"DO YOU KNOW," THE DEVIL CONFIDED, "NOT EVEN THE BEST MATHEMATICIANS ON OTHER PLANETS - ALL FAR AHEAD OF YOURS - HAVE SOLVED IT? WHY, THERE IS A CHAP ON SATURN - HE LOOKS SOMETHING LIKE A MUSHROOM ON STILTS - WHO SOLVES PARTIAL DIFFERENTIAL EQUATIONS MENTALLY; AND EVEN HE'S GIVEN UP."
ARTHUR PORGES, "THE DEVIL AND SIMON FLAGG"

IN ORDER TO SOLVE THIS DIFFERENTIAL EQUATION YOU LOOK AT IT TILL A SOLUTION OCCURS TO YOU.
GEORGE PÓLYA

JARED C. BRONSKI

DIFFERENTIAL EQUATIONS

UNIVERSITY OF ILLINOIS MATHEMATICS

PUBLISHED BY UNIVERSITY OF ILLINOIS MATHEMATICS

TUFTE-LATEX.GOOGLECODE.COM

First printing, July 2021

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Dedicated to Somebody

Introduction

This is a collection of lecture notes for an introductory undergraduate differential equations class based on a course taught by the author at the University of Illinois. The focus is primarily on linear equations with some consideration of more qualitative results for nonlinear equations. The text is divided into three parts. The first part covers scalar equations, focusing on initial value problems but with some consideration of boundary value problems near the end. The second part considers systems of linear differential equations, mainly constant coefficients. The third part considers eigenvalue problems, partial differential equations and separation of variables.

Part I

Ordinary Differential Equations

1

First Order Differential Equations

1.1 What is a Differential Equation?

1.1.1 Basic terminology

An ordinary differential equation is any relationship between a function (usually denoted $y(t)$) and its derivatives up to some order:

$$F(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \ldots, \frac{d^ky}{dt^k}) = 0$$

Example 1.1.1. *The following are all ordinary differential equations.*

$$\frac{dy}{dt} = -ky \tag{1.1}$$

$$\frac{d^2y}{dt^2} = -y \tag{1.2}$$

$$\frac{d^5y}{dt^5} = -y^3 \tag{1.3}$$

$$\frac{d^8y}{dt^8} + y\frac{dy}{dt} = y \tag{1.4}$$

$$y^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{d^{11}y}{dt^{11}}\right)^2 \tag{1.5}$$

Some basic terminology:

The order of a differential equation is the order of the highest derivative which appears in the equation. The **order** of the equations above are $1, 2, 5, 8$ and 11 respectively. It is important to be able to determine the order of a differential equation since this determines the way in which the differential equation is posed. In particular it determines how many pieces of initial data we must provide in order to have a unique solution.

Note that the term **order** is used in many different ways in mathematics. In the context of differential equations order refers exclusively to the order of the derivative. Don't confuse the order with the highest power that appears, or any other usage of order.

In addition to the differential equation itself we typically need to specify a certain number of initial conditions in order to find a unique solution. The number of initial conditions that we need to specify is generally equal to the order of the differential equation. For instance for a second order differential equation we would typically specify two pieces of initial data. Usually this would be the value of the function and the value of the first derivative at the initial point t_0. For instance an example might be

$$\frac{d^2y}{dt^2} = y^3 - y \qquad y(0) = 1 \quad \frac{dy}{dt}(0) = -1.$$

The equation is second order, and we specify two pieces of initial data: the value of the function at the point $t = 0$ and the value of the derivative at the (same) point $t = 0$. This is called an **initial value problem**.

In many applications we specify the values of the function and the first $k-1$ derivatives at the point t_0, so this will be assumed throughout the first part of the text. In a few important applications, however, it is necessary to specify function values at more than one point. For instance one might specify the function value(s) at $t = 0$ as well as $t = 1$. This is called a boundary value problem and will be discussed in a subsequent section of the notes.

Exercise 1.1. *For the initial value problem*

$$\frac{d^2y}{dt^2} = y^3 - y \qquad y(0) = 1 \quad \frac{dy}{dt}(0) = -1.$$

compute the value of $\frac{d^2y}{dt^2}(0)$ *as well as* $\frac{d^3y}{dt^3}(0)$. *Notice that this can be continued indefinitely: given the value of* $y(0)$ *and* $\frac{dy}{dt}(0)$ *we can compute the value of any other derivative at* $t = 0$. *This explains why, for a second order equation, two pieces of initial data are the right amount of data – once we have specified the first two derivatives we can in principle calculate all of the higher order derivatives.*

An equation is linear if the dependent variable and all its derivatives enter into the equation *linearly*. The dependence on the independent variable can be arbitrary. Of the five equations in Example (1.1.1) above the first two are linear, since y, y', y'' all enter linearly. The third equation is nonlinear because there is a y^3 term (which is nonlinear). The fourth equation is nonlinear because of the $y\frac{dy}{dt}$ term.

A linear equation can be written in the form

$$a_n(t)\frac{d^ny}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots a_0(t)y = f(t).$$

In the special case where the righthand side $f(t)$ is zero

$$a_n(t)\frac{d^ny}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots a_0(t)y = 0.$$

the equation is said to be homogeneous. A linear equation where the righthand side is non-zero is said to be non-homogeneous. We will see in later chapters that, if we are able to solve the homogeneous equation, then we will be able to solve the non-homogeneous equation.

A great deal of time in this class will be spend learning to solve linear differential equations. Solving nonlinear differential equations is in general quite difficult, although certain special kinds can be solved exactly.

1.1.2 *Examples of differential equations*

DIFFERENTIAL EQUATIONS ARISE in many physical contexts when the rate of change of some quantity can be expressed in terms of the quantity itself. This usually takes the form of some physical law.

Here i is still measured in years, and r is still the yearly rate, so the above represents the interest gained over one half-year period. If the interest is compounded n times a year then P_i satisifes

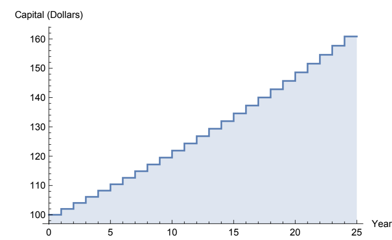


Figure 1.1: An initial investment of 100$ at 2% ($r = .02$) interest compounded yearly.

Example 1.1.2 (Compound Interest). *If interest on an account is compounded yearly at a rate r, and P_i denotes the amount of money in the account after i years then (assuming that no other money is added to or removed from the account in this period) P_i satisfies*

$$P_{i+1} = (1+r)P_i.$$

which has the solution

$$P_i = (1+r)^i P_0.$$

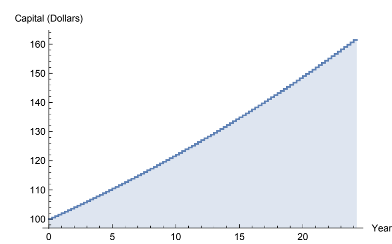


Figure 1.2: An initial investment of 100$ at 2% ($r = .02$) interest compounded quarterly.

If, instead of compounding yearly the interest is compounded twice a year then the amount of money P_i satisfies

$$P_{i+1/2} = (1+\frac{r}{2})P_i$$
$$P_{i+1} = (1+\frac{r}{2})P_{i+1/2} = (1+\frac{r}{2})^2 P_i$$

which has the solution

$$P_i = (1+r/2)^{2i} P_0$$

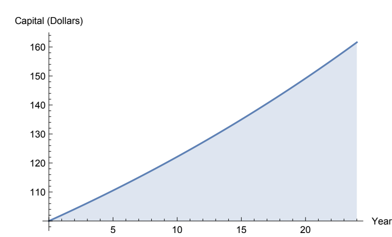


Figure 1.3: An initial investment of 100$ at 2% ($r = .02$) interest compounded continuously.

Notice that compounding twice annually gives you a bit more in interest than compounding once a year, since $(1+r/2)^2 = 1+r+r^2/4 > (1+r)$. In other words you are getting interest on the interest. If the interest is compounded n times per year we have

$$P_{i+\frac{1}{n}} = (1+\frac{r}{n})P_i \qquad P_i = (1+\frac{r}{n})^{ni} P_0$$

The continuum limit consists of letting n tend to infinity, and assuming that P_i goes over to a continuous function $P(t)$. Using the finite difference *approximation to the derivative,* $\frac{P_{i+\Delta}-P_i}{\Delta} \approx \frac{dP}{dt}$ *we find the differential equation*

$$\frac{dP}{dt} = rP.$$

It is not hard to see that this differential equation has the solution

$$P(t) = P_0 e^{rt}.$$

The graphs in the side margin on the previous depict the growth of an initial investment of 100$ *earning* 2% *interest per year over a twenty-four year period when in the cases where the interest is compounded yearly, quarterly (*$4\times$ *per year), and continuously. You can see that the graphs look very similar.*

Similar models governs radioactive decay except that in the case of radioactive decay the constant r *is negative, since one* loses *a fixed fraction of the population at each step.*

Newton's law of cooling is an approximation where we neglect the distribution of temperature inside the body, and assume that it is all at the same temperature. If we are interested in the temperature at different points inside the body we would have to solve the heat equation, a partial differential equation. We will do this in a later chapter.

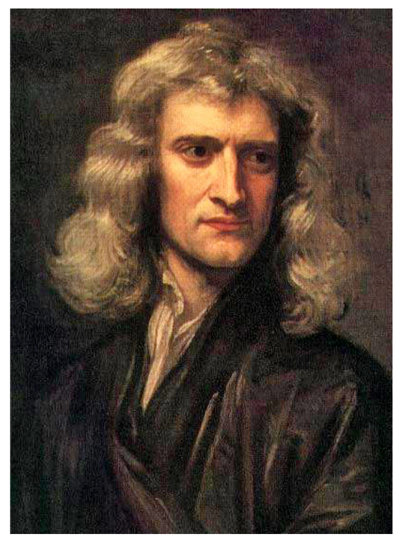
Example 1.1.3 (Newton's Law of Cooling)**.** *Newton's law of cooling states that the rate of change of the temperature of a body is proportional to the difference in temperature between the body and the surrounding medium.What is the differential equation which governs the temperature of the body?*

Let T *denote the temperature of the body. The rate of change of the temperature is obviously* $\frac{dT}{dt}$*. If the temperature of the surrounding medium is denoted by* T_0 *then we have the equation*

$$\frac{dT}{dt} = -k(T - T_0)$$

where $-k$ *is the constant of proportionality. We put the* $-$ *sign in there because it is clear that the temperature of the body should converge to that of the medium: If* $T > T_0$ *then the body should be getting cooler (the rate of change should be negative*

Another example comes from classical physics:



It would be impossible to discuss differential equations without discussing Isaac Newton, probably the greatest mathematician/physicist in history. One of Newton's first applications of calculus was to the problem of planetary motion, where he showed that Kepler's empirical laws of planetary motion followed from the assumption that the planets move under the influence of gravity. Illustration by After Godfrey Kneller - http://www.newton.cam.ac.uk/art/portrait.html

Example 1.1.4 (Newton's Law of Motion)**.** *Let* $x(t)$ *denote the position of a particle. Newton's second law of motion states that* $\mathbf{f} = m\mathbf{a} = m\frac{d^2\mathbf{x}}{dt^2}$*. If we assume that the force is conservative, meaning that the work done on a particle is independent of the path, then it follows that force is given by minus the derivative of the potential energy* $V(\mathbf{x}(t))$*.*

In this case we have

$$m\frac{d^2\mathbf{x}}{dt^2} = F = -\nabla_{\mathbf{x}} V(\mathbf{x}(t)).$$

This is, of course, a differential equation for the position $\mathbf{x}(t)$*. It is second order and is generally nonlinear.*

Example 1.1.5 (Growth of a Raindrop). *A simple model for the growth of raindrops is this: the rate of change of the* **volume** *of the raindrop is proportional to the surface area of the drop. The basic intuition is this: the drop changes its volume by (1) evaporation and (2) by merging with other droplets. The rate at which both of these processes happen should be proportional to the surface area.*

The relevant equations are

$$V = \frac{4\pi}{3} r(t)^3$$
$$\frac{dV}{dt} = \underbrace{4\pi r^2 \frac{dr}{dt}}_{\text{Rate of change of volume}} = k \underbrace{4\pi r^2}_{\text{Surface Area}}$$
$$\frac{dr}{dt} = k$$

so the rate of change of the radius of the droplet is constant, so the radius is expected to grow linearly in time.

On the importance of verification: While it can be very difficult to solve a general ordinary differential equation it is usually pretty easy to check whether or not a given function $y = f(t)$ solves the equation, since one can simply compute all derivatives and check whether or not the equation is satisfied.

Example 1.1.6. *Verify that the function*

$$y(t) = A\cos(t) + B\sin(t)$$

satisfies the equation

$$\frac{d^2y}{dt^2} = -y$$

Since this is a second order equation we should be able to specify two initial conditions, the value of the function and the value of the derivative. Thus it should be no surprise that this solution involves two arbitrary constants A, B.

Example 1.1.7. *Check that the function*

$$y(t) = \frac{1}{\cos(t - t_0)}$$

satisfies the equation

$$\frac{d^2y}{dt^2} = 2y^3 - y$$

This function only involves one undetermined constant, t_0. Since this equation is second order we expect that the general solution should involve two constants. The most general solution to this equation is something called an **elliptic function**. These are special functions originally studied by Gauss, and are not generally expressible in terms of the elementary transcendental functions such as $\sin t, \cos t, e^t$, although for certain special values they reduce to elementary functions.

Practice Exercises:

1. Write down a differential equation which is different from any of the examples from this chapter. Give the order of the equation, and state whether it is linear or nonlinear.

2. Check that the function $y(t) = 1 - t^2$ satisfies the differential equation $y - ty' = 1 + t^2$

3. Suppose that $y(t)$ satisfies $y'' = y$ together with the initial conditions $y(0) = 1, y'(0) = 0$. What is $\frac{d^k y}{dt^k}(0)$ as a function of k?

1.2 *Basic solution techniques.*

There are essentially only two methods for solving ordinary differential equations (ODEs). The first is to recognize the equation as representing the exact derivative of something, and integrate up using the fundamental theorem of calculus. The second is to guess. This text will cover a number of techniques for solving ordinary differential equations, but they all reduce to one of these two basic methods.

This section will mainly focus on the first technique: recognizing an equation as representing a exact derivative. There are a number of types of equation that often arise in practice where this is possible.

The simplest example is when one has some derivative of y equal to a function of x. This occasionally arises in applications. For such equations we can simply integrate up a number of times.

Example 1.2.1. *Suppose that the height $y(t)$ of a falling body evolves according to*

$$\frac{d^2y}{dt^2} = -g$$

Find the height as a function of time.

Solution 1.1. *Integrating up once we find that*

$$\frac{dy}{dt} = -gt + v$$

where v is a constant of integration (representing the velocity of the body at time $t = 0$: $v = y'(0)$. Integrating up a second time gives the equation

$$y = -\frac{gt^2}{2} + vt + h$$

where $h = y(0)$ is a second constant of integration representing the initial height of the body.

We see again a lesson from the previous section: we have a second order differential equation the solution of which involves two arbitrary constants of integration, v and h. The general solution of an equation of n^{th} order will *typically* involve n arbitrary constants. Here the constants enter linearly, since the equation is linear. For a nonlinear equation the dependence on the constants could be much more complicated.

In the previous example the equation was given in the form of an exact derivative. While this is not always the case one can frequently manipulate the equation so that it takes the form of an exact derivative.

Example 1.2.2. *Solve the equation*

$$\frac{dT}{dt} = -k(T - T_0)$$

arising in Newton's law of cooling.

Solution 1.2. *If we divide the equation through by $T - T_0$ we find*

$$\frac{\frac{dT}{dt}}{T - T_0} = -k$$

The left-hand side is the derivative of $\ln |T - T_0|$*. Integrating up gives*

$$\ln |T - T_0| = -kt + A$$

where A is a constant of integration. Exponentiating give

$$\begin{aligned} |T - T_0| &= e^{-kt+A} = e^A e^{-kt} \\ T - T_0 &= \pm e^A e^{-kt} \\ T &= T_0 + Ce^{-kt}. \end{aligned}$$

The form of this solution should not be surprising. As $t \to \infty$ the exponential term decays away and we have $T(t) \to T_0$. This implies that, at long times, the temperature of the body tends to the equilibrium temperature.

In the last step we have a prefactor $\pm e^A$, where the $\pm$ comes from eliminating the absolute value. This is, in the end, just a constant so we call $C = \pm e^A$.

Here is a similar example that involves both the independent variable x and the dependent variable y:

Example 1.2.3. *Solve the equation*

$$\frac{dy}{dt} = ty \qquad y(0) = 1$$

Solution 1.3. *We can rewrite this as*

$$\frac{dy}{y} = tdt.$$

Once we have done this we are in a situation where we have only y on the left-hand side of the equation and only t on the right. Since the equation has separated" *in this way we can integrate it. Integrating gives*

$$\int \frac{dy}{y} = \int tdt \tag{1.6}$$

$$\ln(y(t)) = \frac{t^2}{2} + C \tag{1.7}$$

Imposing the initial condition $y(0) = 1$ gives $C + 0 = \ln 1 = 0$, *therefore*

$$\begin{aligned} \ln |y(t)| &= \frac{t^2}{2} \\ y(t) &== e^{\frac{t^2}{2}}. \end{aligned}$$

The most general first order equation which can be solved this way is called a separable equation, which we define as follows:

Definition 1.2.1. *A first order equation is said to be separable if it can be written in the form*

$$\frac{dy}{dt} = f(y)g(t)$$

If $f(y) \neq 0$ then this equation can be integrated in the following manner.

$$\int \frac{dy}{f(y)} = \int g(t)dt + C$$

This procedure gives y implicitly as a function of t via the implicit function theorem. It may or may not be possible to actually solve for y explicitly as a function of t, as we were able to do in the preceeding two examples.

Example 1.2.4. *Solve the equation*

$$\frac{dy}{dt} = (y^2 + y)t \qquad y(2) = 1$$

Solution 1.4. *We can solve this by writing it in the form*

$$\frac{dy}{y^2 + y} = tdt$$

and integrating up to get

$$\int_{y(2)}^{y(t)} \frac{dy}{y^2 + y} = \int_2^t tdt \tag{1.8}$$

$$\int_{y(2)}^{y(t)} \frac{1}{y} - \frac{1}{y+1} dy = \int_2^t tdt \tag{1.9}$$

$$\ln(y) - \ln(y+1) - \ln(1) + \ln(2) = \frac{t^2}{2}|_2^t \tag{1.10}$$

Exponentiating the above gives

$$\frac{y}{y+1} = e^{\frac{t^2}{2} - 2 - \ln(2)} = 2e^{\frac{t^2}{2} - 2}$$

This can be solved for y to give

$$y = \frac{2e^{\frac{t^2}{2} - 2}}{1 - 2e^{\frac{t^2}{2} - 2}}$$

Another example where it is not really possible to solve for y explicitly as a function of t is given by the following:

Example 1.2.5. *Solve the equation*

$$\frac{dy}{dt} = \frac{t^2 + 5}{y^3 + 2y} \qquad y(0) = 1$$

One has to be careful in applying this idea. Usually to put the equation in the form of an exact derivative one must divide through by some quantity. As you are probably familiar with from algebra, if one multiplies through by something you must be careful that you are not introducing extraneous roots, or removing relevant roots. If one is not careful sometimes one can miss solutions by making such implicit assumptions. Here is an example

Example 1.2.6. *Solve the differential equation*

$$\frac{dy}{dt} = y^{\frac{1}{3}} \qquad y(0) = 0$$

Since this is a separable equation we can write it as

$$\frac{dy}{y^{\frac{1}{3}}} = dt$$

Which integrates up to

$$\frac{3}{2}y^{\frac{2}{3}} = t + c$$

when $t = 0$ we have $y = 0$, giving $c = 0$. Thus we have

$$y = (\frac{2t}{3})^{\frac{3}{2}}$$

There is only one problem: This is NOT *the only solution - we missed the solution $y(t) = 0$.*

This should be a little bit unsettling. If a differential equation is modeling a physical system, such as a trajectory, there should be a **unique** *solution: physically there is only one trajectory. Here we have a differential equation with more than one solution. So what happened?*

Well, when we divide through by $y^{\frac{1}{3}}$ we are making the implicit assumption that this quantity is not identically zero. So we need to go back and check that $y = 0$ is not a solution to the equation. In this case it is *a solution to the equation.*

Later we will state a fundamental existence and uniqueness theorem. This will show that for "nice" differential equations this simply does not happen: given an initial condition there is one and only one solution.

Practice Exercises:

- Solve the differential equation $y' = ty \ln |y|$
- Suppose that a projectile is fired upwards at 100 meters per second, and that the acceleration due to gravity is $10\mathrm{m\,s^{-2}}$. At what time does the projectile hit the ground?
- If the projectile has a horizontal velocity of 20 meters per second how far has it traveled horizontally when it strikes the ground?

1.3 Slope fields for first order equations

SLOPE FIELDS give a geometric interpretation for differential equations that is analogous to the familiar interpretation from calculus of the derivative as the slope of the tangent line. If one considers a first order differential equation $\frac{dy}{dt} = f(y,t)$ the left-hand side gives the derivative of the function, which we interpret as the slope of the tangent line, as a function of y and t.

This *tangent line* interpretation will also be useful for deriving numerical methods for solving differential equations such as the Euler approximation.

$$\underbrace{\frac{dy}{dt}}_{\text{slope}} = \underbrace{f(y,t)}_{\text{function of }(t,y)}$$

In other words the differential equation gives the slope of the curve as a function of (t,y). This suggests a graphical construction for solution curves. At each point (t,y) in the plane one can draw a small line segment of slope $f(y,t)$. This is known as a vector field or slope field. By "following" the slope lines we can generate a solution curve to the differential equation. This method is excellent for giving a qualitative idea as to the behavior of solutions to differential equations.

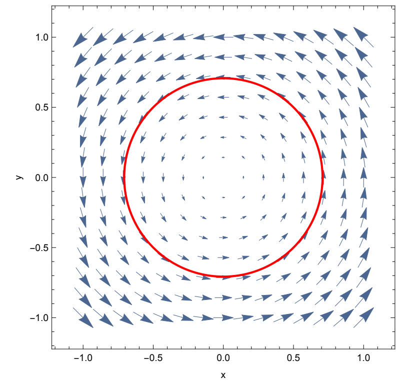


Figure 1.4: A slope field for $\frac{dy}{dt} = -\frac{t}{y}$ (blue) together with a solution curve (red).

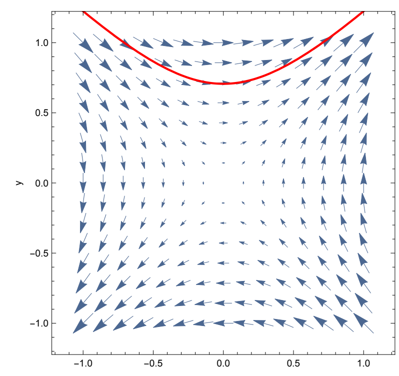


Figure 1.5: A slope field for $\frac{dy}{dt} = \frac{t}{y}$ (blue) together with a solution curve (red).

Example 1.3.1. *Consider the differential equation*

$$\frac{dy}{dt} = -\frac{t}{y}$$

The slope field associated to this equation, along with one solution curve, are shown in the margin. One can see that the curve is tangent to the line segments. Since this equation is separable it can be solved explicitly

$$\begin{aligned}\frac{dy}{dt} &= -\frac{t}{y}\\ ydy &= -tdt\\ \frac{y^2}{2} &= c - \frac{t^2}{2}\\ y^2 + t^2 &= 2c\end{aligned}$$

from which it follows that the solution curves are circles.

A similar looking example is provided by

$$\frac{dy}{dt} = \frac{t}{y}$$

while this differential equation looks quite similar to the previous one the slope field as well as the solutions appear quite different. The solution curve looks like a hyperbola, and this can be verified by integrating the equation.

Often wants a qualitative understanding of the behavior of a differential equation, without necessarily wanting to know the solution formula in great detail. There are a number of equations that are amenable to analysis by this method. One famous one is the logistic model

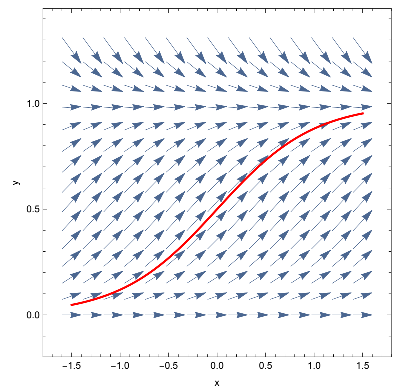


Figure 1.6: The slope field for the logistic equation $\frac{dy}{dt} - 2y(1-y)$

Example 1.3.2. Logistic Model *A common model for the growth of populations is called the "logistic model".*

The logistic model posits that the growth of a population is according to the equation

$$\frac{dP}{dt} = -k(\frac{P}{P_0} - 1)P.$$

Here k and P_0 are positive constants. The quantity k is a growth rate and quantity P_0 is known as the carrying capacity. Note that for small

populations, $P < P_0$ the growth rate is positive, but for populations above the maximum sustainable one ($P > P_0$) the growth rate becomes negative and the population decreases.

The slope field is shown in Figure (1.6) for parameter values $k = 2$ and $P_0 = 1$, along with a typical solution curve (red). The solution grows in an exponential fashion for a while but the population saturates at the carrying capacity $P_0 = 1$.

This model has also been applied to study the question of "peak oil." In this context the quantity $P(t)$ represents the total amount of oil pumped from the ground from the beginning of the oil industry to time t. This is assumed to follow a logistic curve, with the carrying capacity P_o representing the total amount of accessible oil.

$$\frac{dP}{dt} = k_0(P_0 - P(t))P(t)$$

The logistic nature of the curve is meant to reflect the fact that as more oil is pumped the remainder becomes harder to recover. This makes a certain amount of sense - there is only a finite amount of oil on Earth, so it makes sense that $P(t)$ should asymptote to a constant value. There is considerable debate as to whether this hypothesis is correct, and if so how to estimate the parameters k, P_0.

One way to estimate the constant P_0, which represents the total amount of oil, is to look at $P'(t)$, the rate at which oil is being pumped from the ground, as this is something for which we have data. It is easy to see from the graph of $P(t)$ that $P'(t)$ has a single maximum (hence the phrase peak oil) and decays away from there (in the graph above the maximum of $P'(t)$ occurs at $t = 0$. It is not hard to calculate that $P'(t)$, the rate of oil production, has a maximum when $P = P_0/2$, which is to say half of all the oil is gone. The easiest way to see this is to differentiate the differential equation

$$\begin{aligned} P'(t) &= kP(P_0 - P) \\ P''(t) &= k(P_0 - 2P)P' = k^2 P(P_0 - P)(P_0 - 2P) \end{aligned}$$

The pumping rate $P'(t)$ should have a maximum when $P''(t) = 0$ which occurs when $P = 0, P = \frac{P_0}{2}$ or $P = P_0$. From the equation we have that $P'(t) = 0$ when $P = 0$ or when $P = P_0$, so the maximum of the pumping rate occurs when half of the oil has been pumped from the ground. Thus one way to try to estimate the total amount of oil is to look at the pumping records, try to determine when the peak production occured (or will occur) and conjecture that, at that point, half of all the oil has been pumped.

While the jury is still out on this (for more details see David Goodstein's book "Out of Gas: The End of the Age Of Oil " it seems that the peak oil theory has done a very good job of predicting US oil production but not such a good job predicting natural gas production.

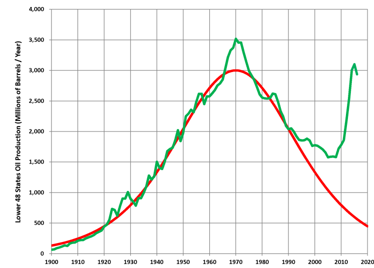


Figure 1.7: A graph of US Oil production as a function of time from 1900 to 2020. The red curve depicts a fit to a logistic curve.

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The graph in the margin shows the crude oil production in the US and a fit of the curve $P'(t)$ to it. It is clear that the peak is around 1970. A crude way to estimate the total area under the curve is to approximate it as a triangle of base $\approx 70 yrs$ and a height of about $3.5 Gb/year$ giving an area of about $125 Gb$ (= 125 Billion barrels) of total production (By eyeball this looks like an overestimate). This suggests that the total amount of crude produced in the US over all time will asymptote to something like $250 Gb$. The US currently consumes 15 Million barrels of oil a day, about $\frac{2}{3}$ of which are imported.

All data and the US crude production graph from Wikipedia

1.4 *Existence-Uniqueness theorem*

In the previous section we saw that the differential equation

$$y' = y^{\frac{1}{3}} \qquad\qquad y(0) = 0$$

does not have a unique solution. There are at least two solutions satisfying the same differential equation and the same initial condition:

$$\begin{aligned} y_1(t) &= (\frac{2}{3}t)^{\frac{3}{2}} \\ y_2(t) &= 0. \end{aligned}$$

This is not good for a physical problem. If one is calculating some physical quantity - the trajectory of a rocket, the temperature of a reactor, etc. - it is important to know that

- The problem has a solution. (Existence)
- There is **only one** solution. (Uniqueness)

This is the problem of existence and uniqueness of solutions. The example above shows us that this is something that cannot just be assumed. In this section we give a theorem that guarantees the existence and uniqueness of solutions to certain differential equations.

Lets begin by stating the following theorem, which is the fundamental existence result for ordinary differential equations

Theorem 1.4.1 (Existence and Uniqueness). *Consider the first order equation*

$$\frac{dy}{dt} = f(y,t) \qquad y(t_0) = y_0$$

- *If $f(y,t)$ is continuous is a neighborhood of the point (y_0, t_0) then a solution exists in some rectangle $|t - t_0| < \delta, |y - y_0| < \epsilon$.*
- *If, in addition, $\frac{\partial f}{\partial y}(y,t)$ is continuous in a neighborhood of the point (y_0, t_0) then the solution is unique.*

The above conditions are sufficient, but not necessary. It may be that one or the other of these conditions fails and the solution still exists/is unique.

Let's consider some examples:

Example 1.4.1. *What can you say about the equation*

$$\frac{dy}{dt} = (y-5)\ln(|y-5|) + t \qquad y(0) = 5$$

the function $(y-5)\ln(|y-5|) + t$ is continuous in a neighborhood of $y = 5, t = 0$. However computing the first partial with respect to y gives

$$\frac{\partial f}{\partial y}(y,t) = 1 + \ln|y-5|$$

which is **NOT** *continuous in a neighborhood of the point $(t = 0, y = 5)$. The above theorem guarantees that there is a solution, but does not guarantee that it is unique. Again the above theorem gives a sufficient condition: The fact that $\frac{\partial f}{\partial y}$ is not continuous does not necessarily imply that there is more than one solution, only that there* **may** *be more than one solution.*

Note that if we had the initial condition $y(0) = 1$ (or anything other than $y(0) = 5$ the theorem would apply, and we would have a unique solution, at least in a neighborhood of the initial point.

The method of proof is interesting. While we won't actually prove it we will mention how it is proved. The method is called "Picard iteration". Iterative methods are important in many aspects of numerical analysis and engineering as well as in mathematics, so it is worthwhile to give some sense of how this particular iteration scheme works. In this case Picard iteration is a useful theoretical tool but is not a very good way to solve equations numerically in practice. Numerical methods will be covered later in the notes.

The idea behind Picard iteration is as follows: suppose that one want to solve

$$y' = f(y,t) \qquad y(t_0) = y_0.$$

The idea is to construct a sequence of functions $y_n(t)$ that get closer and closer to solving the differential equation. The initial guess, $y_0(t)$, will be a function satisfying the right boundary condition. The easiest thing to take is a constant $y_{(0)}(t) = y_0$. The next guess $y_{(1)}(t)$ is defined by

$$\frac{dy_{(1)}}{dt}(t) = f(y_{(0)}(t),t) \qquad y_{(1)}(t_0) = y_0$$

One continues in this way: the n^{th} iterate $y_n(t)$ satisfies

$$\frac{dy_{(n)}}{dt}(t) = f(y_{(n-1)}(t),t) \qquad y_{(n)}(t_0) = y_0.$$

which can be integrated up to give

$$y_{(n)}(t) = y_0 + \int_{t_0}^{t} f(y_{(n-1)}(t),t)dt$$

Picard's existence and uniqueness theorem guarantees that this procedure converges to a solution of the original equation.

Theorem 1.4.2. *Consider the ordinary differential equation*

$$\frac{dy}{dt} = f(y,t) \qquad y(t_0) = y_0.$$

Suppose that $f(y,t)$ and $\frac{\partial f}{\partial y}$ are continuous ina neighborhood of the point (y_0, t_0). Then the differential equation has a unique solution in some interval $[t_0 - \delta, t_0 + \delta]$ for $\delta > 0$, and the Picard iterates $y_n(t)$ converge to this solution as $n \to \infty$.

As an example: Lets try an easy one by hand (NOTE: In practice one would never try to do this by hand - it is primarily a theoretical tool - but this example works out nicely.)

Example 1.4.2. *Find the Picard iterates for*

$$y' = y \qquad y(0) = 1$$

The first Picard iterate is $y_{(0)}(t) = 1$. The second is the solution to

$$\frac{dy_{(1)}}{dt} = y_{(0)}(t) = 1 \qquad y_{(1)}(0) = 1$$

The solution to this is $y_{(1)}(t) = 1 + t$*. Continuing in this fashion* $y_{(2)}(t)$ *solves*

$$\frac{dy_{(2)}}{dt} = y_{(1)}(t) = 1 + t \qquad y_{(2)}(0) = 1$$

so $y_{(2)}(t) = 1 + t + \frac{t^2}{2}$*. Continuing in this fashion we find that*

$$\begin{aligned} y_{(3)}(t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{6} \\ y_{(4)}(t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} \\ y_{(5)}(t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} \end{aligned}$$

It is not difficult to see by induction that the n^{th} *Picard iterate is just the* n^{th} *Taylor polynomial for the function* $y = e^t$*, which is the solution to the original differential equation.*

A second example is as follows:

Example 1.4.3. *Find the first few Picard iterates of the equation*

$$y' = t^2 + y^2 \qquad y(0) = 1$$

They are

$$\begin{aligned} y_{(0)}(t) &= 1 \\ y_{(1)}(t) &= 1 + t + t^3/3 \\ y_{(2)}(t) &= 1 + t + t^2 + \frac{2t^3}{3} + \frac{t^4}{6} + \frac{2t^5}{15} + \frac{t^7}{63} \\ y_{(3)}(t) &= 1 + t + t^2 + \frac{4}{3}t^3 + \frac{5}{6}t^6 + \frac{8}{15}t^5 + \frac{29}{90}t^6 + \frac{47t^7}{315} + \frac{41t^8}{630} \\ &\quad + \frac{299t^9}{11340} + \frac{4t^{10}}{525} + \frac{184t^{11}}{51975} + \frac{t^{12}}{2268} + \frac{4t^{13}}{12285} + \frac{t^{15}}{59535} \end{aligned}$$

The pattern for the coefficients in this example is not so clear.

We will not present the full proof here, but the basic principle behind the proof is something called the contraction mapping principle. The contraction mapping principle is the simplest example of what mathematicians call a fixed point theorem. Leaving aside technical details the contraction mapping principle says the following. Suppose we have a set Ω, and for each element $f \in \Omega$ we have a notion of the size of each element $\|f\|$. This is what is called a normed space. Now suppose that we have a map T from Ω to itself such that for any two elements $f, g \in \Omega$ we have that $\|T(f) - T(g)\| \leq \rho\|f - g\|$ for $\rho < 1$. In other words shrinks the distances between points in the set by a fixed contraction factor ρ that is strictly less than one. Then there is a unique element h that is fixed by the map: $T(h) = h$, and

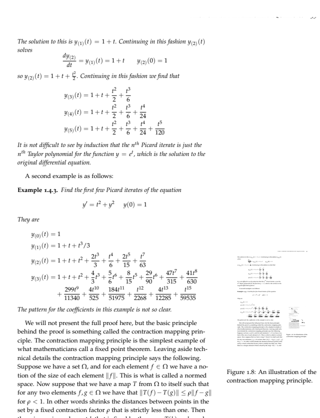


Figure 1.8: An illustration of the contraction mapping principle.

this fixed point can be found by taking any starting point and iterating the map: $\lim_{N\to\infty} T^N(f) = h$. This theorem is illustrated by the margin figure. The proof of the Picard existence theorem amounts to showing that the Picard iterates form (for small enough neighborhood of (t_0, y_0) a contraction mapping, and thus converge to a unique fixed point.

Having some idea of how the existence and uniqueness theorem is proven we will now apply it to some examples.

Example 1.4.4. *Consider the equation*

$$\frac{dy}{dt} = \frac{ty}{t^2 + y^2} \qquad y(0) = 0$$

The function $f(t, y) = \frac{ty}{t^2+y^2}$ *is* **NOT** *continuous in a neighborhood of the origin (Why?), so the theorem does not apply. On the other hand if we had the initial condition*

$$\frac{dy}{dt} = \frac{ty}{t^2 + y^2} \qquad y(1) = 0$$

then the function $f(t, y) = \frac{ty}{t^2+y^2}$ *is continuous in a neighborhood of the point* $y = 0, t = 1$*, so the equation would have a unique solution.*

1.5 First Order Linear Inhomogeneous Equations:

Most of the equations considered in these notes will be linear equations, since linear equations often arise in practice, and are typically much easier to solve than nonlinear equations. A first order linear Inhomogeneous equation takes the form

$$\frac{dy}{dt} + P(t)y = Q(t) \qquad y(t_0) = y_0.$$

If $P(t)$ and $Q(t)$ are continuous functions of t then $f(y, t) = P(t)y + Q(t)$ and $\frac{\partial f}{\partial y} = P(t)$ are continuous functions and hence Picard existence and uniqueness theorem guarantees a unique solution. For linear equations, due to the simple dependence on y , one can do a little better than the Picard theorem. In fact one can show the following.

Theorem 1.5.1. *Given the first order linear inhomogeneous differential equation*

$$\frac{dy}{dt} + P(t)y = Q(t) \qquad y(t_0) = y_0.$$

Suppose that $P(t), Q(t)$ *are continuous on a interval* (a, b) *containing* t_0*. Then the differential equation has a unique solution on* (a, b)*.*

This is stronger than the Picard theorem because it guarantees that the solution exists and is unique in the entire interval over which $P(t), Q(t)$ are continuous. The Picard theorem generally only guarantees existence and uniqueness in some small interval $[t_0 - \delta, t_0 + \delta]$.

It turns out that the above equation can always be solved by what is called the integrating factor method, which is a special case of what is known as variation of parameters.

To begin with we'll consider the special case $Q(t) = 0$ (this is called the "linear" or "linear homogeneous" problem. In this case the equation is separable

$$\frac{dy}{dt} + P(t)y = 0$$

with solution

$$y(t) = y_0 e^{-\int P(t)dt}$$

We're going to introduce a *new* variable $w(t) = y(t)e^{\int P(t)dt}$. Note that

$$\frac{dw}{dt} = (\frac{dy}{dt} + P(t)y)e^{\int P(t)dt}$$

Taking the above equation and multiplying through by $e^{\int P(t)dt}$ gives

$$\frac{dy}{dt} + P(t)y = Q(t) \tag{1.11}$$

$$\frac{dy}{dt} + P(t)y = b(t) \tag{1.12}$$

$$\underbrace{(\frac{dy}{dt} + P(t)y)e^{\int P(t)dt}}_{\frac{dw}{dt}} = Q(t)e^{\int P(t)dt} \tag{1.13}$$

$$\frac{dw}{dt} = Q(t)e^{\int P(t)dt} \tag{1.14}$$

The latter can be integrated up, since it is an exact derivative. This gives

$$w = \int Q(t)e^{\int P(t)dt} + c$$

since $w(t) = y(t)e^{\int P(t)dt}$ we can express this in terms of y as

$$y = \underbrace{e^{-\int P(t)dt}\int b(t)e^{\int P(t)dt}}_{particular} + \underbrace{ce^{-\int P(t)dt}}_{homogeneous}$$

This is called the "integrating factor" method. The basic algorithm is this: given the equation

$$\frac{dy}{dt} + P(t)y = Q(t)$$

Then you should

Here the mathematics terminology differs somewhat from the engineering terminology. In engineering it is customary to split the solution into two parts, the zero input and zero state functions. The zero state solution is a particular solution that satisfies zero boundary conditions at the initial point t_0. This can be achieved by taking t_0 as the lower limit in the integral. The zero state solution is the homogeneous solution satisfying the appropriate boundary conditions. While one does not have to do it this way it is a nice convention that makes things conceptually simpler, as it separates the solution into a part due to the external forcing and a part due to the initial conditions.

- Calculate the integrating factor $\mu(t) = e^{\int P(t)dt}$.
- Multipy through by $\mu(t)$, recognize the lefthand side as derivative
- Integrate up.

There are a couple of things to note about this solution. These are not so important now, but they are going to be important later when we talk about the general solution to a linear inhomogeneous equation.

- The solution splits into a particular solution plus a solution to the homogeneous problem.
- The particular solution can be realized as a integral of the right-hand side times the homogeneous solution. This is a special case of what we will later see as the variation of parameters formula.

Another way to say the first point is as follows: the difference between any two solutions to the linear inhomogeneous problem is a solution to the homogeneous problem. To see this note that if y_1 solves

$$\frac{dy_1}{dt} + P(t)y_1 = Q(t)$$

and y_2 solves

$$\frac{dy_2}{dt} + P(t)y_2 = Q(t)$$

then $y_1 - y_2$ solves

$$\frac{d(y_1 - y_2)}{dt} + P(t)(y_1 - y_2) = 0$$

so the difference between any two solutions of the inhomogeneous problem solves the homogeneous problem.

Let's do some examples:

Example 1.5.1. *Find the general solution to the equation*

$$y' + ty = t^3$$

The integrating factor in this case is

$$\mu(t) = e^{\int t dt} = e^{\frac{t^2}{2}}$$

We can do this another way, by guessing a particular solution. Notice that if y is a polynomial then y' and ty are both polynomials. Assuming that $y = at^2 + bt + c$ then we have

$$y' + ty = at^3 + bt^2 + (c + 2a)t + b$$

for this to equal t^3 I should pick $a = 1, b = 0, c = -2$. This gives a particular solution $y = t^3 - 2$. I know from the above calculation that **ANY** *solution is given by the sum of a particular solution plus the solution to the homogeneous problem. The solution to the homogeneous problem is*

$$y = y_0 e^{-\frac{t^2}{2}}$$

so the general solution is

$$y = y_0 e^{-\frac{t^2}{2}} + t^3 - 2$$

Example 1.5.2. *Find the general solution to*

$$\frac{dy}{dt} + \tan(t)y = \cos(t)$$

The integrating factor is, in this case

$$\mu(t) = e^{\int \frac{sin(t)}{\cos(t)} dt} = e^{-\ln(cos(t))} = \frac{1}{\cos(t)}$$

Multiplying through gives

$$\sec(t)\frac{dy}{dt} + \sec(t)\tan(t)y = 1$$

recognizing the righthand side as $\sec(t)\frac{dy}{dt} + \sec(t)\tan(t)y = \frac{d}{dt}(\sec(t)y)$ *we have*

$$\begin{aligned} \frac{d}{dt}(\sec ty) &= 1 \\ \sec ty &= t + c \\ y &= t\cos t + ct \end{aligned}$$

Example 1.5.3 (A Mixture Problem)**.** *A tank contains 10 kilograms of salt dissolved in 100 liters of water. Water containing .001 kilogram of salt per liter flows into the tank at 10 liters per minute, and* well mixed *water flows out at 10 liters per minute. What is the differential equation governing the amount of salt in the tank at time t? Solve it*

Let $Y(t)$ represent the total amount of salt (in kilograms) in the water. The total amount of water in the tank is a constant 100 liters. The amount of salt flowing into the tank is $.01$ *kg/min. The amount flowing out depends on the concentration of the salt. If the amount of salt is $Y(t)$ then the concentration is $Y(t)/100$, and the amount flowing out is* $10Y(t)/100 = Y(t)/10$. *So the differential equation is*

$$\frac{dY}{dt} = .01 - \frac{Y}{10}$$

The integrating factor is $e^{\int \frac{1}{10} dt} = e^{\frac{t}{10}}$ *giving*

$$\frac{d}{dt}(Ye^{\frac{t}{10}}) = .01e^{\frac{t}{10}} \tag{1.15}$$

$$(Ye^{\frac{t}{10}}) = .1e^{\frac{t}{10}} + c \tag{1.16}$$

$$Y = .1 + ce^{-\frac{t}{10}} \tag{1.17}$$

$$\tag{1.18}$$

Initially we have $Y(0) = 10$ (initially there are 10 kg of salt) thus

$$Y = .1 + 9.9e^{-\frac{t}{10}}$$

and the total amount of salt is asymptotic to .1kg. Can you see why this must be so? Can you show it without explicitly solving the equation?

A more interesting problem occurs when the water flows into the tank at a different rate than it flows out. For instance.

Exercise 1.2. *Suppose now that the water flows out of the tank at 5 liters per minute. What is the equation governing the amount of salt in the tank? Solve it!*

1.6 Exact and other special first order equations

THERE ARE A FEW SPECIAL TYPES of equations that often come up in applications. One of these, which frequently arises in applications, is the exact equation:

$$N(x,y)\frac{dy}{dx} + M(x,y) = 0 \tag{1.19}$$

where M, N are related in a particular way. In order to be exact there must be some function $\psi(x,y)$ (called the potential) such that

$$\frac{\partial \psi}{\partial y} = N(x,y) \tag{1.20}$$

$$\frac{\partial \psi}{\partial x} = M(x,y). \tag{1.21}$$

Exact equations are connected with vector calculus and the problem of determining if a vector field is a gradient, so you may have previously encountered this idea in vector calculus or physics.

To be slightly more careful the equality of mixed partials is guaranteed by Clairaut's theorem.

Theorem 1.6.1 (Clairaut). *Suppose that $\phi(x,y)$ is defined on D, an open subset of $\mathbb{R}^2$. Further suppose that the second order mixed partials ϕ_{xy} and ϕ_{yx} exist and are continuous on $\mathbb{R}^2$. Then the mixed partials ϕ_{xy} and ϕ_{yx} must be equal on D.*

An obvious first question is how to recognize an exact equation. For general functions $N(x,y)$ and $M(x,y)$ there is not always such a potential function $\psi(x,y)$. A fact to recall from vector calculus is the equality of mixed partials: $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$. Differentiating the above equations with respect to x and y respectively gives

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial N}{\partial x} \tag{1.22}$$

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial M}{\partial y} \tag{1.23}$$

So a necessary condition is that $N_x = M_y$. It turns out that this is also a sufficient condition.[1] If we can find such a function ψ then by the

[1] On simply connected domains.

chain rule the above equation is equivalent to

$$\psi_y(x,y)\frac{dy}{dx} + \psi_x(x,y) = 0 \tag{1.24}$$
$$\frac{d}{dx}(\psi(x,y)) = 0 \tag{1.25}$$
$$\psi(x,y) = c \tag{1.26}$$

This makes geometric sense – Equation (1.19) is equivalent to

$$\nabla\psi \cdot (dx, dy) = 0$$

so the directional derivative of ψ along the curve is zero. This is equivalent to saying that ψ is constant along the curve $(x, y(x))$, or that the curves $(x, y(x))$ are level sets of ψ.

Given that we can recognize an exact equation how do we find ψ? The inverse to partial differentiation is partial integration. This is easiest to illustrate with an example

Example 1.6.1. *Solve the equation*

$$(y^3 + 3x^2y)dy + (3y^2x)dx = 0$$

We'd like to find a function $\psi(x,y)$ such that

$$\frac{\partial\psi}{\partial y} = y^3 + 3x^2y \tag{1.27}$$
$$\frac{\partial\psi}{\partial x} = 3y^2x \tag{1.28}$$

Well, first we have to check that this is possible. The necessary condition is that the mixed partials ought to be equal. Differentiating gives

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(y^3 + 3x^2y\right) = 6xy$$
$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(3xy^2\right) = 6xy$$

Thus we know that there exists such a ψ. Now we have to find it. We can start by integrating up

$$\psi = \int y^3 + 3x^2y dy = \frac{y^4}{4} + \frac{3x^2y^2}{2} + f(x).$$

Here we are integrating with respect to y, so the undetermined constant is a function of x. This makes sense: if we take the partial derivative of $f(x)$ with respect to y we will get zero. We still don't know $f(x)$, but we can use the other equation

$$\psi_x = 3xy^2 + f'(x)$$

From the point of view of vector calculus we can define the potential function by the line integral

$$\psi(x,y) = \int_\gamma N(x,y)dy + M(x,y)dx$$

where γ is any curve beginning at an arbitrary point (say $(0,0)$) and ending at (x,y). Because the vector field is conservative the line integral does not depend on the path. In the example we are taking a piecewise path consisting of a line segment from $(0,0)$ to $(0,y)$ followed by a line segment from $(0,y)$ to (x,y).

comparing this to the above gives $f'(x) = 0$ so $f(x) = c$. Thus the solution is given by

$$\psi(x,y) = \frac{y^4}{4} + \frac{3x^2y^2}{2} = c$$

Note that this equation is IMPLICIT. *You can actually solve this for y as a function of x (The above is quadratic in y) but you can also leave it in this form.*

Another example is the following:

Example 1.6.2.

$$\frac{dy}{dx} = \frac{-2xy}{x^2+y^2}$$

This is similar to an example we looked at previously, as an illustration of the existence/uniqueness theorem. We can rewrite this as

$$(x^2+y^2)dy + (2xy)dx = 0$$

*This equation is exact (*check this!*) so we can find a solution $\psi(x,y) = c$. We compute $\psi(x,y)$ as outlined above*

$$\begin{aligned} \psi_x &= 2xy \\ \psi &= x^2y + g(y) \end{aligned}$$

To compute the undetermined function $g(y)$ we use the other equation

$$\begin{aligned} \psi &= x^2y + g(y) \\ \psi_y &= x^2 + g'(y) = x^2 + y.^2 \end{aligned}$$

From this we can conclude that $g'(y) = y^2$ or $g(y) = \frac{y^3}{3}$. Thus the solution is

$$\frac{y^3}{3} + x^2y = c$$

A SECOND IMPORTANT CLASS of exactly solvable first order equations are the separable equations. These are equations that can be written in the form

$$\frac{dy}{dt} = f(t)g(y)$$

Such equations can be solved by *separating* the dependent and independent variables. If we manipulate the equation to get the independent variable on one side and the dependent variable on the other we find that

$$\frac{dy}{g(y)} = f(t)dt.$$

This can be integrated to give

$$\int \frac{dy}{g(y)} = \int f(t)dt.$$

Example 1.6.3 (Logistic Equation)**.** *The logistic equation*

$$\frac{dy}{dt} = y(1-y)$$

is a common model for growth of a population $y(t)$ in a situation where there is a maximum sustainable population, in this case chosen to be 1 *. This is might model a situation in which there is a fixed amount of resource necessary to sustain the population. In the case where the initial population y is small the equation behaves like*

$$\frac{dy}{dt} = y - y^2 \approx y.$$

This is the standard exponential growth model. As y approaches 1*, the maximum sustainable population, the growth rate goes to zero. If the population is above* 1 *then the population will actually decrease towards the maximum sustainable population.*

We can solve this explicitly by using the fact that the equation is separable. With a little algebra we find that

$$\begin{aligned} \frac{dy}{dt} &= y(1-y) \\ \frac{dy}{y(1-y)} &= dt \\ \left(\frac{1}{y} + \frac{1}{1-y}\right) dy &= dt \end{aligned}$$

where the last step follows from the method of partial fractions, from calculus. Assuming that the population y is between 0 *and* 1 *we can integrate this up to find*

$$\begin{aligned} \ln y - \ln(1-y) &= t + c \\ \frac{y}{1-y} &= e^{t+c} \\ y &= e^{t+c} - e^{t+c}y \\ y &= \frac{e^{t+c}}{1+e^{t+c}} \end{aligned}$$

The side margin shows some solutions to the logistic equation for different values of c but they all show the same characteristic shape: the population undergoes what looks like exponential growth for a while but eventually asymptotes to the maximum sustainable population, 1*.*

There are a number of other types of equations that can be integrated by various tricks. One class of equations is those of the form

$$\frac{dy}{dt} = F(\frac{y}{t})$$

These can be solved by the substitution $v = \frac{y}{t}$ or $y = tv$. This leads to

$$\frac{dy}{dt} = t\frac{dv}{dt} + v = F(v)$$

which is a separable equation.

An example of this type is

$$\frac{dy}{dt} = \frac{ty}{t^2 + y^2}$$

Making the substitution $y = tv$ gives

$$t\frac{dv}{dt} + v = \frac{t^2 v}{t^2 + t^2 v} = \frac{v}{1 + v^2}$$

Which is separable.

Example 1.6.4. *Consider the equation*

$$\frac{dy}{dt} = -\frac{t^2 + 2ty + y^2}{1 + (t + y)^2}$$

Well, it is clear that the righthand side is only a function of the sum $t + y$. This suggests making the substitution $v = y + t$. The equation above becomes

$$\frac{dv}{dt} - 1 = -\frac{v^2}{1 + v^2} \tag{1.29}$$

$$\frac{dv}{dt} = \frac{1}{1 + v^2} \tag{1.30}$$

$$(1 + v^2)dv = dt \tag{1.31}$$

Integrating this up gives the equation

$$v + \frac{v^3}{3} = t + c \tag{1.32}$$

$$y + t + \frac{1}{3}(y + t)^3 = t + c \tag{1.33}$$

Another example occurs for equations of second order when either the dependent variable y or the independent variable t is missing.

Example 1.6.5. *Consider the problem*

$$y'' + y'/t = t^5$$

This is a second order equation, but there is no y term. Thus we can make the substitution $v = y'$. This gives

$$v' + v/t = t^5$$

This is a first order linear (inhomogeneous) equation, and thus can be solved by multiplying through by an appropriate integrating factor. In this case the integrating factor is $e^{\int dt/t} = e^{\ln(t)} = t$ Thus

$$\frac{d}{dt}(tv) = t^6$$

Another similar example occurs when the independent variable t is missing.

$$F(y, y', y'') = 0$$

This can be simplified by the substitution

$$y' = p(y)$$

(in other words, instead of thinking of t as the independent variable we think of y as the independent variable. Differentiating the above gives

$$y'' = y' \frac{dp}{dy} = p \frac{dp}{dy}$$

for example

Example 1.6.6. *Solve the equation*

$$y'' y^2 = y'$$

Making the change of variables above gives

$$y^2 p \frac{dp}{dy} = p$$

either $p = 0$ or dividing through by p gives

$$\frac{dp}{dy} = \frac{1}{y^2}$$

or $p(y) = -\frac{1}{y} + c$ but this is the same as

$$\frac{dy}{dt} = c - \frac{1}{y} = \frac{cy - 1}{y} \tag{1.34}$$

$$\frac{y}{cy - 1} dy = dt \tag{1.35}$$

which can be integrated up to get

$$\frac{y}{c} + \frac{\log(1 - cy)}{c^2} = t - t_0.$$

Here we have denoted the second constant of integration by t_0. Note that as this is a second order equation we have two arbitrary constants of integration (c and t_0) but they do not enter linearly. Also note that we have an implicit representation for y as a function of t but we cannot solve explicitly for y as a function of t.

A couple of other examples: A Bernoulli equation is a first order equation of the form

$$\frac{dy}{dt} + P(t)y = Q(t)y^a$$

for some real number $a \neq 0, 1$. If $a = 0$ or $a = 1$ the equation is first order linear. We can divide the equation through by y^a to get

$$y^{-a}\frac{dy}{dt} + P(t)y^{1-a} = Q(t).$$

If we define $w = y^{1-a}$; $\frac{dw}{dt} = (1-a)y^{-a}\frac{dy}{dt}$ we see that the new dependent variable $w(t)$ satisfies a first order linear equation

$$\frac{1}{1-a}\frac{dw}{dt} + P(t)w = Q(t).$$

Example 1.6.7. *Find the general solution to the nonlinear first order equation*

$$\frac{dy}{dt} + y = \cos(t)y^5.$$

Making the change of variables $w(t) = y(t)^{-4}$; $\frac{dw}{dt} = -4y^{-5}\frac{dy}{dt}$ gives the following differential equation for $w(t)$:

$$-1/4\frac{dw}{dt} + w = \cos(t).$$

This is first order linear and can be solved: the general solution is

$$w(t) = Ae^{-4t} + \frac{16}{17}\cos(t) - \frac{4}{17}\sin(t) = y(t)^{-4}$$

which gives

$$y(t) = \left(Ae^{-4t} + \frac{16}{17}\cos(t) - \frac{4}{17}\sin(t)\right)^{-\frac{1}{4}}$$

1.7 Autonomous equations, equilibria, and the phase line.

In this section we will consider first order autonomous equations, where the equation does not explicitly involve the independent variable. While such equations are always separable, and thus solvable, the explicit solutions may be complicated and not particularly useful. It is always possible, on the other hand, to get very concrete and useful qualitative information on the solution.

The general first order autonomous equation is given by

$$\frac{dy}{dt} = f(y). \tag{1.36}$$

Again this equations are always solvable in principle, as they are separable and can be reduced to integration, but such a representation may not be very useful. Even assuming that the integration can be done explicitly the method naturally gives t as a function of y. It may not be possible to explicitly invert the relation to find y as a function of t.

The ideas in this section amount to an elaboration on the idea of a slope field. Equation (1.36) relates the slope of the curve $y(t)$ to the value of $y(t)$. A special role is played by the points where the slope is zero. These are called equilibria.

Definition 1.7.1. *An equilibrium is any constant solution to* $\frac{dy}{dt} = f(y)$*. If* $y(t) = y_0$ *is a solution to Equation (1.36) then* $\frac{dy}{dt} = \frac{dy_0}{dt} = 0 = f(y_0)$*, hence an equilibrium is a solution to* $f(y_0) = 0$*.*

One of the main ideas of this section is this: if we consider drawing the slope field for an autonomous equation it is clear that the slope does not depend on the independent variable t. For this reason we do not really need to draw the whole plane, as any vertical segment is

repeated over and over. We can collapse all of the information into a graphical device known as the phase line as follows. We draw a vertical line representing the y-axis. Along this axis we put an arrow pointing upward where $f(y)$ is positive and an arrow pointing downwards where $f(y)$ is negative. Points where $f(y) = 0$ are, of course, equilibria.

We refer to this kind of diagram as the phase line.

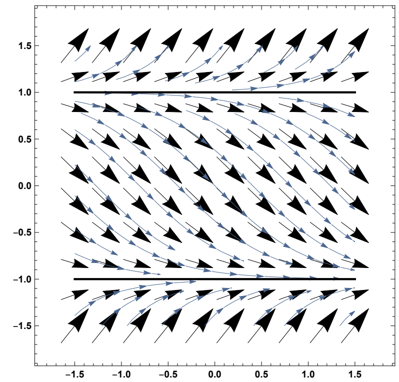


Figure 1.9: A plot of the slope field for $\frac{dy}{dt} = y^2 - 1$.

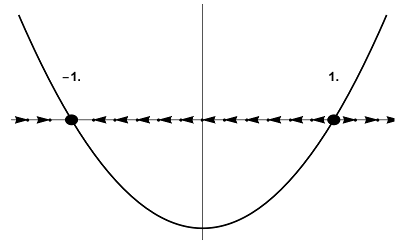


Figure 1.10: A plot of $f(y) = y^2 - 1$ vs. y. The arrows on the y axis indicate the sign of $\frac{dy}{dt} = f(y)$.

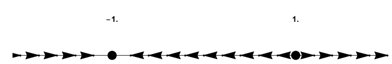


Figure 1.11: A plot of the phase line for $\frac{dy}{dt} = y^2 - 1$. The two equilibria are $y = -1$ and $y = 1$.

Example 1.7.1. *Consider the autonomous equation*

$$\frac{dy}{dt} = y^2 - 1$$

*The equilibria are values of y for which $y^2 - 1 = 0$, or $y = \pm 1$. In drawing the phase line we see that $y^2 - 1 < 0$ if $y \in (-1, 1)$, so this interval would be populated with downward arrows. If, on the other hand, $y > 1$ or $y < -1$ then $y^2 - 1 > 0$, and these two regions would be filled with upwards arrows. This is illustrated in the figure in the three margin figures. Figure 1.9 depicts the slope field. It is clear that the slopes do not depend on t, only one y, so we don't really need a two dimensional figure – all we really need to know is how the slopes depend on y. In the next figure, (Fig. (1.12) we have plotted the slope $\frac{dy}{dt} = y^2 - 1$ as a function of y. Note that this plot is turned 90 degrees from the slope field: in the slope field t is taken to be the independent variable, here y is taken to be the independent variable. On the y axis we have plotted arrows representing the sign of $\frac{dy}{dt}$: arrows pointing to the left indicate $\frac{dy}{dt} < 0$, or y decreasing, while arrows pointing to the right indicate $\frac{dy}{dt} > 0$, or y increasing. We can clearly see the equilibria at $y = \pm 1$. We can condense this plot to just a one-dimensional field of arrows, as we have done in Fig. (**??**). This figure tells us a lot about the qualitative behavior of $y(t)$: we can see that y is decreasing in the interval $y \in (-1, 1)$ and increasing outside that interval. Further* *we can see that there is a real difference in the behavior of the solution near the equilibria $y = \pm 1$. The equilibrium at $y = -1$ has the property that y values that start near it move* towards *it. The equilibrium at $y = 1$ has the property that y values that start near it move* away. *This motivates the next definition.*

Definition 1.7.2. *An equilibrium is said to be* **stable** *if nearby initial points converge to the equilibrium. An equilibrium is* **unstable** *if nearby initial points move away from the equilibrium.*

Stability is a very important concept from the point of view of applications, since it tells us something about the robustness of the solution. Stable equilibria are in some sense "self-correcting". If we start with an initial condition that is close to the equlibrium then the dynamics will naturally act in such a way as to move the solution closer to the equilibrium. If the equilibrium is unstable, on the other hand, the dynamics tends to move the solution away from the equilibrium. This means that unstable equilibria are less robust – we have to get things just right in order to observe them.

It would be nice to have a method to decide whether a given equilibrium is stable or unstable. This is the content of the next theorem.

Theorem 1.7.1 (Hartman-Grobman). *Suppose that we have the equation*

$$\frac{dy}{dt} = f(y)$$

and y^* is an equilibrium ($f(y^*) = 0$). Then

- *If $f'(y^*) > 0$ then the equilibrium is unstable - solutions initially near y^* move away from y^* exponentially.*
- *If $f'(y^*) < 0$ then the equilibrium is stable - solutions initially near y^* move towards y^* exponentially.*
- *If $f'(y^*) = 0$ further analysis is required.*

We won't give a proof of this theorem, which is extremely important in the study of dynamical systems, but we will try to give the idea behind it. If y is near the equilibrium Y^* then we can try to find a solution $y(t) = y^* + \epsilon v(t)$, where ϵ is small. If we substitute this into the equation we find that

$$\epsilon \frac{dv}{dt} = f(y^* + \epsilon v).$$

So far this is exact – all that we have done is to rewrite the equation. At this point, however, we make an approximation. Specifically we will Taylor expand the function f to first order in ϵ and use the fact that $f(y^*) = 0$:

$$\epsilon \frac{dv}{dt} = f(y^* + \epsilon v) \approx f(y^*) + \epsilon f'(y^*) v + O(\epsilon^2) = \epsilon f'(y^*) v.$$

So the quantity $v = \frac{y-y^*}{\epsilon}$ satisfies the approximate equation

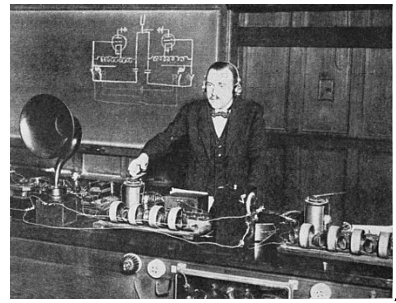
$$\frac{dv}{dt} = f'(y^*)v.$$

We know how to solve this equation: v exhibits exponential *growth* if $f'(y^*) > 0$ and exponential *decay* if $f'(y^*) < 0$.

Example 1.7.2 (Logistic Growth Revisited)**.** *Previously we considered*

$$\frac{dP}{dt} = P(P_0 - P)$$

the equilibria are $P = 0$ *and* $P = P_0$*. In this case* $f'(P) = P_0 - 2P$*, and we have* $f'(0) = P_0 > 0$ *and* $f'(P_0) = -P_0 < 0$*, so the equilibrium with zero population is unstable and the equilibrium with* $P = P_0$ *is stable. The zero population is unstable, so a small population will tend to grow exponentially. As the population approaches* P_0 *the growth rate slows and the population approaches an asymptote. If the population begins above* P_0 *then the population will decrease towards the equilibrium population* P_0*.*



The first flip-flop was built in 1918 by W. Eccles and F.W. Jordan using a pair of vacuum tubes. Vacuum tube flip-flops were used in the Colossus code-breaking computer in Bletchly Park during World War II.

Photo of William Eccles. Uncredited from a photograph that appeared on p. 228 of the June 1925 issue of Radio Broadcast magazine. , Public domain, via Wikimedia Commons

Example 1.7.3 (Bistability)**.** *The differential equation*

$$\frac{dy}{dy} = y(1-2y)(y-1)$$

has a property called "bistability" . In electrical engineering circuits with bistable behavior are commonly called "flip-flops" and have been an important design element since the earliest days of electrical engineering. In this example there are three equilibria, $y = 0, y = \frac{1}{2}, y = 1$*. The middle equilibrium* $y = \frac{1}{2}$ *is unstable, while* $y = 0$ *and* $y = 1$ *are stable. The phase line for this equation is depicted in the margin.*

Flip-flops are ubiquitous in consumer electronics, as they form the basis of logic gates and RAM memory, and one can buy any number of integrated circuits with various numbers and types of flips-flops on a single chip.

Bistability means that they will remain in one of the two stable states (either 0 *or* 1*) until some external force comes along to change the state.*

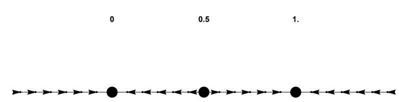


Figure 1.12: The phase-line for a bistable system (flip-flop) $\frac{dy}{dt} = y(1-2y)(y-1)$. The equilibria $y = 0$ and $y = 1$ are stable, the equilibrium $y = \frac{1}{2}$ unstable.

In the study of differential equations and their applications one is often led to consider how the qualitative aspects of an autonomous differential equation – the number and stability of the equilibria – are influenced by various physical parameters in the problem. It turns out that while generally small changes in the physical parameters typically do not change the qualitative properties of the equation there are certain values of the parameters where small changes in the parameters can produce dramatic changes in the qualitative properties of the solutions. This is a circle of ideas known as bifurcation theory. We will explore a relatively simple case of this in the next example.

Example 1.7.4. Fish

In population biology people often try to understand how various forces can effect populations. For instance, if one is trying to manage a resource such as a fishing ground one would like to weigh the economic benefits of increased fishing against the long-term sustainability of the resource. In the absence of fishing the simplest model for the growth of a population of fish would be a logistic type model

$$\frac{dP}{dt} = kP(P_0 - P).$$

As we have observed before this equation has a stable equilibrium population of $P = P_0$ – any initially positive population will converge exponentially to the stable equilibrium P_0– and an unstable equilibrium population $P = 0$.

One might try to model the effect of fishing on a fish population by a model of the following form

$$\frac{dP}{dt} = kP(P_0 - P) - h. \tag{1.37}$$

Here the first term is the standard logistic term and the second term is a constant, which is meant to represent fish being caught at a constant rate. Let's assume that h, the rate at which fish are caught, is something that we can set by a policy decision. We'd like to understand how the fish population is influenced by h, and how large a fishing rate we can allow before it ceases to be sustainable.

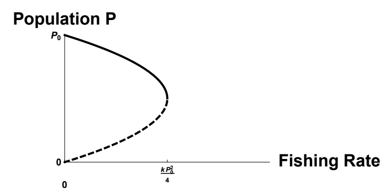


Figure 1.13: The bifurcation diagram for the logistic model with constant harvesting, $\frac{dP}{dt} = kP(P_0 - P) - h$. For low fishing rates, $h < \frac{kP_0^2}{4}$ there are two equilibria, one stable and one unstable. for higher fishing rates $h > \frac{kP_0^2}{4}$ there are no equilibria.

The righthand side is a quadratic, so there are two roots and thus two equilibria. The quadratic formula gives the equilibria as

$$P = \frac{P_0 \pm \sqrt{P_0^2 - 4h/k}}{2}$$

so if $\sqrt{h/k}$ is less than $P_0/2$ there are two positive real roots. The lower one has positive derivative and the upper one has negative derivative so the upper one is stable and the lower one is unstable. So if $\sqrt{h/k} < \frac{P_0}{2}$ there is always a stable fixed point with a population greater than $\frac{P_0}{2}$.

Now consider what happens if $\sqrt{h/k}$ is greater than $P_0/2$, or $h > \frac{kP_0^2}{4}$. In this case the roots $P = \frac{P_0 \pm \sqrt{P_0^2 - 4h/k}}{2}$ are complex, and so there are no equilibria. In this case the righthand side of Equation (1.37) is always negative. This implies that the population rapidly decays to zero and the population crashes.

Consider this from a policy point of view. Imagine that, over time, the fishing rate is slowly increased until h is just slightly less than $\frac{kP_0^2}{4}$. As the fishing rate is increased the stable equilibrium population decreases, but it is always larger than $\frac{P_0}{2}$, and there is a second, unstable equilibrium just below $\frac{P_0}{2}$. There is no obvious indication that the population is threatened, but *if h is further increased then the two equilibria vanish and the population crashes.*

This suddent disappearance of a stable and an unstable equilibrium is known as the "saddle-node bifurcation"'. It is illustrated in the margin Figure (1.13), which plots the locations of the two equilibria (the stable and the unstable) as a function of the fishing rate h. The stable branch is depicted by a solid line, the unstable branch by a dashed line. For small values of h there is one stable equilibrium and one unstable one, but at $h = \frac{kP_0^2}{4}$ *the two equilibria collide and vanish, and for* $h > \frac{kP_0^2}{4}$ *there are no equilibria.*

We will discuss equilibria and qualitative methods further in a later section of the notes. The Hartman-Grobman theorem extends to higher dimensional systems, although the classification of the equilibria is somewhat more involved, and is no longer simply a question of stable or unstable.

1.8 Numerical Methods

We close the chapter on first order equations with an introduction to numerical methods. Just as in calculus, where not every integral can be expressed in terms of elementary functions, we cannot expect to be able to analytically solve every differential equation. Even in the case where the equation can be solved explicitly it may happen that the solution is defined implicitly or in some form that may not be convenient to use. For this reason it is desirable to have numerical methods. In the case of integration you probably learned about the Riemann sum, the trapezoidal rule, Simpson's rule, etc. There are natural analogs of these methods for solving differential equations.

1.8.1 First order (Euler) method.

THE SIMPLEST NUMERICAL METHOD is what is known as a first order or Euler method. Suppose that one would like to solve the first order ordinary differential equation

$$y' = f(y,t) \qquad y(0) = y_0.$$

One can formally solve this equation by integrating and applying the fundamental theorem of calculus. If we integrate this equation from $t = 0$ to $t = \Delta t$ we find that

$$y(\delta t) - y(0) = \int_0^{\Delta t} f(y(t),t)dt.$$

This formula is correct, but it is not terribly useful – it expresses $y(\Delta t)$ in terms of $y(t)$ for $t \in (0, \Delta t)$. So we would have to know $y(t)$ in order to find $y(t)$.

If you think about this from the point of the slope field what we are really doing is replacing the smooth curve with a polygonal path. We pick a point (t_0, y_0), follow the tangent line for a distance Δt to find a new point $(t_0 + \Delta t, y_0 + f(y_0, t_0)\Delta t)$ and continue this process to get a polygonal curve made up of a bunch of straight line segments.

However, if Δt is small then we expect that $y(t)$ will not change very much in the interval $(0, \Delta t)$. In this case we can just approximate $y(t) \approx y(0)$. This gives

$$\begin{aligned} y(\delta t) - y(0) &= \int_0^{\Delta t} f(y(t), t) dt \\ &\approx \int_0^{\Delta t} f(y(0), 0) dt \\ y(\Delta t) &\approx y(0) + f(y(0), 0)\Delta t \end{aligned}$$

We can, of course, continue this process: we can use our approximation of $y(\Delta t)$ to find an approximation for $y(2\Delta t)$, and so on. This method is known as the first order explicit scheme, or (explicit) Euler scheme.

Figure 1.14: The Euler method got some positive press in the movie "Hidden Figures", when Katherine Goble Johnson used it to calculate the orbits of the Mercury astronauts. The trajectories of the first two manned Mercury missions (Freedom 7, piloted by Alan Shepard and Liberty Bell 7, piloted by Gus Grissom) were calculated entirely by hand by Johnson and other computers. Glenn's flight (Friendship 7) was the first to have the orbit calculated by an electronic computer. Glenn refused to fly the mission unless Johnson checked the results of the electronic computation personally. NASA; restored by Adam Cuerden, Public domain, via Wikimedia Commons

Method 1.8.1. *The first order Euler scheme for the differential equation*

$$\frac{dy}{dt} = f(y, t) \qquad y(a) = y_0 \tag{1.38}$$

on the interval (a, b) is the following iterative procedure. For some choice of N we divide the interval (a, b) into $N + 1$ subintervals with $\Delta t = \frac{b-a}{N}$ and $t_i = t_0 + i\Delta t$. We then define y_i by the following iteration

$$y_{i+1} = y_i + f(y_i, t_i)\Delta t.$$

Then the solution to Equation (1.38) at position t_i is approximately given by y_i:

$$y(t_i) \approx y_i + O(\Delta t).$$

Of course we get a better approximation if we take smaller steps, Δt, but it comes at a cost since we have to do **more steps** to compute the function. The Euler scheme is first order because the error scales like the first power of the step size, $(\Delta t)^1$. It is often more desirable to use what is known as a higher order method, where the error scales like $(\Delta t)^k$, with $k > 1$, especially if one requires more accuracy. To see this imagine that three different methods have errors $\Delta t, \Delta t^2$ and δt^4, and one would like the error to be 10^{-6}. The first method would require $N = 1000000$ steps, the second would require $N = 1000$ steps, and the third would require $N = 10^{\frac{3}{2}} \approx 32$ steps. This is a bit naive – the higher order methdos generally require more work, and there are generally different constants multiplying the error – but it is often more efficient to use a higher order method.

We briefly mention a couple of popular higher order methods. If the Euler method is the equivalent of the left-hand rule for Riemann sums then the equivlent of the midterm rule is the following rule:

$$y_{i+1} = y_i + f(y_i + f(y_i, t_i)\Delta t/2, t_i + \Delta t/2).$$

This rule is also called the midpoint rule, and you can see why. Instead of evaluating f at (t_i, y_i) it is evaluated at the midpoint between this and the next point. This is second order accurate: we have that $y_i - y(t_i) = O(\Delta t^2)$. You can also see that this rule requires a bit more computational work: we have to evaluate the function $f(y,t)$ twice for each step that we take. A different second order method is called Heun's method or the improved Euler method. It is defined by

$$y_{i+1} = y_i + f(y_i, t_i)\Delta t/2 + f(y_i + f(y_i, t_i)\Delta t, t_i + \Delta t)\Delta t/2$$

and is the analog of the trapezoidal rule for numerical integration. A more complicated rule is called the fourth order Runge-Kutta method, or RK4. This is defined by

$$\begin{aligned}
k_1 &= f(y_i, t_i)\Delta t \\
k_2 &= f(y_i + k_1/2, t_i + \Delta t/2)\Delta t \\
k_3 &= f(y_i + k_2/2, t_i + \Delta t/2)\Delta t \\
k_4 &= f(y_i + k_3, t_i + \Delta t)\Delta t \\
y_{i+1} &= y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)
\end{aligned}$$

and is the natural analog to Simpson's rule for numerical integration. RK4 is a fourth order method, $y_i - y(t_i) = O(\Delta t^4)$. There are many other methods that one can define – each has its own advantages and disadvantages.

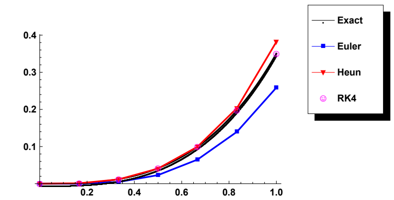


Figure 1.15: A graph of the exact solution to $\frac{dy}{dt} = y^2 + t^2$ with $y(0) = 0$ for $t \in (0,1)$ together with the Euler and improved Euler approximations to the solution with $N = 6$ subdivisions (($\Delta t = \frac{1}{6}$). The step size has been deliberately chosen to be large to exaggerate the difference. It is apparent that the improved Euler method does a better job of approximating the solution than the standard Euler method, and that the fourth order Runge-Kutta can't be distinguished from the exact solution at this scale.

2

Higher Order Linear Equations

2.0.1 Existence and uniqueness theorem.

One important class of differential equations is that of linear differential equations. Recall that a linear differential equation of order n is one that can be written in the form

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + a_{n-2}(t)\frac{d^{n-2}y}{dt^{n-2}} + \ldots a_0(t)y = f(t)$$

In other words a linear differential equation is a linear relationship between y and its first n derivatives. It can depend on the independent variable t in an arbitrary way. To begin with we first mention the fundamental existence and uniqueness theorem. Since the differential equation depends linearly on y and its derivatives the existence theorem becomes simpler, and can be stated solely in terms of the coefficients $a_k(t)$.

Theorem 2.0.1 (Existence and Uniqueness)**.** *Consider the general linear differential equation*

$$\frac{d^n y}{dt^n} + \sum_{k=0}^{n-1} a_k(t)\frac{d^k y}{dt^k} = f(t) \qquad y(t_0) = y_0, y'(t_0) = y_0', \ldots y^{(n-1)}(t_0) = y_0^{(n-1)}$$

If the functions $\{a_k(t)\}_{k=0}^{n-1}, f(t)$ are all continuous in an open set $t \in (a,b)$ containing t_0 then the above equation has a unique solution defined for all $t \in (a,b)$.

Note that this result is much stronger than the general (nonlinear) existence/uniqueness result. The nonlinear result only guarantees the existence of a solution in some small neighborhood about the point. We don't really know apriori how big that interval might be. For linear coefficients we know that as long as the coefficients are well-behaved (continuous) then a solution exists.

It is worth defining a couple of pieces of terminology which will be important.

Definition 2.0.1. *A linear differential equation in which the forcing term $f(t)$ is zero is called "homogeneous":*

$$\frac{d^n y}{dt^n} + \sum_{k=0}^{n-1} a_k(t) \frac{d^k y}{dt^k} = 0 \qquad y(t_0) = y_0, y'(t_0) = y'_0, \dots y^{(n-1)}(t_0)$$

A linear differential equation in which the forcing term $f(t)$is non-zero is called "non-homogeneous" or "inhomogeneous".

	Inhomogeneous	**Homogeneous**
Example 2.0.1.	$\frac{dy}{dt} - 5y = t$	$\frac{dy}{dt} - 3y = 0$
	$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \cos t + e^t$	$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 5y = 0$
	$\frac{d^4y}{dt^4} + \cos t \frac{d^3y}{dt^3} + y = \frac{1}{1+t^3}$	$\frac{d^4y}{dt^4} + \sin t \frac{d^3y}{dt^3} + e^t y = 0$

2.1 *Linear Homogeneous Equations*

2.1.1 *Linear Independence and the Wronskian*

Linear equations are important for a couple of reasons. One is that many of the equations that arise in science and engineering are linear. A second reason is that solutions of linear equations have a lot of structure – much more so than solutions of some arbitrary differential equation. In particular they form a linear vector space. That is the basic idea that we wish to develop in this section. The first important aspect of this structure that we consider is the idea of superposition.

Theorem 2.1.1 (Superposition Principle)**.** *Consider the linear homogeneous ($f(t) = 0$) differential equation*

$$\frac{d^n y}{dt^n} + \sum_{k=0}^{n-1} a_k(t) \frac{d^k y}{dt^k} = 0$$

If $y_1(t), y_2(t), \dots y_n(t)$ are all solutions to the above equation then an arbitrary linear combination, $y(t) = a_1 y_1(t) + a_2 y_2(t) + \dots a_n y_n(t)$ is also a solution.

The proof is simple – it just amounts to noticing that the derivative is a linear operator: $\frac{d}{dt}(a_1 y_1(t) + a_2 y_2(t)) = a_1 \frac{dy_1}{dt} + a_2 \frac{dy_2}{dt}$, together with the fact that the derivatives of y enter into the differential equation linearly.

Example 2.1.1. *Consider the equation*

$$\frac{d^2y}{dt^2} + y = 0$$

It is not hard to guess that the functions $y_1(t) = \sin(t), y_2(t) = \cos(t)$ *both satisfy the equation. By the superposition theorem* $y = a_1 \sin(t) + a_2 \cos(t)$ *is also a solution. If one is given the equation with boundary conditions*

$$y'' + y = 0 \qquad y(0) = y_0 \qquad y'(0) = y_0'$$

we can try the solution $y = a_1 \sin(t) + a_2 \cos(t)$ *and see if we can solve for* a_1, a_2. *Substituting* $t = 0$ *gives*

$$y(0) = a_1 \cos(0) + a_2 \sin(0) = a_1 = y_0$$

Similarly differentiating $y = a_1 \sin(t) + a_2 \cos(t)$ *and substituting* $t = 0$ *gives*

$$\frac{dy}{dt}(0) = -a_1 \sin(0) + a_2 \cos(0) = a_2 = y_0'$$

This gives a solution to the problem which, by the existence and uniqueness theorem, must be the ONLY solution, since solutions are unique.

The same thing holds in general. We'll state this as a "guiding principle".

Guiding Principle: If we can find *n different* solutions to a n^{th} order linear homogeneous differential equation, then the general solution will be given by a linear combination of those solutions.

The important thing here is "different." In the previous example we could have taken a linear combination of $\sin t$ and $2 \sin t$. This would have solved the differential equation. However this *would not have been the most general solution.* For instance the solution to

$$\frac{d^2y}{dt^2} + y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

is $y = cos(t)$, which is *not* a linear combination of $\sin(t)$ and $2\sin(t)$.

This may remind you of the following fact from linear algebra: if we have n linearly independent vectors in an n-dimensional vector space then they form a basis, and any vector in the space can be expressed as a linear combination of these vectors. In fact it is easy to see that the solutions to a linear homogeneous differential equation satisfy all of the axioms of an abstract vector space. So when we are solving linear homogeneous differential equations we are really doing linear algebra.

The next important idea is to distinguish when a collection of solutions are really "different". This is not always so clear, especially in the case of higher order equations. We'll need some facts from linear algebra regarding the solvability of a linear system of equations.

The first is the idea of linear independence. A set of vectors $\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_k$ is linearly indpendent if $\sum c_i \vec{v}_i = 0$ implies that $c_i = 0$ for all i. In other words a set of vectors is linearly independent if the only linear combination of the vectors that adds up to the zero vector is when the coefficient of each vector individually is zero. We extend this same definition to functions

Definition 2.1.1. *A set of functions* $y_1(t), y_2(t), \ldots, y_n(t)$ *is linearly independent if* $\sum c_i y_i(t) = 0$ *implies that* $c_i = 0$ *for all i. A set of functions* $y_1(t), y_2(t), \ldots, y_n(t)$ *is linearly dependent if there is a linear combination* $\sum c_i y_i(t) = 0$ *where the coefficients* c_i *are not all zero.*

Referring back to the earlier example it is easy to see that the functions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are linearly independent. One way to see this is to use the trigonometric identity

$$c_1 \cos t + c_2 \sin t = \sqrt{c_1^2 + c_2^2} \cos(t - \phi) \qquad \phi = \arctan(\frac{c2}{c1}).$$

A second, and probably easier, way is to take the equation $c_1 \cos t + c_2 \sin t = 0$ and substitute in $t = 0$ and $t = \frac{\pi}{2}$ to find

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0. \end{aligned}$$

Similarly the functions $y_1(t) = \sin t$ and $y_2(t) = 2 \sin t$ are linearly dependent, since $2y_1 - y_2 = 0$.

The second is the following fact about the solutions of a set of n linear equations in n unknowns, which is a standard fact from any first course in linear algebra.

Theorem 2.1.2. *Given a set of n linear equations in n unknowns a_i*

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \ldots & M_{1n} \\ M_{21} & M_{22} & M_{23} \ldots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{2n} & M_{3n} \ldots & M_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \\ \vdots \\ y_0^{(n-1)} \end{pmatrix}$$

These equations have a unique solution if and only if the determinant is nonzero:

$$\begin{vmatrix} M_{11} & M_{12} & M_{13} \ldots & M_{1n} \\ M_{21} & M_{22} & M_{23} \ldots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{2n} & M_{3n} \ldots & M_{nn} \end{vmatrix} \neq 0.$$

If the determinant is zero then the equations either have no solution or have infinitely mny solutions.

As a practical matter if we have n solutions $y_1(t), y_2(t), \ldots y_n(t)$ to an n^{th} order linear differential equation

$$\frac{d^n y}{dt^n} + \sum_{k=0}^{n-1} a_k(t) \frac{d^k y}{dt^k} = 0$$

then what we would really like is to be able to express the solution to the equation with arbitrary initial conditions

$$\frac{d^n y}{dt^n} + \sum_{k=0}^{n-1} a_k(t) \frac{d^k y}{dt^k} = 0 \qquad y(t_0) = y_0; y'(t_0) = y_0'; \ldots y^{(n-1)}(t_0) = y_0^{(n-1)}$$

as a linear combination of our basis, $y(t) = a_1 y_1(t) + a_2 y_2(t) + \ldots a_n y_n(t)$. If one tries to solve for the coefficients $a_1, a_2 \ldots, a_n$ then we find a system of linear equations

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) & y_3(t_0)\ldots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & y_3'(t_0)\ldots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & y_3^{(n-1)}(t_0)\ldots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \\ \vdots \\ y_0^{(n-1)} \end{pmatrix}$$

We know that this system has a unique solution if the determinant of this matrix is non-zero. This motivates the following definition, which formalizes the idea that $y_1(t), y_2(t), \ldots y_n(t)$ should all be "different".

Definition 2.1.2. *The Wronskian determinant (or Wronskian) of a set of n functions $y_1(t), y_2(t), \ldots y_n(t)$ is defined to be the determinant*

$$W(y_1, y_2, \ldots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\ldots & y_n(t) \\ y_1'(t) & y_2'(t) & y_3'(t)\ldots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & y_3^{(n-1)}(t)\ldots & y_n^{(n-1)}(t) \end{vmatrix}$$

The following observation shows that the Wronskian is a way to detect the dependence or independence of a set of functions.

Observation 2.1.1. *Suppose that a set of functions $y_1(t), y_2(t), \ldots, y_n(t)$ are linearly dependent. Then the Wronskian is identically zero,*

Proof. If $y_1(t), y_2(t), \ldots, y_n(t)$ are linearly dependent then there exists constants $c_1, c_2, \ldots c_n$ not all zero such that $c_1 y_1(t) + c_2 y_2(t) + \ldots c_n y_n(t) = 0$. Differentiating with respect to t shows that $c_1 y_1'(t) + c_2 y_2'(t) + \ldots c_n y_n'(t) = 0$, and similarly all higher derivatives. Thus we have a linear combination of the columns of the matrix which sums to zero, and thus the determinant is zero. □

It is important that, given a linear differential equation and an arbitrary set of initial data, we be able to find a solution satisfying that initial data. The following theorem shows that the Wronskian is also the correct tool for determining when this is possible. The first theorem states that, if the Wronskian is non-zero at a point then we can find a linear combination satisfying any given initial condition.

Theorem 2.1.3. *If $y_1(t), y_2(t), \ldots, y_n(t)$ are n solutions to an n^{th} order linear homogeneous differential equation*

$$\frac{d^n y}{dt^n} + \sum_{i=0}^{n-1} a_i(t) \frac{d^i y}{dt^i} = 0$$

where the coefficients $a_i(t)$ are all continuous in a neighborhood of t_0 and and the Wronskian $W(y_1, y_2, \ldots, y_n)(t_0) \neq 0$ then **any solution** *to the initial vaue problem*

$$\begin{aligned}
&\frac{d^n y}{dt^n} + \sum_{i=0}^{n-1} a_i(t) \frac{d^i y}{dt^i} = 0 \\
&y(t_0) = y_0 \\
&y'(t_0) = y_0' \\
&\vdots \\
&y^{(n-1)}(t_0) = y_0^{(n-1)}
\end{aligned}$$

can be written as a linear combination of $y_1(t), y_2(t), \ldots, y_n(t)$.

To see this we simply apply the existence and uniqueness theorem. We know that solutions to differential equations are, under the hypothesis above, unique. Since we can always find solution in the form of a linear combination of $y_1(t), y_2(t) \ldots y_n(t)$ then all solutions are of this form. There is a it of subtlety here. Note that the Wronskian is a function of the independent variable, in this case t. If the functions $\{y_i(t)\}_{i=1}^n$ are linearly dependent then the Wronskian is identically zero – in other words zero for all t. The first theorem says that we can find a linear combination that satisfies any initial condition at t_0 provided that the Wronskian is not zero at t_0. The next thing that we will show is that if the functions $y_i(t)$ solve a linear differential equation with continuous coefficients then the Wronskian is either **NEVER** zero or it is **ALWAYS** zero. So a basis for solutions at one point will also be a basis for solutions at a different point.

Theorem 2.1.4 (Abel)**.** *Suppose that $y_1(t), y_2(t) \ldots y_n(t)$ are solutions to the linear homogeneous linear differential equation*

$$\frac{d^n y}{dt^n} + a_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + a_{n-2}(t) \frac{d^{n-2} y}{dt^{n-2}} + \ldots a_0(t) y = 0$$

Then the Wronskian $W(t)$ solves the FIRST ORDER homogeneous linear differential equation

$$W' + a_{n-1}(t) W = 0$$

Proof. We will prove this only for $n = 2$, which follows from a straightforward calculation. we have

$$\begin{aligned}
W(t) &= y_1 y_2' - y_1' y_2 \\
W'(t) &= y_1' y_2' + y_1 y_2'' - y_1' y_2' - y_1'' y_2 \\
&= y_1 y_2'' - y_1'' y_2
\end{aligned}$$

and so

$$\begin{aligned} W' + a_1(t)W &= y_1(y_2'' + a_1(t)y_2') - y_2(y_1'' + a_1(t)y_1') \\ &= y_1(-a_2(t)y_2) - y_2(-a_2(t)y_1) = 0 \end{aligned}$$

□

Corollary 2.1.1. *Suppose that $y_1(t), y_2(t)...y_n(t)$ are solutions to the linear homogeneous linear differential equation*

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + a_{n-2}(t)\frac{d^{n-2}y}{dt^{n-2}} + \ldots a_0(t)y = 0$$

with $a_i(t)$ continuous on the whole real line. Then the Wronskian is either identically zero or it is never zero,

Proof. We know that $W' + a_{n-1}(t)W = 0$, so we have that $W(t) = W(t_0)e^{\int_{t_0}^{2} a_{n-1}(s)ds}$. Since $a_{n-1}(t)$ is continuous we have that $\int_{t_0}^{t} a_{n-1}(s)ds$ is finite. The exponential function is never zero, so $e^{\int_{t_0}^{2} a_{n-1}(s)ds}$ is never zero. Thus the only way that $W(t)$ can be zero is if the constant $W(t_0) = 0$ in which case the Wronskian is identically zero. □

Example 2.1.2. *The differential equation*

$$y'' + y = 0$$

has solutions $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$. It follows from Abel's theorem that the Wronskian solves

$$W' = 0$$

in other words the Wronskian is a constant independent of t. We already know this to be the case – it is easy to compute that $W(t) = \cos^2(t) + \sin^2(t) = 1$.

Thus we know that $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ can satisfy an arbitrary set of initial conditions at any point t.

Example 2.1.3. *It is easy to verify that two solutions to*

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0$$

are $y_1 = t^2$ and $y_2 = t^3$. Notice that the coefficients are continuous everywhere except $t = 0$. It follows that the Wronskian solves

$$W' - \frac{4}{t}W = 0$$

which has the solution $W = ct^4$. Computing the Wronskian of y_1 and y_2 gives

$$W = y_1y_2' - y_1'y_2 = t^2(3t^2) - (2t)(t^3) = t^4$$

. The Wronskian is zero at $t = 0$ and non-zero everywhere else. Thus we can satisfy an arbitrary initial condition everywhere **EXCEPT** $t = 0$

2.2 *Linear constant coefficient equations: the characteristic polynomial*

We saw in the previous section that if one can construct n linearly independent solutions $y_1(t), y_2(t) \dots y_n(t)$ to an n^{th} order linear equation then the general solution is simply a linear superposition of these solutions $y(t) = \sum_i A_i y_i(t)$. This brings us to the question of how we find such solutions. Unfortunately we will generally not be able to solve most differential equations in terms of the familiar functions of calculus. However there is one important situation in which we **CAN** always solve differential equations in terms of functions like trigonometric functions, exponentials, and polynomials. This is the case of "constant coefficient" linear differential equations, in which the coefficients do not depend on the independent variable but are instead constant. The most general constant coefficient linear n^{th} order equation is given by

$$\frac{d^n y}{dt^n} + p_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots p_1\frac{dy}{dt} + p_0 y = 0$$

with $p_0, p_1, \dots p_{n-1}$ constants.

To begin with we can look for a solution of a particular form, that of an exponential function

$$y = Ae^{rt}.$$

If we substitute this into the above differential equation the condition that $y = e^{rt}$ to be a solution is that

$$(r^n + p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \dots p_1 r + p_0) = 0$$

The above is called the characteristic equation (sometimes the characteristic polynomial).

Definition 2.2.1. *Given the n^{th} order constant coefficient linear differential equation*

$$\frac{d^n y}{dt^n} + p_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots p_1\frac{dy}{dt} + p_0 y = 0$$

the characteristic equation is defined to be

$$(r^n + p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \dots p_1 r + p_0) = 0$$

The utility of the characteristic equation is the *for constant coefficient equations the solution can be reduced to the problem of finding the roots of a polynomial.* To illustrate this we first begin with a simple example.

Example 2.2.1. *Find the solution to the differential equation*

$$y'' - 3y' + 2y = 0$$
$$y(0) = 1$$
$$y'(0) = 4$$

Looking for a solution to $y'' - 3y' + 2y = 0$ *in the form* $y = e^{rt}$ *gives a characteristic equation* $r^2 - 3r + 2 = (r-1)(r-2) = 0$. *This gives two roots,* $r = 2$ *and* $r = 1$, *and thus two solutions* $y_1 = e^t$ *and* $y_2 = e^{2t}$. *The general solution is given by* $y = Ae^t + Be^{2t}$ *and* $y' = Ae^t + 2Be^{2t}$. *Imposing the conditions* $y(0) = 1, y'(0) = 4$ *gives two equations*

$$\begin{aligned} y(0) &= A + B = 1 \\ y'(0) &= A + 2B = 4 \end{aligned}$$

The simultaneous solution to these two equations is $A = -2, B = 3$.

We know from the fundamental theorem of algebra that an n^{th} degree polynomial always has exactly n roots, counted according to multiplicity. These roots will give us the n linearly independent solutions to differential equation. However it is a bit tricky to deal with the cases of complex roots and roots of higher multiplicity, so we begin with the simplest possible case, where the polynomial has n real distinct roots.

2.2.1 Real distinct roots

If the above has n distinct real roots then one obviously has n solutions $y_1 = e^{r_1 t}, y_2 = e^{r_2 t}, \ldots$. This motivates the following:

Theorem 2.2.1. *Suppose that the characteristic equation*

$$(r^n + p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \ldots p_1 r + p_0) = 0$$

has n real distinct roots $r_1, r_2, r_3, \ldots, r_n$. *Then the constant coefficient linear differential equation*

$$\frac{d^n y}{dt^n} + p_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots p_1 \frac{dy}{dt} + p_0 y = 0$$

has solutions $y_1 = e^{r_1 t}, y_2 = e^{r_2 t}, \ldots$.

It may not be completely obvious, but these solutions are linearly independent, as is shown in the next theorem:

Theorem 2.2.2. *If* $y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t}, \ldots, y_n(t) = e^{r_n t}$ *with* $r_1, r_2 \ldots r_n$ *distinct, then* $y_1(t), y_2(t) \ldots, y_n(t)$ *are linearly independent. In particular the Wronskian is given by*

$$W(y_1(t), y_2(t), \ldots, y_n(t)) = e^{(r_1 + r_2 + \ldots r_n)t} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (r_j - r_i)$$

We won't prove this, but it follows from an identity for the determinant of what is known as a Vandermonde matrix. The Wronskian is given as the product of an exponential (which never vanishes) times a product over the differences of roots. If all of the roots are distinct the difference between any two roots is non-zero, and thus the product is non-zero, and so the Wronskian is never zero.

Example 2.2.2. *Find three linearly independent solutions to*

$$y''' + 6y'' + 3y' - 10y = 0$$

The characteristic equation is given by

$$r^3 + 6r^2 + 3r - 10 = 0$$

It is easy to see that the roots are $r = 1_1, r_2 = -2$ *and* $r_3 = -5$*. This gives*

$$y_1(t) = e^t \qquad y_2(t) = e^{-2t} \qquad y_3(t) = e^{-5t}.$$

Since the roots are distinct the solutions are guaranteed to be linearly independent. In fact we know that the Wronskian satsifies

$$\begin{aligned} W(y_1, y_2, y_3) &= e^{(1-2-5)t}(-2-1)(-5-1)(-5-2) \\ &= (-3)(-6)(-3)e^{-6t} = -54e^{-6t} \end{aligned}$$

Since the characteristic equation for an n^{th} order linear constant coefficient equation is n^{th} degree polynomial will have n roots counted according to multiplicity. However the roots may be complex, and there may be roots of higher multiplicity – we may have to count certain roots more than once. Recall the following definition of multiplicity

Definition 2.2.2. *A number* r_0 *is a root of the polynomial* $P(r)$ *of multiplicity* k *if* $(r - r_0)^k$ *divides* $P(r)$ *and* $(r - r_0)^{k+1}$ *does not divide* $P(r)$
.

Exercise 2.1. *Below are a list of polynomials together with a root of that polynomial. Find the multiplicity of the root.*

$$\begin{aligned} &P(r) = (1 - r^2) && r = 1 \\ &P(r) = (1 - r)^2 && r = 1 \\ &P(r) = r^3 - 3r^2 + 4 && r = 2 \\ &P(r) = r^3 - 3r^2 + 4 && r = -1 \\ &P(r) = r^5 + r^3 && r = 0 \end{aligned}$$

There are some subtleties connected with complex roots and multiple roots. The next two subsections illustrate this.

2.2.2 *Complex distinct roots*

The case of complex roots is not much more difficult than that of real roots, if we remember the Euler formula.

Theorem 2.2.3 (Euler)**.** *The complex exponential $e^{i\omega t}$ can be expressed in terms of trigonometric functions as follows:*

$$e^{i\sigma t} = \cos \sigma t + i \sin \sigma t.$$

More generally we have that

$$e^{(\mu + i\sigma)t} = e^{\mu t} \cos \sigma t + i e^{\mu t} \sin \sigma t.$$

If r is a complex root of the characteristic equation then one can either work directly with the complex exponential e^{rt}, or one can work with the real and imaginary parts.

Theorem 2.2.4. *Suppose that the characteristic equation*

$$(r^n + p_{n-1} r^{n-1} + p_{n-2} r^{n-2} + \ldots p_1 r + p_0) = 0$$

has a pair of complex conjugate roots $r = \mu + i\omega$ and $\bar{r} = \mu - i\omega$. Then the constant coefficient linear differential equation

$$\frac{d^n y}{dt^n} + p_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots p_1 \frac{dy}{dt} + p_0 y = 0$$

has solutions $y_1(t) = e^{(\mu+i\omega)t}, y_2(t) = e^{(\mu - i\omega)t}$. Alternatively one can work with the real and imaginary parts

$$y_1(t) = e^{\mu t} \cos(\omega t) \qquad y_2(t) = e^{\mu t} \sin(\omega t)$$

Example 2.2.3 (Harmonic Oscillator)**.** *The differential equation*

$$my'' + ky = 0,$$

is an idealized equation for a mass-spring system called the simple harmonic oscillator. Here m represents the mass and k the linear spring constant.

The characteristic equation is

$$mr^2 + k = 0,$$

with roots given by

$$r = \sqrt{-\frac{k}{m}} = i\sqrt{\frac{k}{m}}.$$

This gives two solutions: $y_1 = e^{i\omega t}, y_2 = e^{-i\omega t}$, with $\omega = \sqrt{\frac{k}{m}}$ These are perfectly acceptable complex solutions to the differential equation. In engineering, particularly in electrical engineering, it is standard to work with complex valued solutions. If one would prefer to have real-valued solutions then one can use the fact that any linear combination of solutions is also a solution. Taking the special linear combinations

$$\frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) = \cos \omega t$$
$$\frac{1}{2i}(e^{i\omega t} - e^{-i\omega t}) = \sin \omega t$$

shows that $y_1(t) = \cos \omega t$ and $y_2(t) = \sin \omega t$ are (linearly independent) real solutions to the differential equation $my'' + ky = 0$.

There are a couple of idealizations being made here. First we are neglecting damping and loss terms. We are also assuming a perfectly Hooke's law spring – one in which the restoring force is proportional to the displacement. Real springs have mechanical loss due to friction, air resistance, etc, and are not perfectly linear. Nevertheless it is a very common and useful model.

Figure 2.1: Robert Hooke (1635–1703) was a natural philosopher and polymath. Originally employed at Oxford as an organist he became assistant to Robert Boyle. He first hypothesized

The other complication that can arise is when the characteristic polynomial has roots of higher multiplicity.

2.2.3 *Multiple roots*

The other exceptional case occurs when the characteristic equation has multiple roots. As always we begin with an example.

Example 2.2.4. *Consider the differential equation*

$$y'' - 2y' + y = 0.$$

looking for a solution in the form $y = e^{rt}$ gives the characteristic equation

$$r^2 - 2r + 1 = (r-1)^2 = 0$$

so $r = 1$ is a root of multiplicity two. This gives one solution as $y_1 = e^t$. We need to find a second linearly independent solution. This can be done by the following trick: we can factor the differential operator as follows: if we define $w = y' - y$ then $y'' - 2y' + y = 0$ is equivalent to $w' - w = 0$. The solution to $w' - w = 0$ is given by $w = Ae^t$. The equation $y' - y = w$ is then equivalent to $y' - y = Ae^t$. This can be solved by the integrating factor method detailed in the first chapter to give $y = Ate^t + Be^t$.

Theorem 2.2.5. Constant coefficient equations *Suppose that*

$$\frac{d^n y}{dt^n} + p_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots p_1\frac{dy}{dt} + p_0 y = 0$$

is a constant coefficient n^{th} order differential equation, and that

$$(r^n + p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \ldots p_1 r + p_0) = 0$$

is the characteristic polynomial. Recall that the total number of roots of the polynomial counted according to mutliplicity is n. For each simple root (multiplicity one) r_i we get a solution

$$y_i(t) = \exp(r_i t)$$

If r_i is a repeated root of the characteristic equation of multiplicity k then the differential equation has k linearly independent solutions given by

$$\begin{aligned}
y_1(t) &= e^{r_i t} \\
y_2(t) &= te^{r_i t} \\
y_3(t) &= t^2 e^{r_i t} \\
&\vdots \\
y_k(t) &= t^{k-1} e^{r_i t}
\end{aligned}$$

If $r = \mu + i\sigma, r^* = \mu - i\sigma$ are a complex conjugate pair of roots one can either use the complex exponential solutions

$$y_1(t) = e^{\mu + i\sigma)t}$$
$$y_2(t) = e^{\mu - i\sigma)t}$$

or the real form of the solutions

$$y_1(t) = e^{\mu t}\cos(\sigma t)$$
$$y_2(t) = e^{\mu}\sin(\sigma t)$$

Example 2.2.5. *Solve the boundary value problem*

$$y''' - 3y'' + 3y' - y = 0 \qquad y(0) = 0 \qquad y'(0) = 0 \qquad y''(1) = 1$$

Solution 2.1. *The characteristic equation is $r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0$. There is a root $r = 1$ of multiplicity $k = 3$. This gives three linearly independent solutions $y_1(t) = e^t, y_2(t) = te^t, y_3(t) = t^2e^t$. Taking the general solution as $y(t) = Ae^t + Bte^t + ct^2e^t$ we get*

$$y(t) = Ae^t + Bte^t + ct^2e^t$$
$$y'(t) = Ae^t + Bte^t + Be^t + Ct^2e^t + 2Cte^t$$
$$y''(t) = Ae^t + Bte^t + 2Be^t + Ct^2e^t + 4Cte^t + 2Ce^t$$

This gives three equations in three unknowns

$$y(0) = A = 0$$
$$y'(0) = A + B = 0$$
$$y''(0) = A + 2B + 2C = 1$$

which can be solved to find $A = 0, B = 0, C = \frac{1}{2}$

Example 2.2.6. *Solve the initial value problem*

$$y'''' + 2y''' - 2y' - y = 0 \qquad y(0) = 0 \qquad y'(0) = 0 \qquad y''(0) = 0 \qquad y'''(0) = 1$$

Solution 2.2. *The characteristic equation is*

$$r^4 + 2r^3 - 2r - 1 = 0$$

The roots are $r = -1$ with multiplicity $k = 3$ and $r = 1$ with multiplicity $k = 1$. Therefore a linearly independent set of solutions is given by

$$y_1(t) = e^{-t}$$
$$y_2(t) = te^{-t}$$
$$y_3(t) = t^2e^{-t}$$
$$y_4(t) = e^t$$

Writing the general solution as $y(t) = Ae^{-t} + Bte^{-t} + Ct^2e^{-t} + De^t$ we can solve for the coefficients as follows:

$$\begin{aligned}
&y(t) = Ae^{-t} + Bte^{-t} + Ct^2e^{-t} + De^t \\
&y(0) = A + D = 0 \\
&y'(t) = -Ae^{-t} - Be^{-t}t + Be^{-t} - Ce^{-t}t^2 + 2Ce^{-t}t + De^t \\
&y'(0) = -A + B + D = 0 \\
&y''(t) = Ae^{-t} + Be^{-t}t - 2Be^{-t} + Ce^{-t}t^2 - 4Ce^{-t}t + 2Ce^{-t} + De^t \\
&y''(0) = A - 2B + 2C + D = 0 \\
&y'''(t) = -Ae^{-t} - Be^{-t}t + 3Be^{-t} - Ce^{-t}t^2 + 6Ce^{-t}t - 6Ce^{-t} + De^t \\
&y'''(0) = -A + 3B - 6C + D = 1
\end{aligned}$$

Solving this system of four equations for the unknowns A, B, C, D gives $A = -\frac{1}{8}, B = -\frac{1}{4}, C = -\frac{1}{4}, D = \frac{1}{8}$.

2.2.4 Exercises

Find the general solutions to the following differential equations

$$y''' + 3y'' + 3y' + y = 0 \tag{2.1}$$

$$y'''' + 2y'' + y = 0 \tag{2.2}$$

$$y''' - 3y'' + 4y = 0 \tag{2.3}$$

$$y''' - 5y'' + y - 5 = 0 \tag{2.4}$$

$$y'''' - 8y''' + 16y'' = 0 \tag{2.5}$$

2.3 Non-homogeneous linear equations: Operator notation and the structure of solutions

In the previous section we found a method for solving a general constant coefficient linear homogeneous equation by finding the roots of the characteristic polynomial. We next consider the non-homogeneous case, where the forcing term $f(t)$ is non-zero. We will begin with the method of undetermined coefficients, and the closely related Annihilator Method. These methods only work for certain forcing terms $f(t)$, but when they do work they are typically much easier than method of variation of parameters, which we will learn later.

2.3.1 Operator notation

We are going to use the following notation. $\mathcal{L}$ will be a linear differential operator: something that acts on functions. For instance if we define the linear operator $\mathcal{L}$ to be

$$\mathcal{L} = \frac{d^2}{dt^2} + 5\frac{d}{dt} + 4$$

then the operator $\mathcal{L}$ is something that acts on functions and returns another function. For instance $\mathcal{L}$ acting on a function y gives

$$\mathcal{L}y = y'' + 5y' + 4y.$$

So

$$\begin{aligned}
&\mathcal{L}e^t = e^t + 5e^t + 4e^t = 10e^t, \\
&\mathcal{L}e^{-2t} = 4e^{-2t} - 10e^{-2t} + 4e^{-2t} = -2e^t, \\
&\mathcal{L}\sin t = -\sin t + 5\cos t + 4\sin t = 5\cos t + 3\sin t
\end{aligned}$$

Similarly if we define the linear operator $\mathcal{L}$ to be

$$\mathcal{L} = (1+t^2)\frac{d^2}{dt^2} + 9t\frac{d}{dt} - e^t$$

then

$$\mathcal{L}y = (1+t^2)y'' + 9ty' - e^t y$$

Note that in each of these cases the operator $\mathcal{L}$ has the property that $\mathcal{L}(ay_1(a) + by_2(t)) = a\mathcal{L}y_1(t) + a\mathcal{L}y_2(t)$. This is called a linear operator. This follows from the fact that the derivative (and hence the n^{th} derivative) is a linear operator, and so a linear combination of these is also a linear operator. Notice that a linear homogeneous differential equation can always be written in the form

$$\mathcal{L}y = 0$$

for some choice of linear operator $\mathcal{L}$, while a linear non-homogeneous equation can be written in the form

$$\mathcal{L}y = f(t)$$

It will often be useful to use the operator notation, particularly when we are discussing linear differential equations in the abstract and don't necessarily want to discuss a particular equation.

2.4 *The structure of solutions to a non-homogeneous linear differential equation*

This section is short, but contains an important observation about the structure of the solution to a non-homogeneous linear equation.

Theorem 2.4.1. *Suppose that the non-homogeneous differential*

$$\mathcal{L}y = f(t)$$

has a solution $y_{part}(t)$ (called the particular solution). This solution need not involve any arbitrary constant. Then the general solution to

$$\mathcal{L}y = f(t)$$

is given by

$$y(t) = y_{part}(t) + y_{homog}(t)$$

where $y_{homog}(t)$ is the general solution to

$$\mathcal{L}y = 0.$$

It is common in engineering texts, though not in mathematics texts, to further separate the solution into "zero input response" and "zero state response" functions. The zero input response function solves the homogeneous problem $\mathcal{L}y = 0$ together with the appropriate initial conditions. The zero state response function solves the non-homogeneous problem $\mathcal{L}y = f$ together with zero initial conditions. In other words the zero input solution is the homogeneous solution satisfying the correct initial conditions zero forcing term and the zero state response solution is a particular solution satisfying zero initial conditions and the correct forcing term.

Proof. The proof is easy, given the operator notation. Suppose that $y(t)$ is the general solution to

$$\mathcal{L}y = f(t)$$

while $y_{part}(t)$ is a particular solution, $\mathcal{L}y_{part}(t) = f(t)$. Let's look at $y(t) - y_{part}(t)$. We have that

$$\mathcal{L}(y(t) - y_{part}(t)) = \mathcal{L}y(t) - \mathcal{L}y_{part}(t) = f(t) - f(t) = 0$$

so $y(t) - y_{part}(t) = y_{homog}(t)$ is a solution to the homogeneous problem. □

In summary, if we can solve the homogeneous problem and find one particular solution then we know the general solution.

Example 2.4.1. *Find the general solution to*

$$y'' + y = t$$

It isn't hard to see that if $y(t) = t; y'(t) = 1; y''(t) = 0$ and so $y'' + y = 0 + t = t$. So the general solution is the particular solution $y_p(t) = t$ plus a solution to the homogeneous problem. The homogeneous problem is

$$y'' + y = 0$$

which has the solution $y_{homog}(t) = A\sin t + B\cos t$. Therefore the general solution to

$$y'' + y = t$$

is given by $y = t + A\sin t + B\cos t$.

Example 2.4.2. *Find the zero input response and zero state response to*

$$y' + y = 1 \qquad y(0) = 2$$

The zero input response satisfies the differential equation

$$y_{zi}' + y_{zi} = 0 \qquad y_{zi}(0) = 2$$

which has the solution

$$y_{zi}(t) = 2e^{-t}.$$

The zero state response satisfies the differential equation

$$y_{zs}' + y_{zs} = 1 \qquad y_{zs}(0) = 0$$

It is not hard to guess that a particular solution is $y_{part}(t) = 1$ which gives the zero state response as

$$y_{zs} = 1 - e^{-t}.$$

The complete solution is the sum

$$y(t) = (1 - e^{-t}) + 2e^{-t} = 1 + e^{-t}$$

The question now is how to find particular solutions. The first method that we will present is the the method of undetermined coefficients, which works for constant coefficient differential equations with a righthand side taking a particular form.

2.5 *The method of undetermined coefficients*

The method of undetermined coefficients is a way to solve *certain* non-homogeneous linear differential equations of the form

$$\mathcal{L}y = f(t)$$

where $\mathcal{L}$ is a *constant coefficient* differential operator - in other words

$$\mathcal{L} = \frac{d^n}{dt^n} + p_1 \frac{d^{n-1}}{dt^{n-1}} + \ldots p_n$$

with $p_1, p_2, \ldots p_n$ constant, and $f(t)$ is a forcing term that can be written as an

- Polynomial in t
- $\sin(\omega t)$ or $\cos(\omega t)$
- Exponential e^{at},

or sums and products of terms of this form (but not compositions or ratios!). For instance it would work for $f(t)$ of any of the following forms:

$$f(t) = \cos(t)$$
$$f(t) = t^5$$
$$f(t) = e^{-6t}$$
$$f(t) = e^t \cos(t) + t^{11} \sin(t) - 256 sin^3(3t)$$
$$f(t) = t^3 e^{-t} \cos(t) + t^2 \sin(5t) - 11$$
$$f(t) = e^t - 22t \cos(5t)$$

But it would not work for $f(t)$ of the following forms

$$f(t) = e^{t^2}$$
$$f(t) = \tan(t)$$
$$f(t) = \frac{cos(t)}{t}$$

The basic idea can be summarized as follows:

It takes a bit of experience to be able to see what is the right form. We will shortly talk about the annihilator method, which will *always* give you the correct form to guess, but is a little bit more involved to apply in practice. Note that it is not a problem to include extra terms, as you will find in the course of applying the method that the coefficients of the unnecessary terms are zero.

Method 2.5.1 (Undetermined Coefficients (Naive)). *To solve a solve a constant coefficient linear differential equation of the form*

$$\mathcal{L}y = \left(\frac{d^n}{dt^n} + \sum_{i=0}^{n-1} p_i \frac{d^i}{dt^i}\right)y = f(t)$$

where $f(t)$ is given by sums and products of

- *Exponentials e^{at}*
- *Sine or Cosine $\sin \omega t$ or $\cos \omega t$*
- *Polynomial $P_n(t) = \sum a_i t^i$.*

You should

- *Make a guess (ansatz) for $y(t)$ in* the same form as $f(t)$ *, with undetermined coefficients $A_1, A_2, \ldots A_n$.*
- *Substitute your guess into the differential equation.*
- *Solve for the coefficients A_i. These should be constants.*

As an example consider the differential equation

If you include a $\sin t$ term you should always include the $\cos t$ term, and vice-versa. Basically this is because the form is closed under derivatives – the derivative of a linear combination of *sint* and $\cos t$ is again a linear combination of $\sin t$ and $\cos t$.

$$y'' + 3y' - 4y = \sin t.$$

The forcing term $f(t) = \sin t$, so we guess something in the same form, a trigonometric function $y(t) = A_1 \sin t + A_2 \cos t$. Substituting this guess into the original equation gives

$$
\begin{aligned}
y &= A_1 \sin t + A_2 \cos t \\
y' &= A_1 \cos t - A_2 \sin t \\
y'' &= -A_1 \sin t - A_2 \cos t \\
y'' + 3y' - 4y &= -(5A_1 + 3A_2) \sin t + (-5A_2 + 3A_1) \cos t.
\end{aligned}
$$

We want to have $y'' + 3y' - 4y = \sin t$, so we need

$$
\begin{aligned}
-(5A_1 + 3A_2) &= 1 \\
(-5A_2 + 3A_1) &= 0
\end{aligned}
$$

since that will give us the result that we want. We can solve for A_1 and A_2 to find $A_1 = -\frac{5}{34}; A_2 = -\frac{3}{34}$.

Exercise 2.2. *Suppose that in the example above one tried a solution of the form $y = A_1 \sin t + A_2 \cos t + A_3 e^{2t}$. Show that you must have $A_3 = 0$.*

It can sometimes be a little confusing as to the proper form of the guess, so here is a table that you might find helpful. Note that we will see in a second that these forms occasionally need to be modified depending on the solutions to the homogeneous equation.

(Usually!) The form of solution for method of undetermined coefficients.

$f(t)$	$y(t)$
$f(t) = t^k$	$y(t) = A_0 + A_1 t + A_2 t^2 \ldots A_k t^k = P_k(t)$
$f(t) = e^{\sigma t}$	$y(t) = Ae^{\sigma t}$
$f(t) = \sin \omega t$ or $f(t) = \cos \omega t$	$y(t) = A \sin \omega t + B \cos \omega t$
$f(t) = t^k \sin \omega t$ or $f(t) = t^k \cos \omega t$	$y = P_k(t) \sin \omega t + Q_k(t) \cos \omega t$
$f(t) = \sin \omega t e^{\sigma t}$ or $f(t) = \cos \omega t e^{\sigma t}$	$y(t) = A \sin \omega t e^{\sigma t} + B \cos \omega t e^{\sigma t}$
$f(t) = t^k e^{\sigma t}$	$y = P_k(t) e^{\sigma t}$
$f(t) = t^k e^{\sigma t} \sin \omega t$ or $f(t) = t^k e^{\sigma t} \sin \omega t$	$y = P_k(t) e^{\sigma t} \sin \omega t + Q_k(t) e^{\sigma t} \cos \omega t$

Exercise 2.3. *(1) State the form you should try to guess for the solution to the following equations and (2) solve the equation.*

$$y''' + 5y' - 6y = e^{5t} \cos(4t) \tag{2.6}$$

$$y'' + 2y' + y = \cos(t) + t^2 e^t \tag{2.7}$$

$$y''' + 5y' - 6y = e^t \tag{2.8}$$

$$y'' + 3y' + 2y = e^{-t} \tag{2.9}$$

Unfortunately although the "naive" version of undetermined coefficients does not always work. The last example in the previous exercise shows that sometimes the naive method as described here fails: the solution is not quite of the same form as the righthand side. Rather than being an exponential it is te^{-t} times an exponential. This suggests the following rule of thumb.

Method 2.5.2 (Better Undetermined Coefficients Method)**.** *To solve a constant coefficient differential equation*

$$\mathcal{L}(y) = f(t)$$

guess $y(t)$ of the same form as $f(t)$ unless *one or more of the terms of your guess is itself a solution to the homogeneous equation*

$$\mathcal{L}y = 0$$

in this case multiply these terms by the smallest power of t such that none of the terms in your guess satisfy the homogeneous equation.

Example 2.5.1. *Consider the differential equation*

$$\frac{d^2y}{dt^2} - y = t^2e^t + \sin t$$

Normally when we see t^2e^t we would guess $y = At^2e^t + Bte^t + Ce^t + D\sin t + E\cos t$. In this case the solutions to the homogeneous problem $\frac{d^2y}{dt^2} - y = 0$ has two linearly indpendent solutions $y_1 = e^t$ and $y_2 = e^{-t}$. That means that we should try a solution of the form $y = At^3e^t + Bt^2e^t + Cte^t + D\sin t + E\cos t$. Note that we don't multiply the $\sin t$ or $\cos t$ terms by t, as they are not solutions to the homogeneous equation.

Example 2.5.2. *Consider the differential equation*

$$\frac{d^4y}{dt^4} - 2\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t + 5\sin t + 3e^t + 2te^t$$

The characteristic equation for the homogeneous problem is

$$r^4 - 2r^3 + 2r^2 - 2r + 1 = 0$$

This has four roots. $r = \pm 1$ are simple roots and $r = 1$ is a double root. This gives four solutions to the homogeneous problem

$$\begin{aligned} y_1(t) &= \sin t \\ y_2(t) &= \cos t \\ y_3(t) &= e^t \\ y_4(t) &= te^t t \end{aligned}$$

Normally for the right-hand side $f(t) = t + \sin t + e^t + te^t$ we would guess something of the following form

$$y(t) = \underbrace{A + Bt}_{t} + \underbrace{C\sin t + D\cos t}_{5\sin t} + \underbrace{Ee^t + Fte^t}_{3e^t + 2te^t}.$$

Here the underbraces show the term(s) in $f(t)$ that are responsible for the terms in $y(t)$. However we have some exceptions to make here: some of the *terms in our guess are solutions to the homogeneous problem: The functions* $\sin t, \cos t, e^t$ *and* te^t *all solve the homogeneous equation. We should mulitply the solutions by the smallest power of* t *so that* no terms in the guess solve the homogeneous equation. *The function* $A + Bt$ *doesn't solve the homogeneous problem, so we don't need to change these terms. The functions* $\sin t, \cos t$ *do, but* $t \sin t$ *and* $t \cos t$ *do not, so we multiply these terms by* t*. The functions* e^t *and* te^t *both solve the homogeneous equation. If we multiply by* t *we get* $Ete^t + Ft^2e^t$*. One of these terms still solves the homogeneous problem. If we multiply by* t^2 *we get* $Et^2e^t + Ft^3e^t$*, none of which solves the homogeneous problem. Thus we should guess*

$$y(t) = \underbrace{A + Bt}_{t} + \underbrace{Ct \sin t + Dt \cos t}_{5 \sin t} + \underbrace{Et^2e^t + Ft^3e^t}_{3e^t + 2te^t}.$$

The particular solution works out to be

$$y(t) = \frac{1}{6}t^3e^t + \frac{1}{4}t^2e^t + \frac{5}{4}t \sin t + t + 2.$$

Finding the correct form of the solution in the method of undetermined coefficients becomes a bit cumbersome when the characteristic equation has roots of high multiplicity. There is a variation of this method, usually called the annihilator method, This method is a little more work but it is always clear what the correct form of the solution should be. This is the subject of the next section.

2.6 The Annihilator Method

Now to present this in a slightly different perspective: the right-hand sides that are allowed are those for which the derivatives are in the same form: the derivative of a polynomial is a polynomial, the derivative of an exponential is an exponential, the derivative of $A_1e^t \cos(t) + A_2e^t \sin(t)$ is a linear combination of $e^t \cos(t)$ and $e^t \sin(t)$. Since all of these functions have the property that the derivative is of the same form it follows that a constant coefficient linear combination of them is also of the same form. So it makes sense to look for a solution that is *also* in the same form.

Another way to think of these functions is as the functions $f(t)$ that are allowed are those which can arise from solving a constant coefficient homogeneous differential equation. This is the basic idea of the method of annihilators: one finds an "annihilator", a differential operator that annihilates the righhand side.

The basic algorithm is as follows:

Method 2.6.1 (Annihilator Method)**.** *To solve the linear constant coefficient inhomogeneous differential equation*

$$\mathcal{L}y = f(t)$$

where $f(t)$ is given by sums and products of polynomials, exponentials, sines and cosines.

1. *Find a set of linearly independent solutions to the homogeneous problem $\mathcal{L}y = 0$.*

2. *Find a linear constant coefficient operator $\tilde{\mathcal{L}}$ such that $\tilde{\mathcal{L}}f = 0$*

3. *Act on both sides of the equation with $\tilde{\mathcal{L}}$ to get*

$$\tilde{\mathcal{L}}\mathcal{L}y = \tilde{\mathcal{L}}f = 0$$

4. *Find a set of linearly independent solutions to the homogeneous problem $\tilde{\mathcal{L}}\mathcal{L}y = 0$.*

5. *The solution will be in the form of a linear combination of all functions in (4) that* **DO NOT APPEAR IN** *(1)*

Example 2.6.1. *Solve the equation*

$$\frac{d^2y}{dt^2} + y = e^t$$

using annihilators.

To do this we first find the solution to the homogeneous problem. The characteristic equation for $\frac{d^2y}{dt^2} + y = 0$ is $r^2 + 1 = 0$. Solving for r gives $r = \pm i$ or $y_1(t) = \sin t, y_2(t) = \cos t$

We next need to find an annihilator, something that "kills" the righthand side. The operator $\frac{d}{dt} - 1$ does the trick: if we act on the function e^t with this operator we get zero. Acting on the above equation with $\frac{d}{dt} - 1$ gives

$$y''' - y'' + y' - y = 0$$

The characteristic equation is $r^3 - r^2 + r - 1 = 0$. This has three roots $r = \pm i, r = 1$. Three linearly independent solutions are $y_1 = \sin t; y_2 = \cos t; y_3 = e^t$. The first two are solutions to the original equation, but the third is not. Thus we should try a particular solution of the form $y = Ae^t$. Substituting in to the original equation gives

$$y'' + y = 2Ae^t = e^t$$

Thus $2A = 1$ and $A = \frac{1}{2}$. Note that if we mistakenly included the other terms $B \sin t + C \cos t$ we would find that B and C are arbitrary.

Example 2.6.2. *Find the correct form of the solution for the method of undetermined coefficients for the equation*

$$\frac{d^6y}{dt^6} + 3\frac{d^4y}{dt^4} + 3\frac{d^2y}{dt^2} + y = 2\sin t + 5\cos t$$

The homogeneous equation is

$$\frac{d^6y}{dt^6} + 3\frac{d^4y}{dt^4} + 3\frac{d^2y}{dt^2} + y = 0$$

so that the characteristic equation is

$$r^6 + 3r^4 + 3r^2 + 1 = 0$$

which can be written as $(r^2+1)^3 = 0$*, so* $r = \pm i$ *are roots of multiplicity* 3*. This means that the six linearly independent solutions are*

$$\begin{aligned} y_1 &= \cos t \\ y_2 &= t\cos t \\ y_3 &= t^2\cos t \\ y_4 &= \sin t \\ y_5 &= t\sin t \\ y_6 &= t^2\sin t \end{aligned}$$

The annihilator for $A\sin t + B\cos t$ *is* $\tilde{\mathcal{L}} = \frac{d^2}{dt^2} + 1$*. Acting on both sides of the equation with the annihilator gives*

$$(\frac{d^2}{dt^2} + 1)(\frac{d^6y}{dt^6} + 3\frac{d^4y}{dt^4} + 3\frac{d^2y}{dt^2} + y) = 0.$$

The characteristic equation is

$$(r^2+1)^4 = 0$$

This has eight linearly independent solutions:

$$\begin{aligned} y_1 &= \cos t \\ y_2 &= t\cos t \\ y_3 &= t^2\cos t \\ y_4 &= t^3\cos t \\ y_5 &= \sin t \\ y_6 &= t\sin t \\ y_7 &= t^2\sin t \qquad\qquad y_8 = t^3\sin t \end{aligned}$$

There are two solutions of $(\frac{d^2}{dt^2}+1)(\frac{d^6y}{dt^6} + 3\frac{d^4y}{dt^4} + 3\frac{d^2y}{dt^2} + y) = 0$ *that do not solve* $\frac{d^6y}{dt^6} + 3\frac{d^4y}{dt^4} + 3\frac{d^2y}{dt^2} + y = 0$*. These are* $t^3\sin t$ *and* $t^3\cos t$*. Therefore the particular solution can be aassumed to take the form* $At^3\sin t + Bt^3\cos t$*.*

For most cases this method is overkill – it is typically easier to use the naive method of undetermined coefficients – but it can sometimes be tricky to figure out what the correct guess should be. Here is a list of various forcing terms $f(t)$ and their annihilators.

$f(t)$	Annihilator
1	$\frac{d}{dt}$
$p_k(t)$	$\frac{d^{k+1}}{dt^{k+1}}$
e^{at}	$\frac{d}{dt} - a$
$A\sin\omega t + B\cos\omega t$	$\frac{d^2}{dt^2} + \omega^2$
$A\sin\omega te^{at} + B\cos\omega te^{at}\ p_k(t)e^{at}$	$(\frac{d}{dt} - a)^2 - \omega^2$
$P_k(t)\sin\omega t + Q_k(t)\cos\omega t$	$(\frac{d^2}{dt^2} + \omega^2)^{k+1}$
$P_k(t)\sin\omega te^{at} + Q_k(t)\cos\omega te^{at}$	$((\frac{d}{dt} - a)^2 + \omega^2)^{k+1}$

Note that $P_k(t)$ and $Q_k(t)$ denote polynomials of degree k. Note that the coefficients of the polynomials P, Q should be different.

Exercise 2.4. *Find particular solutions to the following differential equations*

1. $\frac{dx}{dt} + x = e^t$
2. $\frac{dx}{dt} + 3x = \sin(t) + e^{2t}$
3. $4\frac{dx}{dt} + 5x = t\sin(t) + e^{-t}$
4. $\frac{d^2x}{dt^2} + x = te^{-t}$
5. $\frac{d^2x}{dt^2} + 2fracdxdt + 5x = e^{-t} + \cos(t)$

Exercise 2.5. *Find annihilators for the following forcing functions*

1. $f(t) = t^2 + 5t$
2. $f(t) = \sin(3t)$
3. $f(t) = 2\cos(2t) + 3t\sin(2t)$
4. $f(t) = t^2e^t + \sin(t)$

3
Mechanical and electrical oscillations

3.1 Mechanical oscillations

3.1.1 Undamped mass-spring systems

Mechanical systems are ultimately governed by Newton's law, $F = ma$. For simple (one degree of freedom) systems this means that the governing equations are second order, since the acceleration is the second derivative of the position. The force is provided by Hooke's law, which says that the force is proportional to the displacement. This gives the equation for a mass m and spring with spring constant k with no damping or external forcing as

$$mx'' = -kx.$$

The general solution takes the form

$$x(t) = A\cos(\sqrt{\frac{k}{m}}t) + B\sin(\sqrt{\frac{k}{m}}t)$$

This is what we expect: in the absence of damping the solutions oscillate periodically, with a natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$, a quantity that will be important in our later analysis.

In the case where there is an external forcing $f(t)$the equation becomes

$$mx'' = -kx + f(t).$$

Often one can assume that the forcing is some periodic function. For instance for an automobile suspension one source of forcing is the rotation of the wheels, which is a periodic phenomenon. Similarly if one drives over a grooved road at a constant speed this generates a periodic forcing. It is simplest to model this by a simple $\sin(\omega t)$ or $\cos(\omega t)$ term, so we get a model like

$$mx'' = -kx + \sin(\omega t)$$

A couple of things to note about this formula. First a bit of units analysis. The spring constant k has units $\mathrm{kg/s^2} = \mathrm{N/m}$ while the mass m has units kg, so that ω_0 has units s^{-1}. This is what we expect: the quantity $\omega_0 t$ should be in radians, which are unitless.

It is also good to think about the scaling: since $\omega_0 = \sqrt{\frac{k}{m}}$ this implies that large masses oscillate more slowly (lower frequency) as do smaller spring constants. This jibes with our intuition about how an oscillating mass should behave.

Let's try to solve this. We know from the method of undetermined coefficients that the correct guess should (usually!) be to try to find a particular solution of the form

$$x_{\text{part}}(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t)$$

Substituting this into the equation give

$$\begin{aligned} m x''_{\text{part}} + k x_{\text{part}} &= (k - m\omega^2)(A_1 \cos(\omega t) + A_2 \sin(\omega t)) \\ &= \sin(\omega t) \end{aligned}$$

Thus we need to choose $A_1 = 0$ and $A_2 = \frac{1}{k - m\omega^2}$. The general solution is the sum of the particular solution and the solution to the homogeneous problem, so

$$x(t) = A \cos(\sqrt{\frac{k}{m}} t) + B \sin(\sqrt{\frac{k}{m}} t) + \frac{\sin(\omega t)}{(k - m\omega^2)}$$

You should notice a few things about this solution.

- The solution has positive amplitude for ω^2 small - it is in phase with the driver.
- The solution has negative amplitude for ω^2 - it is π out of phase with the driver.
- When the driving frequency ω is close to $\sqrt{\frac{k}{m}}$ the amplitude of the response becomes large.
- The amplitude is undefined at $\omega = \pm\sqrt{\frac{k}{m}} = \pm\omega_0$.

This phenomenon is called "resonance". In engineering and the sciences resonance is an extremely important phenomenon. In some situations resonant is a good thing; if one is designing a concert hall or a musical instrument or a speaker cabinet having the correct resonant frequencies can greatly improve the sound. When designing structures such as buildings that are subject to external forces resonance is usually something to be avoided. Either way it is important to understand the phenomenon and how to cause it (or prevent it). While it may look like it from the formula it is not actually true that the amplitude is infinite when $\omega^2 = \frac{k}{m}$. Rather, since the forcing term is itself a solution to the homogenous equation we need to look for a solution of a different form. We know what to guess in this case: a linear combination of $t \sin(t)$ and $t \cos(t)$. It is easy to check that this works.

For some dramatic incidents of unplanned resonance see the footage of the collapse of the Tacoma Narrows Bridge in Washington (AKA galloping Gerty) or the oscillations of the Millenium Bridge in London.

Exercise 3.1. *Verify that a particular solution to*

$$m x'' = -kx + F_0 \sin(\sqrt{\frac{k}{m}} t)$$

is given by

$$x_{part}(t) = -F_0 \frac{t\cos(\sqrt{\frac{k}{m}}t)}{2\sqrt{km}}$$

Exercise 3.2. *Suppose that the suspension of a car with bad shock absorbers can be modelled as a mass-spring system with no damping. Assume that putting a* 100kg *mass into the trunk of the car causes the car to sink by* 2cm*, and that the total mass of the car is 1000*kg*. Find the resonant frequency of the suspension, in* s^{-1}.

You will first need to find the spring constant of the suspension. For simplicity take $g = 10\mathrm{ms}^{-2}$.

3.1.2 Mass-spring systems with damping

In real physical systems one usually has some form of damping as well. In the context of a mass-spring system damping is usually modelled by a term proportional to the velocity of the system, $\frac{dx}{dt}$, but having the opposite sign. This acts to slow the movement of the mass and decrease oscillations. In mechanical systems a dashpot is a component that provides a damping force, usually assumed to be linear in velocity and of opposite sign. Screen doors on homes, for example, frequently have a dashpot to keep the door from slamming shut. As a second example the suspension of a car has two main components, springs and shocks. The springs are basically Hooke's law springs, and shocks are dashpots that provide a damping force.

In the case where we consider a mass-spring system with linear damping Newton's law becomes

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt} + f(t)$$

where, as always, $f(t)$ represents some external forcing term.

First we look at the homogeneous equation.

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0.$$

The characteristic equation is

$$mr^2 + \gamma r + k = 0$$

which has roots

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$

There are two cases here, with very different qualitative behaviors. in the case $\gamma^2 < 4km$, or equivalently $\frac{\gamma}{\sqrt{km}} < 2$, the characteristic

Again it is worthwhile doing some dimensional analysis. The coefficient γ, when multiplied by a velocity, should give a force, so γ should have units $\mathrm{kg\,s}^{-1}$. Notice that from the three quantities m, γ, k we can form the quantity $\frac{\gamma}{\sqrt{mk}}$, which is dimensionless. This is a measure of how important damping is in the system. If $\frac{\gamma}{\sqrt{mk}}$ is small then damping is less important compared with inertial (mass-spring) effects. If $\frac{\gamma}{\sqrt{mk}}$ is large it means that inertial effects are small compared with damping. This will become important in our later analysis.

polynomial has a complex conjugate pair of roots. In this situation the two linearly independent solutions are given by

$$y_1(x) = e^{-\frac{\gamma}{2m}x}\cos(\sqrt{\frac{k}{m} - \frac{\gamma^2}{4km}}x) \tag{3.1}$$

$$y_2(x) = e^{-\frac{\gamma}{2m}x}\sin(\sqrt{\frac{k}{m} - \frac{\gamma^2}{4km}}x) \tag{3.2}$$

In this case the solutions to the homogeneous problem consist of exponentially decaying oscillations. This is called the underdamped case due to the presence of these sustained oscillations – in mechanical systems such sustained oscillations are *usually* unwanted.

In the case $\gamma^2 > 4km$, or $\frac{\gamma}{\sqrt{km}} > 2$, the roots of the characteristic polynomial are real and negative, and the solutions to the homogeneous problem consist of decaying exponentials, with no sustained oscillations. This is known as the overdamped case. In most mechanical situations one would like to be in the critically damped or slightly overdamped case.

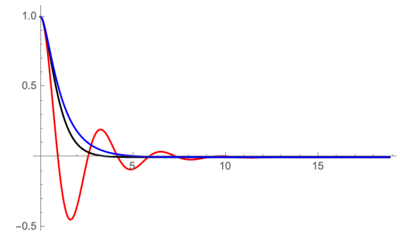


Figure 3.1: The solutions to the equation $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dx} + 4x = 0$ with $x(0) = 1; x'(0) = 0$ for $\gamma = 1, 4, 5$ (shown in red, black, and blue respectively) representing the underdamped, critically damped and overdamped cases. Note that the solution decays fastest in the critically damped case.

Exercise 3.3. *Let us revisit an exercise from the previous subsection. Suppose that the suspension of a car can be modeled as a mass-spring system with damping. Assume that putting a* 100kg *mass into the trunk of the car causes the car to sink by* 2cm*, and that the total mass of the car is 1000*kg*. Assume that for best performance one would like the suspension of the car to be critically damped. How should the value of the damping coefficient* γ*, in* $\mathrm{kg\,s^{-1}}$*, be?*

The borderline case $\frac{\gamma}{\sqrt{km}} = 2$ is called the critically damped case. In this situation the characteristic polynomial has a double root.

In engineering the reciprocal quantity $Q = \frac{\sqrt{km}}{\gamma}$ is sometimes denoted as the quality factor or Q-factor of the system, and measures the quality of the resonator, with a high Q representing a system with strong resonance. Examples of mechanical systems where one would like to have a high Q include musical instruments, tuning forks, etc. The Q for a tuning form is typically of the order of 1000, so it will vibrate for a long time before the sound dies out.

Exercise 3.4. *Verify that, in the critically damped case, the two solutions to the homogeneous problem are given by*

$$x_1(t) = e^{-\frac{\gamma}{2m}t}$$

$$x_2(t) = te^{-\frac{\gamma}{2m}t}$$

Now let's consider the inhomogeneous damped harmonic oscillator with a sinusoidal external forcing term

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dx} + kx = F_0\sin(\omega t).$$

If we look for a solution of the form

$$y = A_1\cos(\omega t) + A_2\sin(\omega t)$$

and substitute this in to the equation we get the following set of equations for A_1 and A_2:

$$(k - m\omega^2)A_1 + \gamma\omega A_2 = 0 \tag{3.3}$$

$$(k - m\omega^2)A_2 - \gamma\omega A_1 = F_0. \tag{3.4}$$

The first equation comes from requiring that the $\cos(\omega t)$ terms sum to zero, the second from demanding that the $\sin(\omega t)$ terms sum to 1. The solution to these two linear equations is given by

$$\left((k - m\omega^2)^2 + \gamma^2\omega^2\right) A_1 = F_0\gamma\omega \tag{3.5}$$

$$\left((k - m\omega^2)^2 + \gamma^2\omega^2\right) A_2 = F_0(k - m\omega^2) \tag{3.6}$$

This solution is often expressed in a somewhat different way. One can use the angle addition formulas

$$\sin(\omega t + \phi) = \sin(\omega t)\cos(\phi) + \cos(\omega t)\sin(\phi)$$
$$\cos(\omega t + \phi) = \cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)$$

to express a linear combination of $\sin(\omega t)$ and $\cos(\omega t)$ in the form

$$A_1\cos(t) + A_2\sin(t) = A\sin(\omega t + \phi).$$

The amplitude A and phase ϕ are given by

$$A = \sqrt{A_1^2 + A_2^2}$$
$$\phi = \arctan(A_1/A_2).$$

One can think of the mechanical system as transforming the forcing term by changing the amplitude and phase of the sinusoid. This point of view is particularly prevalent when talking about RLC circuits, which we will cover next. In our case using the expression for A_1, A_2 derived above we find that

$$A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}}$$
$$\phi = \arctan(\frac{\gamma\omega}{k - m\omega^2}).$$

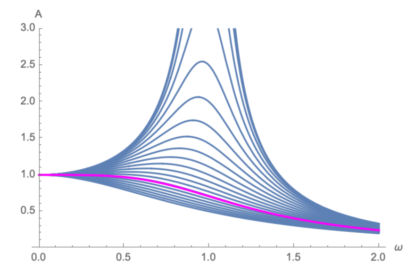


Figure 3.2: A plot of $A = \frac{1}{\sqrt{(k-m\omega^2)^2+\gamma^2\omega^2}}$ as a function of ω for $k = 1, m = 1$ and different values of γ between $\gamma = 0$ and $\gamma = 2$

Figure 3.2 shows the plot of A as a function of ω for $k = 1, m = 1$ and various values of γ between $\gamma = 0$ and $\gamma = 2$. Notice that there are two different behaviors depending on the damping coefficient γ. When $\gamma < \sqrt{2km}$ the curve has a local minimum at $\omega = 0$ and a global maximum at $\omega = \pm\sqrt{\frac{2km-\gamma^2}{2m^2}}$. When $\gamma < \sqrt{2km}$ the curve has a global maximum at $\omega = 0$ and it decreases as $|\omega|$ increases. The magenta curve represents the value $\gamma = \sqrt{2km}$.

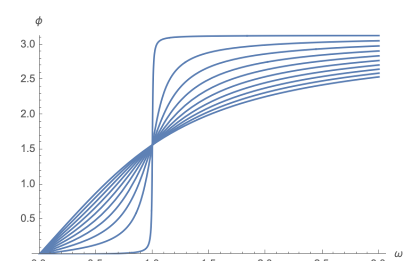


Figure 3.3: A plot of $\phi = \arctan(\frac{\gamma\omega}{k-m\omega^2})$ as a function of ω for $k = 1, m = 1$ and different values of γ between $\gamma = 0$ and $\gamma = 2$

Figure 3.3 shows a graph of the phase shift $phi = \arctan(\frac{\gamma\omega}{k-m\omega^2})$ as a function of ω for $k = 1, m = 1$ and various values of the damping γ. The phase shift varies from 0 to π. It is always $\phi = \frac{\pi}{2}$ at frequency $\omega = \sqrt{\frac{k}{m}}$, and for small damping (high Q) the phase looks approximately like a step function – it is close to zero for $\omega < \sqrt{\frac{k}{m}}$ and close to π for $\omega > \sqrt{\frac{k}{m}}$. In the case of weak damping ($\frac{\gamma}{\sqrt{km}} \ll 1$)

this means that if the system is driven at frequencies below the "natural" frequency $\omega = \sqrt{\frac{k}{m}}$ then the response will be in phase with the forcing term, while if the system is driven at a frequency above the natural frequency then the response will be 180° out of phase with the forcing.

Exercise 3.5. *Suppose that a car with bad shocks can be modeled as a mass-spring system with a mass of* $m = 750\mathrm{kg}$, *a spring constant of* $k = 3.75 \times 10^5\ \mathrm{N/m}$, *and a damping coefficient of* $\gamma = 2 \times 10^4 \mathrm{kg\,s^{-1}}$, *and that the car is subject to a periodic forcing of* $f(t) = 1000\mathrm{N}\sin(150t)$ *due to an unbalanced tire. Find the particular solution. What is the amplitude of the resulting oscillations?*

3.2 RC and RLC circuits

3.2.1 RC Circuits

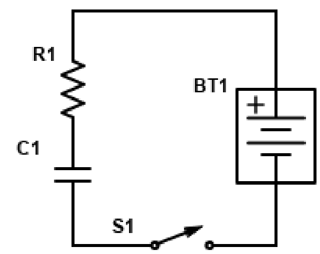


Figure 3.4: A simple circuit consisting of a battery, switch, resistor and capacitor.

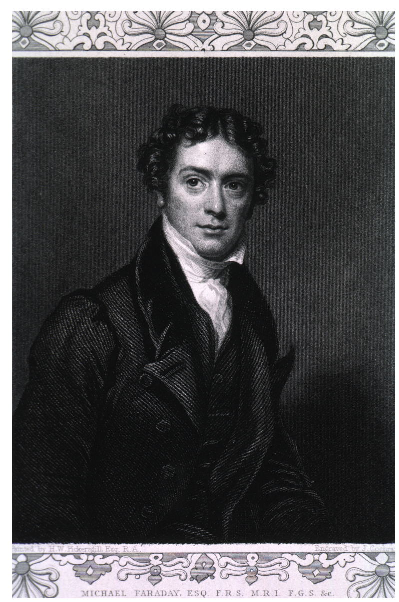


Figure 3.5: The unit of capacitance, the farad, is named after Michael Faraday. Despite growing up poor and leaving school at an early age to apprentice to a bookbinder Faraday became one of the preeminent scientists of his day.

Another important application for second order constant coefficient differential equations is to RLC circuits, circuits containing resistance (R), inductance (L) and capacitance (C). As we will see the role that these quantities play in circuits is analogous to the role that damping coefficient γ, the mass m and the spring constant k play in mass-spring systems.

We begin with a very simple circuit consisting of a resistor and a capacitor in series with a battery and a switch. Suppose that the battery supplies a voltage V_0 and that at time $t = 0$ the switch is closed, completing the circuit. We would like to derive the differential equation governing the system, and solve it. To begin with let V_R denote the voltage drop across the resistor and V_C denote the voltage drop across the capacitor. These should add up (by Kirchhoff's law) to the voltage supplied by the battery:

$$V_R + V_C = V_0.$$

Now both V_R and V_C can be expressed in terms of the current. If $i(t)$ denotes the current at time t then we have that

$$\begin{aligned} V_R &= iR \\ V_C &= \frac{1}{C}\int_0^t i(s)ds \end{aligned}$$

In order to get a differential equation we must differentiate $V_R + V_C = V_0$ and apply the fundamental theorem of calculus to find that

$$\frac{dV_R}{dt} + \frac{dV_C}{dt} = R\frac{di}{dt} + \frac{1}{C}i = \frac{dV_0}{dt} = 0,$$

since the battery supplies a constant voltage. Thus the current satisfies $R\frac{di}{dt} + \frac{1}{C}i = 0$. To find the initial condition we note that at time $t = 0$ the voltage drop across the capacitor is $V_C = \frac{1}{C}\int_0^0 i(t)dt = 0$, and thus $V_R(0) = Ri(0) = V_0$, and the differential equation becomes

$$R\frac{di}{dt} + \frac{1}{C}i = 0 \qquad i(0) = \frac{V_0}{R}.$$

This can be solved to give

$$i(t) = \frac{V_0}{R}e^{-\frac{t}{RC}},$$

from which the other quantities can be derived. For instance $V_R = iR = V_0 e^{-\frac{t}{RC}}$. The voltage drop across the capacitor, V_C can be found either by using $V_C + V_R = V_0$ or by taking $V_C = \frac{1}{C}\int_0^t i(s)ds$ and doing the integral. Both methods give $V_C = V_0(1 - e^{-\frac{t}{RC}})$. Note that a resistance R ohms times a capacitance of C Farads gives time in seconds. This is usually called the RC time constant – one common application of RC circuits is to timing.

Example 3.2.1. *A circuit consists of a* 5V *battery, a* 5 kΩ *resistor, a* 1000μF *capacitor and a switch in series, as in Figure 3.6. At time $t = 0$ the switch is closed. At what time does V_C, the voltage drop across the capacitor, equal* 4V*?*

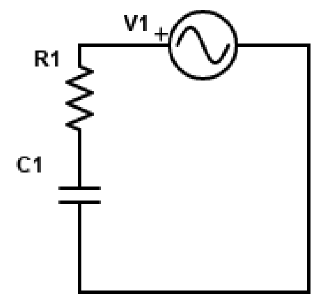


Figure 3.6: A simple circuit consisting of a resistor, a capacitor, and a sinusoidally varying voltage.

A somewhat more interesting problem is when the voltage is not constant (DC) but instead varies in time. If, for instance, the voltage varies sinusoidally time as $V(t) = V_0 \sin(\omega t)$. Essentially the same derivation above gives the equation

$$R\frac{di}{dt} + \frac{1}{C}i = \frac{dV(t)}{dt} = V_0\omega\cos(\omega t).$$

This can be solved by the method of undetermined coefficients, looking for a particular solution of the form $A_1\cos(\omega t) + A_2\sin(\omega t)$. Alternatively one can use the formula derived in the section on mass spring systems with the replacements $m \to 0, \gamma \to R, k \to \frac{1}{C}$ and $F_0 \to V_0\omega$. Either way we find the solution n many problems with damping such as this the homogeneous solution is exponentially decaying. This is often called the "transient response" and can, in many situations, be neglected.

1

$$i_{\text{part}}(t) = \frac{V_0 RC^2\omega^2}{(\omega RC)^2 + 1}\sin(\omega t) + \frac{V_0\omega C}{(\omega RC)^2 + 1}\cos(\omega t).$$

The homogeneous solution $i_{\text{homog}}(t) = Ae^{-\frac{t}{RC}}$ is exponentially decaying, so we will assume that enough time has passes that this term is negligibly small.

From here any other quantity of interest can be found. For instance V_C, the voltage drop across the capacitor, is given by

$$\begin{aligned} V_C = V_R - V(t) &= Ri(t) - V_0 \sin(\omega t) \\ &= \frac{V_0 \sin(\omega t)}{1 + (\omega RC)^2} - \frac{V_0 R\omega C \cos(\omega t)}{1 + (\omega RC)^2} \\ &= \frac{V_0}{\sqrt{1 + (\omega RC)^2}} \sin(\omega t + \phi) \end{aligned}$$

where $\phi = \arctan(-\omega RC)$.

Again note that ω has units s^{-1} and RC has units s so ωRC is dimensionless. Dimensionless quantities are an important way to think about a system, since they do not depend on the system of units used. If ωRC is small it means that the period of the sinusoid is much longer than the time-constant of the RC circuit, and it should behave like a constant voltage. In this case, where $\omega RC \approx 0$ it is not hard to see that $V_C \approx V_0 \sin(\omega t)$. If ωRC is large, on the other hand, the voltage drop across the capacitor will be small, $V_C \approx -\frac{V_0 \cos(\omega t)}{\omega RC}$. This is the simplest example of a low-pass filter. Low frequencies are basically unchanged, while high frequencies are damped out.

3.2.2 *RLC Circuits, Complex numbers and Phasors*

THERE IS A THIRD TYPE OF PASSIVE COMPONENT that is interesting from a mathematical point of view, the inductor. An inductor is a coil of conducting wire: the voltage drop across an inductor is proportional to the rate of change of the current through the inductor. The proportionality constant, denoted by L, is the inductance and is measured in Henrys. The derivation of the response of an RLC circuit to an external voltage is similar to that given for the RC circuit. Kirchhoff's las implies that the total voltage drop across all of the components is equal to the imposed voltage. Denoting the voltage drops across the inductor, resistor and capacitor by V_I, V_R, V_C respectively we have that

$$V_L + V_R + V_C = L\frac{dI(t)}{dt} + RI(t) + \frac{1}{C}\int I = V(t).$$

Taking the derivative gives a second order differential equation for the current $i(t)$

$$L\frac{d^2I(t)}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt}. \tag{3.7}$$

It is again worth thinking a bit about units. The quantity RC has units of time – seconds if R is measured in Ohms and C in Farads. The quantity $\frac{L}{R}$ is also a unit of time, seconds if L is measured in Henrys and R in Ohms. The quantity $\frac{R^2C}{L}$ is dimensionless. The equation is *overdamped*, with exponentially decaying solutions, if $\frac{R^2C}{L} > 4$ and is underdamped, with solutions in the form of an exponentially damped sine or cosine, if $\frac{R^2C}{L} < 4$.

Note that all of the results of the section on mechanical oscillations translates directly to this equation, with inductance being analogous to mass, resistance to damping coefficient, the reciprocal of capacitance to the spring constant, current to displacement and the derivative of voltage to force. All of the results of the previous section apply here, if one translates between from the mechanical quantities to the analogous electrical quantities, so we will not repeat those calculations here. Rather we will take the opportunity to introduce a new, and much easier way to understand these equations through the use of complex numbers and what are called "phasors". While it requires a little bit more sophistication the complex point of view makes the tedious algebraic calculations that we did in the previous section unnecessary: all that one has to be able to do is elementary operations on complex numbers.

In many situations one is interested in the response of a circuit to a sinusoidal voltage. The voltage from a wall outlet, for instance, is sinusoidal. If the voltage is sinusoidal, $V(t) = V_0 \cos(\omega t)$ then from (3.7) the basic differential equation governing the current $I(t)$ is

$$L\frac{d^2I(t)}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt} = -\omega V_0 \sin(\omega t).$$

We could use the method of undetermined coefficients here, as we did in the previous section, and looks for a particular solution in the form $I(t) = A\cos(\omega t) + B\sin(\omega t)$ but it is a lot easier to use the Euler formula and complex arithmetic. Recall the Euler formula says

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t).$$

We will begin by replacing the real voltage $V(t) = V_0 \cos(\omega t)$ with a complex voltage $\tilde{V}(t) = V_0 e^{i\omega t}$, so that $\text{Re}(V_0 e^{i\omega t}) = V_0 \cos(\omega t)$. Here Re denotes the real part. This seems strange but it is just a mathematical trick: since the equations are linear one can solve for the complex current and then take the real part, and that gives the particular solution. In other words in order to solve

$$L\frac{d^2I(t)}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt} = V_0 \,\text{Re}\left(i\omega e^{i\omega t}\right)$$

because the equations are linear this is the same as solving

$$\text{Re}\left(L\frac{d^2\tilde{I}(t)}{dt^2} + R\frac{d\tilde{I}}{dt} + \frac{1}{C}\tilde{I}\right) = \frac{dV}{dt} = \text{Re}\left(i\omega V_0 e^{i\omega t}\right)$$

which is the same as solving

$$L\frac{d^2I(t)}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt} = iV_0\omega e^{i\omega t}$$

and then taking the real part of the complex current $I(t)$. This will be a lot simpler for the following reason: instead of looking for a solution as a linear combination of $\cos(\omega t)$ and $\sin(\omega t)$ we can just look for a solution in the form $\tilde{I}(t) = Ae^{i\omega t}$. The constant A will be complex but we won't have to solve any systems of equations, etc – we will just have to do complex arithmetic. When we are done we just have to take the real part of the complex solution $\tilde{I}(t)$ and we get the solution to the original problem.

To begin we consider the case of an RC circuit with imposed voltage $V(t) = V_0 \cos(\omega t)$:

$$R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt} = -\omega V_0 \cos(\omega t).$$

Complexifying the voltage $V(t) = V_0 e^{i\omega t}$ we get $\frac{dV}{dt} = i\omega V_0 e^{i\omega t}$ and thus

$$R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt} = i\omega V_0 e^{i\omega t}.$$

Using undetermined coefficients we look for a solution in the form $I(t) = I_0 e^{i\omega t}$. Since this guess already contains both the $\sin(\omega t)$ and $\cos(\omega t)$ terms we do not need any additional terms. This is what makes the method easier. Substituting this into the equation gives

It is worth noting here that $RC\omega$ is dimensionless, so $\frac{1}{\omega C}$ has units of Ohms.

$$(i\omega R + \frac{1}{C}) I_0 e^{i\omega t} = i\omega V_0 e^{i\omega t}$$

which is the same as

$$\begin{aligned} I_0 &= \frac{i\omega V_0}{i\omega R + \frac{1}{C}} \\ &= \frac{V_0}{R - \frac{i}{\omega C}} \\ &= V_0 \frac{R + \frac{i}{\omega C}}{R^2 + \frac{1}{\omega^2 C^2}} \end{aligned}$$

So the complex current is just the complex voltage multiplied by the complex number $\frac{R+\frac{i}{\omega C}}{R^2+\frac{1}{\omega^2 C^2}}$. This should remind you of Ohm's law: $V = IR$ or $I = \frac{V}{R}$. In fact we have $V_0 = (R - \frac{i}{\omega C}) I_0$ so it looks like Ohm's law with a complex resistance, called impedance. If the capacitance is really big, so $\frac{1}{\omega C}$ is really small then this is exactly the usual Ohm's law.

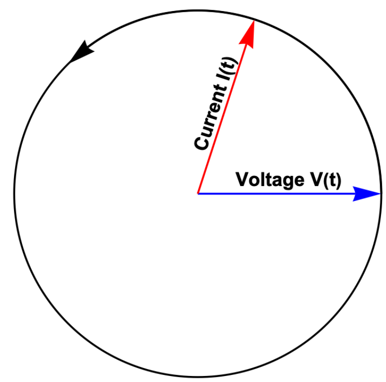


Figure 3.7: The particular solution for an RC-circuit with a sinusoidal voltage. The current $I(t)$ will lead the voltage by an angle between 0 and $\frac{\pi}{2}$. As time increases the picture rotates counterclockwise but the angle between the voltage and the current does not change.

Recall that complex numbers can be identified with points in the plane. The function $V_0 e^{i\omega t}$ represents a point that rotates counterclockwise around a circle of radius V_0 at a uniform angular frequency. Also recall that when we multiply complex numbers the magnitudes multiply and the arguments add. The complex number (impedance) $(R - \frac{i}{\omega C})$ lies in the fourth quadrant. Since $V_0 = (R - \frac{i}{\omega C}) I_0$ this means that the voltage always lags behind the current, or equivalently the current leads the voltage. Remember that we set things up so that the voltage and current rotate counterclockwise. If $R = 0$ then $V_0 = -\frac{i}{\omega C}) I_0$ and the voltage is exactly $\frac{\pi}{2}$ behind the current. If C is large then $V_0 \approx I_0 R$ and the voltage and the current are in phase.

Next let's consider an LR circuit, which is dominated by resistance and inductance, again with a voltage $V(t) = V_0 \cos(\omega t)$. This time we are solving

$$L \frac{d^2 I(t)}{dt^2} + R \frac{dI}{dt} = i V_0 \omega e^{i\omega t}.$$

Again looking for a solution of the form $I(t) = I_0 e^{i\omega t}$ we find

$$(-\omega^2 L + i\omega R) I_0 e^{i\omega t} = i\omega V_0 e^{i\omega t}$$

or

$$V_0 = (R + i\omega L) I_0 \qquad \text{or} \qquad I_0 = \frac{V_0}{(R + i\omega L)} = \frac{R - i\omega L}{R^2 + \omega^2 L^2} V_0$$

Since the complex number (impedance) $(R - i\omega L)$ lies in the fourth quadrant this implies that the voltage leads the current, or equivalently the current lags the voltage. This is illustrated in the marginal figure.

Of course any real circuit will have inductance, resistance and capacitance. In this case we have

$$L\frac{d^2I(t)}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = iV_0\omega e^{i\omega t}.$$

Looking for a solution of the form $I(t) = I_0 e^{i\omega t}$ we find that

$$(-\omega^2 L + i\omega R + \frac{1}{C})I_0 = i\omega V_0 \qquad \text{or} \qquad V_0 = I_0(R + i(\omega L - \frac{1}{\omega C})).$$

The impedance $(R + i(\omega L - \frac{1}{\omega C}))$ can either be in the fourth or the first quadrant, depending on the frequency ω and the sizes of the inductance L and the capacitance C. If $\omega L > \frac{1}{\omega C}$ or $\omega^2 LC > 1$ then $(R + i(\omega L - \frac{1}{\omega C}))$ lies in the first quadrant and the voltage leads the current. If $\omega^2 LC < 1$ then the voltage lags the current. If $\omega^2 LC = 1$ then the impedance is real and the voltage and the current are in phase. We have encountered this condition before – this is (one) definition of effective resonance.

The relative phase between the voltage and the current in an AC system is important, and is related to the "power factor". For reasons of efficiency it is undesirable to have too large an angle between the voltage and the current in a system: the amount of power that can be delivered to a device is $V_{rms}I_{rms}\cos(\theta)$, where θ is the angle between the current and the voltage. If this angle is close to $\frac{\pi}{2}$ then very little power can be delivered to the device since $\cos(\theta$ is small. Many industrial loads, such as electric motors, have a very high inductance due to the windings. This high inductance will usually be offset by a bank of capacitors to keep the angle between the voltage and the current small.

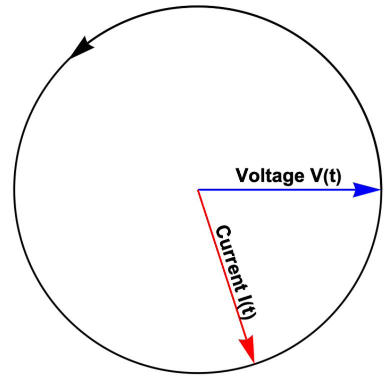


Figure 3.8: The particular solution for an RL-circuit with a sinusoidal voltage. The current $I(t)$ will lag behind the voltage by an angle between 0 and $\frac{\pi}{2}$. As time increases the picture rotates counterclockwise but the angle between the voltage and the current does not change.

There is a mnemonic "ELI the ICE man", to remember the effects of inductance and capacitance. In an inductive circuit (L is inductance) the voltage E leads the current I. In a capacitive circuit (C is capacitance) the current I leads the voltage E.

Example 3.2.2. *Solve the linear constant coefficient differential equation*

$$\frac{d^3y}{dt^3} + 4\frac{dy}{dt} + 5y = \cos(2t)$$

using complex exponentials.

We can replace this with the complex differential equation

$$\frac{d^3z}{dt^3} + 4\frac{dz}{dt} + 5z = e^{2it}$$

and then take the real part. Looking for a solution in the form $z(t) = Ae^{2it}$

we find that

$$\begin{aligned}
&A((2i)^3 + 4(2i) + 5)e^{2it} = e^{2it}\\
&5Ae^{2it} = e^{2it}\\
&A = \frac{1}{5}.
\end{aligned}$$

Notice that in this case the constant A worked out to be purely real, although in general we expect it to be complex. This gives a particular solution. $z = \frac{1}{5}e^{2it}$. Taking the real part gives $y(t) = \mathrm{Re}(z(t)) = \mathrm{Re}(e^{2it}/5) = \frac{1}{5}\cos(2t)$.

The homogeneous solution is given by the solution to

$$\frac{d^3z}{dt^3} + 4\frac{dz}{dt} + 5z = 0.$$

The characteristic polynomial is

$$r^3 + 4r + 5 = 0$$

which has the three roots $r = -1, r = \frac{1 \pm \sqrt{19}\, i}{2}$. Thus the general solution to

$$\frac{d^3y}{dt^3} + 4\frac{dy}{dt} + 5y = \cos(2t)$$

is

$$y = \frac{1}{5}\cos(2t) + A_1e^{-t} + A_2e^{\frac{t}{2}}\cos(\frac{\sqrt{19}t}{2}) + A_3e^{\frac{t}{2}}\sin(\frac{\sqrt{19}t}{2})$$

or alternatively in terms of complex exponentials as

$$y = \frac{1}{5}\cos(2t) + A_1e^{-t} + A_2e^{\frac{1+i\sqrt{19}}{2}t} + A_3e^{\frac{1-i\sqrt{19}}{2}t}$$

Example 3.2.3. *Solve the differential equation*

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = \sin(t)$$

As before we can solve this by solving the complex equation

$$\frac{d^3z}{dt^3} - 3\frac{d^2z}{dt^2} + 2\frac{dz}{dt} = e^{it}$$

and taking the imaginary part. Looking for a solution in the form $z(t) = Ae^{it}$ we find that

$$A((i)^3 - 3(i)^2 + 2(i))e^{it} = e^{it} \tag{3.8}$$

$$A(3 + i) = 1 \tag{3.9}$$

$$A = \tfrac{1}{3+i} = \tfrac{3-i}{10}. \tag{3.10}$$

The particular solution is thus

$$\begin{aligned} y(t) &= \operatorname{Im}(\frac{3-i}{10}e^{it}) \\ &= \operatorname{Im}(\frac{3-i}{10}(\cos(t) + i\sin(t)) \\ &= \frac{3}{10}\sin(t) - \frac{1}{10}\cos(t) \end{aligned}$$

The characteristics polynomial for the homogeneous equation is given by

$$r^3 - 3r^2 + 2r = 0$$

which has roots $r = 0, r = 1, r = 2$. *This gives the general solution as*

$$y(t) = \frac{3}{10}\sin(t) - \frac{1}{10}\cos(t) + A_1 + A_2e^t + A_3e^{2t}$$

4 Systems of ordinary differential equations

4.1 Introduction

Systems of differential equations arise in many ways. In some cases the differential equations of physics and engineering are most naturally posed as systems of equations. For instance in the absence of charges and currents Maxwell's equations for the electric field $\boldsymbol{E}(x,t)$ and the magnetic field $\boldsymbol{B}(x,t)$ are given by

Throughout this section quantities in lower-case bold will generally represent vector quantities, while those in upper case bold represent matrices. In this case, however, we keep with tradition and represent the electric and magnetic fields with upper case $\boldsymbol{E}$ and $\boldsymbol{B}$ respectively.

$$\begin{aligned}\frac{\partial \boldsymbol{B}}{\partial t} &= -\nabla \times \boldsymbol{E}\\ \mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t} &= \nabla \times \boldsymbol{B}.\end{aligned}$$

This is a system of six partial differential equations relating the first partial derivatives of the 3-vectors $\boldsymbol{E}(x,t)$and $\boldsymbol{B}(x,t)$. In other situations it may be desirable to write a higher order equation as a system of first order equations. Take, for instance, the second order equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t).$$

If we introduce the new variable $z(t) = \frac{dy}{dt}$ and using the fact that $\frac{dz}{dt} = \frac{d^2y}{dt^2}$ then the second order equation above can be written as the system of equations

$$\begin{aligned}\frac{dy}{dt} &= z\\ \frac{dz}{dt} &+ p(t)z + q(t)y = f(t).\end{aligned}$$

This system of equations can be written as

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}(t)\boldsymbol{v} + \boldsymbol{g}(t).$$

Here the vector quantities $\boldsymbol{v}(t), \boldsymbol{g}(t)$ and the matrix $\boldsymbol{A}(t)$ are given by

$$\boldsymbol{v}(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \qquad \boldsymbol{g}(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \qquad \boldsymbol{A}(t) = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}.$$

More generally if we have an n^{th} order equation

$$\frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + p_1(t)\frac{dy}{dt} + p_0(t)y = f(t)$$

we can do the change of variables to rewrite this as n first order equations

$$\begin{aligned}
z_1(t) &= \frac{dy}{dt} \\
z_2(t) &= \frac{dz_1}{dt}(= \frac{d^2y}{dt^2}) \\
&\vdots \\
z_{n-1}(t) &= \frac{dz_{n-2}}{dt}(= \frac{d^{n-1}y}{dt^{n-1}}) \\
\frac{dz_{n-1}}{dt} &= -p_{n-1}(t)z_{n-1} - p_{n-2}z_{n-2} - \ldots - p_1 z_1 - p_0 y + f(t).
\end{aligned}$$

This is a set of n first order equations in $y, z_1, z_2, \ldots z_{n-1}$. There are, of course, many ways to rewrite an n^{th} order equation as n first order equations but this is in some sense the most standard.

From this point on we will consider the general system of n first order linear equations

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}(t)\boldsymbol{v} + \boldsymbol{g}(t). \tag{4.1}$$

All of the terminology and results from the section on higher order linear equations carries over naturally to systems of first order equations: in fact in many cases the definitions and formulae look more natural in the setting of a first order system.

4.1.1 *Homogeneous linear first order systems.*

As in the case of scalar equations a homogeneous equation is one for which there is no forcing term – no term that is not proportional to y. If the case of a first-order system a homogeneous equation is one of the form

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}(t)\boldsymbol{v}.$$

We begin with an existence/uniqueness theorem.

Theorem 4.1.1. *Consider the initial value problem*

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}(t)\boldsymbol{v} \qquad \boldsymbol{v}(t_0) = \boldsymbol{v}_0. \tag{4.2}$$

Suppose that $\boldsymbol{A}(t)$ depends continuously on t in some interval I containing t_0. Then (4.2) has a unique solution in the interval I.

Next we define two important notions, the idea of a Wronskian and linear independence and the idea of a fundamental solution matrix.

Definition 4.1.1. *Suppose that $\boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \boldsymbol{v}_3(t) \ldots \boldsymbol{v}_n(t)$ are n solutions to equation 4.2. The Wronskian $W(t)$ of these solutions is defined to be the determinant of the matrix with columns $\boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \boldsymbol{v}_3(t) \ldots \boldsymbol{v}_n(t)$. We say that the vectors are linearly independent if $W(t) \neq 0$.*

At this point it is worth doing an example to verify that this definition of the Wronskian agrees with our previous definition of the Wronskian.

Example 4.1.1. *Suppose that we have a second order equation*

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

with $y_1(t)$ and $y_2(t)$ two linearly independent solutions. Previously we defined the Wronskian to be $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$. We can write this second order equation as a pair of first order equations by introducing a new variable $z(t) = \frac{dy}{dt}$. As we showed earlier $y(t), z(t)$ satisfy

$$\begin{aligned} \frac{dy}{dt} &= z \\ \frac{dz}{dt} &= -p(t)z - q(t)y. \end{aligned}$$

Since $y_1(t), y_2(t)$ are two solutions to the original second order equation we have that two solutions to the system of equations are given by

$$\boldsymbol{v}_1(t) = \begin{bmatrix} y_1(t) \\ y_1'(t) \end{bmatrix} \qquad \boldsymbol{v}_2(t) = \begin{bmatrix} y_2(t) \\ y_2'(t) \end{bmatrix}.$$

The Wronskian of these two solutions is given by

$$W(t) = \det(\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

in agreement with the previous definition. More generally if we take an n^{th} order linear equation and rewrite it as a system of n first order equations in the usual way then the Wronskian of a set of solutions to the n^{th} order equation will agree with the Wronskian of the corresponding solutions to the system.

Definition 4.1.2. *Suppose that $\boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \boldsymbol{v}_3(t) \ldots \boldsymbol{v}_n(t)$ are n solutions to equation 4.2 whose Wronskian is non-zero. We define a fundamental solution matrix $\boldsymbol{M}(t)$ to be a matrix whose columns are given by $\boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \boldsymbol{v}_3(t) \ldots \boldsymbol{v}_n(t)$.*

We note a few facts about $\boldsymbol{M}(t)$.

- The determinant of a fundamental solution matrix $\boldsymbol{M}(t)$ is the Wronskian $W(t)$. A fundamental solution matrix is invertible as long as the Wronskian is non-zero.
- There are many choices of a fundamental solution matrix, depending on how one chooses the basis. Perhaps the most convenient choice is the one where $\boldsymbol{M}(t_0) = \boldsymbol{I}$, the identity matrix. Given any fundamental solution matrix the one which is the identity at $t = t_0$ is given by $\boldsymbol{M}(t)\boldsymbol{M}^{-1}(t_0)$.
- The solution to the homogeneous ode $\boldsymbol{v}' = \boldsymbol{A}(t)\boldsymbol{v}; \quad \boldsymbol{v}(t_0) = \boldsymbol{v}_0$ is given by $\boldsymbol{v}(t) = \boldsymbol{M}(t)\boldsymbol{M}^{-1}(t_0)\boldsymbol{v}_0$.

The next theorem that we need is Abel's theorem. Recall that if $\boldsymbol{A}$ is a matrix then the trace of $\boldsymbol{A}$, denoted $\mathrm{Tr}(\boldsymbol{A})$, is equal to the sum of the diagonal elements of $\boldsymbol{A}$.

Theorem 4.1.2 (Abel's Theorem)**.** *Suppose that $\boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \ldots \boldsymbol{v}_n(t)$ are solutions to the system*

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}(t)\boldsymbol{v}.$$

Let $W(t)$ be the Wronskian of those solutions. Then $W(t)$ satisfies the first order equation

$$\frac{dW}{dt} = \mathrm{Tr}(\boldsymbol{A}(t))W.$$

In the case where the system comes from a higher order linear equation of the form

$$\frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + p_{n-2}(t)\frac{d^{n-2}y}{dt^{n-2}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = 0$$

the matrix $\boldsymbol{A}(t)$ takes the form

$$\boldsymbol{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0\ldots & 0 \\ 0 & 0 & 1 & 0\ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -p_0(t) & -p_1(t) & -p_2(t)\ldots & -p_{n-2}(t) & -p_{n-1}(t) \end{bmatrix}$$

so that $\mathrm{Tr}\left(\boldsymbol{A}(t)\right) = -p_{n-1}(t)$ and Abel's theorem implies that $\frac{dW}{dt} = -p_{n-1}(t)W$.

As in the case of Abel's theorem for a n^{th} order equation the main utility of Abel's theorem is the following corollary, which tells us that (assuming that the coefficients are continuous) the Wronskian is either never zero or it is always zero.

Corollary 4.1.1. *Suppose that $\boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \ldots \boldsymbol{v}_n(t)$ are solutions to the system*

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}(t)\boldsymbol{v}.$$

in some interval I in which $\boldsymbol{A}(t)$ is continuous. Then the Wronskian $W(t)$ is either never zero *in the interval I or it is* always zero *in the interval I.*

Proof. The proof follows simply from Abel's theorem. Suppose that there exists a point $t_0 \in I$ such that $W(t_0) = 0$. From Abel's theorem the Wronskian satisfies the differential equation

$$\frac{dW}{dt} = \mathrm{Tr}(\boldsymbol{A}(t))W \qquad W(t_0) = 0$$

One solution is obviously $W(t) = 0$ but $\boldsymbol{A}(t)$ is continuous, and therefore by our existence and uniqueness theorem this is the only solution. □

It is also worth noting once again that the Wronskian is non-zero at a point t_0 if and only if any vector $\boldsymbol{x}$ can be respresented as a linear combination of $\boldsymbol{v}_1(t_0), \boldsymbol{v}_2(t_0), \ldots \boldsymbol{v}_n(t_0)$.

4.2 *Variation of Parameters*

Generally if one is able to solve a homogeneous linear differential equation then one can solve the corresponding inhomogeneous equation. This fact is known as the "variation of parameters" formula. Variation of parameters is one instance where the results for systems of first order equations look much simpler than the corresponding results for higher order equations.

Here we can see one of the benefits of solving a first order system of equations, rather than a single n^{th} order equation. Both the solution formula itself as well as the proof greatly resemble the integrating factor method for a first order linear inhomogeneous equation. The derivation of the variation of parameters formula for a higher-order scalar equation, on the other hand, is much less transparent.

Theorem 4.2.1 (Variation of Parameters)**.** *Consider the non-homogeneous linear equation*

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}(t)\boldsymbol{v} + \boldsymbol{g}(t) \qquad \boldsymbol{v}(t_0) = \boldsymbol{v}_0. \tag{4.3}$$

Suppose that one can find n linearly independent solutions $\boldsymbol{v}_1(t), \boldsymbol{v}_2(t) \ldots \boldsymbol{v}_n(t)$ to the corresponding homogeneous equation

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}(t)\boldsymbol{v}.$$

Define the matrix $\boldsymbol{M}(t)$ to be the matrix with column $\{\boldsymbol{v}_i(t)\}_{i=1}^n$. Then the general solution to (4.3) is given by

$$\boldsymbol{v}(t) = \boldsymbol{M}(t)\int_{t_0}^t \boldsymbol{M}^{-1}(s)\boldsymbol{g}(s)ds + \boldsymbol{M}(t)\boldsymbol{M}(0)^{-1}\boldsymbol{v}_0$$

Proof. The proof is quite similar to the integrating factor method for a first order scalar equation. We have the equation

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}(t)\boldsymbol{v} + \boldsymbol{g}(t) \qquad \boldsymbol{v}(t_0) = \boldsymbol{v}_0$$

and the fundamental solution matrix $\boldsymbol{M}(t)$, satisfying

$$\frac{d\boldsymbol{M}}{dt} = \boldsymbol{A}(t)\boldsymbol{M}.$$

We define a new variable $\boldsymbol{w}(t)$ via $\boldsymbol{v}(t) = \boldsymbol{M}(t)\boldsymbol{w}(t)$. From this definition it follows that

$$\begin{aligned}\frac{d\boldsymbol{v}}{dt} &= \frac{d\boldsymbol{M}}{dt}\boldsymbol{w}(t) + \boldsymbol{M}(t)\frac{d\boldsymbol{w}}{dt}\\ &= \boldsymbol{A}(t)\boldsymbol{M}\boldsymbol{w}(t) + \boldsymbol{M}(t)\frac{d\boldsymbol{w}}{dt}\\ &= \boldsymbol{A}(t)\boldsymbol{v} + \boldsymbol{M}(t)\frac{d\boldsymbol{w}}{dt}\end{aligned}$$

Substituting this into the differential equation for $\boldsymbol{v}(t)$ gives

$$\begin{aligned}&\boldsymbol{A}(t)\boldsymbol{v} + \boldsymbol{M}(t)\frac{d\boldsymbol{w}}{dt} = \boldsymbol{A}(t)\boldsymbol{v} + \boldsymbol{g}(t)\\ &\boldsymbol{M}(t)\frac{d\boldsymbol{w}}{dt} = \boldsymbol{g}(t)\\ &\frac{d\boldsymbol{w}}{dt} = \boldsymbol{M}^{-1}(t)\boldsymbol{g}(t)\end{aligned}$$

Integrating this equation from t_0 to t and applying the fundamental theorem gives

$$\begin{aligned}&\boldsymbol{w}(t) - \boldsymbol{w}(t_0) = \int_{t_0}^{t} \boldsymbol{M}^{-1}(s)\boldsymbol{g}(s)ds\\ &\boldsymbol{w}(t) = \int_{t_0}^{t} \boldsymbol{M}^{-1}(s)\boldsymbol{g}(s)ds + \boldsymbol{w}(t_0)\end{aligned}$$

Using the fact that $\boldsymbol{v} = \boldsymbol{M}\boldsymbol{w}$ or $\boldsymbol{w} = \boldsymbol{M}^{-1}\boldsymbol{v}$ gives

$$\boldsymbol{v}(t) = \boldsymbol{M}(t)\int_{t_0}^{t} \boldsymbol{M}^{-1}(s)\boldsymbol{g}(s)ds + \boldsymbol{M}(t)\boldsymbol{M}(0)^{-1}\boldsymbol{v}_0$$

□

4.3 *Constant coefficient linear systems.*

There are, unfortunately, very few systems of ordinary differential equations for which one can write down the complete analytical solution. One important instance when we *can* write down the complete solution is the case of constant coefficients.

Definition 4.3.1. *A constant coefficient homogeneous system of differential equations is one that can be written in the form*

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}\boldsymbol{v}. \tag{4.4}$$

We can find a number of solutions to a constant coefficient equation, perhaps even all of them, using basic linear algebra.

Lemma 4.3.1. *Suppose that the matrix $\boldsymbol{A}$ has a set of eigenvectors $\{\boldsymbol{e}_k\}$ with corresponding eigenvalues λ_k:*

$$\boldsymbol{A}\boldsymbol{e}_k = \lambda_k \boldsymbol{e}_k$$

Then the functions

$$\boldsymbol{v}_k(t) = \boldsymbol{e}_k e^{\lambda_k t}$$

are solutions to (4.4). If the eigenvalues are distinct then the solutions $\boldsymbol{v}_k(t)$ are linearly independent.

Example 4.3.1. *Consider the constant coefficient linear homogenous equation*

$$\frac{d\boldsymbol{v}}{dt} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \boldsymbol{v}.$$

The matrix $\mathbf{A}$ has eigenvalues of $\lambda_1 = -1$, with eigenvector $(1,-1)^t$ and $\lambda_2 = 3$, with eigenvector $(1,1)^t$. This gives

$$\boldsymbol{v}_1(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\boldsymbol{v}_2(t) = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Since the eigenvalues are distinct these solutions are linearly independent.

Unfortunately matrices do not in general always have a complete set of eigenvectors. One example of this is the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The characteristic equation is $(\lambda - 1)^2 = 0$ so $\lambda = 1$ is an eigenvalue of (algebraic) multiplicity 2, but there is only one linearly independent vector such that $\boldsymbol{A}\boldsymbol{e} = \boldsymbol{e}$, namely $\boldsymbol{e} = [0,1]^t$. This should not be too surprising – we have already seen that for higher order equations if the characteristic polynomial has a root of multiplicity higher than one then we may have to include terms like $te^{\lambda t}, t^2e^{\lambda t}$, etc. For systems the annihilator method does not really apply, so instead we give an algorithm for directly computing the fundamental solution matrix. First a definition

Definition 4.3.2. *For a constant coefficient homogeneous linear differential equation*

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}\boldsymbol{v}$$

we define the matrix exponential $\boldsymbol{M}(t) = e^{\boldsymbol{A}t}$ to be the fundamental solution matrix satisfying the initial condition $\mathbf{M}(0) = \boldsymbol{I}$, where $\boldsymbol{I}$ is the $n \times n$ identity matrix. Alternatively the matrix exponential can be defined by the power series

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2 t^2}{2!} + \frac{\boldsymbol{A}^3 t^3}{3!} + \ldots = \sum_{k=1}^{\infty} \frac{t^k \boldsymbol{A}^k}{k!}$$

It is usually not practical to directly compute the matrix exponential from the power series, but for simple examples it may be possible.

Example 4.3.2. *Find the matrix exponential for the matrix*

$$\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

using the power series definition. We begin by computing the first few powers.

$$\begin{aligned}
\boldsymbol{A}^2 &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\boldsymbol{I} \\
\boldsymbol{A}^3 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\boldsymbol{A} \\
\boldsymbol{A}^4 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{I} \\
\boldsymbol{A}^5 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \boldsymbol{A} \\
&\ldots
\end{aligned}$$

To summarize even powers of $\boldsymbol{A}$ are $\pm\boldsymbol{I}$ and odd powers of $\boldsymbol{A}$ are $\pm\boldsymbol{A}$. This suggests separating the power series into even and odd powers. This gives

$$\begin{aligned}
e^{t\boldsymbol{A}} &= \boldsymbol{I} + t\boldsymbol{A} + \frac{t^2}{2}\boldsymbol{A}^2 + \frac{t^3}{6}\boldsymbol{A}^3 + \frac{t^4}{24}\boldsymbol{A}^4 + \ldots \\
&= (1 - \frac{t^2}{2} + \frac{t^4}{24} + \ldots)\boldsymbol{I} + (t - \frac{t^3}{6} + \frac{t^5}{120} + \ldots)\boldsymbol{A} \\
&= \cos(t)\boldsymbol{I} + \sin(t)\boldsymbol{A} \\
&= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}.
\end{aligned}$$

This example notwithstanding direct evaluation of the power series is usually not a practical method for finding the matrix exponential. We give two algorithms for computing the matrix exponential: diagonalization and Putzer's method. The diagonalization method works only for the case when the matrix $\boldsymbol{A}$ is diagonalizable – when it has a complete set of eigenvectors. This is not always true but it does hold for some important classes of matrices. If the eigenvalues of $\boldsymbol{A}$ are all distinct, for instance, or if $\boldsymbol{A}$ is a symmetric matrix: $\boldsymbol{A}^t = \boldsymbol{A}$ then we know that $\boldsymbol{A}$ is diagonalizable.

Method 4.3.1 (Diagonalization)**.** *Suppose that the matrix $\boldsymbol{A}$ has n linearly independent eigenvectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots \boldsymbol{e}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$. The eigenvalues may have higher multiplicity as long as the number of linearly independent eigenvectors is the same as the multiplicity of the eigenvalue. Let $\boldsymbol{U}$ denote the matrix with columns given by the eigenvectors, $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots \boldsymbol{e}_n$, and $\boldsymbol{U}^{-1}$ denote the inverse. Note that since the eigenvectors are assumed to be linearly independent we know that $\boldsymbol{U}$ must be invertible. Then the matrix exponential is given by*

$$e^{t\boldsymbol{A}} = \boldsymbol{U} \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 & \ldots & 0 \\ 0 & 0 & e^{\lambda_3 t} & 0 & \ldots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 0 & e^{\lambda_{n-1} t} & 0 \\ 0 & 0 & \ldots & 0 & 0 & e^{\lambda_n t} \end{bmatrix} \boldsymbol{U}^{-1}$$

Notice that in this case, since there are "enough" eigenvectors we are basically just finding n linearly independent solutions and writing down a particular fundamental matrix, the one that satisfies the initial condition $\boldsymbol{M}(t) = \boldsymbol{I}$. This method covers a lot of cases that arise in practice, but not all of them. For instance if the system is derived from a higher order scalar equation that has multiple roots then $\boldsymbol{A}$ will *never* have a full set of eigenvectors.

The second method is a little more involved to describe but it will always work and does not require one to calculate the eigenvectors, only the eigenvalues.

Method 4.3.2 (Putzer's Method)**.** *Given an $n \times n$ matrix $\boldsymbol{A}$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ Define the matrices $\{\mathbf{B}_k\}_{k=0}^{n-1}$ as follows:*

The eigenvalues can be in any order but the same order should be used throughout. Each eigenvalue should be listed according to its multiplicity – in other words if 2 is an eigenvalue of multiplicity five then it should appear five times in the list.

$$\begin{aligned}
&\boldsymbol{B}_0 = \boldsymbol{I}\\
&\boldsymbol{B}_1 = \boldsymbol{A} - \lambda_1 \boldsymbol{I}\\
&\boldsymbol{B}_2 = (\boldsymbol{A} - \lambda_2 \boldsymbol{I})(\boldsymbol{A} - \lambda_1 \boldsymbol{I}) = (\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{B}_1\\
&\boldsymbol{B}_3 = (\boldsymbol{A} - \lambda_3 \boldsymbol{I})(\boldsymbol{A} - \lambda_2 \boldsymbol{I})(\boldsymbol{A} - \lambda_1 \boldsymbol{I}) = (\boldsymbol{A} - \lambda_3 \boldsymbol{I})\boldsymbol{B}_2\\
&\vdots\\
&\boldsymbol{B}_{n-1} = (\boldsymbol{A} - \lambda_{n-1} \boldsymbol{I})(\boldsymbol{A} - \lambda_{n-2} \boldsymbol{I})\ldots(\boldsymbol{A} - \lambda_1 \boldsymbol{I}) = (\boldsymbol{A} - \lambda_{n-1} \boldsymbol{I})\boldsymbol{B}_{n-2}.
\end{aligned}$$

As a check if you compute $\boldsymbol{B}_n = (\boldsymbol{A} - \lambda_n \boldsymbol{I})\boldsymbol{B}_{n-1}$ it should be zero due to a result in linear algebra known as the Cayley-Hamilton theorem. It is uncommon but possible for Define a sequence of functions $\{r_k(t)\}_{k=1}^n$

Again the order of the eigenvalues is not important but the order here should be the same as when you calculated the matrices $\boldsymbol{B}_k$. Note the pattern that holds for every function after $k = 1$: each function satisfies a first order linear equation with the previous function as the forcing term.

$$\begin{aligned}
&\frac{dr_1}{dt} = \lambda_1 r_1 && r_1(0) = 1\\
&\frac{dr_2}{dt} = \lambda_2 r_2 + r_1(t) && r_2(0) = 0\\
&\frac{dr_3}{dt} = \lambda_3 r_3 + r_2(t) && r_3(0) = 0\\
&\vdots\\
&\frac{dr_n}{dt} = \lambda_n r_n + r_{n-1}(t) && r_n(0) = 0
\end{aligned}$$

Then the matrix exponential is given by

$$e^{t\boldsymbol{A}} = r_1(t)\boldsymbol{B}_0 + r_2(t)\boldsymbol{B}_1 + \ldots r_n(t)\boldsymbol{B}_{n-1} = \sum_{k=0}^{n-1} r_{k+1}(t)\boldsymbol{B}_k.$$

The Putzer method tends to require more computations, but the computations required are easy – matrix multiplication and solution of a first order linear differential equation. It also works for all matrices. The diagonalization method tends to require less computations, but the computations that are required use somewhat more linear algebra: calculation of eigenvectors and inversion of a matrix. It may also not work if $\boldsymbol{A}$ is non-symmetric with eigenvalues of higher multiplicity.

Example 4.3.3. *Compute the matrix exponential $e^{t\boldsymbol{A}}$ where $\boldsymbol{A}$ is the matrix*

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

using the diagonalization method and Putzer's method.

The eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 2$. There is a repeated eigenvalue – $\lambda = 2$ has multiplicity two – but since $\boldsymbol{A}$ is symmetric we are guaranteed that there are two linearly independent eigenvectors. The three *eigenvectors are*

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{e}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

and so the matrix $\boldsymbol{U}$ *is given by*

$$\boldsymbol{U} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the inverse by

$$\boldsymbol{U}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we have that the matrix exponential is given by

$$\begin{aligned} e^{t\boldsymbol{A}} &= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{2t}+1}{2} & \frac{e^{2t}-1}{2} & 0 \\ \frac{e^{2t}-1}{2} & \frac{e^{2t}+1}{2} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \end{aligned}$$

Using Putzer's method we have that the matrices $\boldsymbol{B}_{0,1,2}$ *are given by*

$$\begin{aligned} \boldsymbol{B}_0 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \boldsymbol{B}_1 &= \boldsymbol{A} - 0\boldsymbol{I} = \boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ \boldsymbol{B}_2 &= (\boldsymbol{A} - 2\boldsymbol{I})\,\boldsymbol{B}_1 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

In this case $\boldsymbol{B}_2$ happens to be zero. Computing the functions $r_i(t)$ we have

$$\begin{aligned}
\frac{dr_1}{dt} &= 0r_1 & r_1(0) = 1\\
r_1(t) &= e^{0t} = 1\\
\frac{dr_2}{dt} &= 2r_2(t) + 1 & r_2(0) = 0\\
r_2(t) &= \frac{e^{2t}}{2} - \frac{1}{2}
\end{aligned}$$

Ordinarily we would have to compute the solution to

$$\frac{dr_3}{dt} = 2r_3 + r_2(t) = 2r_3 + \frac{e^{2t}}{2} - \frac{1}{2} \qquad r_3(0) = 0$$

(you can check that the solution is $r_3(t) = \frac{t}{2}e^{2t} + \frac{1}{4}e^{2t} - \frac{1}{4}$) but in this case since it is being multiplied by $\boldsymbol{B}_2$ which is the zero matrix we don't really need to calculate this. Thus we have

$$e^{t\boldsymbol{A}} = 1 \cdot \boldsymbol{B}_0 + \frac{e^{2t} - 1}{2}\boldsymbol{B}_1 = \begin{bmatrix} \frac{e^{2t}+1}{2} & \frac{e^{2t}-1}{2} & 0 \\ \frac{e^{2t}-1}{2} & \frac{e^{2t}+1}{2} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

Example 4.3.4. *Find the matrix exponential $e^{t\boldsymbol{A}}$ for*

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

The characteristic polynomial for this matrix is $(\lambda - 1)^3 = 0$ so $\lambda = 1$ is an eigenvalue of multiplicity three. There is only one linearly independent eigenvector, $\boldsymbol{e} = [-1, 1, 0]^t$, so this matrix is not diagonalizable, so Putzer's method is the best option. Computing the $\boldsymbol{B}$ matrices.

$$\begin{aligned}
\boldsymbol{B}_0 = \boldsymbol{I} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\\
\boldsymbol{B}_1 = \boldsymbol{A} - I &= \begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix}\\
\boldsymbol{B}_2 = (\boldsymbol{A} - I)\,\boldsymbol{B}_1 &= \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

In this case all of the $\boldsymbol{B}$ matrices are non-zero, but if we multiply $\boldsymbol{B}_2$ by $\boldsymbol{A} - \boldsymbol{I}$ you will see that we get zero, as implied by the Cayley-Hamilton *theorem. Next we need to compute the functions $r(t)$.*

$$\begin{aligned}
\frac{dr_1}{dt} &= r_1 & r_1(0) &= 1 \\
r_1(t) &= e^t \\
\frac{dr_2}{dt} &= r_2(t) + e^t & r_2(0) &= 0 \\
r_2(t) &= te^t \\
\frac{dr_3}{dt} &= r_3(t) + te^t & r_3(0) &= 0 \\
r_2(t) &= \frac{t^2}{2}e^t
\end{aligned}$$

The matrix exponential is given by

$$\begin{aligned}
e^{t\mathbf{A}} &= e^t \boldsymbol{B}_0 + te^t \boldsymbol{B}_1 + \frac{t^2}{2}e^t \boldsymbol{B}_2 \\
&= \begin{bmatrix} (1+2t+\frac{t^2}{2})e^t & (2t+\frac{t^2}{2})e^t & (t+\frac{t^2}{2})e^t \\ -(t+\frac{t^2}{2})e^t & (1-t-\frac{t^2}{2})e^t & -\frac{t^2}{2}e^t \\ -te^t & -te^t & (1-t)e^t \end{bmatrix}
\end{aligned}$$

Part II

Boundary value problems, Fourier series and the solution of partial differential equations.

5
Boundary value problems

5.1 Examples of boundary value problems

Prior to this point we have been exclusively focused on initial value problems. An initial value problem is one where all of the information is prescribed at some initial time, say $t = 0$. An example of an initial value problem is

$$\frac{d^3y}{dx^3} + y^2\frac{dy}{dx} + cos(y) = 5 \qquad y(0) = 1; \quad \frac{dy}{dx}(0) = 2; \quad \frac{d^2y}{dx^2}(0) = -2.$$

For initial value problems we have an existence and uniqueness theorem that (under some mild technical conditions) guarantees that there exists a unique solution in some small neighborhood of the initial data.

A differential equationa where values are specified at more than one point is called a boundary value problem. For instance the equation

$$y'' + y = 0 \quad y(0) = 1 \quad y(\frac{\pi}{2}) = 0$$

is a boundary value problem. In some cases, such as the above, there is a unique solution (in the above case the solution is $y = \cos(t)$). In other cases there may be no solution, or the solution may not be unique. While many applications of differential equations are posed as initial value problems there are also applications which give rise to two point boundary value problems.

One example of a boundary value problem arises in computing the deflection of a beam:

Example 5.1.1. *The equation for the deflection $y(x)$ of a beam with constant cross-sectional area and elastic modulus is given by*

$$EI\frac{d^4y}{dx^4} = g(x)$$

where E is a constant called the "elastic modulus", and I is the second moment of the cross-sectional area of the beam. A larger value of E represents a stiffer beamer that is more resistant to bending. The function $g(x)$ represents load on the beam: the force per unit of length.

Since this is a fourth order equation it requires four boundary conditions. If the ends of the beam are located at x_1 and x_2 then one typically requires two boundary conditions at x_1 and two at x_2. The nature of the boundary condition condition depends on the way in which the end of the beam is supported:

Simply supported/pinned $\quad y(x_i) = 0; y''(x_i) = 0$

Fixed/clamped $\quad y(x_i) = 0; y'(x_i) = 0$

Free $\quad y''(x_i) = 0; y'''(x_i) = 0$

One can "mix-and-match" these boundary conditions. For instance the equation for a beam which is fixed at $x = 0$ and simply supported at $x = L$ and is subjected to a constant load w is

$$EI\frac{d^4y}{dx^4} = w \quad y(0) = 0; y'(0) = 0; y(L) = 0; y''(0) = 0$$

Integrating this equation up four times gives

$$y(x) = \frac{wx^4}{24} + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D$$

imposing the four boundary conditions $y(0) = 0; y'(0) = 0; y(L) = 0; y''(L) = 0$ gives four equations for the four unknowns A, B, C, D.

$$\begin{aligned}
&D = 0 && y(0) = 0\\
&C = 0 && y'(0) = 0\\
&D + CL + \frac{BL^2}{2} + \frac{AL^3}{6} + \frac{wL^4}{24} = 0 && y(L) = 0\\
&B + AL + \frac{wL^2}{2} = 0 && y''(L) = 0
\end{aligned}$$

The solution of these four linear equations is given by

$$A = -\frac{5Lw}{8EI} \qquad B = \frac{L^2w}{8EI} \qquad C = 0 \qquad D = 0$$

So the deflection of the beam is given by

$$y(x) = \frac{wL^4}{48EI}\left(2(\frac{x}{L})^4 - 5(\frac{x}{L})^3 + 3(\frac{x}{L})^2\right).$$

Since the beam is supported at both $x = 0$ and $x = L$ one expects that the maximum deflection should occur somewhere between those points. To find the point of maximum deflection we can take the derivative and set it equal to zero:

$$y'(x) = \frac{wL^4}{48EI}\left(8\frac{x^3}{L^4} - 15\frac{x^2}{L^3} + 6\frac{x}{L^2}\right) = \frac{wL^3}{48EI}8(\frac{x}{L})((\frac{x}{L})^2 - 15\frac{x}{L} + 6) = 0$$

The roots of this cubic are $\frac{x}{L} = 0$*,* $\frac{x}{L} = \frac{15-\sqrt{33}}{16} \approx 0.58$ *and* $\frac{x}{L} = \frac{15+\sqrt{33}}{16} \approx 1.30$*. The beam has length* L *so the root* $\frac{x}{L} = 1.30$ *is outside of the domain. The root* $\frac{x}{L} = 0$ *is one of the support points: we know that the deflection there is zero, so the maximum deflection must occur at* $x \approx 0.58L$*.*

5.1.1 *Existence and Uniqueness*

Unlike initial value problems which, under some mild assumptions, always have unique solutions a boundary value problem may have a unique solution, no solution, or an infinite number of solutions. Let's consider two examples, which appear quite similar. The two examples are

$$y'' + \pi^2 y = 0 \qquad y(0) = 0 \qquad y(1) = 0 \tag{5.1}$$

$$y'' + \pi^2 y = \sin(\pi x) \qquad y(0) = 0 \qquad y(1) = 0 \tag{5.2}$$

In the first example we can find that the characteristic equation is given by

$$r^2 + \pi^2 = 0$$

so $r = \pm i\pi$ and the general solution to be $y(x) = A\cos(\pi x) + B\sin(\pi x)$ If we impose the conditions $y(0) = 0$ and $y(1) = 0$ this gives two equations

$$\begin{aligned} A\cos(0) + B\sin(0) &= A = 0 \\ A\cos(\pi) + B\sin(\pi) &= -A = 0 \end{aligned} \quad .$$

In this case we get two equations that are not linearly independent, but the are consistent. We know from linear algebra that there are an infinite number of solutions – $A = 0$ and B is undetermined.

In the second case we can solve the inhomogeneous equation using the method of undetermined coefficients or of variation of parameters. We find that the general solution is given by

$$y(x) = A\cos(\pi x) + B\sin(\pi x) - \frac{x\cos(\pi x)}{2\pi}.$$

In this case when we try to solve the boundary value problem we find

$$\begin{aligned} y(0) &= A\cos(0) + B\sin(0) - \frac{0\cos(0)}{2\pi} = A = 0 \\ y(1) &= A\cos(\pi) + B\sin(\pi) - \frac{\pi\cos(\pi)}{2\pi} = -A + \frac{1}{2} = 0. \end{aligned}$$

In this case the boundary conditions lead to an inconsistent set of linear equations: $A = 0$ and $A = \frac{1}{2}$, for which there can be no solution.

The following theorem tells us that these are the only possibilities: a two point boundary value problem can have zero, one or infinitely many solutions. For simplicity we will state and prove the theorem for second order two point boundary value problems; the proof for higher order boundary value problems is basically the same, with minor notational changes.

Theorem 5.1.1. *Consider the second order linear two point boundary value problem*

$$\begin{aligned} &\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = \boldsymbol{L}y = f(x) \\ &\alpha_1 y(a) + \alpha_2 y'(a) = A \\ &\beta_1 y(b) + \beta_2 y'(b) = B, \end{aligned} \tag{5.3}$$

where $p(x), q(x), f(x)$ are continuous on $[a,b]$. The corresponding homogeneous problem is given

$$\begin{aligned} &\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = \boldsymbol{L}y = 0 \\ &\alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ &\beta_1 y(b) + \beta_2 y'(b) = 0, \end{aligned} \tag{5.4}$$

Then the following holds:

1. *Equation (5.3) has either a unique solution $u(x) = 0$ or an infinite number of solutions.*

2. *If Equation (5.3) has a unique solution then Equation (5.4) has a unique solution.*

3. *If Equation (5.3) does not have a unique solution then Equation (5.4) either has no solutions or an infinite number of solutions.*

Proof. This theorem may remind you of the following fact from linear algebra: a system of linear equations

$$\mathbf{M}\mathbf{x} = \mathbf{b}$$

either has no solutions, a unique solution, or infinitely many solutions. It has a unique solution if the only solution to the homogeneous equation $\mathbf{M}\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = 0$. If $\mathbf{M}\mathbf{x} = \mathbf{0}$ has at least one non-zero solution then $\mathbf{M}\mathbf{x} = \mathbf{b}$ either has no solutions or infinitely many solutions. We will use this fact of linear algebra to prove this result.

We can assume without loss of generality that $A = 0 = B$. If this is not the case one can make the change of variables $y(x) = w(x) + u(x)$, where $w(x)$ is any infinitely differentiable function that satisfies the boundary conditions

$$\alpha_1 w(a) + \alpha_2 w'(a) = A \qquad \beta_1 w(a) + \beta_2 w'(a) = B,$$

This will lead to a boundary value problem for $u(x)$ with homogeneous boundary conditions.

We know that the solution to equation (5.4) can be written as a linear combination of two independent solutions

$$y(x) = Cy_1(x) + Dy_2(x).$$

Imposing the boundary conditions gives us the following system of equations for C, D

$$\begin{aligned}(\alpha_1 y_1(a) + \alpha_2 y_1'(a))C + (\alpha_1 y_2(a) + \alpha_2 y_2'(a))D = 0\\(\beta_1 y_1(b) + \beta_2 y_1'(b))C + (\beta_1 y_2(b) + \beta_2 y_2'(b))D = 0.\end{aligned}$$

We know from linear algebra that typically the only solution to this set of homogeous linear equations is $C = 0, D = 0$. There is a non-zero solution if and only if the matrix

$$\boldsymbol{M} = \begin{pmatrix} \alpha_1 y_1(a) + \alpha_2 y_1'(a) & (\alpha_1 y_2(a) + \alpha_2 y_2'(a)) \\ (\beta_1 y_1(b) + \beta_2 y_1'(b)) & (\beta_1 y_2(b) + \beta_2 y_2'(b)) \end{pmatrix} \tag{5.5}$$

is singular, in which case there are an infinite number of solutions.

The inhomogeneous case is similar. There the general solution is

$$y(x) = Cy_1(x) + Dy_2(x) + y_{part}(x),$$

where $y_{part}(x)$ is the particular solution. Applying the boundary conditions leads to the equations for C, D

$$\begin{aligned}(\alpha_1 y_1(a) + \alpha_2 y_1'(a))C + (\alpha_1 y_2(a) + \alpha_2 y_2'(a))D = -\alpha_1 y_{part}(a) + \alpha_2 y_{part}'(a)\\(\beta_1 y_1(b) + \beta_2 y_1'(b))C + (\beta_1 y_2(b) + \beta_2 y_2'(b))D = -\alpha_1 y_{part}(b) + \alpha_2 y_{part}'(b)\end{aligned}$$

or equivalently

$$\boldsymbol{M} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} -(\alpha_1 y_{part}(a) + \alpha_2 y_{part}'(a)) \\ -(\alpha_1 y_{part}(b) + \alpha_2 y_{part}'(b)) \end{pmatrix} \tag{5.6}$$

Noe that the $\boldsymbol{M}$ arising here is the same as the $\boldsymbol{M}$ arising in the solution to the homogeneous problem. Therefore the system of equations (5.6), and hence the boundary value problem, has a unique solution if and only if the matrix $\boldsymbol{M}$ is non-singular, which is true if and only if the homogeneous boundary value problem (5.3) has a unique solution $y(x) = 0$. □

5.2 *Eigenvalue problems*

A special kind of boundary value problem is something called an eigenvalue problem. An eigenvalue problem is a boundary value problem involving an auxiliary parameter, called an eigenvalue parameter. The problem is to determine the values of the eigenvalue parameter for which the problem has a non-trivial solution. These special values of the eigenvalue parameter are called the "eigenvalues."

Definition 5.2.1. *A second order two point boundary value problem consists of a second order linear differential equation involving an unknown parameter λ, called the eigenvalue parameter*

$$\boldsymbol{L}(\lambda)y = 0,$$

where $\boldsymbol{L}(\lambda)$ is a second order linear differential operator, together with homogeneous two point boundary conditions:

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \qquad \beta_1 y(b) + \beta_2 y'(b) = 0.$$

This equation always has the solution $y(x) = 0$. We know from Theorem 5.1.1 that such an equation will either have a unique solution or an infinite number of solutions. The eigenvalues are the values of λ for which the boundary value problem has an infinite number of solutions.

More generally an n^{th} order boundary value problem would consist of an n^{th} order linear differential equation involving an unknown parameter, together with n homogeneous boundary conditions. Most commonly these would be specified at two points, although one could specify boundary conditions at more than two points. In the theory of elastic beams, for instance, one often has to solve a fourth order eigenvalue problem, with two boundary conditions specified at each end of the beam.

We now present several example of eigenvalue problems, along with their solution.

Example 5.2.1. *For what values of λ does the boundary value problem*

$$y'' + \lambda^2 y = 0 \qquad y(0) = 0 \qquad y'(1) = 0$$

have a non-trivial solution (a solution other than $y = 0$).

The solutions to $y'' + \lambda^2 y = 0$ are given by

$$\begin{aligned} y &= A\cos(\lambda x) + B\sin(\lambda x) && \lambda \neq 0 \\ y &= A + Bx && \lambda = 0 \end{aligned}$$

In the first case we have that $y(0) = A = 0$, and $y'(1) = B\lambda\cos(\lambda) = 0$. Since $\lambda \neq 0$ the only solutions are $B = 0$ or $\cos(\lambda = 0)$. We are only interested in non-zero solutions so $\cos(\lambda) = 0$ so $\lambda = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$.

In the second case ($\lambda = 0$) we have $y = A + Bx$. Imposing $y(0) = 0$ tells us that $A = 0$. Imposing $y'(1) = 0$ tells us that $B = 0$. So $\lambda = 0$ is not an eigenvalue – the eigenvalues are $\lambda = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ or $\lambda = (n + \frac{1}{2})\pi$.

Example 5.2.2. *For what values of λ does the boundary value problem*

$$y'' + y' + \lambda^2 y = 0 \qquad y(0) = 0 \qquad y(1) = 0$$

have a non-trivial solution (a solution other than $y = 0$).

The characteristics equation is given by

$$r^2 + r + \lambda^2 = 0$$

which has roots $r = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda^2}$. It is convenient to separate this into three cases:

- $\lambda^2 > \frac{1}{4}$*, when the characteristic equation has complex roots.*
- $\lambda = \frac{1}{4}$*, when the characteristic equation has a double root.*
- $\lambda < \frac{1}{4}$*, when the characteristic equation has real roots.*

Case 1: $\lambda^2 > \frac{1}{4}$

In this case the general solution is given by $y(x) = Ae^{-x/2}\cos(\sqrt{\lambda^2 - \frac{1}{4}}x) + Be^{-x/2}\sin(\sqrt{\lambda^2 - \frac{1}{4}}x)$. The boundary condition $y(0) = 0$ implies that $A = 0$. The boundary condition that $y(1) = 0$ implies that $B\sin(\sqrt{\lambda^2 - \frac{1}{4}}) = 0$. Choosing $B = 0$ means we have only the zero solution, so $\sin(\sqrt{\lambda^2 - \frac{1}{4}}) = 0$ This implies that $\sqrt{\lambda^2 - \frac{1}{4}} = n\pi$ or $\lambda^2 = n^2\pi^2 + \frac{1}{4}$.

Case 2: $\lambda^2 = \frac{1}{4}$

In this case $r = -\frac{1}{2}$ is a double root of the characteristic equation, so the general solution is $y = Ae^{-x/2} + Bxe^{-x/2}$. The condition $y(0) = 0$ implies that $A = 0$, and the condition $y(1) = Be^{-1/2} = 0$ implies that $B = 0$. So $\lambda^2 = 1/4$ is not an eigenvalue.

Case 3: $\lambda^2 < \frac{1}{4}$

In this case the general solution is $y = Ae^{r_1 x} + Be^{r_2 x}$, with $r_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda^2}$ and $r_2 = -frac12 - \sqrt{\frac{1}{4} - \lambda^2}$. IMposing the boundary conditions gives two equations

$$y(0) = A + B = 0$$
$$y(1) = Ae^{r_1} + Be^{r_2} = 0$$

This can be written as

$$\begin{pmatrix} 1 & 1 \\ e^{r1} & e^{r2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The only solution is $A = 0, B = 0$ unless the determinant of the matrix is zero. The determinant is given by $e^{r_2} - e^{r_1}$ which is not zero since $r_1 \neq r_2$.

Example 5.2.3 (Guitar String)**.** *The equation for a one dimensional vibrating string, such as a guitar string, is*

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad y(0) = 0 \qquad y(L) = 0.$$

Here c is a constant representing the wave speed of the string. If we look for a "pure tone" or harmonic we look for a solution of the form

$$y = f(x)\cos(\omega t)$$

Substituting this into the equation gives the following equation for $f(x)$

$$c^2 f_{xx} + \omega^2 f = 0 \qquad f(0) = f(L) = 0.$$

This is an eigenvalue problem for ω, the frequency. The fact that this equation only has a non-zero solution tells us that the string will only vibrate at certain specific frequencies, the eigenvalues. The general solution to the above is given by

$$f(x) = A\cos(\frac{\omega}{c}x) + B\sin(\frac{\omega}{c}x)$$

as long as $\omega \neq 0$. Imposing the boundary conditions we find that

$$f(0) = A = 0 \qquad f(L) = A\cos(\frac{\omega}{c}L) + B\sin(\frac{\omega}{c}L) = 0.$$

Substituting the first equality into the second tells us that $B\sin(\frac{\omega}{c}L) = 0$, so either $B = 0$ or $\sin\frac{\omega}{c}L = 0$. The first gives us the zero solution. The second case happens when $\frac{\omega}{c}L = \pi, 2\pi, 3\pi, \ldots$ These are the frequencies at which the string can vibrate. The lowest frequency $\omega = \frac{\pi c}{L}$ is often called the fundamental while the higher frequencies are known as harmonics or overtones.

5.3 *Fourier Series*

5.3.1 *Background*

In science and engineering it is often desirable to have an efficient way to represent some function of interest. In linear algebra one often works with eigenvectors. The eigenvectors of a matrix $\boldsymbol{A}$ are the non-zero vectors $\boldsymbol{v}_i$ for which

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i.$$

This should be very reminiscent of the preceeding section, 5.1.1, on boundary value problems. In each case we have an equation depending on a parameter λ which has a non-zero solution only for certain values of that parameter. In the case of the eigenvalues of the matrix these are the roots of the characteristic polynomial, while in the case of a differential boundary value problem these are typically the roots of some more complicated function, but the idea is similar. It is worthwhile to recall a couple of results from linear algebra.

Theorem 5.3.1. *Suppose that the matrix $\boldsymbol{A}$ is a real $n \times n$ symmetric matrix. The eigenvalue problem*

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$$

has n linearly independent eigenvectors $\{\boldsymbol{v}_i\}_{i=1}^n$. *These vectors are linearly independent, form a basis for* $\mathbb{R}^n$, *and can be chosen to be orthonormal:*

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

One of the advantages of an orthonormal basis is that it makes many of the operations of linear algebra much simpler. The matrix $\boldsymbol{A}$ itself looks much simpler in the eigenbasis. Recall the following additional results from linear algebra:

Theorem 5.3.2. *Suppose that* $\{\boldsymbol{v}_i\}_{i=1}^n$ *is an orthonormal basis for* $\mathbb{R}^n$. *For any* $\boldsymbol{w} \in \mathbb{R}^n$ *we have that*

$$\boldsymbol{w} = \sum_{i=1}^n \alpha_i \boldsymbol{v}_i \qquad \alpha_i = \boldsymbol{v}_i \cdot \boldsymbol{w}$$

This theorem shows that it is easy to express a given vector in terms of an orthonormal basis: the coefficients are simply the dot product of the vector with the basis vectors. We will give a proof here, since it is simple and will be important for what is to follow

Proof. We are given that $\{\boldsymbol{v}_i\}_{i=1}^n$ is an orthonormal basis for $\mathbb{R}^n$, so for any $\boldsymbol{w}$ there must exists coefficient α_i such that

$$\boldsymbol{w} = \sum_{i=1}^n \alpha_i \boldsymbol{v}_i = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \ldots \alpha_n \boldsymbol{v}_n \tag{5.7}$$

To find (for instance) α_1 we can take the dot product with $\boldsymbol{v}_1$ to get

$$\boldsymbol{v}_1 \cdot \boldsymbol{w} = \sum_{i=1}^n \alpha_i \boldsymbol{v}_i \cdot \boldsymbol{v}_1 = \alpha_1 \boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \ldots \alpha_n \boldsymbol{v}_1 \cdot \boldsymbol{v}_n.$$

However the basis is orthonormal, so $\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 = 1$ and $\boldsymbol{v}_j \cdot \boldsymbol{v}_1 = 0$ for $j \neq 1$, so this reduces to

$$\boldsymbol{w} \cdot \boldsymbol{v}_1 = \alpha_1.$$

In the same spirit taking the dot product of Equation (5.7) with $\boldsymbol{v}_j$ for $j \geq 2$ gives $\alpha_j = \boldsymbol{v}_j \cdot \boldsymbol{w}$ □

Theorem 5.3.3. *Suppose that the matrix* $\boldsymbol{A}$ *is a real* $n \times n$ *symmetric matrix. Assume that the n linearly independent eigenvectors* $\{\boldsymbol{v}_i\}_{i=1}^n$ *of* $\boldsymbol{A}$ *are chosen to be orthonormal. Let* $\boldsymbol{U}$ *be the matrix whose* i^{th} *column is* $\boldsymbol{v}_i$. *Then the matrix* $\boldsymbol{\Lambda}$ *defined by*

$$\boldsymbol{\Lambda} = \boldsymbol{U}^t \boldsymbol{A} \boldsymbol{U}$$

is diagonal, with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ *along the diagonal.*

The topic of most of the rest of the course will be Fourier series. The classical Fourier series, along with the Fourier sine and cosine series, are the analog of the eigenvector expansion for certain second order boundary value problems. The functions $\{\sin(\frac{n\pi x}{L}\}_{n=1}^{\infty}$ or $\{\cos(\frac{n\pi x}{L}\}_{n=0}^{\infty}$ are analogous to the eigenvectors, and most "reasonable" functions can be decomposed in terms of these basis functions.

5.3.2 *The Classical Fourier Series: Orthogonality*

Suppose that we have a function $f(x)$ that is periodic with period L: $f(x+L) = f(x)$. It is often convenient to be able to express $f(x)$ in terms of known functions, such as sines and cosines. This is the idea behind the classical Fourier series. To begin with we state a result about definite integrals that is related to the the orthogonality that we discussed in the previous section.

Lemma 5.3.1. *The functions* $1, \{\sin \frac{2n\pi x}{L}\}_{n=1}^{\infty}, \{\cos \frac{2n\pi x}{L}\}_{n=1}^{\infty}$ *are orthogonal on the interval* $(0, L)$*. Letting* $\delta_{n,m}$ *denote the Kronecker delta symbol*

$$\delta_{n,m} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

then we have the identities (assuming n, m *are integers* ≥ 1*)*

$$\begin{aligned}
&\int_0^L \cos \frac{2n\pi x}{L} \cos \frac{2m\pi x}{L} dx = \frac{L}{2}\delta_{n,m} \\
&\int_0^L \sin \frac{2n\pi x}{L} \cos \frac{2m\pi x}{L} dx = 0 \\
&\int_0^L \sin \frac{2n\pi x}{L} \sin \frac{2m\pi x}{L} dx = \frac{L}{2}\delta_{n,m} \\
&\int_0^L \cos \frac{2n\pi x}{L} dx = 0 \\
&\int_0^L \sin \frac{2n\pi x}{L} dx = 0 \\
&\int_0^L 1 dx = L.
\end{aligned}$$

Note that the functions are orthogonal, not orthonormal! Now if one knew in advance that a given function was expressible as a linear combination of the functions $1, \{\sin \frac{n\pi x}{L}\}_{n=1}^{\infty}, \{\cos \frac{n\pi x}{L}\}_{n=1}^{\infty}$ then it is relatively simple to find what the coefficients must be. Suppose that one knows that

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{2n\pi x}{L}) + \sum_{n=1}^{\infty} B_n \sin(\frac{2n\pi x}{L}).$$

Then, to find (for instance) A_5 we would multiply the above by $\cos(\frac{10\pi x}{L})$ and integrate over the interval $(0, L)$. This would give

$$\int_0^L f(x)\cos(\frac{10\pi x}{L})dx = A_0\int_0^L \cos(\frac{10\pi x}{L})dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos(\frac{10\pi x}{L})\cos(\frac{2n\pi x}{L})dx + \sum_{n=1}^{\infty} B_n \int_0^L \cos(\frac{10\pi x}{L})\sin(\frac{n\pi x}{L})dx. \tag{5.8}$$

Next we should notice that *all of the terms on the righthand side of the equation are zero except for one!* We have that

$$\begin{aligned}
&\int_0^L \cos(\frac{10\pi x}{L})dx = 0 \\
&\int_0^L \cos(\frac{10\pi x}{L})\sin(\frac{2n\pi x}{L})dx = 0 && \text{for all } \ n \\
&\int_0^L \cos(\frac{10\pi x}{L})\cos(\frac{2n\pi x}{L})dx = 0 && \text{for all! } n \neq 5 \\
&\int_0^L \cos(\frac{10\pi x}{L})\cos(\frac{10\pi x}{L})dx = \frac{L}{2}.
\end{aligned}$$

Using these to simplify the righthand side of equation (5.8) we find

$$\int_0^L f(z)\cos(\frac{10\pi x}{L})dx = \frac{L}{2}A_5,$$

or $A_5 = \frac{2}{L}\int_0^L f(z)\cos(\frac{10\pi x}{L})dx$. There is, of course, nothing special about $n = 5$: one can do the same calculation for any value of n. This leads to the following result:

Theorem 5.3.4. *Suppose that $f(x)$ can be represented as a series of the form*

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{2n\pi x}{L}) + \sum_{n=1}^{\infty} B_n \sin(\frac{2n\pi x}{L}). \tag{5.9}$$

Then the coefficients are given by

$$A_0 = \frac{1}{L}\int_0^L f(x)dx \tag{5.10}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{2n\pi x}{L})dx \qquad n \geq 1 \tag{5.11}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{2n\pi x}{L})dx \tag{5.12}$$

The series in Equation (5.8) of Theorem 5.3.4 is called a Fourier series. The somewhat surprising fact is that under some very mild assumptions all periodic functions can be represented as a Fourier series.

Theorem 5.3.5. *Suppose that $f(x)$ is piecewise C^2 (twice differentiable) and L-periodic: $f(x + L) = f(x)$, with jump discontinuities at the points of discontinuity. Then the Fourier series*

$$A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{2n\pi x}{L}) + \sum_{n=1}^{\infty} B_n \sin(\frac{2n\pi x}{L})$$

converges to $f(x)$ at points of continuity of $f(x)$, and to $\frac{1}{2}\left(f(x^-)+f(x^+)\right)$ at the jump discontinuities.

Example 5.3.1 (Square Wave)**.** *Let $f(x)$ be defined to be*

$$f(x) = \begin{cases} 1 & x \in (0, \frac{1}{2}) \\ 0 & x \in (\frac{1}{2}, 1) \end{cases}$$

and repeated periodically (see the margin figure). Find the Fourier series for $f(x)$.

The period in this case is $L = 1$. The Fourier coefficients are easy to compute in this case, and are given as follows:

$$\begin{aligned} A_0 &= \int_0^1 f(x)dx = \int_0^{\frac{1}{2}} dx = \frac{1}{2} \\ A_n &= 2\int_0^{\frac{1}{2}} \cos(2n\pi x)dx = \frac{\sin(2\pi n x}{\pi n}|_0^{\frac{1}{2}} = 0 \\ B_n &= 2\int_0^{\frac{1}{2}} \sin(2n\pi x)dx = -\frac{\cos(2\pi n x}{\pi n}|_0^{\frac{1}{2}} = \frac{1-\cos(\pi n)}{\pi n} \end{aligned}$$

Given the fact that $\cos(\pi n) = 1$ if n is even and $\cos(\pi n) = -1$ if n is odd B_n can also be written as

$$B_n = \begin{cases} \frac{2}{\pi n} & n \;\; odd \\ 0 & n \;\; even \end{cases}$$

The margin figure depicts the square wave function $f(x)$ together with the first twenty-six terms of the Fourier series

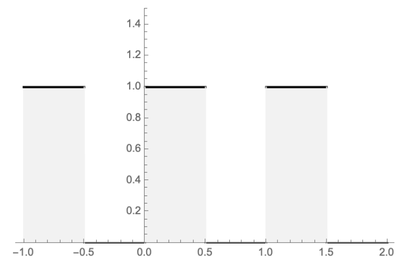


Figure 5.1: The square wave function.

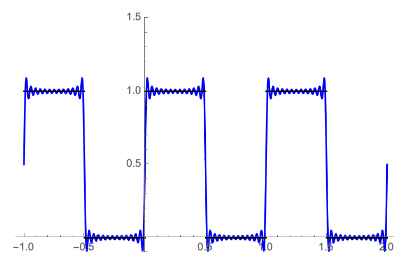


Figure 5.2: The square wave function together with the first twenty-six terms of the Fourier series.

Example 5.3.2. *Find the Fourier series for the function $f(x)$ defined as follows*

$$\begin{aligned} f(x) &= x(1-x) \qquad x \in (0,1) \\ f(x+1) &= f(x) \end{aligned}$$

The graph of this function on the interval $(-1,2)$ is presented in the side margin: the function continues periodically to the rest of the real line.

This is a Fourier series with $L = 1$ The Fourier coefficients can be calculated by integration by parts as

$$\begin{aligned} A_0 &= \frac{1}{L}\int_0^L f(x)dx = \int_0^1 x(1-x)dx = \frac{1}{6} \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos(\frac{2n\pi x}{L})dx = 2\int_0^1 x(1-x)\cos(\frac{2n\pi x}{L})dx = \\ &= 2(\frac{1}{4n^3\pi^3} - \frac{x(1-x)}{2n\pi})\sin(2n\pi x)|_0^1 + \frac{(1-2x)\cos(2n\pi x)}{4n^2\pi^2}|_0^1 \\ &= -\frac{1}{n^2\pi^2} \\ B_n &= 2\int_0^1 x(1-x)\sin(\frac{2n\pi x}{L})dx = 0. \end{aligned}$$

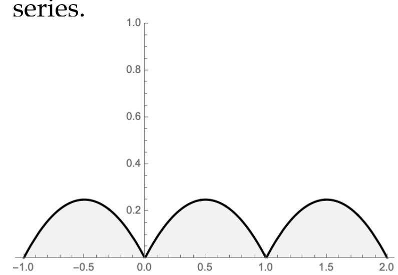


Figure 5.3: The function $f(x) = x(1-x)$ for $x \in (0,1)$ and extended periodically.

The easiest way to see that all coefficients B_n must be zero is to note that $\sin(2n\pi x)$ *is an odd function and the function $f(x)$ is an even function, so they must be orthogonal. This gives the series*

$$f(x) = \frac{1}{6} - \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^2\pi^2}$$

The margin figure shows a comparison between $f(x)$, (in black) the first five terms of the Fourier series (in red) , and the first twenty-five terms of the Fourier series (in blue). It is clear that even five terms of the Fourier series gives a good approximation to the original function (though with visible error), and that it is difficult to distinguish between the original function and the first twenty-five terms of the Fourier series– one can see a very small deviation between the two near $x = 0$ and $x = 1$, the endpoints. Note that since $f(x)$ is a piecewise C^2 continuous function the Fourier series converges to the function at all points.

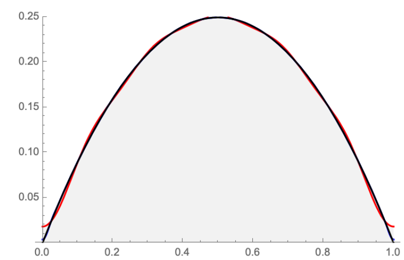


Figure 5.4: The function from Example 5.3.2 (Black) together with first 5 terms (Red) and first twenty five terms (Blue) of the Fourier series.

5.3.3 *Periodic, even and odd extensions*

It is common to have a function that is defined on a finite domain $[0, L]$ that we wish to extend to the whole line. There are a number of different ways to do this, and these are connected with different variations of the Fourier series and to different boundary conditions for partial differential equations. In this section we discuss three of the most common ways to extend a function.

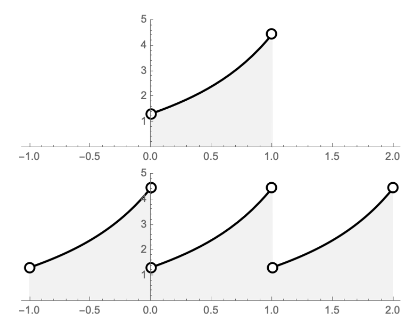


Figure 5.5: The function defined $x \in (0, 1)$ (top) and the same function extended periodically to the whole line (bottom).

5.3.4 *The periodic extension*

If we are given a function $f(x)$ defined on the domain $(0, L)$ the simplest way to extend is is periodically: we define $f(x)$ on the whole line by $f(x + L) = f(x)$. For instance if we want to define $f(x)$ in the interval $(L, 2L)$ we can use the equation $f(x + L) = f(x)$: if $x \in (0, L)$ then $x + L \in [L, 2L)$. Similarly we can use the equation $f(x + L) = f(x)$ again to relate values of the function in $[2L, 3L)$ to values of the function in $(L, 2L)$. Graphically the periodic extension consists of taking the graph of the function $f(x)$ in the interval $[0, L)$ are "repeating" it in the intervals $(-L, 0), (L, 2L), (2L, 3L), \ldots$. Notice that this procedure may introduce discontinuities in the graph if $\lim_{x\to 0^+} f(x) \neq \lim_{x\to L^-}(x)$.

This procedure is illustrated in the marginal figure. Depicted is a function defined on $(0, L)$, together with the function extended periodically to the whole line. Note that the resulting function is, by construction periodic with period L and even if the function $f(x)$ is continuous on $[0, L]$ the resulting periodic extension will typically have jump discontinuities at $x = nL$.

5.3.5 *The even extension*

Recall that a function is even if $f(-x) = f(x)$. An even function is invariant under reflection across the y-axis. More generally we say that $f(x)$ is "even about the point x_0" if $f(2x_0 - x) = f(x)$, with the case $x_0 = 0$ being the usual definition of even. Such a function is invariant under reflection across the line $x = x_0$. A second method of extending a function $f(x)$ defined for $x \in (0, L)$ to the whole line is to extend it so it is even across both boundaries, $x = 0$ and $x = L$. In other words given (x) defined for $x \in (0, L)$ we would define $f(x)$ for $x \in (-L, 0)$ by $f(-x) = f(x)$, we would define it for $x \in (L, 2L)$ by $f(2L - x) = f(x)$, etc. Graphically this amounts to taking the graph of $f(x)$ for $x \in (0, L)$ and repeatedly reflecting it across the boundary points $x = nL$. This is illustrated in the marginal figure for a typical function $f(x)$.

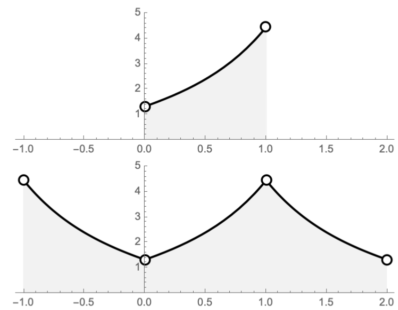


Figure 5.6: The function defined $x \in (0, 1)$ (top) and the even extension to the whole line (bottom).

There are a couple of things to notice here. First note that if the original function $f(x)$ is continuous on the closed interval $[0, L]$ then the even extension is also a continuous function. This differs from the periodic extension, since the periodic extension of a continuous function on $[0, L]$ won't be continuous unless $f(0) = f(L)$. There is, however, typically a jump in the derivative of the function across the boundary. Second notice that the resulting function is periodic, but the period is $2L$, not L. This is because the function is *reflected* across the boundaries. It requires *two* reflections to get back the original function.

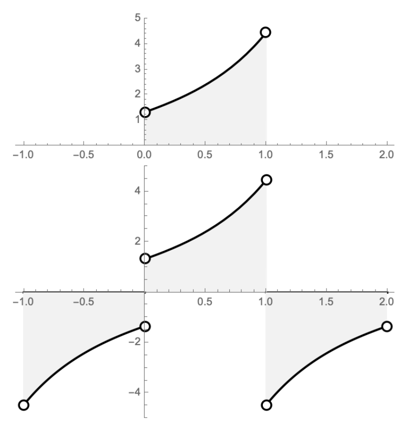


Figure 5.7: The function defined $x \in (0, 1)$ (top) and the odd extension to the whole line (bottom).

5.3.6 *The odd reflection*

The last case that we will discuss is the case of odd reflections. Recall that a function is odd if $f(-x) = -f(x)$, and that an odd function is invariant under reflection across the y-axis followed by reflection across the x-axis. More generally a function is "odd about the point x_0" if $f(2x_0 - x) = -f(x)$. A function is odd if it is invariant under reflection across the line $x = x_0$ followed by reflection across the x-axis. A second method of extending a function $f(x)$ defined for $x \in (0, L)$ to the whole line is to extend it so it is odd across both boundaries, $x = 0$ and $x = L$. In other words given $f(x)$ defined for $x \in (0, L)$ we would define $f(x)$ for $x \in (-L, 0)$ by $f(-x) = -f(x)$, we would define it for $x \in (L, 2L)$ by $f(2L - x) = -f(x)$, etc. Graphically this amounts to taking the graph of $f(x)$ for $x \in (0, L)$ and repeatedly doing odd reflections across the boundary points $x = nL$. This is illustrated in the marginal figure for a typical function $f(x)$.

Note that, as in the case of even reflections the resulting function is periodic with a period of $2L$, twice the width of the original domain. Again this is because it requires two reflections to get back the original function. Also note that the resulting function will typically have jump discontinuities unless the original function tends to zero at $x = 0$ and $x = L$.

5.3.7 *The Fourier cosine and sine series.*

The Fourier cosine and Fourier sine series are connected with the even and the odd extensions of a function defined on $(0, L)$ to the whole line. We'll begin with the even extension. We saw in the previous section that the even extension of a function defined on $(0, L)$ results in a function with a period of $2L$. Thus we can expand this function in a Fourier series of period $2L$. Because the extended function is even we expect that this series will only involve the cosine terms, since cosine is an even function. The formulas for the Fourier coefficients become

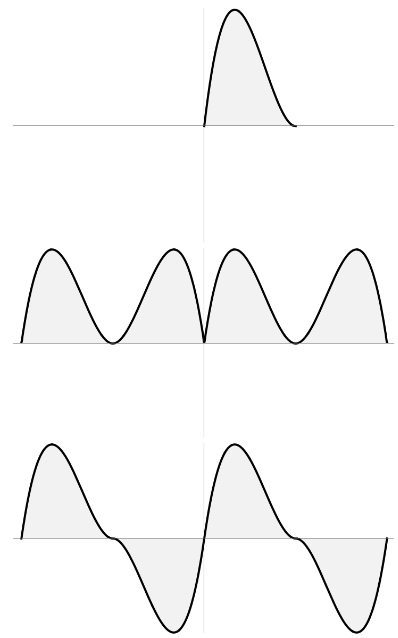


Figure 5.8: The function defined $x \in (0, L)$ (top) and Fourier cosine series (middle) and Fourier sine series (bottom).

$$
\begin{aligned}
A_0 &= \frac{1}{2L}\int_0^{2L} f(x)dx \\
A_k &= \frac{2}{2L}\int_0^{2L} f(x)\cos(\frac{2\pi kx}{2L})dx \\
B_k &= \frac{2}{2L}\int_0^{2L} f(x)\sin(\frac{2\pi kx}{2L})dx \\
f(x) &= A_0 + \sum_{k=1}^{\infty} A_k\cos(\frac{2\pi kx}{2L}) + B_k\sin(\frac{2\pi kx}{2L})
\end{aligned}
$$

Now the original function is defined only on $(0, L)$ so it would be preferable to express everything in terms of the function values in the interval $(0, L)$. For the A_k terms since $f(x)$ and $\cos(\frac{2\pi kx}{2L})$ are both even functions the integrals over $(0, L)$ and over $(L, 2L)$ are equal. For the B_k terms, on the other hand, $f(x)$ is even and $\sin((\frac{2\pi kx}{2L})$ is odd, so the integrals over $(0, L)$ and over $(L, 2L)$ are equal in magnitude and opposite in sign and cancel. This gives

$$A_0 = \frac{1}{L}\int_0^{L} f(x)dx \tag{5.13}$$

$$A_k = \frac{2}{L}\int_0^{L} f(x)\cos(\frac{\pi kx}{L})dx \tag{5.14}$$

$$f(x) = A_0 + \sum_{k=1}^{\infty} A_k\cos(\frac{\pi kx}{L}) \tag{5.15}$$

This is the Fourier cosine series for a function $f(x)$ defined on $(0, L)$.

The Fourier sine series is similar, but we make the odd extension of $f(x)$ across all boundaries. This again results in a function with period $2L$, which can be expanded in a Fourier series

$$\begin{aligned}
A_0 &= \frac{1}{2L}\int_0^{2L} f(x)dx \\
A_k &= \frac{2}{2L}\int_0^{2L} f(x)\cos(\frac{2\pi kx}{2L})dx \\
B_k &= \frac{2}{2L}\int_0^{2L} f(x)\sin(\frac{2\pi kx}{2L})dx \\
f(x) &= A_0 + \sum_{k=1}^{\infty} A_k\cos(\frac{2\pi kx}{2L}) + B_k\sin(\frac{2\pi kx}{2L}).
\end{aligned}$$

This time when we reduce the integrals to integrals over the interval $(0, L)$ we find that when computing A_k the contributions from the integration over $(0, L)$ and the integration over $(L, 2L)$ are equal in magnitude but opposite in sign, since $f(x)$ is odd and cos even. This means that all of the A_k terms are zero. For the B_k terms the contributions from the integration over $(0, L)$ and the integration over $(L, 2L)$ are the same, so the integral is just twice the integral over $(0, L)$. This results in

$$B_k = \frac{2}{L}\int_0^{L} f(x)\sin(\frac{\pi kx}{L})dx \tag{5.16}$$

$$f(x) = \sum_{k=1}^{\infty} B_k\sin(\frac{\pi kx}{L}). \tag{5.17}$$

This is the Fourier sine series.

The difference between the Fourier cosine and sine series is illustrated in the margin figure. The top graph depicts the original function, which is defined only on $(0, L)$. The middle figure depicts the Fourier cosine series, and the bottom the Fourier sine series. Note that all three functions agree in the original domain $(0, L)$ but they differ outside of this region.

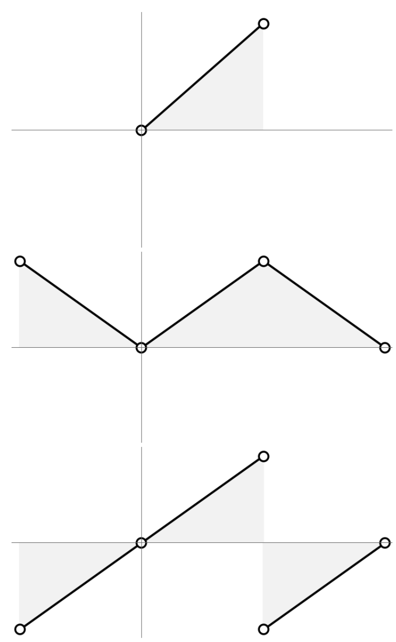


Figure 5.9: The function $f(x) = x$ defined for $x \in (0, 2)$ (top) together with the even (middle) and odd (bottom) extensions.

Example 5.3.3. *We consider the Fourier sine and cosine series for the function $f(x) = x$ on the interval $x \in (0, 2)$. The Fourier cosine series is given by*

$$\begin{aligned}
A_0 &= \frac{1}{2}\int_0^{2} xdx = 1 \\
A_k &= \frac{2}{2}\int_0^{2} x\cos(\frac{\pi kx}{2})dx = \frac{4}{\pi^2k^2}\left(\cos(\pi k) - 1\right) \\
f(x) &= A_0 + \sum_{k=1}^{\infty} A_k\cos(\frac{\pi kx}{2}) \\
&= 1 + \sum_{k=1}^{\infty} \frac{4}{\pi^2k^2}\left(\cos(\pi k) - 1\right)\cos(\frac{\pi kx}{2})
\end{aligned}$$

Note that the integral for A_k is easily done by parts. Similarly the Fourier sine series is given by

$$
\begin{aligned}
B_k &= \frac{2}{2}\int_0^2 x\sin(\frac{\pi k x}{L})dx \\
&= -\frac{4}{\pi k}\cos(\pi k) \\
f(x) &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\pi k}\sin(\frac{\pi k x}{L}).
\end{aligned}
$$

The Fourier cosine and Fourier sine series for $f(x) = x$ defined for $x \in (0,2)$ is depicted in margin figure 5.10. The graphs depict the first fifty terms of the cosine series (top) and sine series (bottom). The fifty term cosine series is essentially indistinguishable from the even extension of the function. The fifty term sine series is a good approximation but one can see a noticeable oscillation. This is the typical artifact associated with a jump discontinuity, and is known as Gibbs phenomenon. This same effect can sometimes be seen in image compression – the jpg algorithm uses a discrete cosine series to do compression. At high compression rates one can sometimes observe this ringing phenomenon in regions where the image has a sharp transition from light to dark.

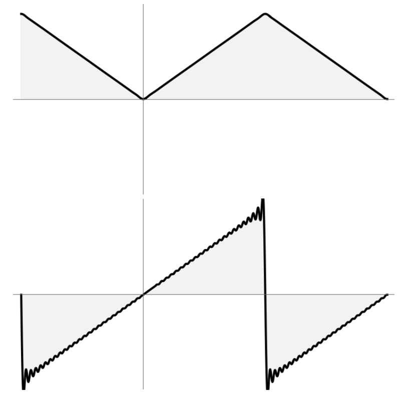


Figure 5.10: The first fifty terms of the Fourier cosine (top) and sine (bottom) series for function $f(x) = x$ defined for $x \in (0,2)$.

5.4 *Separation of variables and boundary value problems: The heat equation*

The heat equation or diffusion equation governs phenomenon such as the propagation of heat in a solid body or diffusion of a passive tracer (such as a dye) in a liquid. In three spatial dimensions the heat equation is the following partial differential equation.

$$\frac{\partial u}{\partial t} = \sigma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

Here σ is a constant known as the (thermal) diffusivity, and $u(x,y,z,t)$ represents the termperature at location (x,y,z) at time t or, in the case of diffusion, the concentration of the tracer at location (x,y,z) at time t. In one spatial dimension the heat equation would be

$$\frac{\partial u}{\partial t} = \sigma\frac{\partial^2 u}{\partial x^2}.$$

The one dimensional heat equation governs propagation of heat in a one-dimensional medium such as a thin rod.

Let's think a little about what this equation means. Imagine that at some time t the temperature profile is given by $u(x,t)$. If $u(x,t)$ has a local minimum (a cool spot) then $u_{xx} > 0$ and hence $u_t > 0$. Conversely if $u(x,t)$ has a local maximum, a hot spot, then $u_{xx} < 0$ and hence $u_t < 0$. So at any given time the hot spots will be getting cooler and the cold spots will be getting warmer. The heat equation is the mathematical expression of the idea that a body will tend to move towards thermal equilibrium.

In solving ordinary differential equations we need to specify initial conditions. For the heat equation we need to specify an initial temperature distribution, $u(x,0) = u_0(x)$, to specify what the initial distribution of temperature in the rod looks like. Finally when we are solving the heat equation we usually are doing so on some finite domain. We generally need to say something about what happens at the boundary of the domain. There are different types of boundary conditions. The two most important types are called Dirichlet and Neumann conditions. A Dirichlet condition consists of specifying $u(x,t)$ at the boundary. This would correspond to specifying the temperature at the boundary. We could imagine, for instance, putting one end of the rod in a furnace at a fixed temperature and studying how the rest of the rod heats up. The end of the rod in the furnace would satisfy a Dirichlet condition.

A Neumann condition means that u_x is specified at the boundary. Physically specifying u_x amounts to specifying the rate at which heat is entering or leaving the body through the boundary. A homogeneous Neumann boundary condition, $u_x = 0$, means that no heat is entering or leaving the body through this boundary. For instance the heat equation

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \sigma \frac{\partial^2 u}{\partial x^2} \\
u(x,0) &= 0 \\
u(L,t) &= T_0 \\
u_x(0,t) &= 0
\end{aligned}$$

describes a rod of length L. Initially the rod is at zero temperature ($u(x,0) = 0$). One end of the rod is maintained at a constant temperature T_0 ($u(L,t) = T_0$) while the other end of the rod is insulated so that no heat is gained or lost through the end ($u_x(0,t) = 0$).

5.4.1 The heat equation with Dirichlet boundary conditions.

In this section we consider the solution to the heat equation with Dirichlet boundary conditions

$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2} \tag{5.18}$$

$$u(x,0) = u_0(x) \tag{5.19}$$

$$u(0,t) = T_{\text{Left}} \tag{5.20}$$

$$u(L,t) = T_{\text{right}}. \tag{5.21}$$

Again this equation describes the evolution of the temperature as a function of x and t, $u(x,t)$, given that the initial temperature distribution in the rod is given by $u_0(x)$ an the ends are maintained at temperatures T_{Left} and T_{Right} respectively. The first step in solving this equation is to do a change of variables to make the boundary conditions homogeneous. The best way to think about this is as the "equilibrium solution". An equilibrium solution is, by definition, one that is not changing in time. Thus it should satisfy the following two point boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} = 0 \tag{5.22}$$

$$u(0,t) = T_{\text{Left}} \tag{5.23}$$

$$u(L,t) = T_{\text{right}}. \qquad . \tag{5.24}$$

Note that this is just Equations (5.18)–(5.21) with $u_t = 0$ (we are looking for an equilibrium, so it should not change in time) and the initial condition removed. This is easy to solve, as it is basically just an ODE. The solution is

$$u_{\text{equilibrium}}(x) = T_{\text{Left}} + \frac{T_{\text{Right}} - T_{\text{Left}}}{L} x.$$

If we now define the new function

$$v(x,t) = u(x,t) - u_{\text{equilibrium}}(x) = u(x,t) - \left(T_{\text{Left}} + \frac{T_{\text{Right}} - T_{\text{Left}}}{L} x \right)$$

then it is straightforward to see that the function $v(x,t)$ satisfies

$$\frac{\partial v}{\partial t} = \sigma \frac{\partial^2 v}{\partial x^2} \tag{5.25}$$

$$v(x,0) = u_0(x) - \left(T_{\text{Left}} + \frac{T_{\text{Right}} - t_{\text{Left}}}{L} x \right) \tag{5.26}$$

$$v(0,t) = 0 \tag{5.27}$$

$$v(L,t) = 0. \tag{5.28}$$

So it is enough to be able to solve the heat equation with homogeneous Dirichlet boundary conditions. We will do this by the method of separation of variables.

5.4.2 *Separation of Variables*

We begin with the heat equation with homogeneous Dirichlet boundary conditions

$$\frac{\partial v}{\partial t} = \sigma \frac{\partial^2 v}{\partial x^2} \tag{5.29}$$

$$v(0,t) = 0 \tag{5.30}$$

$$v(L,t) = 0. \tag{5.31}$$

We will incorporate the initial condition later. We are going to look for a solution of a special form. We look for a solution that can be written as a *product of a function of x and a function of t:*

$$v(x,t) = T(t)X(x).$$

Substituting this into equation 5.29 gives the equation

$$\begin{aligned} \frac{dT}{dt}(t)X(x) &= \sigma T(t)\frac{d^2X}{dx^2} \\ \frac{\frac{dT}{dt}(t)}{T(t)} &= \sigma\frac{\frac{d^2X}{dx^2}(x)}{X(x)}, \end{aligned}$$

where the second equation follows from dividing through by $X(x)T(t)$. Notice that at this point we have the variables separated: all of the t dependence is on one side of the equation and all of the x dependence is on the other side. At this point we make an important observation: if we have a function of t equal to a function of x then *both must be constant*. Considering the righthand side we have

$$\begin{aligned} \frac{\frac{d^2X}{dx^2}(x)}{X(x)} &= -\lambda, \\ \frac{d^2X}{dx^2}(x) &= -\lambda X(x) \end{aligned}$$

Here the minus sign is not necessary, we have included it simply for convenience. Now we would like to incorporate the boundary condition that $v = T(t)X(x)$ vanish at $x = 0$ and $x = L$. This implies that $X(x)$ must vanish at $x = 0$ and $x = L$. This gives us the two point boundary value problem

$$\begin{aligned} \frac{\frac{d^2X}{dx^2}(x)}{X(x)} &= -\lambda, \\ \frac{d^2X}{dx^2}(x) &= -\lambda X(x) \\ X(0) &= 0 \\ X(L) &= 0. \end{aligned}$$

This is an eigenvalue problem, one that we have already solved, in fact. The solution is

$$\begin{aligned} X(x) &= B\sin\left(\frac{n\pi x}{L}\right) \\ \lambda &= \frac{n^2\pi^2}{L^2}. \end{aligned}$$

This means that $T(t)$ solves

$$\frac{dT}{dt} = -\sigma\lambda T = -\sigma\frac{n^2\pi^2}{L^2}$$

which has the solution $T(t) = e^{-\sigma \frac{n^2\pi^2}{L^2} t}$. Thus the separated solution is

$$v(x,t) = Be^{-\sigma \frac{n^2\pi^2}{L^2} t} \sin(\frac{n\pi x}{L}).$$

Note that this is a solution for every integer n. We can use superposition to get a more general solution: since the sum of solutions is a solution we have that

$$v(x,t) = \sum_{n=1}^{\infty} B_n e^{-\sigma \frac{n^2\pi^2}{L^2} t} \sin(\frac{n\pi x}{L}) \tag{5.32}$$

is a solution.

To SUMMARIZE the function $v(x,t) = \sum_{n=1}^{\infty} B_n e^{-\sigma \frac{n^2\pi^2}{L^2} t} \sin(\frac{n\pi x}{L})$ satisfies the heat equation with homogeneous boundary conditions

$$\begin{aligned} v_t &= \sigma v_{xx} \\ v(0,t) &= 0 \\ v(L,t) &= 0. \end{aligned}$$

The only remaining condition is the initial condition $v(x,0) = v_0(x)$. Substituting $t = 0$ into equation (5.32) gives the condition

$$v(x,0) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L}) = v_0(x). \tag{5.33}$$

Note that we know how to solve this problem, and to compute the coefficients B_n so the equation (5.33) holds. This is a Fourier sine series, and thus we know that

$$B_n = \frac{2}{L} \int_0^L v_0(x) \sin(\frac{n\pi x}{L}) dx.$$

We will summarize these results in the form of a theorem.

Theorem 5.4.1. *The solution to the heat equation with homogeneous Dirichlet boundary conditions*

$$\begin{aligned} v_t &= \sigma v_{xx} \\ v(x,0) &= v_0(x) \\ v(0,t) &= 0 \\ v(L,t) &= 0 \end{aligned}$$

is given by the Fourier sine series

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L}) e^{-\sigma \frac{n^2\pi^2 t}{L^2}} \\ B_n &= \frac{2}{L} \int_0^L v_0(x) \sin(\frac{n\pi x}{L}) dx \end{aligned}$$

Example 5.4.1. *A steel rod is* 2 m *long. The thermal diffusivity of steel is about* $\sigma = 20\text{mm}^2\,\text{s}^{-1}$. *The rod is initially at temperature* $v(x,0) = 300$ °C *and the ends are cooled to* 0°C. *Find the temperature profile in the rod as a function of time. Assuming that heat is lost only through the ends of the rods how long until the maximum temperature in the rod is* 100°C?

The heat equation is

$$\begin{aligned} v_t &= \sigma v_{xx} \\ v(x,0) &= 300°\text{C} \qquad x \in (0,2) \\ v(0,t) &= 0°\text{C} \\ v(2\,,t) &= 0°\text{C} \end{aligned}$$

It is easiest to work consistently in meters: a thermal diffusivity of $\sigma = 20\text{mm}^2\,\text{s}^{-1}$ *is equal to* $2 \times 10^{-5}\ \text{m}^2\,\text{s}^{-1}$. *The Fourier coefficients are given by*

$$\begin{aligned} B_n &= \frac{2}{2\ m}\int_0^{2\ m} 300°\text{C}\sin(\frac{n\pi x}{2\ m})dx \\ &= -\frac{1}{1\ m}\frac{2\ m}{n\pi}\cos(\frac{n\pi x}{2\ m})|_0^{2\ m} \\ &= \frac{600°\text{C}}{n\pi}\left(1-\cos(n\pi)\right) \end{aligned}$$

and so the solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} \frac{600°\text{C}}{n\pi}\left(1-\cos(n\pi)\right) e^{-2\times 10^{-5}\text{m}^2\,\text{s}^{-1}\frac{n^2\pi^2}{4\ \text{m}^2}t}\sin(\frac{n\pi x}{2\ \text{m}}).$$

This can be simplified a bit if we notice that $1-\cos(\pi n)$ *is equal to* 0 *for n even and* 2 *for n odd, so the above expression is the same as*

$$v(x,t) = \sum_{n=1}^{\infty} \frac{1200°\text{C}}{(2n-1)\pi} e^{-2\times 10^{-5}\text{m}^2\,\text{s}^{-1}\frac{(2n-1)^2\pi^2}{4\ \text{m}^2}t}\sin(\frac{(2n-1)\pi x}{2\ \text{m}}).$$

It is clear that the maximum temperature in the rod will be at the center. Plotting the function

$$v(1\ m,t) = \sum_{n=1}^{\infty} \frac{1200°\text{C}}{(2n-1)\pi} e^{-2\times 10^{-5}\text{m}^2\,\text{s}^{-1}\frac{(2n-1)^2\pi^2}{4\ \text{m}^2}t}\sin((n-\frac{1}{2})\pi)$$

(Figure 5.11) we find the depicted graph of the temperature at the center of the rod. One can see that the temperature at the center of the rod first dips below 100°C *at time* $t \approx 20,000$s.

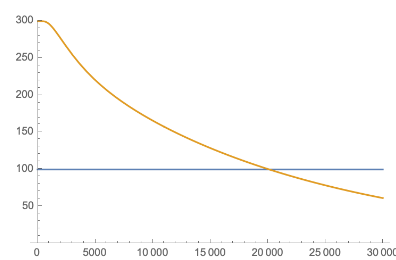


Figure 5.11: The temperature (in °C) at the center of the rod as a function of time.

5.4.3 *The heat equation with Neumann boundary conditions.*

In the previous section we saw that the heat equation with Dirichlet (zero temperature) boundary conditions could be solved by the method of separation of variables. The same basic technique will apply to many different kinds of boundary conditions. Perhaps the next most important type of boundary condition is Neumann, or zero flux boundary conditions. In the heat equation context a homogeneous (zero) Neumann boundary condition means that there is zero flux of heat through the boundary. In other words the boundary is insulated.

A typical heat problem with Neumann boundary conditions might be something like the following:

It is worth thinking about how the solution should behave. It the boundaries are insulated then after a long time the temperature should approach the average of the initial temperature. We will see later that this is true.

$$\begin{aligned}
v_t &= \sigma v_{xx} \\
v(x,0) &= v_0(x) \\
v_x(0,t) &= 0 \\
v_x(L,t) &= 0
\end{aligned}$$

We can solve this by the same method, separation of variables, that we used before. Again we begin by looking for a solution in the form of a product $v(x,t) = X(x)T(t)$. Substituting this in to the equation $v_t = v_{xx}$ gives

$$T'(t)X(x) = \sigma T(t)X''(x) \tag{5.34}$$

$$\frac{T'(t)}{T(t)} = \sigma \frac{X''(x)}{X(x)} \tag{5.35}$$

As in the previous section we notice that if we have a function of only t equal to a function of only x then they must both be constant. This tells us that $X''(x)/X(x) = -\lambda$. At this point we impose the boundary conditions, which gives us an eigenvalue problem for $X(x)$:

We have absorbed the factor of σ into our definition of the constant λ, as well as included the minus sign, for convenience.

$$\begin{aligned}
X''(x) &= -\lambda X(x) \\
X'(0) &= 0 \\
X'(L) &= 0
\end{aligned}$$

As before it is easiest to separate this into three cases: $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ depending on the nature of the roots of the characteristic equation. In the case $\lambda > 0$ the general solution is

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Taking the derivative and substituting $x = 0$ gives

$$X'(0) = B\sqrt{\lambda} = 0,$$

and since $\lambda > 0$ we must have that $B = 0$. Taking the derivative and substituting $x = L$ gives

$$X'(L) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}L) + B\sqrt{\lambda}\cos(\sqrt{\lambda}L) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0,$$

Here we have used the fact that $B = 0$. Once again since $\lambda > 0$ we must have that either $A = 0$ or $\sin(\sqrt{\lambda}L) = 0$. We are interested in finding a **non-zero** solution: if $A = 0$ then our solution is identically zero. So we require that $\sin(\sqrt{\lambda}L) = 0$. This implies that $\sqrt{\lambda}L = k\pi$ or $\lambda = \frac{k^2\pi^2}{L^2}$. This gives $X(x) = A\cos(\frac{k\pi x}{L})$. Substituting this back into Equation (5.35) we find that

$$\frac{T'(t)}{T(t)} = \sigma\frac{X''(x)}{X(x)} = -\sigma\frac{k^2\pi^2}{L^2}$$
$$T'(t) = -\sigma\frac{k^2\pi^2}{L^2}T(t)$$

So the separated solutions for $\lambda > 0$ are $T(t)X(x) = A_k e^{-\frac{\sigma k^2\pi^2 t}{L^2}}\cos(\frac{k\pi x}{L})$.

For $\lambda = 0$ we have $X''(x) = 0$ and so $X(x) = A + Bx$. Imposing the boundary conditions gives

$$X'(0) = B = 0$$
$$X'(L) = B = 0.$$

So there are no conditions on A. Thus $X(x) = A$ is a solution and the corresponding $T(t)$ solves

$$\frac{T'(t)}{T(t)} = \sigma\frac{X''(x)}{X(x)} = 0$$

and so $T(t)$ is also constant. Putting them together we have $T(t)X(x) = A_0$.

Finally for $\lambda < 0$ the general solution is

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{\sqrt{\lambda}x}.$$

It is easy to check that there are no eigenvalues in this case: the only solution satisfying $X'(0) = 0$ and $X'(L) = 0$ is $X(x) = 0$.

Now that we have found all of the separated solutions we can combine them: the equation is linear, so any linear combination of solutions is a solution. This gives us a more general solution

$$v(x,t) = A_0 + \sum A_k\cos(\frac{k\pi x}{L})e^{-\sigma\frac{k^2\pi^2 t}{L^2}}.$$

This function satisfies the partial differential equation $v_t = \sigma v_{xx}$ along with the boundary conditions $v(0,t) = 0$ and $v(L,t) = 0$. The only thing remaining is the initial condition:

$$v(x,0) = A_0 + \sum A_k\cos(\frac{k\pi x}{L}) = v_0(x).$$

This is a Fourier cosine series problem! We already know that

$$A_0 = \frac{1}{L}\int_0^L v_0(x)dx$$
$$A_k = \frac{2}{L}\int_0^L v_0(x)\cos(\frac{k\pi x}{L})dx.$$

We state this as a theorem

Note the behavior when the time t gets large. The exponential terms decay to zero, and all that remains is the constant term. So for long times $t \gg 1$ we have that $v(x,t) \approx A_0 = \frac{1}{L}v_0(x)dx$ – the temperature profile tends to the average of the initial temperature profile. This is exactly what we would expect.

Theorem 5.4.2. *The solution to the heat equation with Neumann boundary conditions*

$$\begin{aligned} v_t &= \sigma v_{xx} \\ v(x,0) &= v_0(x) \\ v(0,t) &= 0 \\ v(L,t) &= 0 \end{aligned}$$

is given by

$$\begin{aligned} v(x,t) &= A_0 + \sum A_k \cos(\frac{k\pi x}{L}) e^{-\sigma \frac{k^2\pi^2 t}{L^2}} \\ A_0 &= \frac{1}{L}\int_0^L v_0(x)dx \\ A_k &= \frac{2}{L}\int_0^L v_0(x)\cos(\frac{k\pi x}{L})dx. \end{aligned}$$

6
Review: Complex numbers and the Euler formula

In this we review some background material on complex numbers and the Euler formula. Recall that the imaginary number i is defined as $i = \sqrt{-1}$ or $i^2 = -1$. A complex number is defined to be a number of the form $a + bi$ where a and b are real numbers. Complex numbers are important for a number of reasons. The first is the fundamental theorem of algebra:

The concept of multiplicity is important. The multiplicity of a root x_i of a polynomial $P(x)$ is the largest power k such that $(x - x_i)^k$ divides $P(x)$. For instance the polynomial $P(x) = x^3 - 3x + 2$ has three roots, $x = -2$ has multiplicity one and $x = 1$ has multiplicity two, since $x^3 - 3x + 2 = (x-1)^2(x+2)$. We sometimes call roots of multiplicity one "simple".

Theorem 6.0.1. *Let $P_n(x)$ be a polynomial of degree n. Then $P_n(x)$ has exactly n roots counted according to multiplicity: there are n complex numbers $x_1, x_2, \ldots x_n$ (not necessarily distinct) such that $P_n(x) = A(x - x_1)(x - x_2)\ldots(x - x_n)$.*

The fundamental theorem of algebra requires that we work over the complex numbers: there are polynomials with no real roots.

Example 6.0.1. *Find the roots of $P(x) = x^4 - 1$.*

Using the identity $(a^2 - b^2) = (a-b)(a+b)$ we have that $p(x) = (x^4-1) = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$. The product is zero if and only if one of the factors is zero. The first factor is zero if $x - 1 = 0$ or $x = 1$. The second is zero if $(x+1) = 0$ or $x = -1$. The third factor is zero if $x^2 + 1 = 0$ or $x = \pm\sqrt{-1} = \pm i$. Thus the four roots are $x = 1, x = -1, x = i, x = -i$.

Of course we need to be able to do algebra with complex numbers. In particular we need to be able to add, subtract, multiply and divide them. Addition and subtraction are defined component-wise:

$$(a+ib)+(c+id) = (a+c)+i(b+d) \qquad (a+ib)-(c+id) = (a-c)+i(b-d)$$

Multiplication is not defined pointwise, but rather it is defined to be consistent with the distributive property: we FOIL it out

$$(a+ib)(c+id) = ac + iad + ibc + (ib)(id) = ac + iad + ibc - bd = (ac-bd) + i(ad+bc).$$

Division is a bit harder to define. We first define the complex conjugate. If $z = a + ib$ is a complex number then $\bar{z}$ (sometimes denoted

z^* is defined to be $\bar{z} = a - ib$. Note that $z\bar{z} = (a+ib)(a-ib) = a^2 - (ib)^2 = a^2 + b^2 \geq 0$. Now we can define division by

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id}\frac{c-id}{c-id} = \frac{(a+ib)(c-id)}{c^2+d^2}.$$

For instance to compute $\frac{2+3i}{5+7i} = \frac{2+3i}{5+7i}\frac{5-7i}{5-7i} = \frac{31+i}{25+49} = \frac{31}{74} + \frac{1}{74}i$. The modulus of a complex number, denoted $|z| = |a+ib|$ is defined by

$$|z| = |a+ib| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}.$$

There are a couple of important geometric interpretations of complex numbers. First is the complex plane

Definition 6.0.1 (Complex Plane)**.** *Given a complex number $z = a + ib$ we can identify this complex number with the point (a, b) in the $x - y$ plane.*

Under this definition complex numbers add in the way that vectors normally do: they add component-wise. We will give an interpretation to multiplication shortly, but first we introduce the polar representation. First we need to recall the Euler formula, which tells us how to exponentiate complex numbers.

Theorem 6.0.2 (Euler)**.** *The complex exponential is defined by*

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

More generally if $z = a + ib$ then we have

$$e^z = e^{a+ib} = e^a(\cos(b) + i\sin(b)).$$

The Euler formula links the transcendental functions e^x, $\cos x$ and $\sin x$ through the imaginary number i. A second way to represent complex numbers is through the polar representation. This is essentially polar coordinates on the $x - y$ plane.

Definition 6.0.2. *Given a complex number $z = a + ib$ we can represent it in polar form as*

$$z = re^{i\theta}$$

The quantities r and θ, referred to as the modulus and the argument, are defined by

$$r = \sqrt{a^2+b^2} = |z| \qquad \tan(\theta) = \frac{b}{a}$$
$$a = r\cos\theta \qquad b = r\sin\theta$$

Going from (r, θ) to (a, b) is straightforward. Going from (a, b) to (r, θ) is straightforward **EXCEPT** *that one must be careful to choose the correct branch of the arctangent. There will be two solutions that differ by π, and you must be careful to choose the correct one. Note that θ is only defined up to multiples of 2π.*

Example 6.0.2. *Express the complex number $-5 - 5i$ in polar form.*

The modulus is given by $r = \sqrt{(-5)^2 + (-5)^2} = 5\sqrt{2}$. The argument θ is defined by

$$\tan\theta = \frac{-5}{-5} = 1$$

Here we have to be careful: there are two angles θ such that $\tan(\theta) = 1$, $\theta = \frac{\pi}{4}$, in the first quadrant and $\theta = \frac{5\pi}{4}$, in the third quadrant. The original point is $-5 - 5i = (-5, -5)$ in the third quadrant, so

$$(-5 - 5i) = 5\sqrt{2}e^{\frac{5\pi}{2}i}$$

The argument $\arg(z) = \theta$ and the modulus $|z|$ have some nice properties under multiplication:

Lemma 6.0.1. *If z_1 and z_2 are complex numbers and $\arg(z_i) = \theta_i$ is the argument or angle in the polar representation then*

$$|z_1 z_2| = |z_1||z_2| \qquad \textit{and} \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

in the second equation we have the understanding that the argument is only defined up to multiples of 2π. In other words when we multiply complex numbers the absolute values multiply and the angles add.

Bibliography