = \mathrm{Re}_2$, then $C_{f1} = C_{f2}$

Two geometrically similar bodies have the same value of friction coefficient at the same Reynolds number.

Functional forms of Friction and Convection Coefficients

Momentum:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L} \frac{\partial^2 u^*}{\partial y^{*2}} - \frac{dp^*}{dx^*}$$

For a given geometry, the solution for u* can be expressed as

$$u^* = f_1\left(x^*, y^*, Re_L\right)$$

Then the shear stress at the surface becomes

$$\tau_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\mu V}{L} \left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} = \frac{\mu V}{L} f_2\left(x^*, Re_L\right)$$

$$C_{f,x} = \frac{\tau_s}{\rho V^2/2} = \frac{\mu V/L}{\rho V^2/2} f_2\left(x^*, Re_L\right) = \frac{2}{Re_L} f_2\left(x^*, Re_L\right) = f_3\left(x^*, Re_L\right)$$

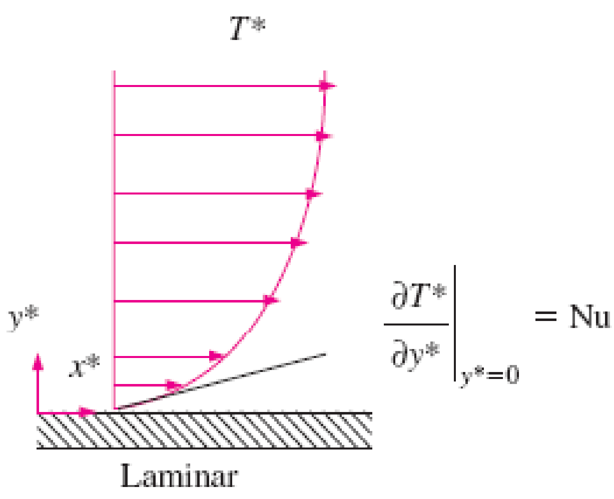
Energy: $$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L \, Pr} \frac{\partial^2 T^*}{\partial y^{*2}}$$

The solution for T* can be expressed as $$T^* = g_1\left(x^*, y^*, Re_L, Pr\right)$$

Using the definition of T*, the convection heat transfer coefficient becomes

$$h = \frac{-k\partial T/\partial y|_{y=0}}{T_s - T_\infty} = \frac{-k(T_\infty - T_s)}{L(T_s - T_\infty)} \partial T^*/\partial y^*\Big|_{y^*=0} = \frac{k}{L} \partial T^*/\partial y^*\Big|_{y^*=0}$$

Nusselt number: $$Nu_x = \frac{hL}{k} = \partial T^*/\partial y^*\Big|_{y^*=0} = g_2(x^*, Re_L, Pr)$$



Note that the Nusselt number is equivalent to the dimensionless temperature gradient at the surface, and thus it is properly referred to as the dimensionless heat transfer coefficient

Average friction coefficient $C_f = f_4(\mathrm{Re}_L)$

Local Nusselt number:

$$\mathrm{Nu}_x = \text{function}\ (x^*, \mathrm{Re}_L, \mathrm{Pr})$$

Average Nusselt number:

$$\mathrm{Nu} = \text{function}\ (\mathrm{Re}_L, \mathrm{Pr})$$

A common form of Nusselt number:

$$Nu = C\,\mathrm{Re}_L^m \mathrm{Pr}^n$$

Reynold Analogy

When Pr =1 (approximately the case for gases) and $\partial P^*/\partial x^* = 0$ (e.g. For flat plate)

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

$$u^*\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L}\frac{\partial^2 T^*}{\partial y^{*2}}$$

$$\Downarrow$$

$$\left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

$$Nu_x = \frac{hL}{k} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

$$\tau_s = \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\mu V}{L}\left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} = \frac{\mu V}{L} Nu_x$$

$$C_{f,x} = \frac{\tau_s}{\rho V^2/2} = \frac{\frac{\mu V}{L} Nu_x}{\rho V^2/2} = \frac{2}{Re_L} Nu_x$$

$$C_{f,x}\frac{Re_L}{2} = Nu_x \qquad \text{(Pr=1)}$$

$$\frac{C_{f,x}}{2} = St_x \qquad \text{(Pr=1)}$$

$$St = \frac{h}{\rho c_p V} = \frac{Nu}{Re_L\, Pr}$$

Clinton-Colburn Analogy

Also called modified Reynolds analogy

$$C_{f,x} = 0.664 Re_x^{-1/2} \qquad Nu_x = 0.332\, Pr^{1/3}\, Re_x^{1/2}$$

Taking the ratio between $C_{f,x}$ and Nu_x

Colburn j-factor

$$C_{f,x}\frac{Re_x}{2} = Nu_x\, Pr^{-1/3} \qquad \frac{C_{f,x}}{2} = \frac{h_x}{\rho c_p V} Pr^{2/3} \equiv j_H$$

Valid for

0.6<Pr<60

Although this relation is developed using relations for laminar flow over a flat plate (for which $\partial P^*/\partial x^* = 0$), experimental studies show that it is also applicable approximately for turbulent flow over a surface, even in the presence of pressure gradients.
For laminar flow, however, the analogy is not applicable unless $\partial P^*/\partial x^* = 0$. Therefore, it does not apply to laminar flow in a pipe