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Math 260

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Math 260 Final

For this project, I have created a population model where growth is present. This is based off the equation u' = λu(1 − u). In this case, two populations that compete with each other are analyzed, u and v. These population competition models, derived from the population dynamics equations given, come out to be:

$$u_t = u(1 - u) - \alpha uv \qquad \textit{Eq. 1}$$

$$v_t = \lambda v(1 - b) - \beta uv \qquad \textit{Eq. 2}$$

For these models, λ represents $\frac{\lambda_1}{\lambda_2}$, the ratio of growth rates. α represents bK_2 and β represents bK_1, bK being the rate of population loss of each species per interaction with eh other species per time multiplied by the carry capacity. Using these equations, I implemented a Runge-Kutta (RK4) model to handle these populations over time. I was tasked with doing this for 4 different scenarios of α, β, and λ and 25 different combinations of starting points for each population. The critical points for the population (u,v) will be (0, 0), (0, 1), (1, 0), ($\bar{u}$, $\bar{v}$), where $\bar{u}$ = λ(α − 1)/(αβ – λ) and $\bar{v}$ = (β – λ)/(αβ − λ).

To do this analysis, I wrote code in which I first created an RK4 function to deal with the population competition equations. This function takes in 7 arguments: α, β, λ, u0, v0, T, and N. α, β, and λ are the set scenario variables, u0 and v0 are the different initial conditions, T is the final time to model until, and N is the number of timesteps. For each case, I set T to 100 years (assuming years are the time units) and N to 10000. T = 100 was chosen to give each population sufficient time to reach its steady state, and N = 10000 was chosen to acquire optimal data for lots of points, but not too many that the code is slow. In the function, I set and time vector for time 0 to 100 with 10000 steps and set Δt = 100/10000. I then created u and v vectors the length of the time vector with all 0 entries, except the first entries are set to the initial conditions. Next, I iterated through the length of these vectors and carried out the RK4 operations. The equations for u are shown below, where 'fu' is the function for equation 1 that was coded into its own function for simplicity:

$$ku_1 = \Delta t * fu(u_n, v_n) \qquad \textit{Eq. 3}$$

$$ku_2 = \Delta t * fu(u_n + \tfrac{ku_1}{2}, v + \tfrac{ku_1}{2}) \qquad \textit{Eq. 4}$$

$$ku_3 = \Delta t * fu(u_n + \tfrac{ku_2}{2}, v + \tfrac{ku_2}{2}) \qquad \textit{Eq. 5}$$

$$ku_4 = \Delta t * fu(u_n + ku_3, v + ku_3) \qquad \textit{Eq. 6}$$

Then:

$$u_{n+1} = x_n + \tfrac{k_1}{6} + \tfrac{k_2}{3} + \tfrac{k_3}{3} + \tfrac{k_4}{6} \qquad \textit{Eq. 7}$$

Staring at the initial condition, each subsequent term for u is found and updated in the previously 0 u vector. The same process is done but for v and using 'fv'. The function returns the u, v, and t vectors.

Next, I plotted a phase portrait of v vs u over time for all 4 scenarios. In each case, I iterated through each of the 25 possible combinations of u0 and v0 initial conditions, as instructed. I then plugged in every iteration with the scenario's specific variables into my RK4 function to return v, u, and t vectors. I plotted all the v's vs u's on the same phase portrait graphs and formatted. I then had 1 graph for each of the 4 scenarios, totaling in 4 graphs, which are shown later. Additionally, I added a marker for the ending point of each population to see where each different starting condition finished. I also plotted points at each critical point for reference, however, only plotting $(\bar{u}, \bar{v})$ if it was applicable, meaning if it was withing the grid of points that I plotted. Next, I chose one starting condition from each plot that looked interesting to me and plotted u and v vs time on the same graph to see how the individual populations increased and/or decreased over time. For these cases, however, I changed the total time looked at, T, to 30 years because, from trial and error, which gave a more zoomed in depiction of the populations while still allowing them to reach their steady states. For further analysis of each scenario, I calculated the eigenvalues and eigenvectors of the Jacobian matrices and sketched them onto the population graph to visualize how these axes pupped of pushed the population lines, which is also shown below. The Jacobian matrixes are found for each critical point using the given equations:

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \qquad \textit{Eq. 8}$$

$$J(1,0) = \begin{pmatrix} -1 & -\alpha \\ 0 & \lambda - \beta \end{pmatrix} \qquad \textit{Eq. 9}$$

$$J(0,1) = \begin{pmatrix} 1-\alpha & 0 \\ -\beta & -\lambda \end{pmatrix} \qquad \textit{Eq. 10}$$

$$J(\bar{u}, \bar{v}) = \begin{pmatrix} \frac{\lambda(1-\alpha)}{\alpha\beta-\lambda} & \frac{\alpha\lambda(1-\alpha)}{\alpha\beta-\lambda} \\ \frac{\beta(\lambda-\beta)}{\alpha\beta-\lambda} & \frac{\lambda(\lambda-\beta)}{\alpha\beta-\lambda} \end{pmatrix} \qquad \textit{Eq. 11}$$

The variables for each scenario at each critical point were plugged into the appropriate Jacobian and the eigenvalues and eigenvectors were found using an online calculator. Further analysis and possible coding for each scenario were done on a case-to-case basis.

For scenario A, the variables given were: α = 0.1, β = 2.1, and λ = 2. Upon running the code for these values, the phase portrait came out to be:

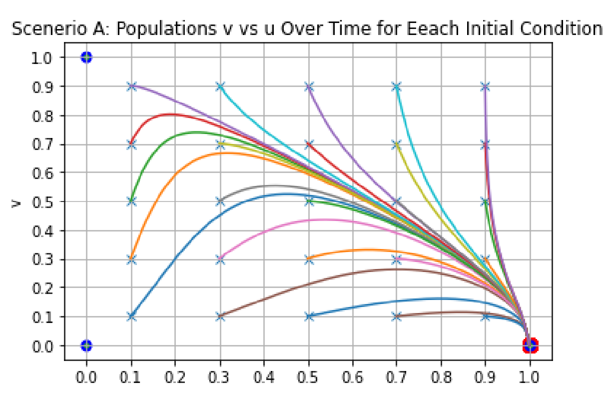

Figure 1: Phase Portrait for Scenario A

It can be seen off the bat these variables lead to a dominant u state, as (1,0) is a stable critical point. All the conditions end at that point, shown in red, with the possible critical points all shown in blue. In this case, $(\bar{u}, \bar{v})$ = (1.006, -.05), so it is off this grid and is not considered. Upon calculation, the (1,0) stable critical point has eigenvalues λ_1 = -1 and λ_2 = -.1, corresponding to eigenvectors of v_1 = (1,0) and v_2 = (-1/9,1). The (0,1) unstable critical point has eigenvalues λ_1 = .9 and λ_2 = -2, corresponding to eigenvectors of v_1 = (-1.38,1) and v_2 = (0,1). The (0,0) unstable critical point has eigenvalues λ_1 = 1 and λ_2 = 2, corresponding to eigenvectors of v_1 = (1,0) and v_2 = (0,1). A critical point is stable if both eigenvalues are negative, meaning that point draws the population towards it in all directions. Otherwise, the point is unstable. The (0,0) point has the same eigenvalues and eigenvectors for each scenario because it is only dependent on λ, which equals 2 for all, and thus it will not be calculated again. Sketching these eigenvalues and eigenvectors on the graph results in:

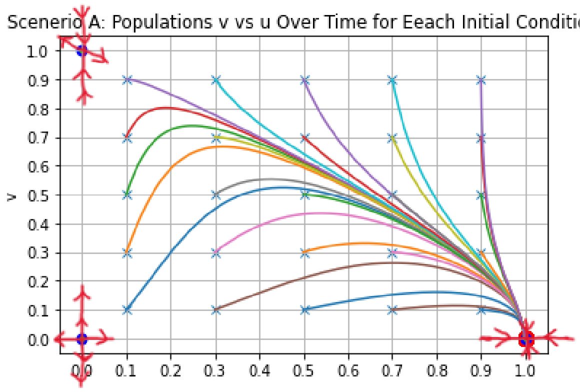

Figure 2: Phase Portrait for Scenario A with Eigen Analysis

Note, the length of the grid lines represents how strong their forces are based off the magnitude of the eigenvalues and the direction of the arrows are from the sign of the eigenvalues, negative meaning pulling in. The axes directions come from the eigenvectors at each point. This analysis gives a better picture of what is going on. It is seen that (1,0) is the only one pulling from all sides, so all populations end at that point. A point starting at (.1,.3), for example, will start moving up due to the (0,0) and (0,1) points' push and pull up the y-axis, but eventually it falls down the the right along the same paths as the other populations, which are due to the diagonal directions of the vectors seen at (0,1) and (1,0).

An initial condition that I decided to further analyze was that starting point previously mentioned, (.1,.3). From the phase portrait, it is seen that u is growing the whole time. Population v, on the other hand, initially increased at a slightly higher rate than u, but then started to decline and eventually died off, leaving just u. These population growths/decays over time are shown below:

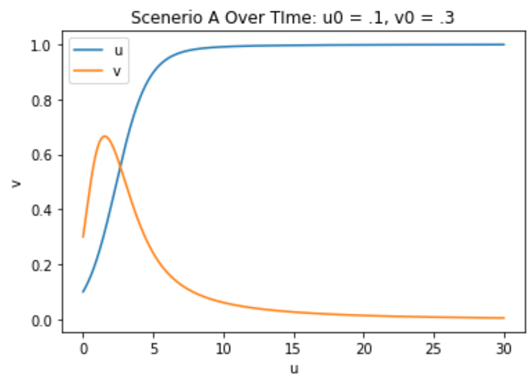

Figure 3: Populations u and v Over Time for Starting Conditions (.1,.3)

This graph exactly depicts the observations previously made. Additionally, it adds some insight to the actual time values associated. It is seen that v begins to decrease only about a couple years after its initial increase, and then becomes miniscule around year 20 while u dominates. The behaviors that we see in the graphs agree with what would be expected in a real population based off the models. Looking at the variables α = 0.1 and β = 2.1 in regards to the initial equations 1 and 2, this means that v will face losses to competition much greater than u because β is so much larger than α. The λ variable will play a factor in that it increases the growth side of the v equation, but as seen, is not enough for v to maintain an increase over time. For this specific case of (.1,.3), the populations are small to start, so the losses of v are too small compared to the growth first half of the equation, so the population grows. But, as u and v become larger, the loss for v becomes larger and eventually it declines and dies off. This is dependent on initial conditions, however, as it can be seen in figure 1 that populations of v that start off high decrease the whole time.

For scenario B, the variables given were: α = 1.1, β = 1, and λ = 2. Upon running the code for these values, the phase portrait came out to be:

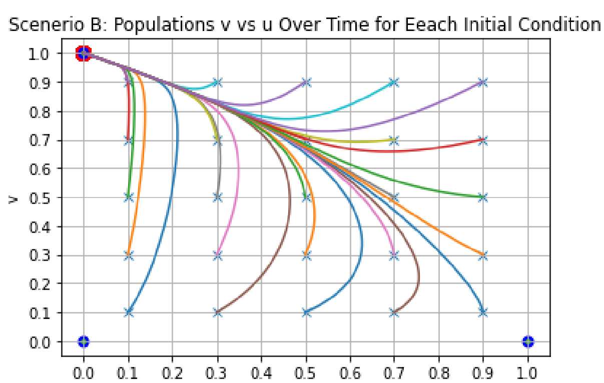

Figure 4: Phase Portrait for Scenario B

These variables, on the contrary to scenario A, lead to a dominant v state, as (0,1) is a stable critical point, where all the starting conditions end at that point. In this case, $(\bar{u}, \bar{v})$ = (-2/9, 10/9), so it is not considered. Upon calculation, the (0,1) stable critical point has eigenvalues λ_1 = -.1 and λ_2 = -2, corresponding to eigenvectors of v_1 = (-1.9,1) and v_2 = (0,1). The (0,1) unstable critical point has eigenvalues λ_1 = -1 and λ_2 = 1, corresponding to eigenvectors of v_1 = (1,0) and v_2 = (-.55,1). Sketching these eigenvalues and eigenvectors on the graph results in:

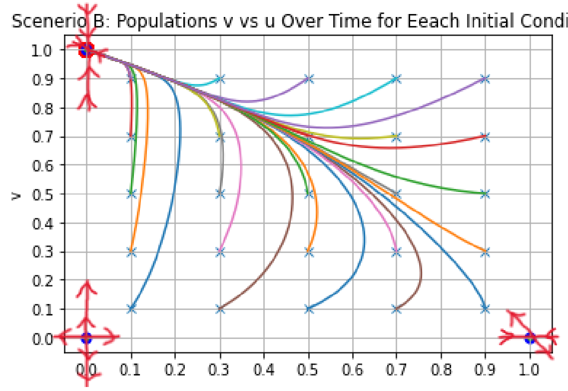

Figure 5: Phase Portrait for Scenario B with Eigen Analysis

Again, the axes allow for a better depiction of why each population moves the way it does. The point (0,1) is the only one pulling from all sides, so all populations end at that point. A point starting at (.5,.1), for example, will start moving both to the right and up due the strong upwards forces from ((0,0) and (0,1) and the rightward forces from (0,0) and (1,0). The upward forces on the left side are the strongest, so any points towards the left are most affected my them. Eventually, the populations are more affected by the upward left moving forces from (1,0) and (0,1) and all move in line with the diagonal axis at point (0,1) until they reach that stable critical point.

Next, I further analyzed the starting condition (.5,.1) which was previously discussed. Looking at the phase portrait for this starting point, v grows the whole time. Population u, however, initially increased at about the same rate as u, but at a certain point, started to decline and eventually died off, leaving just population v to dominate. These population growths/decays over time are shown below:

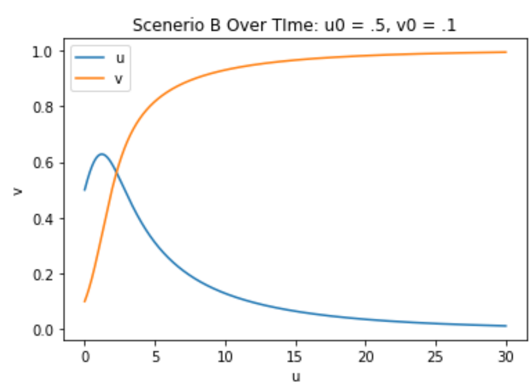

Figure 6: Populations u and v Over Time for Starting Conditions (.5,.1)

This graph confirms that u begins to decrease only a couple years after its initial increase, and then becomes miniscule towards year 30 as v dominates. The behaviors that we see in the graphs again agree with what would be expected in a real population based off the models. Looking at the variables α = 1.1 and β = 1 in regards to the initial equations 1 and 2, this means that u and v will face similar losses to competition because α and β are of similar magnitude. This difference, however, is that λ = 2 is multiplied to the growth of population v, so the grown side of v's equation will be greater than the decay side. For this specific case of (.5,.1), u also grows at first because both populations are small, so the decay rate is small. But, as the populations quickly increase, the losses for u outweigh the gains and it quickly dies off. This is, again, dependent on initial conditions, as in figure 4 that populations that start off large lead to u decreasing right away because its loss starts off greater than its gain.

For scenario C, the variables given were: α = 2, β = 5, and λ = 2. Upon running the code for these values, the phase portrait came out to be:

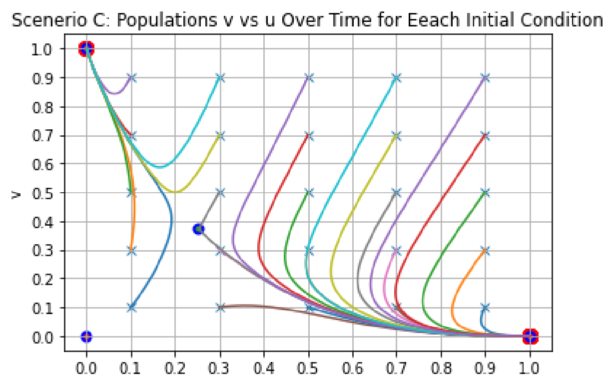

Figure 7: Phase Portrait for Scenario C

This is a unique scenario where the variables can lead to a dominant u state or a dominant v, as both points (1,0) and (0,1) are stable. It is the starting conditions that determine which state the population will go toward. In this case, $(\bar{u}, \bar{v})$ = (.25, .375), so it does play a factor and is shown on the plot. This point is seen to be unstable and is what leads to this split of 2 scenarios. It is difficult to get a good representation of what is happening around the $(\bar{u}, \bar{v})$ critical point from figure 7, so I created a new phase portrait with an additional, tighter grid of starting points around that critical point, shown below:

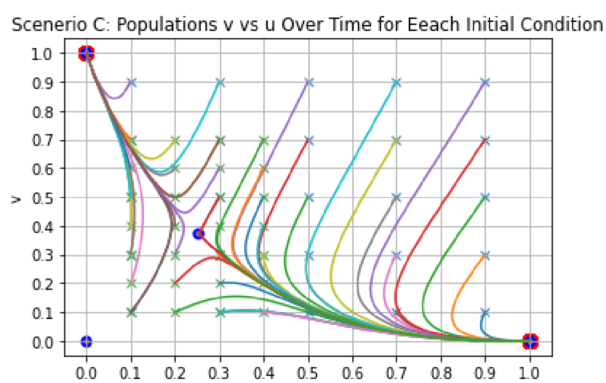

Figure 8: Phase Portrait for Scenario C with Additional Initial Conditions

From this, is appears that any starting points to the right of a diagonal line from (.25,.375) result in the populations ending up u dominant, and any to the left result in it ending up v dominant. It can be reasonable assumed that this is due to the repelling axis around that critical point, which will be seen later. Upon calculation, the (.25,.375) unstable critical point has eigenvalues λ_1 = .5 and λ_2 = -1.5, corresponding to eigenvectors of $v_1 = (-2/3,1)$ and $v_2 = (.4,1)$. The (1,0) stable critical point has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$, corresponding to eigenvectors of $v_1 = (1,0)$ and $v_2 = (1,1)$. The (0,1) unstable critical point has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$, corresponding to eigenvectors of $v_1 = (-.2,1)$ and $v_2 = (0,1)$. Sketching these eigenvalues and eigenvectors on the graph results in:

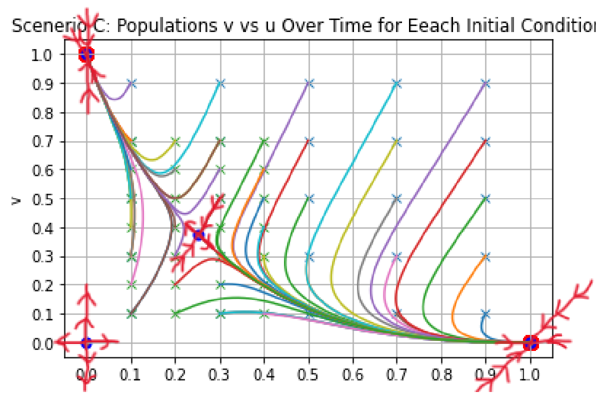

Figure 9: Phase Portrait for Scenario C with Eigen Analysis

This sketches give a great depiction of how the critical points effect u and v. The lines can be seen to all follow axes of the critical points. The critical points (1,0) and (0,1) pull from all sides and so populations finish at one of the two, depending on if they are pushed left or right by the middle critical point. Starting points are pulled to the middle first, but then pushed out to the corners, and all follow one of two finishing trends, as the populations fall on the same line general line based on the ending critical points axes. A point starting at (.9,.7), for example, will start moving down and left as pulled by both (.25,.375) and (1,0), but then goes toward its end condition, pulled/pushed by those same critical points.

Next, I further analyzed the starting condition (.9,.7). Looking at the phase portrait for this starting point, v and u both start with decay, but eventually u starts to grow again until it is dominant, while v decays the whole time. These population growths/decays over time are shown below:

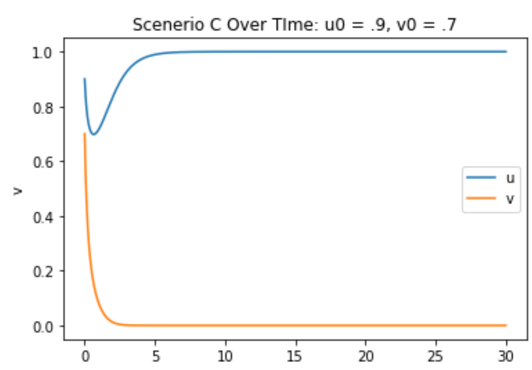

Figure 10: Populations u and v Over Time for Starting Conditions (.9,.7)

It is now better seen that both populations start with initial decrease, but after only a couple years u quickly begins to increase and becomes completely dominant after only about 5 years as v dies out. These behaviors agree with the models used. The variables $\alpha = 2$ and $\beta = 2$, which correspond to the decays of the populations, are both relatively large, and so decay in both populations can happen at first in cases where the populations start large. However, as the populations get smaller, the decay portions get smaller, and one population will begin to dominate. On the other hand, in cases where the populations start small, both will start growing, but then one will begin to dominate as the other dies out. The populations in this scenario have very different tendencies depending on their starting conditions due to the critical point in the middle of the plot.

For scenario D, the variables given were: α = .5, β = 1, and λ = 2. Upon running the code for these values, the phase portrait came out to be:

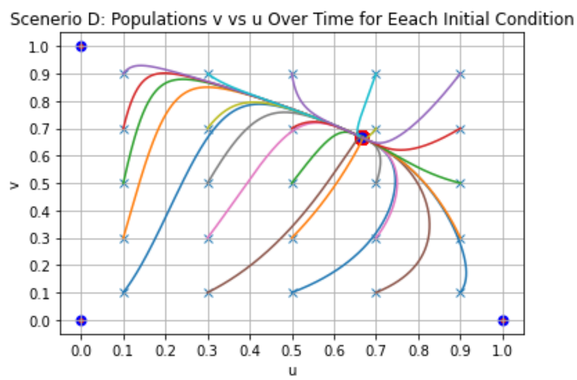

Figure 11: Phase Portrait for Scenario D

These variables, unlike any of the cases previously scene, lead to a coexistence state, where all populations end at the critical point $(\bar{u}, \bar{v})$ = (2/3, 2/3). Upon calculation, this (2/3,2/3) stable critical point has eigenvalues λ_1 = -1.58 and λ_2 = -.42, corresponding to eigenvectors of v_1 = (-.37,1) and v_2 = (-1.37,1). The (1,0) stable critical point has eigenvalues λ_1 = -1 and λ_2 = 1, corresponding to eigenvectors of v_1 = (1,0) and v_2 = (-.25,1). The (0,1) unstable critical point has eigenvalues λ_1 = -.5 and λ_2 = -2, corresponding to eigenvectors of v_1 = (-2.5,1) and v_2 = (0,1). Sketching these eigenvalues and eigenvectors on the graph results in:

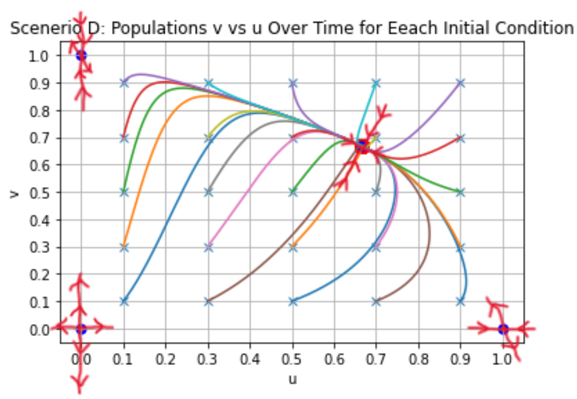

Figure 12: Phase Portrait for Scenario D with Eigen Analysis

It is now seen that the (2/3,2/3) critical point is the biggest factor in determining the trajectories of the populations. From their starting points, populations first move mostly along that critical point's stronger axis and then are pulled along its weaker axis, which is where they all appear to be moving along the same line. The points are still affected by the other critical points, which is better seen when looking at points staring at the top right or top left, such as (.7,.1). This point is mostly affected by the (1,0) critical point at first, but eventually gets closer to (2/3,2/3) and falls in line with the other trajectories.

Next, I further analyzed that (.7,.1) starting condition. Based off the phase portrait, population u initially grows, but then stops growing and declines until it reached 2/3, while v increased toward 2/3 the whole time. These population growths/decays over time are shown below:

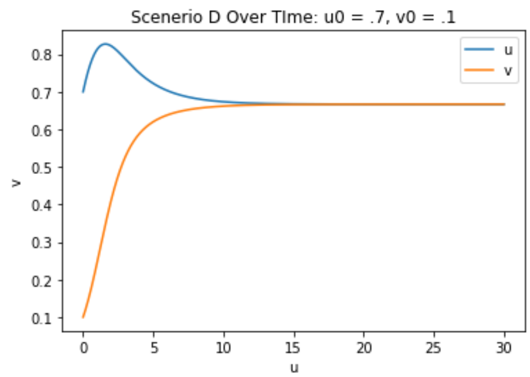

Figure 13: Populations u and v Over Time for Starting Conditions (.7,.1)

This graph confirms that u begins to fall after a couple years of rising and v has a rapid increase that slows right before the 5 year mark. Both populations reach the steady 2/3 state between 10 and 15 years. Because the populations become steady at 2/3, they will either net grow and net decrease to get to that point, however as seen, may include both increases and decreases. The staring conditions will determine which side of the population equation, the growth side or the decay side, starts out larger, but eventually the populations become steady. The variables, α, β, and λ, are what lead to this steady co-existence of populations because of how they balance each other out in the population equations.

From my model, I have been able to analyze two populations under many different scenarios and se and how they behave over time in competition with each other. The phase portraits are a great tool of analysis to compare many different starting conditions, and the populations vs time can provide even further insight. If desired, using this model as a template, even greater analysis could be conducted. This could be done for populations with different competition equations than we saw, as well as for mutualistic or commensalism or other types of relationships. Additionally, this could be done for more than two populations at one to possibly analyze a whole ecosystem, although that would get complicated. Another graph that could be incorporated is one that shows all the starting conditions' population vs time to analyze which starting conditions reach the steady states fastest, because that analysis was not done in my model. Overall, my model showed me a lot about the tendencies of populations in competition and allowed for analysis that would otherwise not be possible.